ON CONNECTIVITY OF SPACES OF EQUIVARIANT CONFIGURATIONS

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Abstract. We provide conditions on a locally smooth G-manifold under which its nonempty spaces of equivariant configurations $\mathrm{Conf}_S^G(X)$ are d-connected for all finite G-sets S. We use this to show that $\mathbb{E}_{dV}$-algebras in a G-symmetric monoidal $(d-1)$-category canonically lift to $\mathbb{E}_{\infty V}$-algebras.

Throughout this paper, we fix G a Lie group.

Definition 1. If $H \subset G$ is a closed subgroup and $S \in \mathbb{F}_H$ a finite H-set, we let

$$\mathsf{Conf}_S^H(X) \subset \mathsf{Map}^H(S, X)$$

be the (topological) subspace of H-equivariant embeddings $S \hookrightarrow M$. ◃

Nonequivariantly, the homotopy type of configurations spaces in X is a rich source of homeomorphism-invariants of X. In this paper, we study some rudiments of an equivariant lift of this in the smooth setting. Namely, in Section 1, we supply sufficient conditions for a smooth G-manifold M such that its nonempty configurations spaces $\mathsf{Conf}_S^G(M)$ are all d-connected.

We have a particular application in mind; the structure spaces of the *little V-disks operad* are configuration spaces in smooth G-manifolds, and connectivity statements of G-operads translate to structural statements about their algebras (see [Ste24a]). For instance, in Section 2, we prove a sharp strengthening of the following theorem.

Theorem 2. *Suppose G is finite. If $\mathcal{C}$ is a G-symmetric monoidal $(d-1)$-category and V a real orthogonal G-representation, then the forgetful functor*

$$\mathrm{Alg}_{\mathbb{E}_{\infty V}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_{dV}}(\mathcal{C})$$

is an equivalence of $(d-1)$-categories.

In particular, $\mathbb{E}_{\infty V}$ is a weak $\mathcal{N}_\infty$-operad, so [Ste24a] and Theorem 2 provide a *homotopical incomplete Mackey functor* model for $\mathbb{E}_{dV}$-algebras in Cartesian G-symmetric monoidal $(d-1)$-categories and [Cno+24, Thm B] provides a *bi-incomplete Tambara functor* model for $\mathbb{E}_{dV}$-rings in the setting of homotopical incomplete Mackey functors valued in a $(d-1)$-category.

1. Configuration spaces in locally smooth G-manifolds

Definition 3 ([Bre72, § IV])**.** If M is a smooth manifold with a continuous G-action, we say that the action is *locally smooth* if, for each point $x \in M$, there exists a real orthogonal $\mathsf{stab}_G(x)$-representation V_x and a trivializing open neighborhood

$$x \in \coprod_{G/\mathsf{stab}_G(x)} V_x \hookrightarrow M,$$

where for a topological H-space X, we write $\coprod_{G/\mathsf{stab}_G(x)} X := G \times_H X$ as a topological G-space. In this case, we say that M with its action is a *locally smooth G-manifold*. ◃

Smooth actions on manifolds admit well-behaved tubular neighborhoods; for example, [Bre72, Cor V.2.4] proves that smooth actions are locally smooth. On the other hand, if M is a locally smooth G-manifold, then the inclusion $M_{(H)} \hookrightarrow M$ of points with orbit isomorphic to G/H is a locally closed topological submanifold [Bre72, Thm IV.3.3], which is smooth if M is smooth [Bre72, Cor VI.2.5].

We begin this section in Section 1.1 by proving the following.

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Theorem 4 (equivariant Fadell-Neuwirth fibration)**.** *Fix M a locally smooth G-manifold, $S, T \in \mathbb{F}_G$ a pair of finite G-sets, and $\iota : S \hookrightarrow M$ a G-equivariant configuration. The following is a homotopy-Cartesian square:*

$$\begin{tikzcd} \mathrm{Conf}^G_T(M - \iota(S)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Conf}^G_{S \sqcup T}(M) \arrow[d, "U"] \\ \{\iota\} \arrow[r, hook] & \mathrm{Conf}^G_S(M) \end{tikzcd}$$

Thus the long exact sequence in homotopy for $T = G/H$ yields means for computing homotopy groups of $\mathrm{Conf}^G_S(M)$ inductively on the cardinality of the orbit set $|S_G|$, with inductive step hinging on homotopy of

$$\mathrm{Conf}^G_{G/H}(M - \iota(S)) \simeq (M - \iota(S))_{(H)}.$$

We denote by $[\mathcal{O}_G]$ the subconjugacy lattice of closed subgroups of G, and we let

$$\mathrm{Istrp}(M) = \{\mathrm{stab}_x(G) \mid x \in M\} \subset [\mathcal{O}_G]$$

be the full subposet spanned by conjugacy classes (H) for which $M_{(H)}$ is nonempty. We are inspired to make the following definition.

Definition 5. A locally smooth G-manifold M is

- $\geq d$*-dimensional at each orbit type* if $M_{(H)}$ is $\geq d$-dimensional for each $(H) \in \mathrm{Istrp}(M)$;
- $(d-2)$*-connected at each orbit type* if $M_{(H)}$ is $(d-2)$-connected for each $(H) \in \mathrm{Istrp}(M)$. ◃

In Section 1.2, we use Theorem 4 to prove the following.

Theorem 6. *If a locally smooth G-manifold M is $\geq d$-dimensional and $(d-2)$-connected at each orbit type, then for all finite G-sets $S \in \mathbb{F}_G$, the configuration space $\mathrm{Conf}^G_S(M)$ is either empty or $(d-2)$-connected.*

In order to identify applications of this theorem, we give sufficient conditions for M to be $(d-2)$-connected at each orbit type. Note by repeatedly applying [Bre72, Thm IV.3.1] that the subspace $M_{\leq(H)} \subset M$ of orbits mapping to G/H is a closed submanifold. In Section 1.3, we use this to prove the following.

Proposition 7. *Suppose that M is a smooth G-manifold satisfying the following conditions:*

(a) M *is* $\geq d$*-dimensional at each orbit type.*
(b) $M_{\leq(H)}$ *is* $(d-2)$*-connected for each* H.
(c) $\mathrm{codim}(M_{\leq(K)} \hookrightarrow M_{\leq(H)}) \geq d$ *for each* $(K) \leq (H)$.
(d) $\mathrm{Istrp}(M)$ *is finite (e.g. G compact and M finite type, c.f. [Bre72, Thm IV.10.5]).*

Then M is $(d-2)$-connected at each orbit type.

1.1. **A Fadell-Neuwirth fibration for equivariant configurations.** Our strategy for Theorem 4 mirrors that of Knudsen in the notes [Knu18]. In particular, we would like to use Quillen's theorem B [Qui73], which requires us to construct $\mathrm{Conf}^H_S(M)$ as a classifying space. In fact, there is a general scheme to do this:

Lemma 8 ([DI04, Thm 2.1], via [Knu18, Thm 4.0.2])**.** *If $\mathcal{B}$ is a topological basis for X such that all elements of $\mathcal{B}$ are weakly contractible, then the canonical map*

$$|\mathcal{B}| = \mathrm{hocolim}_{\mathcal{B}} * \to X$$

is a weak equivalence, where on the left $\mathcal{B}$ is considered as a poset under inclusion.

To use this, define an elementwise-contractible basis for $\mathrm{Conf}^G_S(M)$ by

$$\widetilde{\mathcal{B}}^G_S(M) := \left\{ (X, \sigma) \;\middle|\; \exists (V_x) \in \prod_{[x] \in \mathrm{Orb}_S} \mathbf{Rep}^{\mathrm{orth}}_{\mathbb{R}}(\mathrm{stab}_G([x])), \text{ s.t. } \coprod_{[x]}^{S} V_x \simeq X \subset M,\; \sigma : S \xrightarrow{\sim} \pi_0(U) \right\},$$

where for all tuples $(Y_x) \in \prod_{[x] \in \mathrm{Orb}_S} \mathrm{Top}_{\mathrm{stab}_G([x])}$, we write

$$\coprod_{[x]}^{S} Y_x := \coprod_{[x] \in \mathrm{Orb}(S)} \left(G \times_{\mathrm{stab}_G([x])} Y_x \right) \in \mathrm{Top}_G$$

for the *indexed disjoint union of* Y_x. We fix $\mathcal{B}_S^G(M) \subset \widetilde{\mathcal{B}}_S^G(M)$ the smaller basis consisting of open sets (X,σ) possessing neighborhoods $(X,\sigma) \subset (X',\sigma)$ such that the associated embeddings factor as

$$\begin{tikzcd} \coprod_U^S D(V_U)^\circ \arrow[r, phantom, "\simeq"] \arrow[d, hook, "\exists"'] & \coprod_U^S V_U \arrow[d, hook, "X"] \\ V'_U \arrow[r, hook, "X'"] & M \end{tikzcd} \tag{1}$$

where $D(V_U)^\circ$ denotes the open unit V_U-disk; that is, open sets in $\mathcal{B}_S^G(M)$ consist of collections of configurations possessing a fixed common neighborhood resembling disjoint unions of real orthogonal representations, subject to the condition that there is "space on all sides" of the neighborhood. This is functorial in two ways:

- given a summand inclusion $S \hookrightarrow T \sqcup S$, the forgetful map $\mathrm{Conf}_{T\sqcup S}^G(M) \to \mathrm{Conf}_S^G(M)$ preserves basis elements, inducing a map $\mathcal{B}_{T\sqcup S}^G(M) \to \mathcal{B}_S^G(M)$.
- any open embedding $\iota : M \hookrightarrow N$ induces a map $\mathrm{Conf}_T^G(M) \hookrightarrow \mathrm{Conf}_T^G(N)$ preserving basis elements, inducing a map $\mathcal{B}_S^H(M) \to \mathcal{B}_S^H(N)$.

To summarize, we've observed the proof of following lemma.

Lemma 9. *Given $H \subset G$ and $S, T \in \mathbb{F}_H$, there is an equivalence of arrows*

$$\begin{tikzcd} \left|\mathcal{B}_{T\sqcup S}^G(M)\right| \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Conf}_{T\sqcup S}^G(M) \arrow[d] \\ \left|\mathcal{B}_S^G(M)\right| \arrow[r, phantom, "\simeq"] & \mathrm{Conf}_S^G(M) \end{tikzcd}$$

Thus we can characterize the homotopy fiber of U using Quillen's theorem B and the following.

Proposition 10. *For $(X_S,\sigma_S) \le (X'_S,\sigma'_S) \in \mathcal{B}_S^G(M)$, and an S-configuration $\mathbf{x} \in X_S$, we have a diagram*

$$\begin{tikzcd} \mathcal{B}_T^G(M-\mathbf{x}) & \mathcal{B}_T^G(M-\overline{X}_S) \arrow[l, "\varphi"'] & \mathcal{B}_T^G(M-\overline{X'}_S) \arrow[l] \arrow[ll, bend right, "\varphi"'] \\ & ((X_S,\sigma_S)\downarrow U) \arrow[u, phantom, "\wr"] & \left((X'_S,\sigma'_S)\downarrow U\right) \arrow[l] \arrow[u, phantom, "\wr"] \end{tikzcd}$$

such that the maps φ induce weak equivalences on classifying spaces.

We will power this with the following observation:

Observation 11. Recall that an embedding of topological G-spaces $f : Y \hookrightarrow Z$ is a *G-isotopy equivalence* if there exists another G-equivariant embedding $g : Z \hookrightarrow Y$ and a pair of G-equivariant isotopies $gf \sim \mathrm{id}_Z$, $fg \sim \mathrm{id}_Y$. If $f : Y \to Z$ is a G-isotopy equivalence, then postcomposition with f induces a $G \times \Sigma_n$-isotopy equivalence $\mathrm{Conf}_n(Y) \hookrightarrow \mathrm{Conf}_n(Z)$; indeed, postcomposition with f and g induce G-equivariant embeddings, and postcomposition with the isotopies $gf \sim \mathrm{id}_Z$, $fg \sim \mathrm{id}_Y$ yields equivariant isotopies $\mathrm{Conf}_n(g) \circ \mathrm{Conf}_n(f) \sim \mathrm{Conf}_n(gf) \sim \mathrm{Conf}_n(\mathrm{id}_Z) \sim \mathrm{id}_{\mathrm{Conf}_n(Z)}$ and similar for fg.

In particular, the vertical arrows in the following diagram are isotopy equivalences

$$\begin{tikzcd} \mathrm{Conf}_S^H(X) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Conf}_{|S|}(X)^{\Gamma_S} \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Map}^G\left(G \times \Sigma_{|S|}/\Gamma_S, \mathrm{Conf}_{|S|}(X)\right) \arrow[d] \\ \mathrm{Conf}_S^H(Y) \arrow[r, phantom, "\simeq"] & \mathrm{Conf}_{|S|}(Y)^{\Gamma_S} \arrow[r, phantom, "\simeq"] & \mathrm{Map}^G\left(G \times \Sigma_{|S|}/\Gamma_S, \mathrm{Conf}_{|S|}(Y)\right) \end{tikzcd}$$

where $\Gamma_S = \{(h, \rho_S(h)) \mid h \in H\} \subset G \times \Sigma_{|S|}$ is the graph subgroup corresponding with an H-set S with action map $\rho_S : H \to \Sigma_{|S|}$. Hence f induces a homotopy equivalence $\mathrm{Conf}_S^H(X) \xrightarrow{\sim} \mathrm{Conf}_S^H(Y)$. $\triangleleft$

Proof of Proposition 10. The maps φ are each induced by the open inclusions $M - \overline{X}_S \hookrightarrow M - \mathbf{x}$, so the top horizontal arrows commute. The equivalences $\mathcal{B}_T^G(M - \overline{X_S}) \simeq ((X_S, \sigma_S) \downarrow U)$ simply follow by unwinding definitions. Thus we're left with proving that φ induces an equivalence on classifying spaces

$$\begin{tikzcd} \mathrm{Conf}_T^G(M - \mathbf{x}) \arrow[d, phantom, "\wr"] & \mathrm{Conf}_T^G(M - \overline{X_s}) \arrow[l] \arrow[d, phantom, "\wr"] \\ \left|\mathcal{B}_T^G(M - \mathbf{x})\right| & \left|\mathcal{B}_T^G(M - \overline{X_s})\right| \arrow[l] \end{tikzcd}$$

By Observation 11, it suffices to show that $M - \overline{X_S} \hookrightarrow M - \mathbf{x}$ is a G-isotopy equivalence. In fact, by Eq. (1), it suffices to prove that the inclusion $f : V - D(V) \hookrightarrow V - \{0\}$ is a G-isotopy equivalence. But this is easy; scaling is equivariant, so we may define the G-equivariant embedding $g : V - \{0\} \to V - D(V)$ by $g(x) = \frac{1+|x|}{|x|} \cdot x$. Then, each of the equivariant isotopies $gf \sim \mathrm{id}, fg \sim \mathrm{id}$ can be taken as restrictions of $h(t,x) = \frac{1-t+|x|}{|x|} \cdot x$. $\square$

We are ready to conclude our equivariant homotopical lift of [FN62, Thm 1].

Proof of Theorem 4. By the above analysis, we may replace our diagram with a homotopy equivalent diagram given by the geometric realiztion of the following diagram of posets, and prove that it is homotopy Cartesian

$$\begin{tikzcd} \mathcal{B}_T^G(M - \iota(S)) \arrow[r] \arrow[d] & \mathcal{B}_{T \sqcup S}^G(M) \arrow[d] \\ \{\iota\} \arrow[r, hook] & \mathcal{B}_S^G(M) \end{tikzcd}$$

By Quillen's theorem B [Qui73, Thm B], it suffices to prove two statements:

- for all basis elements (X_S, σ_S), The canonical map $((X_s, \sigma_s) \downarrow U) \to \mathcal{B}_T^G(M - \iota(S))$ induces a weak equivalence on classifying spaces, and
- for all inclusions of basis elements $(X_S, \sigma_S) \subset (X'_S, \sigma'_S)$, the canonical map $\left((X'_S, \sigma'_S) \downarrow U\right) \to ((X_S, \sigma_S) \downarrow U)$ induces a weak equivalence on classifying spaces.

In fact, both statements follow immediately from Proposition 10, with the second using two-out-of-three. $\square$

1.2. **Proof of the main theorem in topology.** To prove Theorem 6, we begin with a lemma.

Lemma 12. *If M is a locally smooth G-manifold which is at least d-dimensional and $(d-2)$-connected at each orbit type and $\iota : G/H \hookrightarrow M$ an embedded orbit, then $M - \iota(G/H)$ is at least d-dimensional and $(d-2)$-connected at each orbit type.*

Proof. We have

$$(M - \iota(G/H))_{(K)} = \begin{cases} M_{(K)} & G/K \neq G/H \\ M_{(H)} - \iota(G/H) & G/K = G/h, \end{cases}$$

so the only nontrivial case is $H = K$, in which case we're tasked with verifying that the complement of a discrete set of points in a d-dimensional $(d-2)$-connected manifold is $(d-2)$-connected. This is a well known classical fact in algebraic topology which follows quickly from the Blakers-Massey theorem. $\square$

Proof of Theorem 6. If $d - 2 < 0$, there is nothing to prove, so assume that $d - 2 \geq 0$. We induct on $|S_G|$ with base case 1, i.e. with $S = G/H$. In this case, $\mathrm{Conf}_{G/H}^G(M) = M_{(H)}$ is $(d-2)$-connected by assumption.

For induction, fix some $S \sqcup G/H \in \mathbb{F}_G$ and inductively assume the theorem when $|T_G| \leq |S_G|$. Then, note that $\mathrm{Conf}_S^G(M)$ is $(d-2)$-connected by assumption and $M - \iota(S)$ is $\geq d$-dimensional and $(d+2)$-connected at each orbit by Lemma 12, so $\mathrm{Conf}_{G/H}^G(M - \iota(S))$ $(d-2)$-connected by the inductive hypothesis. Thus Theorem 4 expresses $\mathrm{Conf}_{S \sqcup G/H}^G(M)$ as the total space of a homotopy fiber sequence with connected base and fiber, so it is connected. Furthermore, examining the long exact sequence associated with Theorem 4, we find that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, phantom, "\wr"] & \pi_k \mathrm{Conf}_{S \sqcup G/H}^G(M) \arrow[r] \arrow[dr] & 0 \arrow[d, phantom, "\wr"] \\ \pi_k \mathrm{Conf}_{G/H}^G(M - \iota(S)) \arrow[ur] & & \pi_k \mathrm{Conf}_S^G(M) \end{tikzcd}$$

is exact for $0 < k \leq d-2$; hence $\mathrm{Conf}_{S \sqcup G/H}^G(M)$ is $(d-2)$-connected, completing the induction. $\square$

1.3. **Some sufficient conditions for connectivity at each orbit.** We begin with the following observation:

Observation 13. If M satisfies the conditions of Proposition 7, then $M_{\le (H)}$ does as well. ◃

We will strengthen Proposition 7. Pick an order on $\mathsf{Istrp}(M) = (G/H_1, \dots, G/H_n, G/G)$, and write

$$\begin{aligned} M_k &= M - \bigcup_{i<k} M_{\le (H_i)} \\ \widetilde{M}_k &= M_{\le (H_k)} - \bigcup_{i<k} M_{\le (H_k)} \cap M_{\le (H_i)} \\ &= M_{\le (H_k)} - \bigcup_{\substack{(K) \le (H_k) \cap (H_i) \\ i<k}} M_{\le (K)} \end{aligned}$$

Lemma 14. *For all k, the space M_k is $(d-2)$-connected.*

Proof. We induct in two ways:

- First, we inductively assume we have proved the lemma at full strength when G is replaced with any proper subgroup $H \subsetneq G$ such that $G/H \in \mathsf{Istrp}(M)$; since $\mathsf{Istrp}(M)$ is finite, this begins with the base case in which case there are no such proper subgroups.
- Second, we inductively assume that we have proved the lemma for all $k' < k$; this begins with the base case that $k = 1$, in which case we have $M_1 = M = M_{\le (G)}$, which is $(d-2)$-connected by assumption.

Under these assumptions, note that $\widetilde{M}_{k-1} \subset M_{k-1}$ is a $(d-2)$-connected closed submanifold of codimension $\ge d$ in a $(d-2)$-connected smooth manifold with complement is M_k. Thus it possesses a tubular neighborhood $\widetilde{M}_{k-1} \subset \tau(\widetilde{M}_{k-1}) \subset M_{k-1}$, and "hemmed gluing" presents a homotopy pushout square

$$\begin{tikzcd} \partial\tau\widetilde{M}_{k-1} \arrow[r] \arrow[d, "\tilde{\iota}"] & M_k \arrow[d, "\iota"] \\ \widetilde{M}_{k-1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & M_{k-1} \end{tikzcd}$$

The boundary $\partial\tau\left(\tilde{M}_{k-1}\right)$ is the total space of a c-sphere bundle over a $(d-2)$-connected space, where

$$c = \operatorname{codim}(M_{\le (H_k)} \hookrightarrow M) - 1 > d - 2.$$

The long exact sequence in homotopy reads

$$\begin{tikzcd} \pi_1(S^c) \arrow[r] & \pi_1\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \arrow[r] & \pi_1\left(\widetilde{M}_{k-1}\right) \arrow[r] \arrow[dr] & 0 \arrow[r] & \pi_0\left(\partial\tau\left(\widetilde{M}_{k-1}\right)\right) \arrow[r] & \pi_0\left(\widetilde{M}_{k-1}\right) \arrow[r] & 0 \\ & & & \pi_0(S^c) \arrow[u, phantom, "\wr"] \arrow[ur] & & & \end{tikzcd}$$

so $\partial\tau\widetilde{M}_{k-1}$ is connected, and when $d-2 \ge 1$, $\partial\tau\widetilde{M}_{k-1}$ is simply connected. Furthermore, at degree $0 < \ell \le (d-2)$ the Gysin sequence reads

$$\begin{tikzcd} 0 \arrow[r] & H^\ell(\partial\tau\widetilde{M}_{k-1}) \arrow[r] \arrow[dr] & 0 \\ H^\ell(\widetilde{M}_{k-1}) \arrow[u, phantom, "\wr"] \arrow[ur] & & H^{\ell-c}(\widetilde{M}_{k-1}) \arrow[u, phantom, "\wr"] \end{tikzcd}$$

so $\partial\tau\tilde{M}_{k-1}$ has vanishing cohomology in degrees $0 < \ell \le d-2$. Hurewicz' theorem then implies that $\partial\widetilde{M}_{k-1}$ is $(d-2)$-connected.

In particular, this together with $(d-3)$-connectivity of the homotopy fiber S^c implies that $\tilde{\iota}$ is a $(d-2)$-connected map, so its homotopy pushout ι is $(d-2)$-connected. Since M_{k-1} is a $(d-2)$-connected space by assumption, this implies that M_k is $(d-2)$-connected, completing the induction. □

Proof of Proposition 7. By Observation 13 it suffices to prove that $M_{(G)}$ is $(d-2)$-connected. This is precisely Lemma 14 when $k = n+1$. □

Warning 15. Neither the conditions of Proposition 7 or of Theorem 6 are stable under restrictions; indeed, for $G = C^2$ and $[C_2]$ a C_2-torsor, the example $[C_2] \cdot D^n$ satisfies the conditions of Proposition 7 for $d = n$, but its underlying manifold does not satisfy the conditions of Theorem 6 for any d, as it is not connected. We will rectify this in the setting of real orthogonal G-representation by introducing stronger sufficient conditions which themselves are stable under restriction. ◁

2. Representations, homotopy-coherent algebra, and configuration spaces

In homotopy-coherent algebra, a prominent role is played by the operads $\mathbb{E}_1 = \mathcal{A}_\infty$ and $\mathbb{E}_\infty$, whose algebras are *homotopy-coherently associative algebras* and *homotopy-coherently commutative algebras*, respectively. Dunn's celebrated "additivity theorem" proved non-homotopically [Dun88] (later made homotopical by Lurie [HA, Thm 5.1.2.2]) that an object possessing n-interchanging $\mathbb{E}_1$-structures may equivalently be presented as an algebra over the $\mathbb{E}_n$*-operad*, whose space of k-ary operations is weakly equivalent to the ordered configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$. Thus, after Dunn and Lurie, a higher-categorical version of the Eckmann-Hilton argument may be phrased as stating that $\mathbb{E}_n$*-algebras in* $(n-1)$*-categories canonically lift to* $\mathbb{E}_\infty$*-algebras*; Lurie showed that this is equivalent to the statement that $\mathrm{Conf}_k(\mathbb{R}^n)$ is $(n-2)$-connected for all n, k [HA, Cor 5.1.1.7], which was a half-century old fact of manifold topology due to [FN62].

We would like to lift this to equivariant higher algebra using the equivariant little disks G-operads $\mathbb{E}_V$; these appear in [Hor19], where they are shown to have S-ary operation space

$$\mathbb{E}_V(S) \simeq \mathrm{Conf}_S^H(V)$$

for all $S \in \mathbb{F}_H$. Thus we are compelled to seek a representation theoretic context lifting the assumptions of Proposition 7. We propose the following.

Definition 16. We say V *has d-codimensional fixed points* if $\left|V^H\right|, \left|V^K/V^H\right| \in \{0\} \cup [d, \infty]$ for all $K \subset H \subset G$. ◁

When $G = e$, this is equivalent to simply being d-dimensional.

Proposition 17. *If a real orthogonal G-representation V has d-codimensional fixed points, then the smooth G-manifold $V - \{0\}$ is at least d-dimensional and $(d-2)$-connected at each orbit type.*

Proof. We may write V as a filtered (homotopy) colimit $V = \bigcup_i V_i$ with V_i a finite dimensional real orthogonal G-representation with $\min(i, d)$-codimensional fixed points; then, if V_i is $(i-2)$-connected for each i, taking a colimit, this implies that V is d-connected. Hence it suffices to prove this in the case we that V is finite dimensional.

In this case, G acts smoothly on V, and we make the following observations:

(a) $V_{(H)} = V^H - \bigcup_{K \le (H)} V^K$ is either empty or $\left|V^H\right| \ge d$-dimensional.
(b) $V_{\le (H)} = V_G^H$ is contractible, hence it is $(d-2)$-connected.
(c) $\mathrm{codim}(V_{\le(K)} \hookrightarrow V^*_{\le(H)}) = \left|V^H\right| - \left|V^K\right| = \left|V^H/V^K\right| \ge d$ by assumption.
(d) $\mathrm{Istrp}(V)$ is finite since V is finite dimensional.

Thus Proposition 7 applies, proving the proposition. □

Corollary 18. *If V has d-codimensional fixed points, then for all closed subgroups $H \subset G$ and finite H-sets $S \in \mathbb{F}_H$, $\mathrm{Conf}_S^H(V)$ is $(d-2)$-connected or empty.*

Proof. We begin by noting

$$\mathrm{Conf}_S^H(V) = \begin{cases} \mathrm{Conf}^H_{S - *_H}(\mathrm{Res}_H^G(V - \{0\})) & S^H \neq \emptyset, \\ \mathrm{Conf}_S^H(\mathrm{Res}_H^G(V - \{0\})) & \text{otherwise.} \end{cases}$$

so it suffices to show $\mathrm{Conf}_S^H(\mathrm{Res}_H^G(V - \{0\}))$ to be $(d-2)$-connected or empty. Noting that the condition of having d-codmimensional fixed points is restriction-stable, this follows by Theorem 6 and Proposition 17. □

In fact, we have a converse to this.

Proposition 19. *If there exists a finite-index inclusion of subgroups $K \subset H$ such that $V^H \hookrightarrow V^K$ is a proper inclusion of codimension $< d$, then there exists some H' such that $\mathrm{Conf}^G_{[G/H']}(V) = V_{(H')}$ is not $(d-2)$-connected.*

Proof. This never occurs when V is 0-dimensional. If V^G is $0 < c < d$-dimensional, then we may directly see $\mathrm{Conf}^G_{2\cdot *_G}(V) = \mathrm{Conf}_2(V^G) = S^{c-1}$ is not $(d-2)$-connected, as it has nontrivial π_{c-1}. Thus we assume that V^G is $\geq d$-dimensional, so that V^H is $\geq d$-dim for all H.

Fix $c := \min_{K \subset H \in \mathrm{Istrp}(V)} \mathrm{codim}(V^H \hookrightarrow V^K)$, and suppose that this minimum is implemented by the inclusion $K \subset H$. We may replace V with the real orthogonal G-representation $V^K = V_{(\geq K)}$ and assume that $V^H \hookrightarrow V$ is a proper inclusion of codimension $< d$. We're left with proving that $V_{(H)} = V - \bigcup_{K \subsetneq H} V^H$ is not $(d-2)$-connected. Pick an order $(H_i)_{1 \leq i \leq n}$ on $\mathrm{Istrp}(V) - \{K\}$ so that $H_1 = H$, and set the notation

$$V_\ell := V - \bigcup_{i=1}^{\ell-1} V^{H_i}$$
$$\widetilde{V}_\ell := V^{H_\ell} - \bigcup_{i=1}^{\ell-1} V^{H_i \cap H_\ell}$$

so that $V_1 = V \simeq *$ and $V_{n+1} = V_{(K)}$. Furthermore, note that $V_2 = V - V^{H_1} \simeq S(|V|) \times S\left(\left|V^{H_1}\right|\right)$; in particular, its reduced homology is

$$\widetilde{H}_m(V_2) = \begin{cases} \mathbb{Z} & n \in \{c-1, \dim V\}; \\ 0 & \text{otherwise.} \end{cases}$$

In the case $c = 1$, this is not connected; V_ℓ is inductively the complement of a positive codimension submanifold of a disconnected manifold, so it is disconnected. Hence $V_{n+1} = V_{(K)}$ is disconnected in this case. As a consequence, we may assume that $\mathrm{codim}(V^H \subset V^J) \geq 2$ for all $J \subset H$ in $\mathrm{Istrp}(V)$.

Now, we're ready to induct in two ways:

(1) We inductively assume the statement has been proved for all inclusions $K' \subset K''$ where $K'' \subsetneq K$; this has base case $K = e$, where V has trivial G-action, and follows from the nonequivariant case.
(2) We inductively assume that $\widetilde{H}_m(V_{\ell-1}) = 0$ when $m < c-1$ and that $\widetilde{H}_{c-1}(V_{\ell-1})$ is nontrivial; this has base case $\ell - 1 = 2$ satisfied by the above computation.

The end of this induction implies the proposition, as Hurewicz' theorem will imply that

$$\pi_{c-1}(V_{(K)})_{\mathbf{Ab}} = \pi_{c-1}(V_{n+1})_{\mathbf{Ab}} \simeq \widetilde{H}_{c-1}(V_{n+1}) \neq 0,$$

and $0 < c-1 \leq d-2$. Note that the normal bundle of $V - V^{H_\ell} \subset V$ is a trivial D^{c_ℓ}-bundle; this restricts to the (trivial) normal bundle of $\widetilde{V}_{\ell-1} \subset V_\ell$, so the bounding $S^{c_\ell - 1}$ sphere bundle $\partial\tau\widetilde{V}_{\ell-1} \to V_\ell$ is trivial. Thus "hemmed gluing" presents a homotopy pushout square

$$\begin{tikzcd} S^{c_{\ell-1}} \times \widetilde{V}_{\ell-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V_\ell \arrow[d] \\ \widetilde{V}_{\ell-1} \arrow[r] & V_{\ell-1} \end{tikzcd}$$

If $c_\ell > c$, the left vertical arrow (hence the right vertical arrow) is a homology isomorphism in degrees $\leq c-1$, proving the inductive step. Furthermore, if $c_\ell = c$, then the vertical arrows are homology isomorphisms in degrees $\leq 2c-2$ and the associated map $\widetilde{H}_c(S^{c-1} \times \widetilde{V}_{\ell-1}) \to \widetilde{H}_c(\widetilde{V}_{\ell-1})$ is an isomorphism. This implies that $H_m(V_\ell) = 0$ when $m < c-1$ and the Mayer-Vietoris sequence restricts to a short exact sequence

$$0 \longrightarrow \widetilde{H}_{c-1}\left(S^{c-1} \times \widetilde{V}_{\ell-1}\right) \longrightarrow \widetilde{H}_{c-1}\left(\widetilde{V}_{\ell-1}\right) \oplus \widetilde{H}_{c-1}(V_\ell) \longrightarrow \widetilde{H}_{c-1}(V_{\ell-1}) \longrightarrow 0$$

so that $\widetilde{H}_{c-1}(V_\ell) \neq 0$, as desired. ☐

To state a corollary, we define the weak indexing system

$$\mathbb{F}_{AV} = \left\{ S \in \mathbb{F}_H \mid \mathrm{Conf}^H_S(V) \neq \varnothing \right\}.$$

as in [Ste24a; Ste24b]. Our main algebraic corollary uses this to marry real representation theory, algebraic topology, homotopy theory, equivariant higher category theory, and equivariant higher algebra.

Theorem 2'. *Let G be a finite group and V a real orthogonal G-representation. Then, the following conditions are equivalent:*

(a) V has d-codimensional fixed points.
(b) For all subgroups $H \subset G$ and finite H-sets S, the space $\mathrm{Conf}_S^H(V)$ is empty or $(d-2)$-connected.
(c) The G-operad $\mathbb{E}_V^{\otimes}$ is $(d-2)$-connected.[1]
(d) The forgetful functor

$$U : \mathrm{CMon}_{AV}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_V}(\mathcal{S})$$

is an equivalence of $(d-1)$-categories.
(e) For all G-symmetric monoidal $(d-1)$-categories, the forgetful functor

$$U : \mathrm{CAlg}_{AV}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{C})$$

is an equivalence of $(d-1)$-categories.

Proof. The equivalence (a) $\iff$ (b) is Corollary 18 and Proposition 19. By [Hor19], the structure spaces $\mathbb{E}_V(S)$ is $\mathrm{Conf}_S^H(V)$, so (b) $\iff$ (c) by definition. The equivalences (c) $\iff$ (d) $\iff$ (e) are recorded in [Ste24a]. □

In particular, note that $\left|k \cdot V^H\right| = k\left|V^H\right|$ and $\left|k \cdot V^K / k \cdot V^H\right| = k \cdot \left|V^K/V^H\right|$; hence if V has d-codmensional fixed points, kV has kd-codimensional fixed points. All representations have 1-codimensional fixed points, so dV has d-codimensional fixed points; hence Theorem 2' specializes to Theorem 2.

Remark 20. Theorem 2' is significantly stronger than Theorem 2; indeed, we may choose $G = C_p$, fix a generator $x \in C_p$, and let λ_i denote the irreducible 2-dimensional real orthogonal C_p-representation on whom x acts by rotation at an angle of $\frac{2\pi i}{p}$. Then, when $d \le p/2$, the (nontrivial) representation $V = d \oplus \bigoplus_{1=i}^d \lambda_i$ has d-codimensional fixed points, but it contains only one copy of each of its nontrivial summands, so it can't be expressed as a direct sum of two copies of a nontrivial representation. ◁

Nevertheless, we specialize the following corollaries to dV for readability. The first yields a natural $RO(G)$-graded AV-Tambara structure on the homotopy groups of an $\mathbb{E}_{2V}$-ring spectrum, and it follows from Theorem 2 in combination with [Cno+24, Thm 4.3.6].[2]

Corollary 21. *If V is a real orthogonal G-representation, then there are factorizations*

$$\begin{tikzcd}
 & \mathrm{Tamb}_{I,AV}\left(\mathbf{Ab}^{RO(G)}\right) \arrow[r] & \mathrm{Tamb}_{I,AV}\left(\mathbf{Ab}^{\mathbb{Z}}\right) \\
 & \mathrm{CAlg}_{AV}\left(\mathbf{Ab}^{RO(G)}\right) \arrow[u, phantom, "\wr"] \arrow[r] \arrow[d, "U"] & \mathrm{CAlg}_{AV}\left(\mathbf{Ab}^{\mathbb{Z}}\right) \arrow[u, phantom, "\wr"] \arrow[d, "U"'] \\
\mathrm{Alg}_{\mathbb{E}_{2V}}\left(\underline{\mathrm{Sp}}_G\right) \arrow[uur, bend left] \arrow[ur, bend left] \arrow[r, "\underline{\pi}_\star"] \arrow[rr, bend right, "\underline{\pi}_*"'] & \mathrm{Mack}_I(\mathbf{Ab})^{RO(G)} \arrow[r] & \mathrm{Mack}_I(\mathbf{Ab})^{\mathbb{Z}}
\end{tikzcd}$$

Finally, we acquire incomplete Mackey structures on $\mathbb{E}_{(n+2)V}$-monoidal n-categories.

Corollary 22. *$\mathbb{E}_{(n+2)V}$-monoidal n-categories canonically lift to AV-symmetric monoidal n categories, i.e.*

$$U : \mathrm{Cat}_{AV,n}^{\otimes} \to \mathrm{Cat}_{\mathbb{E}_{(n+2)V},n}^{\otimes}$$

is an equivalence of $(n+1)$-categories. In particular, when $V = \rho$, the forgetful functor

$$U : \mathrm{Cat}_{G,n}^{\otimes} \to \mathrm{Cat}_{\mathbb{E}_{(n+2)\rho},n}^{\otimes}$$

is an equivalence of 2-categories.

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[1] Recall from [Ste24a] that a G-operad $\mathcal{O}^{\otimes}$ is $(d-2)$-*connected* if its nonempty structure spaces $\mathcal{O}(S)$ are $(d-2)$-connected.
[2] In the case $V^G \neq 0$, AV is an indexing category, so we could simply reference the earlier work of [Cha24; San23].

References

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups*. Vol. 46. Pure and Applied Mathematics. Academic Press, New York-London, 1972, pp. xiii+459. URL: https://jfdmath.sitehost.iu.edu/seminar/Bredon,Introduction_to_Compact_Transformation_Groups.pdf (cit. on pp. 1, 2).

[Cha24] David Chan. "Bi-incomplete Tambara functors as O-commutative monoids". In: *Tunisian Journal of Mathematics* 6.1 (Jan. 2024), pp. 1–47. ISSN: 2576-7658. DOI: 10.2140/tunis.2024.6.1. URL: http://dx.doi.org/10.2140/tunis.2024.6.1 (cit. on p. 8).

[Cno+24] Bastiaan Cnossen et al. *Normed equivariant ring spectra and higher Tambara functors*. 2024. arXiv: 2407.08399 [math.AT]. URL: https://arxiv.org/abs/2407.08399 (cit. on p. 8).

[DI04] Daniel Dugger and Daniel C. Isaksen. "Topological hypercovers and A^1-realizations". In: *Math. Z.* 246.4 (2004), pp. 667–689. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-003-0607-y. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/dugger-hypercover.pdf (cit. on p. 2).

[Dun88] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258. ISSN: 0022-4049,1873-1376. DOI: 10.1016/0022-4049(88)90103-X. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/Dunn.pdf (cit. on p. 6).

[FN62] Edward Fadell and Lee Neuwirth. "Configuration spaces". In: *Math. Scand.* 10 (1962), pp. 111–118. ISSN: 0025-5521,1903-1807. DOI: 10.7146/math.scand.a-10517. URL: https://www.mscand.dk/article/download/10517/8538 (cit. on pp. 4, 6).

[Hor19] Asaf Horev. *Genuine equivariant factorization homology*. 2019. arXiv: 1910.07226 [math.AT] (cit. on pp. 6, 8).

[Knu18] Ben Knudsen. *Configuration spaces in algebraic topology*. 2018. arXiv: 1803.11165 [math.AT]. URL: https://arxiv.org/abs/1803.11165 (cit. on p. 2).

[HA] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf (cit. on p. 6).

[Qui73] Daniel Quillen. "Higher algebraic K-theory. I". In: *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*. Vol. Vol. 341. Lecture Notes in Math. Springer, Berlin-New York, 1973, pp. 85–147. URL: https://sma.epfl.ch/~hessbell/topo_alg/Quillen.pdf (cit. on pp. 2, 4).

[San23] Ivo de los Santos Vekemans. *Bi-Incomplete Tambara Functors As Coherent Monoids*. 2023. URL: https://openresearch-repository.anu.edu.au/server/api/core/bitstreams/4aa63fb4-8f4a-45e6-93fc-165529f326d1/content (cit. on p. 8).

[Ste24a] Natalie Stewart. *On tensor products of equivariant commutative operads*. 2024. URL: https://nataliesstewart.github.io/files/Ninfty_draft.pdf (cit. on pp. 1, 7, 8).

[Ste24b] Natalie Stewart. *Orbital categories and weak indexing systems*. 2024. URL: https://nataliesstewart.github.io/files/windex_draft.pdf (cit. on p. 7).