ON CONNECTIVITY OF SPACES OF EQUIVARIANT CONFIGURATIONS

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Abstract. We provide conditions on a locally smooth $G$-manifold under which its nonempty spaces of equivariant configurations $Conf^G_S(X)$ are $d$-connected for all finite $G$-sets $S$. We use this to show that $E_{dV}$-algebras in a $G$-symmetric monoidal $(d-1)$-category canonically lift to $E_{\infty V}$-algebras.

Throughout this paper, we fix $G$ a Lie group.

Definition 1. If $H \subset G$ is a closed subgroup and $S \in F_H$ a finite $H$-set, we let

$$Conf^H_S(X) \subset Map^H(S,X)$$

be the (topological) subspace of $H$-equivariant embeddings $S \hookrightarrow M$.

Nonequivariantly, the homotopy type of configurations spaces in $X$ is a rich source of homeomorphism-invariants of $X$. In this paper, we study some rudiments of an equivariant lift of this in the smooth setting. Namely, in Section 1, we supply sufficient conditions for a smooth $G$-manifold $M$ such that its nonempty configurations spaces $Conf^G_S(M)$ are all $d$-connected.

We have a particular application in mind; the structure spaces of the little $V$-disks operad are smooth $G$-manifolds, and connectivity statements of $G$-operads translate to structural statements about their algebras (see [Ste24a]). For instance, in Section 2, we prove a strengthening of the following theorem.

Theorem 2. Suppose $G$ is finite. If $C$ is a $G$-symmetric monoidal $(d-1)$-category and $V$ a real orthogonal $G$-representation, then the forgetful functor

$$\text{Alg}_{E_{\infty V}}(C) \to \text{Alg}_{E_{dV}}(C)$$

is an equivalence of $(d-1)$-categories.

In particular, $E_{\infty V}$ is a weak $\mathcal{N}_{\infty}$-operad, so [Ste24a] and Theorem 2 provide a homotopical incomplete Mackey functor model for $E_{dV}$-algebras in Cartesian $G$-symmetric monoidal $(d-1)$-categories and CHLL provides a bi-incomplete Tambara functor model for $E_{dV}$-rings in the setting of homotopical incomplete Mackey functors valued in a $(d-1)$-category.

1. Equivariant configuration spaces in locally smooth manifolds

Definition 3 ([Bre72, § IV]). If $M$ is a smooth manifold with a continuous $G$-action, we say that the action is locally smooth if, for each point $x \in M$, there exists a real orthogonal $\text{stab}_G(x)$-representation $V_x$ and a trivializing open neighborhood

$$x \in M \times \bigcup_{G/\text{stab}_G(x)} V_x$$

where for a topological $H$-space $X$, we write $\bigcup_{G/\text{stab}_G(x)} X := G \times_H X$ as a topological $G$-space. In this case, we say that $M$ with its action is a locally smooth $G$-manifold.

Smooth actions on manifolds admit well-behaved tubular neighborhoods; for example, [Bre72, Cor V.2.4] proves that smooth actions are locally smooth. On the other hand, if $M$ is a locally smooth $G$-manifold, then the inclusion $M_{(H)} \hookrightarrow M$ of points with orbit isomorphic to $G/H$ is a locally closed topological submanifold [Bre72, Thm IV.3.3], which is smooth if $M$ is smooth [Bre72, Cor VI.2.5].

We begin this section in Section 1.1 by proving the following.
Theorem 4 (equivariant Fadell-Neuwirth fibration). Fix $M$ a locally smooth $G$-manifold, $S, T \in \mathcal{F}_G$ a pair of finite $G$-sets, and $i : S \hookrightarrow M$ a $G$-equivariant configuration. The following is a homotopy-Cartesian square:

$$
\begin{array}{ccc}
\text{Conf}^G(M - i(S)) & \longrightarrow & \text{Conf}^G_{S,T}(M) \\
\downarrow j & & \downarrow U \\
\{i\} & \longleftarrow & \text{Conf}^G_S(M)
\end{array}
$$

Thus the long exact sequence in homotopy for $T = G/H$ yields means for computing homotopy groups of $\text{Conf}^G_S(M)$ inductively on the cardinality of the orbit set $|S_G|$, with inductive step hinging on homotopy of $\text{Conf}^G_{G/H}(M - i(S)) \simeq (M - i(S))_{(H)}$.

We denote by $[O_G]$ the subconjugacy lattice of closed subgroups of $G$, and we let

$$
\text{Istrp}(M) = \{\text{stab}_x(G) \mid x \in M\} \subset [O_G]
$$

be the full subposet spanned by conjugacy classes $(H)$ for which $M_{(H)}$ is nonempty. We are inspired to make the following definition.

Definition 5. A locally smooth $G$-manifold $M$ is

- $\geq d$-dimensional at each orbit type if $M_{(H)}$ is $\geq d$-dimensional for each $(H) \in \text{Istrp}(M)$;
- $(d-2)$-connected at each orbit type if $M_{(H)}$ is $(d-2)$-connected for each $(H) \in \text{Istrp}(M)$.

In Section 1.2, we use Theorem 4 to prove the following.

Theorem 6. If a locally smooth $G$-manifold $M$ is $\geq d$-dimensional and $(d-2)$-connected at each orbit type, then for all finite $G$-sets $S \in \mathcal{F}_G$, the configuration space $\text{Conf}^G_S(M)$ is either empty or $(d-2)$-connected.

In order to identify applications of this theorem, we give sufficient conditions for $M$ to be $(d-2)$-connected at each orbit type. Note by repeatedly applying [Bre72, Thm IV.3.1] that the subspace $M_{(H)} \subset M$ of orbits mapping to $G/H$ is a closed submanifold. In Section 1.3, we use this to prove the following.

Proposition 7. Suppose that $M$ is a smooth $G$-manifold satisfying the following conditions:

(a) $M$ is $\geq d$-dimensional at each orbit type.
(b) $M_{(H)}$ is $(d-2)$-connected for each $H$.
(c) $\text{codim}(M_{(H)} \hookrightarrow M) \geq d$ for each $(K) \leq (H)$.
(d) $\text{Istrp}(M)$ is finite (e.g. $G$ compact and $M$ finite type, c.f. [Bre72, Thm IV.10.5]).

Then $M$ is $(d-2)$-connected at each orbit type.

1.1. A Fadell-Neuwirth fibration for equivariant configurations. Our strategy for Theorem 4 mirrors that of Knudsen in the notes [Knu18]. In particular, we would like to use Quillen’s theorem B [Qui73], which requires us to construct $\text{Conf}^G_S(M)$ as a classifying space. In fact, there is a general scheme to do this:

Lemma 8 ([DI04, Thm 2.1], via [Knu18, Thm 4.0.2]). If $\mathcal{B}$ is a topological basis for $X$ such that all elements of $\mathcal{B}$ are weakly contractible, then the canonical map

$$
|\mathcal{B}| = \text{hocolim}_{\mathcal{B}} \to X
$$

is a weak equivalence, where on the left $\mathcal{B}$ is considered as a poset under inclusion.

To use this, define an elementwise-contractible basis for $\text{Conf}^G_S(M)$ by

$$
\mathcal{B}^G_S(M) := \left\{(X, \sigma) \mid \exists (V_x) \in \prod_{[x] \in \text{Orb}_S} \text{Rep}_{G}^\text{orth}(\text{stab}_G([x])), \text{ s.t. } \sum_{[x]} V_x \simeq X \subset M, \sigma : S \Rightarrow \pi_0(U)\right\},
$$

where for all tuples $(Y_x) \in \prod_{[x] \in \text{Orb}_S} \text{Top}_{\text{stab}_G([x])}$, we write

$$
\sum_{[x]} Y_x := \prod_{[x] \in \text{Orb}(S)} (G \times_{\text{stab}_G([x])} Y_x) \in \text{Top}_G
$$
for the indexed disjoint union of $Y_x$. We fix $\mathcal{B}_G^f(M) \subset \mathcal{B}_S^f(M)$ the smaller basis consisting of open sets $(X, \sigma)$ possessing neighborhoods $(X, \sigma) \subset (X', \sigma)$ such that the associated embeddings factor as

$$
\begin{align*}
\bigcup_U D(V_U) &
\xrightarrow{\phi} \bigcup_U V_U \\
\downarrow &
\downarrow \\
V_U &
\xleftarrow{x} \rightarrow M
\end{align*}
$$

where $D(V_U)$ denotes the open $V_U$-disk; that is, open sets in $\mathcal{B}_G^f(M)$ consist of collections of configurations possessing a fixed common neighborhood resembling disjoint unions of real orthogonal representations, subject to the condition that there is “space on all sides” of the neighborhood. This is functorial in two ways:

- given a summand inclusion $S \hookrightarrow T \sqcup S$, the forgetful map $\text{Conf}^G_T(M) \to \text{Conf}^G_S(M)$ preserves basis elements, inducing a map $\mathcal{B}_G^T(M) \to \mathcal{B}_G^S(M)$.

- any open embedding $i : M \hookrightarrow N$ induces a map $\text{Conf}^G_T(M) \hookrightarrow \text{Conf}^G_T(N)$ preserving basis elements, inducing a map $\mathcal{B}_G^T(M) \to \mathcal{B}_G^N(N)$.

To summarize, we’ve observed the proof of following lemma.

**Lemma 9.** Given $H \subset G$ and $S, T \in \mathcal{F}_H$, there is an equivalence of arrows

$$
\left| \mathcal{B}_H^G(M) \right| \simeq \text{Conf}^G_{T \sqcup S}(M)
$$

$$
\downarrow
$$

$$
\left| \mathcal{B}_H^S(M) \right| \simeq \text{Conf}^G_S(M)
$$

Thus we can characterize the homotopy fiber of $U$ using Quillen’s theorem B and the following.

**Proposition 10.** For $(X, \sigma) \leq (X', \sigma') \in \mathcal{B}_G^f(M)$, and an $S$-configuration $x \in X$, we have a diagram

$$
\begin{align*}
\mathcal{B}_T^f(M - x) &
\xleftarrow{\phi} \mathcal{B}_T^f(M - X_S) \\
\downarrow &
\downarrow \\
\mathcal{B}_T^f(M - X_S) &
\xleftarrow{(X, \sigma) \downarrow U} \rightarrow (X', \sigma') \downarrow U
\end{align*}
$$

such that the maps $\phi$ induce weak equivalences on classifying spaces.

**Proof.** The maps $\phi$ are each induced by the open inclusions $M - X \xhookleftarrow{} M - x$, so the top horizontal arrows commute. The equivalences $\mathcal{B}_T^f(M - X_S) \simeq ((X, \sigma) \downarrow U)$ simply follow by unwinding definitions. Thus we’re left with proving that $\phi$ induces an equivalence on classifying spaces

$$
\begin{align*}
\text{Conf}^G_T(M - x) &
\xleftarrow{\phi} \text{Conf}^G_T(M - X) \\
\downarrow &
\downarrow \\
\left| \mathcal{B}_T^f(M - x) \right| &
\xleftarrow{\phi} \left| \mathcal{B}_T^f(M - X) \right|
\end{align*}
$$

By Eq. (1), it suffices to prove that the map $\text{Conf}^G_T(V - D(V)) \to \text{Conf}^G_T(V - \{0\})$ is a weak equivalence, which follows by the standard linearl (hence equivariant) deformation retract of each onto a thickening of the sphere $S(V) \subset V$ of points of norm 2. □

We are ready to conclude our equivariant homotopical lift of [FN62, Thm 1].

**Proof of Theorem 4.** By the above analysis, we may replace our diagram with a homotopy equivalent diagram given by the geometric realization of the following diagram of posets, and prove that it is homotopy Cartesian

$$
\begin{align*}
\mathcal{B}_T^f(M - \iota(S)) &
\xrightarrow{\phi} \mathcal{B}_T^f(M) \\
\downarrow &
\downarrow \\
\left| \iota \right| &
\xleftarrow{\phi} \left| \mathcal{B}_T^f(M) \right|
\end{align*}
$$
By Quillen’s theorem B [Qui73, Thm B], it suffices to prove two statements:

- for all basis elements \((X_\sigma, \sigma)\), The canonical map \((X_\sigma, \sigma) \downarrow U \to B^G_G(M - i(S))\) induces a weak equivalence on classifying spaces, and
- for all inclusions of basis elements \((X_\sigma, \sigma) \subset (X_\sigma', \sigma')\), the canonical map \((X_\sigma', \sigma') \downarrow U \to ((X_\sigma', \sigma') \downarrow U)\) induces a weak equivalence on classifying spaces.

In fact, both statements follow immediately from Proposition 10, with the second using two-out-of-three. □

1.2. Proof of the main theorem in topology. To prove Theorem 6, we begin with a lemma.

Lemma 11. If \(M\) is a locally smooth \(G\)-manifold which is at least \(d\)-dimensional and \((d - 2)\)-connected at each orbit type and \(i : G/H \hookrightarrow M\) an embedded orbit, then \(M - i(G/H)\) is at least \(d\)-dimensional and \((d - 2)\)-connected at each orbit type.

Proof. We have

\[
(M - i(G/H))_{(K)} = \begin{cases} M_{(K)} & G/K \neq G/H \\ M_{(H)} - i(G/H) & G/K = G/K, \end{cases}
\]

so the only nontrivial case is \(H = K\), in which case we’re tasked with verifying that the complement of a discrete set of points in a \(d\)-dimensional \((d - 2)\)-connected manifold is \((d - 2)\)-connected. This is a well known classical fact in algebraic topology which follows quickly from the Blakers-Massey theorem. □

Proof of Theorem 6. If \(d - 2 < 0\), there is nothing to prove, so assume that \(d - 2 \geq 0\). We induct on \(|S_G|\) with base case 1, i.e. with \(S = G/H\). In this case, \(\text{Conf}^G_G(M) = M_{(H)}\) is \((d - 2)\)-connected by assumption.

For induction, fix some \(S \cup G/H \in \mathcal{P}_G\) and inductively assume the theorem when \(|T_G| \leq |S_G|\). Then, note that \(\text{Conf}^G_G(M)\) is \((d - 2)\)-connected by assumption and \(M - i(S)\) is \(\geq d\)-dimensional and \((d + 2)\)-connected at each orbit by Lemma 11, so \(\text{Conf}^G_G(M - i(S))\) \((d - 2)\)-connected by the inductive hypothesis. Thus Theorem 4 expresses \(\text{Conf}^G_{S_G/G/H}(M)\) as the total space of a homotopy fiber sequence with connected base and fiber, so it is connected. Furthermore, examining the long exact sequence associated with Theorem 4, we find that

\[
\begin{array}{ccc}
0 & \to & \pi_k \text{Conf}^G_{S_G/G/H}(M) \\
\pi_k \text{Conf}^G_{G/H}(M - i(S)) & \to & 0 \\
\end{array}
\]

is exact for \(0 < k \leq d - 2\); hence \(\text{Conf}^G_{S_G/G/H}(M)\) is \((d - 2)\)-connected, completing the induction. □

1.3. Some sufficient conditions for connectivity at each orbit. We begin with the following observation:

Observation 12. If \(M\) satisfies the conditions of Proposition 7, then \(M_{S(H)}\) does as well. □

We will strengthen Proposition 7. Pick an order on \(\mathrm{Istrp}(M) = (G/H_1, \ldots, G/H_n, G/G)\), and write

\[
M_k = M - \bigcup_{i < k} M_{s(H_i)}
\]

\[
\tilde{M}_k = M_{s(H_k)} - \bigcup_{i < k} M_{s(H_i)} \cap M_{s(H_i)}
\]

\[
= M_{s(H_k)} - \bigcup_{(K) \in \mathrm{Istrp}(H_k) \cap \mathrm{Istrp}(H_i)} M_{s(K)}
\]

Lemma 13. For all \(k\), the space \(M_k\) is \((d - 2)\)-connected.

Proof. We induct in two ways:

- First, we inductively assume we have proved the lemma at full strength when \(G\) is replaced with any proper subgroup \(H \subset G\) such that \(G/H \in \mathrm{Istrp}(M)\); since \(\mathrm{Istrp}(M)\) is finite, this begins with the base case in which case there are no such proper subgroups.
- Second, we inductively assume that we have proved the lemma for all \(k' < k\); this begins with the base case that \(k = 1\), in which case we have \(M_1 = M = M_{s(G)}\), which is \((d - 2)\)-connected by assumption.
Under these assumptions, note that $\widetilde{M}_{k-1} \subset M_{k-1}$ is a $(d-2)$-connected closed submanifold of codimension $\geq d$ in a $(d-2)$-connected smooth manifold with complement $M_k$. Thus it possesses a tubular neighborhood $\widetilde{M}_{k-1} \subset \tau(M_{k-1}) \subset M_{k-1}$, and “hemmed gluing” presents a homotopy pushout square

\[
\begin{array}{ccc}
\widetilde{M}_{k-1} & \longrightarrow & M_k \\
\downarrow & & \downarrow \\
\partial \tau(M_{k-1}) & \longrightarrow & M_{k-1}
\end{array}
\]

The boundary $\partial \tau(M_{k-1})$ is the total space of a $c$-sphere bundle over a $(d-2)$-connected space, where

\[c = \text{codim}(M_{d(H)} \hookrightarrow M) - 1 > d - 2.\]

The long exact sequence in homotopy reads

\[
\pi_1(S^c) \longrightarrow \pi_1(\partial \tau(\widetilde{M}_{k-1})) \longrightarrow \pi_1(\widetilde{M}_{k-1}) \longrightarrow 0 \rightarrow \pi_0(\partial \tau(\widetilde{M}_{k-1})) \longrightarrow \pi_0(\widetilde{M}_{k-1}) \longrightarrow 0
\]

so $\partial \tau \widetilde{M}_{k-1}$ is connected, and when $d - 2 \geq 1$, $\partial \tau \widetilde{M}_{k-1}$ is simply connected. Furthermore, at degree $0 < \ell \leq (d-2)$ the Gysin sequence reads

\[
\begin{array}{ccc}
0 & \longrightarrow & H^\ell(\partial \tau \widetilde{M}_{k-1}) \\
\downarrow & & \downarrow \\
H^\ell(\widetilde{M}_{k-1}) & \longrightarrow & H^\ell-\cdot(\widetilde{M}_{k-1})
\end{array}
\]

so $\partial \tau \widetilde{M}_{k-1}$ has vanishing cohomology in degrees $0 < \ell \leq d - 2$. Hurewicz’ theorem then implies that $\partial \widetilde{M}_{k-1}$ is $(d - 2)$-connected when $d - 2 \neq 1$.

In particular, this together with $(d-3)$-connectivity of the homotopy fiber $S^c$ implies that $i$ is a $(d-2)$-connected map, so its homotopy pushout $\tilde{i}$ is $(d-2)$-connected. Since $M_{k-1}$ is a $(d-2)$-connected space by assumption, this implies that $M_k$ is $(d-2)$-connected, completing the induction. \hfill \Box

**Proof of Proposition 7.** By Observation 12 it suffices to prove that $M_G$ is $(d - 2)$-connected. This is precisely Lemma 13 when $k = n + 1$. \hfill \Box

**Warning 14.** Neither the conditions of Proposition 7 or of Theorem 6 are stable under restrictions; indeed, for $G = S^2$ and $[C_2]$ a $C_2$-torsor, the example $[C_2] \cdot D^n$ satisfies the conditions of Proposition 7 for $d = n$, but its underlying manifold does not satisfy the conditions of Theorem 6 for any $d$, as it is not connected. We will rectify this in the setting of real orthogonal $G$-representation by introducing stronger sufficient conditions which themselves are stable under restriction. \hfill \triangledown

## 2. Representations, homotopy-coherent algebra, and configuration spaces

In homotopy-coherent algebra, a prominent role is played by the operads $E_1 = A_\infty$ and $E_\omega$, whose algebras are homotopy-coherently associative algebras and homotopy-coherently commutative algebras, respectively. Dunn’s celebrated “additivity theorem” proved non-homotopically [Dun88] (later made homotopical by Lurie [HA, Thm 5.1.2.2]) that an object possessing $n$-interchanging $E_1$-structures may equivalently be presented as an algebra over the $E_n$-operad, whose space of $k$-ary operations is weakly equivalent to the ordered configuration space $\text{Conf}_k(\mathbb{R}^n)$. Thus, after Dunn and Lurie, a higher-categorical version of the Eckmann-Hilton argument may be phrased as stating that $E_n$-algebras in $(n-1)$-categories canonically lift to $E_\omega$-algebras; Lurie showed that this is equivalent to the statement that $\text{Conf}_k(\mathbb{R}^n)$ is $(n-2)$-connected for all $n,k$ [HA, Cor 5.1.1.7], which was a half-century old fact of manifold topology due to [FN62].

We would like to lift this to equivariant higher algebra using the equivariant little disks $G$-operads $E_v$; these appear in [Hor19], where they are shown to have $S$-ary operation space

\[E_v(S) \simeq \text{Conf}_S^H(V)\]

for all $S \in \Gamma_H$. Thus we are compelled to seek a representation theoretic context lifting the assumptions of Proposition 7. We propose the following.

**Definition 15.** We say $V$ has $d$-codimensional fixed points if $|V^H|$, $|V^K/V^H| \in \{0\} \cup \langle d, \infty \rangle$ for all $K \subset H \subset G$. \hfill \triangledown
When $G = e$, this is equivalent to simply being $d$-dimensional.

**Proposition 16.** If a real orthogonal $G$-representation $V$ has $d$-codimensional fixed points, then the smooth $G$-manifold $V - \{0\}$ is at least $d$-dimensional and $(d-2)$-connected at each orbit type.

**Proof.** We may write $V$ as a filtered (homotopy) colimit $V = \bigcup_i V_i$ with $V_i$ a finite dimensional real orthogonal $G$-representation with $\min(i,d)$-codimensional fixed points; then, if $V_i$ is $(i-2)$-connected for each $i$, taking a colimit, this implies that $V$ is $(d-2)$-connected. Hence it suffices to prove this in the case we that $V$ is finite dimensional.

In this case, $G$ acts smoothly on $V$, and we make the following observations:

(a) $V_{(H)} = V^H - \bigcup_{K \leq (H)} V^K$ is either empty or $\left|V^H\right| \geq d$-dimensional.

(b) $V_{s(H)} = V^H_G$ is contractible, hence it is $(d-2)$-connected.

(c) $\text{codim}(V_{s(K)} \hookrightarrow V^S_{s(H)}) = \left|V^K\right| - \left|V^K\right| = \left|V^H/V^K\right| \geq d$ by assumption.

(d) $\text{Istrp}(V)$ is finite since $V$ is finite dimensional.

Thus Proposition 7 applies, proving the proposition. \qed

**Corollary 17.** If $V$ has $d$-codimensional fixed points, then for all closed subgroups $H \subset G$ and finite $H$-sets $S \in \Gamma_H$, $\text{Conf}^H_S(V)$ is $(d-2)$-connected or empty.

**Proof.** We begin by noting

$$\text{Conf}^H_S(V) = \begin{cases} \text{Conf}^H_S(\text{Res}^G_H(V - \{0\})) & S^H \neq \emptyset, \\ \text{Conf}^H_S(\text{Res}^G_H(V - \{0\})) & \text{otherwise.} \end{cases}$$

so it suffices to show $\text{Conf}^H_S(\text{Res}^G_H(V - \{0\}))$ to be $(d-2)$-connected or empty. Noting that the condition of having $d$-codimensional fixed points is restriction-stable, this follows by Theorem 6 and Proposition 16. \qed

**Remark 18.** Let $G = C_p^N$ be the $p^N$th cyclic group for some $N \in \mathbb{N} \cup \{\infty\}$. Then, we have

$$\text{Conf}_{C_p^N/C_p^M}^C(V) = V_{C_p^M} - V_{C_p^{M-1}} \cong S(V_{C_p^{M-1}}) \times S(V_{C_p^M/V_{C_p^{M-1}}});$$

when $V$ embeds $C_p^N/C_p^M$. In particular, this has non-vanishing homotopy group in degrees $\left|V_{C_p^{M-1}}\right| - 1$ and $\left|V_{C_p^M/C_p^{M-1}}\right| - 1$. Thus when $G = C_p^N$, if $V$ does not have $d$-codimensional fixed points, then there exists some $S \in \Gamma_H$ such that $\text{Conf}^H_S(V)$ is neither $(d-2)$-connected nor empty. In particular Corollary 17 is sharp among connectivity bounds using fixed point codimension. \qed

To state a corollary, we define the weak indexing system

$$\mathbb{F}_{AV} = \left\{ S \in \Gamma_H \mid \text{Conf}^H_S(V) \neq \emptyset \right\},$$

as in [Ste24a, Ste24b]. Our main algebraic corollary is the following.

**Theorem 2’.** If $V$ has $d$-codimensional fixed points and $C$ is a $G$-symmetric monoidal $(d - 1)$-category, then

$$U : \text{CAlg}_{AV}(C) \rightarrow \text{Alg}_{\mathbb{E}_V}(C)$$

is an equivalence of $(d - 1)$-categories.

**Proof.** By [Ste24a], this is equivalent to the property that $E_p$ is a $(d-2)$-connected $G$-operad, i.e. its nonempty structure spaces are $(d-2)$-connected. By [Hor19], these structure spaces are $\text{Conf}^H_S(V)$, so the result follows from Corollary 17. \qed

In particular, note that $\left|k \cdot V^H\right| = k \left|V^H\right|$ and $\left|k \cdot V^K/k \cdot V^H\right| = k \cdot \left|V^K/V^H\right|$; hence if $V$ has $d$-codimensional fixed points, $kV$ has $kd$-codimensional fixed points. All representations have 1-codimensional fixed points, so $dV$ has $d$-codimensional fixed points; hence Theorem 2’ specializes to Theorem 2.

**Remark 19.** Theorem 2’ is significantly stronger than Theorem 2; indeed, we may choose $G = C_p$, fix a generator $x \in C_p$, and let $\lambda_i$ denote the irreducible 2-dimensional real orthogonal $C_p$-representation on whom $x$ acts by rotation at an angle of $\frac{2\pi i}{p}$. Then, when $d < p/2$, the (nontrivial) representation $V = d \oplus \bigoplus_{i=1}^d \lambda_i$ has $d$-codimensional fixed points, but it contains only one copy of each of its nontrivial summands, so it can’t be expressed as a direct sum of two nontrivial representations. \qed
Nevertheless, we specialize the following corollaries to $dV$ for readability. The first yields a simple canonical RO($G$)-graded incomplete Tambara structures on $E_{2V}$-ring spectra, and it follows from Theorem 2 in combination with CHLL.

**Corollary 20.** If $V$ is a real orthogonal $G$-representation with positive-dimensional fixed points and $(I, AV)$ a compatible pair of indexing systems (e.g. $I$ complete), then there are factorizations

\[
\begin{align*}
\text{Tamb}_{I,AV}(\text{Ab}^{\text{RO}(G)}) & \longrightarrow \text{Tamb}_{I,AV}(\text{Ab}^{Z}) \\
\text{CAlg}_{AV}(\text{Ab}^{\text{RO}(G)}) & \longrightarrow \text{CAlg}_{AV}(\text{Ab}^{Z}) \\
\text{Alg}_{E_{2V}}(\text{Sp}_{G}) & \underset{U}{\longleftarrow} \text{Mack}_{I}(\text{Ab})^{\text{RO}(G)} \underset{U}{\longrightarrow} \text{Mack}_{I}(\text{Ab})^{Z}
\end{align*}
\]

i.e. the stable homotopy groups of an $E_{2V}$-ring spectrum have natural $AV$-Tambara structures.

Finally, we acquire incomplete Mackey structures on $E_{(n+2)V}$-monoidal $n$-categories.

**Corollary 21.** $E_{(n+2)V}$-monoidal $n$-categories canonically lift to $AV$-symmetric monoidal $n$ categories, i.e.

\[U : \text{Cat}^{\text{sec}}_{AV,n} \rightarrow \text{Cat}^{\text{sec}}_{E_{(n+2)V},n}\]

is an equivalence of $(n+1)$-categories. In particular, when $V = \rho$, the forgetful functor

\[U : \text{Cat}^{\text{sec}}_{G,n} \rightarrow \text{Cat}^{\text{sec}}_{E_{(n+2)\rho},n}\]

is an equivalence of 2-categories.

### References


