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The Kervaire invariant problem*

Michael J. Hopkins

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Abstract. The history and solution of the Kervaire invariant problem is discussed, along with some of the future prospects raised by its solution.

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1. Introduction

The question of existence of framed manifolds of Kervaire invariant one has a long history in differential and algebraic topology, and has played an important

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M.J. Hopkins
Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
(e-mail: mjh@math.harvard.edu)

role in the classification of smooth structures on manifolds in dimension greater than four and in the homotopy groups of spheres. It has been one of the oldest open issues in algebraic topology. My aim in these lectures is to describe the origins and history of this famous problem, and following recent result of Mike Hill, myself, and Doug Ravenel [11].

Theorem 1. *If M is a stably framed smooth, closed manifold of Kervaire invariant one, then the dimension of M is* 2, 6, 14, 30, 62, *or* 126.

Framed manifolds of Kervaire invariant one are known to exist in dimension 2, 6, 14, 30, 62. Theorem 1 therefore resolves the Kervaire invariant one problem in all dimensions except 126.

2. Pontryagin's work of the 1930s

By the mid 1930s, the subject of algebraic topology had reached the state usually described at the beginning of a first year graduate course in the subject. Homology and cohomology groups had been defined, and Brouwer had carefully defined the degree of a map and he and Hopf had proved the theorem that two maps from an oriented n-manifold M to S^n are homotopic if and only if they have the same degree. In the mid 1930s Pontryagin introduced a famous generalization of Brouwer's work and created a new link between homotopy theory and geometry.

Following Hopf, Pontryagin undertook to study the maps

$$f : S^{n+k} \longrightarrow S^n \tag{2.1}$$

in terms of the geometry of the inverse image of a regular value $x \in S^n$. Since x is a regular value, the space $M_x = f^{-1}(x)$ is a closed, smooth manifold of dimension k. Fixing a trivialization of the tangent bundle to S^n at x gives a trivialization of the normal bundle to M_x in S^{n+k}, making M_x into a *stably framed manifold*.

If $y \in S^n$ is another regular value then $M_y = f^{-1}(y)$ is another stably framed manifold. As in Brouwer's work on the degree, the manifolds M_x and M_y can be related by choosing a path $\gamma : [0,1] \to S^n$ which is transverse to f, and with the property that $\gamma(0) = x$ and $\gamma(1) = y$. The inverse image of γ is then a stably framed manifold N with "incoming" boundary equal to M_x and "outgoing" boundary M_y. The manifold N is a *framed cobordism* between M_x and M_y, and two stably framed manifolds are said to be *framed cobordant* if there is a framed cobordism between them.

The relation "framed cobordant" is an equivalence relation, and Pontryagin showed that the construction just described leads to a description of the set $\pi_{n+k}S^n$ of homotopy classes of maps (2.1) in terms of framed cobordism classes of manifolds. The statement takes on its simplest form when n becomes very large. Write

$$\pi_k^{\mathrm{st}} S^0 = \lim_{n \to \infty} \pi_{n+k} S^n$$

for the k^{th} stable homotopy group of spheres, and let Ω_k^{fr} be the set of cobordism classes of stably framed k-manifolds. Pontryagin's correspondence gives an isomorphism of abelian groups

$$\pi_k^{\mathrm{st}} S^0 \approx \Omega_k^{\mathrm{fr}}$$

in which the set of cobordism classes of framed manifolds is made into a group using disjoint union. This work established a very deep relationship between homotopy theory and geometry; one that was destined to shape the subject over the next 50 years.

Using the classification of manifolds, Pontryagin computed the homotopy groups $\pi_k^{\mathrm{st}} S^0$ for $k \leq 2$. The case $k = 0$ reproduces the theorem of Brouwer and Hopf that two maps from the n-sphere to itself of the same degree are homotopic. A compact 0-manifold is a finite set of points, and the framing attaches an element $\mathrm{GL}_n(\mathbb{R})$ to each point. Elements of $\mathrm{GL}_n(\mathbb{R})$ in the same path component correspond to cobordant manifolds, so the only information retained upon passing to cobordism classes is the path component of the framing, which is given by a sign ± 1. The boundary of the unit interval in any framing is the disjoint union of a positively framed point and a negatively framed point. It follows from this that $\pi_n S^n = \mathbb{Z}$.

Up to framed cobordism, there are two connected framed 1-manifolds, corresponding to the fact that for $n > 2$, the fundamental group of $\mathrm{GL}_n(\mathbb{R})$ is cyclic of order 2. This leads to an isomorphism $\pi_{n+1} S^n = \mathbb{Z}/2$. In this case Pontryagin's ideas build on and slightly generalize Hopf's discovery of the Hopf invariant.

Things got interesting in dimension 2, and in his first announcement [22,23] Pontryagin made a famous error, concluding that $\pi_2^{\mathrm{st}} S^0 = 0$. The group was later proved by George Whitehead [31], and by Pontryagin himself [24] to be cyclic of order 2. Pontryagin's error was subtle, and the both geometric maneuvers he introduced, and the algebraic method for correcting his error proved to be of lasting importance.

Here is how the argument went. Let Σ be a stably framed 2-manifold, such as the one drawn in Fig. 1. In the figure, the arrow indicates the trivialization of the normal bundle. Suppose first (unlike in Fig. 1) that the genus of Σ is zero, or in other words that Σ is the two-sphere S^2. Since S^2 is the boundary of a ball, there is *some* stable framing on Σ which represents 0. But any other framing differs from this one by an element of $\pi_2 \, \mathrm{GL}_n(\mathbb{R})$ which is 0. It follows that any stably framed 2-sphere bounds a ball, and so represents 0.

Pontryagin then introduced a technique for replacing a framed 2-manifold of genus $g > 0$ with one of smaller genus. It is depicted in Fig. 2. You choose an

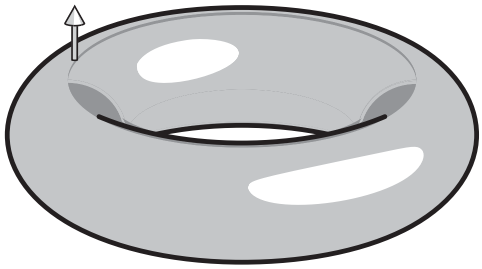

Fig. 1. A stably framed 2-manifold

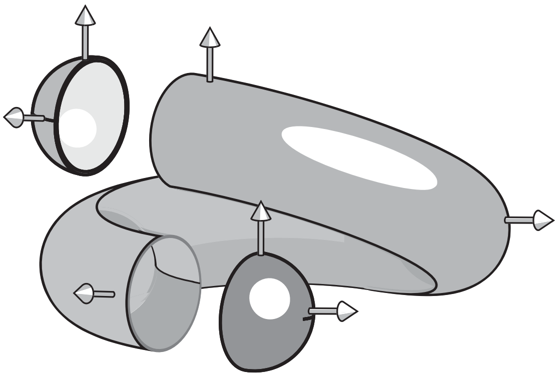

Fig. 2. Framed surgery

embedded circle on the surface, transverse to a handle. Then cut out the circle and glue in two disks. This maneuver, known as *framed surgery* lowers the genus by one, and it's not difficult to check that it leaves the framed cobordism class of Σ fixed. But there is a condition. In order to perform framed surgery on an embedded circle one needs to check that the framing it inherits from Σ can be extended to a framing of a disk. In other words, the choice of embedded circle defines an element of $\Omega_1^{\mathrm{fr}} = \mathbb{Z}/2$, and one can only do framed surgery on the elements representing 0.

In turns out that the element of Ω_1^{fr} attached to an embedded circle depends only on the mod 2 homology class represented by the circle in Σ. The stable framing of Σ therefore defines a map

$$\phi : H_1(\Sigma;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2.$$

Pontryagin argued that since the dimension of $H_1(\Sigma;\mathbb{Z}/2)$ is even, there must always be an element in the kernel of ϕ. One can then perform framed surgery on this cycle, and lower the genus until it is zero. This proves that the group $\pi_2^{\mathrm{st}} S^0$ is trivial.

Pontryagin's error was that the map ϕ is not linear. It is quadratic, with underlying bilinear form

$$\phi(x+y) - \phi(x) - \phi(y) = I(x,y)$$

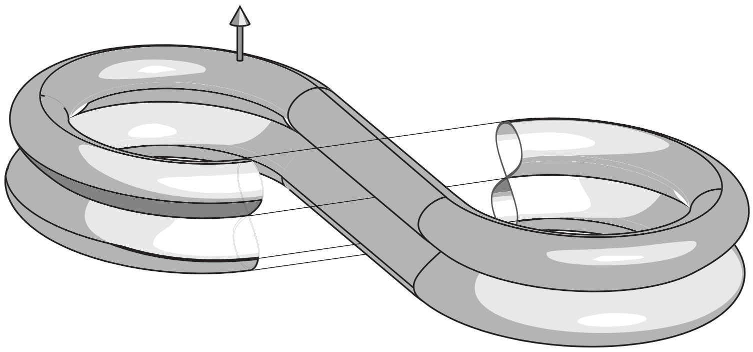

Fig. 3. A non-zero element of Ω_2^{fr}

the intersection form. The *Arf invariant* of ϕ (which works out to be 1 if ϕ takes the value 1 more often than the value 0 and 0 otherwise) is then a cobordism invariant, and gives an isomorphism

$$\Omega_2^{\mathrm{fr}} \longrightarrow \mathbb{Z}/2.$$

A picture of the non-trivial element of Ω_2^{fr} is shown in Fig. 3. It represents an immersion of the torus into $\mathbb{R}^3$ and a trivialization of the normal bundle whose associated quadratic function ϕ has Arf invariant 1.

Around 30 years later these ideas of Pontryagin were picked up and extended by Thom, Milnor, Kervaire and others, leading to triumphant advances in the classification of manifolds in high dimension. Looking back, an important question has it's origin in Pontryagin's argument that $\Omega_2^{\mathrm{fr}} = 0$.

Question 2.2. In which dimensions is every framed manifold cobordant to a homotopy sphere?

Theorem 1 gives a nearly complete answer to this question. It shows that in all but the six dimensions listed, every framed manifold is framed cobordant to a topological sphere. As we will see later, in dimensions 2, 6, 14, 30, and 62 there are framed manifolds which are not framed cobordant to a homotopy sphere. The situation in dimension 126 is still open.

3. Topology around 1960: smooth structures

In his 1956 paper [19], Milnor announced the amazing result that there are manifolds homeomorphic, but not diffeomorphic to the 7-sphere. He followed this with [20] in which he introduced and applied the technique of "surgery" on manifolds. Milnor's ideas greatly generalized the construction of Pontryagin, and soon became the fundamental tools for investigating classification problems in differential topology.

Using the new theory of surgery, Kervaire developed an important generalization of Pontryagin's function ϕ. In [15] he showed how one could associate to every 4-connected combinatorial manifold M of dimension 10 a quadratic function

$$\phi : H^{2k+1}(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

refining the bilinear intersection form $\langle xy, [M]\rangle$. He then showed that

$$\Phi(M) := \text{Arf invariant of } \phi$$

is a cobordism invariant, which vanishes whenever M is a smooth manifold. Finally, he constructed an example of a combinatorial manifold M of dimension 10 for which $\Phi(M) = 1$, in this way giving an example of a combinatorial manifold which does not admit a smooth structure.

Subsequently, Kervaire was able to define a quadratic function

$$\phi : H_{2k+1}(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

for every smooth manifold M of dimension $(4k + 2)$ equipped with a stable framing of the complement of a point in M (an "almost framing"). The *Kervaire invariant* of M is the Arf invariant of ϕ, and denoted $\Phi(M)$. It is one of the fundamental invariants of differential topology. Kervaire's work raised the following question.

Question 3.1. In which dimensions is $\Phi(M)$ zero for every smooth, stably framed manifold M?

Theorem 1 shows that, except in 6 possible dimensions, Kervaire's invariant is always zero for stably framed manifolds.

The Kervaire invariant and Question 3.1 play an important role in the classification theorems in differential topology in dimensions greater than 4. Both came to prominence right away, in the classification of smooth structures on spheres.

Shortly after Milnor's introduction of his 7-spheres, Kervaire and Milnor [21,16] announced a classification of smooth structures on spheres, up to h-cobordism (which, by the h-cobordism theorem, is the same as the set of smooth structures up to diffeomorphism in dimensions greater than 5). They introduced the group Θ_n of h-cobordism classes of n-manifolds homotopy equivalent to S^n, with the group operation of connected sum. For n congruent to 0 or 3 mod 4 they were able to determine Θ_n in terms of the homotopy groups of spheres. For n congruent to 1 or 2 mod 4 they were unable to settle a factor of 2 in the order of Θ_n.

Kervaire and Milnor introduced an exact sequence

$$0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \longrightarrow \operatorname{coker} J_n$$

in which $\operatorname{coker} J_n$ is the cokernel of the stable J-homomorphism $J : \pi_n O \to \pi_n S^0$, and bP_{n+1} is the subgroup of Θ_n consisting of manifolds which bound a stably parallelizable $(n+1)$-manifold. The group $\operatorname{coker} J_n$ can be interpreted as the quotient of the group of stably framed n-manifolds by the subgroup consisting of S^n in its possible stable framings. The rightmost map is constructed by showing that every homotopy sphere Σ admits a stable framing, choosing a framing F and sending Σ to the stably framed manifold Σ with framing F. Kervaire and Milnor showed that the map

$$\Theta_n \longrightarrow \operatorname{coker} J_n \tag{3.2}$$

is surjective unless n is of the form $4k+2$, and that there is an exact sequence

$$\Theta_{4k+2} \longrightarrow \operatorname{coker} J_n \longrightarrow \mathbb{Z}/2$$

in which the rightmost map is given by the Kervaire invariant. In geometric terms, the surjectivity of (3.2) is equivalent to the assertion that every stably framed n-manifold is framed cobordant to a homotopy sphere (Question 2.2). Theorem 1 shows that this map is surjective in all dimensions except 2, 6, 14, 30, 62 and possibly 126. It is known not to be surjective in dimensions 2, 6, 14, 30 and 62.

Kervaire and Milnor were also able to determine the group bP_{n+1}. They showed that $bP_{n+1} = 0$ if $(n+1)$ is odd, and that bP_{4k} is cyclic of order

$$a_k \, 2^{2k-2} \, (2^{2k-1} - 1) \, \text{numerator}(B_k/4k),$$

where B_k is the k^{th} Bernoulli number, and a_k is 1 if k is even and 2 if k is odd. Building on work of Adams, they produced the formula

$$|\Theta_{4m-1}| = a_m \, |\pi_{4m-1} S^0| \, 2^{2m-4}(2^{2m-1} - 1) B_m/m.$$

They were unable to determine the group bP_{4n+2}, and could only show that it is isomorphic to the cokernel of the Kervaire invariant map.

$$\operatorname{coker} J_{4k+2} \longrightarrow \mathbb{Z}/2.$$

Theorem 1 shows that $bP_{4k+2} = \mathbb{Z}/2$ except possibly when $4k+2$ is one of 2, 6, 14, 30, 62, and 126. As we will see below, it is known by the work of Barratt, Jones and Mahowald that $bP_{4k+2} = 0$ in the first five of these six exceptional cases. The situation of bP_{126} is still open.

In this way Theorem 1 completes the Kervaire–Milnor classification of exotic spheres, except in dimensions 125 and 126. Roughly speaking, the conclusion is that the groups Θ_{4k+2} and Θ_{4k+1} are twice as large as they might have been.

At the time the Kervaire–Milnor paper was written the status of the Kervaire invariant problem was far from certain. There were known to be framed manifolds of Kervaire invariant one in dimensions 2, 6 and 14, and there were known to be no framed manifolds of Kervaire invariant one in dimensions 10 and 18. Kervaire and Milnor ended their paper [16] with a remark expressing the guess that the Kervaire invariant of a framed manifold was zero in all dimensions except 2, 6, and 14. As we will see in the next section, this guess looked to be wrong shortly after. Theorem 1 shows that they were almost correct.

4. Browder's work and the connection to homotopy theory

The previous section described in part the important role played by the Kervaire invariant in the classification problems of differential topology. But in Pontryagin's work it originated in connection with the homotopy groups of spheres. In the early 1960s the relationship of Kervaire's invariant to the homotopy group of spheres was unclear, and very little was known about the dimensions in which it could be non-zero. It was known to be non-zero in dimensions 2, 6 and 14, and it was known to always be zero in dimensions 10 and 18. It was the methods of homotopy theory that unlocked the next piece of the puzzle. The first step was taken by Brown and Peterson [7,8] who showed

Theorem 4.1 (Brown, Peterson). *The Kervaire invariant $\Theta(M)$ is zero if the dimension of M is of the form $8k+2$, with $k>0$.*

But the definitive result came in the 1969 paper [6] of Browder

Theorem 4.2 (Browder). *The Kervaire invariant of a framed n-manifold is zero unless n is of the form $2^{k+1}-2$, and in that case there is a framed manifold of Kervaire invariant 1 if and only if there is an element $\theta_j \in \pi_{2^{j+1}-2}$ represented at the E_2-term of the classical Adams spectral sequence by the class h_j^2.*

For $j>0$, the element h_j represents a potential element of $\pi_{2^j-1}S^0$ of Hopf invariant one. Only the h_j with $j\le 3$ survive the spectral sequence. The work of Barratt, Jones, Mahowald, and Tangora [18,5,4] showed that elements θ_j exist for $j\le 5$. Their method, roughly, amounts to computing all of the homotopy groups $\pi_i S^0$ for $i\le 62$ using the Adams spectral sequence, and then observing that the appropriate classes h_j^2 survive.

5. The homotopy groups of spheres

Part of what made the Kervaire invariant one problem so lasting was that it also played an important role in the homotopy groups of spheres. Here is one way it arises.

In the 1950s a remarkable inductive device was discovered by George Whitehead [32] and Ioan James [12]. It was a long exact sequence, known as the EHP sequence

$$\begin{tikzcd} \pi_k S^n \arrow[rr, "E"] & & \pi_{k+1} S^{n+1} \arrow[dl, "H"] \\ & \pi_{k+1} S^{2n+1} \arrow[ul, dashed, "P"] & \end{tikzcd}$$

in which the map P goes to $\pi_{k-1} S^n$. The groups in question are to be localized at the prime 2. The sequence raises the possibility that the homotopy groups of spheres could be computed inductively, starting with the homotopy groups of S^1. In fact this procedure was carried out by Toda [29] who used it to determine $\pi_{n+k} S^n$ for $k \leq 19$. It's interesting to focus on the point at which the sphere S^{2n+1} first makes an appearance. The "degree" identifies the group $\pi_{2n+1} S^{2n+1}$ with the group of integers, and the map P sends the map of degree 1 to the Whitehead square

$$[\iota_n, \iota_n] \in \pi_{2n-1} S^n.$$

Computing with the EHP sequence leads one to two fundamental questions.

Question 5.1. For which k is $[\iota_n, \iota_n]$ in the image of E^k?

Question 5.2. For which n is $[\iota_n, \iota_n]$ divisible by 2?

Answers to these questions have immediate computational impact on our knowledge of the homotopy groups of spheres. But they are also equivalent to other famous problems in topology. James showed in [13] that $[\iota_n, \iota_n]$ is in the image of E^k if an only if S^n admits k linearly independent vector fields. So Question 5.1 is equivalent to the vector field problem, solved by Adams in the early 1960s [2]. Question 5.2 breaks into two cases, depending on the parity of n. When n is even, the class $[\iota_n, \iota_n]$ has Hopf invariant 2, so Question 5.1 is equivalent to a strong form of the Hopf invariant one problem, which was solved by Adams [1] around 1960. When n is odd it is equivalent to the assertion that a Kervaire invariant one class exists in $\pi_{n-1} S^0$ and has order 2. By Barratt, Jones, Mahowald and Tangora this is the case when $j \leq 5$. By Theorem 1 it is not the case when $j > 6$. So again, the only remaining open case is in dimension 126.

In the front piece to his book on Stiefel manifolds, James [14] gives an elementary formulation of the Kervaire invariant problem. Let V_n be the space of pairs of points $x, y \in S^n$ with the property that $x \neq \pm y$.

Question 5.3 (James [14]). For which n is the identity map of V_n homotopic to the map which switches x and y.

The homotopy cannot exist if n is odd. When n is even, James [14, Theorem 1.13] shows that such a homotopy exists if and only if $[\iota_{n+1}, \iota_{n+1}]$ is divisible by 2.

6. Brief sketch of the proof

Let me make a few brief comments on the proof of Theorem 1, given in [11]. The result proved is actually the following

Theorem 6.1. *For $j \geq 7$ the class $h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ does not represent an element of the stable homotopy groups of spheres. In other words, the Kervaire invariant elements θ_j do not exist for $j \geq 7$.*

By Browder's Theorem 4.2 this implies Theorem 1.

Our proof builds on the strategy used by Ravenel in [25] and on the homotopy theoretic refinement developed by the author and Haynes Miller (see [27]). Let $MU_{\mathbb{R}}$ be the $\mathbb{Z}/2$-equivariant *real bordism* spectrum of Landweber [17] and Fujii [10]. Roughly speaking one can think of $MU_{\mathbb{R}}$ as describing the cobordism theory of *real manifolds*, which are stably almost complex manifolds equipped with a conjugate linear action of $\mathbb{Z}/2$. Write

$$MU^{((4))} = MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$

for the $\mathbb{Z}/8$-equivariant spectrum gotten by smashing 4 copies of $MU_{\mathbb{R}}$ together and letting $\mathbb{Z}/8$ act by

$$(a, b, c, d) \longmapsto (\bar{d}, a, b, c).$$

Roughly speaking $MU^{((4))}$ can be thought of as the cobordism theory of stably almost manifolds equipped with a $\mathbb{Z}/8$-action, with the property that the restriction of the action to $\mathbb{Z}/2 \subset \mathbb{Z}/8$ determines a real structure. To get somewhere we need to invert an equivariant analogue of the Bott periodicity class to form the $\mathbb{Z}/8$-equivariant spectrum $\widetilde{\Omega} = D^{-1}MU^{((4))}$. Finally, we define Ω to be the homotopy fixed point spectrum of the $\mathbb{Z}/8$-action on $\widetilde{\Omega}$. The proof of Theorem 6.1 is assembled from the following results.

Theorem 6.2 (The Detection Theorem). *If θ_j exists, then it has a non-zero image in $\pi_{2^{j+1}-2}\Omega$.*

Theorem 6.3 (The Periodicity Theorem). *The groups $\pi_*\Omega$ are periodic, with period 256: $\pi_*\Omega \approx \pi_{*+256}\Omega$.*

Theorem 6.4 (The Gap Theorem). *The group $\pi_{-2}\Omega$ is zero.*

The reader can easily check that these three results imply Theorem 6.1.

The proof of the Detection Theorem is a straightforward check through the inventory of possible representatives for h_j^2 in the E_2-term of the Adams–Novikov spectral sequence, as can be found in Shimomura [28] or Ravenel [26].

The spectrum $MU^{((4))}$ and the class D are chosen so that the Detection Theorem holds. The other two results exploit relatively simple fact that the map

$$\widetilde{\Omega}^{\mathbb{Z}/8} \longrightarrow \widetilde{\Omega}^{h\mathbb{Z}/8}$$

from the fixed point spectrum to the homotopy fixed point spectrum is a homotopy equivalence. The Periodicity Theorem is proved for the homotopy fixed point spectrum, while the Gap Theorem for the actual fixed point spectrum. Both proofs involve the *slice filtration* which is a novel equivariant refinement of the Postnikov tower. It is analogous to the slice filtration in motivic homotopy theory [30] and generalizes the filtration described by Dugger in [9], and builds on the unpublished work of the author and Fabien Morel.

7. Open questions

Here are a few questions raised by this work.

The possibility that smooth manifolds of Kervaire invariant one exist in infinitely many dimensions left no clear suggestion as to how one might construct them. But the fact that they exist in only five or six dimensions brings to mind "special" constructions as opposed to systematic ones. When this work was announced Atiyah asked whether or not the five or six Kervaire invariant one manifolds were related to other exceptional phenomena in mathematics, such as the projective planes of exceptional Jordan algebras.

Question 7.1. Can "exceptional phenomena" in mathematics be used to construct smooth manifolds of Kervaire invariant one?

The proof of Theorem 1 loses connection with geometry, and so does not point to geometric structures which may have been overlooked in surgery theory. This is because of the Detection Theorem, which requires detailed knowledge of

$$\operatorname{Ext}^{2,2^j}_{MU_*MU}(MU_*, MU_*)$$

(the "Adams–Novikov 2-line").

Question 7.2. Can the Detection Theorem be proved without reference to the Adams–Novikov spectral sequence?

Even though the proof of Theorem 1 is not geometric, the main computation can be described in geometric terms. Suppose that X is a space with a $\mathbb{Z}/2$-action. Recall from [3] that a *real vector bundle* on X is a complex vector bundle $V \to X$ equipped with a compatible, conjugate linear $\mathbb{Z}/2$-action. A *real manifold* is a manifold M equipped with an action of $\mathbb{Z}/2$, a real vector bundle V on M, and equivariant isomorphism $TM \oplus V \approx M \times \mathbb{C}^N$. This condition forces the dimension of M to be even and the dimension of the fixed point submanifold to be one half of the dimension of M. A cobordism of real manifolds is a cobordism N equipped with real vector bundle W extending the given one on the boundary, and an isomorphism

$$TN \oplus V \approx N \times \mathbb{R} \times \mathbb{C}^N$$

compatible with the given structures on the boundary.

Question 7.3. Suppose that M is a real manifold of dimension $2n$ whose fixed point space Z bounds an unoriented manifold. Can one describe a cobordism invariant of M which in the case in which Z is empty is

$$\int_{M/(\mathbb{Z}/2)} w_1^{2d}$$

where w_1 is the first Stiefel–Whitney class of the double cover $M \to M/\mathbb{Z}/2$?

There is an analogue of the problem in which $\mathbb{Z}/2$ is replaced by $\mathbb{Z}/2^n$. The proof of Theorem 1 makes use of the case $n = 3$.

The homotopy theoretic formulation of the Kervaire invariant problem has an analogue at odd primes. For primes greater than 3 it was solved by Ravenel, whose paper [25] was part of the inspiration for the present work. The situation at the prime 3 is still open. The methods of [11] do not apply because of the lack of an analogue of real bordism for primes $p > 2$. What one actually needs is an analogue of "real" BP-theory $BP_{\mathbb{R}}$. This should be a $\mathbb{Z}/p$-equivariant spectrum $BP^{((\mathbb{Z}/p))}$ whose underlying, non-equivariant spectrum has the homotopy type of the $(p-1)$-fold smash product $BP^{(p-1)}$ of BP with itself. Write σ for a generator of $\mathbb{Z}/p$ and let $V = Z_{(p)}[Z/p]/(1 + \sigma + \cdots + \sigma^{p-1})$ be the reduced regular representation. Write $V[n]$ for the graded representation of $\mathbb{Z}/p$ consisting of V in degree n and 0 everywhere else. As an algebra with an action of $\mathbb{Z}/p$, the ring $\pi_* BP^{(p-1)}$ should be isomorphic to the symmetric algebra on

$$\bigoplus_{k \geq 0} \operatorname{Sym} V[2p^k - 2].$$

Question 7.4. For each prime p can one construct the "p-analogue" of $BP_{\mathbb{R}}$ described above?

References

[1] J.F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math. (2), **72** (1960), 20–104.

[2] J.F. Adams, Vector fields on spheres, Ann. of Math. (2), **75** (1962), 603–632.

[3] M.F. Atiyah, K-theory and reality, Quart. J. Math. Oxford Ser. (2), **17** (1966), 367–386.

[4] M.G. Barratt, J.D.S. Jones and M.E. Mahowald, Relations amongst Toda brackets and the Kervaire invariant in dimension 62, J. London Math. Soc. (2), **30** (1984), 533–550.
[5] M.G. Barratt, M.E. Mahowald and M.C. Tangora, Some differentials in the Adams spectral sequence. II, Topology, **9** (1970), 309–316.
[6] W. Browder, The Kervaire invariant of framed manifolds and its generalization, Ann. of Math. (2), **90** (1969), 157–186.
[7] E.H. Brown, Jr. and F.P. Peterson, The Kervaire invariant of $(8k+2)$-manifolds, Bull. Amer. Math. Soc., **71** (1965), 190–193.
[8] E.H. Brown, Jr. and F.P. Peterson, The Kervaire invariant of $(8k+2)$-manifolds, Amer. J. Math., **88** (1966), 815–826.
[9] D. Dugger, An Atiyah–Hirzebruch spectral sequence for KR-theory, K-Theory, **35** (2005), 213–256.
[10] M. Fujii, Cobordism theory with reality, Math. J. Okayama Univ., **18** (1976), 171–188.
[11] M.A. Hill, M.J. Hopkins and D.C. Ravenel, On the non-existence of elements of Kervaire invariant one, preprint, arXiv:0908.3724.
[12] I.M. James, On the suspension sequence, Ann. of Math. (2), **65** (1957), 74–107.
[13] I.M. James, Whitehead products and vector-fields on spheres, Proc. Cambridge Philos. Soc., **53** (1957), 817–820.
[14] I.M. James, The Topology of Stiefel Manifolds, London Math. Soc. Lecture Note Ser., **24**, Cambridge Univ. Press, Cambridge, 1976.
[15] M.A. Kervaire, A manifold which does not admit any differentiable structure, Comment. Math. Helv., **34** (1960), 257–270.
[16] M.A. Kervaire and J.W. Milnor, Groups of homotopy spheres. I, Ann. of Math. (2), **77** (1963), 504–537.
[17] P.S. Landweber, Conjugations on complex manifolds and equivariant homotopy of MU, Bull. Amer. Math. Soc., **74** (1968), 271–274.
[18] M. Mahowald and M. Tangora, Some differentials in the Adams spectral sequence, Topology, **6** (1967), 349–369.
[19] J.W. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math. (2), **64** (1956), 399–405.
[20] J.W. Milnor, A procedure for killing homotopy groups of differentiable manifolds, In: Proc. Sympos. Pure Math., Vol. III, Amer. Math. Soc., Providence, RI, 1961, pp. 39–55.
[21] J.W. Milnor and M.A. Kervaire, Bernoulli numbers, homotopy groups, and a theorem of Rohlin, In: Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York, NY, 1960, pp. 454–458.
[22] L.S. Pontryagin, A classification of continuous transformations of a complex into a sphere. 1, Dokl. Akad. Nauk SSSR (N.S.), **19** (1938), 147–149.
[23] L.S. Pontryagin, A classification of continuous transformations of a complex into a sphere. 2, Dokl. Akad. Nauk SSSR (N.S.), **19** (1938), 361–363.
[24] L.S. Pontryagin, Homotopy classification of mappings of an $(n+2)$-dimensional sphere on an n-dimensional one. (Russian), Dokl. Akad. Nauk SSSR (N.S.), **19** (1950), 957–959.
[25] D.C. Ravenel, The non-existence of odd primary Arf invariant elements in stable homotopy, Math. Proc. Cambridge Philos. Soc., **83** (1978), 429–443.
[26] D.C. Ravenel, Complex Cobordism and Stable Homotopy Groups of Spheres, Pure Appl. Math., **121**, Academic Press, Orlando, FL, 1986.
[27] C. Rezk, Notes on the Hopkins–Miller theorem, In: Homotopy Theory via Algebraic Geometry and Group Representations, Evanston, IL, 1997, Contemp. Math., **220**, Amer. Math. Soc., Providence, RI, 1998, pp. 313–366.
[28] K. Shimomura, Novikov's Ext^2 at the prime 2, Hiroshima Math. J., **11** (1981), 499–513.
[29] H. Toda, Composition Methods in Homotopy Groups of Spheres, Ann. of Math. Stud., **49**, Princeton Univ. Press, Princeton, NJ, 1962.

[30] V. Voevodsky, Open problems in the motivic stable homotopy theory. I, In: Motives, Polylogarithms and Hodge Theory. Part I, Irvine, CA, 1998, Int. Press Lect. Ser., **3**, Int. Press, Somerville, MA, 2002, pp. 3–34.
[31] G.W. Whitehead, The $(n + 2)^{\text{nd}}$ homotopy group of the n-sphere, Ann. of Math. (2), **52** (1950), 245–247.
[32] G.W. Whitehead, On the Freudenthal theorems, Ann. of Math. (2), **57** (1953), 209–228.