MACKEY FUNCTORS AND THE TOM DIECK SPLITTING

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Abstract. We introduce the category of *G-Mackey functors* along with a functor $\Sigma^\infty_{\underline{\rho}} : \mathscr{C}_G(\mathcal{C}) \to \mathscr{M}_G(\mathcal{C})$ from coefficient systems from Mackey functors. Using isotropy separation, we construct an equivalence

$$\left(\Sigma^\infty_{\underline{\rho}} X\right)^G \simeq \prod_{H \in \mathrm{Conj}(G)} \Sigma^\infty X^H_{hW_G H}$$

and derive as a corollary the classical computation

$$A(G) \simeq \pi_0^G(\mathbb{S}),$$

where $A(G) \simeq \mathbb{Z}[\mathrm{Conj}(G)]$ is the Burnside ring, strengthening the Segal conjecture.

Our goal in this talk[1] is to develop some of the techniques central to stable equivariant homotopy theory, naturally generalizing along the way from G-spectra to more general categories of Mackey functors, as in [Bar14; Bar+16; Gla17; Gla18; NS22]. We are interested in proving the *Tom Dieck splitting*, originally proved for homotopy groups in [Die75].

Theorem A. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{F}$ be a family of subgroups. Then, the free functor $\Sigma^\infty_{\underline{\rho},+} : \mathrm{Fun}(\mathcal{F}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{\mathcal{F}}), \mathrm{CMon}(\mathcal{C}))$ satisfies the splitting*

$$\left(\Sigma^\infty_{\underline{\rho},+} X\right)^G \simeq \prod_{(H) \in \mathcal{F}^\simeq} \Sigma^\infty X^H_{hW_{\mathcal{F}} H}$$

We will derive the following well-known corollary for finite groups:

Corollary B. *The 0th genuine stable homotopy group is the Burnside ring*

$$A(G) \simeq \mathbb{Z}[\mathrm{Conj. classes of } G] \xrightarrow{\sim} \pi_0^G(\mathbb{S}).$$

Much of the contents of this paper are original only as synthesis of old thought and new technology. The old thought (i.e. the content of our proof of the tom Dieck splitting) can be sourced at [Deb17, § 2.7], and the new technology can be sourced at [Gla17].

1. Mackey functors, Coefficient systems, and equivariant homotopy theory

1.1. Unstable equivariant homotopy theory.

We begin with some motivational content:

Definition 1.1. Let G be a finite group, and let X, Y be topological spaces with G-action.

(1) a *G-homotopy equivalence* is a G-equivariant map $f : X \to Y$ possessing a G-equivariant map $Y \to X$ and G-equivariant homotopies $1_X \simeq gf$ and $1_Y \simeq fg$.
(2) a *G-weak homotopy equivalence* is a G-equivariant map $f : X \to Y$ whose point-set fixed points $X^H \to Y^H$ is a weak equivalence for every subgroup $H \subset G$.
(3) a *naive G-weak equivalence* is a G-equivariant map $f : X \to Y$ whose underlying $X^e \to Y^e$ is a weak equivalence.

We'd like to repeat all of classical homotopy theory in this setting. To do so, let's define CW complexes:

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[1]Delivered to the zygotop seminar.

Definition 1.2. Let G be a finite group, and let $\mathbf{Set}_G$ be the category of sets with G-action. Then, a *G-CW complex* is a sequence $X_{-1} = \varnothing, X_0, X_1, \ldots$ together with pushout squares

$$\begin{array}{ccc} T_n \times S^{n-1} & \hookrightarrow & T_n \times D^n \\ \downarrow & \lrcorner & \downarrow \\ X_{n-1} & \longrightarrow & X_n \end{array}$$

of topological G-spaces, where $T_n \in \mathbf{Set}_G$ and G acts on S^{n-1} and D^n trivially.

These notions satisfy a large collection of generalizations of the classical theorems of unstable homotopy theory, which we list here:

- Every compactly generated and weakly Hausdorff G-space is G-weakly equivalent to a G-CW complex [May96, Thm 3.6].
- Every compact smooth G-manifold is G-weakly equivalent to a finite G-CW complex [Ill83].
- Continuous maps between (relative) G-CW complexes are homotopic to cellular maps [May96, Thm 3.4].
- A map between G-CW complexes is a G-weak equivalence if and only if it's a G-homotopy equivalence [May96, Cor 3.3].

For our purposes, we collect these together with some other results into an *Omnibus theorem*. Define the ∞-category $\mathcal{S}_G$ as the homotopy-coherent nerve of the topological category whose objects are CGWH topological spaces with G-action and whose mapping spaces are $\mathrm{Map}(X,Y)^G$, i.e. the space of G-equivariant maps.

The *orbit category* is the subcategory $\mathcal{O}_G \subset \mathcal{S}_G$ spanned by the homogeneous G-spaces G/H for $H \subset G$ ranging over the subgroups. The co-yoneda embedding restricted to $\mathcal{O}_G$ yields a functor

$$\gamma : \mathcal{S}_G \to \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}).$$

Note that

$$\mathrm{Map}(G/H, X)^G \simeq X^H,$$

so γ recovers precisely the homotopy types of the fixed points as well as the restriction maps between them. The following combines Elmendorf's theorem and the equivariant Whitehead's theorem

Theorem 1.3 ([Elm83; May96])**.** *γ along with the functors $\mathrm{G-CW} \to \mathbf{Top}_G$ and $\mathrm{G-CW} \to \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S})$ induce equivalences*

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{G-CW}[\mathrm{G-WEQ}^{-1}] \simeq \mathbf{Top}_G[\mathrm{G-EQ}^{-1}]$$

Remark. There is an *unstraightening functor*

$$\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}}$$

landing in cocartesian fibrations to $\mathcal{O}_G^{\mathrm{op}}$ with fibers which are spaces; the total space of this cocartesian fibration is called the *underlying space*. Elmendorf's theorem endows this space with the structure of a genuine G-action.

As a corollary, we acquire a conservative functor

$$\underline{\pi}_* : \mathcal{S}_G \to \mathscr{C}_G(\mathbf{Set})^{\mathbb{N}},$$

where $\mathscr{C}_G(\mathbf{Set}) := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Set})$ is the *category of G-coefficient systems valued in* $\mathbf{Set}$. One might refer to these as the *coefficient system homotopy groups*; it's not too hard to see that $\underline{\pi}_n$ naturally lifts to functors $\mathcal{O}_G^{\mathrm{op}} \to \mathbf{Grp}$ when $n \geq 1$ and $\mathcal{O}_G^{\mathrm{op}} \to \mathbf{Ab}$ when $n \geq 2$.

1.2. **Stable equivariant homotopy.** Let V be a real orthogonal G-representation. Then, we may form the one-point compactification G-space S^V by taking the unit disk $D(V) \subset V$ and taking the quotient by the unit sphere

$$S^V := D(V)/S(V)$$

This, for instance, satisfies $\left(S^V\right)^H \simeq S^{V^H}$. When X is a based G-space, we denote the smash product and mapping based G-space as

$$\Sigma^V X := S^V X, \qquad \Omega^V X := \mathrm{Map}_*(S^V, X)$$

As in motivic homotopy theory, we may consider the stabilization under these functors, or equivalently, under Ω^ρ or Σ^V, where ρ is the regular representation. An early indication of what one might acquire was the *Wirthmüller isomorphism*. In order to state it, temporarily define the *extended Spanier-Whitehead category*

$$\mathcal{SW}_G := \operatorname{colim}\left(\cdots \to \mathcal{S}_{G,*} \xrightarrow{\Sigma^\rho} \mathcal{S}_{G,*} \to \cdots\right)$$

Theorem 1.4 (Wirthmüller isomorphism). *For $V \gg 0$, there is a based transfer map $\tau : S^V \to S^V \wedge G/H_+$ and restriction map $r : S^V \wedge G/H_+ \to S^V$ which together witness G/H_+ as self-dual in $\mathcal{SW}_G$. If $\mathcal{SW}_{\mathcal{O}_G} \subset \mathcal{SW}_G$ denotes the full subcategory spanned by orbits, then the restriction and transfer maps yield an equivalence*

$$\mathcal{SW}_{\mathcal{O}_G} \simeq \operatorname{Span}(\mathcal{O}_G).$$

In particular, if we form the category

$$\mathrm{Sp}_G := \lim\left(\cdots \leftarrow \mathcal{S}_{G,*} \xleftarrow{\Omega^\rho} \mathcal{S}_{G,*} \leftarrow \cdots\right)$$

then similar to Elmendorf's theorem, mapping spectra out of the unreduced suspension spectra of finite G-sets together with restriction and transfer maps yield a functor

$$\gamma : \mathrm{Sp}_G \to \mathrm{Fun}^\times(\operatorname{Span}(\mathbb{F}_G), \mathrm{Sp}),$$

where $\mathbb{F}_G$ denotes the category of finite G-sets. The stable version of Elmendorf's theorem states that this is an equivalence:

Theorem 1.5 ([GM11; Nar16]). *The functor $\gamma : \mathrm{Sp}_G \to \mathrm{Fun}^\times(\operatorname{Span}(\mathbb{F}_G), \mathrm{Sp})$ is an equivalence.*

It is further established in [Nar16] that Sp_G may be computed intrinsically as the *G-stabilization* of $\mathcal{S}_G$, i.e. the universal stable category out of $\mathcal{S}_G$ for which *indexed* products and coproducts agree. This theorem immediately constructs for us a conservative functor

$$\underline{\pi}_* : \mathrm{Sp}_G \to \mathcal{M}_G(\mathbf{Ab})$$

where

$$\mathcal{M}_G(\mathcal{C}) := \mathrm{Fun}^\times(\operatorname{Span}(\mathbb{F}_G), \mathcal{C}).$$

These are called the *Mackey functor stable homotopy groups*. Before developing more of equivariant stable homotopy theory, we take a brief diversion into some of the combinatorics of coefficient systems and Mackey functors.

1.3. **Coefficient systems.** We take this opportunity to depart from the setting of equivariant homotopy theory in our first way: *we take the G out of genuine*.

Definition 1.6. A category $\mathcal{T}$ is *orbital* if the finite coproduct completion $\mathbb{F}_{\mathcal{T}} := \mathcal{T}^{\amalg}$ has all pullbacks. An orbital category $\mathcal{T}$ is *atomic* if every map attaining a section is an equivalence.

If $\mathcal{F} \subset \mathcal{O}_G$ is an initial subcategory of the orbit category of a profinite group (i.e. a collection of subgroups closed under subcojugation), then $\mathcal{F}^{\mathrm{op}}$ is an atomic orbital ∞-category. In order to avoid annoying details while talking about profiniteness or families of subgroups, we simply work in the setting of atomic orbital ∞-categories.

Example 1.7:
Let $G = C_p$. Then, the orbit category of G is given by

$$C_p \circlearrowright [C_p/e] \longrightarrow [C_p/C_p]$$

where $[G/H]$ denotes the homogeneous G-space.

Inspired by this, we make the following structural statement:

Proposition 1.8. *Fix G a group. Let C_G be the category whose objects are the subgroups $H \subset G$ and whose morphism objects* $\mathrm{Hom}(H, H')$ *are given by the morphisms $H \to H'$ computed by conjugation by an element of G, i.e.*

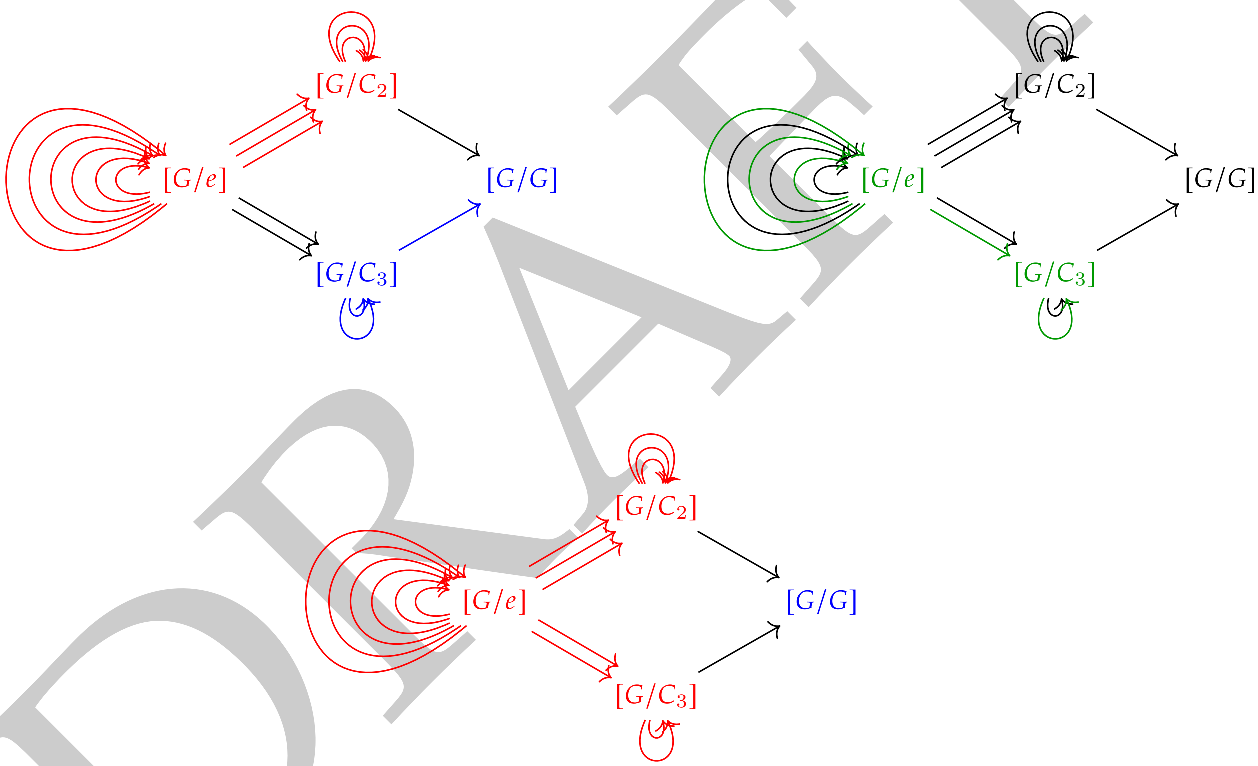
$$\mathrm{Hom}_{C_G}(H, H') := \left\{ g \mid gHg^{-1} \subset H' \right\} / C_G(H).$$

Then, the functor which sends H to the homogeneous space G/H yields an equivalence

$$C_G \xrightarrow{\sim} \mathcal{O}_G^{\mathrm{op}}.$$

In particular, $\mathcal{O}_G$ is atomic orbital, and we may view it as a glued together version of the various conjugacy classes of subgroups of G, with endomorphism monoids recording the Weyl groups of the various subgroups of G. To see this in action, we write down a less trivial example:

Example 1.9:
Let $G = C_6$. We illustrate several orbital subcategories of $\mathcal{O}_G$:



$$i_{\mathcal{F}} : \mathcal{F} \hookrightarrow \mathcal{O}_G,$$
$$i_H : \mathcal{O}_H \hookrightarrow \mathcal{O}_G,$$
$$j_{\mathcal{F}^\perp} : \mathcal{F}^\perp \hookrightarrow \mathcal{O}_G.$$

As illustrated here, if $H \subset G$ is a subgroup, the subgroups of G yield a family $\mathcal{F}_{H \subset G} \subset \mathcal{O}_G$, and when $H \subset G$ is normal, $\mathcal{F}_{H \subset G} \simeq W_H G \times \mathcal{O}_H$; any family possesses a *right $+$-semiorthogonal complement*

$$\mathcal{F}^\perp := \{ V \in \mathcal{O}_G \mid \forall U \in \mathcal{F}, \mathrm{Map}(V, U) = \varnothing \}.$$

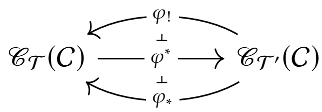
and when $H \subset G$ is normal, $\mathcal{F}^\perp_{H \subset G} \simeq \mathcal{O}_{G/H}$. We will use the inclusion $j_{G/H} : \mathcal{O}_{G/H} \hookrightarrow \mathcal{O}_G$ to construct *fixed points* soon.

Additionally, for any subgroup $H \subset G$, there is a (non-full) subcategory $i_H : \mathcal{O}_H \to \mathcal{O}_G$ spanned by orbits G/H' for $H' \subset H$ with morphisms which are computed by conjugation by elements of H.

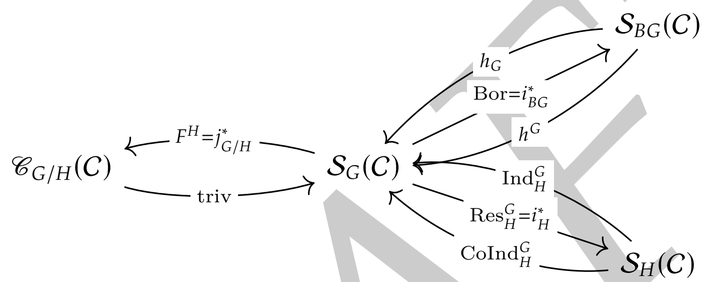
Before moving on to a construction, we write down one more example:

Example 1.10:
Let X be a space. Then, X is atomic orbital by construction, since $X^{\amalg}$ is a groupoid. The category $\mathcal{S}_X := \mathrm{Fun}(X, \mathcal{S})$ is the category of *parameterized spaces over H* from e.g. [ABGHR].

If C is an ∞-category and $\mathcal{T}$ is an atomic orbital ∞-category, let $\mathscr{C}_{\mathcal{T}}(C) := \mathrm{Fun}(\mathcal{T}, C)$ denote the *category of coefficient systems valued in C*. Then, whenever $\varphi : \mathcal{T}' \to \mathcal{T}$ is a functor between atomic orbital ∞-categories, pullback together with left and right Kan extension yield a double adjunction

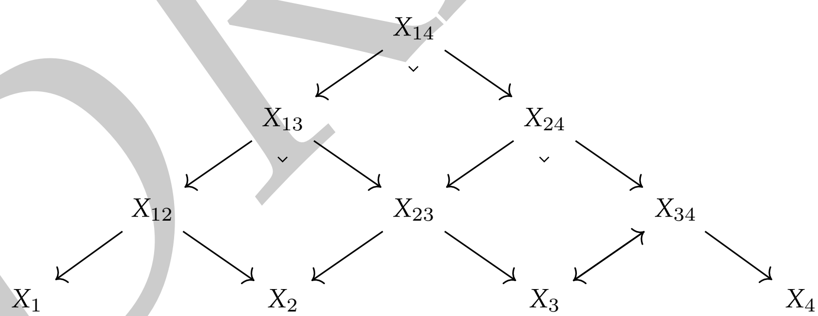


Using these, we define the following *change of universe data*:



We can similarly replace Bor with pullback to an arbitrary family and F^H with pullback to the right-complement of an arbitrary family.

1.4. **Mackey functors and the isotropy separation sequence.** In [GR17], the construction of a *category of correspondences* was sketched, which was latter fully carried out in [Bar14], there called the *effective Burnside category*. When C is a category with pullbacks, we refer to this as $\mathrm{Span}(C)$. This is constructed initially as a *complete Segal space* whose n-simplices can be epitimized by the case $n = 3$:



Here, X_i and X_{ij} are objects in C, all arrows are morphisms in C, and all squares are Cartesian.This recovers the traditional category of spans when C is a 1-category. In general, when $\mathcal{T}$ is an orbital ∞-category, $\mathbb{F}_{\mathcal{T}}$ has pullbacks, so we may make define the *category of $\mathcal{T}$-Mackey functors valued in C*:

$$\mathscr{M}_{\mathcal{T}}(C) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}}), C).$$

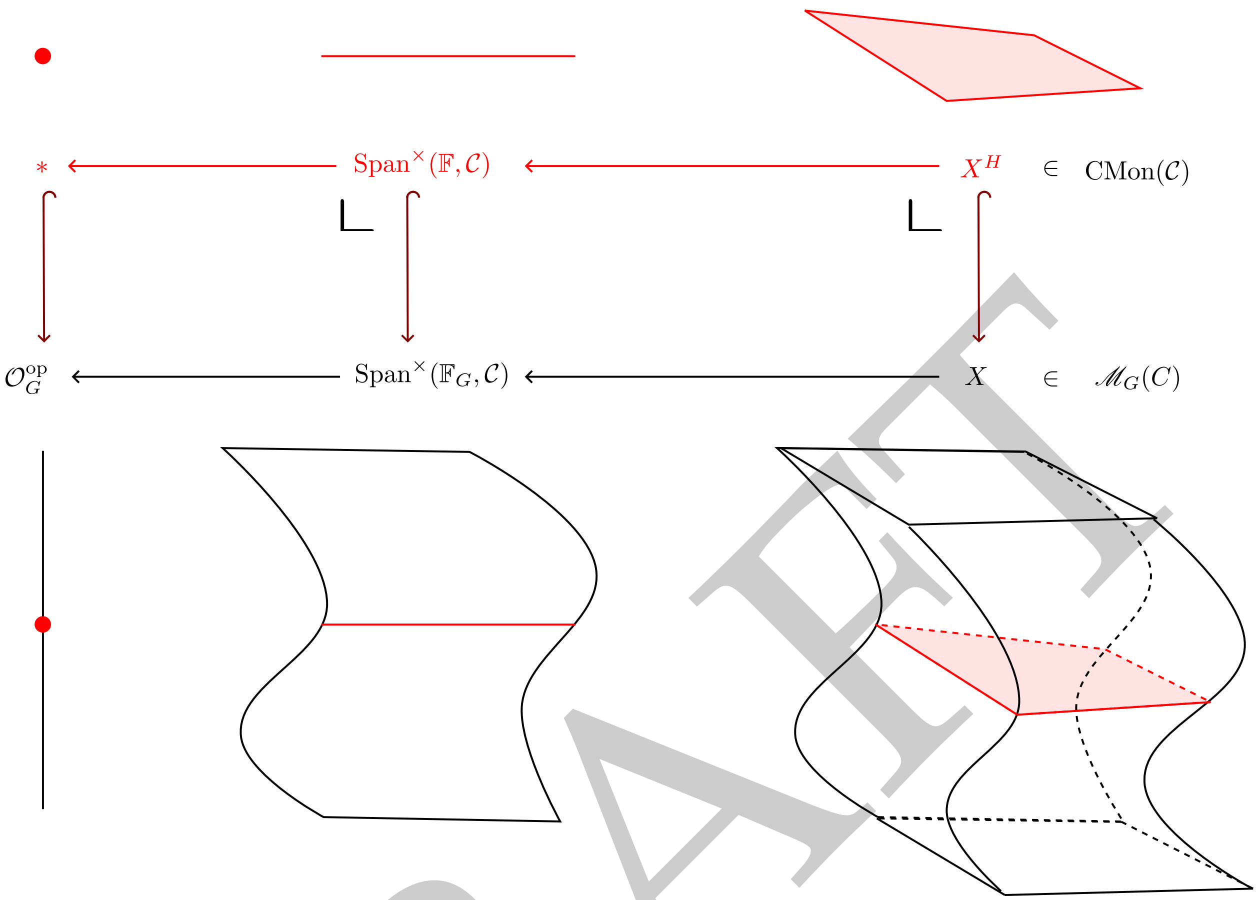
When $\mathcal{T} = O_G^{\mathrm{op}}$, we write $\mathscr{M}_G(C) := \mathscr{M}_{O_G^{\mathrm{op}}}(C)$.

This has a description which may be familiar to representation theorists (see [Deb17]):

Theorem 1.11. *A G-Mackey functor valued in C is equiivalent to the data of two functors*

$$R : \mathbb{F}_G \to C,$$
$$T : \mathbb{F}_G^{\mathrm{op}} \to C,$$

subject to the following conditions:



(i) there is an (unnatural) equivalence $R(S) = T(S)$ *for all finite G-sets* S*;*
(ii) for every pair of finite G-sets S, T*, the canonical map* $R(X) \times R(Y) \to R(X \sqcup Y)$ *is an equivalence; and*
(iii) for every pullback square

$$\begin{array}{ccc} \coprod_{x \in [H \backslash J / K]} G/(H \cap xKx^{-1}) & \xrightarrow{\alpha} & G/H \\ \downarrow \beta & & \downarrow \gamma \\ G/K & \xrightarrow{\delta} & G/J \end{array}$$

there is an identity $R(\beta)T(\alpha) = T(\delta)R(\gamma)$*, i.e. the* double coset formula *holds:*

$$R_K^J T_H^J = \prod_{x \in K \backslash J / H} T^K_{K \cap xHx^{-1}} c_{x,H} R^H_{H \cap x^{-1}Kx}$$

where $c_{x,H} : G/H \to G/(xHx^{-1})$ *is the conjugation action.*

For instance, this theorem immediately allows one to construct a functor

$$\mathbf{Rep}_G(R) \to \mathscr{M}_G(R - \mathbf{Mod}).$$

One quickly arrives at several other algebraic examples, such as $K(\mathbb{Z}[G])$ (or more general equivariant algebraic K-theory), group (co)homology, algebraic K theory of (intermediate extensions of) a Galois extension of fields, etc. In this talk, we will squarely avoid such things, instead choosing to think in terms of the span category description of $\mathscr{M}_{\mathcal{T}}(\mathcal{C})$. However, there is one example where thinking explicitly is quite easy. Let $\iota : \mathcal{T} \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ be the canonical inclusion.

Theorem 1.12. *Let* X *be a space. Then, the functor*

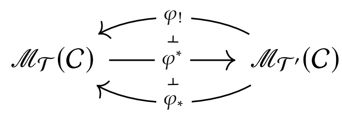
$$\iota^* : \mathscr{M}_X(\mathcal{C}) \to \mathscr{C}_X(\mathrm{CMon}(\mathcal{C}))$$

is an equivalence.

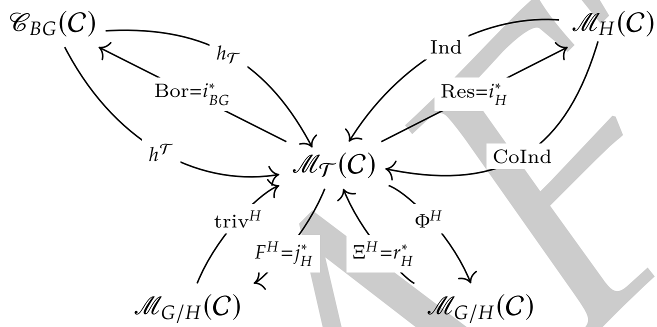
In particular, when $C = \mathrm{Sp}$, this establishes the fact that X-equivariant stable homotopy theory is precisely stable homotopy theory parameterized over X; setting $X := BG$, this establishes that BG-equivariant stable homotopy theory is precisely Borel G-equivariant stable homotopy.

We'd like to use this to construct *change of universe data*. To do so, the following follows from lax limits citation :

Theorem 1.13. *Suppose $\varphi : \mathcal{T}' \to \mathcal{T}$ is a functor between atomic orbital ∞-categories. Then, the functors $\varphi^*, \varphi_*, \varphi_!$ preserve the property of being Mackey functors; in particular, they supply a double adjunction*



Using this, we have change of universe data:



Assume C has an initial object $\varnothing \in C$, and let $\mathcal{U} \subset \mathcal{T}$ be an upward-closed full subcategory. We say that a $\mathcal{T}$-Mackey functor M is *supported on* $\mathcal{N}$ if $M(X) = \varnothing$ for all $x \notin \mathcal{N}$> The following theorem was proved in [Gla17].

Theorem 1.14. *Let $\mathcal{U} \subset \mathcal{T}$ be an upwardly closed subcategory of an atomic orbital ∞-category. Then, the functor $\Xi^{\mathcal{U}}$ is the inclusion of a localizing subcategory consisting of Mackey functors supported on $\mathcal{U}$.*

It is clear by inspection that, when $\mathcal{L} \subset \mathcal{T}$ is a downwardly-closed subcategory and $\mathcal{U}$ is its upwardly-closed complement, the above theorem is equivalent to the statement that $(h_{\mathcal{L}}, \Xi^{\mathcal{U}})$ presents a semiorthogonal decomposition of $\mathcal{M}_{\mathcal{T}}(C)$. The theorem takes the following equivalent form.

Corollary 1.15 (The isotropy separation sequence)**.** *Let $\mathcal{L} \subset \mathcal{T}$ be a downwardly-closed subcategory of an atomic orbital ∞-category, let $\mathcal{U}$ be its upwardly-closed complement, and let C be a stable ∞-category. Then, there is a cofiber sequence of $\mathcal{M}_{\mathcal{T}}(C)$-endofunctors*

$$h_{\mathcal{L}}\mathrm{Bor}^{\mathcal{L}} \to \mathrm{id} \to \Xi^{\mathcal{U}}\Phi^{\mathcal{U}} \tag{1}$$

Proof idea. First, the theory of *nonabelian derived categories* proves that

$$\mathcal{M}_{\mathcal{T}}(C) \simeq \mathrm{Fun}^L(\mathcal{M}_{\mathcal{T}}(\mathrm{CMon}(\mathcal{S})), C).$$

Furthermore, $\Phi^{\mathcal{N}}$ is a localization onto the full subcategory consisting of functors $F : \mathcal{M}_{\mathcal{T}}(\mathrm{CMon}(\mathcal{S}), C)$ taking $\Phi^{\mathcal{N}}$-equivalences to equivalences; since C is stable, if we can prove the theorem in the case $C \simeq \mathrm{CMon}(\mathcal{S})$, then we may conclude that a functor $F : \mathcal{M}_{\mathcal{T}}(\mathrm{CMon}(\mathcal{S}), C)$ takes $\Phi^{\mathcal{N}}$-equivalences to equivalences if and only if it takes Mackey functors concentrated away from $\mathcal{N}$ to 0, i.e. the image of $\Xi^{\mathcal{U}}$ consists of the Mackey functors supported on $\mathcal{U}$.

To prove the corollary in the case that $C = \mathrm{CMon}(\mathcal{S})$, we use cocontinuity of all of the functors involved in order to reduce to checking on corepresentable Mackey functors; there, we may check by hand, as in [Gla17, § B]. □

In particular, if C is compactly generated, this establishes $\Xi\Phi$ as a *finite localization* away from the class of compact C-Mackey functors whose fixed points are concentrated at an object in $\mathcal{L}$. This comes with the following corollary.

Corollary 1.16. *Let $\mathcal{U} \subset \mathcal{T}$ be an upwardly-closed subcategory of an atomic orbital ∞-category and let $\mathcal{C}$ be a compactly generated stable ∞-category. Then, the comparison map*

$$X \otimes \Xi^{\mathcal{U}} 1_{\mathcal{C}} \to \Xi^{\mathcal{U}} X^{\Phi\mathcal{U}}$$

is an equivalence for all $X \in \mathcal{M}_{\mathcal{T}}(\mathcal{C})$.

2. The tom-Dieck splitting

Given $\mathcal{U} \subset \mathcal{T}$ an upwardly-closed subcategory, the $\mathcal{U}$-geometric and genuine fixed points interact. One such way is compatibility with adjoints:

Proposition 2.1. *There is a canonical natural equivalence $F^{\mathcal{U}}\Xi^{\mathcal{U}} \simeq \mathrm{id}_{\mathcal{M}_{\mathcal{U}}(\mathcal{C})}$.*

Proof. By ref , the map $j : \mathcal{U} \to \mathcal{T}$ is a section of $r : \mathcal{T} \to \mathcal{U}$. The functoriality of star pullback then provides natural equivalences

$$\begin{aligned} F^{\mathcal{U}}\Xi^{\mathcal{U}} &= j^* r^* \\ &\simeq (r \circ j)^* \\ &= (\mathrm{id}^*) \\ &\simeq \mathrm{id} \end{aligned}$$

□

Using this, we define a comparison natural transformation

$$\gamma^{\mathcal{U}} : X^{F\mathcal{U}} \xrightarrow{\eta^{F\mathcal{U}}_{(\Phi,\Xi)}} \left(\Xi^{\mathcal{U}} X^{\Phi\mathcal{U}}\right)^{F\mathcal{U}} \simeq X^{\Phi\mathcal{U}}.$$

In this section, we characterize precisely the failure of $\gamma^{\mathcal{U}}$ to be an equivalence on equivariant suspension spectra. We will define the notion of *inductive atomic orbital ∞-categories* and prove the following theorem.

Theorem 2.2. *Suppose $\mathcal{T}$ is an inductive atomic orbital ∞-category and $\mathcal{C}$ is an ∞-category. Write $\Sigma^\infty_\rho : \mathcal{C}_{\mathcal{T}}(\mathcal{C}) \to \mathcal{M}_{\mathcal{T}}(\mathrm{Sp}\,\mathcal{C})$. Then, there is a natural equivalence*

$$\Sigma^\infty_\rho X \simeq \prod_{V \in \mathcal{T}} \Sigma^\infty X^V_{h\,\mathrm{Aut}_{\mathcal{T}} V}$$

We say that an atomic orbital ∞-category $\mathcal{T}$ is *inductive* if there exists a well-ordered ordinal ω and an ω-indexed filtration $\mathrm{Fil}^*\mathcal{T}$ of $\mathcal{T}$ by downward-closed subcategories such that $prn\mathrm{Fil}^i\mathcal{T}^\perp \subset \left(\mathrm{Fil}^{i+1}\mathcal{T}\right)$ is a connected groupoid for all $i \in \mathcal{T}$ and $\mathrm{colim}_{i\in I}\,\mathrm{Fil}^i\mathcal{T} \simeq \mathrm{Fil}^{\sup I}\mathcal{T}$ for all totally ordered chains $I \subset \omega$; that is, *$\mathcal{T}$ is inductive if it is constructed under transfinite induction by adding in upwardly-closed objects*, i.e. it is inductive if we can induct up the collection of downwardly-closed subcategories using the isotropy separation sequence. Examples of this include every family of subgroups of a profinite group.

For the remainder of this section, let $(\mathcal{L}, \mathcal{U})$ be complementary downward and upward closed subcategories of $\mathcal{T}$ and let $\mathcal{X}$ be a final subspace of $\mathcal{L}$. Let $\tilde{\mathcal{U}} := \mathcal{U} \cup \mathcal{X}$ and let $\tilde{\mathcal{L}} := \mathcal{L} - \mathcal{X}$. We have a comparison map

$$\psi^{\mathcal{U}}_X : \Sigma^\infty_\rho \left(X^{F\mathcal{U}}\right) \to \left(\Sigma^\infty_\rho X\right)^{F\mathcal{U}} \xrightarrow{\gamma} \left(\Sigma^\infty_\rho X\right)^{\Phi\mathcal{U}}$$

Lemma 2.3. *Suppose $Y = h_{\mathcal{L}} Z$. Then, there are equivalences making the following diagram commute:*

$$\begin{array}{ccc} \Sigma^\infty_\rho Y^{F\tilde{\mathcal{U}}} & \xrightarrow{\Psi^{\tilde{\mathcal{U}}}_Y} & \left(\Sigma^\infty_\rho Y\right)^{\Phi\tilde{\mathcal{U}}} \\ \downarrow\sim & & \downarrow\sim \\ h_{\mathcal{L}}\Sigma^\infty_\rho Z^{F\mathcal{X}} & \xrightarrow{h_{\mathcal{L}}\Psi^{\mathcal{X}}_Z} & h_{\mathcal{L}}\left(\Sigma^\infty_\rho Z\right)^{\Phi\mathcal{X}} \end{array}$$

Prove this

Proposition 2.4. *When $Y \simeq h_{\mathcal{L}} Z$ for some $Z \in \mathcal{S}_{\mathcal{L}}$, the comparison map $\psi_Y^{\tilde{\mathcal{U}}}$ is an equivalence; in particular, for all objects $V \in \mathcal{T}$, we have*

$$\Sigma^\infty_\rho \left(X^V\right) \simeq \left(\Sigma^\infty_\rho X\right)^{\Phi V}.$$

Proof. By lemma 2.3, it suffices to prove this in the case that $\mathcal{L} = \mathcal{T}$, so that $\tilde{\mathcal{U}} \simeq \mathcal{X}$. In fact, we have $\psi_Y^{\mathcal{X}} \simeq \bigoplus_{V \in \mathcal{X}} \psi_Y^V$, so it suffices to prove that

$$\psi_Y^V : \Sigma^\infty_\rho Y^V \to \left(\Sigma^\infty_\rho Y\right)^{\Phi V}$$

when V is a weakly terminal object. Since everything involved is compatible with colimits, it suffices to prove this in the case that Y is representable, i.e. $Y = T \in \mathcal{T}$.

We compute on the left side and use universal properties on the right side; on the left, we have

$$T^V \simeq \begin{cases} \mathrm{End}(T) & T = V, \\ \varnothing & \text{otherwise,} \end{cases}$$

so that

$$\Sigma^\infty T^V \simeq \begin{cases} \Sigma^\infty \mathrm{End}(T) & T = V, \\ * & \text{otherwise.} \end{cases}$$

On the right, note that by smashing ref and Wirthmüler ref , we have an equivalence

$$\begin{aligned} \Xi^V \left(\Sigma^\infty T\right)^{\Phi V} &\simeq \Sigma^\infty T \otimes \Xi^V \mathbb{S}, \\ &\simeq \mathrm{map}(T, \Xi^V \mathbb{S}); \end{aligned}$$

in particular, we have

$$\begin{aligned} \left(\Sigma^\infty T\right)^{\Phi V} &\simeq \left(\Xi^V \left(\Sigma^\infty T\right)^{\Phi V}\right)^V \\ &\simeq \mathrm{map}(T, \Xi^V \mathbb{S})^V \\ &\simeq \left(\Xi^V \mathbb{S}\right)^{V \times T} \\ &\simeq \begin{cases} \Sigma^\infty \mathrm{End}(T) & T = V, \\ * & \text{otherwise.} \end{cases} \end{aligned}$$

□

In particular, ref provides a splitting of the following sequence, which is the $\mathcal{U}$-fixed points of (1)

$$\begin{array}{ccccc} \left(h_{\tilde{\mathcal{L}}} \mathrm{Bor}^{\tilde{\mathcal{L}}} \Sigma^\infty M\right)^{\mathcal{U}} & \longrightarrow & \left(h_{\mathcal{L}} \mathrm{Bor}^{\mathcal{L}} \Sigma^\infty M\right)^{\mathcal{U}} & \longrightarrow & \left(h_{\mathcal{L}} \mathrm{Bor}^{\mathcal{L}} \Sigma^\infty_\rho M\right)^{\Phi \mathcal{U}} \\ & & & \nwarrow & \downarrow \wr \\ & & & & \left(h_{\mathcal{L}} \mathrm{Bor}^{\mathcal{L}} \Sigma^\infty_\rho M^{\mathcal{U}}\right) \end{array}$$

By the pointwise formula for left Kan extension, we additionally find that

$$\left(\left(\Sigma^\infty h_{\mathcal{L}} M\right)^{\Phi \tilde{\mathcal{U}}}\right)^{\mathcal{T}} \simeq \bigoplus_{V \in \mathcal{X}} \Sigma^\infty M^V_{h\,\mathrm{End}(V)},$$

Altogether, these facts prove the following proposition.

Proposition 2.5. *There is an equivalence*

$$\left(\Sigma^\infty h_{\mathcal{L}} \mathrm{Bor}^{\mathcal{L}} M\right)^{\mathcal{T}} \simeq \left(\Sigma^\infty h_{\tilde{\mathcal{L}}} \mathrm{Bor}^{\tilde{\mathcal{L}}} M\right)^{\mathcal{T}} \oplus \bigoplus_{V \in X} \Sigma^\infty M^V_{h\,\mathrm{Aut}(V)}.$$

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