EQUIVARIANT OPERADS, SYMMETRIC SEQUENCES, AND BOARDMAN-VOGT TENSOR PRODUCTS

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Abstract. Let Op_G be Nardin-Shah's ∞-category of $\mathcal{O}_G$-∞-operads (henceforth G-operads). We construct the *underlying G-symmetric sequence* of a (one color) G-operad, yielding a monadic functor; we use this to lift Bonventre's genuine operadic nerve to a conservative functor of ∞-categories, restricting to an equivalence between categories of discrete G-operads.

We then go on to define and characterize closed *Boardman-Vogt tensor product* $\overset{\mathrm{BV}}{\otimes}$ on Op_G; in particular, this specializes to a G-symmetric monoidal ∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebras in a G-symmetric monoidal ∞-category $\mathcal{C}$. We show that the category of G-symmetric monoidal ∞-categories possesses a canonical symmetric monoidal structure whose tensor products are compatible with the Boardman-Vogt tensor product via the G-symmetric monoidal envelope.

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Introduction

Fix G a finite group. Within the burgeoning study of algebraic structures in G-equivariant homotopy theory, relatively little is known about *G-operads*. In this paper, we use ∞-categorical foundations to advance the study of G-operads in several ways. This concerns structural statements both about Nardin-Shah's

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∞-category of (colored) $\mathcal{O}_G$-∞-operads Op_G (henceforth just G-operads) and about the ∞-categories of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebras for various examples of interest.[1]

Our first contribution concerns generalizing the theory of G-symmetric monoidal ∞-categories to *I-symmetric monoidal ∞-categories*, for I a *weak* indexing category in the sense of [Ste24b]; these posses indexed tensor products over a collection of arities only under the assumptions that they can be restricted and composed. We go on to generalize G-operads to *I-operads*, which occur as a fully faithful subcategory $\mathrm{Op}_I \subset \mathrm{Op}_G$ with a terminal object $\mathcal{N}_{I\infty}^{\otimes}$, which we refer to as a *weak $\mathcal{N}_\infty$-operad*; in particular, an I-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ has an underlying (colored) I-operad of the same name, and $\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$ correspond with maps of G-operads $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$.

Additionally, we define a monadic functor

$$\mathrm{sseq}\colon \mathrm{Op}_G^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathcal{S}),$$

the former being the *one-colored* G-operads and the latter being the *∞-category of G-symmetric sequences*. The objects of $\mathrm{Tot}\underline{\Sigma}_G$ are identified with pairs (H,S) where $H \subset G$ is a subgroup and $S \in \mathbb{F}_H$ is a finite H-set; given this data, we write $\mathcal{O}(S) := \mathrm{sseq}\,\mathcal{O}^{\otimes}(S)$, which we call the *$S$-ary structure space of $\mathcal{O}^{\otimes}$*. This intertwines with Bonventre's genuine operadic nerve, so the nerve lifts to a conservative functor of ∞-categories.

We use this data to characterize the compatible $(d+1)$-categories of *G-symmetric monoidal d-categories* and *G-d-operads*: a G-operad $\mathcal{O}^{\otimes}$ is a *G-d-operad* if the S-ary structure space $\mathcal{O}(S)$ is $(d-1)$-truncated for all subgroups $H \subset G$ and finite H-sets $S \in \mathbb{F}_H$. These form a localizing subcategory, with localization functor $h_d\colon \mathrm{Op}_G \to \mathrm{Op}_{G,d}$. When $d \le 1$, we show that the inclusion of d-operads intertwines with Bonventre's nerve.

Having done this, we define a homotopy-commutative tensor product on Op_G called the *Boardman-Vogt tensor product*. We show that this tensor product is *closed*, i.e. it has an associated *(colored) G-operad of algebras* $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$. When $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category, we show that $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ underlies an I-symmetric monoidal ∞-category, which we give the same name; in particular, $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is an I-symmetric monoidal ∞-category whose $\mathcal{P}$-algebras are characterized by the formula

$$\mathrm{Alg}_{\mathcal{P}}\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{P}\otimes\mathcal{O}}(\mathcal{C}).$$

We thus interpret $\mathcal{P}\otimes\mathcal{O}$-algebras as *homotopy coherently interchanging pairs of $\mathcal{P}$-algebras and $\mathcal{O}$-algebras*; indeed we give a "bifunctor" presentation generalizing [HA, § 2.2.5.3].

We end by developing an "inflation and fixed points" adjunction $\mathrm{Infl}_e^G\mathrm{Op} \rightleftarrows \mathrm{Op}_G\colon \Gamma^G$ and showing that it is compatible with Boardman-Vogt tensor products.

We now move on to a more careful accounting of the background and main results of this paper.

Background and motivation. Let $\mathcal{C}$ be a semiadditive category, i.e. a pointed 1-category whose *norm* map $X \sqcup Y \to X \times Y$ is an isomorphism for all $X, Y \in \mathcal{C}$. Let G be a finite group and let $\mathcal{O}_G$ be the orbit category of G. Recall that a *semi Mackey functor* valued in $\mathcal{C}$ is the data of:

- a contravariant functor $R\colon \mathcal{O}_G^{\mathrm{op}} \to \mathcal{C}$, and
- a covariant functor $N\colon \mathcal{O}_G \to \mathcal{C}$

subject to the conditions that

(a) for all $H \subset G$, the values $R([G/H])$ and $N([G/H])$ are isomorphic, and
(b) writing $R_K^H\colon R([G/H]) \to R([G/K])$ for the contravariant functoriality and $N_K^H\colon N([G/K]) \to N([G/H])$ for the covariant functoriality, R and N satisfy the *double coset formula*

$$R_J^H N_K^H(-) \simeq \sum_{g \in J\backslash H/K} N_{H\cap gKg^{-1}}^H \operatorname{Res}_K^H(-)_g$$

where $(-)_g$ denotes the covariant conjugation action and $J\backslash G/K$ is the set of *double cosets*.

[1] In this paper we will call ∞-categories *∞-categories* and ∞-categories with discrete mapping spaces *1-categories*, as their theory is equivalent to the traditional theory of categories. More generally, we will call ∞-categories whose mapping spaces are $(d-1)$-truncated *d-categories*.

Let $\mathsf{Span}(\mathbb{F}_G)$ be the effective Burnside 1-category, whose objects are finite G-sets, whose morphisms $R_{XY}\colon X \to Y$ are given by spans $X \leftarrow R_{XY} \to Y$, and whose composition is given by pullback of spans

$$\begin{tikzcd}[column sep=small] & & R_{XZ} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & R_{XY} \arrow[dl] \arrow[dr] & & R_{YZ} \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd}$$

It is an observation due to Lindner [Lin76] that (semi)-Mackey functors valued in $\mathcal{C}$ are equivalently given by biproduct preserving functors

$$\mathsf{Span}(\mathbb{F}_G) \to \mathcal{C}.$$

This appears as a straightforward generalization of the Lawvere theory $\mathsf{Span}(\mathbb{F})$ for commutative monoids, so we will refer to semi-Mackey functors as *G-commutative monoids*.

Moreover, *any* $\mathcal{C}$ admits a universal map from a semiadditive category, given by the forgetful functor $U\colon \mathcal{C} \to \mathsf{CMon}(\mathcal{C})$; since $\mathsf{Span}(\mathbb{F}_G)$ possesses an identity-on-objects anti-involution, it is semiadditive, and so U induces an equivalence

$$\mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}_G), \mathsf{CMon}(\mathcal{C})) \xrightarrow{\sim} \mathsf{Fun}^{\oplus}(\mathsf{Span}(\mathbb{F}_G), \mathcal{C});$$

in fact, replacing $\mathsf{Span}(\mathbb{F}_G)$ with the effective Burnside *2*-category of [Bar14] (whose 2-cells are isomorphisms of spans), $\mathcal{C}$ with an ∞-category, and interpreting $\mathsf{CMon}(\mathcal{C})$ as $\mathbb{E}_\infty$-monoids in $\mathcal{C}$, the semiadditivization result for $\mathsf{CMon}(\mathcal{C})$ still holds [GGN15], and $\mathsf{Span}(\mathbb{F}_G)$ is still semiadditive. Thus we are justified in making the following definition.

Definition. The *∞-category of G-commutative monoids in $\mathcal{C}$* is the product-preserving functor ∞-category

$$\mathsf{CMon}_G(\mathcal{C}) := \mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}_G), \mathcal{C});$$

the *∞-category of small G-symmetric monoidal ∞-categories* is

$$\mathsf{Cat}_G^{\otimes} := \mathsf{CMon}_G(\mathsf{Cat}). \qquad \triangleleft$$

This recovers the notion of [NS22], which generalizes the notion of [HH16]. Recall that, we define G-∞-categories to be categorical coefficient systems

$$\mathsf{Cat}_G := \mathsf{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathcal{C}\right);$$

the $[G/H]$-value of a G-∞-category $\mathcal{C}$ will be written $\mathcal{C}$, and the contravariant functoriality along $[G/K] \to [G/H]$ will be written $\mathrm{Res}_K^H\colon \mathcal{C}_H \to \mathcal{C}_K$. G-symmetric monoidal ∞-categories $\mathcal{C}^{\otimes}$ have underlying G-∞-categories $\mathcal{C}$ defined by the precomposition

$$\mathcal{C}\colon \mathcal{O}_G^{\mathrm{op}} \to \mathsf{Span}(\mathbb{F}_G) \xrightarrow{\mathcal{C}^{\otimes}} \mathsf{Cat}.$$

Given a subgroup $H \subset G$ and a finite H-set S, we will write the value of $\mathcal{C}^{\otimes}$ on $\mathrm{Ind}_H^G S$ as $\mathcal{C}_S$, noting that there is a canonical equivalence $\mathcal{C}_S \simeq \prod_{[H/K] \in \mathrm{Orb}(S)} \mathcal{C}_K$.

We may induce the canonical map of H-sets $S \to *_H$ to G to construct a structure map $\mathrm{Ind}_H^G S \to [G/H]$,[2] and covariant functoriality yields a natural *S-indexed tensor product* operation

$$\bigotimes^S\colon \mathcal{D}_S \to \mathcal{D}_H.$$

We may induce the *orbit set* factorization $S \to \coprod_{[H/K] \in \mathrm{Orb}(S)} *_H \to *_H$ to yield a natural equivalence

$$\bigotimes_K^S X_K \simeq \bigotimes_{[H/K] \in \mathrm{Orb}(S)} N_K^H X_K.$$

Similarly, contravariant functoriality yields an *S-indexed diagonal* $\Delta^S\colon \mathcal{D}_H \to \mathcal{D}_S$ satisfying

$$\Delta^S X \simeq \left(\mathrm{Res}_K^H X\right)_{[H/K] \in \mathrm{Orb}(S)}.$$

[2] See [Die09] for a discussion of induced G-sets.

This allows us to define S*-indexed tensor power* of an object $X_H \in \mathcal{D}_H$ by

$$X_H^{\otimes S} := \bigotimes^S \Delta^S X_H \simeq \bigotimes_K^S \mathrm{Res}_K^H X_H \simeq \bigotimes_{[H/K]\in \mathrm{Orb}(S)} N_K^H \mathrm{Res}_K^H X_H.$$

Akin to the discrete case, these satisfy a double coset formula by functoriality under the composite span

$$\begin{tikzcd}[column sep=small] & & \coprod_{g\in[J\backslash H/K]} G/(K\cap gJg^{-1}) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\ & G/J \arrow[dl, equal] \arrow[dr] & {} & G/K \arrow[dl] \arrow[dr, equal] & \\ G/J & & G/H & & G/K \end{tikzcd}$$

We are concerned with algebraic structures *inside* G-symmetric monoidal ∞-categories, which we will control with a version of Nardin-Shah's ∞-category Op_G of $\mathcal{O}_G$-∞-operads, which we simply call G*-operads*. Work of Barkan, Haugseng, and Steinebrunner [BHS22] identifies these with functors of ∞-categories $\pi_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathsf{Span}(\mathbb{F}_G)$ with cocartesian lifts over backwards maps and satisfying a pair Segal conditions, which we may summarize in two cases of interest:

(1) in the case that $\mathcal{O}^{\otimes}$ additionally has $\pi_{\mathcal{O}}$-cocartesian lifts over forward maps, $\mathcal{O}^{\otimes}$ is a G-operad if and only if it is is the unstraightening of a G-symmetric monoidal ∞-category;
(2) in the case that the fibers $\pi_{\mathcal{O}}^{-1}(S)$ are contractible for all $S \in \mathbb{F}_G$ (i.e. $\mathcal{O}^{\otimes}$ *has one color*), cocartesian lifts over the backwards maps $(S \leftarrow [G/H] = [G/H])_{[G/H]\in \mathrm{Orb}(S)}$ furnish an equivalence

$$\mathrm{Map}_{\pi_{\mathcal{O}}}^{T\to S}(iT, iS) \simeq \prod_{[G/H]\in \mathrm{Orb}(S)} \mathrm{Map}_{\pi_{\mathcal{O}}}^{T_H \to [G/H]}(iT_H, i[G/H]),$$

where we set $T_H := T \times_S [G/H]$ and we write iS for the unique object of $\pi_{\mathcal{O}}^{-1}(S)$.[3]

These span a localizing subcategory [BHS22, Cor 4.2.3].

$$\begin{tikzcd} \mathrm{Op}_G \arrow[r, phantom, "\perp"] \arrow[r, bend right, hook] & \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_G)} \arrow[l, bend right, "L_{\mathrm{Op}_G}"'] \end{tikzcd} \tag{1}$$

Given $\mathcal{O}^{\otimes}$ a one-color G-operad, $H \subset G$ a subgroup and $S \in \mathbb{F}_H$ a finite H-set, we write

$$\mathcal{O}(S) := \mathrm{Map}_{\pi_{\mathcal{O}}}^{\mathrm{Ind}_H^G S \to [G/H]}(i\mathrm{Ind}_H^G S, i[G/H])$$

for the S*-ary structure space of* $\mathcal{O}^{\otimes}$. An $\mathcal{O}$-algebra in $\mathcal{C}^{\otimes}$ is defined to be a map of G-operads $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$; these posses an underlying G-object $X_\bullet$ (i.e. cocartesian section of $\mathcal{C} \to \mathcal{O}_G^{\mathrm{op}}$)) together with action maps

$$\mathcal{O}(S) \to \mathrm{Map}_{\mathcal{C}_H}\left(X_H^{\otimes S}, X_H\right)$$

which are suitably functorial and compatible with cocartesian lifts of backwards maps. In fact, as in [NS22], we may lift these to a G*-∞-category* $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$ whose H-value consists of algebras over the restricted H*-operad*:

$$\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})_H \simeq \mathrm{Alg}_{\mathrm{Res}_H^G \mathcal{O}}(\mathrm{Res}_H^G \mathcal{C}).$$

Summary of main results. Write $\underline{\Sigma}_G$ for the G-space core of the G-∞-category of finite G-sets $\underline{\mathbb{F}}_G$; write $\mathrm{Tot}\colon \mathsf{Cat}_G \to \mathsf{Cat}$ for the functor taking a G-∞-category to the total ∞-category of its corresponding cocartesian fibration. We identify objects with $\mathrm{Tot}\,\underline{\Sigma}_G$ with pairs (H,S) where $(H) \subset G$ is a conjugacy class and $S \in \mathbb{F}_H$ is a finitie H-set.

Theorem A. *There exists a monadic functor*

$$\mathrm{sseq}\colon \mathrm{Op}_G^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_G, \mathcal{S})$$

whose composite functor $\mathrm{Op}_G \xrightarrow{\mathrm{sseq}} \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_G, \mathcal{S}) \xrightarrow{\mathrm{ev}_{(H,S)}} \mathcal{S}$ *recovers* $\mathcal{O}(S)$.

[3] Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, and $\psi\colon FX \to FY$ a map in $\mathcal{D}$, we write $\mathrm{Map}_F^{\psi}(X,Y) \subset \mathrm{Map}_{\mathcal{C}}(X,Y)$ for the disjoint union of the connected components consisting of maps $\varphi\colon X \to Y$ such that $F\varphi$ is homotopic to ψ.

In parallel, Bonventre-Pereira developed a model category $s\mathrm{Op}_G$ of *colored genuine G-operads*, and the one-color variant $s\mathrm{Op}_{G,*}$ is right-transferred along a monadic *underlying G-symmetric sequence* functor $U\colon s\mathrm{Op}_{G,*_G} \xrightarrow{\mathrm{Fun}} (\mathrm{Tot}\,\underline{\Sigma}_G, \mathrm{sSet}_{\mathrm{Quillen}})$ [BP21, Thm II].[4] We refer to the associated ∞-categories as

$$g\mathrm{Op}_G \coloneqq s\mathrm{Op}_G[\mathrm{weq}^{-1}]; \qquad\qquad g\mathrm{Op}_{G,*} \coloneqq s\mathrm{Op}_{G,*}[\mathrm{weq}^{-1}].$$

Unwinding definitions, we will see that $\mathbf{sseq}$ is total right derived from a functor of 1-categories out of Nardin-Shah's model structure [NS22] which preserves and reflects weak equivalences between fibrant objects, and Bonventre's *genuine operadic nerve* $N^{\otimes}$ satisfies $\mathcal{P}(S) \simeq (N^{\otimes}\mathcal{O})(S)$. We conclude by two-out-of-three that $N^{\otimes}$ preserves and refelcts weak equivalences between fibrant objects. In Section 2.6 we extend this to the multiple-color setting, yielding the following.

Corollary B. *Bonventre's genuine operadic nerve $N^{\otimes}$ possesses a conservative total right derived functor of ∞-categories*

$$N^{\otimes}\colon g\mathrm{Op}_G \to \mathrm{Op}_G;$$

when $\mathcal{O}$ is a one color genuine G-operad, this satisfies $\mathcal{O}(S) \simeq (N^{\otimes}\mathcal{O})(S)$.

Moreover, in Section 2.6, we will see that this restricts to an equivalence between their respective full subcategories of G-operads with discrete structure spaces.

Having done this, we move on to develop a notion of *equivariant homotopy-coherent interchange* via the *Boardman-Vogt tensor product*

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \coloneqq L_{\mathrm{Op}}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathrm{Span}(\mathbb{F}_G) \times \mathrm{Span}(\mathbb{F}_G) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_G)\right).$$

where L_{Op_G} is as in Eq. (1). We verify many basic properties of this.

Theorem C. *The bifunctor $\overset{BV}{\otimes}\colon \mathrm{Op}_G \times \mathrm{Op}_G \to \mathrm{Op}_G$ enjoys the following properties.*

(1) In the case $G = e$ is the trivial group, $\overset{BV}{\otimes}$ is naturally equivalent to the Boardman-Vogt tensor product of [HM23; HA].

(2) The functor $- \overset{BV}{\otimes} \mathcal{O}\colon \mathrm{Op}_G \to \mathrm{Op}_G$ possesses a right adjoint $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(-)$, whose underlying G-∞-category is the G-∞-category of algebras $\underline{\mathrm{Alg}}_{\mathcal{O}}(-)$; the associated ∞-category is the ∞-category of algebras $\mathrm{Alg}_{\mathcal{O}}(-)$.

(3) The $\overset{BV}{\otimes}$-unit of Op_G is the G-operad $\mathrm{triv}^{\otimes}_G$ of [NS22]; hence $\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_G}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$.

(4) When $\mathcal{C}^{\otimes}$ is a G-symmetric monoidal ∞-category, $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is a G-symmteric monoidal ∞-category; furthermore, when $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a map of G-operads, the pullback lax G-symmetric monoidal functor

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$$

is G-symmetric monoidal; in particular, if $\mathcal{O}^{\otimes}$ has one object, then pullback along the unique map $\mathrm{triv}^{\otimes}_G \to \mathcal{P}^{\otimes}$ presents the unique natural transformation of operads

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \to \mathcal{C}^{\otimes},$$

and this is G-symmetric monoidal when $\mathcal{C}$ is G-symmetric monoidal.

(5) When $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a G-symmetric monoidal functor, the induced lax G-symmetric monoidal functor

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D})$$

is G-symmetric monoidal.

(6) The adjunction $\mathrm{Infl}^G_e\colon \mathrm{Op} \rightleftarrows \mathrm{Op}_G\colon \Gamma^G$ enjoys the following (natural) equivalences:

$$\mathrm{Infl}^G_e \mathrm{triv}^{\otimes} \simeq \mathrm{triv}^{\otimes}_G;$$

$$\Gamma^G \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}^{\mathcal{T}}_e \mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}^{\otimes}_{\mathcal{O}}\left(\Gamma^G \mathcal{C}\right);$$

$$\mathrm{Infl}^G_e(\mathcal{O}) \overset{BV}{\otimes} \mathrm{Infl}^G_e(\mathcal{P}) \simeq \mathrm{Infl}^G_e(\mathcal{O} \otimes \mathcal{P}).$$

[4] When we say a model category $\mathcal{C}$ is *right-transferred along* $F\colon \mathcal{C} \to \mathcal{D}$, we mean that F preserves and reflects weak equivalences and fibrations.

Hence, writing $\mathbb{E}_n$ for the little n_G-disks G-operad,[5] *the maps $\mathbb{E}_n, \mathbb{E}_m \to \mathbb{E}_{n+m}$ induce an equivalence*

$$\mathbb{E}_n^{\otimes} \overset{BV}{\otimes} \mathbb{E}_m^{\otimes} \xrightarrow{\sim} \mathbb{E}_{n+m}$$

(7) The G-symmetric monoidal envelope of [BHS22; NS22] intertwines localized Day convolution with Boardman-Vogt tensor products, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathrm{Op}_G^2 \arrow[rr, "\overset{BV}{\otimes}"] \arrow[d, "\mathrm{Env}^2"] & & \mathrm{Op}_G \arrow[d, "\mathrm{Env}"] \\
\mathrm{Fun}(\mathrm{Span}(\mathbb{F}_G), \mathrm{Cat})^2 \arrow[r, "\circledast"] & \mathrm{Fun}(\mathrm{Span}(\mathbb{F}_G), \mathrm{Cat}) \arrow[r, "L_{\mathrm{Seg}}"] & \mathrm{Cat}_G^{\otimes}
\end{tikzcd}$$

References. Statement (1) is Corollary 3.13. Statement (2) is Observation 2.52, Proposition 3.6, and Corollary 3.19. Statement (3) is Proposition 3.16. Statements (4) and (5) are Corollary 3.11. Statement (6) is Propositions 3.25 and 3.28 and Corollaries 3.26 and 3.27. Statement (7) is Proposition 3.9. □

Notation and conventions. We assume that the reader is familiar with the technology of higher category theory and higher algebra as developed in [HTT] and [HA, § 2-3], though we encourage the reader to engage with such technologies via a "big picture" perspective akin to that of [Gep19, § 1-2] and [Hau23, § 1-3].

Throughout this paper, we frequently describe conditions which may be satisfied by objects parameterized over some ∞-category $\mathcal{T}$. If P is a property, in the instance where there exists Borelification adjunctions

$$E_{\mathcal{F}}^{\mathcal{T}} : \mathcal{C}_{\mathcal{F}} \rightleftarrows \mathcal{C}_{\mathcal{T}} : \mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}}$$

along family inclusions $\mathcal{F} \subset \mathcal{T}$, we say that $X \in \mathcal{C}_{\mathcal{T}}$ is *essentially* P (or E-P) when there exists some $\overline{X} \in \mathcal{C}_{\mathcal{F}}$ which is P such that $X \simeq E_{\mathcal{F}}^{\mathcal{T}} \overline{X}$. We say that X is *almost essentially* P (or aE-P) if $\mathcal{C}_{\mathcal{F}}$ has a terminal object $*_{\mathcal{F}}$ for all $\mathcal{F}$, and there is a pushout expression

$$X \simeq *_{\mathcal{F}'} \sqcup_{*_{\mathcal{F}}} *_{\mathcal{F}'}$$

for some $\mathcal{F}' \subset \mathcal{F}$ we say that X is *almost* P (or a-P) if it's almost essentially P and $\mathcal{F}' = \mathcal{T}$ in the above.

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1. Equivariant symmetric monoidal categories

In this section, we review and advance the equivariant ∞-category theory of of *homotopical incomplete (semi)-Mackey functors* for a weak indexing system I, which we call *I-commutative monoids*. To that end, we begin in Section 1.1 by reviewing our equivariant higher categorical setup. We go on to cite and prove some basic facts about I-commutative monoids in Section 1.2. In Section 1.3 we then endow the $\mathcal{T}$-∞-category of I-commutative monoids with its *mode* symmetric monoidal structure, and prove that this is uniquely determined as a presentable symmetric monoidal structure by the free functor from coefficient systems; we use this to identify the resulting symmetric monoidal structure with the *localized Day convolution structure.* Following this, in Section 1.4 we quickly develop a framework for $\mathcal{T}$-symmetric monoidal d-categories.

[5] Here, n_G is the n-dimensional trivial orthogonal G-representation.

1.1. Recollections on $\mathcal{T}$-∞-categories.

We center on the following definition.

Definition 1.1. An ∞-category $\mathcal{T}$ is

(1) *orbital* if the finite coproduct completion $\mathbb{F}_{\mathcal{T}} := \mathcal{T}^{\coprod}$ has all pullbacks, and
(2) *atomic orbital* if it is orbital and every map in $\mathcal{T}$ possessing a section is an equivalence. ◁

We view the setting of atmoic orbital ∞-categories as a natural axiomatic home for higher algebra centered around the Burnside category (see [Nar16, § 4]), generalizing the orbit categories of a finite groups. The reader who is exclusively interested in equivariant homotopy theory is encouraged to assume every atomic orbital ∞-category is the orbit category of a family of subgroups of a finite group.

Definition 1.2. Let $\mathcal{T}$ be an ∞-category. Then, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is a *$\mathcal{T}$-family* if whenever $V \in \mathcal{F}$ and $W \to V$ is a map, we have $W \in \mathcal{F}$.[6] The poset of $\mathcal{T}$-families under inclusion is denoted $\mathrm{Fam}_{\mathcal{T}}$.

Similarly, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is a *$\mathcal{T}$-cofamily* if its opposite $\mathcal{F}^{\mathrm{op}} \subset \mathcal{T}^{\mathrm{op}}$ is a $\mathcal{T}^{\mathrm{op}}$-family. ◁

Example 1.3. Let G be a topological group, let $\mathcal{S}_G$ be the ∞-category of G-spaces, and let $\mathcal{O}_G \subset \mathcal{S}_G$ be the full subcategory spanned by homogeneous G-spaces $[G/H]$, where $H \subset G$ is a closed subgroup. The following are all atomic orbital ∞-categories (see [Ste24b]).

(1) For G is a topological group, the full subcategory $\mathcal{O}_G^{fin} \subset \mathcal{O}_G$ spanned by G/H for H finite.
(2) If G is a topological group, the wide subcategory $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ whose morphisms are projections $G/K \to G/H$ for $K \subset H$ finite index inclusion of closed subgroups.
(3) If G is a topological group, the full subcategory $\mathcal{O}_G^{f.i.sb} \subset \mathcal{O}_G^{f.i.}$ spanned by G/H for $H \subset G$ a finite-index closed subgroup.
(4) X a space, considered as an ∞-category.
(5) P a meet semilattice.
(6) If $\mathcal{T}$ is an atomic orbital ∞-category, $\mathrm{ho}(T)$.
(7) If $\mathcal{T}$ is an atomic orbital ∞-category, $\mathcal{F} \subset \mathcal{T}$ a full subcategory satisfying the following conditions:
   (a) For all $U, W \in \mathcal{F}$ and paths $U \to V \to W$ in $\mathcal{T}$, $V \in \mathcal{F}$.
   (b) For all $U, W \in \mathcal{F}$ and cospans $U \to V \leftarrow W$ in $\mathcal{T}$, there is a span $U \leftarrow V' \to V$ in $\mathcal{F}$.

   For instance, $\mathcal{F}$ may be the intersection of a family and a cofamily whose connected components have weakly initial objects, such as $\mathcal{T}_{\geq V}$.
(8) If $\mathcal{T}$ is an atomic orbital ∞-category and $V \in \mathcal{T}$, the ∞-category $\mathcal{T}_{/V}$. ◁

In this section, we briefly summarize some relevant elements of parameterized and equivariant higher category theory in the setting of atomic orbital ∞-categories. Of course, this theory has advanced far past that which is summarized here; for instance, further details can be found in the work of Barwick-Dotto-Glasman-Nardin-Shah [BDGNS16a; BDGNS16b; Nar16; Sha22; Sha23], Cnossen-Lenz-Linskens [CLL23a; CLL23b; CLL24; Lin24; LNP22], Hilman [Hil24], and Martini-Wolf [Mar22a; Mar22b; MW22; MW23; MW24].

1.1.1. *The $\mathcal{T}$-∞-category of small $\mathcal{T}$-∞-categories.*

Example 1.4. Let G be a finite group, $\mathcal{F} \subset \mathcal{O}_G$ a G-family of subgroups, and $\mathcal{S}_{\mathcal{F}}$ be the ∞-category of $\mathcal{F}$-spaces, constructed e.g. by inverting $\mathcal{F}$-weak equivalences between topological G-spaces. Then, a version of Elmendorf's theorem [Elm83] for families (see [DK84, Thm 3.1]) states that the *total $\mathcal{F}$-fixed points* functor yields an equivalence

$$\mathcal{S}_{\mathcal{F}} \simeq \mathrm{Fun}(\mathcal{F}^{\mathrm{op}}, \mathcal{S}). \qquad ◁$$

This motivates the following definition.

Definition 1.5. The *∞-category of small $\mathcal{T}$-∞-categories* is

$$\mathrm{Cat}_{\mathcal{T}} := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}),$$

where Cat is the ∞-category of small ∞-categories. If $\widehat{\mathrm{Cat}}$ is the (very large) ∞-category of *arbitrary* ∞-categories, then the *very large ∞-category of $\mathcal{T}$-∞-categories* is

$$\widehat{\mathrm{Cat}}_{\mathcal{T}} := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \widehat{\mathrm{Cat}}). \qquad ◁$$

[6] These are named *families* after subconjugacy closed families of subgroups, which frequently occur in equivariant homotopy; these are referred to as *sieves* in [BH15; NS22] and *upwards-closed subcategories* in [Gla17].

Notation 1.6. Fix $\mathcal{C} \in \mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. We refer to the value of $\mathcal{C}$ at $V \in \mathcal{T}^{\mathrm{op}}$ as the *V-value category of $\mathcal{C}$*, written as $\mathcal{C}_V$; given $f : V \to W$, we refer to the associated functor as *restriction*

$$\mathrm{Res}_V^W : \mathcal{C}_W \to \mathcal{C}_V. \qquad \triangleleft$$

Remark 1.7. We show in Example 2.15 that $\mathrm{Cat}_{\mathcal{T}}$ is equivalently presented as *complete Segal obects* in the ∞-topos $\mathcal{S}_{\mathcal{T}} := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$. $\triangleleft$

Remark 1.8. The Grothendieck construction, imported to ∞-category theory as the straightening-unstraightening equivalence in [HTT, Thm 3.2.0.1], produces an equivalence

$$\mathrm{Cat}_{\mathcal{T}} \simeq \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}^{\mathrm{cocart}},$$

the latter denoting the (non-full) subcategory of $\mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}$ whose objects are cocartesian fibrations and whose morphisms are functors over $\mathcal{T}^{\mathrm{op}}$ which preserve cocartesian arrows. Under this identification, the fiber of $\mathrm{Un}(\mathcal{C}) \to \mathcal{T}^{\mathrm{op}}$ over V is identified with the V-value $\mathcal{C}_V$, and the restriction functors are identified with cocartesian transport. $\triangleleft$

Given $\mathcal{C}, \mathcal{D}$ a pair of $\mathcal{T}$-∞-categories, we may define the *$\mathcal{T}$-functor category* to be the full subcategory

$$\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}^{\mathrm{cocart}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{C}, \mathcal{D})$$

consisting of functors over $\mathcal{T}^{\mathrm{op}}$ which preserve cocartesian lifts of the structure maps.

Example 1.9. For any object $V \in \mathcal{T}$, the forgetful functor $(\mathcal{T}_{/V})^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$ is a cocartesian fibration classified by the representable presheaf $\mathrm{Map}_{\mathcal{T}}(-, V)$. We refer to the associated $\mathcal{T}$-category as $\underline{V}$. This is covariantly functorial in V, since postcomposition yields functors $f_! : \mathcal{T}_{/V} \to \mathcal{T}_{/W}$ for all maps $f : V \to W$. $\triangleleft$

The representable $\mathcal{T}$-categories have total categories of a particularly nice form.

Proposition 1.10 ([NS22, Prop 2.5.1])**.** *If an atomic orbital ∞-category $\mathcal{T}$ has a terminal object, then it is a 1-category; in particular, $\mathcal{T}_{/V}$ is a 1-category.*

These play an important role in equivariant higher category theory.

Notation 1.11. Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, we define the *restricted* $\mathcal{T}_{/V}$-category by

$$\mathcal{C}_{\underline{V}} := \mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} (\mathcal{T}_{/V})^{\mathrm{op}}. \qquad \triangleleft$$

Proposition 1.12 ([BDGNS16b, Thm 9.7])**.** *$\mathrm{Cat}_{\mathcal{T}}$ has exponential objects $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ classified by the functor*

$$V \mapsto \mathrm{Fun}_{\mathcal{T}_{/V}}\left(\mathcal{C}_{\underline{V}}, \mathcal{D}_{\underline{V}}\right).$$

We refer to monomorphisms in $\mathrm{Cat}_{\mathcal{T}}$ as *$\mathcal{T}$-subcategories*, and $\mathcal{T}$-functors which are fiberwise-fully faithful as *full $\mathcal{T}$-subcategories*, or *$\mathcal{T}$-fully faithful functors.*

Observation 1.13. By the fiberwise expression for limits in functor categories (c.f. [HTT, Cor 5.1.2.3]), a $\mathcal{T}$-functor $F : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-subcategory inclusion if and only if $F_V : \mathcal{C}_V \to \mathcal{D}_V$ is a subcategory inclusion for all $V \in \mathcal{T}$. $\triangleleft$

Example 1.14. The terminal $\mathcal{T}$-∞-category $\underline{*}_{\mathcal{T}}$ is classified by the constant functor $V \mapsto *$. The poset of *sub-terminal objects in* $\mathrm{Cat}_{\mathcal{T}}$ (i.e. monomorphisms with codomain $\underline{*}_{\mathcal{T}}$) is isomorphic to $\mathrm{Fam}_{\mathcal{T}}$; the $\mathcal{T}$-∞-category $\underline{*}_{\mathcal{F}}$ associated with $\mathcal{F}$ is determined by the values

$$\underline{*}_{\mathcal{F},V} \simeq \begin{cases} * & V \in \mathcal{F}; \\ \varnothing & \text{otherwise.} \end{cases} \qquad \triangleleft$$

The ∞-category $\mathrm{Cat}_{\mathcal{T}}$ participates in an adjunction

$$\mathrm{Tot} \colon \mathrm{Cat}_{\mathcal{T}} \rightleftarrows \mathrm{Cat} \colon \underline{\mathrm{Coeff}}^{\mathcal{T}}$$

whose left adjoint Tot is the total category of cocartesian fibrations, and whose right adjoint has V-value

$$(\underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C})_V \simeq \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{C}\right)$$

where the functoriality on f is given by $(f_!)^*$ (see [BDGNS16b, Thm 7.8]). We refer to $\underline{\mathrm{Coeff}}^{\mathcal{T}}$ as the *$\mathcal{T}$-∞-category of coefficient systems in $\mathcal{C}$.*[7]

Example 1.15. There is an equivalence $\underline{*}_{\mathcal{T}} = \underline{\mathrm{Coeff}}^{\mathcal{T}} * \in \mathrm{Cat}_{\mathcal{T}}$, since right adjoints preserve terminal objects. ◁

We may additionally construct the *associated ∞-category*

$$\Gamma^{\mathcal{T}}\mathcal{C} := \mathrm{Fun}_{\mathcal{T}}(\underline{*}, \mathcal{C}),$$

whose objects consist of cocartesian sections of the structure functor $\mathcal{C} \to \mathcal{T}^{\mathrm{op}}$. We refer to this as the *∞-category of $\mathcal{T}$-objects in $\mathcal{C}$*. For instance, if $\mathcal{T}$ has a terminal object V, [BDGNS16b, Lemma 2.12] shows that we have an equivalence

$$\Gamma^{\mathcal{T}}\mathcal{C} \simeq \mathcal{C}_V;$$

more generally, this implies that $\Gamma^{\mathcal{T}}\mathcal{C} \simeq \lim_{V \in \mathcal{T}^{\mathrm{op}}} \mathcal{C}_V$, i.e. it is the *$\mathcal{T}$-fixed points* (or the limit of $\mathcal{C}$ viewed as a $\mathcal{T}^{\mathrm{op}}$ functor). Defining the *$\mathcal{T}$-inflation* to have V-values

$$\left(\mathrm{Infl}_e^{\mathcal{T}} \mathcal{D}\right)_V := \mathcal{D}$$

for any $\mathcal{D} \in \mathrm{Cat}$ and $V \in \mathcal{T}$, the adjunction between limits and diagonals immediately yields the following.

Proposition 1.16. *The functor $\mathrm{Infl}_e^{\mathcal{T}} : \mathrm{Cat} \to \mathrm{Cat}_{\mathcal{T}}$ is left adjoint to $\Gamma^{\mathcal{T}} : \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}$.*

Using this adjunction, given $\mathcal{C} \in \mathrm{Cat}$, we define the ∞-category

$$\mathrm{Coeff}^{\mathcal{T}}\mathcal{C} := \Gamma^{\mathcal{T}}\, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C});$$

then, we have $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Coeff}^{\mathcal{T}}\mathrm{Cat}$, and Elmendorf's theorem states that $\mathcal{S}_G \simeq \mathrm{Coeff}^{\mathcal{O}_G}\mathcal{S}$, motivating the following.

Definition 1.17. The *$\mathcal{T}$-∞-category of small $\mathcal{T}$-∞-categories* is $\underline{\mathrm{Cat}}_{\mathcal{T}} := \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathrm{Cat})$; the $\mathcal{T}$-∞-category of $\mathcal{T}$-spaces is $\underline{\mathcal{S}}_{\mathcal{T}} := \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{T})$, and the ∞-category of $\mathcal{T}$-spaces is $\mathcal{S}_{\mathcal{T}} := \mathrm{Coeff}^{\mathcal{T}}(\mathcal{S}) \simeq \Gamma^{\mathcal{T}}\underline{\mathcal{S}}_{\mathcal{T}}$. ◁

Observation 1.18. The *V-value of $\underline{\mathrm{Cat}}_{\mathcal{T}}$* is $\left(\underline{\mathrm{Cat}}_{\mathcal{T}}\right)_V = \mathrm{Cat}_{\mathcal{T}_{/V}}$; we henceforth refer to this as Cat_V. The restriction functor $\mathrm{Res}_V^W : \mathrm{Cat}_W \to \mathrm{Cat}_V$ is presented from the perspective of cocartesian fibrations by the pullback

$$\begin{tikzcd} \mathrm{Res}_W^V \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ (\mathcal{T}_{/V})^{\mathrm{op}} \arrow[r] & (\mathcal{T}_{/W})^{\mathrm{op}} \end{tikzcd}$$

In particular, given a map $U \to V \to W$, abusively referring to $(U \to V) \in \mathcal{T}_{/V}$ as U, this is characterized by the formula

$$\left(\mathrm{Res}_W^V \mathcal{C}\right)_U \simeq \mathcal{C}_U. \qquad ◁$$

1.1.2. *Language in the case $\mathcal{T} = \mathcal{O}_G$.* When G is a finite group, the category $\mathcal{O}_G$ has objects the homogeneous G-spaces $[G/H]$ and morphisms the G-equivariant maps $[G/K] \to [G/H]$; tracking the image of the identity, the hom set from $[G/K]$ to $[G/H]$ may alternatively be presented as

$$\mathrm{Hom}([G/K], [G/H]) \simeq \frac{\left\{a \in G \mid aKa^{-1} \subset H\right\}}{a \sim b \quad \text{when} \quad ab^{-1} \in K}$$

(see e.g. [Die09, Prop 1.3.1] for details). In particular, the endomorphism monoid of $[G/K]$ is the Weyl group $W_G H = N_G(H)/H$. Using this, one may see that when G is a finite group, the map $\mathrm{Ind}_H^G : \mathcal{O}_H \to \mathcal{O}_{G,/(G/H)}$ is an equivalence of categories. Thus we may set the following notation without creating clashes.

Notation 1.19. In the setting that $\mathcal{T} = \mathcal{O}_G$, we use the following notation:

(1) we refer to $\underline{[G/H]}$ as $\underline{H}$;
(2) we refer to $\underline{\mathcal{O}_G}$-$\infty$-categories as G-∞-categories and $\underline{\mathrm{Cat}}_{\mathcal{O}_G}$ as $\underline{\mathrm{Cat}}_G$;
(3) we refer to $\mathcal{C}_{[G/H]}$ as $\mathcal{C}_H$ and $\mathrm{Res}_{[G/K]}^{[G/H]}$ as Res_K^H;

---

[7] These are referred to as the *cofree* parameterization $\underline{\mathrm{CoFree}}(\mathcal{C})$ in [Hil24] and as the *$\mathcal{T}$-∞-category of $\mathcal{T}$-objects* $\underline{\mathcal{C}}_{\mathcal{T}}$ in [Nar17]. We avoid the former for clarity (as we do not view Tot as a forgetful functor), and we avoid the latter as it conflicts with the $\mathcal{T}$-∞-category of $\mathcal{T}$-spectra $\underline{\mathrm{Sp}}_{\mathcal{T}}$; instead, our name is chosen to evoke the *coefficient systems* used in equivariant cohomology.

(4) we refer to $\mathcal{O}_G$-spaces as G-spaces and $\underline{\mathcal{S}}_{\mathcal{O}_G}$ as $\underline{\mathcal{S}}_G$. ◁

1.1.3. *Join, slice, and (co)limits.* We now summarize some elements of [Sha22; Sha23].

Definition 1.20 ([Sha23, Def 4.1])**.** Let $\iota\colon \mathcal{T}^{\mathrm{op}} \times \partial\Delta^1 \hookrightarrow \mathcal{T}^{\mathrm{op}} \times \Delta^1$ be the evident inclusion. Then, the *$\mathcal{T}$-join* is the top horizontal functor

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{T}}^2 \arrow[rr, "- \star_{\mathcal{T}} -"] \arrow[d, hook] & & \mathrm{Cat}_{\mathcal{T}} \arrow[d, hook] \\ \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}} \times \partial\Delta^1} \arrow[r, "\iota^*"] & \mathrm{Cat}_{/\mathcal{T} \times I} \arrow[r, "\pi_!"] & \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}} \end{tikzcd}$$

which exists by [Sha22, Prop 4.3]. We write

$$K^{\underline{\triangleright}} := K \star_{\mathcal{T}} \underline{*}_{\mathcal{T}} \qquad \text{and} \qquad K^{\underline{\triangleleft}} := \underline{*}_{\mathcal{T}} \star_{\mathcal{T}} K$$ ◁

Definition 1.21. If $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\mathcal{T},\mathcal{E}/}$ are $\mathcal{T}$-∞-categories under $\mathcal{E}$, the *$\mathcal{T}$-∞-category of $\mathcal{T}$-functors under $\mathcal{E}$* is defined by the pullback of $\mathcal{T}$-categories

$$\begin{tikzcd} \underline{\mathrm{Fun}}_{\mathcal{T},\mathcal{E}/}(\mathcal{C},\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D}) \arrow[d, "(\pi_{\mathcal{C}})^*"] \\ \underline{*} \arrow[r, "\pi_{\mathcal{D}}"'] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{E},\mathcal{D}) \end{tikzcd}$$

If $p : K \to \mathcal{C}$ is a $\mathcal{T}$-functor, then the *$\mathcal{T}$-undercategory and $\mathcal{T}$-overcategory* are the functor ∞-categories

$$\mathcal{C}^{(p,\mathcal{T})/} := \underline{\mathrm{Fun}}_{\mathcal{T},K/}(K^{\underline{\triangleright}}, \mathcal{C});$$
$$\mathcal{C}^{/(p,\mathcal{T})} := \underline{\mathrm{Fun}}_{\mathcal{T},K/}(K^{\underline{\triangleleft}}, \mathcal{C})$$ ◁

In the case $p : \underline{*}_{\mathcal{T}} \to \mathcal{C}$ corresponds with the $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$, we simply write $\mathcal{C}^{X/} := \mathcal{C}^{(p,\mathcal{T})/}$ and similar for overcategories. In general, the categories $\mathcal{C}^{(p,\mathcal{T})/}$ take part in a functor out of $\mathrm{Cat}_{\mathcal{T},K/}$. Of fundamental importance is the adjoint relationship between these functors:

Theorem 1.22 ([Sha23, Cor 4.27])**.** *The $\mathcal{T}$-join forms the left adjoint in a pair of adjunctions*

$$K \star_{\mathcal{T}} - \colon \mathrm{Cat}_{\mathcal{T}} \rightleftarrows \mathrm{Cat}_{\mathcal{T},K/} \colon (-)^{(-,\mathcal{T})/},$$
$$- \star_{\mathcal{T}} K \colon \mathrm{Cat}_{\mathcal{T}} \rightleftarrows \mathrm{Cat}_{\mathcal{T},K/} \colon (-)^{/(-,\mathcal{T})}.$$

We say a $\mathcal{T}$-functor $\underline{p} : K^{\underline{\triangleleft}} \to \mathcal{C}$ extends $p : K \to \mathcal{C}$ if the composite $K \to K^{\underline{\triangleleft}} \to \mathcal{C}$ is homotopic to p.

Definition 1.23. Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. A $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ is *final* if for all $V \in \mathcal{T}$, the object $X_V \in \mathcal{C}_V$ is final; if $\underline{p} : K^{\underline{\triangleleft}} \to \mathcal{C}$ is a $\mathcal{T}$-functor extending $p : K \to \mathcal{C}$ and the corresponding cocartesian section $\sigma_{\underline{p}} : *_{\mathcal{T}} \to \mathcal{C}^{/(p,\mathcal{T})}$ is a final $\mathcal{T}$-object, then we say $\underline{p}$ is a *limit diagram for* p. ◁

The *fiberwise opposite* (or vertical opposite) functor $\mathrm{op}\colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ is the $\mathcal{T}$ functor induced under $\mathrm{Coeff}^{\mathcal{T}}$ by the *opposite category* functor $\mathrm{op}\colon \mathrm{Cat} \to \mathrm{Cat}$; the notions of initial $\mathcal{T}$-objects and $\mathcal{T}$-colimits are defined dually as final $\mathcal{T}$-objects and $\mathcal{T}$-limits in the fiberwise opposite.

In many cases, these are familiar; for instance, *trivially indexed* colimits are non-equivariant in nature.

Proposition 1.24 ([Sha22, Thm 8.6])**.** *Suppose K is a $\mathcal{T}$-category such that, for all morphisms $V \to W$ in $\mathcal{T}$, the associated restriction (i.e. cocartesian transport) functor $K_W \to K_V$ is an equivalence. Then, a diagram $\underline{p} : K^{\underline{\triangleleft}} \to \mathcal{C}$ is a limit diagram for $p : K \to \mathcal{C}$ if and only if for all V, the associated diagram $\underline{p}_V : K_V^{\triangleleft} \to \mathcal{C}_V$ is a limit diagram for p_V.*

Definition 1.25. Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category and let $\underline{\mathcal{K}}_{\mathcal{T}} = (\mathcal{K}_V)_{V \in \mathcal{T}} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ be a restriction-stable collection of V-categories. We say that $\mathcal{C}$ *strongly admits $\mathcal{K}$-shaped limits* if for each $V \in \mathcal{T}$, each C-category $K \in \mathcal{K}_V$ and each V-functor $p : K \to \mathcal{C}_{\underline{V}}$, there exists a limit diagram for p. We say $\mathcal{C}$ is *$\mathcal{T}$-complete* if it strongly admits $\underline{\mathrm{Cat}}_{\mathcal{T}}$-shaped limits.

If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{T}$-∞-categories which strongly admit all $\mathcal{K}$-shaped limits and $F : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$, functor, we say F *strongly preserves $\mathcal{K}$-shaped limits* if for all $V \in \mathcal{T}$ and all $K \in \mathcal{K}_V$, postcomposition with the $\underline{V}$-functor $F_{\underline{V}} : \mathcal{C}_{\underline{V}} \to \mathcal{D}_{\underline{V}}$ sends K-shaped limits diagrams to limits diagrams.

If $\mathcal{C} \subset \mathcal{D}$ is a full $\mathcal{T}$-subcategory whose inclusion strongly preserves $\mathcal{K}$-shaped limits, we say that $\mathcal{C}$ is *strongly closed under $\mathcal{K}$-shaped limits.* ◁

An important class of examples is *indexed (co)products*.

Definition 1.26. Consider $S \in \mathbb{F}_V$, considered as a V-category under the unique coproduct-preserving inclusion $\mathrm{Set}_V \hookrightarrow \mathrm{Cat}_V$. Then, we refer to S-shaped V-limits as *S-indexed products* and S-shaped V-colimits as *S-indexed coproducts*.

If $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, we refer to $\mathcal{T}$-colimits of the corresponding class as *$\mathcal{C}$-indexed coproducts*; similarly, following [Ste24b], if $I \subset \mathrm{Set}_{\mathcal{T}}$ is a pullback-stable subcategory, we define the full $\mathcal{T}$-subcategory $\underline{\mathrm{Set}}_I \subset \underline{\mathrm{Set}}_{\mathcal{T}}$ of *I-admissible sets* by

$$\left(\underline{\mathrm{Set}}_I\right)_V := \mathrm{Set}_{I,V} := \left\{S \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I\right\} \subset \mathrm{Set}_V.$$

We refer to the class of $\underline{\mathrm{Set}}_I$-indexed coproudcts as *$I$-indexed coproducts*, and use the dual language for I-indexed products. If $\mathcal{D}$ strongly admits $\underline{\mathrm{Set}}_I$-shaped limits, we simply say $\mathcal{D}$ *admits I-indexed coproducts*; if $I = \mathbb{F}_{\mathcal{T}}$, we say that $\mathcal{D}$ *admits finite indexed coproducts*, and if $I = \mathrm{Set}_{\mathcal{T}}$, we say that $\mathcal{D}$ *admits small indexed coproducts.* ◁

Notation 1.27. Given $\mathcal{C}$ a $\mathcal{T}$-category and $S \in \mathrm{Set}_{\mathcal{T}}$, we write

$$\mathcal{C}_S := \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U,$$

where $\mathrm{Orb}(S)$ is the set of *orbits* expressing S as a disjoint union of elements of $\mathcal{T}$. Given $S \in \mathrm{Set}_{I,V}$, and $(X_U) \in \mathcal{C}_S$, we denote the S-indexed products and coproducts as

$$\prod_U^S X_U \in \mathcal{C}_V, \qquad \coprod_U^S X_U \in \mathcal{C}_V.$$

In particular, in the case that S has one orbit U, we write $\mathrm{Ind}_U^V(-)$ and $\mathrm{CoInd}_U^V(-)$ for S-indexed coproducts and products, respectively. ◁

Indexed coproducts may be decomposed into coproducts of inductions:

Observation 1.28. If $\mathcal{C} \in \mathrm{Cat}_{\mathcal{T}}$ admits all indexed coproducts, $S \in \mathrm{Set}_V$, and $(X_U) \in \mathcal{C}_S$, then $\coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V X_U$ satisfies the universal property for S-indexed coproducts; hence there is a natural equivalence

$$\coprod_U^S X_U \simeq \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V X_U.$$

and the dual argument characterizes indexed products similarly. ◁

In nonequivariant higher category theory, all colimits are geometric realizations of coproducts. The equivariant version of this states that $\mathcal{T}$-colimits are geometric realizations of indexed coproducts, hence of coproducts of inductions. An example is the following result of Shah.

Proposition 1.29 ([Sha23, Cor 12.15])**.** *Let $\mathcal{T}$ be an orbital ∞-category. Then, $\mathcal{C}$ is $\mathcal{T}$-cocomplete if and only if it admits all geometric realizations and indexed coproducts.*

Given $\mathcal{K} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ a restriction-stable collection of V-categories and $W \in \mathcal{T}$, we let $\mathcal{K}_{\underline{W}} \subset \underline{\mathrm{Cat}}_W$ be the corresponding restriction-stable collection V-categories, where V ranges over $\mathcal{T}_{/W}$. We will use the following notation for strongly (co)limit-presereving functors.

Notation 1.30. Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a pullback-stable subcategory. Following and slightly extending [Sha22, Notn 1.15], we use the following notation for the described distinguished full $\mathcal{T}$-subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$:

(1) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\mathcal{K}-L}(\mathcal{C}, \mathcal{D})$: the V-functors which strongly preserve $\mathcal{K}_{\underline{V}}$-indexed colimits;
(2) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\mathcal{K}-R}(\mathcal{C}, \mathcal{D})$: the V-functors which strongly preserve $\mathcal{K}_{\underline{V}}$-indexed limits;
(3) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{L}(\mathcal{C}, \mathcal{D})$: the V-functors which strongly preserve small V-colimits;
(4) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{R}(\mathcal{C}, \mathcal{D})$: the V-functors which strongly preserve small V-limits;
(5) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\sqcup}(\mathcal{C}, \mathcal{D})$: the V-functors which (strongly) preserve I-indexed coproducts;

(6) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\times}(\mathcal{C},\mathcal{D})$: the V-functors which (strongly) preserve I-indexed products.
(7) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\sqcup}(\mathcal{C},\mathcal{D})$: the V-functors which (strongly) preserve finite ordinary coproducts;
(8) $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\times}(\mathcal{C},\mathcal{D})$: the V-functors which (strongly) preserve finite ordinary products. ◁

In many cases of interest, it is easy to verify these properties. Given $\mathcal{K} \subset \mathrm{Cat}$, define $\underline{\mathcal{K}}_V \subset \mathrm{Cat}_{\mathcal{T}^V}$ to consist of $\underline{V}$-categories whose fibers lie in $\mathcal{K}$, and define $\underline{\mathcal{K}} := (\underline{\mathcal{K}}_V) \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$.

Proposition 1.31 ([Sha22, Thm 8.6])**.** *Let $\mathcal{C},\mathcal{D}$ be ∞-categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.*

(1) $\underline{\mathrm{Coeff}}^G\mathcal{C}$ strongly admits $\underline{\mathcal{K}}$-shaped limits if and only if $\mathcal{C}$ admits $\mathcal{K}$-shaped limits, and
(2) $\underline{\mathrm{Coeff}}^G F\colon \underline{\mathrm{Coeff}}^G\mathcal{C} \to \underline{\mathrm{Coeff}}^G\mathcal{D}$ strongly preserves $\underline{\mathcal{K}}$-shaped limits if and only if F preserves $\mathcal{K}$-shaped limits.

Some important examples of indexed (co)limit preserving functors come from *parameterized adjunctions.*

Definition 1.32. A $\mathcal{T}$-functor $L\colon \mathcal{C} \to \mathcal{D}$ is *left adjoint* to $R\colon \mathcal{D} \to \mathcal{C}$ if the associated functors $L_V\colon \mathcal{C}_V \to \mathcal{D}_V$ are left adjoint to $R_V\colon \mathcal{D}_V \to \mathcal{C}_V$ for all $V \in \mathcal{T}$. ◁

These are the same as *relative adjunctions* over $\mathcal{T}^{\mathrm{op}}$ by [HA, Prop 7.3.2.1]; $\mathcal{T}$-left adjoints strongly preserve small $\mathcal{T}$-colimits and $\mathcal{T}$-right adjoints strongly preserve small $\mathcal{T}$-limits [Hil24, Thm 3.1.10], and they satisfy a parameterized version of the adjoint functor theorem [Hil24, Thm 6.2.1]. Additionally: they are plentiful.

Lemma 1.33. *Suppose $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ is an adjunction of ∞-categories. Then,*

$$\underline{\mathrm{Coeff}}^{\mathcal{T}} L\colon \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C} \rightleftarrows \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\colon \underline{\mathrm{Coeff}}^{\mathcal{T}} R$$

is an adjunction of $\mathcal{T}$-∞-categories.

Proof. This follows from the fiberwise description of $\underline{\mathrm{Coeff}}^{\mathcal{T}}(-)$; indeed, the V-values

$$L_*\colon \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}},\mathcal{C}\right) \rightleftarrows \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}},\mathcal{D}\right)\colon R_*$$

are adjoint. □

Example 1.34. We may use Lemma 1.33 to e.g. realize the full subcategory of $\mathcal{T}$-spaces whose fixed points are d-truncated and d-connected as (co)localizing subcategories

$$\mathcal{S}_{\mathcal{T},\geq d} \rightleftarrows \mathcal{S}_{\mathcal{T}} \rightleftarrows \mathcal{S}_{\mathcal{T},\leq d}.$$

Under the assumption that $\mathcal{T}$ is orbital, the author believes that most of the results of [LM06] may be carried out on this level of generality; later on, we will use this line of thought to understand *truncatedness and connectedness of $\mathcal{T}$-operads and $\mathcal{T}$-symmetric monoidal categories.* ◁

Example 1.35. By Lemma 1.33, the *classifying space and core* double adjunction $(-)_{\simeq} \dashv \iota \dashv (-)^{\simeq}$ yields

$$\begin{tikzcd} \underline{\mathrm{Cat}}_{\mathcal{T}} \arrow[r, bend left, "(-)_{\simeq}"] \arrow[r, bend right, "(-)^{\simeq}"'] & \underline{\mathcal{S}}_{\mathcal{T}} \arrow[l, hook'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Cat}_{\mathcal{T}} \arrow[r, bend left, "(-)_{\simeq}"] \arrow[r, bend right, "(-)^{\simeq}"'] & \mathcal{S}_{\mathcal{T}} \arrow[l, hook'] \end{tikzcd}$$

a double $\mathcal{T}$-adjunction and double adjunction. ◁

In the case that $K = \underline{*}_{\mathcal{T}}$, the results [HTT, Lem 6.1.1.1], Proposition 1.24, and Proposition 1.31 together with [Sha23, Lem 4.8] immediately imply the following.

Lemma 1.36. *The $\mathcal{T}$-functor $\mathrm{Ar}(\mathcal{C}) \xrightarrow{\mathrm{ev}_1} \mathcal{C}$ is a Cartesian fibration if and only if $\mathcal{C}$ admits $\mathcal{T}$-pullbacks; in this case, for $\alpha\colon X \to Y$ a morphism of $\mathcal{T}$-objects in $\mathcal{C}$, there exists an adjunction*

$$\alpha_!\colon \mathcal{C}^{/X} \rightleftarrows \mathcal{C}^{/Y}\colon \alpha^*$$

where $\alpha^*(Z) \simeq Z \times_Y X$.

Additionally, we can make genuine adjunction *non-genuine* using [HA, Prop 7.3.2.1].

Proposition 1.37. *If $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ are adjoint $\mathcal{T}$-functors, then* $\mathrm{Tot}\, L\colon \mathrm{Tot}\,\mathcal{C} \rightleftarrows \mathrm{Tot}\,\mathcal{D}\colon \mathrm{Tot}\, R$ *and* $\Gamma L\colon \Gamma\mathcal{C} \rightleftarrows \Gamma\mathcal{D}\colon \Gamma R$ *are adjoint pairs.*

Proof. The adjunction on Tot is [HA, Prop 7.3.2.1], and it induces an adjunction

$$\mathrm{Tot}\, L_* \colon \mathrm{Fun}_{/\mathcal{T}}(\mathcal{T}, \mathrm{Tot}\,\mathcal{C}) \rightleftarrows \mathrm{Fun}_{/\mathcal{T}}(\mathcal{T}, \mathrm{Tot}\,\mathcal{D}) \colon \mathrm{Tot}\, R_*,$$

which restricts to the full subcategories of cocartesian sections, and hence yields an adjunction

$$\Gamma^{\mathcal{T}} L \colon \Gamma^{\mathcal{T}} \mathcal{C} \rightleftarrows \Gamma^{\mathcal{T}} \mathcal{D} \colon \Gamma^{\mathcal{T}}. \qquad \square.$$

We will need the following lemmas later.

Lemma 1.38. *Suppose a $\mathcal{T}$-functor $F \colon \mathcal{C} \to \mathcal{D}$ has $F_V : \mathcal{C}_V \to \mathcal{D}_V$ conservative for all $V \in \mathcal{T}$; then, $\Gamma^{\mathcal{T}} F$ is conservative.*

Proof. Suppose $f_\bullet : X_\bullet \to Y_\bullet$ is a map of $\mathcal{T}$-objects in $\mathcal{C}$, i.e. a natural transformation of cocartesian sections of $\mathrm{Tot}\,\mathcal{C} \to \mathcal{T}^{\mathrm{op}}$. Then, $f_\bullet$ is an equivalence if and only if f_V is an equivalence for each V; by conservativity of F_V, this is true if and only if $F_v f_v$ is an equivalence for each V, i.e. if and only if $F f_\bullet$ is an equivalence, so $\Gamma^{\mathcal{T}} F$ is conservative. $\square$

Lemma 1.39. *Suppose $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ is a $\mathcal{T}$-adjunction such that R_V is monadic for all $V \in \mathcal{T}$; Then, $\Gamma^{\mathcal{T}} R \colon \Gamma^{\mathcal{T}} \mathcal{D} \to \Gamma^{\mathcal{T}} \mathcal{C}$ is monadic.*

Proof. We verify that $\Gamma^{\mathcal{T}} R$ satisfies the conditions of the ∞-categorical Barr-Beck theorem [HA, Thm 4.7.3.5(c)]. First, by Lemma 1.38, $\Gamma^{\mathcal{T}} R$ is conservative. Second, note that a simplicial object $Z_\bullet(-)$ in $\Gamma^{\mathcal{T}} \mathcal{D}$ corresponds to a family of simplicial objects $Z_V(-)$ in $\mathcal{D}_V$, and a $\Gamma^{\mathcal{T}} R$-splitting of $Z_\bullet(-)$ corresponds with a restriction-stable family of R_V-splittings of $Z_V(-)$. Thus R_V creates a colimit of Z_V for all V, and the resulting cocartesian section creates a colimit for $Z_\bullet$. Unwinding definitions, we've argued that $\Gamma^{\mathcal{T}} R$ creates colimits for $\Gamma^{\mathcal{T}} R$-split simplicial diagrams, we've verified the conditions of the ∞-categorical Barr-Beck theorem; hence $\Gamma^{\mathcal{T}} R$ is monadic, as desired. $\square$.

1.2. ***I*-commutative monoids.** Following [Bar14], we say that an *adequite triple* is the data of two core-preserving wide subcategories $\mathcal{X}_b \subset \mathcal{X} \supset \mathcal{X}_f$ of an ∞-category such that cospans $X \xrightarrow{\varphi_f} Y \xleftarrow{\varphi_b} Z$ satisfying $\varphi_f \in \mathcal{X}_f$ and $\varphi_b \in \mathcal{X}_b$ lift to pullback diagrams

$$\begin{tikzcd} & X \times_Y Z \arrow[dl, "\psi_b"'] \arrow[dr, "\psi_f"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ X \arrow[dr, "\varphi_f"'] & & Z \arrow[dl, "\varphi_b"] \\ & Y & \end{tikzcd}$$

satisfying $\psi_b \in \mathcal{X}_b$ and $\psi_f \in \mathcal{X}_f$. Given an adequate triple $\mathcal{X}_b \subset \mathcal{X} \supset \mathcal{X}_f$, we define the *span category* to be

$$\mathrm{Span}_{b,f}(\mathcal{X}) := A^{eff}(\mathcal{X}, \mathcal{X}_b, \mathcal{X}_f).$$

In particular, the objects of $\mathrm{Span}_{b,f}(\mathcal{X})$ are precisely those of $\mathcal{X}$, and the morphisms from X to Z are the spans $X \xleftarrow{\varphi_b} Y \xrightarrow{\varphi_f} Z$ with $\varphi_b \in \mathcal{X}_b$ and $\varphi_f \in \mathcal{X}_f$, with composition defined by taking pullbacks. [8]

Example 1.40. For $\mathcal{T}$ an orbital ∞-category and $I \subset \mathbb{F}_{\mathcal{T}}$ a pullback-stable wide subcategory, $\mathbb{F}_{\mathcal{T}} = \mathbb{F}_{\mathcal{T}} \hookleftarrow I$ is an adequate triple; write

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := \mathrm{Span}_{all,I}(\mathbb{F}_{\mathcal{T}}). \qquad \triangleleft$$

Warning 1.41. Even when $\mathbb{F}_{\mathcal{T}}$ is a 1-category (i.e. $\mathcal{T}$ is a 1-category), $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ will seldom be a 1-category; indeed, in this case, $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is a 2-category whose 2-cells given by the isomorphisms of spans

$$\begin{tikzcd} & Y' \arrow[dl] \arrow[dr] \arrow[dd, "\sim"] & \\ X \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd} \qquad \triangleleft$$

[8] Those readers more familiar with [EH23] may note that this specializes to the notion of a *span pair*, when backwards maps are $\mathcal{X}_b = \mathcal{X}$, in which case $\mathrm{Span}_f(\mathcal{X})$ recovers that of [EH23], and hence lifts to an $(\infty, 2)$-category with a universal property that we will not use.

In this subsection, we review the cartesian algebraic theory $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ corepresents, called *I-commutative monoids*. We will find that, in the same way that CMon is easily characterized via *semiadditivity* (c.f. [GGN15]), CMon_I is easily characterized via *I-semiadditivity*. Little of this subsection is original; instead, it forms a slight generalization of [Nar16] and a massive specialization of [CLL24].

1.2.1. *Weak indexing systems.* We briefly review the setting of *weak indexing systems* introduced in [Ste24b], which we view as the combinatorial context for the intersection of category theoretic and algebraic notions of I-commutative monoids.

Definition 1.42. A *$\mathcal{T}$-weak indexing category* is a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ satisfying the following conditions:

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\underline{\mathbb{F}}_{\mathcal{T}}$;
(IC-b) (segal condition) $T \to S$ and $T' \to S$ are both in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I; and
(IC-c) ($\Sigma_{\mathcal{T}}$-action) if $S \in I$, then all automorphisms of S are in I.

A *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ satisfying the following conditions:

(IS-a) whenever the V-value $\mathbb{F}_{I,V} := (\underline{\mathbb{F}}_I)_V$ is nonempty, we have $*_V \in \underline{\mathbb{F}}_{I,V}$; and
(IS-b) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under $\underline{\mathbb{F}}_I$-indexed coproducts. ◁

Observation 1.43. By a basic inductive argument, condition (IC-b) is equivalent to the condition that $S \to T$ is in I if and only if $T_U = T \times_S U \to U$ is in I for all $U \in \mathrm{Orb}(S)$; in particular, I is determined by its slice categories over *orbits*. ◁

We denote the *I-admissible sets* by $\underline{\mathbb{F}}_I := \underline{\mathrm{Set}}_I \subset \mathbb{F}_{\mathcal{T}}$ as in Definition 1.26. This is a full $\mathcal{T}$-subcategory.

Remark 1.44. By Observation 1.43, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the condition that for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{tikzcd} T \times_V U \arrow[r] \arrow[d, "\alpha'"] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\alpha"] \\ U \arrow[r] & V \end{tikzcd} \tag{2}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◁

Inspired by Observation 1.43 and Remark 1.44, in [Ste24b, Thm A] we prove the following.

Proposition 1.45. *The assignment $I \mapsto \underline{\mathbb{F}}_I$ implements an equivalence between the posets of $\mathcal{T}$-weak indexing categories and $\mathcal{T}$-weak indexing systems.*

We additionally recall the following conditions, which may equivalently be restated for weak indexing categories by [Ste24b, Thm A]. In view of [Ste24b, § 2.4], we encourage the reader to think primarily of *unitality*.

Definition 1.46. We say that $\underline{\mathbb{F}}_I$:

(i) has one color if for all $V \in \mathcal{T}$, we have $\mathbb{F}_{I,V} \neq \emptyset$;
(ii) is almost essentially unital (or aE-unital) if for all non-contractible V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$;
(iii) is essentially unital (or E-unital) if, for all V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$; and
(iv) is an *indexing system* if the subcategory $\underline{\mathbb{F}}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

We say that $\underline{\mathbb{F}}_I$ *almost unital* if it's almost essentially unital and has one color, and we say that $\underline{\mathbb{F}}_I$ is *unital* if it is essentially unital and has one color. These lie in a diagram of embedded sub-posets

$$\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}}, \mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}.$$ ◁

We say that $\underline{\mathbb{F}}_I$ is *unital* if it contains the V-set $\emptyset_V$ for all $V \in \mathcal{T}$; we say that $\underline{\mathbb{F}}_I$ is an *indexing system* if $n \cdot *_V$ is I-admissible for all $V \in \mathcal{T}$ and all $n \in \mathbb{N}$. When $\mathcal{T} = \mathcal{O}_G$, this recovers the notion given the same name in [BH15]; see [Ste24b] for details. Some useful invariants of these include

$$\begin{aligned} c(I) &:= \{V \in \mathcal{T} \mid *_V \in \mathbb{F}_{I,V}\}; \\ v(I) &:= \{V \in \mathcal{T} \mid \emptyset_V \in \mathbb{F}_{I,V}\}; \\ \nabla(I) &:= \{V \in \mathcal{T} \mid 2 \cdot *_V \in \mathbb{F}_{I,V}\}. \end{aligned} \tag{3}$$

These are each families [Ste24b, § 1.2], which we call the families of *colors*, *units*, and *fold maps* in I.

These will show in Proposition 2.33, where they parameterize a family of $\mathcal{T}$-operads called the *weak $\mathcal{N}_\infty$ operads*. We will see in forthcoming work on tensor products of weak $\mathcal{N}_\infty$-operads [Ste24a] that these play an important structural role in the theory of $\mathcal{T}$-operads. Narrowly, this role comes down to the fact that I-indexed coproducts in $\underline{\mathbb{F}}_I$ appear as the arities of *compositions* of I-indexed algebraic structures, so weak indexing systems occur as the possible "arity supports" that $\mathcal{T}$-equivariant algebraic theories can have, so long as they possess identity operations and they allows for the formation of composite operations. Indeed, weak $\mathcal{N}_\infty$-operads will represent a *support* stratification on $\mathrm{Op}_{\mathcal{T}}$.

1.2.2. *Indexed semiadditivity.* One central source of weak indexing categories is *indexed semiadditivity.*

Definition 1.47. Given $\mathcal{F} \subset \mathcal{T}$ a $\mathcal{T}$-family, we say that $\mathcal{D}$ is *$\mathcal{F}$-pointed* if $\mathcal{D}_V$ is pointed for all $V \in \mathcal{F}$. ◃

Given $S \in \mathbb{F}_V$ a finite V-set with a distinguished orbit $W \subset S$, $\mathcal{D}$ a $\mathcal{T}_{\leq V}$-pointed $\mathcal{T}$-∞-category admitting S-indexed products and coproducts, and $(X_U) \in \mathcal{D}_U$, [Nar16, Cons 5.2] constructs a map

$$\chi_W \colon \mathrm{Res}_W^V \coprod_U^S X_U \to X_W$$

by distinguishing a "diagonal" X_W-summand on the left hand side and dictating the map to be the indentity on this summand and 0 elsewhere; then, the *norm map*

$$\mathrm{Nm}_S \colon \coprod_U^S X_U \to \prod_U^S X_W$$

has projected map $\coprod_U^S X_U \to \mathrm{CoInd}_W^V X_W$ adjunct to χ_W.

Definition 1.48. Given $\mathcal{D}$ a $\mathcal{T}$-∞-category and $S \in \mathbb{F}_V$ a finite V-set, we say that S is *$\mathcal{D}$-ambidextrous* if $\mathcal{D}$ admits S-indexed products and coproducts, is $\mathcal{T}_{\leq V}$-pointed, and for all $(X_U) \in \mathcal{D}_S$, the norm map is an equivalence

$$\coprod_U^S X_U \xrightarrow{\sim} \prod_U^S X_U.$$

Given I a $\mathcal{T}$-weak indexing category, we say that $\mathcal{D}$ is *I-semiadditive* if S is $\mathcal{D}$-ambidextrous for all $S \in \mathbb{F}_I$. ◃

Remark 1.49. We've given an elementary presentation of this notion; this has been generalized to encapsulate Hopkins-Lurie's *higher semiadditivity* in [CLL24] (see Example 3.37 there). In particular, we find that $T \to S$ is $\mathcal{D}$-ambidextrous in the sense of [CLL24] if and only if the U-set $T \times_S U$ is $\mathcal{D}$-ambidextrous for all orbits $U \subset S$, so we adpot their language for *ambidextrous maps.* In particular, by [Cno23, Prop 3.13, Prop 3.16], ambidextrous maps are closed under composition and base change. ◃

Given $\mathcal{D}$ a $\mathcal{T}$-∞-category, we define the *semiadditive locus*

$$s(\mathcal{D}) = \{f \colon T \to S \mid f \text{ is } \mathcal{D}\text{-ambidextrous}\} \subset \mathbb{F}_{\mathcal{T}}.$$

This is closed under composition by Remark 1.49; furthermore, it's clear that an equivalence $T \simeq S$ is $\mathcal{D}$-ambidextrous if and only if $\mathcal{D}$ is $\mathcal{T}_{\leq V}$-pointed, so $s(\mathcal{D}) \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory satisfying Condition (IC-c). In fact, we may say more.

Proposition 1.50. *$s(\mathcal{D})$ is a weak indexing category, and $\mathcal{D}$ is I-semiadditive if and only if $I \leq s(\mathcal{D})$.*

Proof. By Observation 1.43 and Remark 1.49, $s(\mathcal{D})$ satisfies Condition (IC-b). In fact, by Remark 1.49, ambidextrous maps are closed under base change, i.e. $s(\mathcal{D})$ satisfies Condition (IC-a). We're left with verifying that $\mathcal{D}$ is I-semiadditive if and only if $I \leq s(\mathcal{D})$, but this follows immediately by unwinding definitions. ☐

By [Ste24b], the poset $\mathrm{wIndexCat}_{\mathcal{T}}$ has joins, which we write as $- \vee -$. The following is immediate.

Corollary 1.51. *$\mathcal{D}$ is $I \vee J$-semiadditive if and only if it is I-semiadditive and J-semiadditive.*

1.2.3. *I-commutative monoids as the I-semiadditivization.* Let $\mathrm{Trip}^{\mathrm{adeq}} \subset \mathrm{Fun}(\bullet \to \bullet \leftarrow \bullet, \mathrm{Cat})$ be the full subcategory spanned by adequate triples. By definition [Bar14, Def 3.6], $\mathrm{Span}_{-,-}(-)$ forms a functor $\mathrm{Trip}^{\mathrm{adeq}} \to \mathrm{Cat}$. Fix I a one-object weak indexing category. Write $\underline{\mathbb{F}}_V := \underline{\mathbb{F}}_{\mathcal{T},/V} \simeq \underline{\mathbb{F}}_{\mathcal{T}_{/V}}$ and let $\underline{\mathbb{F}}_{\mathcal{T}}^I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ be the wide subcategory whose V-value is $\left(\underline{\mathbb{F}}_{\mathcal{T}}^I\right)_V := I_V \subset \mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T},/V}$ is the wide subcategory of maps whose underlying map in $\mathbb{F}_{\mathcal{T}}$ lies in I.

The wide $\mathcal{T}$-subcategory inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is fiberwise given by a (one object) weak indexing category [Ste24b, § 2.1], so in particular, this yields a functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{Trip}^{\mathrm{adeq}}$ (c.f. [CLL24, § 4.1]). We use this to define the composite $\mathcal{T}$-functor

$$\underline{\mathrm{Span}}_I(\underline{\mathbb{F}}_{\mathcal{T}}) : \mathcal{T}^{\mathrm{op}} \xrightarrow{(\underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}}^I)} \mathrm{Trip}^{\mathrm{adeq}} \xrightarrow{\mathrm{Span}} \mathrm{Cat}.$$

Definition 1.52. If $\mathcal{C}$ is a $\mathcal{T}$-∞-category admitting I-indexed products, then the *$\mathcal{T}$-∞-category of I-commutative monoids in $\mathcal{C}$* is

$$\underline{\mathrm{CMon}}_I(\mathcal{C}) := \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\times}\left(\underline{\mathrm{Span}}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}\right). \quad \triangleleft$$

Definition 1.53. We say that a $\mathcal{T}$-functor $F : \mathcal{D} \to \mathcal{C}$ is the *I-semiadditive completion of $\mathcal{C}$* if $\mathcal{D}$ is I-semiadditive and for all I-semiadditive $\mathcal{T}$-categories $\mathcal{E}$, postcomposition along F yields an equivalence

$$\underline{\mathrm{Fun}}^{I-\times}(\mathcal{E}, \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{I-\times}(\mathcal{E}, \mathcal{C}). \quad \triangleleft$$

The following theorem is of fundamental importance in the theory of equivariant higher algebra.

Theorem 1.54 ([CLL24, Thm B])**.** *$U : \underline{\mathrm{CMon}}_I(\mathcal{C}) \to \mathcal{C}$ is the I-semiadditive completion.*

1.2.4. *Commutative monoids in $\mathcal{T}$-objects.* Let $I^\infty \subset \mathbb{F}_{\mathcal{T}}$ denote the smallest core-preserving wide subcategory containing the fold maps $n \cdot V \to V$ for all $V \in \mathcal{T}$ and $n \in \mathbb{N}$; this is precisely the indexing category corresponding with the minimal indexing system. We set the notation

$$\underline{\mathrm{CMon}}_\nabla(\mathcal{C}) := \underline{\mathrm{CMon}}_{I^\infty}(\mathcal{C}).$$

Observation 1.55. I^∞-indexed products are precisely *trivially* indexed products; by Proposition 1.24 the I^∞-indexed product preserving functors are precisely the fiberwise product-preserving $\mathcal{T}$-functors. Furthermore, a $\mathcal{T}$-category is ∇-semiadditive if and only if, for each $V \in \mathcal{T}$, the ∞-category $\mathcal{C}_V$ is semiadditive. Thus we have equivalences $\mathrm{Cat}_{\mathcal{T}}^\times \simeq \mathrm{Coeff}^{\mathcal{T}}(\mathrm{Cat}^\times)$ and $\mathrm{Cat}_{\mathcal{T}}^\oplus \simeq \mathrm{Coeff}^{\mathcal{T}}(\mathrm{Cat}^\oplus)$ compatible with the inclusions. $\triangleleft$

Lemma 1.33 and Observation 1.55 directly imply that the I^∞-semiadditive closure satisfies

$$\underline{\mathrm{CMon}}_\nabla(\mathcal{C}) \simeq \left(\mathcal{T}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Cat}^\times \xrightarrow{\mathrm{CMon}} \mathrm{Cat}^\oplus\right);$$

Cnossen-Lenz-Linsken's semiadditive closure theorem (i.e. Theorem 1.54) then yields the following.

Corollary 1.56. *There is a canonical equivalence $\mathrm{CMon}_\nabla(\mathcal{C}) \simeq \mathrm{CMon}(\Gamma\mathcal{C})$.*

1.2.5. *I-commutative monoids in ∞-categories.* We recall a special case of Cnossen-Lenz-Linsken's Mackey functor theorem.

Theorem 1.57 ([CLL24, Thm C])**.** *For every presentable ∞-category $\mathcal{C}$, there are canonical equivalences*

$$\mathrm{CMon}_I(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})) \simeq \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C});$$

$$\underline{\mathrm{CMon}}_I(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C}))_V \simeq \mathrm{Fun}^\times(\mathrm{Span}_{I_V}(\mathbb{F}_V), \mathcal{C}.$$

Furthermore, given a map $f : V \to W$, the associated restriction functor

$$\mathrm{Res}_V^W : \mathrm{Fun}(\mathrm{Span}_{I_W}(\mathbb{F}_W), \mathcal{C}) \to \mathrm{Fun}(\mathrm{Span}_{I_V}(\mathbb{F}_V), \mathcal{C})$$

is given by precomposition along $\mathrm{Span}(\mathrm{Ind}_V^W(-))$.

This motivates us to make the following definition.

Definition 1.58. If $\mathcal{C}$ is an ∞-category with finite products, then the *$\mathcal{T}$-∞-category of I-commutative monoids in $\mathcal{C}$* is

$$\underline{\mathrm{CMon}}_I(\mathcal{C}) := \underline{\mathrm{CMon}}_I(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})). \quad \triangleleft$$

Similar to the case of $\underline{\mathrm{Coeff}}^{\mathcal{T}}$, this construction is compatible with adjunctions.

Lemma 1.59. *Let $I \subset \mathcal{T}$ be a pullback-stable wide subcategory of an orbital ∞-category.*

(1) If $f : \mathcal{C} \to \mathcal{D}$ is a product-preserving functor, then postcomposition yields a $\mathcal{T}$-functor

$$f_* \colon \underline{\mathrm{CMon}}_I \mathcal{C} \to \underline{\mathrm{CMon}}_I \mathcal{D}.$$

(2) If $L : \mathcal{C} \rightleftarrows : R$ is an adjunction whose right adjoint R is product preserving, then

$$L_* \colon \underline{\mathrm{CMon}}_I \mathcal{C} \rightleftarrows \underline{\mathrm{CMon}}_I \mathcal{D} : R_*$$

is a $\mathcal{T}$-adjunction.

Proof. (1) follows by noting that f_* exists since f is product preserving, and it is compatible with restriction because postcomposition and precomposition commute. (2) follows by noting that the associated functors

$$L_* \colon (\mathrm{CMon}_I \mathcal{C})_V \simeq \mathrm{Fun}^\times\left(\mathrm{Span}_{I_V}(\mathbb{F}_V), \mathcal{C}\right) \rightleftarrows \mathrm{Fun}^\times\left(\mathrm{Span}_{I_V}(\mathbb{F}_V), \mathcal{D}\right) = (\mathrm{CMon}_I \mathcal{D})_V) : R_*$$

are adjoint. $\square$

We may unpack the structure of I-commutative monoids more using the following.

Construction 1.60. Let $X \in \mathrm{CMon}_I \mathcal{C}$ be a a I-commutative monoid, and let $V \in \mathcal{T}$ be an orbit. Let $\iota_V : \mathbb{F} \to \mathbb{F}_{\mathcal{T}}$ be the coproduct-preserving functor sending $* \mapsto V$. Then, the *V-value* is the pullback

$$\begin{tikzcd}
\mathrm{CMon}_I \mathcal{C} \arrow[rr, "(-)_V"] \arrow[d, phantom, "\wr"] & & \mathrm{CMon}_{I_V} \mathcal{C} \arrow[d, phantom, "\wr"] \\
\mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}) \arrow[rr, "\iota_V^*"] & & \mathrm{Fun}^\times(\mathrm{Span}_{I \times_{\mathbb{F}_{\mathcal{T}}, \iota_V} \mathbb{F}}(\mathbb{F}), \mathcal{C})
\end{tikzcd}$$

In particular, when I contains all fold maps (i.e. I is an *indexing category* in the sense of [BH15; Ste24b]) and X is an I-commutative monoid, X_V is a commutative monoid in $\mathcal{C}$. ◁

Construction 1.61. Fix $X \in \mathrm{CMon}_I(\mathcal{C})$ and $f : V \to W$ a map in I. There exists a natural transformation $\alpha_f : \iota_V \to \iota_W$ whose value on n is the copower map $n \cdot V \to n \cdot W$; this induces a natural transformation $N_V^W : (-)_V \implies (-)_W$, which we refer to as the *norm map*. ◁

1.2.6. *I-symmetric monoidal ∞-categories.* We refer to

$$\mathrm{Cat}_I^\otimes := \underline{\mathrm{CMon}}_I \mathrm{Cat}$$

as the *$\mathcal{T}$-∞-category of I-symmetric monoidal ∞-categories*. In the case $I = \mathbb{F}_{\mathcal{T}}$, we refer to these simply as *$\mathcal{T}$-symmetric monoidal ∞-categories* and write $\mathrm{Cat}_{\mathcal{T}}^\otimes := \mathrm{Cat}_{\mathbb{F}_{\mathcal{T}}}^\otimes$.

Notation 1.62. Suppose $S \in \underline{\mathbb{F}}_I$. Associated with the structure map $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ we have functors

$$\bigotimes_U^S : \mathcal{C}_S \to \mathcal{C}_V, \qquad \Delta^S : \mathcal{C}_V \to \mathcal{C}_S$$

called the *S-indexed tensor product and S-indexed diagonal*. We refer to the composite $(-)^{\otimes S} : \mathcal{C}_V \xrightarrow{\Delta^S} \mathcal{C}_S \xrightarrow{\otimes_U^S} \mathcal{C}_V$ as the *S-indexed tensor power*. In the case $\mathrm{Ind}_V^{\mathcal{T}} S = W$ is an orbit (i.e. S is a *transitive V-set*), we write

$$N_W^V := \bigotimes_U^W : \mathcal{C}_W \to \mathcal{C}_V.$$

In general, we will use the inset notation $-\otimes-$ for $\otimes_U^{2 \cdot *_V}$, and when $\varnothing_V \in \underline{\mathbb{F}}_I$, we will refer to the $\varnothing_V$-ary operation $* \to \mathcal{C}_V$ as the *V-unit* and denote it as 1_V. ◁

Observation 1.63. Suppose S, $|\mathrm{Orb}(S)| \cdot *_V$, and all orbits of S are is I-admissible V-sets. Then, the following path lies in I:

$$\mathrm{Ind}_V^{\mathcal{T}} S \to |\mathrm{Orb}(S)| \cdot V \to V.$$

In algebra, this yields the formula

$$\begin{tikzcd} \mathcal{C}_S \arrow[rr, "\bigotimes_U^S"] \arrow[dr, "(N_U^V -)"'] & & \mathcal{C}_V \\ & \mathcal{C}_V^{\times \mathrm{Orb}(S)} \arrow[ur, "\otimes"'] & \end{tikzcd}$$

i.e. $\bigotimes_U^S X_U \simeq \bigotimes_{U \in \mathrm{Orb}(S)} N_U^V X_U$. Thus, when I is an indexing category, the indexed tensor products in an I-symmetric monoidal ∞-category is are determined by their binary tensor products and norms. Furthermore, in [Ste24b, § 1.2], we see that I-symmetric monoidal ∞-categories satisfy a version of the *double coset formula*

$$\mathrm{Res}_W^V N_U^V Z \simeq \bigotimes_X^{U \times_V W} \mathrm{Res}_X^U Z$$

for all cospans $U \to V \leftarrow W$ in $\mathcal{T}$ such that $U \to W$ is in I. ◁

Construction 1.64. Right Kan extensions preserve product preserving functors; applying this to the *orbits* functor $F_{\mathcal{T}} : \mathbb{F}_{\mathcal{T}} \to \mathbb{F}$ yields a functor

$$\Gamma := \mathrm{Span}(F_{\mathcal{T}})_* : \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}), \mathcal{C}) \to \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}), \mathcal{C}).$$

In particular, Γ is right adjoint to $\mathrm{Infl}_e^{\mathcal{T}} := \mathrm{Span}(F_{\mathcal{T}})^*$. When $\mathcal{C} = \mathrm{Cat}$, the counit of this adjunction is a natural $\mathcal{T}$-symmetric monoidal functor.

$$\mathrm{Infl}_e^{\mathcal{T}} \Gamma \mathcal{C}^\otimes \to \mathcal{C}^\otimes$$

We refer to the (symmetric monoidal) V-value of this as the *symmetric monoidal V-evaluation*

$$\mathrm{ev}_V : \Gamma \mathcal{C}^\otimes \to \mathcal{C}_V^\otimes.$$ ◁

1.2.7. *Symmetric monoidal $\mathcal{T}$-∞-categories.* The ∞-category of *symmetric monoidal $\mathcal{T}$-∞-categories* is

$$\mathrm{Cat}_{I^\infty, \mathcal{T}}^\otimes \simeq \mathrm{Coeff}^{\mathcal{T}} \mathrm{Cat}_\infty^\otimes \simeq \mathrm{CMonCat}_{\mathcal{T}}.$$

Definition 1.65. Suppose $L\mathcal{C} \subset \mathcal{C}$ is a localizing $\mathcal{T}$-subcategory of a symmetric monoidal $\mathcal{T}$-∞-category. We say that L is *compatible with the symmetric monoidal structure* if for each $V \in \mathcal{T}$, the localization L_V is compatible with the symmetric monoidal structure on $\mathcal{C}_V$ in the sense of [HA, Def 2.2.1.6]. ◁

We will crucially use the following proposition in Section 1.3.

Proposition 1.66. *If L is compatible with the symmetric monoidal structure, there exists a commutative diagram of $\mathcal{T}$-∞-categories*

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[rr, "L^\otimes"] \arrow[dr, "p"] & & L\mathcal{C}^\otimes \arrow[dl] \\ & (\mathbb{F}_*)_{\mathrm{triv}} & \end{tikzcd}$$

satisfying the following conditions:

(a) $L\mathcal{C}^\otimes$ is a symmetric monoidal $\mathcal{T}$-∞-category and $L^\otimes$ is a symmetric monoidal $\mathcal{T}$-functor,
(b) the underlying $\mathcal{T}$-functor of $L^\otimes$ is $L \colon \mathcal{C} \to L\mathcal{C}$, and
(c) $L^\otimes$ possesses a fully faithful and lax symmetric monoidal right $\mathcal{T}$-adjoint extending the inclusion $L\mathcal{C} \subset \mathcal{C}$.

Proof. This is the specialization of [NS22, Thm 2.9.2] to $\mathcal{O}^\otimes := \mathbb{E}_\infty^\otimes$. □

1.3. **The canonical symmetric monoidal structure on I-commutative monoids.** We now explore the observation that the parameterized presentability results of [Hil24] are sufficiently strong to power non-indexed lifts of [GGN15] in the I-semiadditive setting.

Definition 1.67 (c.f. [Hil24, Thm 3.1.9(2), Thm 6.1.2])**.** A (large) $\mathcal{T}$-∞-category $\mathcal{C}$ is *$\mathcal{T}$-presentable* if it strongly admits finite $\mathcal{T}$-coproducts and its straightening factors as

$$\mathcal{C}\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Pr}^{L,\kappa} \to \widehat{\mathrm{Cat}}$$

for some regular cardinal κ. The (nonfull) subcategory

$$\mathrm{Pr}^L_{\mathcal{T}} \subset \widehat{\mathrm{Cat}}_{\mathcal{T}}$$

has objects given by $\mathcal{T}$-presentable ∞-categories and morphisms given by $\mathcal{T}$-left adjoints. ◁

Observation 1.68. The conditions of factoring through $\mathrm{Pr}^{L,\kappa}$, of strongly admitting finite $\mathcal{T}$-coproducts, and of being $\mathcal{T}$-left adjoints are preserved by restriction; hence $\mathrm{Pr}^L_{\mathcal{T}}$ canonically lifts to a (nonfull) $\mathcal{T}$-subcategory

$$\underline{\mathrm{Pr}}^L_{\mathcal{T}} \subset \underline{\widehat{\mathrm{Cat}}}_{\mathcal{T}}$$ ◁

These satisfy an adjoint functor theorem [Hil24, Thm 6.2.1] and have analogous characterizations to the non-equivariant case; in particular, $\mathrm{Pr}^L_{\mathcal{T}} \subset \widehat{\mathrm{Cat}}_{\mathcal{T}}$ is closed under functor categories from small categories [Hil24, Lem 6.7.1] and by Definition 1.67, $\mathrm{Pr}^L_{\mathcal{T}}$ is closed under fiberwise κ-accessible $\mathcal{T}$-localizations. Hence $\underline{\mathrm{CMon}}_I(\mathcal{C})$ is $\mathcal{T}$-presentable when $\mathcal{C}$ is $\mathcal{T}$-presentable.

Additionally, in [Nar17], a $\mathcal{T}$-symmetric monoidal structure was constructed on $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$. In order to characterize this structure, we use the following definition (c.f. [QS19, § 5.1]).

Definition 1.69 ([QS19, Def 5.14])**.** Fix S a finite V-set, $(\mathcal{C}_U)$ an S-∞-category, $\mathcal{D}$ a V-∞-category, and $F\colon \prod_U^S \mathcal{C}_U \to \mathcal{D}$ a V-functor. Denote by $(-)_*$ the indexed products in $\mathrm{Cat}_{\mathcal{T}}$ and $(-)^*$ the restriction. We say that F is *S-distributive* if, for every pullback diagram

$$\begin{tikzcd} T \times_V S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "g"] \\ S \arrow[r, "f"] & V \end{tikzcd}$$

and S-colimit diagram $\overline{p}\colon K^{\trianglerighteq} \to g'^*\mathcal{C}$ for $p\colon K \to g'^*\mathcal{C}$, the composite T-functor

$$(f'_*K)^{\trianglerighteq} \xrightarrow{\mathrm{can}} f'_*(K^{\trianglerighteq}) \xrightarrow{f'_*\overline{p}} f'_*g'^*\mathcal{C} \simeq g^*f_*\mathcal{C} \xrightarrow{g^*F} g^*\mathcal{D}$$

is a T-colimit diagram for the associated composite $f'_*K \to g^*\mathcal{D}$. We denote by

$$\mathrm{Fun}^{\delta}_{\mathcal{T}}(f_*\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}_{\mathcal{T}}(f_*\mathcal{C}, \mathcal{D})$$

the full subcategory spanned by S-distributive functors. ◁

By the proof of [Nar17, Prop 3.25], Nardin's $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$ has V unit $\underline{\mathcal{S}}_V$ and indexed tensor products characterized by the universal property

$$\mathrm{Fun}^L_{\mathcal{T}}\left(\bigotimes_U^S \mathcal{C}, \mathcal{E}\right) \simeq \mathrm{Fun}^{\delta}_{\mathcal{T}}\left(\prod_U^S \mathcal{C}, \mathcal{D}\right).$$

Definition 1.70. The ∞-category of *presentably $\mathcal{T}$-symmetric monoidal ∞-categories* is the (non-full) subcategory $\mathrm{CAlg}_{\mathcal{T}}(\underline{\mathrm{Pr}}^{L,\otimes}_{\mathcal{T}}) \subset \widehat{\mathrm{Cat}}^{\otimes}_{\mathcal{T}}$; the ∞-category of *presentably symmetric monoidal $\mathcal{T}$-∞-categories* is the (non-full) subcategory $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{T}}) \subset \mathrm{CMon}(\widehat{\mathrm{Cat}}_{\mathcal{T}})$. ◁

Observation 1.71. By definition, a $\mathcal{T}$-symmetric monoidal ∞-category whose underlying $\mathcal{T}$-∞-category is presentable factors through the inclusion $\underline{\mathrm{Pr}}^L_{\mathcal{T}} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ if and only if its structure maps $\mathcal{C}_V^{\times S} \to \mathcal{C}_V$ are in $\mathrm{Fun}^{\delta}_V(\mathcal{C}_V^{\times S}, \mathcal{C}_V)$; in the language of [NS22], a presentably $\mathcal{T}$-symmetric monoidal ∞-category is precisely a *distributive* $\mathcal{T}$-symmetric monoidal ∞-category whose underlying $\mathcal{T}$-∞-category is presentable. ◁

Example 1.72. By [NS22, Prop 3.2.5], if $\mathcal{C}$ is a cocomplete ∞-category with finite products such that finite products preserve colimits separately in each variable, then the cartesian symmetric monoidal structures on $\mathrm{Coeff}^V\mathcal{C}$ lift to a distributive symmetric monoidal $\mathcal{T}$-∞-category $\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}^{\times}$. It follows from Hilman's characterization of parameterized presentability [Hil24, Thm 6.1.2] that $\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}$ is presentable, so Observation 1.71 implies that $\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}^{\times}$ is presentably symmetric monoidal. ◁

Hilman used the universal property of $\otimes$ in [Hil24, Prop 6.7.5] to prove the formula

$$\mathcal{C} \otimes \mathcal{D} \simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}).$$

Using this, for any $\mathcal{T}$-presentable $\mathcal{T}$-∞-category $\mathcal{C}$, we have

$$\begin{aligned}
\underline{\mathrm{CMon}}_I(\mathcal{C}) &\simeq \underline{\mathrm{Fun}}^{I-\times}_{\mathcal{T}}(\underline{\mathrm{Span}}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}) \\
&\simeq \underline{\mathrm{Fun}}^{I-\times}_{\mathcal{T}}(\underline{\mathrm{Span}}_I(\mathbb{F}_{\mathcal{T}}), \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})) \\
&\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Fun}}^{I-\times}_{\mathcal{T}}(\underline{\mathrm{Span}}_I(\mathbb{F}_{\mathcal{T}}), \underline{\mathcal{S}}_{\mathcal{T}})) \\
&\simeq \mathcal{C} \otimes \underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}}).
\end{aligned}$$

In particular, this implies that the functor $\mathcal{C} \mapsto \underline{\mathrm{CMon}}_I(\mathcal{C})$ is *smashing*. In fact, we can say more.

Notation 1.73. We say that a presentable $\mathcal{T}$-∞category is *I-semiadditive* if its underlying $\mathcal{T}$-∞-category is I-semiadditive, and we let $\mathrm{Pr}^{L,I-\oplus}_{\mathcal{T}} \subset \mathrm{Pr}^L_{\mathcal{T}}$ be the full subcategory spanned by I-semiadditive presentable $\mathcal{T}$-categories. ◁

It follows from Cnossen-Lenz-Linsken's semiadditive closure theorem [CLL24, Thm B] that a $\mathcal{T}$-presentable $\mathcal{T}$-∞-category is fixed by $\underline{\mathrm{CMon}}_I(-)$ if and only if it's I-semiadditive, i.e. $\underline{\mathrm{CMon}}_I(-)$ implements the localization functor

$$\mathrm{Pr}^L_{\mathcal{T}} \to \mathrm{Pr}^{L,I-\oplus}_{\mathcal{T}}$$

left adjoint to the evident inclusion. By the above argument, we find that $\underline{\mathrm{CMon}}_I(-)$ is a *smashing localization*, hence a symmetric monoidal localization; by [GGN15, Lemma 3.6], this implies that given $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{T}})$, there is a unique compatible commutative algebra structure on its localization $\underline{\mathrm{CMon}}_I(\mathcal{C})$. In other words, we've shown the following.

Theorem 1.74. *The localizing subcategory*

$$\underline{\mathrm{CMon}}_I \colon \mathrm{Pr}^L_{\mathcal{T}} \rightleftarrows \mathrm{Pr}^{L,I-\oplus}_{\mathcal{T}} \colon \iota$$

is smashing; in particular, if $\mathcal{D}^{\otimes}$ is a presentably symmetric monoidal $\mathcal{T}$-category, then there is an essentially unique presentably symmetric monoidal $\mathcal{T}$-∞-category $\underline{\mathrm{CMon}}^{\otimes-\mathrm{mode}}_I(\mathcal{D})$ possessing a (necessarily unique) symmetric monoidal lift

$$\mathrm{Fr}^{\otimes} \colon \mathcal{D}^{\otimes} \to \underline{\mathrm{CMon}}^{\otimes-\mathrm{mode}}_I(\mathcal{D})$$

of $\mathrm{Fr} \colon \mathcal{D} \to \underline{\mathrm{CMon}}_I(\mathcal{D})$.

Warning 1.75. Theorem 1.74 is not as *genuinely equivariant* as the user may want, as it constructs *symmetric monoidal structures*, but never norm maps. The author is content with this for the purposes of this paper, as the algebraic interpretation of indexed tensor products of $\mathcal{T}$-operads is unclear. She hopes to address the indexed case in forthcoming work. ◁

Remark 1.76. Under the equivalence of Theorem 1.57, writing $\mathcal{D} = \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})$, Theorem 1.74 constructs an essentially unique presentably symmetric monoidal structure on $\underline{\mathrm{CMon}}_I(\mathcal{C})$ subject to the condition that the free functor $\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C} \to \underline{\mathrm{CMon}}_I(\mathcal{C})$ is bears a symmetric monoidal structure. ◁

Observation 1.77. The $\mathcal{T}$-∞-category $\underline{\mathcal{S}}_{\mathcal{T}}$ is freely generated under $\mathcal{T}$-colimits by one $\mathcal{T}$-point, in the sense that evaluation at the V-units $(*_V)$ yields an equivalence [Sha23, Thm 11.5]

$$\mathrm{Fun}^L_{\mathcal{T}}(\underline{\mathcal{S}}_{\mathcal{T}}, \mathcal{C}) \simeq \Gamma\mathcal{C}.$$

In particular, every symmetric monoidal $\mathcal{T}$-∞-category receives at most one symmetric monoidal $\mathcal{T}$-left adjoint from $\underline{\mathcal{S}}_{\mathcal{T}}$; in the case $\mathcal{C} = \underline{\mathcal{S}}^{\times}_{\mathcal{T}}$ the condition of Theorem 1.74 then may be read as saying that there is a unique presentably symmetric monoidal structure on $\underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$ with V-unit $1^{\mathrm{mode}}_V = \mathrm{Fr}(*_V)$ for all $V \in \mathcal{T}$.

Furthermore, by Yoneda's lemma, these V-units are characterized by the property that

$$\mathrm{Map}_V(1^{\mathrm{mode}}_V, X_V) \simeq \mathrm{Map}(*_V, X_V(*_V)) \simeq X_V(*_V). \qquad ◁$$

We'd like to identify this symmetric monoidal structure via a familiar formula. We have a candidate:

Proposition 1.78 ([BS24b, Prop 4.24], via [CHLL24, Prop 3.3.4])**.** *If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category, then the Day convolution structure on $\mathrm{Fun}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C})$ with respect to the smash product on $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is compatible with the localization*

$$L_{\mathrm{Seg}} : \mathrm{Fun}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}) \to \mathrm{CMon}_I(\mathcal{C})$$

Proof. By the general criterion [CHLL24, Prop 3.3.4], it suffices to verify that $A_+ \wedge - : \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is product-preserving, which follows by the fact that it is colimit preserving and $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is semiadditive. □

By Proposition 1.66, Proposition 1.78 constructs a symmetric monoidal structure on $\underline{\mathrm{CMon}}_I(\mathcal{C})$. We will show that this agrees with the mode symmetric monoidal structure.

Theorem 1.79. *Let $\mathcal{C}^{\otimes}$ be a presentably symmetric monoidal ∞-category. Then, there is a unique equivalence between the Day convolution and mode symmetric monoidal structures on $\underline{\mathrm{CMon}}_I(\mathcal{C})$ lifting the identity.*

The proof of [BS24b, Lemma 4.21] and [CSY20, Lemma 5.2.1] apply identically to the following.

Lemma 1.80. *Fix $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{T}})$ and $L : \mathcal{A}_0 \to \mathcal{A}_1$ a $\mathcal{T}$-localization functor which is compatible with the symmetric monoidal structure on $\mathcal{A}_0$. Then, $L \otimes \mathrm{id}_{\mathcal{B}} : \mathcal{A}_0 \otimes \mathcal{B} \to \mathcal{A}_1 \otimes \mathcal{B}$ is a $\mathcal{T}$-localization functor which is compatible with the symmetric monoidal structure on $\mathcal{A}_0 \otimes \mathcal{B}$.*

Proof of Theorem 1.79. Set the temporary notation $\underline{\mathrm{PCMon}}_I(-) \coloneqq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\mathrm{Span}}_I(\mathbb{F}_{\mathcal{T}}), -\right)$. Our argument follows along the lines of [BS24b, Thm 4.26]. Repeating the argument of Theorem 1.74, for all presentably symmetric monoidal $\mathcal{T}$-∞-categories $\mathcal{D}$, we acquire a diagram

$$\begin{tikzcd}
\underline{\mathrm{PCMon}}_I(\mathcal{D}) \arrow[r, phantom, "\simeq"] & \underline{\mathrm{PCMon}}_I(\mathcal{S}) \otimes \mathcal{D} \\
\underline{\mathrm{CMon}}_I(\mathcal{D}) \arrow[u, hook] \arrow[r, phantom, "\simeq"] & \underline{\mathrm{CMon}}_I(\mathcal{S}) \otimes \mathcal{D} \arrow[u, hook]
\end{tikzcd}$$

Furthermore, the associated map $\underline{\mathrm{PCMon}}_I(\mathcal{S}) \to \underline{\mathrm{PCMon}}_I(\mathcal{D})$ is postcomposition along the canonical symmetric monoidal left adjoint $\mathcal{S}_{\mathcal{T}} \to \mathcal{D}$, and the associated map $\mathcal{D} \to \underline{\mathrm{PCMon}}_I(\mathcal{D})$ is the Yoneda lemma; the former bears a symmetric monoidal structure for the Day convolution symmetric monoidal structure and the latter bears an I-symmetric monoidal structure by [NS22, Prop 6.0.2]. Thus the top arrow can be lifted to a *symmetric monoidal* equivalence. We may take adjoint functors to find the diagram

$$\begin{tikzcd}
\underline{\mathrm{PCMon}}_I(\mathcal{D}) \arrow[d, "L_{\mathrm{Seg}}"] \arrow[r, phantom, "\simeq"] & \underline{\mathrm{PCMon}}_I(\mathcal{S}) \otimes \mathcal{D} \arrow[d, "L_{\mathrm{Seg}}"] \\
\underline{\mathrm{CMon}}_I(\mathcal{D}) \arrow[r, phantom, "\simeq"] & \underline{\mathrm{CMon}}_I(\mathcal{S}) \otimes \mathcal{D}
\end{tikzcd}$$

of [CHLL24, Prop 3.3.4]. The bottom functor is a symmetric monoidal localization of the top. In particular, choosing $\mathcal{D} = \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})$, by Lemma 1.80, it suffices to prove this in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$.

The $\mathcal{T}$-Yoneda embedding is $\mathcal{T}$-symmetric monoidal for the $\mathcal{T}$-Day convolution by [NS22, Thm 6.0.12], so $1_V^{\mathrm{Day}} \simeq y(*_V)$. Hence Yoneda's lemma yields that

$$\mathrm{Map}_V(1_V^{\mathrm{Day}}, X_V) \simeq \mathrm{Map}(y(*_V), X_V) \simeq X_V(*_V),$$

which implies that $1^{\mathrm{Day}} \simeq 1^{\mathrm{mode}}$, and hence the theorem, by Observation 1.77. □

Remark 1.81. It is not likely that it is necessary for $\mathcal{T}$ to be atomic orbital in the above argument; indeed, for $\underline{\mathrm{CMon}}_I(\mathcal{C}) \coloneqq \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C})$ to implement I*-semiadditivization*, it suffices to assume that I is a weak indexing category with respect to an implicit *atomic orbital subcategory* $\mathcal{P} \subset \mathcal{T}$ (c.f. [CLL23b; CLL24]). Unfortunately, the author is not aware of a symmetric monodial structure on partially presentable $\mathcal{T}$-categories, and developing such a thing would lead us far afield from our current operadic goals. ◁

1.4. **The homotopy I-symmetric monoidal d-category.** Recall that, a space is (-2)-*truncated* if it is empty, (-1)-*truncated* if it is empty or contractible, and for $d \geq 0$, a space X is d-*truncated* if it is a disjoint union of connected spaces $(X_\alpha)_{\alpha \in A}$ such that $\pi_m(X_\alpha) = 0$ for all $m > d$ and $\alpha \in A$.

Recall that a $(d+1)$-*category* is an ∞-category $\mathcal{C}$ such that the space $\mathrm{Map}(X,Y)$ is d-truncated for all $X, Y \in \mathcal{C}$. We say that an ∞-category is a -1-*category* if it is either $*$ or empty. In general, we write $\mathrm{Cat}_d \subset \mathrm{Cat}$ for the full subcategory spanned by the ∞-categories with the property that they are d-categories.

Definition 1.82. The $\mathcal{T}$-∞-category of small $\mathcal{T}$-d-categories is

$$\underline{\mathrm{Cat}}_{\mathcal{T},d} := \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathrm{Cat}_d.$$

A $\mathcal{T}$-*poset* is a $\mathcal{T}$-0-category. If $I \subset \mathbb{F}_{\mathcal{T}}$ is pullback-stable, the $\mathcal{T}$-∞-*category of small I-symmetric monoidal d-categories* is

$$\underline{\mathrm{Cat}}^{\otimes}_{I,d} := \underline{\mathrm{CMon}}_I \mathrm{Cat}_d.$$

We write $\mathrm{Cat}_{\mathcal{T},d} := \Gamma^{\mathcal{T}} \underline{\mathrm{Cat}}_{\mathcal{T},d}$ and $\mathrm{Cat}^{\otimes}_{I,d} := \Gamma^{\mathcal{T}} \underline{\mathrm{Cat}}^{\otimes}_{I,d}$. ◁

By the following lemma, $\underline{\mathrm{Cat}}_{\mathcal{T},d}$ is a $\mathcal{T}$-$(d+1)$-category and $\mathrm{Cat}_{\mathcal{T},d}$ is a $(d+1)$-category.

Lemma 1.83 ([HTT, Cor 2.3.4.8, Prop 2.3.4.12, Cor 2.3.4.19])**.** Cat_d *is a $(d+1)$-category and the inclusion*

$$\mathrm{Cat}_d \hookrightarrow \mathrm{Cat}$$

has a right adjoint $h_d \colon \mathrm{Cat} \to \mathrm{Cat}_d$.

Construction 1.84. By Lemmas 1.33 and 1.83, the functor $\underline{\mathrm{Cat}}_{\mathcal{T},d} \hookrightarrow \underline{\mathrm{Cat}}_{\mathcal{T}}$ is an inclusion of a localizing $\mathcal{T}$-subcategory; let $h_{\mathcal{T},d} \colon \underline{\mathrm{Cat}}_{\mathcal{T}} \to \underline{\mathrm{Cat}}_{\mathcal{T},d}$ be the associated $\mathcal{T}$-left adjoint.

The mapping spaces in a product of categories are the product of the mapping spaces; in particular, the inclusion $\mathrm{Cat}_d \hookrightarrow \mathrm{Cat}$ is product-preserving. Hence Lemmas 1.59 and 1.83 construct an adjunction

$$h_{\mathcal{T},d} \colon \mathrm{Cat}^{\otimes}_I \rightleftarrows \mathrm{Cat}^{\otimes}_{I,d} \colon \iota.$$

whose right adjoint is fully faithful. We refer to $h_{\mathcal{T},d}$ as the *homotopy I-symmetric monoidal d-category.* ◁

The remainder of this subsection will be dedicated to recognition results for $\mathcal{T}$-symmetric monoidal d-categories, which will be useful throughout the remainder of the paper. We first reduce this consideration to that of plain $\mathcal{T}$-∞-categories; the following proposition follows by unwinding definitions and noting that $\mathrm{Cat}_d \hookrightarrow \mathrm{Cat}$ is closed under products.

Proposition 1.85. *If $I \subset \mathbb{F}_{\mathcal{T}}$ is a one-object weak indexing system, then $\mathcal{C}^{\otimes} \in \mathrm{Cat}^{\otimes}_I$ is a I-symmetric monoidal d-category if and only if its underlying $\mathcal{T}$-∞-category $\mathcal{C}$ is a $\mathcal{T}$-d-category.*

Often in equivariant higher algebra, we will find that our objects come with natural maps to $\mathcal{T}$-1-categories, and we'd like to develop a recognition theorem in this case in terms of mapping spaces.

Proposition 1.86. *A $\mathcal{T}$-∞-category $\mathcal{C}$ is a $\mathcal{T}$-d-category if and only if*

$$\mathrm{Mor}_V(\mathcal{C}) := \mathrm{Fun}(\Delta^1, \mathcal{C}_V)^{\simeq}$$

is $(d-2)$-truncated for all $V \in \mathcal{T}$.

Proof. By definition, it suffices to prove this in the case $\mathcal{T} = *$. Fix $f, g \in \mathrm{Mor}_V(\mathcal{C})$. Then, we may present $\mathrm{Map}(f,g)$ as a disjoint union over a,b of homotopies

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & X \arrow[d, "b"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

For fixed a, b, this is either empty or equivalent to the component of the space $\mathrm{Map}(S^1, \mathrm{Map}(W,Z))$ whose underlying map is homotopic to bf. If $\mathcal{C}$ is a d-category, then this is $(d-2)$-truncated; conversely, choosing $a, b = \mathrm{id}$ and $f = g$, if this is $(d-2)$-truncated for all f, then the mapping spaces of $\mathcal{C}$ are $(d-1)$-truncated, i.e. $\mathcal{C}$ is a d-category. □

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a map $\psi\colon \Delta^1 \to \mathcal{C}_V$ and $F\colon \mathcal{C} \to \mathcal{D}$, define the pullback space

$$\begin{tikzcd} \mathrm{Mor}_F^{\psi}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mor}_V(\mathcal{C}) \arrow[d] \\ B\mathrm{Aut}_{\psi} \arrow[r, hook] & \mathrm{Mor}_V(\mathcal{D}) \end{tikzcd}$$

so that $\mathrm{Mor}_F^{\psi}(\mathcal{C})$ is the disjoint union of the connected components of $\mathrm{Mor}_V(\mathcal{C})$ whose image in $\mathrm{Mor}_V(\mathcal{D})$ is equivalent to ψ. We say that F has $(d-1)$-*truncated mapping fibers* if $\mathrm{Mor}_F^{\psi}(\mathcal{C})$ is $(d-2)$-truncated for all $V \in \mathcal{T}$ and $\psi \in \mathrm{Mor}_V(\mathcal{C})$.

Corollary 1.87. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-functor and $\mathcal{D}$ is a $\mathcal{T}$-1-category. Then, the following are equivalent for $d \geq 1$:*

(1) F has $(d-1)$-truncated mapping fibers.
(2) $\mathcal{C}$ is a $\mathcal{T}$-d-category.

Additionally, the following are equivalent.

(1') $F^{\simeq}\colon \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is fully faithful and F has (-1)-truncated mapping fibers.
(2') F includes $\mathcal{C}$ as a (replete) $\mathcal{T}$-subcategory of $\mathcal{D}$.

Proof. After Proposition 1.86, the only remaining part is the equivalence between (1') and (2'). Note that $B\mathrm{Aut}_{\psi}$ is -1-truncated by Proposition 1.86, so (1') is equivalent to the conditions that $\mathcal{C}$ is a $\mathcal{T}$-1-category and $F_V : \mathcal{C}_V \to \mathcal{D}_V$ is a faithful functor which is fully faithful on cores, i.e. it is a (replete) subcategory inclusion; by Observation 1.13, this is equivalent to (2'). □

2. Equivariant operads and symmetric sequences

In Section 2.1, we begin by recalling rudiments of the theory of *algebraic patterns and Segal objects* of [CH21] and the theory of *fibrous patterns and the Segal envelope* of [BHS22]; in the case of $\mathfrak{O} = \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, we show in Appendix A.1 that this recovers the theory of $\mathcal{T}$-symmetric monoidal ∞-categories, $\mathcal{T}$-∞-operads (henceforth $\mathcal{T}$-operads), and the $\mathcal{T}$-symmetric monoidal envelope of [NS22]. We go on in Section 2.2 to specialize several results of [BHS22; CH21] to this setting and construct the family of *weak $\mathcal{N}_\infty$-operads.*

After this, we go on to study the *underlying $\mathcal{T}$-symmetric monoidal sequence* functor in Section 2.3, showing in Corollary 2.62 that it forms a fiberwise-monadic $\mathcal{T}$-functor

$$\underline{\mathrm{sseq}}_{\mathcal{T}} : \underline{\mathrm{Op}}_{\mathcal{T}} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}});$$

in particular, this implies that it is a conservative right $\mathcal{T}$-adjoint and confirms an atomic orbital lift of Theorem A. In Section 2.6.3, we use this to confirm Corollary B.

In Section 2.4 we go on to compute the monad $T_{\mathcal{O}}(-)$ for $\mathcal{O}$-algebras in arbitrary $\mathcal{T}$-symmetric monoidal ∞-categories; in particular, when $\mathcal{C} \simeq \underline{\mathcal{S}}_{\mathcal{T}}$ for a structure whose indexed tensor products are indexed products, we naturally split off a $\mathcal{O}(S)$-summand from $T_{\mathcal{O}}(S)$; using our atomic orbital lift of Theorem A, we conclude that $\underline{\mathrm{Alg}}_{(-)}(\underline{\mathcal{S}}_{\mathcal{T}})\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ is conservative.

Last, in preparation for forthcoming work, we initiate in Section 2.5 the study of the localizing subcategory of $\mathcal{T}$-operads whose underlying $\mathcal{T}$-symmetric sequence is $(d-1)$-truncated, called *$\mathcal{T}$-d-operads*; we show in particular that the full $\mathcal{T}$-subcategory of $\underline{\mathrm{Op}}_{\mathcal{T}}$ spanned by $\mathcal{T}$-operads whose S-ary spaces are empty or contractible form a *$\mathcal{T}$-poset.* We use this in Section 2.6 to prove that Bonventre's nerve restricts to an equivalence between categories of G-1-operads.

We assure the reader exclusively interested in *using* $\mathcal{T}$-operads that the relevant interpretations of the results of Section 2.1 will be restated throughout the following subsections, so these sections may be black-boxed at the cost of completeness of proofs.

2.1. **Recollections on algebraic patterns.** An algebraic pattern is a collection of data encoding *Segal conditions* for the purpose of homotopy-coherent algebra. Given an algebraic pattern $\mathfrak{O}$ and a complete ∞-category $\mathcal{C}$, there is an ∞-category of *Segal $\mathfrak{O}$-objects in* $\mathcal{C}$, which we view as $\mathfrak{O}$-monoids in $\mathcal{C}$; these are presented as functors $\mathfrak{O} \to \mathcal{C}$ satisfying a Segal condition.

We may view $\mathcal{O}$-objects in $\mathbf{Cat}$ (aka Segal $\mathcal{O}$-∞-categories) as $\mathcal{O}$-monoidal ∞-categories; these straighten to cocartesian fibrations over $\mathcal{O}$ satisfying conditions. As in [HA, § 2], the condition of *being a cocartesian fibration* may be relaxed to construct a form of operads parameterized by $\mathcal{O}$, called *fibrous $\mathcal{O}$-patterns.*

In contrast to the categorical patterns of [HA, § B], these are manifestly ∞-categorical, and it is relatively easy to construct push-pull adjunctions between categories of fibrous patterns over different algebraic patterns; we found our theory of I-operads in this syntax for this reason, as the Boardman-Vogt tensor product is most easily defined in terms of pushforward along maps of algebraic patterns.

The author would like to emphasize that the program surrounding algebraic patterns has achieved many results not mentioned here, as fibrous patterns only play a foundational role. For a significantly more thorough and elegant treatment, we recommend [BHS22; CH21; CH23].

2.1.1. *Algebraic patterns, Segal objects, and fibrous patterns.*

Definition 2.1. An *algebraic pattern* is a triple $(\mathfrak{B}, (\mathfrak{B}^{\mathrm{in}}, \mathfrak{B}^{\mathrm{act}}), \mathfrak{B}^{\mathrm{el}})$, where $(\mathfrak{B}^{\mathrm{in}}, \mathfrak{B}^{\mathrm{act}})$ is a factorization system on $\mathfrak{B}$ and $\mathfrak{B}^{\mathrm{el}} \subset \mathfrak{B}^{\mathrm{in}}$ is a full subcategory.[9] The ∞-category $\mathrm{AlgPatt} \subset \mathrm{Fun}(\mathbf{Q}, \mathrm{Cat})$ is the full subcategory spanned by algebraic patterns, where

$$\mathbf{Q} := \bullet \to \bullet \to \bullet \leftarrow \bullet. \tag{4}$$ ◁

We refer to the morphisms in $\mathfrak{B}^{\mathrm{in}}$ as "inert morphisms," morphisms in $\mathfrak{B}^{\mathrm{act}}$ as "active morphisms," and objects in $\mathfrak{B}^{\mathrm{el}}$ as "elementary objects." When it is clear from context, we will abusively refer to the triple $\left(\mathfrak{B}, (\mathfrak{B}^{\mathrm{in}}, \mathfrak{B}^{\mathrm{act}}), \mathfrak{B}^{\mathrm{el}}\right)$ simply as $\mathfrak{B}$. The following is our a primary source of examples.

Construction 2.2. An *adequate quadruple* is the data of an adequate triple $\mathcal{X}_b, \mathcal{X}_f \subset \mathcal{X}$ in the sense of Section 1.2 together with a full subcategory $\mathcal{X}_0 \subset \mathcal{X}_b$; the *$\infty$-category of adequate quadruples* is the full subcategory

$$\mathrm{Quad}^{\mathrm{adeq}} \subset \mathrm{Fun}(\mathbf{Q}, \mathrm{Cat})$$

spanned by adequate quadruples, where $\mathbf{Q}$ is defined by Eq. (4).

Given an adequate quadruple $\mathcal{X}_0 \subset \mathcal{X}_b \subset \mathcal{X} \supset \mathcal{X}_f$, let $\mathcal{X}_b^{\mathrm{op}} \subset \mathrm{Span}_{b,f}(\mathcal{X})$ be the wide subcategory spanned by the spans $X \xleftarrow{\psi_b} R \xrightarrow{\psi_f} Y$ with ψ_f an equivalence, and similarly $\mathcal{X}_f \subset \mathrm{Span}_{b,f}(\mathcal{X})$ the side subcategory of spans with ψ_b an equivalence. This yields a factorization cystem [HHLN23, Prop 4.9]

$$\mathcal{X}_b^{\mathrm{op}} \hookrightarrow \mathrm{Span}_{b,f}(\mathcal{X}) \hookleftarrow \mathcal{X}_f.$$

We define the span pattern $\mathrm{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ via the data

- underlying ∞-category $\mathrm{Span}_{b,f}(\mathcal{X})$,
- inert morphisms $\mathcal{X}_b^{\mathrm{op}} \subset \mathrm{Span}(\mathcal{X})$,
- active morphisms $\mathcal{X}_f \subset \mathrm{Span}(\mathcal{X})$, and
- elementary objects $\mathcal{X}_0^{\mathrm{op}} \subset \mathcal{X}_b^{\mathrm{op}}$.

Given a map of adequate quadruples $\left(\mathcal{X}, (\mathcal{X}_b, \mathcal{X}_f), \mathcal{X}_0\right) \to \left(\mathcal{Y}, (\mathcal{Y}_b, \mathcal{Y}_f), \mathcal{Y}_0\right)$ the associated functor $\mathrm{Span}_{b,f}(\mathcal{X}) \to \mathrm{Span}_{b,f}(\mathcal{Y})$ preserves inert morphisms, active morphisms, and elementary objects by defintiion; hence the functor $\mathrm{Span}_{-,-}(-;-)\colon \mathrm{Quad}^{\mathrm{adeq}} \to \mathrm{Fun}(\mathbf{Q}, \mathrm{Cat})$ descends to a functor

$$\mathrm{Span}_{-,-}(-;-)\colon \mathrm{Quad}^{\mathrm{adeq}} \to \mathrm{AlgPatt}.$$ ◁

The central example for equivariant higher algebra is the following.

Example 2.3. When $\mathcal{T}$ is an orbital ∞-category, $I \subset \mathbb{F}_{\mathcal{T}}$ is a $\mathcal{T}$-weak indexing system (e.g. $I = \mathbb{F}_{\mathcal{T}}$), and $c(I)$ its *color family* in the sense of Eq. (3), we define the *effective I-Burnside pattern*

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := \mathrm{Span}_{\mathrm{all},I}\left(\mathbb{F}_{c(I)}; c(I)\right)$$ ◁

[9] Throughout this paper, we adopt the definition of *factorization system* used in [CH21, Rmk 2.2], which does not assert any lifting properties; that is, a facorization system on $\mathcal{C}$ is a pair of wide subcategories $\mathcal{C}^L, \mathcal{C}^R \subset \mathcal{C}$ satisfying the condition that, for all maps $X \xrightarrow{f} X'$, the space of factorizations $X \xrightarrow{l} Y \xrightarrow{r} X'$ with $l \in \mathcal{C}^L$ and $r \in \mathcal{C}^R$ is contractible.

Example 2.4. Given $\mathcal{T}$ an orbital ∞-category, we may define the *∞-category of finite pointed $\mathcal{T}$-sets* as

$$\mathbb{F}_{\mathcal{T},*} := \mathrm{Span}_{\mathrm{si,all}}(\mathbb{F}_{\mathcal{T}}),$$

where $\mathbb{F}_{\mathcal{T}}^{\mathrm{si}} \subset \mathbb{F}_{\mathcal{T}}$ is the wide subcategory of summand inclusions. In fact, the class of summand inclusions is restriction-stable, so this lifts to an algebraic pattern

$$\mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T},*} \simeq \mathrm{Span}_{\mathrm{si,all}}(\mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T}};\mathcal{T}^{\mathrm{op}});$$

together with a map of algebraic patterns

$$\varphi : \mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T},*} \hookrightarrow \mathrm{Span}_{\mathrm{all,all}}(\mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T}};\mathcal{T}^{\mathrm{op}}) \xrightarrow{U} \mathrm{Span}(\mathbb{F}_{\mathcal{T}}). \tag{5}$$ ◁

Algebraic patterns provide a general framework for algebraic structures satisfying the associated *Segal conditions*, which are encoded in the notion of *Segal objects*.

Definition 2.5. Let $\mathcal{C}$ be a complete ∞-category and let $\mathfrak{O}$ be an algebraic pattern. Then, the ∞-category of *Segal $\mathfrak{O}$-objects in $\mathcal{C}$* is the full subcategory $\mathrm{Seg}_{\mathfrak{O}}(\mathcal{C}) \subset \mathrm{Fun}(\mathfrak{O},\mathcal{C})$ consisting of functors F such that, for every object $O \in \mathcal{O}$, the natural map

$$F(O) \to \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{O/}} F(E)$$

is an equivalence, where $\mathfrak{O}^{\mathrm{el}}_{O/} := \mathfrak{O}^{\mathrm{el}} \times_{\mathfrak{O}^{\mathrm{in}},\mathrm{ev}_1} \mathfrak{O}^{\mathrm{in}}_{O/}$ is the ∞-category whose objects consist of inert morphisms from O to an elementary object. ◁

Remark 2.6. By [CH21, Lem 2.9], a functor $F : \mathfrak{O} \to \mathcal{C}$ is a Segal $\mathfrak{O}$-object if and only if the associated functor $F|_{\mathfrak{O}^{\mathrm{int}}}$ is right Kan extended from $F|_{\mathfrak{O}^{\mathrm{el}}}$ along the inclusion $\mathfrak{O}^{\mathrm{el}} \to \mathfrak{O}^{\mathrm{int}}$. ◁

Example 2.7. We show in Lemma A.5 that $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{Z/} = (\mathbb{F}_{\mathcal{T},/Z})^{\mathrm{op}}$ contains the set of orbits $\mathrm{Orb}(Z)$ as an initial subcategory. Hence there is an equivalence of full subcategories

$$\mathrm{Seg}_{\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{C}) \simeq \mathrm{CMon}_I(\mathcal{C}) \qquad \subset \qquad \mathrm{Fun}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}),\mathcal{C}).$$ ◁

One benefit of the framework of Segal objects is their general monadicity result.

Proposition 2.8 ([CH21, Cor 8.2])**.** *if $\mathfrak{O}$ is an algebraic pattern and $\mathcal{C}$ a presentable ∞-category, then the forgetful functor*

$$U : \mathrm{Seg}_{\mathfrak{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathfrak{O}^{\mathrm{el}},\mathcal{C})$$

is monadic; in particular, it is conservative.

Corollary 2.9. *A morphism of I-commutative monoids is an equivalence if and only if its underlying morphism of $c(I)$-objects is an equivalence; in particular, an I-symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is an equivalence if and only if the underlying $c(I)$-functor is an equivalence.*

Another benefit of Segal objects is a rich framework for functoriality.

Definition 2.10. A morphism of algebraic patterns $f : \mathfrak{P} \to \mathfrak{O}$ is a called a:

- *Segal morphism* if pullback $f^* : \mathrm{Fun}(\mathfrak{O},\mathcal{C}) \to \mathrm{Fun}(\mathfrak{P},\mathcal{C})$ preserves Segal objects in any complete ∞-category $\mathcal{C}$.
- *strong Segal morphism* if the associated functor $f^{\mathrm{el}}_{X/} : \mathfrak{P}^{\mathrm{el}}_{X/} \to \mathfrak{O}^{\mathrm{el}}_{f(X)/}$ is initial for all $X \in \mathfrak{P}$. ◁

Observation 2.11. The conditions for Segal morphisms and strong Segal morphisms are each compatible with compositions and equivalences; that is, there are *core-preserving* wide subcategories $\mathrm{AlgPatt}^{\mathrm{Seg}}, \mathrm{AlgPatt}^{\mathrm{Strong-Seg}} \subset \mathrm{AlgPatt}$ whose morphisms are the Segal morphisms and strong Segal morphisms, respectively. ◁

Remark 2.12. [CH21, Lem 4.5] concludes that f is a Segal morphism if f^* preserves Segal objects in *spaces*. ◁

Example 2.13. We show in Proposition A.12 that, given any functor $\mathcal{T} \to \mathcal{T}'$ of atomic orbital ∞-categories, the associated functor

$$\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}'})$$

is a Segal morphism. Additionally, in Corollary A.8, we show that the map φ of Eq. (5) is a segal morphism, constructing a pullback map

$$\mathrm{CMon}_{\mathcal{T}}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{C}) \to \mathrm{Seg}_{\mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C}).$$

In [Bar23, Cor 2.64], conditions for a strong Segal morphism were developed concerning when their pullback maps are equivalences, and these conditions were checked in [BHS22, Prop 5.2.14] in the case $\mathcal{T} = \mathcal{O}_G^{\mathrm{op}}$; we review their argument and extend it to arbitrary atomic orbital ∞-categories in Appendix A.1. The existence of such an equivalence (not necessarily induced by a pattern) is not new, and to the author's knowledge, first appeared as [Nar16, Thm 6.5]. $\triangleleft$

Lemma 2.14 ([CH21, Cor 5.5])**.** *$\mathrm{AlgPatt} \subset \mathrm{Fun}(\mathbf{Q}, \mathrm{Cat})$ is a localizing subcategory; in particular, $\mathrm{AlgPatt}$ has small limits.*

Example 2.15. In particular, AlgPatt has products. By [CH21, Ex 5.7], there is an equivalence

$$\mathrm{Seg}_{\mathfrak{B}\times\mathfrak{B}'}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathfrak{B}}\mathrm{Seg}_{\mathfrak{B}'}(\mathcal{C}).$$

In particular, this combined with Example 2.7 gives a complete segal space model for I-symmetric monoidal categories; indeed, the pattern $\Delta^{\mathrm{op},\natural}$ of [CH21, Ex 5.8] has Segal $\Delta^{\mathrm{op},\natural}$-objects in $\mathcal{C}$ given by *complete Segal objects in* $\mathcal{C}$, specializing to the fact that $\mathrm{Seg}_{\Delta^{\mathrm{op},\natural}}(\mathcal{S}) \simeq \mathrm{Cat}$, and hence

$$\mathrm{Seg}_{\Delta^{\mathrm{op},\natural}}(\mathcal{S}_{\mathcal{T}}) \simeq \mathrm{Seg}_{\mathcal{T}^{\mathrm{op,el}}\times\Delta^{\mathrm{op},\natural}}(\mathcal{S}) \simeq \mathrm{Seg}_{\mathcal{T}^{\mathrm{op,el}}}(\mathrm{Cat}) \simeq \mathrm{Cat}_{\mathcal{T}},$$

where $\mathcal{T}^{\mathrm{op,op,el}}$ is the algberaic pattern with $\left(\mathcal{T}^{\mathrm{op,el}}\right)^{\mathrm{el}} \simeq \left(\mathcal{T}^{\mathrm{op,el}}\right)^{\mathrm{int}} \simeq \mathcal{T}^{\mathrm{op}} \simeq \left(\mathcal{T}^{\mathrm{op,el}}\right)^{\mathrm{act}}$. Additionally,

$$\mathrm{Seg}_{\Delta^{\mathrm{op},\natural}}(\mathrm{CMon}_{\mathcal{T}}(\mathcal{S})) \simeq \mathrm{Seg}_{\Delta^{\mathrm{op},\natural}\times\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{S}) \simeq \mathrm{Seg}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathrm{Cat}) \simeq \mathrm{CMon}_{\mathcal{T}}(\mathrm{Cat}).$$ $\triangleleft$

Cartesian products of patterns play nicely with well-structured maps of patterns.

Lemma 2.16. *Suppose $f : \mathfrak{O} \to \mathfrak{P}$ and $f' : \mathfrak{O}' \to \mathfrak{P}'$ are (resp. strong) Segal morphisms. Then,*

$$f \times f' : \mathfrak{O} \times \mathfrak{O}' \to \mathfrak{P} \times \mathfrak{P}'$$

is a (strong) Segal morphism.

Proof. The case of Segal morphisms follows immediately from Example 2.15, so we assume that f, f' are strong Segal. Then, the induced map

$$f^{\mathrm{el}}_{X/} \times f'^{\mathrm{el}}_{X'/} = (f \times f')^{\mathrm{el}}_{(X,X')/} : (\mathfrak{O}\times\mathfrak{O}')^{\mathrm{el}}_{(X,X')/} \to (\mathfrak{P}\times\mathfrak{P}')^{\mathrm{el}}_{(fx,fx')/}$$

is a product of initial maps; it then follows that it is initial, since limits in product categories are computed pointwise. $\square$

The unstraightening functor of [HTT] realizes $\mathrm{Seg}_{\mathfrak{O}}(\mathrm{Cat})$ as a non-full subcategory of $\mathrm{Cat}_{/\mathfrak{O}}$ consisting of cocartesian fibrations satisfying Segal conditions; we relax this for the following definition, which is equivalent to the original definition stated in [BHS22, Def 4.1.2] by [BHS22, Prop 4.1.6].

Definition 2.17. Let $\mathfrak{B}$ be an algebraic pattern. A *fibrous $\mathfrak{B}$-pattern* is a functor $\pi : \mathfrak{O} \to \mathfrak{B}$ such that

(1) (inert morphisms) $\mathfrak{O}$ has π-cocartesian lifts for inert morphisms of $\mathfrak{B}$,
(2) (Segal condition for colors) For every active morphism $\omega \colon V_0 \to V_1$ in $\mathfrak{B}$, the functor

$$\mathfrak{O}_{V_0} \to \lim_{\alpha\in\mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathfrak{O}_{\omega_{\alpha,!}V_1}$$

induced by cocartesian transport along ω_α is an equivalence, where $\omega_{(-)} \colon \mathfrak{B}^{\mathrm{el}}_{Y/} \to \mathfrak{B}^{\mathrm{int}}_{X/}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and
(3) (Segal condition for multimorphisms) for every active morphism $\omega \colon V_0 \to V_1$ in $\mathfrak{B}$ and all pairs of objects $X_i \in \mathfrak{O}_{\mathfrak{B}_{V_i}}$, the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathfrak{O}}(X_0, X_1) \arrow[r] \arrow[d] & \lim_{\alpha\in\mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_{\mathfrak{O}}(X_0, \omega_{\alpha,!}X_1) \arrow[d] \\
\mathrm{Map}_{\mathfrak{B}}(V_0, V_1) \arrow[r] & \lim_{\alpha\in\mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_{\mathfrak{B}}(V_0, \omega_{\alpha,!}V_1)
\end{tikzcd}$$

is cartesian.

We denote by $\mathrm{Fbrs}(\mathfrak{B}) \subset \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathfrak{B}}$ the full subcategory spanned by the fibrous $\mathfrak{B}$-patterns, where the latter category has objects the functors to $\mathfrak{B}$ possessing cocartesian lifts over inert morphisms and morphisms the functors preserving such cocartesian lifts. $\triangleleft$

Remark 2.18. As noted in [BHS22, Rmk 4.1.8], in the presence of condition (3) above, condition (2) may be weakened to assert that the functor $\mathfrak{O}_{V_0} \to \lim_{\alpha \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathfrak{O}_{\omega_{\alpha,!}V_1}$ is a π_0-surjection without changing the resulting notion. To match [BHS22, Prop 4.1.6], we may even take the intermediate assumption that this functor induces an equivalence on cores. ◁

Example 2.19. Fibrous $\mathbb{F}_*$-patterns are equivalent to ∞-operads (c.f. [HA]), and in Appendix A.1 we will extend a proof due to [BHS22] (in the case $\mathcal{T} = \mathcal{O}_G$) that fibrous $\underline{\mathbb{F}}_{\mathcal{T},*}$-patterns are equivalent to the $\mathcal{T}$-∞-operads of [NS22]. ◁

A fibrous pattern $\pi : \mathfrak{O} \to \mathfrak{B}$ inherits a structure of an algebraic pattern whose inert morphisms consist of π-cocartesian lifts of inert morphisms in $\mathfrak{B}$, whose active morphisms are arbitrary lifts of active morphisms in $\mathfrak{B}$, and whose elementary objects are spanned by lifts of elementary objects. This is canonical:

Proposition 2.20 ([BHS22, Cor 4.1.7])**.** *Fibrous patterns are closed under composition for the above pattern structure, inducing an equivalence*

$$\mathrm{Fbrs}(\mathfrak{O}) \simeq \mathrm{Fbrs}(\mathfrak{B})_{/\mathfrak{O}}.$$

Furthermore, the fully faithful functor $U : \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{AlgPatt}_{/\mathfrak{B}}$ is well behaved.

Proposition 2.21 ([BHS22, Cor 4.2.3])**.** *The fully faithful functor U participates in an adjunction*

$$\begin{tikzcd} \mathrm{Fbrs}(\mathfrak{B}) \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathrm{AlgPatt}_{/\mathfrak{B}} \arrow[l, bend left, "L_{\mathrm{Fbrs}}"] \end{tikzcd}$$

We construct many Segal morphisms in Appendix A.3. Many more are constructed in the following.

Proposition 2.22 ([BHS22, Obs 4.1.14])**.** *Fibrous patterns are strong Segal morphisms.*

2.1.2. *The Segal envelope.* In [BHS22, Lem 4.2.4] it was verified that a fibrous $\mathfrak{O}$-pattern is a cocartesian fibration if and only if it's the straightening of a Segal $\mathfrak{O}$-category under the condition of *soundness*; this lifts the fact that an operad $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category if and only if the corresponding functor $\mathcal{C}^{\otimes} \to \mathbb{F}_*$ is a cocartesian fibration. We would like to describe adjunctions relating fibrous patterns to Segal objects, but to do so, we need a few constructions.

Definition 2.23. Given $\mathfrak{O} \to \mathfrak{B}$ a map of algebraic patterns, the *Segal envelope of* $\mathfrak{O}$ *over* $\mathfrak{B}$ is the horizontal composite

$$\begin{tikzcd} \mathrm{Env}_{\mathfrak{B}}\mathfrak{O} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B}) \arrow[r, "t"] \arrow[d, "s"] & \mathfrak{B} \\ \mathfrak{O} \arrow[r] & \mathfrak{B} & \end{tikzcd}$$

Where $\mathrm{Ar}_{\mathrm{act}}(\mathfrak{B}) \subset \mathrm{Ar}(\mathfrak{B}) = \mathrm{Fun}(\Delta^1, \mathfrak{B})$ is the full subcategory spanned by active arrows and s, t are the *source and target* functors. We denote the envelope of the terminal $\mathfrak{B}$-pattern as

$$\mathscr{A}_{\mathfrak{B}} := \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B}) \xrightarrow{t} \mathfrak{B}. \qquad ◁$$

Let $\mathfrak{O}$ be an algebraic pattern and $\omega : X \to Y$ an active map. Define the pullback square

$$\begin{tikzcd} \mathfrak{O}^{\mathrm{el}}(\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}(\mathfrak{O}^{\mathrm{int}}_{X/}) \arrow[d, "{(s,t)}"] \\ \mathfrak{O}^{\mathrm{el}}_{Y/} \times \mathfrak{O}^{\mathrm{el}}_{X/} \arrow[r, "{(\omega_{(-)}, \mathrm{id})}"] & \mathfrak{O}^{\mathrm{int}}_{X/} \times \mathfrak{O}^{\mathrm{int}}_{X/} \end{tikzcd}$$

where $\omega_{(-)} : \mathfrak{O}^{\mathrm{el}}_{Y/} \to \mathfrak{O}^{\mathrm{int}}_{X/}$ sends $\alpha : Y \to E$ to the inert map ω_a of the inert-active factorization of $X \xrightarrow{\omega} Y \xrightarrow{a} E$.

Definition 2.24. $\mathfrak{O}$ is *sound* if, for all $\omega : X \to Y$ active, the associated map $\mathfrak{O}^{\mathrm{el}}(\omega) \to \mathfrak{O}^{\mathrm{el}}_{X/}$ is initial. A sound pattern $\mathfrak{O}$ is *soundly extendable* if $\mathscr{A}_{\mathfrak{O}}$ is a Segal $\mathfrak{O}$-∞-category. ◁

Soundness as a condition allows one to simplify Segal conditions, yielding functoriality results for the categories of Segal objects and fibrous patterns; sound extendibility reduces many instances of *relative Segal objects* in the sense [BHS22, Def 3.1.8] to a morphism with Segal domain by [BHS22, Obs 3.1.9]. To that end, we prove the following in Proposition A.11, extending [BHS22, Lem 4.1.19].

Proposition 2.25. *Suppose $f\colon \mathfrak{P} \to \mathfrak{O}$ is a Segal morphism and either $\mathfrak{O}$ is soundly extendable or f is strong Segal. Then, the pullback functor $f^*\colon \mathrm{Cat}_{/\mathfrak{P}} \to \mathrm{Cat}_{/\mathfrak{O}}$ preserves fibrous patterns; furthermore, the functor*

$$f^*\colon \mathrm{Fbrs}(\mathfrak{O}) \to \mathrm{Fbrs}(\mathfrak{P})$$

has a left adjoint given by $L_{\mathrm{Fbrs}} f_!$.

Example 2.26. We verify in Lemma A.7 that $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is soundly extendable; hence Example 2.13 and Proposition 2.25 together yield a functor

$$\mathrm{Op}_{\mathcal{T}} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}});$$

we review a proof that this is an equivalence (originally due to [BHS22] when $\mathcal{T} = \mathcal{O}_G$) in Corollary A.8. ◁

Given $f\colon \mathfrak{P} \to \mathfrak{O}$ a Segal morphism between algebraic patterns, we then define the composite functor

$$f^{\circledast}\colon \mathrm{Seg}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{O}}} \xrightarrow{f^*} \mathrm{Seg}_{\mathfrak{O}}^{/f^*\mathscr{A}_{\mathfrak{O}}} \xrightarrow{q^*} \mathrm{Seg}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{P}}}$$

where q is the map fitting into the following diagram:

$$\begin{tikzcd}
\mathscr{A}_{\mathfrak{P}} \arrow[dr, "q"] \arrow[drr, "\mathscr{A}_f"] \arrow[ddr, "p"'] & & \\
& f^*\mathscr{A}_{\mathfrak{O}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{A}_{\mathfrak{O}} \arrow[d, "p"] \\
& \mathfrak{P} \arrow[r, "f"'] & \mathfrak{O}
\end{tikzcd}$$

This participates in the following theorem, which was proved under a *strong Segal* assumption which is rendered unnecessary by Proposition 2.25.

Theorem 2.27 ([BHS22, Prop 4.2.1, Prop 4.2.5, Thm 4.2.6, Rem 4.2.8])**.** *Let $\mathfrak{O}$ be a soundly extendable pattern. Then, $\mathrm{Env}_{\mathfrak{O}}$ is the left adjoint in an adjoint pair*

$$\begin{tikzcd}
\mathrm{Fbrs}(\mathfrak{O}) \arrow[r, bend left, "\mathrm{Env}_{\mathfrak{O}}"] \arrow[r, phantom, "\perp"] & \mathrm{Seg}_{\mathfrak{O}}(\mathrm{Cat}) \arrow[l, bend left, "\mathrm{Un}"]
\end{tikzcd}$$

By taking slice categories, this induces an adjunction

$$\begin{tikzcd}
\mathrm{Fbrs}(\mathfrak{O}) \arrow[r, hook, bend left, "\mathrm{Env}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{O}}}"] \arrow[r, phantom, "\perp"] & \mathrm{Seg}_{\mathfrak{O}}(\mathrm{Cat}) \arrow[l, bend left]
\end{tikzcd}$$

whose left adjoint is fully faithful. Furthermore, if $f: \mathfrak{O} \to \mathfrak{P}$ is a Segal morphism between soundly extendable patterns, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Seg}_{\mathfrak{O}}(\mathrm{Cat}_\infty) \arrow[r, "\mathrm{Un}"] \arrow[d, "f^*"'] & \mathrm{Fbrs}(\mathfrak{O}) \arrow[r, hook, "\mathrm{Env}_{\mathfrak{O}}^{/\mathscr{A}_{\mathfrak{O}}}"] \arrow[d, "f^*"'] & \mathrm{Seg}_{\mathfrak{O}}(\mathrm{Cat}_\infty)_{/\mathscr{A}_{\mathfrak{O}}} \arrow[r, "\mathrm{Un}"] \arrow[d, "f^{\circledast}"'] & \mathrm{Fbrs}(\mathfrak{O}) \arrow[d, "f^*"] \\
\mathrm{Seg}_{\mathfrak{P}}(\mathrm{Cat}_\infty) \arrow[r, "\mathrm{Un}"'] & \mathrm{Fbrs}(\mathfrak{P}) \arrow[r, hook, "\mathrm{Env}_{\mathfrak{P}}^{/\mathscr{A}_{\mathfrak{P}}}"'] & \mathrm{Seg}_{\mathfrak{P}}(\mathrm{Cat}_\infty)_{/\mathscr{A}_{\mathfrak{P}}} \arrow[r, "\mathrm{Un}"'] & \mathrm{Fbrs}(\mathfrak{P})
\end{tikzcd}$$

We will make frequent use of product patterns, so we observe their interaction with Segal envelopes.

Observation 2.28. If $\mathfrak{O}, \mathfrak{P}$ are fibrous $\mathfrak{B}$-patterns, then their Segal envelopes satisfy

$$\begin{aligned}
\mathrm{Env}_{\mathfrak{B}\times\mathfrak{B}}(\mathfrak{O} \times \mathfrak{P}) &\simeq (\mathfrak{O} \times \mathfrak{P}) \times_{\mathfrak{B}\times\mathfrak{B}} \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B} \times \mathfrak{B}) \\
&\simeq (\mathfrak{O} \times_{\mathfrak{B}} \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B})) \times (\mathfrak{P} \times_{\mathfrak{B}} \mathrm{Ar}_{\mathrm{act}}(\mathfrak{B})) \\
&\simeq \mathrm{Env}_{\mathfrak{B}}(\mathfrak{O}) \times \mathrm{Env}_{\mathfrak{B}}(\mathfrak{P})
\end{aligned}$$ ◁

Observation 2.29. Suppose $\mathfrak{B}$, $\mathfrak{B}'$ are soundly extendable algebraic patterns. Unwinding definitions, the projection $p : \mathfrak{B} \times \mathfrak{B}' \to \mathfrak{B}$ is a flat inner fibration and a Segal morphism; in particular, this yields a functor

$$p_*\colon \mathrm{Fbrs}(\mathfrak{B} \times \mathfrak{B}') \xrightarrow{U} \mathrm{Cat}_{/\mathfrak{B}\times\mathfrak{B}'} \xrightarrow{p_*} \mathrm{Cat}_{/\mathfrak{B}} \xrightarrow{C_{\mathrm{Fbrs}}} \mathrm{Fbrs}(\mathfrak{B})$$

which is right adjoint to $p^*\colon \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{Fbrs}(\mathfrak{B} \times \mathfrak{B}')$ by definition. ◃

2.2. $\mathcal{T}$-operads and I-operads.

We're finally ready to specialize to equivariant operads.

Definition 2.30. The ∞-category of $\mathcal{T}$-operads is

$$\mathrm{Op}_{\mathcal{T}} := \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}})).$$

More generally, when $I \subset \mathbb{F}_{\mathcal{T}}$ is a weak indexing category, the *∞-category of I-operads* is

$$\mathrm{Op}_I := \mathrm{Fbrs}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})).$$ ◃

By Proposition 2.20, if $\mathcal{O}^{\otimes}$ is an I-operad, then it has a natural pattern structure s.t. $\mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is a morphism of patterns; the inert morphisms are cocartesian lifts of backwards maps, and the active maps are *arbitrary* lifts of forwards maps.

Definition 2.31. The *∞-category of $\mathcal{O}$-monoidal ∞-categories is*

$$\mathrm{Cat}^{\otimes}_{\mathcal{O},I} := \mathrm{Seg}_{\mathcal{O}^{\otimes}}(\mathrm{Cat}).$$ ◃

When $\mathcal{O}^{\otimes} \in \mathrm{Op}_I$ is terminal, we write $\mathrm{Cat}^{\otimes}_I := \mathrm{Cat}^{\otimes}_{\mathcal{O},I}$; Corollary A.6 yields an equivalence

$$\mathrm{Cat}^{\otimes}_I \simeq \mathrm{CMon}_I(\mathrm{Cat}).$$

when I is clear from context, we will simply write $\mathrm{Cat}^{\otimes}_{\mathcal{O}}$ for $\mathrm{Cat}^{\otimes}_{\mathcal{O},I}$.

Definition 2.32. If $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are I-operads, then an *$\mathcal{O}$-algebra in $\mathcal{P}$* is a map of I-operads $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$; the ∞-category of $\mathcal{O}$-algebras in $\mathcal{P}$ is written

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) := \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}).$$ ◃

For us, the appropriate degree of generality for I will be that for which the pushforward functor $\mathrm{Op}^{\otimes}_I \to \mathrm{Op}^{\otimes}_{\mathcal{T}}$ is simply given by postcomposition along the canonical functor $\iota_I^{\mathcal{T}}\colon \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$; this turns out to be a familiar setting (c.f. [NS22, Ex 2.4.7]).

Proposition 2.33. *Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a core-full, pullback-stable subcategory. Then, the functor*

$$\mathcal{N}^{\otimes}_{I\infty} := \left(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\pi_I} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right)$$

is presents a $\mathcal{T}$-operad if and only if I is a weak indexing category in the sense of Definition 1.42.[10]

We will delay the proof of this until Page 31. If $\mathcal{O}^{\otimes} \simeq \mathcal{N}^{\otimes}_{I\infty}$ arises from Proposition 2.33, we say that $\mathcal{O}^{\otimes}$ is a *weak $\mathcal{N}_\infty$ $\mathcal{T}$-operad*, and if I is an indexing category, then we say that $\mathcal{N}^{\otimes}_{I\infty}$ is an *$\mathcal{N}_\infty$-operad*; in either case, we write

$$\mathrm{CAlg}_I(\mathcal{C}) := \mathrm{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C})$$

for the ∞-category of *I-commutative algebras in $\mathcal{C}$*. This fits nicely into the theory of I-operads:

Corollary 2.34. *There exists a canonical equivalence of categories $\mathrm{Op}_I \simeq \mathrm{Op}_{\mathcal{T},/\mathcal{N}^{\otimes}_{I\infty}}$.*

Proof. Unwinding definitions, this is Proposition 2.20 applied with $\mathfrak{O} := \mathcal{N}^{\otimes}_{I\infty}$. □

In forthcoming work [Ste24a], we will show that the morphism $\mathcal{N}^{\otimes}_{I\infty} \to \mathrm{Comm}^{\otimes}_{\mathcal{T}}$ is monic, so pushforward $\mathrm{Op}_I \to \mathrm{Op}_{\mathcal{T}}$ is fully faithful. Until then, we will largely consider Op_I and $\mathrm{Op}_{\mathcal{T}}$ separately.

[10] The conditions that $I \subset \mathbb{F}_{\mathcal{T}}$ is core-full and pullback-stable are necessary to define the ∞-category $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ in the first place; this is the most general this result can reasonably be made to be.

Example 2.35. The terminal $\mathcal{T}$-operad is presented by $\mathrm{Comm}_{\mathcal{T}}^{\otimes} = \left(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\mathrm{id}} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right)$, and hence it is a weak $\mathcal{N}_\infty$-operad; we write $\mathrm{CAlg}_{\mathcal{T}}(\mathcal{C}) := \mathrm{CAlg}_{\mathbb{F}_{\mathcal{T}}}(\mathcal{C})$, and call these *$\mathcal{T}$-commutative algebras*. For any $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$, pullback along the unique map $\mathcal{O}^{\otimes} \to \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ determines a unique natural transformation

$$\mathrm{CAlg}_{\mathcal{T}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}),$$

so we view $\mathcal{T}$-commutative algebras as a *universal $\mathcal{T}$-equivariant algebraic structure*. ◁

Fix I a weak indexing category. If $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_I^{\otimes}$ are I-symmetric monoidal ∞-categories, we say that a *lax I-symmetric monoidal functor* $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a map of their underlying $\mathcal{T}$-operads; this is an I-symmetric monoidal functor if and only if it lands in $\mathrm{Cat}_I^{\otimes}$, i.e. if and only if it preserves cocartesian lifts for arbitrary maps in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$.

2.2.1. *The structure of $\mathcal{T}$-operads.* The Segal conditions for fibrous $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$-patterns were characterized in [BHS22] in the case $\mathcal{T} = \mathcal{O}_G$; we generalize this to weak indexing systems over general atomic orbital ∞-categories in Lemma A.5, and summarize the results here.

Construction 2.36. Given $\pi_{\mathcal{O}} : \mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ an I-operad and $S \in \mathbb{F}_{\mathcal{T}}$ a finite $\mathcal{T}$-set, we define

$$\mathcal{O}_S := \pi_{\mathcal{O}}^{-1}(S).$$

Then, inert cocartesian lifts endow on $(\mathcal{O}_V)_{V \in \mathcal{T}}$ the structure of a $\mathcal{T}$-∞-category, formally given by the pullback

$$\begin{tikzcd} U(\mathcal{O}^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d] \\ \mathcal{T}^{\mathrm{op}} \arrow[r, hook] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \end{tikzcd}$$

We call this the *underlying $\mathcal{T}$-∞-category of $\mathcal{O}^{\otimes}$*, and refer to it as $\mathcal{O}$ when this won't cause confusion. ◁

Proposition 2.37. *A functor $\pi : \mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if the following are satisfied:*

(a) $\mathcal{O}^{\otimes}$ has π-cocartesian lifts for backwards maps in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$;

(b) (Segal condition for colors) for every $S \in \mathbb{F}_{\mathcal{T}}$, cocartesian transport along the π-cocartesian lifts lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ together induce an equivalence

$$\mathcal{O}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U;$$

(c) (Segal condition for multimorphisms) for every map of orbits $T \to S$ in I and pair of objects $(\mathbf{C}, \mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_U$, postcomposition with the π-cocartesian lifts $\mathbf{D} \to D_U$ lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \to S}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \leftarrow T_U \to U}(\mathbf{C}, D_U).$$

where $T_U := T \times_S U$.

Furthermore, a cocartesian fibration $\pi : \mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if its unstraightening $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Cat}$ is an I-symmetric monoidal category.

Proof. Each of our conditions nearly matches with that of Definition 2.17, with the exception being that we evaluate the limits on the sub-diagram $\mathrm{Orb}(S) \subset \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})_{S/}^{\mathrm{el}}$; we show in Lemma A.2 that this is an initial subcategory, proving the proposition. □

Remark 2.38. Cocartesian lifts over backwards maps furnish an equivalence

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \leftarrow T_U \to U}(\mathbf{C}, D_U) \simeq \mathrm{Map}_{\mathcal{O}^{\otimes}}^{T_U \to U}(\mathbf{C}_{T_U}, D_U),$$

where $\mathbf{C}_{T_U} \in \mathcal{O}_{T_U}$ is the T_U-tuple of colors underlying $\mathbf{C}$. Hence in the presence of Conditions (a) and (b), Condition (c) may equivalently stipulate that the map

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}^{T \to S}(\mathbf{C}, \mathbf{D}) \to \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}_{\mathcal{O}^{\otimes}}^{T_U \to U}(\mathbf{C}_{T_U}, D_U)$$

is an equivalence. We will generally prefer this version, as the data of a $\mathcal{T}$-operad is most naturally viewed as living over the *active* (i.e. forward) maps. ◁

Remark 2.39. Practicioners of [HA, Def 2.1.10] should note that, by Remark 2.18, we may weaken Condition (b) to assert only that cocartesian transport induces a π_0-surjection $\mathcal{O}_S \to \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U$; with this modification, Proposition 2.37 recovers Lurie's definition of ∞-operads when $\mathcal{T} = *$. ◁

We're finally ready to prove Proposition 2.33

Proof of Proposition 2.33. Note that Conditions (IC-a) and (IC-c) of Definition 1.42 are true by assumption (they were forced on us in order to make $\mathrm{Span}_I(\mathbb{F}_\mathcal{T})$ definable). We verify the conditions of Proposition 2.37 for $\mathcal{T}$-operads.

Note that $\mathrm{Span}_I(\mathbb{F}_\mathcal{T})$ has *unique* lifts for backwards maps, so condition (a) follows always. Furthermore, $\mathrm{Span}_I(\mathbb{F}_\mathcal{T})$ always satisfies condition (b) by construction. Lastly, by unwiding definitions and noting that there exists a map of spaces $X \to Y \times \varnothing = \varnothing$ if and only if X is empty, Observation 1.43 implies that (c) is equivalent to Condition (IC-b). □

Using Proposition 2.37, we gain access to the *structure spaces* of $\mathcal{T}$-operads.

Construction 2.40. Let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-operad. When $\mathbf{C}, \mathbf{D} \in \mathcal{O}^\otimes$ are objects, define

$$\mathrm{Mul}_\mathcal{O}(\mathbf{C}, \mathbf{D}) := \coprod_{\substack{\psi : \pi(C) \to \pi(D) \\ \text{active}}} \mathrm{Map}^{\psi}_{\pi_\mathcal{O}}(\mathbf{C}, \mathbf{D}).$$

In the case $D \in \mathcal{O}^\otimes_V$, $S \in \mathbb{F}_V$, and $\mathbf{C} \in \mathcal{O}^\otimes_{\mathrm{Ind}_V^\mathcal{T} S}$, we write

$$\mathcal{O}(\mathbf{C}; D) := \mathrm{Map}_\mathcal{O}^{\mathrm{Ind}_V^\mathcal{T} S \to V}(\mathbf{C}; D).$$

Similarly, given $S \in \mathbb{F}_V$, we write

$$\mathcal{O}(S) := \coprod_{(\mathbf{C}, D) \in \mathcal{O}_{\mathrm{Ind}_V^\mathcal{T} S} \times \mathcal{O}_V} \mathcal{O}(\mathbf{C}; D);$$

we refer to this is the *space of S-ary operations in $\mathcal{O}$*. ◁

We use this to define a litany of useful full subcategories of $\mathrm{Op}_\mathcal{T}$.

Definition 2.41. A $\mathcal{T}$-operad $\mathcal{O}^\otimes$ is:

- *at most one-colored* if $\mathcal{O}_V \in \{\varnothing, *\}$ for all $V \in \mathcal{T}$, i.e. $\mathcal{O}(*_V) \in \{\varnothing, *\}$ for all $V \in \mathcal{T}$,
- *at least one-colored* if $\mathcal{O}_V \neq \varnothing$ for all $V \in \mathcal{T}$, i.e. $\mathcal{O}(*_V) \neq \varnothing$ for all $V \in \mathcal{T}$,
- *one-colored* if $\mathcal{O}^\otimes$ is at least one-colored and at-most one colored,
- *almost essentially unital* (or aE-unital) if $\mathcal{O}(\varnothing_V) = *$ whenever there exists some $S \neq *_V \in \mathbb{F}_V$ such that $\mathcal{O}(S) \neq \varnothing$.
- *unital* if $\mathcal{O}(\varnothing_V) = *$ for all $V \in \mathcal{T}$m
- *almost essentially reduced* (or aE-reduced) if $\mathcal{O}^\otimes$ is almost-E-unital and at-most one colored,
- *reduced* if $\mathcal{O}^\otimes$ is unital and one-colored. ◁

We denote the associated full subcategories by

$$\begin{tikzcd}
 & \mathrm{Op}_\mathcal{T}^{\mathrm{uni}} \arrow[r] \arrow[dr] & \mathrm{Op}_\mathcal{T}^{\geq \mathrm{oc}} \arrow[dr] & \\
\mathrm{Op}_\mathcal{T}^{\mathrm{red}} \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{Op}_\mathcal{T}^{\mathrm{oc}} \arrow[ur] \arrow[dr] & \mathrm{Op}_\mathcal{T}^{aE\mathrm{uni}} \arrow[r] & \mathrm{Op}_\mathcal{T} \\
 & \mathrm{Op}_\mathcal{T}^{aE\,\mathrm{red}} \arrow[r] \arrow[ur] & \mathrm{Op}_\mathcal{T}^{\leq \mathrm{oc}} \arrow[ur] &
\end{tikzcd}$$

Warning 2.42. An almost essentially unital $\mathcal{T}$-operad with at least one object need not be unital (and likewise for reducedness); they satisfy the more general notion of *almost unitality* following [Ste24b], but we suppress this notion for the time being. ◁

Construction 2.43. Given $\mathcal{O}^\otimes$ a one-colored $\mathcal{T}$-operad, $V \in \mathcal{T}$ an orbit, and $S \in \mathbb{F}_V$ a finite V-set, we write $\mathcal{O}_S \simeq \{iS\}$. For any $T \leftarrow \mathrm{Ind}_V^{\mathcal{T}} S$, we have an equivalence

$$\mathcal{O}(S) \simeq \mathrm{Map}_{\pi_{\mathcal{O}}}^{T \leftarrow \mathrm{Ind}_V^{\mathcal{T}} S \to V}(iS; iV)$$

due to the existence of cocartesian lifts for inert morphisms. Given a map $U \to V$ in $\mathcal{T}$ and a finite V-set $S \in \mathbb{F}_V$, composition of the cospan $\mathrm{Ind}_V^{\mathcal{T}} S \to V \leftarrow U$ in $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ induces a restriction map

$$\begin{tikzcd} \mathcal{O}(S) \arrow[r, "\mathrm{Res}_U^V"] \arrow[d, phantom, "\wr"] & \mathcal{O}(\mathrm{Res}_U^V S) \arrow[d, phantom, "\wr"] \\ \mathrm{Map}_{\pi_{\mathcal{O}}}^{\mathrm{Ind}_V^{\mathcal{T}} S \to V}(iS; iV) \arrow[r] & \mathrm{Map}_{\pi_{\mathcal{O}}}^{\mathrm{Ind}_V^{\mathcal{T}} S \leftarrow \mathrm{Ind}_V^{\mathcal{T}} S \times_V U \to U}(i\,\mathrm{Res}_U^V S; iU) \end{tikzcd} \tag{6}$$

Furthermore, given a map of V-sets $\varphi_{TS} : T \to S$, write $T_U \simeq T_U \times_S U \to U$ for the pullback, and write $\varphi_{TV} : \mathrm{Ind}_V^{\mathcal{T}} T \to V$ for the structure map of T. Composition in $\mathcal{O}^\otimes$ restricts to a map

$$\begin{tikzcd} \mathcal{O}(S) \times \prod\limits_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, phantom, "\wr"] & \mathcal{O}(T) \arrow[d, phantom, "\wr"] \\ \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(iS; iV) \times \mathrm{Map}_{\pi_{\mathcal{O}}}^{\varphi_{TS}}(iT, iS) \arrow[r] & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{TV}}(iT; iV) \end{tikzcd} \tag{7}$$

Lastly, note that every V-equivariant automorphism of S yields an automorphism of $\mathrm{Ind}_V^{\mathcal{T}} S$ over V, which are backwards maps by definition; cocartesian transport then yields an action

$$\rho_S : \mathrm{Aut}_V(S) \times \mathcal{O}(S) \longrightarrow \mathcal{O}(S). \tag{8}$$

We refer to Res_U^V as *restriction*, γ as the *composition*, and ρ_S as Σ-*action*. ◁

Example 2.44. Let I be a weak indexing category. Recall the example $\mathcal{N}_{I\infty}^\otimes = (\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}}))$ of Proposition 2.33, and write

$$c(I) := \{V \in \mathcal{T} \mid V \in I\}$$

as in [Ste24b]. Then, it follows by definition that $U\mathcal{N}_{I\infty}^\otimes \simeq *_{c(I)}$; that is, $\mathcal{N}_{I\infty}$ always has at most one color, and it has one color if and only if I has one color in the sense of [Ste24b].

Moreover, we have

$$\mathcal{N}_{I\infty}(S) \simeq \begin{cases} * & S \in \mathbb{F}_{I,V}; \\ \varnothing & S \notin \mathbb{F}_{I,V}. \end{cases}$$

Thus we see that $\mathcal{N}_{I\infty}^\otimes$ is almost essentially unital (hence almost essentially reduced) if and only if I is almost essentially unital in the sense of [Ste24b]; likewise, $\mathcal{N}_{I\infty}^\otimes$ is unital (hence reduced) if and only if I is unital. Unwinding definitions, each of the maps $\mathrm{Res}_U^V, \gamma, \rho_S$ are canonical, as they have codomain either $*$ or $\varnothing$. ◁

Observation 2.45. The structures of Eqs. (6) to (8) are compatible in the following ways:

(1) The restriction maps are Borel equivariant, i.e. the following commutes:

$$\begin{tikzcd} \{\text{cocart lifts of } \mathrm{Aut}_V(S)\} \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(iS, iV) \arrow[rrr, "\circ"] \arrow[dd, bend right, "\mathrm{Res}_W^V"'] \arrow[dr, equal] & & & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(iS, iV) \arrow[dd, bend left, "\mathrm{Res}_W^V"] \\ & \mathrm{Aut}_V(S) \times \mathcal{O}(S) \arrow[r, "\rho"] \arrow[d, "\mathrm{Res}_W^V"] & \mathcal{O}(S) \arrow[ur, equal] \arrow[d, "\mathrm{Res}_W^V"] & \\ & \mathrm{Aut}_W(\mathrm{Res}_W^V S) \times \mathcal{O}(\mathrm{Res}_W^V S) \arrow[r, "\rho"] \arrow[dl, equal] & \mathcal{O}(\mathrm{Res}_W^V S) \arrow[dr, equal] & \\ \left\{\text{cocart lifts of } \mathrm{Aut}_W(\mathrm{Res}_W^V S)\right\} \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(i\,\mathrm{Res}_W^V S, iW) \arrow[rrr, "\circ"] & & & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(i\,\mathrm{Res}_W^V S, iW) \end{tikzcd}$$

(2) The composition maps are Borel $\mathrm{Aut}_V(S) \times \prod_{U \in \mathrm{Orb} S} \mathrm{Aut}_U(T_U)$-equivariant in an analogous way.
(3) The identity map on $*_V$ yields an element $1_V \in *_V$ which is taken to 1_V by Res^V_U.
(4) The map γ is unital, i.e. the following commutes.

$$\begin{tikzcd}
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^\otimes}(iS, iV) \arrow[rrrr, "(\mathrm{id},\{\mathrm{id}\})"] \arrow[dr, equal] \arrow[dddd, bend right, "(\{\mathrm{id}\},\mathrm{id})"'] & & & & \mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^\otimes}(iS, iV) \times \mathrm{Map}^{\mathrm{id}}_{\mathcal{O}^\otimes}(iS, iS) \arrow[dl, equal] \arrow[dddd, bend left, "\circ"] \\
 & \mathcal{O}(S) \arrow[rr, "(\mathrm{id},(\{1_U\}))"] \arrow[d, "(\{1_V\},\mathrm{id})"'] \arrow[drr, equal] & & \mathcal{O}(S) \otimes \bigotimes_{U \in \mathrm{Orb}(S)} \mathcal{O}(*_U) \arrow[d, dashed, "\gamma"] & \\
 & \mathcal{O}(*_V) \otimes \mathcal{O}(S) \arrow[rr, "\gamma"] \arrow[dl, equal] & & \mathcal{O}(S) \arrow[dr, equal] & \\
 & & & & \\
\mathrm{Map}^{\mathrm{id}}_{\mathcal{O}^\otimes}(iV, iV) \times \mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^\otimes}(iS, iV) \arrow[rrrr, "\circ"] & & & & \mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^\otimes}(iS, iV)
\end{tikzcd}$$

(5) The map γ is compatible with restriction, i.e. given a composable pair of morphisms

$$\begin{tikzcd}
 & \mathrm{Ind}^{\mathcal{T}}_V S \arrow[dr, "\varphi_{SV}"] & \\
\mathrm{Ind}^{\mathcal{T}}_V T \arrow[ur, "\varphi_{TS}"] \arrow[rr, "\varphi_{TV}"] & & V,
\end{tikzcd}$$

and $W \to V$ a map in $\mathcal{T}$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^\otimes}(iS, iV) \times \mathrm{Map}^{\varphi_{TS}}_{\mathcal{O}^\otimes}(iT, iS) \arrow[rrr, "\circ"] \arrow[dr, equal] \arrow[ddd, bend right, "\mathrm{Res}^V_W"'] & & & \mathrm{Map}^{\varphi_{TV}}_{\mathcal{O}^\otimes}(iT, iV) \arrow[dl, equal] \arrow[ddd, bend left, "\mathrm{Res}^V_W"] \\
 & \mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, "\mathrm{Res}^W_V"] & \mathcal{O}(T) \arrow[d, "\mathrm{Res}^W_V"] & \\
 & \mathcal{O}\Big(\mathrm{Res}^V_W S\Big) \times \prod_{U' \in \mathrm{Orb}(\mathrm{Res}^V_W S)} \mathcal{O}(T_{U'}) \arrow[r, "\gamma"] \arrow[dl, equal] & \mathcal{O}\Big(\mathrm{Res}^V_W T\Big) \arrow[dr, equal] & \\
\mathrm{Map}^{\mathrm{Res}^V_W \varphi_{SV}}_{\mathcal{O}^\otimes}\Big(i \mathrm{Res}^V_W S, iW\Big) \times \mathrm{Map}^{\mathrm{Res}^V_W \varphi_{TS}}_{\mathcal{O}^\otimes}(i \mathrm{Res}^V_W T, i \mathrm{Res}^V_W S) \arrow[rrr, "\circ"] & & & \mathrm{Map}^{\mathrm{Res}^V_W \varphi_{TV}}_{\mathcal{O}^\otimes}(i \mathrm{Res}^V_W T, iW)
\end{tikzcd}$$

(6) The map γ is associative, i.e. given a collection of maps and composites

$$\begin{tikzcd}
\mathrm{Ind}^{\mathcal{T}}_V R \arrow[r, "\varphi_{RT}"] \arrow[rrr, bend left=30, "\varphi_{RV}"] \arrow[rr, bend right, "\varphi_{RS}"'] & \mathrm{Ind}^{\mathcal{T}}_V T \arrow[r, "\varphi_{TS}"] \arrow[rr, bend left, "\varphi_{TV}"] & \mathrm{Ind}^{\mathcal{T}}_V S \arrow[r, "\varphi_{SV}"] & V,
\end{tikzcd}$$

the following commutes:

$$\begin{tikzcd}
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^\otimes}(iS,iV)\times \mathrm{Map}^{\varphi_{TS}}_{\mathcal{O}^\otimes}(iT,iS)\times \mathrm{Map}^{\varphi_{RT}}_{\mathcal{O}^\otimes}(iR,iT) \arrow[r, "\circ"] \arrow[d, equal] \arrow[dddd, bend right, "\circ"'] & \mathrm{Map}^{\varphi_{TV}}_{\mathcal{O}^\otimes}(iT,iV)\times \mathrm{Map}^{\varphi_{RT}}_{\mathcal{O}^\otimes}(iR,iT) \arrow[d, equal] \arrow[dddd, bend left, "\circ"] \\
\left(\mathcal{O}(S)\times \prod\limits_{U\in \mathrm{Orb}(S)} \mathcal{O}(T_U)\right)\times \prod\limits_{\substack{U\in\mathrm{Orb}(S)\\ W\in \mathrm{Orb}(T_U)}} \mathcal{O}(R_W) \arrow[r, "\gamma"] \arrow[d, equal] & \mathcal{O}(T)\times \prod\limits_{W\in \mathrm{Orb}(T)} \mathcal{O}(R_W) \arrow[d, "\gamma"] \\
\mathcal{O}(S)\times \prod\limits_{U\in\mathrm{Orb}(S)}\left(\mathcal{O}(T_U)\times \prod\limits_{W\in\mathrm{Orb}(T_U)} \mathcal{O}(R_W)\right) \arrow[d, "\gamma"] & \mathcal{O}\left(\coprod\limits_{W}^{T} R_W\right) \arrow[d, equal] \\
\mathcal{O}(S)\times \prod\limits_{U\in\mathrm{Orb}(S)} \mathcal{O}\left(\coprod\limits_{W}^{T_U} R_W\right) \arrow[r, "\gamma"] \arrow[d, equal] & \mathcal{O}(R) \arrow[d, equal] \\
\mathrm{Map}^{\varphi_{SV}}_{\mathcal{O}^\otimes}(iS,iV)\times \mathrm{Map}^{\varphi_{RS}}_{\mathcal{O}^\otimes}(iR,iS) \arrow[r, "\circ"] & \mathrm{Map}^{\varphi_{RV}}_{\mathcal{O}^\otimes}(iR,iV)
\end{tikzcd}$$

Thus, passing to the homotopy category, the data of a $\mathcal{T}$-operad supplies a discrete genuine $\mathcal{T}$-operad in $\mathrm{ho}\mathcal{S}$ in the sense of Definition 2.79. ◁

Remark 2.46. The assumption that $\mathcal{O}^\otimes$ has one color is not strictly necessary in Construction 2.43 and Observation 2.45; for instance, in general we may choose a V-color B, a S-color $\mathbf{C} = (C_U)$, and for every $U \in \mathrm{Orb}(S)$ a T_U-color $\mathbf{D}_U$. Then, writing $\mathbf{D}$ for associated T-color associated with $(\mathbf{D}_U)$, composition in $\mathcal{O}^\otimes$ yields an analogous map

$$\gamma\colon \mathcal{O}(\mathbf{C};B)\times \prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(\mathbf{D}_U;C_U) \longrightarrow \mathcal{O}(\mathbf{D};B),$$

which is associative in an analogous way to Observation 2.45. In particular, if $\mathcal{O}^\otimes$ merely has *at most one color*, then all statements in Construction 2.43 and Observation 2.45 apply whenever $\mathcal{O}^\otimes$ has colors over the appropriate orbits. We do not explore this further here, as it is not necessary for our present purposes. ◁

2.2.2. *The $\mathcal{T}$-∞-category of $\mathcal{T}$-operads.* Recall the map of algebraic patterns $\varphi\colon \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ of Eq. (5). In Proposition A.1 and Corollary A.8, we prove the following generalization of the contents of [BHS22, §5.2], which identifies our $\mathcal{T}$-operads with those of [NS22].

Proposition 2.47. *Suppose $\mathcal{T}$ is an atomic orbital ∞-category. Then, pullback along $\varphi\colon \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ implements equivalences of categories*

$$\mathrm{Cat}_{\mathcal{T}} \simeq \mathrm{Seg}_{\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C});$$
$$\mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Fbrs}\left(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}\right),$$

and $\mathrm{Fbrs}\left(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},}\right)$ is equivalent to the ∞-category of $\mathcal{T}$-∞-categories of [NS22].

Remark 2.48. The functor $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is natural in $\mathcal{T}$; in particular, applying this for $\mathcal{T}_{/V} \to \mathcal{T}$, we acquire a commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}\underline{\mathbb{F}}_{V,*} \arrow[r] \arrow[d] & \mathrm{Span}(\mathbb{F}_V) \arrow[d] \\
\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \arrow[r] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}})
\end{tikzcd}$$

Functoriality of pullbacks witnesses the fact that $\mathrm{Res}^{\mathcal{T}}_V\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_V$ is implemented by pullback along $\mathrm{Tot}\underline{\mathbb{F}}_{V,*} \to \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$. ◁

By assumption, if $\mathcal{O}^\otimes$ is a fibrous $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$-pattern, it possesses cocartesian lifts over *all* morphisms in the composite $\mathcal{O}^\otimes \to \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathcal{T}^{\mathrm{op}}$. Thus, fibrous $\underline{\mathbb{F}}_{\mathcal{T},*}$-patterns possess total $\mathcal{T}$-∞-*categories*; we refer to the associated functor as

$$\mathrm{Tot}_{\mathcal{T}}\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}.$$

Definition 2.49. Let $\mathcal{O}^\otimes, \mathcal{P}^\otimes$ be $\mathcal{T}$-operads, Then, the $\mathcal{T}$-∞-*category of* $\mathcal{O}$-*algebras in* $\mathcal{P}$ is the full subcategory

$$\begin{aligned}\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P}) &\coloneqq \underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T},*}}\left(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes, \mathrm{Tot}_{\mathcal{T}}\mathcal{P}^\otimes\right)\\ &\subset \underline{\mathrm{Fun}}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T},*}}\left(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes, \mathrm{Tot}_{\mathcal{T}}\mathcal{P}^\otimes\right)\end{aligned}$$

with V-values spanned by the V-functors $\mathrm{Res}^{\mathcal{T}}_V \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes \to \mathrm{Res}^{\mathcal{T}}_V \mathrm{Tot}_{\mathcal{T}}\mathcal{P}^\otimes$ preserving cocartesian lifts over inert arrows in $\underline{\mathbb{F}}_{V,*}$. ◁

We lift $\mathrm{Op}_{\mathcal{T}}$ to a $\mathcal{T}$-∞-category by the following.

Definition 2.50. We show in Proposition A.13 that $\mathrm{Span}\left(\mathrm{Ind}_U^V\right)\colon \mathrm{Span}(\mathbb{F}_U) \to \mathrm{Span}(\mathbb{F}_V)$ is a Segal morphism for all maps $U \to V$ in $\mathcal{T}$. We refer to the resulting $\mathcal{T}$-∞-category

$$\underline{\mathrm{Op}}_{\mathcal{T}}\colon \mathcal{T}^{\mathrm{op}} \xrightarrow{\mathrm{Span}(\mathbb{F}_{(-)})} \mathrm{AlgPatt}^{\mathrm{SE,Seg,op}} \xrightarrow{\mathrm{Fbrs}} \mathrm{Cat}.$$

as the $\mathcal{T}$-∞-*category of* $\mathcal{T}$-*operads,* where $\mathrm{AlgPatt}^{\mathrm{SE,Seg}} \subset \mathrm{AlgPatt}^{\mathrm{Seg}}$ is the full subcategory spanned by soundly extendable patterns. ◁

Observation 2.51. The V-value of $\underline{\mathrm{Op}}_{\mathcal{T}}$ is $\mathrm{Op}_V \coloneqq \mathrm{Op}_{\mathcal{T}_{/V}}$; the restriction functor $\mathrm{Res}^V_U\colon \mathrm{Op}_V \to \mathrm{Op}_U$ is implemented by the pullback

$$\begin{tikzcd}
\mathrm{Res}^V_U \mathcal{O}^\otimes \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{O}^\otimes \arrow[d] \\
\mathrm{Span}(\mathbb{F}_U) \arrow[r] & \mathrm{Span}(\mathbb{F}_V).
\end{tikzcd}$$

with bottom functor is $\mathrm{Span}(\mathrm{Ind}_U^V)$. ◁

Observation 2.52. Via Proposition 2.47, we find that $\Gamma^{\mathcal{T}}\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$. Furthermore, we find that

$$\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P})_V \simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_V)}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}^\otimes, \mathrm{Res}^{\mathcal{T}}_V \mathcal{P}^\otimes) \simeq \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}}(\mathrm{Res}^{\mathcal{T}}_V \mathcal{O})$$

with restriction functors induced by functoriality of Res^V_U. ◁

2.2.3. *Envelopes.* In [NS22], a left adjoint to the inclusion $\mathrm{CMon}_{\mathcal{T}}\mathrm{Cat} \to \mathrm{Op}_{\mathcal{T}}$ was constructed, called the $\mathcal{T}$-*symmetric monoidal envelope.* This was greatly generalized by Theorem 2.27 in view of Propositions 2.37 and 2.47. For convenience, we spell this out here.

Corollary 2.53. *If $\mathcal{P}^\otimes \to \mathcal{O}^\otimes$ is a map of $\mathcal{T}$-operads, then the following diagram consists of maps of $\mathcal{T}$-operads*

$$\begin{tikzcd}
\mathrm{Env}_{\mathcal{O}}\mathcal{P}^\otimes \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}^{\mathrm{act}}(\mathcal{O}^\otimes) \arrow[r, "t"] \arrow[d, "s"] & \mathcal{O}^\otimes \\
\mathcal{P}^\otimes \arrow[r] & \mathcal{O}^\otimes &
\end{tikzcd}$$

and the top horizontal composition is an $\mathcal{O}$-monoidal ∞-category. The corresponding functor

$$\mathrm{Env}_{\mathcal{O}}\colon \mathrm{Op}_{\mathcal{T},/\mathcal{O}^\otimes} \to \mathrm{Cat}^\otimes_{\mathcal{O}}$$

is left adjoint to the inclusion of $\mathcal{O}$-monoidal ∞-categories into $\mathcal{T}$-operads over $\mathcal{O}^\otimes$, and the induced functor

$$\mathrm{Env}^{/\mathscr{A}_{\mathcal{O}}}_{\mathcal{O}}\colon \mathrm{Op}_{\mathcal{T},/\mathcal{O}^\otimes} \to \mathrm{Cat}^\otimes_{\mathcal{O},/\mathscr{A}_{\mathcal{O}}}$$

is fully faithful, with image spanned by equifibrations in the sense of [BHS22, Thm C].

We will simply write $\mathrm{Env}_I(-) \coloneqq \mathrm{Env}_{\mathcal{N}_{I\infty}}(-)$ and $\mathrm{Env}(-) \coloneqq \mathrm{Env}_{\mathrm{Comm}_{\mathcal{T}}}(-)$.

Example 2.54. Let I be a weak indexing category. Then, unwinding definitions, we find that

$$\mathrm{Env}_I \mathcal{N}^{\otimes}_{I\infty} \simeq \underline{\mathbb{F}}_I^{I-\sqcup},$$

where $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is the full $\mathcal{T}$-subcategory defined in Section 1.2, i.e. it is the I-symmetric monoidal subcategory generated by $\{*_V \mid V \in c(I)\}$. ◁

Remark 2.55. Suppose $\mathbb{F}_{\mathcal{T}}^{\simeq} \subset I \subset \mathbb{F}_{\mathcal{T}}$ is a core-preserving wide subcategory which is *not* a weak indexing category. We've already seen in Proposition 2.33 that $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is not a $\mathcal{T}$-operad, so we can't specialize from $\mathcal{T}$-operads to a theory of I-operads; in fact, Example 2.54 is a prominent example where I-operads would act quite differently. Indeed, since I does not satisfy Condition (IC-b), $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is not closed under I-indexed coproducts, so $\underline{\mathbb{F}}_I$ can not even be endowed with a generalization of the above I-symmetric monoidal structure. ◁

We record a convenient property of $\mathrm{Env}_I(-)$ here, which follows by unwinding definitions.

Lemma 2.56 ([HA, Rmk 2.4.4.3])**.** *If $\mathcal{O}^{\otimes} \in \mathrm{Op}_I$ and $\psi\colon T \to S$ is a map of V-sets, then there is an equivalence*

$$\begin{aligned}\mathrm{Mor}^{\psi}_{\mathrm{Env}_I(\mathcal{O})_V \to \mathbb{F}_{I,V}}(\mathrm{Env}_I(\mathcal{O})_V) &\simeq \coprod_{(\mathbf{C},\mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_S} \mathrm{Map}^{\psi}_{\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathbf{C},\mathbf{D}) \\ &\simeq \coprod_{(\mathbf{C},\mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_S} \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(\mathbf{C}_U; D_U)\end{aligned}$$

In particular, if $\mathcal{O}^{\otimes}$ has one color, then

$$\mathrm{Map}^{\psi}_{\mathrm{Env}_I(\mathcal{O})_V \to \mathbb{F}_{I,V}}(iT; iS) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U).$$

2.3. **The underlying $\mathcal{T}$-symmetric sequence.** Set the notation $\underline{\Sigma}_{\mathcal{T}} := \underline{\mathbb{F}}_{\mathcal{T},*}^{\simeq}$, where the latter is the $\mathcal{T}$-space core of Example 1.35. We refer to this as the *$\mathcal{T}$-symmetric $\mathcal{T}$-category*, and we refer to $\mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \mathcal{C})$ as the *∞-category of $\mathcal{T}$-symmetric sequences in $\mathcal{C}$*; in the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$, we refer to $\mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ simply as the *∞-category of $\mathcal{T}$-symmetric sequences*.

Observation 2.57. For any adequate triple $(\mathcal{X}, \mathcal{X}_b, \mathcal{X}_f)$, the inclusion

$$\mathcal{X} \hookrightarrow \mathrm{Span}_{b,f}(\mathcal{X})$$

induces an equivalence on cores. In particular, choosing $(\underline{\mathbb{F}}_{\mathcal{T}}, \underline{\mathbb{F}}_{\mathcal{T}}^{s.i.}, \underline{\mathbb{F}}_{\mathcal{T}})$, we find that the inclusion $(-)_+ : \underline{\mathbb{F}}_{\mathcal{T}} \to \underline{\mathbb{F}}_{\mathcal{T},*}$ induces an equivalence

$$\underline{\mathbb{F}}_{\mathcal{T}}^{\simeq} \simeq \underline{\mathbb{F}}_{\mathcal{T},*}^{\simeq} \simeq \underline{\Sigma}_{\mathcal{T}}.$$

In particular, unwinding definitions, we have the computation

$$\Sigma_V := \underline{\Sigma}_{\mathcal{T},V} \simeq \mathbb{F}_V^{\simeq} \simeq \coprod_{S \in \mathbb{F}_V} B\mathrm{Aut}_V S$$

and that the restriction map $\Sigma_V \to \Sigma_W$ is induced by the forgetful maps $B\mathrm{Aut}_V S \to B\mathrm{Aut}_W S$. ◁

Observation 2.58. Under the equivalence $\mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Fbrs}(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*})$, by Proposition 3.16, $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is modeled by the inclusion $\underline{\Sigma}_{\mathcal{T}} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T},*}$. Every morphism in the associated factorization system on $\underline{\Sigma}_{\mathcal{T}}$ is equivalent to an inert morphism; hence there exist equivalences

$$\mathrm{Cat}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}} \simeq \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathrm{Cat}) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathrm{Cat}}_{\mathcal{T}}).$$ ◁

Construction 2.59. Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}^{\mathrm{red}}$, there is a structure map

$$\mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}} \simeq \mathrm{triv}_{\mathcal{T}}^{\otimes} \times_{\mathrm{Comm}_{\mathcal{T}}^{\otimes}} \mathrm{Ar}^{\mathrm{act,/el}}(\mathcal{O}) \to \mathrm{triv}_{\mathcal{T}}^{\otimes}$$

which is an inert-cocartesian fibration by pullback-stability of inert-cocartesian fibrations [BHS22, Obs 2.1.7]. The *underlying $\mathcal{T}$-symmetric sequence of $\mathcal{O}^{\otimes}$* is

$$\mathcal{O}^{\otimes}_{\mathrm{sseq}} := \mathrm{Un}_{\mathrm{triv}_{\mathcal{T}}}\mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}} \in \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathrm{Cat}).$$

Unwinding definitions, we find that there exists a cartesian square

$$\begin{tikzcd}
\mathcal{O}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Env}_{\mathcal{O}}\mathrm{triv} \arrow[r, equal] \arrow[d] & \mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \arrow[d] \\
* \arrow[r, "S"] & \mathrm{triv}^{\otimes} \arrow[r, equal] & \mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}
\end{tikzcd}$$

so that $\mathcal{O}^{\otimes}_{\mathrm{sseq}}$ is indeed a $\mathcal{T}$-symmetric sequence. The associated functor is denoted

$$\mathrm{sseq} : \mathrm{Op}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}). \qquad \triangleleft$$

We will often use the following to reduce questions about $\mathcal{T}$-operads to $\mathcal{T}$-symmetric sequences.

Proposition 2.60. *Suppose a functor of $\mathcal{T}$-operads $\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ satisfies the following conditions:*

(a) φ induces surjective maps $\pi_0\mathcal{O}_V \to \pi_0\mathcal{P}_V$ for all $V \in \mathcal{T}$, and

(b) for all $V \in \mathcal{T}$, all $S \in \mathbb{F}_V$, all $\mathbf{C} \in \mathcal{O}_S$, and all $D \in \mathcal{O}_V$, the map φ induces equivalences $\varphi : \mathcal{O}(\mathbf{C}; D) \xrightarrow{\sim} \mathcal{P}(\varphi\mathbf{C}; \varphi D)$.

Then φ is an equivalence of $\mathcal{T}$-operads; in particular, the restricted functor

$$\mathrm{sseq} : \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$$

is conservative.

To prove this, we proceed by reduction to the following observation.

Observation 2.61. If $\mathcal{C} \to \mathcal{D}$ is an equivalence of categories over $\mathcal{E}$, then it preserves and reflects cocartesian lifts of arrows in $\mathcal{E}$; in particular, if $\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a morphism of $\mathcal{T}$-operads who induces an equivalence $\mathrm{Tot}\,\varphi \colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ between the total ∞-categories of the associated functors to $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, then its inverse is also a morphism of $\mathcal{T}$-operads. Said another way, we've observed that the functor $U \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ is an isofibration, so $\mathrm{Tot} \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}$ is conservative.

Similar arguments show that $U \colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T},*}} \to \mathrm{Cat}_{/\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}}$ is an isofibration. $\triangleleft$

Proof of Proposition 2.60. In view of Construction 2.59, the second statement follows immediately from the first, since morphisms of reduced $\mathcal{T}$-operads are automatically π_0-isomorphisms by two-out-of-three. Fixing φ satisfying (a) and (b), we will prove that φ is an equivalence of $\mathcal{T}$-operads. Using Observation 2.61, it suffices to prove that $\mathrm{Tot}\,\varphi$ is an equivalence of ∞-categories.

By the Segal condition for colors, we have an equivalence of arrows

$$\begin{tikzcd}
\pi_0\mathcal{O}_S \arrow[r, phantom, "\simeq"] \arrow[d, "\varphi_S"] & \prod_{V \in \mathrm{Orb}(S)} \pi_0\mathcal{O}_V \arrow[d, "\prod \varphi_V"] \\
\pi_0\mathcal{P}_S \arrow[r, phantom, "\simeq"] & \prod_{V \in \mathrm{Orb}(S)} \pi_0\mathcal{P}_V
\end{tikzcd}$$

Since $\pi_0\mathcal{O} \simeq \coprod_S \pi_0\mathcal{O}_S$, (a) implies that φ is essentially surjective. Furthermore, the Segal condition for multimorphisms yields isomorphisms of arrows

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}^{\otimes}}(\mathbf{C}, \mathbf{D}) \arrow[r, phantom, "\simeq"] \arrow[d, "\varphi"] & \coprod_{f : \pi C \to \pi D} \mathrm{Map}^{f}_{\mathcal{O}}(\mathbf{C}; \mathbf{D}) \arrow[r, phantom, "\simeq"] \arrow[d, "\coprod \varphi"] & \coprod_{f} \prod_{V \in \mathrm{Orb}(\pi(D))} \mathrm{Map}^{f_V}_{\mathcal{O}}(\mathbf{C}_{f_V^{-1}}; D_V) \arrow[r, phantom, "\simeq"] \arrow[d, "\coprod\prod \varphi"] & \coprod_f \prod_V \mathcal{O}(\mathbf{C}_{f^{-1}V}; D_V) \arrow[d, "\coprod\prod \varphi(T_V)"] \\
\mathrm{Map}_{\mathcal{P}^{\otimes}}(\varphi\mathbf{C}, \varphi\mathbf{D}) \arrow[r, phantom, "\simeq"] & \coprod_{f : \pi \mathbf{C} \to \pi \mathbf{D}} \mathrm{Map}^{f}_{\mathcal{P}}(\varphi\mathbf{C}; \varphi\mathbf{D}) \arrow[r, phantom, "\simeq"] & \coprod_{f} \prod_{V \in \mathrm{Orb}(S)} \mathrm{Map}^{f'}_{\mathcal{P}}(\varphi\mathbf{C}_{f^{-1}V}, \varphi D_V) \arrow[r, phantom, "\simeq"] & \coprod_f \prod_V \mathcal{P}(\varphi\mathbf{C}_{f^{-1}V}; \varphi D_V).
\end{tikzcd}$$

the right arrow is an equivalence by (b), so the leftmost arrow is an equivalence, hence φ is fully faithful. $\square$

The author learned the $U_{\circ}$ portion of the following argument from Thomas Blom.

Corollary 2.62. *The functor $\mathrm{sseq}_{\mathcal{T}} : \mathrm{Op}^{oc}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ is monadic and preserves sifted colimits.*

Proof. By [BHS22, Cor 4.2.2], $\mathrm{Op}_{\mathcal{T}}^{\mathrm{red}}$ and $\mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}},\mathcal{S})$ are presentable, so by Barr-Beck [HA, Thm 4.7.3.5] and the adjoint functor theorem [HTT, Cor 5.5.2.9], it suffices to prove that sseq is conservative and preesrves limits and sifted colimits. Conservativity is Proposition 2.60, and (co)limits in functor categories are computed pointwise by [HTT, Prop 5.1.2.2], so it suffices to prove that $\mathcal{O}\mapsto\mathcal{O}(S)$ preseres limits and sifted colimits. We separate this into manageable chunks via the following diagram:

$$\begin{tikzcd}
\mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} \arrow[r, "\mathcal{O}\mapsto\mathcal{O}(S)"] \arrow[d, "U_{\mathrm{Seg}}"] & \mathcal{S} & & \mathcal{S}^{\pi_0\mathrm{Map}(\mathrm{Ind}_V^{\mathcal{T}}S,V)} \arrow[ll, "\pi"'] \\
\mathrm{Cat}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}^{\mathrm{Int-cocart,core-iso}} \arrow[r, "U_{\mathrm{cocart}}"] & \mathrm{Cat}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}^{\mathrm{core-iso}} \arrow[rr, "U_\circ"] & & \mathrm{Fun}\left((\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq})^{\times 2},\mathcal{S}\right) \arrow[u, "\mathrm{ev}_{\mathrm{Ind}_V^{\mathcal{T}}S,V}"']
\end{tikzcd}$$

π and $\mathrm{ev}_{\mathrm{Ind}_V^{\mathcal{T}}S,V}$ preserve (co)limits since they are evaluation of functor categories [HTT, Prop 5.1.2.2]. U_{Cocart} preserves limits and sifted colimits by [BHS22, Cor 2.1.5]. U_{Seg} preserves limits and sifted colimits, as each commute with finite products.

By [Hau20, Prop 3.12], $U_\circ$ is equivalent to the forgetful functor

$$\mathrm{Alg}(\mathcal{S}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq},\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq}}) \to \mathcal{S}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}}),\mathrm{Span}(\mathbb{F}_{\mathcal{T}})},$$

where $\mathcal{S}^{\otimes}_{/Y,Y}$ is a symmetric monoidal structure on $\mathcal{S}_{/Y,Y}\simeq\mathcal{S}_{Y\times Y}\simeq\mathrm{Fun}(Y\times Y,\mathcal{S})$. This functor preserves limits and sifted colimits by [HA, Prop 3.2.3.1], completing the argument. □

In particular, this constructs a left adjoint

$$\mathrm{Fr}:\mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}},\underline{\mathcal{S}}_{\mathcal{T}})=\mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}},\mathcal{S})\to\mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}}$$

to sseq. We lift this to a $\mathcal{T}$-adjunction in the following construction.

Construction 2.63. The functor sseq is associated with a $\mathcal{T}$-functor $\underline{\mathrm{sseq}}$ as in the following diagram

$$\begin{tikzcd}
 & & & & \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}^{\otimes}) \arrow[d, dashed, "s"] \\
\mathcal{O}^{\otimes} \arrow[r, mapsto, bend left] & \mathrm{triv}_{\mathcal{T}}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes} \arrow[rr, mapsto, bend left] & \mathrm{triv}_{\mathcal{T}}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes} \\
\Cap & \Cap & & \Cap & \\
\underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{oc}} \arrow[r, hook] \arrow[d, "\underline{\mathrm{sseq}}"'] & \underline{\mathrm{Op}}_{\mathcal{T},\mathrm{triv}_{\mathcal{T}}^{\otimes}/} \arrow[rr] & & \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Lambda_2^2,\mathrm{Op}_{\mathcal{T}}\right)\times_{\underline{\mathrm{Op}}_{\mathcal{T}}}\left\{\mathrm{triv}_{\mathcal{T}}^{\otimes}\right\} \arrow[d] & \\
\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\Sigma}_{\mathcal{T}},\underline{\mathcal{S}}_{\mathcal{T}}\right) \arrow[r, equal] & \mathrm{Op}_{\mathcal{T},/\mathrm{triv}_{\mathcal{T}}^{\otimes}} & & \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Delta_1^2,\underline{\mathrm{Op}}_{\mathcal{T}}\right)\times_{\underline{\mathrm{Op}}_{\mathcal{T}}}\left\{\mathrm{triv}_{\mathcal{T}}^{\otimes}\right\} \arrow[ll, "U"] & \\
\Cup & \Cup & & \Cup & \\
\mathrm{sseq}\,\mathcal{O}^{\otimes} & \mathrm{Env}_{\mathcal{O}}\mathrm{triv} \arrow[l, mapsto] \arrow[d] & & \mathrm{Env}_{\mathcal{O}}\mathrm{triv} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}^{\otimes}) \arrow[d, dashed, "s"] \\
 & \mathrm{triv}_{\mathcal{T}}^{\otimes} & & \mathrm{triv}_{\mathcal{T}}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes}
\end{tikzcd}$$

By [HA, Prop 7.3.2.1], the pointwise left adjoints Fr lifts to a $\mathcal{T}$-adjunction

$$\underline{\mathrm{sseq}}:\underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{red}}\leftrightarrows\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}},\underline{\mathcal{S}}_{\mathcal{T}}):\underline{\mathrm{Fr}},$$

i.e. $\underline{\mathrm{Fr}}$ is compatible with restriction. ◁

2.4. **The monad for $\mathcal{O}$-algebras.** Fix $\mathcal{O}^{\otimes}$ a one-object $\mathcal{T}$-operad, fix $\mathcal{C}^{\otimes}$ a distributive $\mathcal{O}$-monoidal category in the sense of [NS22] (e.g. it may be presentably $\mathcal{O}$-monoidal) and let $\mathrm{triv}_{\mathcal{T}}^{\otimes}\to\mathcal{C}^{\otimes}$ be the functor of operads associated with a $\mathcal{T}$-object $X\in\Gamma\mathcal{C}$. Denote by $X^{\otimes}:\mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}}^{\otimes}\to\mathcal{C}^{\otimes}$ the associated $\mathcal{O}$-symmetric monoidal functor, and denote by

$$\mathcal{O}_{\mathrm{sseq}}(X):\mathrm{Env}_{\mathcal{O}}\mathrm{triv}_{\mathcal{T}}\to\mathcal{C}$$

the underlying $\mathcal{T}$-functor. Recall that

$$X^{\otimes S} \simeq \bigotimes_{V \in \mathrm{Orb}(S)} N_V^{\mathcal{T}} X_V \in \Gamma \mathcal{C}.$$

Proposition 2.64 ("Equivariant [SY19, Lem 2.4.2]")**.** *The forgetful $\mathcal{T}$-functor $U : \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ is monadic, and the associated monad $T_{\mathcal{O}}$ acts on $X \in \mathcal{C}$ by the indexed colimit*

$$T_{\mathcal{O}} X := \underline{\mathrm{colim}} \mathcal{O}_{\mathrm{sseq}}(X).$$

In particular, we have

$$(T_{\mathcal{O}} X)_V \simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \cdot X^{\otimes S} \right)_{h \mathrm{Aut}_V S}. \tag{9}$$

Proof. Monadicity is precisely [NS22, Cor 5.1.5], so it suffices to compute the associated monad.

By [NS22, Rem 4.3.6], the left adjoint $\mathrm{Fr} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is computed on X by $\mathcal{T}$-operadic left Kan extension of the corresponding map $\mathrm{triv}^{\otimes} \xrightarrow{X} \mathcal{C}^{\otimes}$ along the canonical inclusion $\mathrm{triv}^{\otimes} \to \mathcal{O}^{\otimes}$, and the underlying $\mathcal{T}$-functor of this is computed by the $\mathcal{T}$-left Kan extension

$$\begin{tikzcd}
\mathrm{Env}_{\mathcal{O}} \mathrm{triv} \arrow[r, equal] \arrow[dd] & \underline{\Sigma}_{\mathcal{T}} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \arrow[rr, "X"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & & \mathcal{C} \\
& \underline{\Sigma}_{\mathcal{T}} \arrow[urr, dashed, "\widetilde{T}_{\mathcal{O}} X"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & & \\
\mathcal{O} \arrow[r, equal] & *_{\mathcal{T}} \arrow[uurr, dashed, bend right, "T_{\mathcal{O}} X"'] & &
\end{tikzcd}$$

$\mathcal{T}$-left Kan extension diagrams to $\underline{*}_{\mathcal{T}}$ are $\mathcal{T}$-colimit diagrams by definition (see [Sha23, Def 10.1] when $D = \underline{*}_{\mathcal{T}}$), so the underlying $\mathcal{T}$-object is

$$T_{\mathcal{O}} X \simeq \underline{\mathrm{colim}} \mathcal{O}_{\mathrm{sseq}}(X).$$

More generally, the $\mathcal{T}$-left Kan extension $\widetilde{T}_{\mathcal{O}} X$ has values

$$\begin{aligned}
\widetilde{T}_{\mathcal{O}} X(S) &\simeq \operatorname*{colim}_{\{S\} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O})} X^{\otimes} \\
&\simeq \operatorname*{colim}_{\pi_{\mathcal{O}}^{-1}(S)} X^{\otimes S} \\
&\simeq \mathcal{O}(S) \cdot X^{\otimes S}.
\end{aligned}$$

By composition of left Kan extensions and [Sha23, Prop 5.5], we then have

$$\begin{aligned}
(T_{\mathcal{O}} X)_V &\simeq \operatorname*{colim}_{S \in \mathbb{F}_V^{\simeq}} \widetilde{T}_{\mathcal{O}} X^{\otimes S} \\
&\simeq \operatorname*{colim}_{S \in \mathbb{F}_V^{\simeq}} \mathcal{O}(S) \cdot X^{\otimes S} \\
&\simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \cdot X^{\otimes S} \right)_{h \mathrm{Aut}_V S}.
\end{aligned}$$

□

Remark 2.65. Let $\mathcal{O}_{G \times \Sigma_n, \Gamma_n} \subset \mathcal{O}_{G \times \Sigma_n}$ be the full subcategory spanned by $G \times \Sigma_n / \Gamma_S$ for $\phi_S : H \to \Sigma_n$ with associated graph subgroup $\Gamma_S = \{(h, \phi_S(h)) \mid h \in H\} \subset H \times \Sigma_{|S|}$. Then, a G-equivalence

$$\coprod_{n \in \mathbb{N}} \mathcal{O}_{G \times \Sigma_n, \Gamma_n} \simeq \underline{\Sigma}_G$$

was constructed in [NS22, Ex 4.3.7], and in particular, this provides a formula akin to Eq. (9) in the language of *graph families*. ◁

By [NS22, Prop 3.2.5] (noting that all colimits involved are finite), the Cartesian $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})$ is distributive whenever $\mathcal{C}$ is a cocomplete Cartesian closed category. We apply this to $\underline{\mathcal{S}}_{\mathcal{T}} := \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{S}$.

Example 2.66. Fix $\mathcal{C} := \mathrm{Coeff}^{\mathcal{T}}(\mathcal{S})$ with the Cartesian structure, and recall that $\mathcal{C}$ is distributive. The monad formula of Proposition 2.64 says that the free $\mathcal{O}$-algebra on a $\mathcal{T}$-space $X_{\mathcal{T}}$ has restriction

$$\mathrm{Res}_V^{\mathcal{T}} T_{\mathcal{O}} X_{\mathcal{T}} \simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \cdot \left(\mathrm{Res}_V^{\mathcal{T}} X_{\mathcal{T}}\right)^{\otimes S}\right)_{h\mathrm{Aut}_V S}.$$

In particular, its genuine V-fixed points is the space

$$\left(T_{\mathcal{O}} X_{\mathcal{T}}\right)^V \simeq \coprod_{S \in \mathbb{F}_V} \left(\mathcal{O}(S) \cdot \left(X_{\mathcal{T}}^V\right)^{\otimes S}\right)_{h\mathrm{Aut}_V S}. \qquad \triangleleft$$

Corollary 2.67. *The functor $\mathrm{Alg}_{(-)}(\underline{\mathcal{S}}_{\mathcal{T}}) : \mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} \to \mathrm{Cat}$ is conservative.*

Proof. Suppose $\varphi : \mathcal{O} \to \mathcal{P}$ induces an equivalence $\mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$. Then φ induces a natural equivalence $T_{\mathcal{O}} \implies T_{\mathcal{P}}$ respecting the summand decomposition in Proposition 2.64. Choosing $X = S \in \mathbb{F}_V$, there is a natural coproduct decomposition

$$\begin{aligned}\left(\mathcal{O}(S) \times S^{\times S}\right)_{h\mathrm{Aut}_V S} &\simeq \left(\mathcal{O}(S) \times \mathrm{Aut}_V S\right)_{h\mathrm{Aug}_V S} \sqcup J_{\mathcal{O},S} \\ &\simeq \mathcal{O}(S) \sqcup J_{\mathcal{O},S},\end{aligned}$$

for some $J_{\mathcal{O},S}$; hence the summand-preserving equivalence $T_\varphi : T_{\mathcal{O}} S \implies T_{\mathcal{P}} S$ implies that $\varphi(S) : \mathcal{O}(S) \to \mathcal{P}(S)$ is an equivalence for all S, i.e. $\mathrm{sseq}\,\varphi : \mathrm{sseq}\,\mathcal{O} \to \mathrm{sseq}\,\mathcal{P}$ is an equivalence of $\mathcal{T}$-symmetric sequences. Thus Proposition 2.60 implies that φ is an equivalence. $\square$

We also point out a straightforward consequence of the fact that the forgetful functor is a right $\mathcal{T}$-adjoint.

Corollary 2.68. *The I-indexed tensor products in $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right)$ are products.*

Proof. The forgetful functor $U : \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right) \to \mathcal{C}$ is conservative, preserves $\mathcal{T}$-limits, and preserves tensor products; for all $(X_W) \in \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}\left(\mathcal{C}^{I-\times}\right)_S$, the canonical map

$$U\left(\bigotimes_W^S X_W\right) \simeq \bigotimes_W^S U(X_W) \to \prod_W^S U(X_W) \simeq U\left(\prod_W^S X_W\right)$$

is an equivalence, so $\bigotimes_W^S X_W \to \prod_W^S X_W$. $\square$

To finish the section, we repeat the above work without the one-color assumption.,

Observation 2.69. By either [NS22, Lem 2.4.4] or [CH21, Lem 2.9], we find that $\underline{\Sigma}_{\mathcal{T}}$-fibrous patterns are right Kan extended from their underlying $\mathcal{T}^{\mathrm{op}}$-fibrous patterns. Unwinding definitions, this expresses

$$\pi_0 \mathrm{triv}(\mathcal{O})_V \simeq \{(\mathbf{C}, D) \in \mathcal{O}_S \times \mathcal{O}_V \mid S \in \mathbb{F}_V\}$$

$\triangleleft$

Observation 2.70. Analogously to the above, for $\mathcal{O}^{\otimes}$ an *arbitrary* $\mathcal{T}$-operad, the operadic left kan extension formula of [NS22, Rmk 4.3.6] expresses the values of the associated monad as the left Kan extension

$$\begin{tikzcd}
\mathrm{Env}_{\mathcal{O}}\mathrm{triv}(\mathcal{O}) \arrow[r, equal] \arrow[dd] & \mathrm{Tot}\,\mathrm{triv}(\mathcal{O})^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \arrow[r, "X"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{C} \\
& \mathrm{Tot}\,\mathrm{triv}^{\otimes}_{(}\mathcal{O})^{\otimes} \arrow[d] \arrow[ur, dashed, "\widetilde{T}_{\mathcal{O}} X"'] \arrow[dr, phantom, "\Rightarrow"] & \\
\mathcal{O} \arrow[r, equal] & \mathcal{O} \arrow[uur, dashed, bend right, "T_{\mathcal{O}} X"'] &
\end{tikzcd}$$

The $\mathcal{T}$-functor $\tilde{T}_{\mathcal{O}}(X)$ sends

$$(\mathbf{C}, D) \mapsto \left(\mathcal{O}(\mathbf{C}; D) \otimes \bigotimes_U^S X_U\right)_{h\mathrm{Aut}_V S} \tag{10}$$

$\triangleleft$

Corollary 2.71 ("Equivariant [HM23, Thm 4.1.1]"). *A map of $\mathcal{T}$-operads $\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an equivalence if and only if it satisfies the following conditions:*

(a) $U(\varphi) : \mathcal{O} \to \mathcal{P}$ is $\mathcal{T}$-essentially surjective, and
(b) the pullback functor $\varphi^* : \mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is an equivalence of ∞-categories.

Proof. The fact that φ being an equivalence implies the above conditions is obvious, so assume the above conditions. The result follows by using an identical argument to Corollary 2.67, using Eq. (10) instead of Eq. (9) to show that $\varphi : \mathcal{O}(\mathbf{C}; D) \to \mathcal{P}(\varphi\mathbf{C}; \varphi D)$ is an equivalence for all C, concluding the equivalence from Proposition 2.60. □

2.5. **$\mathcal{O}$-algebras in I-symmetric monoidal d-categories.** Recall that a space X is said to be *d-truncated* if it is empty or $\pi_n(X, x) = *$ for all $x \in X$ and $n > 0$; in particular, X is (-1)-truncated precisely if it is either empty or contractible. In Section 1.4, we applied this to mapping spaces to define *$\mathcal{T}$-symmetric monoidal d-categories*. In this section, we define a compatible notion of *$\mathcal{T}$-d-operads*, centered on the following result.

Proposition 2.72. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and let $d \geq -1$. Then, the following conditions are equivalent:*

(a) $\mathcal{O}(S)$ is d-truncated for all $S \in \mathbb{F}_V$.
(b) The $\mathcal{T}$-functor $\mathrm{Env}\mathcal{O} \to \underline{\mathbb{F}}_{\mathcal{T}}$ has d-truncated mapping fibers.

Proof. Let $\psi : T \to S$ be a map of $\mathcal{T}$-sets over V. Then, by Lemma 2.56, we have an equivalence

$$\begin{aligned} \mathrm{Mor}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathrm{Env}\mathcal{O}) &\simeq \coprod_{\mathbf{C}\in\mathcal{O}_T, \mathbf{D}\in\mathcal{O}_S} \mathrm{Map}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathbf{C}, \mathbf{D}) \\ &\simeq \coprod_{\mathbf{C}\in\mathcal{O}_T, \mathbf{D}\in\mathcal{O}_S} \prod_{U\in\mathrm{Orb}(S)} \mathrm{Map}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathbf{C}_U, D_U) \\ &\simeq \coprod_{\mathbf{C}\in\mathcal{O}_T, \mathbf{D}\in\mathcal{O}_S} \prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(\mathbf{C}_U; D_U) \end{aligned} \tag{11}$$

First, in the case $d = -1$, note that conditions (a) and (b) both imply that $\mathcal{O}$ has at most one color, so Eq. (11) specializes to

$$\mathrm{Mor}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathrm{Env}\mathcal{O}) \simeq \prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(S).$$

Thus it suffices to note that a product is -1-truncated if and only if its factors are.

Next, in the case $d \geq 0$, note that a coproduct of spaces is d-truncated if and only if its factors are; hence Eq. (11) shows that (b) is equivalent to the condition that $\prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(\mathbf{C}_U; D_U)$ is d-truncated for all $S, \mathbf{C}, \mathbf{D}$. In fact, the equation

$$\mathcal{O}(S) \simeq \coprod_{(\mathbf{C}, D)\in\mathcal{O}_S\times\mathcal{O}_V} \mathcal{O}(\mathbf{C}; D)$$

shows that this (b) equivalent to the condition that $\mathcal{O}(S)$ is d-truncated for all $S \in \mathbb{F}_V$, as desired. □

We define the full subcategory of *d-operads*

$$\iota_d : \mathrm{Op}_{\mathcal{T},d} \hookrightarrow \mathrm{Op}_{\mathcal{T}}$$

to be spanned by $\mathcal{T}$-operads satisfying the condition that $\mathcal{O}(S)$ is $(d-1)$-truncated for all $S \in \mathbb{F}_V$ as in Proposition 2.72.

The following corollary immediately follows from Proposition 2.72 and the mapping fiber truncation characterizations of Corollary 1.87.

Corollary 2.73. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and let $d \geq 1$. The following conditions are equivalent:*

(a) $\mathcal{O}$ is a d-operad, and
(b) $\mathrm{Env}\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-symmetric monoidal d-category.

Furthermore, the following conditions are equivalent:

(a') $\mathcal{O}$ is a 0-operad, and
(b') the $\mathcal{T}$-symmetric monoidal functor $\mathrm{Env}\mathcal{O}^{\otimes} \to \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ is a $\mathcal{T}$-symmetric monoidal subcategory inclusion.

Corollary 2.74. *The inclusion $\mathrm{Op}_{\mathcal{T},d} \hookrightarrow \mathrm{Op}_{\mathcal{T}}$ has a left adjoint $h_{\mathcal{T},d}$ satisfying*

$$(h_{\mathcal{T},d}\mathcal{O})(S) \simeq \tau_{\leq d}\mathcal{O}(S).$$

Furthermore, when $d \geq 1$, this fits into the following diagram

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, "h_{\mathcal{T},d}"] \arrow[d] & \mathrm{Op}_{\mathcal{T},d} \arrow[d] \\ \mathrm{Cat}_{\mathcal{T}}^{\otimes} \arrow[r, "h_{\mathcal{T},d}"] & \mathrm{Cat}_{\mathcal{T},d}^{\otimes} \end{tikzcd}$$

In particular, when $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-symmetric monoidal d-category, the canonical map $\mathcal{O}^{\otimes} \rightarrow h_{\mathcal{T},d}\mathcal{O}^{\otimes}$ induces an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{h_{\mathcal{T},d}\mathcal{O}}(\mathcal{C}).$$

Proof. By [BHS22, Prop 4.2.1], the image of the fully faithful functor $\mathrm{Op}_{\mathcal{T}} \hookrightarrow \mathrm{Cat}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}$ is spanned by the *equifibered* $\mathcal{T}$-symmetric monoidal ∞-categories, i.e. $\mathcal{C}^{\otimes}$ such that, given $T \rightarrow S$ a map of finite $\mathcal{T}$-sets, the associated diagram

$$\begin{tikzcd} \mathcal{C}_T \arrow[r] \arrow[d] & \mathcal{C}_S \arrow[d] \\ \mathbb{F}_T \arrow[r] & \mathbb{F}_S \end{tikzcd}$$

is cartesian. We separately argue in the case $d \geq 1$ and $d = 0$ that the image of this is closed under $h_{\mathcal{T},d}$; this will imply that $h_{\mathcal{T},d}\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ corresponds with a $\mathcal{T}$-d-operad $h_{\mathcal{T},d}\mathcal{O}^{\otimes}$, which computes the left adjoint to the inclusios $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ by fully faithfulness of $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$.

We first consider the case $d \geq 1$. In this case, since $h_{\mathcal{T},d} : \mathrm{Cat}_{\mathcal{T}}^{\otimes} \rightarrow \mathrm{Cat}_{\mathcal{T},d}^{\otimes}$ is applied pointwise, it preserves equifibrations, so $h_{\mathcal{T},d}\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ corresponds with a d-operad $h_{\mathcal{T},d}\mathcal{O}^{\otimes}$.

The case $d = 0$ is similar, except that we are tasked with replacing equifibered $\mathcal{T}$-symmetric monoidal functors with an equifibered subcategory. In fact, subcategories are precisely (-1)-truncated maps in Cat, so we may do this by taking the pointwise (-1)-truncation functor and applying [HTT, Prop 5.5.6.5] to see that the result is equifibered. $\square$

Corollary 2.75. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-d-operad.*

(1) if $d \geq 1$, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is a d-category; hence $\mathrm{Op}_{\mathcal{T},d}$ is a $(d+1)$-category.
(2) if $d = 0$, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is either empty or contractible; hence $\mathrm{Op}_{\mathcal{T},0}$ is a poset.

Proof. In each case, the second statement follows from the first by noting that the mapping spaces in $\mathrm{Op}_{\mathcal{T}}$ are $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq}$. For the first statements, note that

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Alg}_{h_d\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Fun}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}(\mathrm{Env}h_d\mathcal{O}^{\otimes}, \mathrm{Env}\mathcal{P}^{\otimes});$$

if $d \geq 1$, then this is a subcategory of a d-category, so it's a d-category. If $d = 0$, then this category is either empty or contractible since we verified that the map $\mathrm{Env}\mathcal{O}^{\otimes} \rightarrow \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ is monic. $\square$

Corollary 2.76. *If $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-0-operad, then it is a sub-terminal object of $\mathrm{Op}_{\mathcal{T}}$.*

Proof. The mapping space criterion of monomorphisms shows that this is equivalent to the condition that

$$\mathrm{Alg}_{h_0\mathcal{O}}(\mathcal{P})^{\simeq} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq} \rightarrow \mathrm{Alg}_{\mathcal{O}}(\mathrm{Comm}_{\mathcal{T}}^{\otimes})^{\simeq} \simeq *$$

is a monomorphism, i.e. $\mathrm{Alg}_{h_0\mathcal{O}}(\mathcal{P})^{\simeq} \in \{\varnothing, *\}$; this follows from Corollary 2.75. $\square$

Corollary 2.77. *Let $I \leq J$ be related weak indexing categories. Then, the unslicing functor*

$$\mathrm{Op}_I \simeq \mathrm{Op}_{J,/\mathcal{N}_{I\infty}^{\otimes}} \rightarrow \mathrm{Op}_J$$

is fully faithful.

Proof. Fully faithful functors satisfy two-out-of-three, so we may replace $\mathrm{Op}_I \to \mathrm{Op}_J$ with the composite unslicing functor $\mathrm{Op}_I \to \mathrm{Op}_J \to \mathrm{Op}_{\mathcal{T}}$, and assume $I = \mathbb{F}_{\mathcal{T}}$. The corollary is then equivalent to the statement that $\mathcal{N}_{I\infty}^{\otimes} \to \mathrm{Comm}_{\mathcal{T}}^{\otimes}$ is a monomorphism [HTT, § 5.5.6]. In fact, by Example 2.44, $\mathcal{N}_{I\infty}^{\otimes}$ is a $\mathcal{T}$-0-operad, so this follows from Corollary 2.76. $\square$

We finish the subsection with a recognition result highly connected maps; we say that a map $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is *n-connected* if any of the following equivalent conditions hold.

Proposition 2.78. *Let $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ be a morphism of $\mathcal{T}$-operads. Then, the following are equivalent:*

(a) The underlying $\mathcal{T}$-functor $U\varphi\colon \mathcal{O} \to \mathcal{P}$ is fiberwise-essentially surjective and for all $V \in \mathcal{T}$ and $S \in \mathbb{F}_{\mathcal{T}}$, the induced map $\mathcal{O}(S) \to \mathcal{P}(S)$ is n-connected.
(b) φ is an $h_{\mathcal{T},n+1}$-equivalence.
(c) For all $\mathcal{T}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}$, the pullback $\mathcal{T}$-symmetric monoidal functor

$$\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$$

is an equivalence.
(d) The pullback functor

$$\mathrm{Alg}_{\mathcal{P}}(\mathcal{S}_{\le n+1}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}_{\le n+1})$$

is an equivalence.

Proof. Suppose (a); in view of Proposition 2.60, to prove (b), we're tasked with proving that the maps $h_{\mathcal{T},n+1}\mathcal{O}(\mathbf{C};D) \to h_{\mathcal{T},n+1}\mathcal{P}(\mathbf{C};D)$ are equivalences. But by the natural equivalence

$$\mathcal{O}(S) \simeq \coprod_{(\mathbf{C},D)\in \mathcal{O}_S\times\mathcal{O}_V} \mathcal{O}(\mathbf{C};D),$$

it suffices to verify that $h_{\mathcal{T},n+1}\mathcal{O}(S) \to h_{\mathcal{T},n+1}\mathcal{P}(S)$ is an equivalence for each S. This follows from (a) by Corollary 2.74.

Suppose (b); by the factorization

$$\mathrm{Cat}_{\mathcal{T},n+1}^{\otimes} \hookrightarrow \mathrm{Op}_{\mathcal{T},n+1} \hookrightarrow \mathrm{Op}_{\mathcal{T}}$$

of Corollary 2.74, given $\mathcal{C} \in \mathrm{Cat}_{\mathcal{T},n+1}^{\otimes}$, the top map in the following is an equivalence

$$\begin{tikzcd} \mathrm{Alg}_{h_{\mathcal{T},n+1}\mathcal{P}}(\mathcal{C}) \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & \mathrm{Alg}_{h_{\mathcal{T},n+1}\mathcal{O}}(\mathcal{C}) \arrow[d, phantom, "\wr"] \\ \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[r] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \end{tikzcd}$$

the bottom arrow is an equivalence from two-out-of-three, and (c) follows from Corollary 2.9. Furthermore, (c) implies (d) by setting $\mathcal{C}^{\otimes} \coloneqq \underline{\mathcal{S}}_{\mathcal{T},\ge n+1}^{\mathcal{T}-\times}$.

Finally, suppose (d). Note that $\mathcal{O}(S)\cdot 1 \simeq \tau_{\le n+1}\mathcal{O}(S)$, so using the same argument as Proposition 2.64, we naturally split off the map

$$\tau_{\le n+1}\varphi(S)\colon \tau_{\le n+1}\mathcal{O}(S) \to \tau_{\le n+1}\mathcal{P}(S)$$

from the map $T_{\varphi}S$ between monads over $\mathcal{S}_{\mathcal{T},\le n+1}$. By assumption, $T_{\varphi}S$ is an equivalence so $\tau_{\le n+1}\varphi(S)$ is an equivalence, implying (a). $\square$

This suggests a notion of *n-connected $\mathcal{T}$-operads*, who satisfy the property that the truncation unit $\mathcal{O}^{\otimes} \to h_{\mathcal{T},0}\mathcal{O}^{\otimes}$ is n-connected. In forthcoming work [Ste24a], we will classify $\mathcal{T}$-0-operads, and gain characterization of n-connected $\mathcal{T}$-operads as a corollary.

2.6. The genuine operadic nerve.

2.6.1. *The 1-categorical nerve.* [BP21] introduced a variant of the following.

Definition 2.79. A one-color genuine $\mathcal{T}$-operad in a symmetric monoidal 1-category $\mathcal{V}$ the data of:

(1) a $\mathcal{T}$-symmetric sequence $\mathcal{O}(-) : \mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}} \to \mathcal{V}$,
(2) for all $V \in \mathcal{T}$, a distinguished "identity" element $1_V \in \mathcal{O}(*_V)$, and
(3) for all $S \in \mathbb{F}_V$ and $U \in \mathbb{F}_S$, a Borel $\Sigma_S \times \prod_{U \in \mathrm{Orb}(S)} \Sigma_{T_U}$-equivariant "composition" map

$$\gamma : \mathcal{O}(S) \otimes \bigotimes_{U \in \mathrm{Orb}(S)} (T_U) \to \mathcal{O}\left(\coprod_U^S T_U\right)$$

subject to the following compatibilities for all :

(a) (restriction-stability of the identity) for all $U \to V$, the map $\mathrm{Res}_U^V : \mathcal{O}(*_V) \to \mathcal{O}(*_U)$ sends 1_V to 1_U;
(b) (restriction-stability of composition) for all $U \to V$, the following commutes

$$\begin{tikzcd}
\mathcal{O}(S) \times \prod\limits_{U \in \mathrm{Orb}(s)} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, "\mathrm{Res}_V^W"] & \mathcal{O}(T) \arrow[d, "\mathrm{Res}_V^W"] \\
\mathcal{O}\left(\mathrm{Res}_W^V S\right) \times \prod\limits_{U' \in \mathrm{Orb}(S)} \mathcal{O}(T_{U'}) \arrow[r, "\gamma"] & \mathcal{O}\left(\mathrm{Res}_W^V S\right)
\end{tikzcd}$$

(c) (unitality) for all $S \in \mathbb{F}_V$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(S) \arrow[r, "(\mathrm{id},(\{1_U\}))"] \arrow[d, "(1_V\text{,id})"'] \arrow[dr, equal] & \mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S)} \mathcal{O}(*_U) \arrow[d, "\gamma"] \\
\mathcal{O}(*_V) \otimes \mathcal{O}(S) \arrow[r, "\gamma"] & \mathcal{O}(S)
\end{tikzcd}$$

(d) (associativity) For all $S \in \mathbb{F}_V$, $(T_U) \in \mathbb{F}_S$ writing $T := \coprod\limits_U^S T_U$, and $(R_W) \in \mathbb{F}_T$ writing $R := \coprod\limits_W^T R_W$, the following diagram commutes

$$\begin{tikzcd}
\left(\mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S_U)} \mathcal{O}(T_U)\right) \otimes \bigotimes\limits_{\substack{U \in \mathrm{Orb}(S) \\ W \in \mathrm{Orb}(T_U)}} \mathcal{O}(T_U) \arrow[r, "\gamma"] \arrow[d, equal] & \mathcal{O}(T) \otimes \bigotimes\limits_{W \in \mathrm{Orb}(T)} \mathcal{O}(R_W) \arrow[d, "\gamma"] \\
\mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S)} \left(\mathcal{O}(T_U) \otimes \bigotimes\limits_{W \in \mathrm{Orb}(T_U)} \mathcal{O}(R_U)\right) \arrow[d, "\gamma"'] & \mathcal{O}\left(\coprod\limits_W^T R_W\right) \arrow[d, equal] \\
\mathcal{O}(S) \otimes \bigotimes\limits_{U \in \mathrm{Orb}(S)} \mathcal{O}\left(\coprod\limits_W^{T_U} R_W\right) \arrow[r, "\gamma"] & \mathcal{O}(R)
\end{tikzcd}$$

A morphism of one-color discrete $\mathcal{T}$-operads in $\mathcal{V}$ is a map of $\mathcal{T}$-symmetric sequences in $\mathcal{V}$ preserving 1_V and intertwining γ; we refer to the resulting 1-category as $g\mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}}(\mathcal{V})$. ◁

We write $s\mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} := g\mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}}(s\mathrm{Set})$. In [BP21], a *many-colors* variant $g\mathrm{Op}_{\mathcal{T}}(\mathcal{V})$ was introduced, and a model structure was given to $s\mathrm{Op}_G := g\mathrm{Op}_{\mathcal{T}}(s\mathrm{Set})$; this was later shown to be Quillen equivalent to several other model categorical variations on G-operads (e.g. [BP20, Tab 1]). This was used in [Bon19] to construct a *genuine operadic nerve* functor of 1-categories

$$N^{\otimes} \colon g\mathrm{Op}_G(s\mathrm{Set}) \to s\mathrm{Set}^+_{/(\mathrm{Tot}\underline{\mathbb{F}}_{G,*},\mathrm{Ne})}$$

whose restriction $g\mathrm{Op}_G(\mathrm{Kan})$ lands in fibrant objects in Nardin-Shah's model structure [NS22, § 2.6], and hence presents G-operads.

Moreover, $g\mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}}(\mathrm{Kan})$ agrees with the *fibrant simplicial colored $\mathcal{T}$-operads* of [NS22, Def 2.5.4] subject to the condition that the underlying $\mathcal{T}$-set of colors is contractible; thus Nardin-Shah construct an analogous nerve functor

$$N^{\otimes}\colon g\mathrm{Op}_{\mathcal{T}}(\mathrm{Kan}) \to s\mathrm{Set}^{+}_{/(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}),\mathrm{NE}}$$

whose specialization to $\mathcal{T} = \mathcal{O}_G$ agrees with the one-color version of Bonventre's nerve.

These nerves can be understood as taking $\mathcal{O} \in g\mathrm{Op}_{\mathcal{T}}(\mathrm{Kan})$ with underlying $\mathcal{T}$-coefficient system $\mathfrak{C}$ to the Kan-enriched category over $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$ with $\mathrm{Ob}\,\mathcal{O}_S = \mathfrak{C}_S$ and with mapping space

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}(\mathbf{C}, \mathbf{D}) \simeq \coprod_{\pi_{\mathcal{O}}\mathbf{C} \to \pi_{\mathcal{O}}\mathbf{D}} \prod_{U \in \mathrm{Orb}(\pi_{\mathcal{O}}(\mathbf{D}))} \mathcal{O}(\mathbf{C}_U; D_U)$$

mapping down to $\mathrm{Map}_{\underline{\mathbb{F}}_{\mathcal{T},*}}(\pi_{\mathcal{O}}\mathbf{C}, \pi_{\mathcal{O}}\mathbf{D})$ via the evident forgetful map.

2.6.2. *Restriction and the nerve.* $N^{\otimes}$ interacts with restrictions.

Construction 2.80. Let $W \in \mathcal{T}$ be a distinguished object. Then, the *restriciton functor*

$$\mathrm{Res}^{\mathcal{T}}_{W}\colon g\mathrm{Op}_{\mathcal{T}}(\mathcal{V}) \to g\mathrm{Op}_{W}(\mathcal{V}) \coloneqq g\mathrm{Op}_{\mathcal{T}_{/W}}(\mathcal{V})$$

acts on underlying $\mathcal{T}$-symmetric sequences via pullback along the map $\mathrm{Tot}\underline{\Sigma}_W \to \mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}$, with the data 1_V and γ defined in $\mathrm{Res}^{\mathcal{T}}_{W}\mathcal{O}^{\otimes}$ by restriction from $\mathcal{O}^{\otimes}$. ◁

We define restriction $\mathrm{Res}^{\mathcal{T}}_{W}\colon \mathrm{Cat}_{s\mathrm{Set},/\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}} \to \mathrm{Cat}_{s\mathrm{Set},/\mathrm{Tot}\underline{\mathbb{F}}_{W,*}}$ by pullback along $\mathrm{Tot}\underline{\mathbb{F}}_{W,*} \to \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$.

Proposition 2.81. *There is a natural isomorphism of simplicial categories $N^{\otimes}\mathrm{Res}^{\mathcal{T}}_{W} \simeq \mathrm{Res}^{\mathcal{T}}_{W} N^{\otimes}$ over $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$.*

Proof. Let $\mathcal{O}^{\otimes}$ be a one-color simplicial genuine $\mathcal{T}$-operad. We may construct a functor $N^{\otimes}\mathrm{Res}^{\mathcal{T}}_{W}\mathcal{O}^{\otimes} \to N^{\otimes}\mathcal{O}^{\otimes}$ sending the object over a $(V \to W)$-set $S_{V\to W}$ to it underlying V-set S and acting on mapping spaces by taking coproducts of the equivalence $\mathrm{Res}^{\mathcal{T}}_{W}\mathcal{O}(S_{V\to W}) \simeq \mathcal{O}(S_V)$. This constructs a natural diagram

$$\begin{tikzcd}
\mathrm{Tot}_{\mathcal{T}} N^{\otimes}\mathrm{Res}^{\mathcal{T}}_{W}\mathcal{O}^{\otimes} \arrow[dr, dashed, "F"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathrm{Tot}_{\mathcal{T}}\,\mathrm{Res}^{\mathcal{T}}_{W} N^{\otimes}\mathcal{O}^{\otimes} \arrow[r] \arrow[d] & \mathrm{Tot}_{\mathcal{T}} N^{\otimes}\mathcal{O}^{\otimes} \arrow[d] \\
& \mathrm{Tot}\underline{\mathbb{F}}_{W,*} \arrow[r] & \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}
\end{tikzcd}$$

since $\pi_{N^{\otimes}\mathrm{Res}^{\mathcal{T}}_{W}\mathcal{O}^{\otimes}}$ and $\pi_{\mathrm{Res}^{\mathcal{T}}_{W}N^{\otimes}\mathcal{O}^{\otimes}}$ are both π_0-isomorphisms, F is as well; hence F is essentially surjective. It follows by unwinding definitions that F is fully faithful, and hence an equivalence of simplicial categories over $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$, as desired. □

Pullback along $\mathrm{Tot}\underline{\mathbb{F}}_{W,*} \subset \mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$ implements restriction of $\mathcal{T}$-operads Remark 2.48, yielding the following.

Corollary 2.82. *There is a natural equivalence of W-operads $\mathrm{Res}^{\mathcal{T}}_{W} N^{\otimes}\mathcal{O}^{\otimes} \simeq N^{\otimes}\mathrm{Res}^{\mathcal{T}}_{W}\mathcal{O}^{\otimes}$.*

The main reason we went to this trouble is for the following example.

Example 2.83. Let G be a finite group and V be a real orthogonal G-representation. Let D_V be a genuine G-operad which is equivalent to the little V-disks operad (see [Hor19, § 3.9]). Then, given $K \subset H \subset G$, and $S \in \mathbb{F}_K$, we have a tautological equivalence

$$\mathrm{Res}^{G}_{H} D_V(S) \simeq \mathrm{Conf}^{K}_{S}(\mathrm{Res}^{G}_{K} V) \simeq \mathrm{Conf}^{K}_{S}(\mathrm{Res}^{H}_{K}\mathrm{Res}^{G}_{H} V) \simeq D_{\mathrm{Res}^{G}_{H} V}(S)$$

which intertwines with the composition rule in D_V; writing $\mathbb{E}^{\otimes}_{V} \coloneqq N^{\otimes}D^{V}$, we acquire an equivalence

$$\mathrm{Res}^{G}_{H}\mathbb{E}^{\otimes}_{V} \simeq \mathbb{E}^{\otimes}_{\mathrm{Res}^{G}_{H} V}$$ ◁.

2.6.3. *The conservative ∞-categorical lift.* $N^\otimes$ has homotopical structure.

Proposition 2.84. *$N^\otimes$ preserves and reflects weak equivalences between one-color locally fibrant genuine equivariant G-operads.*

Proof. By [BP21, Thm II, Prop 4.31], the functor $U : s\mathrm{Op}_G^{\mathrm{oc}} \to \mathrm{Fun}(\underline{\Sigma}_G, \mathrm{sSet})$ is monadic and $s\mathrm{Op}_G^{\mathrm{oc}}$ possesses the (right-)transferred model structure from the projective model structure on $\mathrm{Fun}\left(\underline{\Sigma}_G, \mathrm{sSet}_{\mathrm{Quillen}}\right)$; in particular, U preserves and reflects weak equivalences.

It is not hard to see that sseq may be presented as total right-derived from a functor

$$s\mathrm{sseq} : \mathrm{sSet}^{+,\mathrm{oc}}_{/(\underline{\mathbb{F}}_{\mathcal{T}}, Ne)} \to \mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_G, \mathrm{sSet}_{\mathrm{Quillen}}\right)_{\mathrm{Proj}}$$

setting $\mathcal{O}_{\mathrm{sseq}}(S) := \pi_{\mathcal{O}}^{-1}(\mathrm{Ind}_H^G S \to G/H)$; by Proposition 2.60 sseq is conservative, so $s\mathrm{sseq}$ preserves and reflects weak equivalences between fibrant objects. Hence it suffices unwind definitions and note that the following diagram commutes

$$\begin{tikzcd}
s\mathrm{Op}_G^{\mathrm{oc}} \arrow[r, "N^\otimes"] \arrow[dr, "U"'] & \mathrm{sSet}^{+,\mathrm{oc}}_{/(\underline{\mathbb{F}}_G, Ne)} \arrow[d, "s\mathrm{sseq}"] \\
& \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathrm{sSet})
\end{tikzcd}$$

□

In fact, the one-color assumption was not necessary.

Proposition 2.85. *$N^\otimes$ preserves and reflects weak equivalences between arbitrary locally fibrant genuine equivariant G-operads.*

Proof. It is not too hard to see that $N^\otimes$ preserves and reflects the property of *inducing bijections on sets of colors*, so we may fix a coefficient system of sets of colors $\mathfrak{C}$. Then, we are tasked with proving that $N^\otimes_{\mathfrak{C}} : s\mathrm{Op}_{G,\mathfrak{C}} \to \mathrm{Op}_{G,\mathfrak{C}} := (\pi_0 U)^{-1}(\mathfrak{C})$ preserves and reflects weak equivalences between fibrant objects. Thankfully, we have the same tools as in the one-color case; writing $\mathrm{Tot}\underline{\Sigma}_{\mathfrak{C}}$ for the 1-category of [BP22, Def 3.1], $s\mathrm{Op}_{G,\mathfrak{C}}$ possesses the right-transferred model structure from along a monadic functor $U : s\mathrm{Op}_{G,\mathfrak{C}} \to \mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_{\mathfrak{C}}, s\mathrm{Set}_{\mathrm{Quillen}}\right)$ by [BP22, § 5.2]. Furthermore, Proposition 2.60 constructs a functor $s\,\mathrm{sseq} : \mathrm{sSet}^{+,\mathfrak{C}}_{/(\underline{\mathbb{F}}_{\mathcal{T}}, Ne)} \to \mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_{\mathfrak{C}}, \mathrm{sSet}_{\mathrm{Quillen}}\right)$ which preserves and reflects weak equivalences between fibrant objects, and such that $N^\otimes$ is a functor over $\mathrm{Fun}\left(\mathrm{Tot}\underline{\Sigma}_{\mathfrak{C}}, \mathrm{sSet}_{\mathrm{Quillen}}\right)$; by two-out-of-three for weak equivalences, $N^\otimes$ preserves and reflects weak equivalences between fibrant objects. □

The theory of total right derived functors (e.g. [Rie14, § 2]) then immediately yields Corollary B.

2.6.4. *The discrete genuine nerve is an equivalence.* Recall that whenever $\mathcal{O}^\otimes$ is a $\mathcal{T}$-operad and $\mathcal{C}^\otimes$ is a $\mathcal{T}$-1-category, there is an equivalence of $\mathcal{T}$-1-categories

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{h_1\mathcal{O}}(\mathcal{C});$$

because of this, for the rest of this subsection, we assume all $\mathcal{T}$-operads are $\mathcal{T}$-1-operads.

Note that the (fully faithful) inclusion of discrete simplicial sets $\mathrm{Set} \hookrightarrow \mathrm{sSet}$ is product-preserving, so it induces a fully faithful functor $g\mathrm{Op}_{\mathcal{T}}(\mathrm{Set}) \hookrightarrow g\mathrm{Op}_{\mathcal{T}}(\mathrm{sSet})$. We refer to these as *discrete genuine $\mathcal{T}$-operads*. We're concerned with relating this to $\mathcal{T}$-1-categories, beginning with the following.

Observation 2.86. For all $\mathcal{O} \in g\mathrm{Op}_{\mathcal{T}}(\mathrm{Set})$, $N^\otimes\mathcal{O}$ is a $\mathcal{T}$-1-operad. ◁

Conversely, from the data of a $\mathcal{T}$-1-operad $\mathcal{O}$, the data of a discrete genuine $\mathcal{T}$-operad $\mathcal{O}(-)$ is supplied by Observation 2.45.

Proposition 2.87. *$N^\otimes$ descends to a functor $g\mathrm{Op}_{\mathcal{T}}(\mathrm{Set}) \to \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T},1}$ with quasi-inverse $\mathcal{O}(-)$.*

Proof. By Observation 2.86, $N^\otimes$ restricts as above. Thus it suffices to prove that the compositions $g\mathrm{Op}_{\mathcal{T}}(\mathrm{Set}) \to g\mathrm{Op}_{\mathcal{T}}(\mathrm{Set})$ and $\mathrm{Op}^{\mathrm{oc}}_{\mathcal{T},1} \to \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T},1}$ are homotopic to the identity; this follows immediately after unwinding definitions. □

Now having an explicit combinatorial model for $\mathcal{T}$-1-operads, we focus on algebras. We need the following.

Construction 2.88. Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and $\mathcal{P} \subset \mathcal{O}$ a full $\mathcal{T}$-subcategory. Then, we define the full subcategory $\mathcal{P}^{\otimes} \subset \mathcal{O}^{\otimes}$ to be spanned by the tuples $\mathbf{C} \in \mathcal{O}_S$ such that, for each $U \in \mathrm{Orb}(S)$, $C_U \in \mathcal{P}$. $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-operad and $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ a map of $\mathcal{T}$-operads [NS22, § 2.9].

In particular, if $X \in \Gamma^{\mathcal{T}}\mathcal{O}$ is a $\mathcal{T}$-object in $\mathcal{O}$, we define the *endomorphism $\mathcal{T}$-operad* $\mathrm{End}_X^{\otimes} \subset \mathcal{C}^{\otimes}$ *of* X to be the full $\mathcal{T}$-operad of $\mathcal{O}^{\otimes}$ spanned by $\{X\}$. ◁

Observation 2.89. Suppose $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category and $X \in \Gamma^{\mathcal{T}}\mathcal{C}$. Then, End_X has underlying $\mathcal{T}$-symmetric sequence $\mathrm{End}_X(S) \simeq \mathrm{Map}(X_V^{\otimes S}, X_V)$ for $S \in \underline{\mathbb{F}}_I$, identity element $1_V = \mathrm{id}_{X_V}$, and composition map given by composition of maps

$$\gamma(\mu_S;(\mu_{T_U}))\colon X_V^{\otimes T} \simeq \bigotimes_U^S X_U^{\otimes T_U} \xrightarrow{\bigotimes_U^S \mu_{T_U}} X_V^{\otimes S} \xrightarrow{\mu_S} X_V. \quad ◁$$

In general, an $\mathcal{O}$-algebra in $\mathcal{C}^{\otimes}$ may be viewed as the information of its underlying object X together with the factored map $\mathcal{O}^{\otimes} \to \mathrm{End}_X^{\otimes} \hookrightarrow \mathcal{C}^{\otimes}$. The following proposition follows by unwinding definitions.

Proposition 2.90. *If* $\mathcal{C}^{\otimes}$ *is a* $\mathcal{T}$*-1-category and* X, Y *are* $\mathcal{O}$*-algebras in* $\mathcal{C}^{\otimes}$*, then the hom set* $\mathrm{Hom}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ *consists of those maps such that the following diagram of operads commutes:*

$$\begin{tikzcd} & \mathrm{End}_X^{\otimes} \arrow[d] \\ \mathcal{O}^{\otimes} \arrow[ur] \arrow[dr] & \\ & \mathrm{End}_Y^{\otimes} \end{tikzcd}$$

For the sake of comparison, we will propose one more model for discrete I-commutative algebras.

Definition 2.91. Let I be a one-color weak indexing category. Then, a *strict I-commutative algebra in* $\mathcal{C}$ is the data of a $\mathcal{T}$-object X together with $\mathrm{Aut}_V S$-equivariant maps $\mu_S : X_V^{\otimes S} \to X_V$ for all $S \in \mathbb{F}_{I,V}$ subject to the following conditions:

(1) (restriction-stability) The functor Res_U^V takes μ_S to $\mu_{\mathrm{Res}_U^V S}$.
(2) (unitality) for all maps $S \sqcup *_V \in \mathbb{F}_{I,V}$, the following diagram commutes:

$$\begin{tikzcd} & X_V^{\otimes S \sqcup *_V} \arrow[dr] & \\ X_V \arrow[ur] \arrow[rr, equal] & & X_V \end{tikzcd}$$

(3) (associativity) for all S-tuples $(T_U) \in \mathbb{F}_{I,S}$, writing $T = \coprod_U^S T_U$, the following diagram commutes:

$$\begin{tikzcd} \bigotimes_U^S X_U^{\otimes T_U} \arrow[r, "(\mu_{T_U})"] \arrow[d, phantom, "\wr"] & X_V^{\otimes S} \arrow[d, "\mu_S"] \\ X_V^{\otimes T} \arrow[r, "\mu_T"] & X_V \end{tikzcd}$$

◁

Proposition 2.92. *If* $\mathcal{C}^{\otimes}$ *is a* $\mathcal{T}$*-symmetric monoidal 1-category, then the categories of* I*-commutative algebras and strict* I*-commutative algebras in* $\mathcal{C}$ *agree.*

Proof. This follows from Observation 2.89, noting that $\mathrm{Map}(\mathcal{N}_{I\infty}^{\otimes}, \mathrm{End}_X^{\otimes}) \simeq \mathrm{Map}(\mathcal{N}_{I\infty}^{\otimes}, \mathrm{Bor}_I^{\mathcal{T}} \mathrm{End}_X^{\otimes})$ and unwinding definitions using Proposition 2.87. □

Let X, Y be I-commutative algebras and $f\colon X \to Y$ a morphism between their underlying $\mathcal{T}$-objects. For the rest of this subsection, we assume familiarity with the techniques of [Ste24b]. We will say that f

intertwines at $S \in \mathbb{F}_{I,V}$ if the following diagram commutes:

$$\begin{tikzcd} X_V^{\otimes S} \arrow[r] \arrow[d] & X_V \arrow[d] \\ Y_V^{\otimes S} \arrow[r] & Y_V \end{tikzcd}$$

Define the collection $\underline{\mathbb{F}}_{t(f)} \subset \underline{\mathbb{F}}_I$ by

$$\mathbb{F}_{t(f),V} := \{S \mid f \text{ intertwines at } S\} \subset \mathbb{F}_{I,V}$$

The fact that f is a map of $\mathcal{T}$-objects implies that $\underline{\mathbb{F}}_{t(f)}$ is restriction stable. Hence $\underline{\mathbb{F}}_{t(F)} \subset \underline{\mathbb{F}}_I$ is a full $\mathcal{T}$-subcategory.

Proposition 2.93. $\underline{\mathbb{F}}_{t(f)}$ *is a weak indexing system.*

Proof. It follows by unwinding definitions that $c(t(f)) = c(I)$, so we're left with proving that $\underline{\mathbb{F}}_{t(f)}$ is closed under self-indexed coproducts. To that end, fix $S \in \mathbb{F}_{t(f),V}$ and $T \in \mathbb{F}_{t(f),S}$. By the associativity condition, we're tasked with proving that the outer rectangle of the following diagram commutes

$$\begin{tikzcd} X_V^{\otimes T} \arrow[d] & \simeq & \bigotimes_U^S X_U^{T_U} \arrow[r] \arrow[d] & X_V^{\otimes S} \arrow[r] \arrow[d] & X_V \arrow[d] \\ Y_V^{\otimes T} & \simeq & \bigotimes_U^S Y_U^{T_U} \arrow[r] & Y_V^{\otimes S} \arrow[r] & Y_V \end{tikzcd}$$

The left inner rectangle is commutative by definition; the right inner rectangle is commutative by the assumption $S \in \mathbb{F}_{t(f),V}$; the middle inner rectangle is commutative by taking a (pointwise) S-indexed tensor product of the commutativity diagrams for each T_U. ☐

Recall that a *sparse* V*-set* is a V-set of the form

$$\epsilon \cdot *_V \sqcup W_1 \sqcup \cdots \sqcup W_n$$

where $\epsilon \in \{0,1\}$ and there exist no maps $W_i \to W_j$ over V for $i \neq j$.

Corollary 2.94. *Let I be an almost essentially unital weak indexing system. Then,*

(1) f is a map of I-commutative algebras if and only if it intertwines at all sparse I-admissible V-sets.
(2) If I is an indexing system, then f is a map of I-commutative algebras if and only if it intertwines at $2 \cdot *_V$ and at all I-admissible transitive *V-sets for all $V \in \mathcal{T}$.*

Proof. In each case, it suffices to show that the applicable V-sets generate $\underline{\mathbb{F}}_I$ as a weak indexing category. Case (1) is shown in [Ste24b] and case (2) follows by noting that every V-set is an $n \cdot *_V$-indexed coproduct of transitive V-sets for some $n \in \mathbb{N}$, and $n \cdot *_V$ is generated by $2 \cdot *_V$ under $2 \cdot *_V$-indexed coproducts. ☐

Corollary 2.95. *If $\mathcal{C}$ is a G-symmetric monoidal 1-category and I is an indexing system, then I-commutative algebras in $\mathcal{C}$ are equivalent to [Cha24, Def 5.6]'s "I-commutative monoids" over $\mathcal{C}$.*

Proof. This follows by matching Corollary 2.94 with [Cha24, Def 5.6]. ☐

3. Equivariant Boardman-Vogt tensor products

Using the language of fibrous patterns, in Section 3.1 we define the *Boardman Vogt tensor product*, and we show that it's closed and compatible with the Segal envelope in Propositions 3.6 and 3.9. Following this, in Section 3.2 we specialize this to $\mathrm{Op}_{\mathcal{T}}$. Then, in Section 3.3, we characterize the $\overset{\mathrm{BV}}{\otimes}$-unit of Op_I and leverage this to compute the $\mathcal{T}$-∞-categories underlying operads of algebras in the unital case. Finally, in Section 3.4, we define the inflation adjunction $\mathrm{Infl}_e^{\mathcal{T}} : \mathrm{Op}_{\mathcal{T}} \rightleftarrows \mathrm{Op} : \Gamma^{\mathcal{T}}$ and characterize its relationship with the Boardman-Vogt tensor product.

3.1. **Boardman-Vogt tensor products of fibrous patterns.** If $\mathcal{C}$ is an ∞-category, we refer to the data of an object $M \in \mathcal{C}$ and a map $M \times M \to M$ as a *magma in* $\mathcal{C}$. We refer to magmas in the nonfull subcategory $\mathrm{AlgPatt}^{\mathrm{Seg,se}} \subset \mathrm{AlgPatt}$ of soundly extendable patterns and Segal morphisms as *magmatic patterns.*

Construction 3.1. Let $(\mathfrak{B}, \wedge)$ be a magmatic pattern. Then, the $\mathfrak{B}$-*Boardman-Vogt tensor product* is the bifunctor $- \overset{\mathrm{BV}}{\otimes} -\colon \mathrm{Fbrs}(\mathfrak{B}) \times \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{Fbrs}(\mathfrak{B})$ defined by

$$\mathfrak{O} \overset{\mathrm{BV}}{\otimes} \mathfrak{P} := L_{\mathrm{Fbrs}}\left(\mathfrak{O} \times \mathfrak{P} \to \mathfrak{B} \times \mathfrak{B} \xrightarrow{\wedge} \mathfrak{B}\right). \qquad \triangleleft$$

We defined this in order to have a mapping out property with respect to the following construction.

Definition 3.2. Let $(\mathfrak{B}, \wedge)$ be a magmatic pattern and $\mathfrak{O}, \mathfrak{P}, \mathfrak{Q}$ fibrous $\mathfrak{B}$-patterns. Then, a *bifunctor of fibrous* $\mathfrak{B}$ *patterns* $\mathfrak{O} \times \mathfrak{P} \to \mathfrak{Q}$ is a commutative diagram in AlgPatt

$$\begin{tikzcd} \mathfrak{O} \times \mathfrak{P} \arrow[r] \arrow[d] & \mathfrak{Q} \arrow[d] \\ \mathfrak{B} \times \mathfrak{B} \arrow[r, "\wedge"] & \mathfrak{B} \end{tikzcd}$$

where $\mathfrak{O} \times \mathfrak{P} \to \mathfrak{B} \times \mathfrak{B}$ is induced by the structure maps of $\mathfrak{O}$ and $\mathfrak{P}$. $\triangleleft$

The collection of bifunctors fits into a full subcategory

$$\mathrm{BiFun}_{\mathfrak{B}}(\mathfrak{O}, \mathfrak{P}; \mathfrak{Q}) \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{AlgPatt})$$

Example 3.3. Let $\mathfrak{O}, \mathfrak{P}$ be fibrous $\mathfrak{B}$-patterns, and consider $\mathfrak{B}$ to be a fibrous $\mathfrak{B}$-pattern via the identity. Then, the ∞-category of bifunctors $\mathfrak{O} \times \mathfrak{P} \to \mathfrak{B}$ is contractible, as it is equivalent to composite arrows $\mathfrak{O} \times \mathfrak{P} \to \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$. $\triangleleft$

Observation 3.4. There are natural equivalences

$$\begin{aligned} \mathrm{BiFun}_{\mathfrak{B}}(\mathfrak{O}, \mathfrak{P}; \mathfrak{Q}) &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B} \times \mathfrak{B}}(\mathfrak{O} \times \mathfrak{P}, \wedge^* \mathfrak{Q}) \\ &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}}(\wedge_!(\mathfrak{O} \times \mathfrak{P}), \mathfrak{Q}) \\ &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}}(\mathfrak{O} \overset{\mathrm{BV}}{\otimes} \mathfrak{P}, \mathfrak{Q}). \qquad \triangleleft \end{aligned}$$

Following in the tradition started by the namesake [BV73, § 2.3], in forthcoming work [Ste24a] we will interpret $\mathrm{BiFun}_{\mathfrak{B}}(\mathfrak{O}, \mathfrak{P}; \mathfrak{Q})$ in the context of $\mathcal{T}$-1-operads as *interchanging* $\mathfrak{O}$ *and* $\mathfrak{P}$*-algebra structures*; as in [BV73, Prop 2.19] and the variety of recontextualizations of their ideas (e.g. [HA; Wei11], we additionally recognize this as $\mathfrak{O}$*-algebras in* $\mathfrak{P}$*-algebras*, making $\overset{\mathrm{BV}}{\otimes}$ into a closed tensor product.

Construction 3.5. Fix $(\mathfrak{B}, \wedge)$ a magmatic pattern, let $F\colon \mathfrak{O} \times \mathfrak{P} \to \mathfrak{Q}$ be a bifunctor of fibrous $\mathfrak{B}$-patterns, and let $\mathfrak{C}$ be a fibrous $\mathfrak{Q}$-pattern. We have a diagram

$$\mathfrak{O} \xleftarrow{p} \mathfrak{O} \times \mathfrak{P} \xrightarrow{F} \mathfrak{Q};$$

admitting push-pull adjunctions $p^* \dashv p_*$ and $L_{\mathrm{Fbrs}} F_! \dashv F^*$ on fibrous patterns, with compatible adjunctions on Segal objects by Propositions 2.20 and 2.22 and Observation 2.29. We define the pattern

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C}) := p_* F^* \mathfrak{C} \in \mathrm{Fbrs}(\mathfrak{O});$$

this is the *fibrous* $\mathfrak{O}$*-pattern of* $\mathfrak{P}$*-algebras in* $\mathfrak{C}$ *over* $\mathfrak{Q}$. In most cases, we will have $\mathfrak{Q} = \mathfrak{O} = \mathfrak{B}$, in which case the information of a bifunctor $\mathfrak{B} \times \mathfrak{P} \to \mathfrak{B}$ is simply that of a fibrous $\mathfrak{B}$-pattern $\mathfrak{P}$ by Example 3.3. In this case, we simply write

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}}(\mathfrak{C}) := \underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{B}}(\mathfrak{C}) \in \mathrm{Fbrs}(\mathfrak{B});$$

this is the *fibrous* $\mathfrak{B}$*-pattern of* $\mathfrak{P}$*-algebras in* $\mathfrak{C}$. $\triangleleft$

In the case $\mathfrak{Q} = \mathfrak{O} = \mathfrak{B}$, the above diagram refines to

$$\mathfrak{B} \xleftarrow{p} \mathfrak{B} \times \mathfrak{P} \xrightarrow{\mathrm{id} \times \pi} \mathfrak{B} \times \mathfrak{B} \xrightarrow{\wedge} \mathfrak{B},$$

so the functor $\mathfrak{P} \mapsto \underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}}(\mathfrak{C})$ has a left adjoint computed by $L_{\mathrm{Fbrs}} \wedge_! (\mathrm{id} \times \pi)_! p^*$; explicitly, this is computed on $\mathfrak{P}'$ by the fibrous localization of the diagonal composite

$$\begin{tikzcd}
\mathfrak{P}' \times \mathfrak{P} \arrow[r, phantom, "\simeq"] \arrow[dr, dashed, "\pi_{\mathfrak{Q}} \times \mathrm{id}"] \arrow[ddr, bend right, "\pi_{\mathfrak{Q}} \times \pi_{\mathfrak{P}}"'] & \pi^* \mathfrak{P}' \arrow[d] & \\
& \mathfrak{B} \times \mathfrak{P} \arrow[d, dashed, "\mathrm{id} \times \pi_{\mathfrak{P}}"] \arrow[dr] & \\
& \mathfrak{B} \times \mathfrak{B} \arrow[r, "\wedge"'] & \mathfrak{B}
\end{tikzcd}$$

By definition, this is precisely $\mathfrak{P}' \otimes^{\mathrm{BV}} \mathfrak{P}$, so we've proved the following.

Proposition 3.6. *The functor* $(-) \overset{BV}{\otimes} \mathfrak{O} : \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{Fbrs}(\mathfrak{B})$ *is left adjoint to* $\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{O}}(-)$.

We additionally spell out a few useful characteristics of $\overset{\mathrm{BV}}{\otimes}$ here. First, we describe functoriality.

Observation 3.7. Fix the fibrous $\mathfrak{B}$-pattern $\mathfrak{Q}$. Suppose we have bifunctors of fibrous $\mathfrak{B}$-patterns

$$F : \mathfrak{O} \times \mathfrak{P} \to \mathfrak{Q} \leftarrow \mathfrak{O} \times \mathfrak{P}' : G$$

together with a morphism of fibrous $\mathfrak{B}$-patterns $\varphi : \mathfrak{P} \to \mathfrak{P}'$ making the following diagram commute:

$$\begin{tikzcd}
& \mathfrak{O} \times \mathfrak{P} \arrow[dl, "\pi"] \arrow[dr, "F"] \arrow[dd] & \\
\mathfrak{O} & & \mathfrak{Q} \\
& \mathfrak{O} \times \mathfrak{P}' \arrow[ul, "\pi'"] \arrow[ur, "G"'] &
\end{tikzcd}$$

The left triangle possesses a Beck-Chevalley transformation

$$\pi^* \varphi_! \implies \mathrm{id}_! \pi'^* = \pi'^*,$$

which possesses a mate natural transformation $\pi'_* \implies \pi_* \varphi^*$; precomposing with G^*, this yields a "pullback" natural transformation

$$\mathrm{Alg}^{\otimes}_{\mathfrak{P}'/\mathfrak{Q}}(-) \implies \mathrm{Alg}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(-). \qquad \triangleleft$$

.We observe that, in all of the work above, we may have instead assumed that $\mathfrak{C} \in \mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})$, in which case all of our constructions land in $\mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})$. Spelled out, this yields the following.

Proposition 3.8. *Fix* $\mathfrak{O}, \mathfrak{P}, \mathfrak{Q}, \mathfrak{C}$ *as in Construction 3.5. Then*

(1) if $\mathfrak{C}$ *is a Segal* $\mathfrak{Q}$*-∞-category, then* $\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C})$ *is a Segal* $\mathfrak{O}$*-∞-category;*

(2) if $\mathfrak{C} \to \mathfrak{D}$ *is a morphism of Segal* $\mathfrak{Q}$*-∞-categories, then the induced map* $\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{D})$ *is a morphism of Segal* $\mathfrak{O}$*-∞-categories; and*

(3) if $\mathfrak{P} \to \mathfrak{P}'$ *is a morphism of fibrous* $\mathfrak{B}$*-patterns and* $\mathfrak{C}$ *is a Segal* $\mathfrak{Q}$*-∞-category, then the induced map of fibrous patterns*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}'/\mathfrak{Q}}(\mathfrak{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C})$$

is a functor of Segal $\mathfrak{O}$*-∞-categories, i.e. it preserves cocartesian lifts for inert morphisms.*

Finally, in analogy to [BS24a] we show that this tensor product is compatible with Segal envelopes.

Proposition 3.9. *The following diagram commutes*

$$\begin{tikzcd}
\mathrm{Fbrs}(\mathfrak{B})^2 \arrow[rr, "\overset{BV}{\otimes}"] \arrow[d, "\mathrm{Env}"] & & \mathrm{Fbrs}(\mathfrak{B}) \arrow[d, "\mathrm{Env}"] \\
\mathrm{Fun}(\mathfrak{B}, \mathrm{Cat})^2 \arrow[r, "\circledast"] & \mathrm{Fun}(\mathfrak{B}, \mathrm{Cat}) \arrow[r, "L_{\mathrm{Seg}}"] & \mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})
\end{tikzcd}$$

Proof. Fix $\mathfrak{C}$ a Segal $\mathfrak{B}$-∞-category. Then, there are natural equivalences

$$\begin{aligned}
\mathrm{Fun}_{\mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})}\left(\mathrm{Env}\left(\mathfrak{O} \overset{\mathrm{BV}}{\otimes} \mathfrak{P}\right), \mathfrak{C}\right) &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}\times\mathfrak{B}}(\mathfrak{O}\times\mathfrak{P}, \wedge^*\mathfrak{C}) \\
&\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}\times\mathfrak{B}}(\mathrm{Env}_{\mathfrak{B}\times\mathfrak{B}}(\mathfrak{O}\times\mathfrak{P}), \wedge^*\mathfrak{C}) \\
&\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}\times\mathfrak{B}}(\mathrm{Env}_{\mathfrak{B}}(\mathfrak{O})\times\mathrm{Env}_{\mathfrak{B}}(\mathfrak{P}), \wedge^*\mathfrak{C}) \qquad (12)\\
&\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}}\left(L_{\mathrm{Seg}}\wedge_!(\mathrm{Env}_{\mathfrak{B}}(\mathfrak{O})\times\mathrm{Env}_{\mathfrak{B}}(\mathfrak{P})), \mathfrak{C}\right) \\
&\simeq \mathrm{Fun}_{\mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})}\left(L_{\mathrm{Seg}}(\mathrm{Env}_{\mathfrak{B}}(\mathfrak{O})\circledast\mathrm{Env}_{\mathfrak{B}}(\mathfrak{P})), \mathfrak{C}\right) \qquad (13)
\end{aligned}$$

Equivalence Eq. (12) is Observation 2.28; Eq. (13) follows by symmetric monoidality of the Grothendieck construction [Ram22, Thm B]. The result then follows by Yoneda's lemma. ☐

3.2. **Boardman-Vogt tensor products of $\mathcal{T}$-operads.** Recall that $\mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$. We specialize the results of Section 3.1 to this case.

Construction 3.10. We show in Proposition A.15 that the Cartesian product in $\mathbb{F}_{\mathcal{T}}$ endows $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ with the structure of a magmatic pattern in the sense of Section 3.1 via the *smash product*

$$\wedge := \mathrm{Span}(\times) : \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\times\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}});$$

we refer to the resulting bifunctor as the *Boardman-Vogt tensor product*

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Fbrs}}\left(\mathcal{O}^{\otimes}\times\mathcal{P}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\times\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})\right). \qquad \triangleleft$$

The $\mathcal{T}$*-operad of $\mathcal{O}$-algebras in $\mathcal{P}$* is given by the right adjoint $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \in \mathrm{Op}_{\mathcal{T}}$ to the Boardman-Vogt tensor product constructed in Proposition 3.6.

Proposition 3.8 immediately implies the following.

Corollary 3.11. *Fix $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ a map of $\mathcal{T}$-operads and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a map of $\mathcal{T}$-symmetric monoidal ∞-categories. Then, $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is a $\mathcal{T}$-symmetric monoidal category, and the canonical lax $\mathcal{T}$-symmetric monoidal functors*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}), \qquad \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D})$$

are $\mathcal{T}$-symmetric monoidal.

Proposition 3.9 specializes to the following.

Corollary 3.12. *The $\mathcal{T}$-symmetric monoidal envelope intertwines with the mode structure:*

$$\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}\left(\mathcal{O}^{\otimes}\right) \otimes^{\mathrm{Mode}} \mathrm{Env}\left(\mathcal{P}^{\otimes}\right).$$

In particular, [BS24a, Thm E] shows that this property identifies the Boardman-Vogt tensor product, so we acquire the following.

Corollary 3.13. *When $\mathcal{T} \simeq *$, $\overset{BV}{\otimes}$ is naturally equivalent to the Boardman-Vogt tensor product of [BS24a; HM23; HA].*

In forthcoming work [Ste24a], we will use a variant of Barkan-Steinebrunner's strategy to lift $\overset{\mathrm{BV}}{\otimes}$ to a canonical symmetric monoidal structure.

3.3. **$\mathcal{T}$-∞-categories underlying $\mathcal{T}$-operads of algebras.** Recall the *underlying $\mathcal{T}$-∞-category* functor

$$U : \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$$

of Construction 2.36. In this subsection, we characterize the relationship of U with $\underline{\mathrm{Alg}}^{\otimes}_{-}(-)$. One significant reason to study the underlying $\mathcal{T}$-∞-category is the following.

Observation 3.14. In the case $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal category, $\mathcal{C}^{\otimes}$ is a Segal $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$-pattern and $U(\mathcal{C}^{\otimes})$ its underlying $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}$-pattern. Hence the composite functor

$$\mathrm{Cat}^{\otimes}_I \to \mathrm{Op}_I \to \mathrm{Cat}_{\mathcal{T}}$$

is conservative by Proposition 2.8. $\triangleleft$

Warning 3.15. The functor U is *not* conservative on $\mathrm{Op}_{\mathcal{T}}$; indeed, users of $\mathcal{T}$-operads will find that they are often describing distinct algebraic theories as corepresented by *one-object* $\mathcal{T}$-operads, yet every map between one-object $\mathcal{T}$-operads is a U-equivalence. ◁

Let $\mathrm{triv}_{\mathcal{T}}^{\otimes} := \mathcal{N}^{\otimes}_{\mathbb{F}_{\mathcal{T}}^{\simeq}\infty}$. Nardin-Shah showed the following.

Proposition 3.16 ([NS22, Cor 2.4.5]). *U induces an equivalence*

$$\mathrm{Op}_{\mathcal{T},/\mathrm{triv}_{\mathcal{T}}^{\otimes}} \simeq \mathrm{Cat}_{\mathcal{T}};$$

writing $\mathrm{triv}^{\otimes}(\mathcal{C}) := U^{-1}_{/\mathrm{triv}^{\otimes}}(\mathcal{C})$, *these are identified by the property*

$$\underline{\mathrm{Alg}}_{\mathrm{triv}^{\otimes}(\mathcal{C})}(\mathcal{P}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, U(\mathcal{P}^{\otimes}));$$

in particular, $\mathrm{triv}^{\otimes}(-) : \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ *is a fully faithful left adjoint to the underlying $\mathcal{T}$-category.*

These are weak $\mathcal{N}_\infty$-operads $\mathcal{T}$-operads if and only if $\mathcal{C}$ has at most one V-object for each V, i.e. $\mathcal{C} = *_{\mathcal{F}} \subset *_{\mathcal{T}}$ for a $\mathcal{T}$-family $\mathcal{F}$. In this case, we write

$$\mathrm{triv}_{\mathcal{F}}^{\otimes} := \mathrm{triv}^{\otimes}(*_{\mathcal{F}}) \simeq \mathcal{N}^{\otimes}_{\mathbb{F}_{\mathcal{F}}^{\simeq}\infty}$$

under the evident embedding $\mathbb{F}_{\mathcal{F}}^{\simeq} \subset \mathbb{F}_{\mathcal{T}} \simeq \subset \mathbb{F}_{\mathcal{T}}$.

Observation 3.17. Proposition 3.16 directly implies that

$$\mathrm{triv}^{\otimes}(\mathcal{C}) \simeq L_{\mathrm{Fbrs}}(\mathcal{C} \to \mathcal{T}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}}));$$

furthermore, if $\mathcal{T}$ posseses a terminal object V, then we have

$$\mathrm{triv}_{\mathcal{T}}^{\otimes} \simeq L_{\mathrm{Fbrs}}(\{V\} \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})).$$ ◁

An important property of $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is that it is the $\overset{\mathrm{BV}}{\otimes}$-unit.

Proposition 3.18. *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, we have $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes}$; hence there exists a natural equivalence*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{O}) \to \mathcal{O}^{\otimes}.$$

Proof. The first statement implies the second by the usual folklore argument:

$$\begin{aligned}\mathrm{Map}(\mathcal{O}^{\otimes}, \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{P})) &\simeq \mathrm{Map}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes}, \mathcal{P}^{\otimes}\right),\\ &\simeq \mathrm{Map}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}),\end{aligned}$$

so Yoneda's lemma yields a natural equivalence $\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{P}) \simeq \mathcal{P}^{\otimes}$. The same argument in reverse shows that the second statement implies the first. Furthermore, in view of Observation 2.52, it suffices to verify the second statement (hence the first) over the base $\mathcal{T}_{/V}$, so we may assume that $\mathcal{T}$ has a terminal object.

In the case that $\mathcal{T}$ has a terminal object, by Observation 3.17, bifunctors $\mathrm{triv}_{\mathcal{T}}^{\otimes} \times \mathcal{O} \to \mathcal{P}$ correspond canonically with functors of $\mathcal{T}$-operads $\mathcal{O} \to \mathcal{P}$; put another way, using the bifunctor presentation for algebras of Observation 3.4, this demonstrates that the forgetful natural transformation

$$\mathrm{Alg}_{\mathcal{O} \otimes^{\mathrm{BV}} \mathrm{triv}}(\mathcal{P}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$$

is a natural equivalence; Yoneda's lemma then demonstrates that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes} \simeq \mathcal{O}^{\otimes}$. □

Using this, we have a sequence of natural equivalences

$$\begin{aligned} U\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) &\simeq \underline{\mathrm{Alg}}_{\mathrm{triv}_{\mathcal{T}}} \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P})\\ &\simeq \underline{\mathrm{Alg}}_{\mathcal{O} \otimes \mathrm{triv}_{\mathcal{T}}}(\mathcal{P})\\ &\simeq \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P});\end{aligned}$$

in particular, we've proved the following corollary.

Corollary 3.19. *There exists a natural equivalence*

$$U\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P}).$$

We've shown in Proposition 3.9 that Env intertwines $\overset{\mathrm{BV}}{\otimes}$ with $\circledast$, and we've now seen that $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is the $\overset{\mathrm{BV}}{\otimes}$-unit. In fact, Env intertwines units.

Proposition 3.20. $\mathrm{Env}_I(\mathrm{triv}_{\mathcal{T}})$ *is the* $\circledast$*-unit in* $\mathrm{CMon}_I(\mathrm{Cat})^{\circledast}$*.*

Proof. Recall from Observation 1.77 that, when $\mathcal{C}^{\times}$ is cartesian, the free object $\mathrm{Fr}_I(*) \in \mathrm{CMon}_I(\mathcal{C})$ is the unit; thus

$$\begin{aligned}
\mathrm{Fun}_I^{\otimes}(\mathrm{Env}_I(\mathrm{triv}_{\mathcal{T}})^{\otimes}, \mathcal{D}^{\otimes}) &\simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{D}^{\otimes}) && 2.53\\
&\simeq \mathcal{D} && 3.16\\
&\simeq \mathrm{Fun}_I^{\otimes}(\mathrm{Fr}_I *, \mathcal{D}^{\otimes})\\
&\simeq \mathrm{Fun}_I^{\otimes}\left(1^{\circledast}, \mathcal{D}^{\otimes}\right),
\end{aligned}$$

so the result follows from Yoneda's lemma. □

3.4. **Inflation and the Boardman-Vogt tensor product.** Recall that the $\mathcal{T}$*-fixed points* of a $\mathcal{T}$-category $\Gamma^{\mathcal{T}}$ are right adjoint to *inflation*. We briefly discuss an operadic version of this and relate it to $\overset{\mathrm{BV}}{\otimes}$.

Construction 3.21. Given $\mathcal{O}^{\otimes}$ a $\mathcal{T}$-operad, and $V \in \mathcal{T}$, we form the V*-value operad*

$$\Gamma^V \mathcal{O}^{\otimes} := i_V^* \mathcal{O}^{\otimes},$$

where $i_V\colon \mathrm{Span}(\mathbb{F}) \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is the map of patterns extending the coproduct preserving functor $\mathbb{F} \hookrightarrow \mathbb{F}_{\mathcal{T}}$ sending $* \mapsto *_V$. Using this, we may set

$$\Gamma^{\mathcal{T}} \mathcal{O}^{\otimes} := \lim_{V \in \mathcal{T}} \mathcal{O}^{\otimes},$$

noting that this recovers Γ^V if V is terminal in $\mathcal{T}$. ◁

Remark 3.22. In the case that $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-symmetric monoidal ∞-category, the structure map of the operad $\Gamma^V \mathcal{C}$ is the pullback of a cocartesian fibration, so it is a cocartesian fibration, i.e. it presents a symmetric monoidal ∞-category; unwinding definitions, this agrees with the construction $\Gamma^V \mathcal{C}$ of Construction 1.64. Since the forgetful functor $\mathrm{Cat} \to \mathrm{Op}$ is a right adjoint, it preserves limits, so the two constructions of $\Gamma^{\mathcal{T}} \mathcal{C}$ also agree. ◁

Unwinding definitions, we find that Corollary 1.56 implies that the map of patterns $\mathcal{T}^{\mathrm{op}} \times \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{I^{\infty}}(\mathbb{F}_{\mathcal{T}})$ induces equivalences on Segal objects, hence on fibrous patterns. Further unwinding definitions, this yields an equivalence

$$\mathrm{Op}_{I^{\infty}} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Op}).$$

In particular, this yields the following.

Proposition 3.23. *The functor* $\Gamma^{\mathcal{T}} : \mathrm{Op}_{I^{\infty}} \to \mathrm{Op}$ *has a fully faithful left adjoint* $\mathrm{Infl}^{\mathcal{T}} : \mathrm{Op} \to \mathrm{Op}_{I^{\infty}}$ *whose image is spanned by the* I^{∞}*-operads whose corresponding functors* $\mathcal{T}^{\mathrm{op}} \to \mathrm{Op}$ *are constant.*

In particular, we find that $\mathbb{E}_{\infty}^{\otimes} \simeq \mathrm{Infl}^{\mathcal{T}} \mathbb{E}_{\infty}^{\otimes}$. The map of patterns i_V induces a push-pull adjunction $E_{I^{\infty}}^{\mathcal{T}} \colon \mathrm{Op}_{I^{\infty}} \rightleftarrows \mathrm{Op}_{\mathcal{T}} \colon \mathrm{Bor}_{I^{\infty}}^{\mathcal{T}}$, and we will write $\mathrm{Infl}^{\mathcal{T}} : \mathrm{Op} \rightleftarrows \mathrm{Op}_{\mathcal{T}} : \Gamma^{\mathcal{T}}$ for the composite adjunction as well.

Example 3.24. Let G be a finite group and n_G the trivial n-dimensional real orthogonal G-representation. Note that the bottom map

$$\begin{tikzcd}
\mathbb{E}_{n_G}(m \cdot *_H) \arrow[r] \arrow[d, phantom, "\wr"] & \mathbb{E}_{n_G}(m \cdot *_K) \arrow[d, phantom, "\wr"] \\
\mathrm{Conf}^H_{m \cdot *_H}(n_G) \arrow[r, "\sim"'] & \mathrm{Conf}^H_{m \cdot *_H}(n_G)
\end{tikzcd}$$

is an equivalence for all $K \subset H \subset G$, as it intertwines the tautological identification of each side with $\mathrm{Conf}_m(\mathbb{R}^n)$. In particular, the map $\mathbb{E}_{n_G}^{\otimes} \to \mathbb{E}_{\infty_G}^{\otimes} \simeq \mathbb{E}_{\infty}^{\otimes}$ witnesses $\mathbb{E}_{n_G}$ as an I^{∞}-operad in the image of Infl_e^G; unwinding definitions, we have an equivalence $\mathrm{Infl}_e^G \mathbb{E}_n^{\otimes} \simeq \mathbb{E}_{n_G}$. ◁

In general, we define the $\mathcal{T}$-operad $\mathbb{E}_n^{\otimes} := \mathrm{Infl}_e^{\mathcal{T}} \mathbb{E}_n^{\otimes}$. We will explore such adjunctions at greater length in forthcoming work [Ste24a], but for now, we concern ourselves with Boardman-Vogt tensor products.

Proposition 3.25. *There exists a natural equivalence* $\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathrm{Infl}_e^{\mathcal{T}} \mathcal{P}^{\otimes} \simeq \mathrm{Infl}_e^{\mathcal{T}} \left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right)$.

Proof. We can verify that $\mathrm{Infl}_e^{\mathcal{T}}$ is product-preserving, so we acquire a zigzag of maps

$$\begin{aligned}
\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{Infl}_e^{\mathcal{T}} \mathcal{P}^{\otimes} \xleftarrow{\eta_{\mathrm{Op}_{\mathcal{T}}}} \;& \wedge_! \left(\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}^{\otimes} \times \mathrm{Infl}_e^{\mathcal{T}} \mathcal{P}^{\otimes}\right) \\
\simeq \;& \wedge_! \mathrm{Infl}_e^{\mathcal{T}} \left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}\right) \\
\simeq \;& \mathrm{Infl}_e^{\mathcal{T}} \wedge_! \left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}\right) \\
\xrightarrow{\mathrm{Infl}_e^{\mathcal{T}} \eta_{\mathrm{Op}}} \;& \mathrm{Infl}_e^{\mathcal{T}} \left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right),
\end{aligned}$$

with $\eta_{\mathrm{Op}_{\mathcal{T}}}$ an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence. We're tasked with proving that η_{Op} is an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence; then, the desired equivalence can be gotten by applying $L_{\mathrm{Op}_{\mathcal{T}}}$ and inverting arrows as needed. In fact, if $\mathcal{Q}^{\otimes}$ is a $\mathcal{T}$-operad, then pullback along η_{Op} furnishes an equivalence

$$\begin{aligned}
\mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}\left(\mathrm{Infl}_e^{\mathcal{T}} \left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right), \mathcal{Q}^{\otimes}\right) &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F})}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}, \Gamma^{\mathcal{T}} \mathcal{Q}^{\otimes}\right) \\
&\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F})}\left(\wedge_! \left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}\right), \Gamma^{\mathcal{T}} \mathcal{Q}^{\otimes}\right) \\
&\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F})}\left(\mathrm{Infl}_e^{\mathcal{T}} \wedge_! \left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}\right), \mathcal{Q}^{\otimes}\right)
\end{aligned}$$

so $\mathrm{Infl}_e^{\mathcal{T}} \eta_{\mathrm{Op}}$ is an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence, yielding the desired natural equivalence. □

Corollary 3.26 (Trivially eqivariant Dunn additivity)**.** *There is an equivalence* $\mathbb{E}_n^{\otimes} \overset{BV}{\otimes} \mathbb{E}_m^{\otimes} \simeq \mathbb{E}_{n+m}^{\otimes}$.

Proof. By Corollary 3.13 and Proposition 3.25, it suffices to construct an equivalence of operads $\mathbb{E}_n^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_m^{\otimes} \simeq \mathbb{E}_{n+m}^{\otimes}$; this is nonequivariant Dunn additivity [HA, Thm 5.1.2.2]. □

Corollary 3.27. *There exists a natural equivalence of operads*

$$\Gamma^{\mathcal{T}} \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}^{\otimes}_{\mathcal{O}}(\Gamma^{\mathcal{T}} \mathcal{C})$$

Proof. Once more, there is a string of natural equivalences

$$\begin{aligned}
\mathrm{Alg}_{\mathcal{P}} \Gamma^{\mathcal{T}} \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}}(\mathcal{C}) &\simeq \mathrm{Alg}_{\mathrm{Infl}_e^{\mathcal{T}} \mathcal{P}} \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}}(\mathcal{C}) \\
&\simeq \mathrm{Alg}_{\mathrm{Infl}_e^{\mathcal{T}} \mathcal{P} \otimes \mathrm{Infl}_e^{\mathcal{T}} \mathcal{O}}(\mathcal{C}) \\
&\simeq \mathrm{Alg}_{\mathrm{Infl}_e^{\mathcal{T}} (\mathcal{P} \otimes \mathcal{O})}(\mathcal{C}) \\
&\simeq \mathrm{Alg}_{(\mathcal{P} \otimes \mathcal{O})}(\Gamma^{\mathcal{T}} \mathcal{C}) \\
&\simeq \mathrm{Alg}_{\mathcal{P}} \mathrm{Alg}^{\otimes}_{\mathcal{O}}(\Gamma^{\mathcal{T}} \mathcal{C}),
\end{aligned}$$

so the result follows by Yoneda's lemma. □

A similar statement to Proposition 3.25 for $\mathrm{triv}^{\otimes}$ follows by either symbol pushing or examining the various localizations; we take the former approach, constructing a string of natural equivalences

$$\begin{aligned}
\mathrm{Alg}_{\mathrm{Infl}_e^{\mathcal{T}} \mathrm{triv}_{\mathcal{C}}}(\mathcal{O}) &\simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{C}}}(\Gamma^{\mathcal{T}} \mathcal{O}) \\
&\simeq \mathrm{Fun}(\mathcal{C}, \Gamma^{\mathcal{T}} \mathcal{O}) \\
&\simeq \mathrm{Fun}_{\mathcal{T}}(\mathrm{Infl}_e^{\mathcal{T}} \mathcal{C}, \mathcal{O}) \\
&\simeq \mathrm{Alg}_{\mathrm{triv}_{\mathrm{Infl}_e^{\mathcal{T}} \mathcal{C}}}(\mathcal{O}).
\end{aligned}$$

That is, we've proved the following.

Proposition 3.28. *Let* $\mathcal{C}$ *be an* ∞*-category. Then, there is a canonical natural equivalence*

$$\mathrm{Infl}_e^{\mathcal{T}} \mathrm{triv}_{\mathcal{C}}^{\otimes} \simeq \mathrm{triv}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}}}.$$

Appendix A. Burnside algebraic patterns: the atomic orbital case

The following appendices are not written to be particularly original; most of their contents appear as straightforward technical extensions of beloved works in higher algebra, and they are included for the sake of mathematical completeness.

A.1. I-operads as fibrous patterns.

This subsection deviates only slightly from [BHS22, § 5.2], so we suggest that the reader first read their work. We're interested in proving Proposition 2.47, so we freely use its notation.

A.1.1. *The pattern $\underline{\mathbb{F}}_{\mathcal{T},*}$.*

Our first step is to prove the following proposition.

Proposition A.1. *There are equivalences of categories*

$$\mathrm{Seg}_{\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C}) \simeq \mathrm{CMon}_{\mathcal{T}}(\underline{\mathcal{C}}),$$
$$\mathrm{Fbrs}(\underline{\mathbb{F}}_{\mathcal{T},*}) \simeq \mathrm{Op}_{\mathcal{T},\infty},$$

the latter denoting Nardin-Shah [NS22]'s ∞-category of $\mathcal{T}$-∞-categories.

To prove this, we must understand the associated Segal conditions. The following lemma characterizes their indexing category.

Lemma A.2 ([BHS22, Obs 5.2.9])**.** *Fix $[S \to U]$ an object in $\underline{\mathbb{F}}_{\mathcal{T},*}$. Then, there are equivalences*

$$\left(\left(\underline{\mathbb{F}}_{\mathcal{T},*}\right)^{\mathrm{el}}_{[S\to U]/}\right)^{\mathrm{op}} \simeq \mathcal{T} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \underline{\mathbb{F}}^{si}_{\mathcal{T},*,/[S\to U]} \tag{14}$$
$$\simeq \mathcal{T} \times_{\underline{\mathbb{F}}_{\mathcal{T}}} \underline{\mathbb{F}}_{\mathcal{T},*,/[S\to U]}. \tag{15}$$

Furthermore, the full subcategory of $\mathcal{T} \times_{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{T},*,/[S\to U]}$ consisting of morphisms $f : T \to S$ such that f is a summand inclusion is an initial subcategory equivalent to the set $\mathrm{Orb}(S)$.

Proof. (14) follows by definition. For (15), this follows by noting that whenever $[U = U] \to [S \to V]$ is a morphism in $\underline{\mathbb{F}}_{\mathcal{T}}$ out of an orbit, the associated morphism $U \to S \times_V U$ is a summand inclusion, as it's split by the projection $S \times_V U \to U$.

For the remaining statement, the inclusion $\mathrm{Orb}(S) \hookrightarrow \mathcal{T} \times_{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{T},*,/[S\to U]}$ has a right adjoint sending $f : T \to S$ to $f(T) \to S$, so it is initial. □

Lemma A.3 ([BHS22, Footnote 6])**.** *The pattern $\underline{\mathbb{F}}_{\mathcal{T},*}$ is sound.*

Proof. We verify the conditions of [BHS22, Prop 3.3.23]. First, we must verify that $\left(\underline{\mathbb{F}}^{si}_{\mathcal{T}}\right)_{/S} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T},/S}$ is fully faithful, i.e. if there is a diagram

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d] & S_1 \arrow[r] \arrow[d] & S_0 \arrow[d] \\ U_2 \arrow[r] & U_1 \arrow[r] & U_0 \end{tikzcd}$$

such that the associated maps $S_2 \to S_0 \times_{U_0} U_2$ and $S_1 \to S_0 \times_{U_0} U_1$ are summand inclusions, the map $S_2 \to S_1 \times_{U_1} U_2$ is a summand inclusion. In fact, the associated map $S_2 \to S_0 \times_{U_0} U_2$ may be decomposed as

$$S_2 \to S_1 \times_{U_1} U_2 \to S_0 \times_{U_0} U_1 \times_{U_1} U_2 \simeq S_0 \times_{U_0} U_2.$$

The composition and second map are each summand inclusions, or equivalently, split monomorphisms; this implies that the first map is a split monomorphism, so $S \to S_1 \times_{U_1} U_2$ must be a summand inclusion as well, i.e. $\left(\underline{\mathbb{F}}^{si}_{\mathcal{T}}\right)_{/S} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T},/S}$ is fully faithful.

Last, we must verify that

$$\underline{\mathbb{F}}^{si,\mathrm{el}}_{\mathcal{T},/[S\to U]} \hookrightarrow \underline{\mathbb{F}}^{\mathrm{el}}_{\mathcal{T},/[S\to U]}$$

is final for all $[S \to U] \in \underline{\mathbb{F}}_{\mathcal{T}}$; in fact, it is an equivalence by Lemma A.2. □

Proof of Proposition A.1. For the first statement, note by Lemma A.2 that a Segal $\underline{\mathbb{F}}_{\mathcal{T},*}$-object in $\mathcal{C}$ is equivalent to a functor

$$M : \underline{\mathbb{F}}_{\mathcal{T},*} \to \mathcal{C}$$

satisfying $M(\prod_i U_i) \simeq \prod_i M(U_i)$; this is precisely the condition that M is product preserving, i.e. it is a $\mathcal{T}$-commutative monoid object.

For the second statement, Lemma A.3 together with [BHS22, Prop 4.1.7] reduce the Segal conditions of a fibrous pattern to precisely the conditions of [NS22, Def 2.1.7]. $\square$

We now turn to the remaining statements of Proposition 2.47 making use of the following theorem:

Theorem A.4 ([BHS22, Prop 3.1.16, Thm 5.1.1])**.** *Suppose $\mathcal{O} \to \mathcal{P}$ is a strong Segal morphism of algebraic patterns such that the following conditions hold:*

(1) $f^{\mathrm{el}} : \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ is an equivalence, and

(2) for every $O \in \mathcal{O}$, the functor $\left(\mathcal{O}^{\mathrm{act}}_{/O}\right)^{\simeq} \to \left(\mathcal{P}^{\mathrm{act}}_{/f(O)}\right)^{\simeq}$ is an equivalence.

Then, the functor $f^* : \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence. Furthermore, if $\mathcal{P}$ is soundly extendable, then $f^* : \mathrm{Fbrs}(\mathcal{P}) \to \mathrm{Fbrs}(\mathcal{O})$ is an equivalence.

For posterity, we temporarily increase in generality.

A.1.2. *Global effective burnside patterns.* Let $\mathcal{T}$ be an ∞-category and $I \subset \mathbb{F}^{P}_{\mathcal{T}} \subset \mathbb{F}_{\mathcal{T}}$ a one-object weak indexing category of an atomic orbital subcategory of $\mathcal{T}$ in the sense of [CLL24]; write

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := \mathrm{Span}_{all,I}(\mathbb{F}_{\mathcal{T}}; \mathcal{T}^{\mathrm{op}})$$

for the resulting pattern. There is a span pattern analog to Lemma A.2 which is proved identically.

Lemma A.5. *For $\mathcal{T}$ an* arbitrary *∞-category, the full subcategory of $\left(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{/S}\right)^{\mathrm{op}} \simeq \mathcal{T} \times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}_{\mathcal{T},/S}$ consisting of morphisms $f : T \to S$ such that f is a summand inclusion is an initial subcategory equivalent to the set* $\mathrm{Orb}(S)$.

Unwinding definitions, this demonstrates the following.

Corollary A.6. *The forgetful functor*

$$\mathrm{Seg}_{\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{C}) \to \mathrm{Fun}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C})$$

is fully faithful with image spanned by the product preserving functors.

Global effective Burnside patterns are generally well behaved:

Lemma A.7. *The pattern $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is soundly extendable.*

Proof. It is sound by [BHS22, Cor 3.3.24]. To see that $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is extendable, it is equivalent to prove that $\mathscr{A}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ is a Segal $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$-$\infty$-category, i.e. for every $S \in \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$, the associated functor φ of

$$\begin{tikzcd}
\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}}_{/S} \arrow[r, "\sim"] \arrow[d] & I_{/S} \arrow[r, "\sim"] \arrow[d] & \prod\limits_{V \in \mathrm{Orb}(S)} I_{/V} \arrow[dl, "\varphi"] \\
\lim\limits_{V \in \mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{S/}} \mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}}_{/V} \arrow[r, "\sim"] & \lim\limits_{V \in \mathcal{T} \times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}_{\mathcal{T},/S}} I_{/V} &
\end{tikzcd}$$

is an equivalence. In fact, it is an equivalence by Lemma A.5. $\square$

A.1.3. *The equivalence.* We resume our original assumption that $\mathcal{T}$ is atomic orbital.

Corollary A.8. *The source functor $s : \underline{\mathbb{F}}_{\mathcal{T},*} \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ induces equivalences of categories*

$$\mathrm{Seg}_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{C}) \simeq \mathrm{Seg}_{\underline{\mathbb{F}}_{\mathcal{T},*}}(\mathcal{C});$$

$$\mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}})) \simeq \mathrm{Fbrs}(\underline{\mathbb{F}}_{\mathcal{T},*}).$$

Proof. It is clear that s is a morphism of algebraic patterns, as it is induced by a morphism of quadruples. The pattern $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is soundly extendable by Lemma A.7. In order to verify that s is a strong Segal morphism, we must verify that $s^{\mathrm{el}}_{[S\to V]/}$ is initial. In fact, by the following diagram,

$$\begin{tikzcd}
\underline{\mathbb{F}}^{\mathrm{el}}_{\mathcal{F},*,[S\to V]/} \arrow[r, "\sim"] \arrow[d, "\iota^{\mathrm{el}}_{[S\to V]/}"] & \left(\mathcal{F}\times_{\mathbb{F}_{\mathcal{F}}} \underline{\mathbb{F}}^{si}_{\mathcal{F},/[S\to V]}\right)^{\mathrm{op}} \arrow[r, "\sim"] \arrow[d] & \prod_{U\in \mathrm{Orb}(S)} (B\,\mathrm{Aut}_{\mathcal{F}}(U))^{\mathrm{op}} \arrow[d, "\varphi"] \\
\mathrm{Span}(\mathbb{F}_{\mathcal{T}};\mathcal{F})^{\mathrm{el}}_{S/} \arrow[r, "\sim"] & \left(\mathcal{F}\times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}_{\mathcal{T},/S}\right)^{\mathrm{op}} \arrow[r] & \prod_{U\in \mathrm{Orb}(S)} (\mathcal{F}_{/U})^{\mathrm{op}}
\end{tikzcd}$$

it suffices to verify that the functor φ is final. Indeed, since $\mathcal{T}$ is atomic, the subcategory $B\,\mathrm{Aut}_{\mathcal{T}}(U)\hookrightarrow \mathcal{T}_{/U}$ is downwards closed, i.e. initial. This implies φ is a product of opposites of initial functors, hence it is final.

It remains to check that s satisfies the conditions of Theorem A.4. We check this in parts. Condition 1 follows immediately by construction. Condition 2 follows by noting that the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathbb{F}}^{\mathrm{act}}_{\mathcal{T},*,/[S\to V]} \arrow[r, "\sim"] \arrow[d] & \underline{\mathbb{F}}_{\mathcal{T},/[S\to V]} \arrow[r, "\sim"] \arrow[d] & \mathbb{F}_{\underline{V},/S} \arrow[r, "\sim"] \arrow[d] & \prod_{U\in \mathrm{Orb}(S)} \underline{V}_{/U} \arrow[d, "\varphi"] \\
\mathrm{Span}(\mathbb{F}_{\mathcal{T}};\mathcal{F})^{\mathrm{act}}_{/S} \arrow[r, "\sim"] & \mathbb{F}_{\mathcal{T},/S} \arrow[r, equal] & \mathbb{F}_{\mathcal{T},/S} \arrow[r, "\sim"] & \prod_{U\in \mathrm{Orb}(S)} \mathcal{T}_{/U}
\end{tikzcd}$$

and by noting that φ is an equivalence, since $\underline{V}\subset \mathcal{T}$ is a full subcategory containing any element attaining a map to V, and there exists a map $U\to S\to V$. $\square$

In fact, we may say something more general; define the pullback pattern

$$\begin{tikzcd}
\underline{\mathbb{F}}_{I,*} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbb{F}}_{\mathcal{T},*} \arrow[d] \\
\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \arrow[r] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}})
\end{tikzcd}$$

so that $\mathbb{F}_{I,V,*}$ corresponds with pointed I-admissible V-sets.

Observation A.9. By Lemma A.2, $\underline{\mathbb{F}}_{I,*}$-Segal objects in $\mathcal{C}$ are precisely I-semiadditive functors $\underline{\mathbb{F}}_{I,*}\to \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$. ◁

The conditions of Theorem A.4 follow from the case $I=\mathcal{T}$, so we have the following.

Corollary A.10. *If I is a weak indexing category, then pullback along the map $\underline{\mathbb{F}}_{I,*}\simeq \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ induces an equivalence*

$$\mathrm{Op}_I \simeq \mathrm{Fbrs}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}))\simeq \mathrm{Fbrs}(\underline{\mathbb{F}}_{I,*})$$

A.2. **Pullback of fibrous patterns along Segal morphisms and sound extendability.**

Proposition A.11. *Suppose $\varphi:\mathfrak{O}\to\mathfrak{P}$ is morphism of algebraic patterns and $\mathfrak{P}$ is soundly extendable. Then,*

(1) If the precomposition functor

$$\varphi^*:\mathrm{Fun}(\mathfrak{P},\mathrm{Cat})\to \mathrm{Fun}(\mathfrak{O},\mathrm{Cat})$$

preserves Segal objects, then the pullback functor

$$\varphi^*:\mathrm{Cat}_{/\mathfrak{P}}\to \mathrm{Cat}_{/\mathfrak{O}}$$

preserves fibrous patterns.

(2) If φ is an inert-cocartesian fibration and the left Kan extension functor

$$\varphi_!:\mathrm{Fun}(\mathfrak{O},\mathrm{Cat})\to \mathrm{Fun}(\mathcal{P},\mathrm{Cat})$$

preserves Segal objects, then postcomposition

$$\varphi_!:\mathrm{Cat}_{/\mathfrak{O}}\to \mathrm{Cat}_{/\mathfrak{P}}$$

preserves fibrous patterns.

In particular, if φ is an inert-cocartesian Segal morphism between soundly extendable patterns whose left Kan extension preserves Segal categories, then pullback and postcomposition restrict to an adjunction on fibrous patterns

$$\varphi_! : \mathrm{Fbrs}(\mathfrak{O}) \rightleftarrows \mathrm{Fbrs}(\mathfrak{P}) : \varphi^*$$

Proof. Our argument mirrors that of [BHS22, Lem 4.1.19]. In either case, the property of being an inert-cocartesian fibration is always preserved, either by assumption or by [BHS22, Obs 2.2.6].

We prove (1) first. Fixing $\mathscr{F} \in \mathrm{Fbrs}(\mathfrak{P})$, by [BHS22, Obs 4.1.3], it suffices to prove that the left vertical arrow in the following pullback diagram is a relative Segal $\mathfrak{O}$-∞-category.

$$\begin{tikzcd}
\mathrm{St}^{\mathrm{int}}_{\mathfrak{O}}(\varphi^*\mathscr{F}) \arrow[r] \arrow[d] & \varphi^*\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F} \arrow[d] \\
\mathscr{A}_{\mathfrak{O}} \arrow[r] & \varphi^*\mathscr{A}_{\mathfrak{P}}
\end{tikzcd}$$

By [BHS22, Lem 3.1.10], relative Segal $\mathfrak{O}$-∞-categories are pullback-stable, so it suffices to prove that the right vertical arrow is a relative Segal $\mathfrak{O}$-∞-category. By sound extendability $\mathscr{A}_{\mathfrak{P}}$ is a Segal $\mathfrak{P}$-∞-category, and since φ^* preserves Segal ∞-categories, $\varphi^*\mathscr{A}_{\mathfrak{P}}$ is a Segal $\mathfrak{O}$-∞-category; by [BHS22, Obs 3.1.8] it then suffices to prove that $\varphi^*\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F}$ is a Segal $\mathfrak{O}$-∞-category. Since φ^* preserves Segal ∞-categories, it suffices to prove that $\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F}$ is a Segal $\mathfrak{P}$-category, which follows by the assumption that $\mathscr{F}$ is a fibrous pattern.

(2) is similar; this time, by taking left adjoints to the commutative square of [BHS22, Prop 4.2.5], it suffices to prove that the composition

$$\varphi_!\mathrm{St}^{\mathrm{int}}_{\mathfrak{O}}\mathscr{F} \to \varphi_!\mathscr{A}_{\mathfrak{O}} \to \mathscr{A}_{\mathfrak{P}}$$

is relative Segal; since $\mathfrak{P}$ is soundly extendable, [BHS22, Obs 3.1.8] again reduces this to verifying that $\varphi_!\mathrm{St}^{\mathrm{int}}_{\mathfrak{O}}\mathscr{F}$ is Segal; this follows from the facts that $\mathscr{F}$ is a fibrous pattern and $\varphi_!$ preserves Segal ∞-categories. □

A.3. **Segal morphisms between effective Burnside patterns.** In this section, we fill our grab bag full of a wide variety of Segal morphisms between effective Burnside patterns.

Proposition A.12. *Suppose $F \subset F' \subset \mathbb{F}_{\mathcal{T}}$ are wide subcategories. Then, the inclusion*

$$\iota : \mathrm{Span}_F(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}_{F'}(\mathbb{F}_{\mathcal{T}})$$

is a Segal morphism.

Proof. We are tasked with verifying that precomposition with ι preserves product-preserving functors, i.e. that ι is a product-preserving functor. In fact, this is immediate, since a functor $\mathrm{Span}_F(\mathbb{F}_{\mathcal{T}}) \to \mathcal{C}$ is product-preserving if and only if the backwards maps $(S \leftarrow U)_{U \in \mathrm{Orb}(S)}$ together map to a product diagram, which is obviously true of ι. □

Proposition A.13. *Suppose $\varphi : V \to W$ is a morphism in $\mathcal{T}$. Then, the associated functor* $\mathrm{Span}(\mathrm{Ind}_V^W)$: $\mathrm{Span}(\mathbb{F}_V) \to \mathrm{Span}(\mathbb{F}_W)$ *is a Segal morphism.*

Proof. We're tasked with proving that precomposition along $\mathrm{Span}(\mathrm{Ind}_V^W)$ preserves product-preserving functors, i.e. it is a product-preserving functor. Since $\mathrm{Span}(\mathbb{F}_V)$ and $\mathrm{Span}(\mathbb{F}_W)$ are semiadditive, it is equivalent to prove that $\mathrm{Span}(\mathrm{Ind}_V^W)$ is coproduct-preserving; since coproducts in $\mathrm{Span}(\mathbb{F}_V)$ are computed in $\mathbb{F}_V$, it's equivalent to prove that $\mathrm{Ind}_V^W : \mathbb{F}_V \to \mathbb{F}_W$ is coproduct-preserving, which follows from the fact that it's a left adjoint. □

Proposition A.14. *If $f : \mathcal{T}' \to \mathcal{T}$ is a functor of atomic orbital ∞-categories, then the associated functor* $\mathrm{Span}(\mathbb{F}_{\mathcal{T}'}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ *is a Segal morphism.*

Proof. By [CH21, Rem 4.3], it suffices to verify that $f^{\mathrm{el}}_{X/}$ induces an equivalence on the left vertical arrow

$$\begin{tikzcd} \lim\limits_{\mathsf{Span}(\mathcal{T})^{\mathrm{el}}_{f(X)/}} F \arrow[r, "\sim"] \arrow[d] & \prod\limits_{U\in \mathrm{Orb}(f(X))} F(U) \arrow[d] \\ \lim\limits_{\mathsf{Span}(\mathcal{T}')^{\mathrm{el}}_{X/}} F\circ f^{\mathrm{el}} \arrow[r, "\sim"] & \prod\limits_{V\in \mathrm{Orb}(X)} Ff(V) \end{tikzcd}$$

whenever F is restricted from a Segal $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ space. This follows by noting that the horizontal arrows are equivalences by construction, and $\mathsf{Span}(f)$ sends the set of orbits of X bijectively onto the set of orbits of $f(X)$. □

Proposition A.15. *The map* $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})\times\mathsf{Span}(\mathbb{F}_{\mathcal{T}})\xrightarrow{\wedge}\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ *is a Segal morphism.*

Proof. By [CH21, Ex 5.7], a functor $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})\times\mathsf{Span}(\mathbb{F}_{\mathcal{T}})\to\mathcal{C}$ is a Segal object if and only if it preserves products separately in each variable. Hence we're tasked with verifying that $\wedge^*F$ preserves products separately in each variable whenever F preserves products. In fact, this follows by distributivity of products and coproducs in $\mathbb{F}_{\mathcal{T}}$; indeed, we have

$$\begin{aligned} \wedge^* F\left((X_+\oplus Z_+, Y_+)\right) &\simeq F\left((X\sqcup X')\times Y\right)_+ \\ &\simeq F\left((X\times Y)\sqcup(X'\times Y)\right)_+ \\ &\simeq F\left((X_+\wedge Y_+)\oplus(X'_+\wedge Y_+)\right) \\ &\simeq F\left(X_+\wedge Y_+\right)\oplus F\left(X'_+\wedge Y_+\right) \\ &\simeq \wedge^*F\left(X_+,Y_+\right)\oplus\wedge^*F\left(X'_+,Y_+\right). \end{aligned}$$

□

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