You can't make the borromean rings out of chainmaille

Natalie Stewart*

September 11, 2023

These notes are unfinished!

1 Phenomenon

1.1 Common-radius weaves

Construction 1.1. Let $U := \mathbb{RP}^2$ be the topological space of pointed planes in $\mathbb{R}^3$. We refer to this as the space of *geometric unit circles in* $\mathbb{R}^3$, as it possesses an embedding

$$c : U \times \mathbb{R} \hookrightarrow \text{Links}$$

into the space of embeddings of S^1 into $\mathbb{R}^3$ by restricting to the circle of radius r around the point. Furthermore, define

$$\widetilde{\mathcal{W}}_n := \left\{(r, (p_i)_{i\in[n]}) \mid \exists \varepsilon > 0, \;\; \forall\, i, j \;\; c(r, p_i) \cap c(r, p_j) > \varepsilon\right\} \subset \mathbb{R} \times U^{\times n}_{\text{Aut}\,[n]}.$$

to be the space of tame configurations of n nonintersecting unit circles of the same radius in $\mathbb{R}^3$.[1] Denote by $\widetilde{\mathcal{W}} := \coprod_{n\in\mathbb{N}} \widetilde{\mathcal{W}}$ the union across all n.

The space $\widetilde{\mathcal{W}}_n$ possesses an embedding into Links. The following proposition is physical motivation for our work:

Proposition 1.2. *Let $\widetilde{\mathcal{W}}_{n,\varepsilon}$ denote the topological space of component-wise scaled isometric embeddings of $n \cdot T_\varepsilon$, where T_ε is the metric space formed by taking the ball of radius ε around a standard unit circle in $\mathbb{R}^3$. Then, whenever $\varepsilon \geq \varepsilon'$, there is an embedding $\widetilde{\mathcal{W}}_{n,\varepsilon} \hookrightarrow \widetilde{\mathcal{W}}_{n,\varepsilon'}$, and together these form a colimit diagram*

$$\text{colim}_{\varepsilon\to 0}\, \widetilde{\mathcal{W}}_{n,\varepsilon} \xrightarrow{\sim} \widetilde{\mathcal{W}}_n.$$

For the purpose of these notes, let $\mathbb{G}$ denote the group of isometries of $\mathbb{R}^3$. We give $\widetilde{\mathcal{W}}_n$ a $\mathbb{G}$-action in the following proposition:

Proposition 1.3. *Endowing on S^1 a trivial $\mathbb{G}$-action, there is an evident action of the group $\mathbb{G}$ on* Links*; this action preserves the image of the embedding*

$$\widetilde{\mathcal{W}}_n \hookrightarrow \text{Links},$$

and hence it endows $\widetilde{\mathcal{W}}_n$ with a $\mathbb{G}$-action.

Proof. First note that a map $\mathbb{R}^2 \to \mathbb{R}^3$ is an affine transformation if and only if it is an embedding of a totally geodeisc submanifold; the property of such an embedding being totally geodesic is invariant under the action of the isometry group of $\mathbb{R}^3$, so U is preserved under the action of $\mathbb{G}$ on Links, hence so is $\mathbb{R} \times U^{\times n}$. Since isometries are injective, the locus $\widetilde{\mathcal{W}}_n$ is preserved as well. ☐

We can now make a central construction.

---

*Harvard university.

[1]Here, the tameness assumption only comes into play when $n = \infty$.

Construction 1.4. Let H be a discrete group acting on $\mathbb{R}^3$ by isometries. Let $\mathbf{Set}_H^{\mathrm{cbl}}$ be the category of at-most countable H-sets. The $W(H)$ space $\coprod^H_{n \in \pi_0 \mathbf{Set}^{\mathrm{cbl}}} \mathrm{TopEmb}(S^1, \mathbb{R}^3)$ has a decomposition indexed by isomorphism classes of at-most countable $\mathbb{G}$-sets given by the induced $\mathbb{G}$-action on the set of componens; denote the pullback decomposition on $\widetilde{\mathcal{W}}$ by

$$\widetilde{\mathcal{W}}^H = \coprod_{S \in \pi_0 \mathbf{Set}_H^{\mathrm{cbl}}} \widetilde{\mathcal{W}}_S^H.$$

Then, the *set of S-indexed H-weaves* is $W_S^H := \pi_0 \widetilde{\mathcal{W}}_S^H$.

We note the following consequence of tameness here:

Proposition 1.5. *Suppose $H \subset G$ is a closed subgroup which is not discrete. Then, $\mathcal{W}^H$ is contractible.*

1.2 Unit-radius weaves

Let $\mathcal{W} \subset \widetilde{\mathcal{W}}$ be the topological space of weaves with radius 1. This is our most physically intuitive space. We recover results about physical chainmaille in this section.

The positive $\mathbb{R}_{>0}^+$ acts on $\mathbb{G}$ by conjugation in the group of scaled isometries of $\mathbb{R}^3$, where $\mathbb{G}_a$ acts on $\mathbb{R}^3$ by dilation about the origin. In fact, if τ_x is the translational symmetry taking the origin to $x \in \mathbb{R}^3$, we have $r \cdot \tau_x = \tau_{rx}$. We relate the genuine equivariant homotopy of $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ using this action, via the following easy proposition

Proposition 1.6. $\widetilde{\mathcal{W}}^H \simeq \cup_{r \in \mathbb{R}_{>0}^+} \mathcal{W}^{r \cdot H}$.

There's more we can say about this, but let's let that be for now.

2 Formalism

2.1 G-spaces

The main object of study is W_S^H, which occurs as a canonical decomposition of the 0th homotopy group of the H-fixed points of a topological $\mathbb{G}$-space. A natural way to throwing away some information of a topological space without throwing away π_0 is through *homotopy theory*; when there's a group action floating around, we use something called *genuine equivariant homotopy theory*.

Definition 2.1. Let G be a topological group, and let $\mathbf{Top}_G$ denote the category of topological spaces[2] with G-actions. Then, the *∞-category of G-spaces* is the simplicial localization

$$\mathcal{S}_G := \mathbf{Top}[G - EQ^{-1}]$$

where $G - EQ$ is the class of homotopy equivalences f which are G-equivariant, and for which the homotopies $ff^{-1} \sim \mathrm{id}$ and $f^{-1}f \sim \mathrm{id}$ can be chosen to be G-equivariant at all times.

This recovers the homotopy types of the fixed points:

Proposition 2.2. *Let $H \subset G$ be a closed subgroup. Then, the orbit space G/H has a canonical G-action, and there is a homotopy equivalence*

$$\mathrm{Map}(G/H, X)^G \simeq X^H.$$

In fact, we can do more:

Definition 2.3. The *orbit category* is the subcategory $\mathcal{O}_G \subset \mathbf{Top}_G$ spanned by the homogeneous spaces G/H.

This provides a functor $\mathcal{S}_G \xrightarrow{X^{(-)}} \mathrm{Fun}(\mathcal{O}_G, \mathcal{S})$. One of the most important foundational theorems of genuine equivariant homotopy theory suggests that we can go the other way:

[2] Compactly generated, weakly hausdorff

Theorem 2.4 (Elmendorf's theorem)**.** *The fixed point functor* $\mathcal{S}_G \to \mathrm{Fun}(\mathcal{O}_G, \mathcal{S})$ *is an equivalence of* ∞*-categories.*

There's a simple interpretation of this in terms of chainmail weaves; if X is a $\mathbb{G}$-set, then an *X-valued $\mathbb{G}$-invariant of weaves* is a map of $\mathbb{G}$-spaces $\widetilde{\mathcal{W}} \to X$. Elmendorff's theorem says that X is simply a functor $\mathcal{O}_{\mathbb{G}} \to \mathcal{S}$, and an X-valued invariant is simply a collection of maps $\widetilde{\mathcal{W}}^H \to X^H$ compatible with restriction and conjugation.

In particular, the functor $\mathcal{O}_G \to \mathcal{S}$ whose value on G/H is $\mathbf{Set}_H^{\mathrm{cbl}}$ corresponds with a unique G-space, which we refer to as $\underline{\pi}_0\mathbf{Set}_{\mathbb{G}}^{\mathrm{cbl}}$. We use this to define graded objects:

Definition 2.5. Let G be a topological group. Then, the *category of countably-graded G-spaces* is the overcategory

$$\mathcal{S}_G^{\mathrm{Gr\,cbl}} := (\mathcal{S}_G)_{/\underline{\pi}_0\mathbf{Set}_{\mathbb{G}}^{\mathrm{cbl}}}.$$

The decomposition $\widetilde{\mathcal{W}}^H \simeq \coprod_{S \in \pi_0 \mathbf{Set}_H^{\mathrm{cbl}}} \widetilde{\mathcal{W}}_S^H$ provides a countable grading on $\widetilde{\mathcal{W}}$. A *graded $\mathbb{G}$-invariant of weaves* is a map of countably-graded $\mathbb{G}$-spaces out of $\mathcal{W}$. An archetypical example of this is the *underlying link*, which is graded by the $\mathbb{G}$-set of components.

2.2 G-weak equivalences

Definition 2.6. An *unstable Mackey functor valued in* **Set** is a functor $\mathcal{S}_G \to \mathbf{Set}$. The *unstable Mackey functor homotopy groups* of a G-space are

$$\underline{\pi}_n X := \mathrm{Map}(S^n \times (-), X) : \mathcal{S}_G \to \mathbf{Set}.$$

where G acts trivially on S^n. A *G-weak equivalence* is a map of G-spaces inducing isomorphisms on $\underline{\pi}_*$.

It's not too hard to show that $\left(\underline{\pi}_n X\right)(H) = \pi_n X^H$. For completeness, we'd like to state a version of whitehead's theorem using this, but first we need a version of the theory of CW complexes.

Definition 2.7. A *G-CW complex X* is a G-space with a distinguished decomposition $X = \mathrm{colim}_n \, \mathrm{sk}_n X$ together with expressions of $\mathrm{sk}_n X$ as a pushout of $\mathrm{sk}_{n-1} X$ along $S_n \times \left(S^{n-1} \hookrightarrow D^n\right)$, where S_n is a G-set.

Theorem 2.8 (Equivariant skeletal approximation and whitehead)**.** *Every G-space X is G-weakly equivalent to a G-CW complex; furthermore, there is a canonical equivalence*

$$\mathcal{S}_G \simeq G - \mathrm{CW}[G - WEQ^{-1}]$$

3 Application

Recall that Links is a $\mathbb{G}$-space. We'd like to construct a graded invariant on this summarizing all of the linking information.

Definition 3.1. The $\mathbb{G}$-space Graphs is the space of equivalence classes of embedded graphs in $\mathbb{R}^3$ with evident $\mathbb{G}$-action. This space is countably graded under the forgetful map to the underlying $\mathbb{G}$-set of vertices.

Construction 3.2. Let L be a countable link. Then, the *Linking graph of L* is the graph embedded in $\mathbb{R}^3$ whose underlying set of vertices consists of the centers of the components of L, with straight lines drawn between two vertices precisely when their corresponding link components have nonzero linking number.

Proposition 3.3. *The linking graph* G : Links $\to$ Graphs *is $\mathbb{G}$-equivariant and countably graded.*

Define the pullback countably graded $\mathbb{G}$-space

$$\begin{array}{ccccc}
\widetilde{\mathcal{W}}^{\mathrm{triv}} & & \xrightarrow{\qquad\qquad} & & \underline{\pi}_0\mathbf{Set}_{\mathbb{G}} \\
\downarrow & \lrcorner & & & \downarrow{\scriptstyle \mathrm{discrete}} \\
\widetilde{\mathcal{W}} & \longrightarrow & \mathrm{Links} & \xrightarrow{G} & \mathrm{Graphs}
\end{array}$$

The main theorem of this talk is as follows. We say that a G-space X is *G-connected* if $\underline{\pi}_0 X$ is the constant unstable Mackey functor on $*$, and we say that a (countably) graded G-space *degreewise G-connected* if each graded piece is G-connected.

Theorem 3.4. *The countably graded $\mathbb{G}$-space $\widetilde{\mathcal{W}}^{\mathrm{triv}}$ is degreewise $\mathbb{G}$-connected.*

We prove this theorem in several parts. Let $r : \widetilde{\mathcal{W}} \to \mathbb{R}_{>0}$ be projection onto the radius and let $f : \widetilde{\mathcal{W}} \to \mathbb{R}_{>0}$ be the infemum distance between the centers of components of a weave; both of these are $\mathbb{G}$-equivariant for the trivial $\mathbb{G}$-action on $\mathbb{R}_{>0}$. Define $\widetilde{\mathcal{W}}_\varepsilon$ to be the pullback $\mathbb{G}$-space

$$\begin{array}{ccc} \widetilde{\mathcal{W}}_\varepsilon^{\mathrm{triv}} & \longrightarrow & \widetilde{\mathcal{W}}^{\mathrm{triv}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle r/f} \\ (0,\varepsilon] & \hookrightarrow & \mathbb{R}_{>0} \end{array}$$

consisting of dilations of trivial linking weaves whose components are at separated by distance at least 1 while having radius at most ε. The main geometric content of the theorem is the following proposition:

Proposition 3.5. *The inclusion $\widetilde{\mathcal{W}}_\varepsilon^{\mathrm{triv}} \hookrightarrow \widetilde{\mathcal{W}}^{\mathrm{triv}}$ is an equivalence.*

Proof. Fix the isometric embedding $\iota : \mathbb{R}^3 \simeq \partial \mathbb{H}^4 \hookrightarrow \mathbb{H}^4$, where $\mathbb{H}^4$ is the half-plane model of hyperbolic 4-space, with boundary. By the classification of totally geodesic (TG) submanifolds in $\mathbb{H}^4$, for every configuraiton of geometric circles in $\mathbb{R}^3$, there exist unique totally geodesic properly embedded hemispheres in $\mathbb{H}^4$ whose boundary circles make the weave. This provides a continuous map from $\widetilde{\mathcal{W}}$ from configurations of tg hemispheres in $\mathbb{H}^4$ such that the "vertical cross sections" map

$$\varphi : \widetilde{\mathcal{W}} \to \coprod_n \mathrm{TG}(n \cdot H^2, \mathbb{H}^4) \to \coprod_n \mathbb{R}_{\geq 0} \times \mathrm{Map}(n \cdot S^1, \mathbb{R}^3)$$

is evidently $\mathbb{G}$-equivariant. It was essentially claimed by Freedman and Skora that φ provides a (nonequivariant) deformation retract of $\widetilde{\mathcal{W}}^{\mathrm{triv}}$ onto $\widetilde{\mathcal{W}}$; we prove this statement, and note that the deformation retract is manifestly $\mathbb{G}$-equivariant.

To prove this, we have a few things to check:

1. The associated map $\varphi(t) : \widetilde{\mathcal{W}} \to \mathbb{H}^4 \to \{t\} \times \mathrm{Map}(n \cdot S^1, \mathbb{R}^3)$ has image contained in $\widetilde{\mathcal{W}}$ when $t \neq r$.
2. there exists a "stopping time" function $s : \widetilde{\mathcal{W}} \to \mathbb{R}_{\geq 0}$ such that the map
$$\varphi_s(w)(t) = \varphi(w)(\min(s(w), t))$$
3. $\varphi(w)(s(w)) \in \widetilde{\mathcal{W}}_\varepsilon^{\mathrm{triv}}$.
4. φ_s restricts to the identity on $\widetilde{\mathcal{W}}_\varepsilon^{\mathrm{triv}}$.

The first one is the only interesting point, so it is the only one we'll check. We may do this by extremely hands-on intersecction theory: it suffices to prove that the hemisphere configurations associated with trivial-linking weaves are all nonintersecting.

Fix $w \in \widetilde{\mathcal{W}}$ a weave, and suppose the associated hemisphere configuration has intersections. Such intersections can't be transverse; transverse intersections would be geodesic, and all compact geodesics of $\mathbb{H}^4$, intersect the boundary, which would yield intersection of the components of w.

Non-transverse intersections hit a single point; such intersections correspond with "crossing switch" singularities in the space of singular links. These turn non-linked pairs into linked pairs; since all of the links in $\mathcal{W}_\varepsilon^{\mathrm{triv}}$ and $\mathcal{W}^{\mathrm{triv}}$ are pairwise unlinked, this can't occur. This is horribly written! ☐

Proposition 3.6. *There is an equivalence $\left(\widetilde{\mathcal{W}}_\varepsilon^{\mathrm{triv}}\right)_S^H \simeq \left(\mathbb{RP}^2\right)^{S_H} \times (\mathbb{R}_3)_H$.*

Corollary 3.7. *There is a single S-indexed H-weave with no linking; in particular, every H-weave with no linking is equivalent to one with trivial underlying H-link.*

Corollary 3.8. *No nontrival H-link with trivial linking graph is in the image of $\widetilde{\mathcal{W}}$.*