From $\mathbb{Q}$ to $\mathbb{R}$: a leisurely review of Cauchy completion

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Abstract

For $\mathcal{V}$ a fixed Bénabou cosmos, we review the construction of the $\mathcal{V}\mathbf{Cat}$-enriched categories $\underline{\mathcal{V}\mathbf{Cat}} \subset \underline{\mathcal{V}\mathbf{Prof}}$ of small $\mathcal{V}$-enriched categories and functors or profunctors, respectively. For $\mathcal{C}$ a $\mathcal{V}$-category, we go on to define the *Cauchy completion* $\overline{\mathcal{C}} \subset \underline{\mathcal{V}\mathbf{Prof}}(I, \mathcal{C})$ to be the full $\mathcal{V}$-subcategory of $\mathcal{V}$-profunctors admitting a right adjoint. For $\mathcal{V} = ([0,\infty], \geq, +)$, this recovers the Cauchy completion of Lawvere metric spaces, and after symmetrization, this recovers the Cauchy completion of ordinary metric spaces.

We define a type of weighted colimit, called an *absolute colimit*, which categorifies the property that all short maps preserve limits of Cauchy sequences. We state a universal property realizing the Cauchy completion as the *absolute cocompletion*, recovering the traditional definition of Cauchy complete metric spaces and the traditional universal property satisfied by the Cauchy completion of a metric space.

Time-permitting, we go on to characterize the Cauchy completion of categories, which is computed by the idempotent-splitting completion. We further go on to characterize the Cauchy completion of a ring as a preadditive category, which is computed by the category of finitely generated projective modules.

1 Metric spaces and Cauchy completion

The primary reference for this section of the talk is [2], though the structure of the talk is meant to diverge heavily from this reference, and contain perspectives not described therein. We begin with some motivation for a convenient category of metric spaces.

One disadvantage of the category **Met** of metric spaces and short maps is that it fails to have even *weak coproducts*; for M, N nonempty metric spaces, there are many metrics on $M \sqcup N$ restricting to the given metrics on M and N, but for any such metric, there exists a metric space P and a pair of short maps $M, N \to P$ such that the disjoint-union map $M \sqcup N \to P$ is not short;[1] no matter what metric you choose, M and N will have points which are "too close to each other!" To rectify this, we have to allow points of M and N to have infinite distance from each other.

Another disadvantage is given simply by the fact that some situations are naturally viewed as distance, but are not symmetric; one prominent example is travel time. We can rectify this by relaxing the symmetry requirement of a metric.

A third is that all metric spaces are Hausdorff, but we sometimes encounter topological spaces which themselves are not Hausdorff, and hence they can't possibly be metrizable; this is ubiquitous in algebraic geometry. We can rectify this by relaxing the requirement that $x, y \in M$ distinct must be separated by nonzero distance.

We first define the convenient category of metric spaces alluded to above, called *Lawvere metric spaces*. We go on to define Cauchy completeness in this setting, and construct a left adjoint to the inclusion of Cauchy complete Lawvere metric spaces into Lawvere metric spaces, called *Cauchy completion*. After this, we define the $\mathcal{V}\mathbf{Cat}$-enriched categories $\underline{\mathcal{V}\mathbf{Cat}} \subset \underline{\mathcal{V}\mathbf{Prof}}$ of $\mathcal{V}$-functors and $\mathcal{V}$-profunctors. Then, we define weighted colimits and absolute colimits, and state the universal property of the Cauchy completion of a category.

1.1 Lawvere metric spaces

Let $\mathcal{R} := [0, \infty]$ be the extended nonnegative reals.

Definition 1.1. A *Lawvere metric space* is a set X together with a function $d : X \times X \to \mathcal{R}$ such that

1. (identity) for all $x \in X$,
$$d(x, x) = 0,$$
and

[1] To construct this, pick points $m \in M, n \in N$. Then, we construct the metric space $2 = \{*_m, *_n\}$ with $d(*_m, *_n) = 2d(m, n)$.

2. (triangle inequality) for all $x, y, z \in X$,

$$d(y,z) + d(x,y) \leq d(x,z).$$

If (X, d_X) and (Y, d_Y) are Lawvere metric spaces, then a map $f : X \to Y$ is called a *short map* if for all $x, y \in X$,

$$d_X(x,y) \geq d_Y(f(x), f(y)).$$

A Lawvere metric space is a *quasimetric space* if, in addition, it satisfies the following conditions:

3. (finite distance)

$$d(X,X) \subset [0,\infty),$$

and

4. (separation) for all distinct points $x, y \in X$,

$$d(x,y) > 0.$$

A quasimetric space is an *ordinary metric space* if, in addition, it satisfies the following condition:

5. (symmetry) for all $x, y \in X$,

$$d(x,y) = d(y,x).$$

For reasons that will become clear later, let $\mathcal{R}\mathbf{Cat}$ denote the category of Lawvere metric spaces and short maps. Let $\mathbf{Met}, q\mathbf{Met} \subset \mathcal{R}\mathbf{Cat}$ denote the full subcategory of ordinary metric spaces and quasimetric spaces, respectively. Note that an isomorphism in $\mathcal{R}\mathbf{Cat}$ is precisely an isometry, i.e. a distance-preserving bijection.

There is an involution $(-)^{\mathrm{op}} : \mathcal{R}\mathbf{Cat} \to \mathcal{R}\mathbf{Cat}$ taking a Lawvere metric space L to the Lawvere metric space L^{op} with the same underlying set as L, and with distances

$$d_{L^{\mathrm{op}}}(x,y) := d_L(y,x).$$

The category $\mathcal{R}\mathbf{Cat}$ is a monoidal closed category, with monoidal product $\otimes : \mathcal{R}\mathbf{Cat} \times \mathcal{R}\mathbf{Cat} \to \mathcal{R}\mathbf{Cat}$ where $K \otimes L$ has points $K \times L$ and distance

$$d_{K \times L}(k \times \ell, k' \times \ell') := d_K(k,k') + d_L(\ell,\ell').$$

This has right adjoint $\underline{\mathcal{R}\mathbf{Cat}}(L,M) = [L,M]$ of short maps with distance given by the asymmetric sup norm

$$d(\varphi,\theta) = \max\left(0, \sup_{x \in X}\{\theta(x) - \varphi(x)\}\right)$$

1.2 Cauchy completion

Lawvere metric spaces, like ordinary metric spaces, may have *holes*:

Definition 1.2. Let L be a Lawvere metric space. A *Cauchy sequence in* L is a map of sets $p_{(-)} : \mathbb{N} \to L$ such that, for all ε, there exists some $N_\varepsilon \in \mathbb{N}$ such that

$$d(p_n, p_m) < \varepsilon$$

whenever $n, m > N_\varepsilon$. L is said to be *Cauchy complete* if every Cauchy sequence in L converges to a point in L.

Remark. Let $\{p_n\}$ be a Cauchy sequence in L. Then, $\{p_n\}$ defines a pair of short maps $h_p : L^{\mathrm{op}} \to \mathcal{R}$ and $h^p : L \to \mathcal{R}$ computed by the formulas

$$h_p(x) = \lim_n d(x, p_n);$$
$$h^p(x) = \lim_n d(p_n, x).$$

These form the missing data in a Lawvere metric space $L \cup \{p\}$ where $\{p_n\}$ attains a limit at point p. In particular, a map from L to a Cauchy complete metric space factors uniquely through $L \cup \{p\}$.

Let $\mathcal{R}\mathbf{Cat}_{\text{Cauchy}} \subset \mathcal{R}\mathbf{Cat}$ be the full subcategory of Cauchy complete Lawvere metric spaces, and similarly for $q\mathbf{Met}_{\text{Cauchy}}$ and $\mathbf{Met}_{\text{Cauchy}}$. There is a commutative diagram of categories

$$\begin{array}{ccccc} \mathbf{Met}_{\text{Cauchy}} & \hookrightarrow & q\mathbf{Met}_{\text{Cauchy}} & \hookrightarrow & \mathcal{R}\mathbf{Cat}_{\text{Cauchy}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Met} & \hookrightarrow & q\mathbf{Met} & \hookrightarrow & \mathcal{R}\mathbf{Cat} \end{array}$$

A left adjoint to a vertical arrow in this diagram will be called *Cauchy completion*. Explicitly, if L is a Lawvere metric space, then the *Cauchy completion of* L will be the Cauchy complete Lawvere metric space $L \to \overline{L}$ under L such that all maps $L \to C$ from L to a Cauchy complete Lawvere metric space C factor uniquely through $\overline{L}$:

$$\begin{array}{ccc} L & \longrightarrow & \overline{L} \\ & \searrow & \downarrow \exists! \\ & & C \end{array}$$

Let $\sigma : q\mathbf{Met} \to \mathbf{Met}$ be the *symmetrization* functor taking a quasimetric space Q to the metric space σQ on the same space, with metric $d_{\sigma Q}(x, y) = \min(d_Q(x, y), d_Q(y, x))$. The following more-or-less classical theorem establishes existence of Cauchy completion.

Theorem 1.3.

1. *The Cauchy completion* $\overline{(-)} : \mathcal{R}\mathbf{Cat} \to \mathcal{R}\mathbf{Cat}_{Cauchy}$ *is given on objects by the subspace*

$$\overline{L} \subset \mathcal{R}\mathbf{Cat}(L^{\text{op}}, \mathcal{R})$$

   of short maps $h_p : L^{\text{op}} \to \mathcal{R}$ *such that there exists a short map* $h^p : L \to \mathcal{R}$ *satisfying the properties*
   - *(triangle inequality)* $h_p(x) + h^p(y) \geq d(x, y)$.
   - *(existence of a Cauchy sequence)* $\inf_{x \in X} h_p(x) + h^p(x) = 0$.
2. *The Cauchy completion on Lawvere metric spaces restricts to the Cauchy completion on quasimetric spaces.*
3. *The Cauchy completion of* $\mathbf{Met}$ *is the symmetrization of Cauchy completion of* $q\mathbf{Met}$*:*

$$\begin{array}{ccc} \mathbf{Met}_{Cauchy} & \xleftarrow{\sigma} & q\mathbf{Met}_{Cauchy} \\ \uparrow & & \uparrow \\ \mathbf{Met} & \hookrightarrow & q\mathbf{Met} \end{array}$$

Proof sketch. Part 1 should be viewed intuitively via the *Enriched Yoneda lemma*; there is an isometric embedding $h_{(-)} : L \hookrightarrow \mathcal{R}\mathbf{Cat}(L^{\text{op}}, \mathcal{R})$ sending point p to the "representable function" $h_p(q) = d(q, p)$ and $\mathcal{R}\mathbf{Cat}(L^{\text{op}}, \mathcal{R})$ is a Cauchy complete category. This factors through $\overline{L}$, with the function h^p is given by $h^p(q) = d(p, q)$.

The space $\overline{L}$ is Cauchy complete by a simple argument reminiscent of the classical case; short maps $L \to C$ to a Cauchy sequence factor uniquely through $\overline{L}$ as sketched by the above remark.

Part 2 follows by a simple argument: for $Q \subset L$ a quasimetric subspace of a Lawvere metric space satisfying separation, let $Q \subset \widetilde{Q} \subset L$ be the subspace set of points of L which are finite distance from a point of Q. This is itself a quasimetric space by the triangle inequality; further, any Cauchy sequence in Q attaining a limit in L must have its limit contained in Q, as

$$d(x, L) = \lim d(x_n, L) < \infty$$

for $x \in Q$. In particular, for $L = \overline{Q}$, this implies that $\widetilde{Q}$ is Cauchy complete, so $\widetilde{Q} = \overline{Q}$ and $\overline{Q}$ is a quasimetric space.

Part 3 follows by inspection; alternatively, noting that σ is left adjoint to the inclusion $\mathbf{Met} \hookrightarrow q\mathbf{Met}$, part 3 follows by composition of left adjoints. □

This argument can be reframed as an elegant classical construction: if working directly with ordinary metric spaces, we instead could have embedded M into the metric space

$$\sigma \underline{\mathcal{R}\mathbf{Cat}}\left(M^{\text{op}}, [0, \infty)\right) = \sigma \underline{\mathcal{R}\mathbf{Cat}}\left(M, [0, \infty)\right) = \underline{\mathbf{Met}}\left(M, [0, \infty)\right),$$

i.e. the short maps $M \to [0, \infty)$ under the sup norm; this is a complete metric space, and we could have taken the Cauchy complete closure of M within it.

2 Enriched categories and Cauchy completion

The notation $\mathcal{R}\mathbf{Cat}$ ought to have been suggestive: a Lawvere metric space is *sort of like a category*, but with hom sets replaced with elements of $\mathcal{R}$! This falls into the realm of *Enriched category theory*; a comprehensive reference for the basic theory can be found at [1].

2.1 Enriched categories

Definition 2.1. A *Bénabou cosmos* is a complete and cocomplete closed symmetric monoidal category.

Examples of this include any Grothendieck topos, $(\mathcal{R}\mathbf{Mod}, \otimes, R)$, and $(\mathcal{R}, \leq)$ under addition.

Definition 2.2. Fix $\mathcal{V}$ a Bénabou cosmos. A $\mathcal{V}$-enriched category is the data of

- a class $\mathrm{Ob}\,\mathcal{C}$ of *objects*,
- for each pair $x, y \in \mathrm{Ob}\,\mathcal{C}$, a *hom object* $\mathcal{C}(x, y) \in \mathcal{V}$,
- for each triple $x, y, z \in \mathrm{Ob}\,\mathcal{C}$, a *composition morphism*

$$\circ : \mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z),$$

  and

- for each object $x \in \mathrm{Ob}\,\mathcal{C}$, an *identity arrow*

$$\mathrm{id}_x : I \to \mathrm{Ob}\,\mathcal{C},$$

subject to associativity and unitality of composition up to coherent natural isomorphism.

We will fit these into a $\mathcal{V}\mathbf{Cat}$-enriched category, called $\underline{\mathcal{V}\mathbf{Cat}}$.

Definition 2.3. Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{V}$-categories A $\mathcal{V}$*-functor* $F : \mathcal{C} \to \mathcal{D}$ is a function $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ together with morphisms $F : \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ for each $x, y \in \mathrm{Ob}\,\mathcal{C}$, subject to unitality and compatibility with composition. A $\mathcal{V}$*-natural transformation* $\alpha : F \implies G$ is a class of morphisms $\{\alpha_c : 1 \to \mathcal{D}(Fc, Gc)\}_{c \in C}$ such that $\alpha_{c'} \circ Ff = Ff \circ \alpha_c$ for all $f : 1 \to \mathcal{C}(c, c')$.

The notion of $\mathcal{V}$-naturality generalizes ordinary naturality: when $\mathcal{V} = \mathbf{Set}$, this corresponds with the commutative diagrams

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha} & Gx \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ Fy & \xrightarrow{\alpha} & GY \end{array}$$

for each $f : x \to y$.

With these in mind, we can define the category $\mathcal{V}\mathbf{Cat}$ to have objects given by $\mathcal{V}$-categories and morphisms given by $\mathcal{V}$-functors. Endow on this category the "hom-wise tensor product" monoidal structure $\otimes : \mathcal{V}\mathbf{Cat} \times \mathcal{V}\mathbf{Cat} \to \mathcal{V}\mathbf{Cat}$, where $\mathcal{B} \otimes \mathcal{C}$ has objects $\mathrm{Ob}\,\mathcal{B} \times \mathrm{Ob}\,\mathcal{C}D$ and morphisms

$$\mathcal{B} \otimes \mathcal{C}(b \times b', c \times c') := \mathcal{B}(b, b') \otimes \mathcal{C}(c, c').$$

This has right adjoint $\underline{\mathcal{V}\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) = [\mathcal{C}, \mathcal{D}]$ whose objects are given by the $\mathcal{V}$-functors, with hom-objects given by the ends

$$\begin{aligned} \underline{\mathcal{V} - \mathbf{Cat}}(F, G) : &= \int_c \mathcal{D}(Fc, Gc) \\ &= \mathrm{eq}\left(\prod_{c \in C} \mathcal{D}(Fc, Gc) \rightrightarrows \prod_{c \to c'} \mathcal{D}(Fc, Gc') \right). \end{aligned}$$

Where the equalizer arrows correspond with pre- and post-composition. This endows $\underline{\mathcal{V}\mathbf{Cat}}$ with the structure of a $\mathcal{V}\mathbf{Cat}$-enriched category.

There is a lax monoidal functor called the *underlying category*, $(-)_0 : \mathcal{V}\mathbf{Cat} \to \mathbf{Cat}$ sending $\mathcal{C}$ to the category having objects $\mathrm{Ob}\,\mathcal{C}$ and morphisms $x \to y$ given by arrows $I \to \mathcal{C}(x, y)$ in $\mathcal{V}$. Via change of base, $\mathcal{V}\mathbf{Cat}$ enriched categories have underlying 2-categories, and $(-)_0$ is 2-functorial.

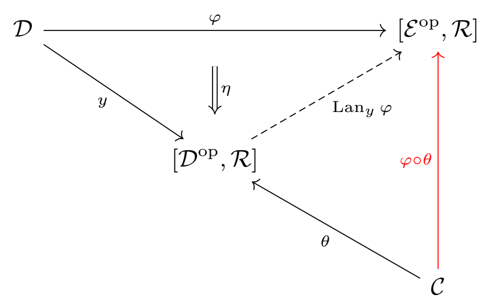
The underlying 2-category of $\mathcal{V}\mathbf{Cat}$ has objects given by $\mathcal{V}$-categories, morphisms given by $\mathcal{V}$-functors, and 2-cells given by $\mathcal{V}$-natural transformations.

2.2 Enriched profunctors

We've covered enriched categorical constructions generalizing the space of short maps, but we haven't considered the *Yoneda perspective* employed for metric spaces yet; to do so, we reframe this perspective a bit. Given a point $p : * \to L$ of a Lawvere metric space, we replace p with a *propoint* $* \to \underline{\mathcal{R}\mathbf{Cat}}(L^{\text{op}}, \mathcal{R})$. In order to describe the subspace of these maps that constitutes the Cauchy completion, we spend a bit of time defining the generalization of these.

Definition 2.4. Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-profunctor* $F : \mathcal{C} \nrightarrow \mathcal{D}$ is a $\mathcal{V}$-functor $\mathcal{C} \to \underline{\mathcal{V}\mathbf{Cat}}(\mathcal{D}^{\text{op}}, \mathcal{R})$, or equivalently, a $\mathcal{V}$-functor $\mathcal{C} \otimes \mathcal{D}^{\text{op}} \to \mathcal{R}$. The $\underline{\mathcal{V}\mathbf{Cat}}$-category $\underline{\mathcal{V}\mathbf{Prof}}$ has:

- Objects given by $\mathcal{V}$-categories.
- Morphisms given by
$$\underline{\mathcal{V}\mathbf{Prof}}(\mathcal{C}, \mathcal{D}) = \underline{\mathcal{V}\mathbf{Cat}}(\mathcal{C} \otimes \mathcal{D}^{\text{op}}),$$
  and
- compositions of morphisms given on objects by the left Kan extension



where y refers to the enriched Yoneda embedding. In particular, it's computed by the coend

$$\begin{aligned}
\varphi \circ \theta(-,-) &= \int^d \varphi(d,-) \otimes \theta(-,d) \\
&= \text{coeq}\left(\coprod_{d \to d'} \varphi(d,-) \otimes \theta(-,d') \rightrightarrows \coprod_{d \in \mathcal{D}} \varphi(d,-) \otimes \theta(-,d) \right)
\end{aligned}$$

This may be motivated in the case that $\mathcal{V} = \mathbf{Ab}$ and $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are rings, in which case $\underline{\mathbf{AbProf}}(\mathcal{C}, \mathcal{D})$ is the preadditive category $\mathcal{C} - \mathbf{Mod} - \mathcal{D}$ of $(\mathcal{C}, \mathcal{D})$-bimodules, and composition corresponds with the tensor product.

Now that we've categorified propoints, we can attempt to categorify propoints within the Cauchy closure of L; the propoints
$$h_p : * \to [L^{\text{op}}, \mathcal{R}] = \underline{\mathcal{R}\mathbf{Prof}}(*, L)$$
in $\overline{L}$ are precisely those with complimentary propoints

$$h^p : * \to [L, \mathcal{R}] = \underline{\mathcal{R}\mathbf{Prof}}(L, *)$$

satisfying some conditions:

1. (existence of a Cauchy sequence)
$$0 \geq h^p \circ h_p(*) = \inf_x h_p(x) + h^p(x).$$
   This is equivalent to existence of a $\mathcal{V}$-natural transformation
$$\begin{pmatrix} * \\ \big\downarrow{\scriptstyle h_*(*)} \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} * \\ \big\downarrow{\scriptstyle h^p \circ h_p(*)} \\ \inf_x h_p(x) + h^p(x) \end{pmatrix}$$

This is precisely a *unit map*

$$\begin{pmatrix} * \\ \not\downarrow \\ * \end{pmatrix} \Longrightarrow \begin{pmatrix} * \\ \not\downarrow \\ L \\ \not\downarrow \\ * \end{pmatrix}$$

in the bicategory of profunctors.

2. (the triangle inequality) $h_p(x) + h^p(y) \geq d(x,y)$; this can be reformulated as the existence of a $\mathcal{V}$-natural transformation

$$\begin{pmatrix} (x,y) \\ \big\downarrow{\scriptstyle h_p \circ h^p(x,y)} \\ h_p(x) + h^p(y) \end{pmatrix} \Longrightarrow \begin{pmatrix} (x,y) \\ \big\downarrow{\scriptstyle y} \\ d(x,y) \end{pmatrix}$$

This is precisely a *counit map*

$$\begin{pmatrix} L \\ \not\downarrow \\ * \\ \not\downarrow \\ L \end{pmatrix} \Longrightarrow \begin{pmatrix} L \\ \not\downarrow \\ L \end{pmatrix}$$

in the bicategory of profunctors.

Together, these establish that the Cauchy completion $\overline{L} \subset \underline{\mathcal{R}\mathbf{Prof}(*, L)}$ is precisely the subspace on propoints who have a right adjoint in $\mathcal{R}\mathbf{Prof}(*, L)$.[2] This motivates the following definition:

Definition 2.5. Let $\mathcal{C}$ be a $\mathcal{V}$-category. The *Cauchy completion of* $\mathcal{C}$ is the full $\mathcal{V}$-subcategory

$$\overline{\mathcal{C}} \subset \underline{\mathcal{V}\mathbf{Prof}}(*, L)$$

of $\mathcal{V}$-profunctors which admit a right adjoint.

Now we're in business! This behaves nicely:

Theorem 2.6. *Let $\mathcal{C}$ be a $\mathcal{V}$-category. Then,*

$$[\mathcal{C}^{op}, \mathcal{V}] = \left[\overline{\mathcal{C}}^{\mathrm{op}}, \mathcal{V}\right].$$

We won't prove this largely formal fact; it does immediately yield some useful corollaries. Following the Australian terminology in viewing profunctors as *modules*, the latter corollary may be viewed as a characterization of *Morita equivalence*:

Corollary 2.7. *Fix $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$.*

1. $\overline{\overline{\mathcal{C}}} = \overline{\mathcal{C}}$.
2. $[\mathcal{C}, \mathcal{V}] \simeq [\mathcal{D}, \mathcal{V}]$ *if and only if* $\overline{\mathcal{C}} = \overline{\mathcal{D}}$.

To continue with the unexplained metaphor, call a profunctor $h \in \underline{\mathcal{V}\mathbf{Prof}(*, L)}$ a *small-projective* if it represents a functor which preserves small weighted colimits. The following is true, and will eventually establish the Cauchy completion of a ring as the associated category of projective modules:

Theorem 2.8. *The Cauchy completion $\overline{\mathcal{C}} \subset \underline{\mathcal{V}\mathbf{Prof}(*, L)}$ is the full $\mathcal{V}$-subcategory consisting of small projectives.*

This is an incredibly useful tool in wrangling examples. But examples are lame; let's discuss a universal property first instead.

[2]We don't need to address commuttativity as $\mathcal{R}$ is a poset, so all diagrams in $\mathcal{R}$ commute.

2.3 Another universal property: limit points as weighted colimits

We recall the correct notion of *enriched colimits*:

Definition 2.9. Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $F : J \to \mathcal{C}$ and $W : J^{\mathrm{op}} \to \mathcal{V}$ be $\mathcal{V}$-functors. Then, a *colimit of F weighted by W* is an object $\operatorname{colim}_W F \in \mathcal{C}$ together with a $\mathcal{V}$-natural isomorphism

$$\mathcal{C}(\operatorname{colim}_W F, c) \simeq [J^{\mathrm{op}}, \mathcal{V}]\,(W, \mathcal{C}(F-, c)).$$

Say that a $\mathcal{V}$-category is *cocomplete* if all small diagrams in it have colimits for all weights. Say that a $\mathcal{V}$-functor is *cocontinuous* if it preserves all small weighted colimits.

When $\mathcal{V} = \mathbf{Set}$ and $W = *$ is the constant functor at the monoidal unit $*$, this is precisely the statement that the cocones $* \implies \mathcal{C}(F-, c)$ are in natural bijection with the morphisms $\operatorname{colim}_* F \to c$, i.e. we recover the notion of ordinary colimits. It's known that all weighted colimits of ordinary categories are computed by ordinary colimits, but this fails to be true in the enriched case.

These correspond with something concrete in the $\mathcal{R}$-enriched case:

Example 2.10:

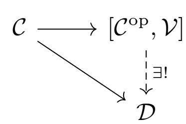
Let L be a Lawvere metric space, $p \in \overline{L}$ a point, and $f : L \to M$ a short map. Then, the profunctor $h_p : L^{\mathrm{op}} \to \mathcal{V}$ together with the short map f correspond with a diagram in M together with a weight; if $p_n \to p$ is a sequence witnessing p as an element of the Cauchy completion of L, then $f(p_n)$ is a Cauchy sequence, which attains a limit if and only if f attains a h_p-weighted colimit, in which case the two coincide. Furthermore, if $p \in L$, then $\operatorname{colim}_{h_p} f = f(p)$.

Choosing $f = \mathrm{id}$, this illustrates that limits of Cauchy sequences occur as instances of weighted colimits, which happen to be preserved by every $\mathcal{R}$-functor.

Remark. We continue to emphasize the space $[L^{\mathrm{op}}, \mathcal{R}]$, at the expense of requiring some translation between $[\mathbb{N}, L]$ and $[L^{\mathrm{op}}, \mathcal{R}]$; alternatively, a sequence $p_{(-)} : \mathbb{N} \to L$ is Cauchy if and only if there exist weights witnessing it as a Cauchy sequence in a "forward" and "backwards" way, in which case the limit and colimit weighted by these limits both correspond with the traditional limit of $p_{(-)}$; see [3].

With this in mind, we can cite the following theorem:

Theorem 2.11 (The yoneda functor is the free cocompletion)**.** *Suppose $\mathcal{C}$ is a small $\mathcal{V}$-category. Then, for every $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$ into a cocomplete category, there is a cocontinuous $\mathcal{V}$-functor $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{D}$, unique up to $\mathcal{V}$-natural isomorphism, causing the following diagram to commute up to $\mathcal{V}$-natural isomorphism:*



This sledgehammer is capable of proving that $[L^{\mathrm{op}}, \mathcal{R}]$ is Cauchy complete! We may take a subspace consisting only of those colimits which correspond with limits of Cauchy sequences, i.e. those colimits that are preserved by every short map, motivating the following definition:

Definition 2.12. A colimit diagram is *absolute* if it is preserved by every $\mathcal{V}$-functor. A functor $W : J \to \mathcal{V}$ is an *absolute weight* if every W-weighted colimit is absolute. A $\mathcal{V}$-category $\mathcal{C}$ is *absolutely cocomplete* if all diagrams in $\mathcal{C}$ together with absolute weights attain colimits.

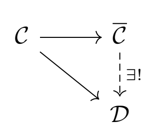
The following theorem is a general fact:

Theorem 2.13. *Let $F : \mathcal{C} \nrightarrow \mathcal{D}$ be a $\mathcal{V}$-profunctor. The following are equivalent:*

1. *F is a weight for absolute colimits.*
2. *F has a right adjoint in $\mathcal{V}$**Prof**.*

Applying this for $\mathcal{C} = I$ and shuffling things around, we obtain the following universal property:

Corollary 2.14 (The Cauchy completion is the absolute cocompletion)**.** *Suppose $\mathcal{C}$ is a small $\mathcal{V}$-category. Then, for every $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$ into an absolutely cocomplete category, there is a $\mathcal{V}$-functor $\overline{\mathcal{C}} \to \mathcal{D}$, unique up to $\mathcal{V}$-natural isomorphism, causing the following diagram to commute up to $\mathcal{V}$-natural isomorphism:*



2.4 Some examples

Theorem 2.15. *Let $\mathcal{C}$ be a small ordinary category. Then, the Cauchy completion of $\mathcal{C}$ is the idempotent completion of $\mathcal{C}$.*

Proof sketch. The idempotent completion is computed by the subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ consisting of retracts of representables. Representables are small-projective since the contravariant hom is cocontinuous, and retracts of cocontinuous functors are cocontinuous, so the idempotent completion is contained in $\overline{\mathcal{C}}$.

The converse proceeds by Yoneda trickery: for $W : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ a small-projective presheaf, we have

$$[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(W, W) = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(W, \mathrm{colim}_W\, y) = \mathrm{colim}_W\, [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(W, y);$$

the identity must correspond with some map

$$I \to \mathrm{colim}_W\, [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]\,(W, y).$$

In particular, since colimits in **Set** are computed by quotients of disjoint unions, this must correspond with some map of presheaves $W \to h_x$ for some object x; naturality then implies that this is a section of the coprojection $h_x \to W$, so W is a retract of a representable. □

According to nLab, this holds whenever $\mathcal{V}$ is a cartesian cosmos where the terminal object is small-projective.

Theorem 2.16. *Let $\mathcal{C}$ be an* **Ab***-category. Then, the Cauchy completion of $\mathcal{C}$ is the idempotent and finite direct-sum cocompletion of $\mathcal{C}$.*

Proof sketch. First note that finite direct sums and idempotent splittings are absolute colimits, so the described cocompletion is contained in $\overline{\mathcal{C}}$. As in the strategy over **Set**, we write the expression

$$[\mathcal{C}^{\mathrm{op}}, \mathbf{Ab}]\,(W, W) = [\mathcal{C}^{\mathrm{op}}, \mathbf{Ab}]\,(W, \mathrm{colim}_W\, y) = \mathrm{colim}_W\, [\mathcal{C}^{\mathrm{op}}, \mathbf{Ab}]\,(W, y);$$

colimits over **Ab** are quotients of direct sums, so there are finitely many objects $x_1, \ldots, x_n$ such that this expresses W as a retract of $\bigoplus_i h_{x_i}$, as desired. □

In particular, for R a ring, $\overline{R}$ is the category of finitely generated projective R-modules.

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