HOMOTOPICAL ADDITIVITY OF EQUIVARIANT LITTLE DISK OPERADS

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Abstract. Given a finite group G, a pair of orthogonal G-representations V and W, and a G-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, we prove that the natural G-symmetric monoidal forgetful functor

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_V}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_W}(\mathcal{C})$$

is an equivalence, In fact, we define a proper equivariant version of $\mathbb{E}_V$ with finite-index restriction and transfers, and prove the we prove same property for finite-index proper $\mathbb{E}_{V\oplus W}$-algebras.

Moreover, we extend these results to equivariant little disk operads with linear G-tangential structure.

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Foreword to this draft

This is an incomplete draft, made public so that I could reasonably speak on the results herein. These results are intended to be mathematically complete, but I do not claim that they are well written or well-proofread, and I do claim that they do not fulfill the full aspirations of this project; when finished, this article will hopefully also prove additivity for a few stratified equivariant versions of $\mathbb{E}_n$, including equivariant swiss cheese and the evident "subrepresentation" analogue of $\mathbb{E}_n \overset{\mathrm{BV}}{\otimes} \mathrm{BiMod}^{\otimes}$. Additionally, I'm a bit stuck on the nonlinear equivariant Kister-Mazur theorem– if I find a proof, the tangential structures will be given by equivariant microbundles instead of equivariant vector bundles. As such, the reader should not expect many readability-oriented updates until these parts are finished. The current goal is to finish this and revise for arxiv submission by September of 2025.

Date: April 5, 2025.

Introduction

In [May72], the study of homotopy-coherent algebraic structures crystallized into a definition of *operads and algebras*, encapsulating Boardman-Vogt's *little cubes* action on iterated loop spaces as a factorization

$$\begin{tikzcd} & \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}_*) \arrow[d, "U"] \\ \mathcal{S}_* \arrow[ur, dashed] \arrow[r, "\Omega^n"] & \mathcal{S}_* \end{tikzcd}$$

Moreover, the *recognition principle* of [May72] essentially proved that the restriction of Ω^n to $(n+1)$-connected pointed spaces is fully faithful with essential image the connected $\mathbb{E}_n$-spaces. (see [HA] for this formulation). This is evidently additive in n; indeed, after Stasheff's $\mathbb{E}_1 = \mathbb{A}_\infty$-recognition principle [Sta63], this formulation of May's recognition principle is easily seen to be *equivalent* to the fact that the forgetful functor

$$\mathrm{Alg}_{\mathbb{E}_{n+m}}(\mathcal{S}_{*,\geq 1}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_n}\mathrm{Alg}^{\otimes}_{\mathbb{E}_m}(\mathcal{S}_{*,\geq 1})$$

is an equivalence for all $n, m \in \mathbb{N}$.

Naturally, one wonders about additivity for algebras in more general symmetric monoidal ∞-categories. If we 1-categorically model $\mathbb{E}_n^{\otimes}$ via the *little cubes topological operad* $C_n \in \mathrm{AlgFun}^{\circ}(\coprod_{n\in\mathbb{N}} B\Sigma_n, \mathrm{Top})$, Boardman and Vogt [BV73] constructed a point-set tensor product on topological operads with mapping property

$$\mathrm{Alg}^{1-\mathrm{cat}}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}^{1-\mathrm{cat}}_{\mathcal{O}}\mathrm{Alg}^{1-\mathrm{cat},\otimes}_{\mathcal{P}}(\mathcal{C})$$

for a pointwise symmetric monoidal structure on $\mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$. The evident operadic conjecture then demands an equivalence $C_n \otimes C_m \simeq C_{n+m}$; this was supplied by Dunn [Dun88].

Unfortunately, in making the corresponding additivity statement for algebras homtopical, we encounter difficulty; the point-set Boardman-Vogt tensor product is not known to admit a left derived functor, let alone satisfy a derived mapping property. One potential remedy in the 1-categorical setting (due to [FV13]) begins by verifying by hand that $\otimes$ is homotopical on weak equivalences between cofibrant operads which are additionally equivalent to C_k, but to the authors knowledge, further headway has not been made.

Instead, Lurie defined a manifestly homotopical tensor product $\overset{\mathrm{BV}}{\otimes}$ on the ∞-category of operads Op directly in terms of its mapping property, then constructed equivalences $\mathbb{E}_n^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_m^{\otimes} \simeq \mathbb{E}_{n+m}^{\otimes}$ and

$$\mathrm{Alg}_{\mathbb{E}_{n+m}}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_n}\mathrm{Alg}^{\otimes}_{\mathbb{E}_m}(\mathcal{C}),$$

the latter natural in the symmetric monoidal ∞-category $\mathcal{C}$. We'll call this result *Dunn-Lurie additivity.*

In the years since, Dunn-Lurie additivity has become a basic tool of homotopy theory and algebraic K-theory; for instance, it underlies the $\mathbb{E}_{n-1}$-monoidal structure on left modules over an $\mathbb{E}_n$-algebra, as well as the $\mathbb{E}_{n-1}$-structure on the value of various $\mathbb{E}_1$-invariants on $\mathbb{E}_n$-algebras, such as topological Hochschild homology, algebraic K-theory, and various approximations therebetween. Crucially, Dunn-Lurie additivity is an indispensable tool in studying $\mathbb{E}_n$-algebras, as it often inductively reduces questions about $\mathbb{E}_n^{\otimes} \simeq \mathbb{E}_1^{\otimes n}$ to questions about $\mathbb{E}_1^{\otimes}$, whose algebras are simply presented as associative algebra objects.[1]

We are interested in an equivariant version of this result; the definition of $\mathbb{E}_n^{\otimes}$ naturally extends to a *little V-disks G-operad* $\mathbb{E}_V^{\otimes}$ for any orthogonal G-representation V, which satisfy an approximation theorem and recognition principle for S^V-loop spaces [GM11; RS00; SW03]. Dunn's result has been lifted to G-operads in [Szc24] for a tensor product of operads in $s\mathrm{Set}_G$ with no known homotopical properties.

Moreover, Lurie's theory of *∞-operads and Boardman-Vogt tensor products* was extended to G-operads in [Ste25b; Ste25d], whose notation we adopt. Our main theorem shows that $\mathbb{E}_V^{\otimes}$ is additive in V under the (homotopical) Boardman-Vogt tensor product.

Theorem A. *The natural G-operad maps $\mathbb{E}_V^{\otimes}, \mathbb{E}_W^{\otimes} \to \mathbb{E}_{V\oplus W}^{\otimes}$ extend to an equivalence*

$$\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \xrightarrow{\sim} \mathbb{E}_{V\oplus W}^{\otimes};$$

in particular, for all G-symmetric monoidal ∞-categories $\mathcal{C}$, the forgetful functor yields an equivalence

$$\mathrm{Alg}_{\mathbb{E}_{V\oplus W}}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_V}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_W}(\mathcal{C}).$$

[1] This can be interpreted ∞-categorically or via Schwede-Shipley's right-transferred model structure on strict associative algebra structures; see e.g. [HA, Thm 4.1.8.4].

For instance, by results of [Ste25d], this constructs an $\mathbb{E}_V$-algebra structure on the Real topological Hochschild homology of an $\mathbb{E}_{V\oplus\sigma}$-algebra and a natural $\mathbb{E}_V$-monoidal structure on right-modules over an $\mathbb{E}_{V\oplus 1}$-algebra, as well as verifying that this is the universal such structure.

In general, when $G = C_2$, this reduces questions about $\mathbb{E}_V$-algebras into questions about $\mathbb{E}_1$-algebras and $\mathbb{E}_\sigma$-algebras, which are relatively well understood [Hil22], as their structure spaces are discrete (indeed, when V is 0-dimensional, its S-ary structure space is either empty or an $\mathrm{Aut}_H S$-torsor [Ste25c]).

Now, we describe a variation of Theorem A in the following setting.

Definition. Let $B_G\mathrm{Top}(n)$ the classifying G-space for equivariant (orthogonal) vector bundles. A *linear G-tangential structure* is a G-space X with a distinguished map $T\colon X \to B_G O(n)$. ◁

Given a linear G-tangential structure X, we define a G-operad $\mathbb{E}_X^\otimes$ of *X-framed little disk embeddings.*

Example. If V is an orthogonal G-representation, a V-framed smooth G-manifold M comes equipped with a map $M \xrightarrow{TM} B_G O(\dim V)$; in particular, V with its tautological self-framing yields a G-tangential structure. The resulting G-operad is $\mathbb{E}_V^\otimes$. ◁

Using this example, we construct a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathbb{E}_X}(\mathcal{C}) \xrightarrow{\sim} \underline{\lim}_{x\in X} \underline{\mathrm{Alg}}_{\mathbb{E}_{T_x}}(\mathcal{C}).$$

which we use to show the following.

Corollary B. *Given a pair of linear G-tangential structures $X \to B_G O(k)$ and $Y \to B_G\mathrm{Top}(k')$, the natural G-operad maps $\mathbb{E}_X^\otimes, \mathbb{E}_Y^\otimes \to \mathbb{E}_{X\times Y}^\otimes$ extend to an equivalence*

$$\mathbb{E}_X^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_Y^\otimes \xrightarrow{\sim} \mathbb{E}_{X\times Y}^\otimes.$$

In Section 3.3 we relate this to Dwyer-Hess-Knudsen's conjecture on skew little cubes operads [DHK18, Conj 4.18] and interpret our results in terms of Horev-Miladinovic's G-∞-categories of B-framed G-disks.

For another application, in [DHLSW25] the (nonequivariant) equivalence $\mathbb{E}_{BU(1)}^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_n \simeq \mathbb{E}_{BU(1)+n}^\otimes$ was explicitly used to construct a natural $\mathbb{E}_n$-S^1-equivariant map $\mathrm{THH}(A) \to A$ when A is an $\mathbb{E}_{\mathrm{BU}(1)+n}$-algebra, which was contributed to Horev; the author expects their techniques and Corollary B to lift *mutatis mutandis* to a construction of a (Borel) twisted $\mathbb{T}$-equivariant $\mathbb{E}_V$-map $\mathrm{THR}(A) \to A$ whenever A is an $\mathbb{E}_{V\times B_G\mathbb{T}_\sigma}$-algebra.[2] We leave this open for clarity's sake.

The strategy. Our strategy is strongly related to that of [Har] when $G = *$. The heart of this strategy reduces to the case of *prefactorization algebras*: we define a (multi-colored) G-0-operad $\mathbb{P}_V^\otimes$ whose algebras are a genuine equivariant version of prefactoriation algebras for the Weiss cover of *disjoint unions of (affinely)* $\coprod_{[G/H]} D\left(\mathrm{Res}_H^G V\right)$*-shaped invariant subspaces.* We define a comparison map $\alpha_V\colon \mathbb{P}_V^\otimes \to \mathbb{E}_V^\otimes$ corepresenting an *underlying V-prefacotrization algebra* functor. In Proposition 2.29, we show that α_V is a *weak approximation*, i.e. it fully faithfully embeds $\mathbb{E}_V$-algebras as locally constant V-prefactorization algebras.

From here, we're forced to use model-specific strategies; writing $\mathrm{POp}_G := \mathrm{Cat}^{\mathrm{Backwards-cocart}}_{/\mathrm{Span}(\mathbb{F}_G)}$ for the ∞-category of *G-preoperads*, we have a collection of functors:

$$\begin{tikzcd} \mathrm{Cat} & \mathrm{Op}_G \arrow[l, "\mathrm{Mon}_{(-)}(\mathcal{S})"'] \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{POp}_G \arrow[l, bend right, "L_{\mathrm{Op}_G}"'] \arrow[r, "\mathrm{Tot}"] & \mathrm{AlgPatt} \arrow[r, "\mathrm{Seg}_{(-)}(\mathcal{S})"] & \mathrm{Cat} \end{tikzcd}$$

In [Ste25d] we constructed a natural equivalence $\mathrm{Mon}_{L_{\mathrm{Op}_G}\mathcal{O}}(\mathcal{S}) \simeq \mathrm{Seg}_{\mathrm{Tot}\mathcal{O}^\otimes}(\mathcal{S})$; since $\mathrm{Mon}_{(-)}(\mathcal{S})$ detects L_{Op_G}-equivalences [Ste25b], a map of G-preoperads is an L_{Op_G}-equivalence if and only if its map of patterns is a *Morita equivalence*, which can be verified using traditional higher category theory.

[2] The only structural aspect of factorization homology which is missing to replicate this argument in the equivariant setting is the *trivial interval Fubini property* $\int_M(-) \simeq \int_{M\times(0,1)}(-)$. But indeed this follows in the relevant case $M = \rho_{C_2} - \{0\}$ by $\otimes$-excision for the "thickening" of the usual C_2-collar decomposition $S^\sigma \simeq D^1 \sqcup_{\mathrm{Ind}_e^{C_2} D^1} D^1$:

$$\int_{\rho_{C_2}-\{0\}} A \simeq \int_{I\times D^1} A \otimes_{\int_{I\times \mathrm{Ind}_e^{C_2} A}} \int_{I\times D^1} A \simeq A \otimes_{N_e^{C_2} A} A \simeq \mathrm{THR}(A).$$

What's left is to construct a convenient G-preoperad model for the Boardman-Vogt tensor product, construct a Dunn map, and verify that it is a Morita equivalence. For the former, we define a *wreath product* G-preoperad connected by a string of natural Morita equivalences $\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes \leftarrow \mathcal{O}^\otimes \times \mathcal{P}^\otimes \to \mathcal{O}^\otimes \wr \mathcal{P}^\otimes$; this has the crucially convenient property that $\mathcal{O}^\otimes \wr \mathcal{P}^\otimes$ has n-truncated structure spaces whenever $\mathcal{O}^\otimes$ and $\mathcal{P}^\otimes$ do.

Using this description, we explicitly construct Dunn maps $\varphi_{\mathbb{P}} \colon \mathbb{P}_V^\otimes \wr \mathbb{P}_W^\otimes \to \mathbb{P}_{V\oplus W}^\otimes$ and $\varphi_{\mathbb{E}} \colon \mathbb{E}_V^\otimes \wr \mathbb{E}_W^\otimes \to \mathbb{E}_{V\oplus W}^\otimes$, the former admitting a canonical factorization through a sub-G-preoperad of *decomposed little cube embeddings* $\mathbb{P}_{V|W}^\otimes$, yielding a commutative diagram of G-preoperads

$$\begin{tikzcd} \mathbb{P}_V^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{P}_W^\otimes \arrow[d, "\alpha_V \otimes \alpha_W"'] & \mathbb{P}_V^\otimes \times \mathbb{P}_W^\otimes \arrow[l, "L_{\mathbb{P}}"'] \arrow[r, "M_{\mathbb{P}}"] \arrow[d, "\alpha_V \times \alpha_W"'] & \mathbb{P}_V^\otimes \wr \mathbb{P}_W^\otimes \arrow[r, "\widetilde{\varphi}_{\mathbb{P}}"] \arrow[d, "\alpha_V \wr \alpha_W"'] & \mathbb{P}_{V|W}^\otimes \arrow[r, hook] \arrow[d, "\alpha_{V|W}"'] & \mathbb{P}_{V\oplus W}^\otimes \arrow[d, "\alpha_{V\oplus W}"] \\ \mathbb{E}_V^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^\otimes & \mathbb{E}_V^\otimes \times \mathbb{E}_W^\otimes \arrow[l, "L_{\mathbb{E}}"] \arrow[r, "M_{\mathbb{E}}"'] & \mathbb{E}_V^\otimes \wr \mathbb{E}_W^\otimes \arrow[r, "\varphi_{\mathbb{E}}"'] & \mathbb{E}_{V\oplus W}^\otimes \arrow[r, equal] & \mathbb{E}_{V\oplus W}^\otimes \end{tikzcd} \tag{1}$$

It suffices to verify that $\varphi_{\mathbb{E}} \circ M_{\mathbb{E}}$ is a Morita equivalence. We begin by using the 1-category theory of wreath products together with a specialization of Harpaz' theory of *strong approximations* [Har] to verify that $\widetilde{\varphi}_{\mathbb{P}}$ is a Morita equivalence; moreover, using an equivariant analog of the topological strategy of [Dun88] we verify that $\alpha_{V|W}$ is itself a weak approximation.[3] Under this, we find that pullback along $\varphi_{\mathbb{E}} \circ M_{\mathbb{E}}$ is an inclusion of full subcategories

$$\mathrm{Mon}_{\mathbb{E}_{V\oplus W}}(\mathcal{S}) \subset \mathrm{Mon}_{\mathbb{E}_V \times \mathbb{E}_W}(\mathcal{S}) \subset \mathrm{Mon}_{\mathbb{P}_{V|W}}(\mathcal{S}),$$

each characterized by a local constancy condition. Some simple bookkeeping shows that these conditions agree for $\mathbb{E}_{V\oplus W}^\otimes$ and $\mathbb{E}_V^\otimes \times \mathbb{E}_W^\otimes$, yielding Theorem A.

We prove Corollary B by reduction to Theorem A using the tensor product disintegration theorem of [Ste25d]; indeed, we construct a G-natural transformation

$$\begin{tikzcd} \underline{\mathcal{S}}_{G,/B_G O(k)} \times \underline{\mathcal{S}}_{G,/B_G O(k')} \arrow[r, "\mathbb{E}_{(-)}^\otimes \times \mathbb{E}_{(-)}^\otimes"] \arrow[d, "\times"'] \arrow[dr, phantom, "\Leftarrow \varphi ="] & \underline{\mathrm{Op}}_G \times \underline{\mathrm{Op}}_G \arrow[d, "\overset{\mathrm{BV}}{\otimes}"] \\ \underline{\mathcal{S}}_{G,/B_G O(k+k')} \arrow[r, "\mathbb{E}_{(-)}^\otimes"'] & \underline{\mathrm{Op}}_G \end{tikzcd}$$

which, on a pair of orbits $[G/H] \to B_G O(k)$ and $[G/H] \to B_G O(k')$ classifying representations V, W, specializes to the Dunn map $\varphi \colon \mathbb{E}_V^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^\otimes \to \mathbb{E}_{V\oplus W}^\otimes$. We use Horev-Miladinovic's equivariant Kister-Mazur theorem to verify that $U\mathbb{E}_X \simeq X$ and that functoriality of $\mathbb{E}_{(-)}^\times$ induces, for a point $x \in X$, an equivalence with the reduced endomorphism $\mathrm{stab}(x)$-operad

$$\mathbb{E}_{T_x X}^\otimes \xrightarrow{\sim} \mathrm{End}_x^{\mathrm{red}} \mathbb{E}_X^\otimes;$$

the tensor disintegration theorem of [Ste25d] then implies that φ is an equivalence.

Relationship to surrounding literature. This paper is the sixth installment in a series of papers [Ste24; Ste25a; Ste25b; Ste25c; Ste25d] which are intended to extend the parameterize and equivariant higher algebra of [BDGNS16; NS22] for use in equivariant homotopy theory and K-theory. As such, it is closely related to similar work in both different foundations and in the non-equivariant case. Moreover, the author intends to use it foundationally in further work concerning obstruction theory of $\mathbb{E}_V$-algebras.

As mentioned before, the strategy of the proof of Theorem A is not essentially new in the case $G = e$, as the ideas in that case are strongly present in [Har]. Moreover, it follows by unwinding definitions that the wreath product model is related to Lurie's wreath product [HA, § 2.4.4] by the comparison functor of [BHS22]. Additionally, the deduction of Corollary B from Theorem A is similar to that of Lurie [HA] assuming the statement of the disintegration and assembly theorem, though the proof of this theorem is quite different between [Ste25d] and [HA].

[3] On the level of prefactorization algebras we may view this as saying that, absent a local constancy or cosheaf condition, additivity for V-disk prefactorization algebras may fail, as the prefactorization Dunn map only surjects onto the basis of *decomposable* elements of the Weiss cover corresponding with $\mathbb{E}_{V\oplus W}$; Dunn's argument [Dun88] essentially boils down to verifying that this basis is *coinitial* in the usual one, so the disparity disappears in presence of local constancy, and Berry's argument [Ber21] uses this to show that the disparity also disappears in presence of a cosheaf condition.

It is worthwhile to note that the result that strong approximations are Morita equivalences is *almost* a special case of Barkan's recognition result for Morita equivalences of algebraic patterns [Bar23]; the discrepancy lies in Condition (WA-a), where Barkan's result requires that the homotopy fibers have contractible cores.

In the equivariant case, the reader ought consider a comparison between our results and Sczczesny's point-set version [Szc24] to be a difficult open problem; indeed, to the author's knowledge, it is still an open problem whether the relevant tensor products are comparable in the *nonequivariant* case (c.f. [FV13]).

Moreover, to the author's knowledge, it is an open problem (in progress by other authors) to compare the ∞-categories of $\mathbb{E}_V$ algebras herein with the ∞-categories presented by the right-transferred model structure of e.g. [Hil22]. These are seen to be plausibly equivalent by the similarity between their associated monads (c.f. [GM17; Ste25b]).

The infinitary cases of Theorem A were already known by previous work of the author [Ste25d].

Notation and conventions. We broadly adopt the terminology of higher category theory from [HTT], equivariant higher category theory from [Sha22; Sha23], algebraic patterns from [CH21], and equivariant operads from [Ste25b; Ste25d]. When it will not cause confusion, we abusively refer to the total ∞-category of a G-preoperad $\mathcal{O}^\otimes$ simply as $\mathcal{O}^\otimes$, and we will abusively refer to the pushforward G-preoperad $\wedge_!(\mathcal{O}^\otimes \times \mathcal{P}^\otimes)$ simply as $\mathcal{O}^\otimes \times \mathcal{P}^\otimes$; the reader should be warned that that this is not the categorical product in G-preoperads, but instead it is the algebraic operation on G-preoperads induced by the cartesian product of ∞-categories under the smash symmetric monoidal structure on $\mathrm{Span}(\mathbb{F}_G)$, i.e. it is the usual ($G$-)preoperadic model for the Boardman-Vogt tensor product.

Throughout this article, the notion of *homotopical categories* will match [DHKS04]. In particular, when a 1-category admits a "conventional" model structure $\mathcal{C}$, we will implicitly refer to $\mathcal{C}$ and its full subcategories of (co)fibrant objects as homotopical categories under the associated weak equivalences.

Moreover, "topological space" will always mean *compactly generated weakly-Hausdorff topological space*, and Top to the category of such topological spaces; all model-categorical aspects of Top will be defined with respect to the Quillen model structure. The word "space" will be reserved for objects in the ∞-category $\mathcal{S} \simeq \mathrm{Top}\left[WEQ^{-1}\right]$, or equivalently, objects in hoTop. If X, Y are topological spaces, $\mathrm{Map}(X,Y)$ will be denote the internal hom in Top, i.e. the compact-open topology.

Lastly, "enriched category" will refer to the 1-categorical notion, $\mathrm{Cat}_{\mathcal{V}}$ will refer to the *1-category* of $\mathcal{V}$-categories, and when $\mathcal{V}$ is a homotopical category, a *Dwyer-Kan equivalence of $\mathcal{V}$-categories* will refer to a $\mathcal{V}$-functor which is fully faithful and whose hom mmorphisms $\mathrm{Map}(X,Y) \to \mathrm{Map}(\varphi X, \varphi Y)$ are weak equivalences.

Acknowledgements. I would like to thank Thomas Blom, Mike Hopkins, and Jan Steinebrunner for helpful conversations on this topic. Additionally, I would be remiss to not mention the work which inspired this paper: I learned the main strategy for this paper by analogy to the beautiful proof of Yonatan Harpaz in [Har], and the combinatorics only became tractable after the work of Shaul Barkan, Rune Haugseng, and Jan Steinebrunner in [BHS22] on *fibrous patterns*. Additionally, I'm thankful to Maxime Ramzi for disabusing me of an early fatally flawed strategy for the proof of Theorem A.

While developing this material, the author was supported by the NSF Grant No. DGE 2140743.

1. Wreath products and weak approximations

The theory developed in this section is of independent interest, so we temporarily replace G with an arbitrary atomic orbital ∞-category $\mathcal{T}$. The reader is encouraged to replace $\mathcal{T}$ with $\mathcal{O}_G$ (or simply G) and elements of $\mathcal{T}$ with homogeneous G-sets. We adopt the notation $\mathrm{POp}_{\mathcal{T}} := \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ for the ∞*-category of $\mathcal{T}$-preoperads*. Recall that $\mathcal{T}$-preoperads have functorial underlying algebraic patterns; explicitly, given a $\mathcal{T}$-preoperad $\mathcal{O}^\otimes$, the total ∞-category $\mathrm{Tot}\mathcal{O}^\otimes$ has a factorization system whose inert maps consist of cocartesian lifts of backwards maps in $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ and whose active maps consist of arbitrary lifts of forward maps. We designate the corresponding wide subcategories as $\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}} \subset \mathrm{Tot}\mathcal{O}^\otimes$; moreover, we designate the full subcategory $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{int}}$ of *elementary objects* as those objects lying over finite $\mathcal{T}$-sets of a single orbit. We will study $\mathcal{T}$-preoperads via their underlying patterns.

Definition 1.1. A morphism of $\mathcal{T}$-preoperads $\varphi\colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ is called a *weak approximation* if:

(WA-a) For every $O \in \mathcal{O}^\otimes$, the homotopy fibers of $\mathcal{O}^{\mathrm{act}}_{/O} \to \mathcal{P}^{\mathrm{act}}_{/\varphi O}$ are weakly contractible, and

(WA-b) The $\mathcal{T}$-functor $U\varphi\colon U\mathcal{O} \to U\mathcal{P}$ attains a fully faithful $\mathcal{T}$-right adjoint.

We say that φ is a *strong approximation* if it is a weak approximation and $U\varphi$ is an equivalence. $\triangleleft$

In this section, we construct a *wreath product* model for the Boardman-Vogt tensor product together with a map of $\mathcal{T}$-preoperads $\mathcal{O}^\otimes \times \mathcal{P}^\otimes \to \mathcal{O}^\otimes \wr \mathcal{O}^\otimes$, culminating in the following procedure for pushing forward Boardman-Vogt tensor product computations along weak approximations.

Theorem 1.2. *Suppose we have maps of $\mathcal{T}$-preoperads α_i and $\varphi_{\mathcal{O}}, \varphi_{\mathcal{P}}$ making the following diagram commute*

$$\begin{tikzcd} \mathcal{O}_1^\otimes \wr \mathcal{O}_2^\otimes \arrow[r, "\varphi_{\mathcal{O}}"] \arrow[d, "\alpha_1 \wr \alpha_2"'] & \mathcal{O}_3^\otimes \arrow[d, "\alpha_3"'] \\ \mathcal{P}_1^\otimes \wr \mathcal{P}_2^\otimes \arrow[r, "\varphi_{\mathcal{P}}"] & \mathcal{P}_3^\otimes \end{tikzcd}$$

and satisfying the following conditions:

(BK-a) Each α_i is a weak approximation;
(BK-b) $\varphi_{\mathcal{O}}$ is a strong approximation; and
(BK-c) the induced $\mathcal{T}$-functor $U\varphi_{\mathcal{P}} \colon U\mathcal{P}_1 \times U\mathcal{P}_2 \to U\mathcal{P}_3$ is an equivalence.

Then, the composite map $\mathcal{P}_1^\otimes \times \mathcal{P}_2^\otimes \to \mathcal{P}_1^\otimes \wr \mathcal{P}_2^\otimes \to \mathcal{P}_3^\otimes$ is an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence, yielding an equivalence

$$L_{\mathrm{Op}_{\mathcal{T}}} \mathcal{P}_1^\otimes \overset{\mathrm{BV}}{\otimes} L_{\mathrm{Op}_{\mathcal{T}}} \mathcal{P}_2^\otimes \simeq L_{\mathrm{Op}_{\mathcal{T}}} \mathcal{P}_3^\otimes.$$

1.1. **Terminology surrounding the combinatorics of equivariant mutli-arity.** We center the following.

Definition 1.3. A $\mathcal{T}$*-multi-arity* is a morphism $T \to S$ in $\mathbb{F}_{\mathcal{T}}$; a $\mathcal{T}$*-arity* is a multi-arity $T \to V$ where $V \in \mathcal{T}$, or equivalently, an element of the ∞-category $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T}}$. $\triangleleft$

These are the *types* one inputs to an operation in the algebraic theory described by a *one-color $\mathcal{T}$-operad*; unfortunately, the main strategy of this paper shifts the data of a $\mathcal{T}$-operad into a complicated $\mathcal{T}$-∞-category of colors, so we take careful account of the types involved in the multi-colored theory. We will use the language of *multi-profiles* associated with a *coefficient system.*

Definition 1.4. Let $\mathcal{C}$ be an ∞-category. A $\mathcal{T}$*-coefficient system in* $\mathcal{C}$ is a functor $\mathcal{T}^{\mathrm{op}} \to \mathcal{C}$.

Given a $\mathcal{T}$-coefficient system of sets $\mathfrak{C}_\bullet \colon \mathcal{T}^{\mathrm{op}} \to \mathsf{Set}$ and a $\mathcal{T}$-equivariant multi-arity $\psi \colon T \to S$, the set of $\mathfrak{C}$*-multi-profiles lying over* ψ is

$$\mathrm{Prf}_{\mathfrak{C}}^{\psi} \coloneqq \prod_{V \in S} \mathcal{C}_{T_V} \times \mathcal{C}_V,$$

where $T_V \coloneqq T \times_S V$. We refer to a $\mathfrak{C}$-multi-profile as a $\mathfrak{C}$*-profile* if its underlying $\mathcal{T}$-multi-arity is a $\mathcal{T}$-arity. We will denote a generic $\mathfrak{C}$-multi-profile by $(\mathbf{C};\mathbf{D})$ and a generic $\mathfrak{C}$-profile as $(\mathbf{C};D)$. $\triangleleft$

Construction 1.5. Let $T \xrightarrow{\psi} S \xrightarrow{\varphi} R$ be composable morphisms. By "forgetting the middle arity," we acquire an operation $\circ$ fitting into the following diagram:

$$\mathrm{Prf}_{\mathfrak{C}}^{\varphi} \times_{\mathcal{C}_S} \mathrm{Prf}_{\mathfrak{C}}^{\psi} \xrightarrow{\circ} \mathrm{Prf}_{\mathfrak{C}}^{\psi \circ \varphi}$$

Moreover, given a pullback diagram

$$\begin{tikzcd} f^*T \arrow[r, "f'"] \arrow[d, "\varphi'"] \arrow[dr, phantom, very near start, "\lrcorner"] & T \arrow[d, "\varphi"] \\ Q \arrow[r, "f"] & S \end{tikzcd}$$

we define a *pullback* operation $f^* \colon \mathrm{Prf}_{\mathfrak{C}}^{\varphi} \to \mathrm{Prf}_{\mathfrak{C}}^{f^*\varphi}$ by setting

$$f^*((C_U);(D_U)) \coloneqq \Big(\big(C_{f'(W)} \mid W \in \mathrm{Orb}(f^*T)\big);\big(D_{f(W)} \mid W \in \mathrm{Orb}(Q)\big)\Big);$$

In particular, when $Q = W$ and $S = V$ are orbits, we write this as $\left(\mathrm{Res}_V^W \mathbf{C}; \mathrm{Res}_V^W D\right)$. $\triangleleft$

Now, this allows for convenient description of $\mathfrak{C}$*-symmetric sequences* and the corresponding theory of $\mathfrak{C}$*-colored genuine $\mathcal{T}$-operads*. To describe this, we need the $\mathfrak{C}$-colored version of Σ, beginning with the following notion.

Definition 1.6. Given $(\mathbf{C};D)$ an S-ary $\mathfrak{C}$-profile, the group of *$\mathfrak{C}$-colored automorphisms of S* is the subgroup

$$\mathrm{Aut}_V(\mathbf{C}) \subset \mathrm{Aut}_V(S)$$

consisting of automorphisms σ of $S \in \mathbb{F}_V$ such that, for all $U \in \mathrm{Orb}(S)$, $C_U = C_{\sigma U}$. ◁

In [Ste25b] we found a cocartesian fibration of $\mathcal{T}$-groupoids $F\colon \underline{\Sigma}_{\mathfrak{C}} \to \underline{\Sigma}_{\mathcal{T}} := \underline{\mathbb{F}}_{\mathcal{T}}^{\simeq}$ with V-value groupoid

$$\Sigma_{\mathfrak{C},V} = \coprod_{\substack{S \in \mathbb{F}_V \\ (\mathbf{C};D) \in \mathfrak{C}^S \times \mathfrak{C}^V}} B\mathrm{Aut}_V(\mathbf{C}),$$

with restriction maps given by applying B to the evident maps $\mathrm{Aut}_V(\mathbf{C}) \to \mathrm{Aut}_U(\mathrm{Res}_U^V \mathbf{C})$ and with F given by applying B to the defining maps $\mathrm{Aut}_V(\mathbf{C}) \to \mathrm{Aut}_V(S)$.

Definition 1.7. Given $\mathcal{V}$ an ∞-category, the ∞-category of *$\mathfrak{C}$-symmetric sequences in $\mathcal{V}$* is $\mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathfrak{C}}, \mathcal{V})$. Given a $\mathfrak{C}$-multi-profile $(\mathbf{C};\mathbf{D})$ lying over multi-arity $\psi\colon T \to S$ and a $\mathfrak{C}$-symmetric sequence $\mathcal{O}$, we define the *$\mathcal{V}$-object of $(\mathbf{C};\mathbf{D})$ multi-operations* to be

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(\mathbf{C};\mathbf{D}) := \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(\mathbf{C}_U; D_U). \quad ◁$$

Definition 1.8 ([NS22, Def 2.5.4])**.** Let $\mathcal{V}$ be a symmetric monoidal 1-category. A *$\mathfrak{C}$-colored genuine $\mathcal{T}$-operad $\mathcal{O}$ in $\mathcal{V}$* is the data of

1. A $\mathfrak{C}$-symmetric sequence $\mathcal{O}$;
2. For every color $C \in \mathfrak{C}$, an identity operation
$$1_C\colon 1_{\mathcal{V}} \to \mathrm{Mul}_{\mathcal{O}}^{\mathrm{id}}(C;C)$$
3. For every composable pair of multi-profiles $(\mathbf{B};\mathbf{C};\mathbf{D})$, a composition map
$$\gamma\colon \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{C};\mathbf{D}) \otimes \mathrm{Mul}_{\mathcal{O}}^{g}(\mathbf{B};\mathbf{C}) \longrightarrow \mathrm{Mul}_{\mathcal{O}}^{g\circ f}(\mathbf{B};\mathbf{D})$$

subject to the following conditions:

(OP-a) (unitality) for every $\mathfrak{C}$-profile $(\mathbf{B};\mathbf{C})$, the following commutes

$$\begin{tikzcd} \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{B};C) \arrow[r, "\mathrm{id}\times\{1\}"] \arrow[d, "\{1\}\times\mathrm{id}"'] \arrow[dr, equal] & \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{B};C) \otimes \mathrm{Mul}_{\mathcal{O}}^{\mathrm{id}}(\mathbf{B};\mathbf{B}) \arrow[d, "\gamma"] \\ \mathrm{Mul}_{\mathcal{O}}^{\mathrm{id}}(C;C) \otimes \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{B};C) \arrow[r, "\gamma"'] & \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{B};C) \end{tikzcd}$$

(OP-b) (associativity) for every composable triple of $\mathfrak{C}$-multi-profiles $(\mathbf{A};\mathbf{B};\mathbf{C};\mathbf{D})$, the following commutes

$$\begin{tikzcd} \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{C};D) \otimes \mathrm{Mul}_{\mathcal{O}}^{g}(\mathbf{B};\mathbf{C}) \otimes \mathrm{Mul}_{\mathcal{O}}^{h}(\mathbf{A};\mathbf{B}) \arrow[r] \arrow[d] & \mathrm{Mul}_{\mathcal{O}}^{g\circ f}(\mathbf{B};D) \otimes \mathrm{Mul}_{\mathcal{O}}^{h}(\mathbf{A};\mathbf{B}) \arrow[d] \\ \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{C};D) \otimes \mathrm{Mul}_{\mathcal{O}}^{h\circ g}(\mathbf{A};\mathbf{C}) \arrow[r] & \mathrm{Mul}_{\mathcal{O}}^{h\circ g\circ f}(\mathbf{A};D) \end{tikzcd}$$

(OP-c) (restriction and units) The restriction map $\mathcal{O}(C;C) \to \mathcal{O}(\mathrm{Res}_U^V C; \mathrm{Res}_U^V C)$ takes 1_C to $1_{\mathrm{Res}_U^V C}$.

(OP-d) (restriction and composition) For every composable pair of $\mathfrak{C}$-multi-profile $(\mathbf{B};\mathbf{C};D)$ living over an arity $f\colon \mathrm{Ind}_H^G T \mathrm{Ind}_H^G S \to V$ and orbit map $U \to V$, the following diagram commutes

$$\begin{tikzcd} \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{C};D) \otimes \mathrm{Mul}_{\mathcal{O}}^{g}(\mathbf{B};\mathbf{C}) \arrow[r, "\gamma"] \arrow[d, "\mathrm{Res}"] & \mathrm{Mul}_{\mathcal{O}}^{g\circ f}(\mathbf{B};D) \arrow[d, "\mathrm{Res}"] \\ \mathrm{Mul}_{\mathcal{O}}^{\mathrm{Res}_U^V f}\left(\mathrm{Res}_U^V \mathbf{C}; \mathrm{Res}_U^V D\right) \otimes \mathrm{Mul}_{\mathcal{O}}^{\mathrm{Res}_U^V g}\left(\mathrm{Res}_U^V \mathbf{B}; \mathrm{Res}_U^V \mathbf{C}\right) \arrow[r, "\gamma"] & \mathrm{Mul}_{\mathcal{O}}^{\mathrm{Res}_U^V g\circ f}\left(\mathrm{Res}_U^V \mathbf{B}; \mathrm{Res}_U^V D\right) \end{tikzcd}$$

(OP-e) ($\mathrm{Aut}_H S$-equivariance) suppose $(\mathbf{C};\mathbf{B};A)$ is a composable multi-profile and profile living over the multi-arities $f\colon \mathrm{Ind}_V^{\mathcal{T}} T \to \mathrm{Ind}_V^{\mathcal{T}} S \to V$. Then, γ is Borel $\mathrm{Aut}_V(\mathbf{B}) \times \prod_{U \in \mathrm{Orb}(S)} \mathrm{Aut}_U(\mathbf{C}_U)$-equivariant.

A *morphism of $\mathfrak{C}$-colored genuine $\mathcal{T}$-operads in $\mathcal{V}$* is a morphism $\varphi\colon \mathcal{O} \to \mathcal{P}$ of $\mathfrak{C}$-symmetric sequences such that $\varphi(1_C) = 1_{\varphi(c)}$ and such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{O}(\mathbf{C};D) \times \mathrm{Mul}^g_{\mathcal{O}}(\mathbf{B};\mathbf{C}) \arrow[r, "\varphi\times\varphi"] \arrow[d, "\gamma"] & \mathcal{P}(\varphi\mathbf{C};\varphi D) \times \mathrm{Mul}^{\varphi g}_{\mathcal{P}}(\varphi\mathbf{B};\varphi\mathbf{C}) \arrow[d, "\gamma"] \\ \mathcal{O}(\mathbf{B};D) \arrow[r, "\varphi"] & \mathcal{P}(\varphi\mathbf{B};\varphi D) \end{tikzcd}$$

In the situation that $\mathcal{V}$ is a *homotopical category*, i.e. we've supplied a wide subcategory $\mathcal{V}^{\mathrm{WEQ}} \subset \mathcal{V}$ of "weak equivalences" satisfying two-out-of-six, a *weak equivalence of $\mathfrak{C}$-colored genuine $\mathcal{T}$-operads in $\mathcal{V}$* is a morphism $\mathfrak{O} \to \mathfrak{P}$ whose values $\mathcal{O}(\mathbf{C};D) \to \mathcal{P}(\varphi\mathbf{C};\varphi D)$ are each weak equivalences. In particular, when $\mathcal{V} = \mathrm{Top}$, a *weak equivalence of topological genuine $\mathcal{T}$-operads* is a weak equivalence with respect to pointwise π_*-isomorphisms. ◁

Construction 1.9. Given $f : \mathfrak{C} \to \mathcal{D}$ a morphism of coefficient systems of sets, we acquire a $\mathcal{T}$-functor $\underline{\Sigma}_{\mathfrak{C}} \to \underline{\Sigma}_{\mathfrak{D}}$, and therefore a pullback functor

$$f^*\colon \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathfrak{D}}, \mathcal{V}) \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathfrak{C}}, \mathcal{V});$$

given $\mathcal{O}^{\otimes} \in g\mathrm{Op}_{\mathcal{T}}^{\mathfrak{C}}(\mathcal{V})$, we define the *pullback $\mathfrak{C}$-colored genuine $\mathcal{T}$-operad $f^*\mathcal{O}$* to have underlying $\mathfrak{C}$-symmetric sequence $f^*\mathcal{O}$, unit given by the map $1_C\colon 1_{\mathcal{V}} \to \mathcal{O}(\varphi C;\varphi C) \simeq f^*\mathcal{O}(C;C)$. The composition map is defined similarly. ◁

Definition 1.10. A *morphism of genuine $\mathcal{T}$-operads in $\mathcal{V}$* from a $\mathfrak{C}$-colored genuine $\mathcal{T}$-operad $\mathcal{P}$ to a $\mathfrak{D}$-colored genuine $\mathcal{T}$-operad $\mathcal{O}$ is a morphism of $\mathcal{T}$-coefficient systems $\varphi_c\colon \mathfrak{C} \to \mathfrak{D}$ together with a morphism of $\mathfrak{C}$-colored $\mathcal{T}$-operads $\varphi\colon \mathcal{O}^{\otimes} \to \varphi_c^*\mathcal{P}$. ◁

In the case that $\mathcal{T} = \mathcal{O}_G$ is the orbit category of a finite group and $\mathcal{V} = \mathrm{sSet}$ under the Quillen homotopical structure, this agrees with the definition employed in [Bon19]. We will generally prefer the topological setting, but the distinction is easy to ignore in the locally fibrant setting by the following easy proposition, whose proof we omit.

Proposition 1.11. *Suppose $\mathcal{V} \to \mathcal{V}'$ is a functor of homotopical categories which reflects weak equivalences. Then, the corresponding functor $g\mathrm{Op}_{\mathcal{T}}(\mathcal{V}) \to g\mathrm{Op}_{\mathcal{T}}(\mathcal{V}')$ reflects weak equivalences.*

1.2. Recollections on $\mathcal{T}$-operads.

1.2.1. *The basic definitions.* Let $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ be the *effective Burnside 2-category*, e.g. as in [HHLN23].

Definition 1.12. A *$\mathcal{T}$-preoperad* is a functor $\pi : \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ satisfying the following condition.

(a) $\mathcal{O}^{\otimes}$ has π-cocartesian lifts for backwards maps in $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$.

A *$\mathcal{T}$-operad* is a $\mathcal{T}$-preoperad satisfying the following additional conditions.

(b) (Segal condition for colors) for every $S \in \mathbb{F}_{\mathcal{T}}$, cocartesian transport along the π-cocartesian lifts lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ together induce an equivalence

$$\mathcal{O}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U;$$

(c) (Segal condition for multimorphisms) for every map of orbits $T \to S$ in I and pair of objects $(\mathbf{C},\mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_U$, postcomposition with the π-cocartesian lifts $\mathbf{D} \to D_U$ lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ induces an equivalence

$$\mathrm{Map}^{T\to S}_{\mathcal{O}^{\otimes}}(\mathbf{C},\mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}^{T \leftarrow T_U \to U}_{\mathcal{O}^{\otimes}}(\mathbf{C}, D_U).$$

where $T_U \coloneqq T \times_S U$.

We the ∞-category of $\mathcal{T}$-preoperads and functors over $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ preserving backwards-cocartesian arrows, together with the full subcategory of $\mathcal{T}$-operads, as

$$\mathrm{Op}_{\mathcal{T}} \subset \mathrm{POp}_{\mathcal{T}} \coloneqq \mathrm{Cat}^{\mathrm{Backwards-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}.$$

We will refer to a morphism of $\mathcal{T}$-preoperads $\mathcal{O}^\otimes \to \mathcal{P}^\otimes$ as a *$\mathcal{O}$-algebra in $\mathcal{P}$*, and we let

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \coloneqq \mathrm{Fun}^{\mathrm{Backwards-cocart}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^\otimes, \mathcal{P}^\otimes) \subset \mathrm{Fun}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathcal{O}^\otimes, \mathcal{P}^\otimes)$$

be the full subcategory spanned by $\mathcal{O}$-algebras in $\mathcal{P}$. ◁

Now, these fit into an *algebraic pattern* framework:

Definition 1.13. An *algebraic pattern* is an ∞-category $\mathfrak{B}$ together with a factorization system $(\mathfrak{B}^{\mathrm{int}}, \mathfrak{B}^{\mathrm{act}})$ and a full subcategory $\mathfrak{B}^{\mathrm{el}} \subset \mathfrak{B}^{\mathrm{int}}$; given $\mathcal{C}$ an ∞-category, a *segal $\mathfrak{B}$-object in $\mathcal{C}$* is a functor $\mathfrak{B} \to \mathcal{C}$ whose restricted functor $\mathfrak{B}^{\mathrm{int}} \to \mathcal{C}$ is right Kan-extended along the inclusion $\mathfrak{B}^{\mathrm{el}} \hookrightarrow \mathfrak{B}^{\mathrm{int}}$. ◁

Example 1.14. The *effective Burnside algebraic pattern* is $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, together with the backwards-forwards factorization system $\mathbb{F}_{\mathcal{T}}^{\mathrm{op}}, \mathbb{F}_{\mathcal{T}} \subset \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ of [HHLN23, Prop 4.9] and elementary objects $\mathcal{T}^{\mathrm{op}} \subset \mathbb{F}_{\mathcal{T}}^{\mathrm{op}}$.

In [Ste25b] we verified that the argument of [BHS22] extends to show that Segal $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$-objects in $\mathcal{C}$ are precisely product-preserving functors $\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \to \mathcal{C}$, i.e. *$\mathcal{T}$-commutative monoids in $\mathcal{C}$*. ◁

Now, [BHS22] developed a convenient framework of *fibrous patterns* (slightly modifying the earlier *weak Segal fibrations* of [CH21], which are higher categorical analogs of the fibrant objects in the model structure of [HA, § B] and generalize the weak ∞-operads of [Har]); we verified in [Ste25b] that their argument extends to show that $\mathcal{T}$-operads are *precisely* fibrous $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$-patterns; it then follow from general category theory that the inclusion $\mathrm{Op}_{\mathcal{T}} \subset \mathrm{POp}_{\mathcal{T}}$ admits a left adjoint $L_{\mathrm{Op}_{\mathcal{T}}} \colon \mathrm{POp}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T}}$ [BHS22], which we call *$\mathcal{T}$-operadic localization.*

1.2.2. *The total $\mathcal{T}$-∞-category.* Unfortunately, in this paper we must occasionally use another equivalent model for $\mathcal{T}$-operads, which we will refer to as the *Γ-space model*.

Construction 1.15 ([NS22])**.** Consider the ∞-category $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \simeq \mathrm{Span}_{s.i.,tdeg}(\mathbb{F}_{\mathcal{T}}^{\vee})$, with its factorization system whose left arrows are backwards and right arrows are forwards [HHLN23]. This lifts to an algebraic pattern $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$ with elementary objects $\mathcal{T}^{\mathrm{op}}$. ◁

Proposition 1.16 ([BHS22; Ste25b])**.** *Pullback along the canonical map* $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ *is an equivalence*

$$\mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_{\mathcal{T}})) \xrightarrow{\sim} \mathrm{Fbrs}(\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*})$$

This Γ-space model recasts a version of $\mathcal{T}$-preoperad theory in the setting of *$\mathcal{T}$-∞-category theory*, which is occasionally useful; for instance, it allows for an *underlying $\mathcal{T}$-∞-category* as follows.

Construction 1.17. Given $\mathcal{O}^\otimes$ a $\mathcal{T}$-preoperad, we define its *total $\mathcal{T}$-∞-category* to have unstraightening the vertical composition

$$\begin{tikzcd}
\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^\otimes \arrow[d] \\
\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*} \arrow[r] \arrow[d, "t"] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \\
\mathcal{T}^{\mathrm{op}} &
\end{tikzcd}$$

That is, $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes \coloneqq \mathrm{St}_{\mathcal{T}^{\mathrm{op}}}\mathrm{Tot}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes$. ◁

Proposition 1.18 ([Ste25b])**.** *The forgetful functors* $\mathrm{Tot}\colon \mathrm{PreOp}_{\mathcal{T}} \to \mathrm{Cat}$ *and* $\mathrm{Tot}_{\mathcal{T}}\colon \mathrm{PreOp}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ *are conservative.*

1.2.3. *The underlying $\mathcal{T}$-symmetric sequence.* Given $\mathcal{O}^\otimes$ a $\mathcal{T}$-preoperad and $(\mathbf{C};\mathbf{D})$ a $\underline{\pi}_0 U\mathcal{O}$-multi-profile lying over the multi-arity ψ, we define

$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(\mathbf{C};\mathbf{D}) \colon \coloneqq \mathrm{Map}^{\psi}_{\mathrm{Tot}\mathcal{O}}(\mathbf{C}, \mathbf{D}).$$

In particular, when ψ is an arity, so $D \coloneqq \mathbf{D}$ is a color, we define

$$\mathcal{O}(\mathbf{C}; D) \coloneqq \mathrm{Mul}_{\mathcal{O}}^{\psi}(\mathbf{C}; D).$$

In particular, when $\mathcal{O}^\otimes$ has at most one color, we write $\mathcal{O}(S) \coloneqq \mathcal{O}(S; V)$.

Proposition 1.19 ([Ste25b])**.** *There is a monadic (and in particular, conservative) functor*

$$\mathrm{sseq}\colon \mathrm{Op}_{\mathcal{T}}^{\leq oc} \to \mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$$

so that $\mathrm{sseq}\,\mathcal{O}(S) \simeq \mathcal{O}(S)$.

1.2.4. *(Co)cartesian structures, $\mathcal{O}$-comonoids.* If $\mathcal{C}$ is a $\mathcal{T}$-∞-category then it admits a canonical *coccartesian* $\mathcal{T}$-operad structure $\mathcal{C}^{\mathcal{T}-\sqcup}$, which is an I-symmetric monoidal ∞-category precisely when $\mathcal{C}$ admits indexed coproducts. This was initially constructed by Nardin-Shah [NS22], and it was subsequently shown to be uniquely determined by the property that its indexed tensor products are indexed coproducts in [Ste25d]. Dually, when $\mathcal{C}$ admits indexed products, it admits an essentially unique $\mathcal{T}$-symmetric monoidal structure $\mathcal{C}^{\mathcal{T}-\times}$ whose indexed tensor products are indexed products.

For concreteness, we recall the following construction of [LLP25].

Construction 1.20 ([LLP25, Def 2.15])**.** Since $\mathbb{F}_{\mathcal{T}}$ is the finite-coproduct closure of $\mathcal{T}$, a $\mathcal{T}$-∞-category $\mathcal{C}\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}$ corresponds with a unique product-preserving functor $\widetilde{\mathcal{C}}\colon \mathbb{F}_{\mathcal{T}}^{\mathrm{op}} \to \mathrm{Cat}$. Let $\mathrm{Un}^{cart}\left(\widetilde{\mathcal{C}}\right) \to \mathbb{F}_{\mathcal{T}}$ be the cartesian unstraightening of $\widetilde{\mathcal{C}}$. Then, we define the functor

$$\mathcal{C}^{\mathcal{T}-\sqcup}\colon \mathrm{Span}_{\mathrm{cart,all}}\left(\mathrm{Un}^{\mathrm{cart}}(\mathbb{F}_{\mathcal{T}})\right) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}}). \qquad \triangleleft$$

This was confirmed to agree with the other notions in [Ste25d]. Regardless, cocartesian structures take on a universal role characterized by the *triviality* of algebra structures:

Proposition 1.21 ([Ste25d])**.** *A unital $\mathcal{T}$-operad $\mathcal{C}^{\otimes}$ is cocartesian if and only if, for all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}}$, the forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(U\mathcal{O}, U\mathcal{C})$$

is an equivalence; in particular, the formation of cocartesian structures yields a right adjoint

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}}^{\mathrm{uni}} \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\mathcal{T}} \arrow[l, bend left, "(-)^{\mathcal{T}-\sqcup}"] \end{tikzcd}$$

We will need *combinatorial* control of the $\mathcal{T}$-preoperad underlying $\mathcal{C}^{I-\sqcup}$ throughout this article, which can be achieved by using an explicit Γ-$\mathcal{T}$-preoperad model generalizing [HA] as in [Ste25d]. However, to keep this article grounded in $\mathcal{T}$-preoperads, we will instead give some independent analysis using the universal property. For this, we must use the *left* adjoint to U.

Proposition 1.22 ([Ste25b])**.** *The* $\mathcal{T}$-∞-category of colors *functor additionally attains a left adjoint:*

$$\begin{tikzcd} \mathrm{Op}_{\mathcal{T}} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\mathcal{T}} \arrow[l, bend right, "\mathrm{triv}(-)^{\otimes}"'] \end{tikzcd}$$

Moreover, the value $\mathrm{triv}_{\mathcal{T}}^{\otimes} \coloneqq \mathrm{triv}(*_{\mathcal{T}})^{\otimes}$ *is presented by the $\mathcal{T}$-preoperad* $\mathrm{Span}_{iso}(\mathbb{F}_{\mathcal{T}}) \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$.

1.2.5. *$\mathcal{O}$-monoids, Morita equivalences.* We now center a definition.

Definition 1.23. Let $\mathfrak{O}, \mathfrak{P}$ be algebraic patterns and $f\colon \mathfrak{O} \to \mathfrak{P}$ a functor between their underlying ∞-categories. We say that f is *compatible with Segal objects* if the pullback functor restricts to Segal objects:

$$\begin{tikzcd} \mathrm{Fun}(\mathfrak{P}, \mathcal{C}) \arrow[r, "f^*"] & \mathrm{Fun}(\mathfrak{O}, \mathcal{C}) \\ \mathrm{Seg}_{\mathfrak{P}}(\mathcal{C}) \arrow[u, hook] \arrow[r, dashed, "f^*_{\mathrm{Seg}}"'] & \mathrm{Seg}_{\mathfrak{O}}(\mathcal{C}) \arrow[u, hook] \end{tikzcd}$$

We additionally say that f is a *Morita equivalence* if f^*_{Seg} is an equivalence. $\triangleleft$

Now, given $\mathfrak{B}$ an algebraic pattern and $\mathfrak{O} \to \mathfrak{B}$ a functor admitting cocartesian lifts over inert morphisms, $\mathfrak{O}$ admits a canonical algebraic pattern structure whose inert morphisms are cocartesian lifts of inert arrows, whose active arrows are are arbitrary lifts of active arrows, and whose elementary objects are arbitrary lifts of elementary objects. In particular, given $\mathcal{O}^\otimes \to \mathrm{Span}(\mathbb{F}_\mathcal{T})$ a $\mathcal{T}$-preoperad, there is a canonical algebraic pattern structure on $\mathcal{O}^\otimes$.

Definition 1.24. Given $\mathcal{C}$ an ∞-category and $\mathcal{O}^\otimes$ a $\mathcal{T}$-preoperad, an *$\mathcal{O}$-monoid in $\mathcal{C}$* is a Segal $\mathcal{O}^\otimes$-object in $\mathcal{C}$; we refer to the associated full subcategory as

$$\mathrm{Mon}_\mathcal{O}(\mathcal{C}) \coloneqq \mathrm{Seg}_{\mathcal{O}^\otimes}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{C}). \qquad \triangleleft$$

Now, in [Ste25d] given a $\mathcal{T}$-∞-category $\mathcal{D}$ with finite indexed products, we characterized a *Cartesian $\mathcal{T}$-symmetric monoidal structure* $\mathcal{D}^{\mathcal{T}-\times}$ which is characterized by the property that $\otimes^S \simeq \prod^S$, and characterized its algebras by the following.

Proposition 1.25 ([Ste25d])**.** *Given $\mathcal{O}^\otimes$ a $\mathcal{T}$-preoperad The forgetful functor*

$$\mathrm{Alg}_\mathcal{O}\left(\underline{\mathrm{Coeff}}^\mathcal{T} \mathcal{C}^{\mathcal{T}-\times}\right) \to \mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{C})$$

is fully faithful with image the $\mathcal{O}$-monoids.

In particular, the atomic orbital counterpart to Elmendorf's theorem defines $\underline{\mathcal{S}}_\mathcal{T} \coloneqq \underline{\mathrm{Coeff}}^\mathcal{T} \mathcal{S}$, so this characterizes Segal $\mathcal{O}$-spaces as $\mathcal{O}$-algebras in $\mathcal{T}$-spaces. Now, by examining the *free $\mathcal{O}$-$\mathcal{T}$-space monad* we have the following convenient property.

Proposition 1.26 ([Ste25b])**.** *Suppose $\varphi\colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ is a morphism of $\mathcal{T}$-operads which induces equivalences*

$$\mathrm{Mon}_\mathcal{P}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Mon}_\mathcal{O}(\mathcal{S})$$

and $U\mathcal{O} \xrightarrow{\sim} U\mathcal{P}$; then, φ is an equivalence.

Corollary 1.27 ([Ste25d])**.** *If $\varphi\colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ is a morphism of $\mathcal{T}$-preoperads inducing an equivalence $UL_{\mathrm{Op}_\mathcal{T}}\mathcal{O}^\otimes \to UL_{\mathrm{Op}_\mathcal{T}}\mathcal{P}^\otimes$, then φ is an $L_{\mathrm{Op}_\mathcal{T}}$-equivalence if and only if it's a Morita equivalence.*

1.2.6. *The Boardman-Vogt tensor product.* in [Ste25b], we used this to define the *Boardman-Vogt tensor product*, which in the case that $\mathcal{T}$ has a terminal object is computed by the $\mathcal{T}$-operadic localization

$$\mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes \simeq L_{\mathrm{Op}_\mathcal{T}}\left(\mathcal{O}^\otimes \times \mathcal{P}^\otimes \xrightarrow{\pi} \mathrm{Span}(\mathbb{F}_\mathcal{T}) \times \mathrm{Span}(\mathbb{F}_\mathcal{T}) \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_\mathcal{T})\right).$$

We verified in [Ste25d] that this endows $\mathrm{Op}_\mathcal{T}$ with the structure of a *presentably symmetric monoidal $\mathcal{T}$-∞-category*, i.e. we may use it to lift $\mathrm{Op}_\mathcal{T}$ to a functor

$$\underline{\mathrm{Op}}_\mathcal{T}^\otimes\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{CAlg}\left(\mathrm{Pr}^L\right).$$

In particular, this yields a *distributivity property* for $\overset{\mathrm{BV}}{\otimes}$ in terms of $\mathcal{T}$-colimits.

1.2.7. *$\mathcal{T}$-operadic disintegration and assembly.* In [Ste25d], given a unital $\mathcal{T}$-operad $\mathcal{O}^\otimes$ and a V-object $X \in \mathcal{O}_V$, we defined the *reduced endomorphism V-operad*

$$\begin{tikzcd}
\mathrm{End}_X^{\mathrm{red}}(\mathcal{O}^\otimes) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Res}_V^\mathcal{T} \mathcal{O}^\otimes \arrow[d] \\
\mathrm{Comm}_V^\otimes \arrow[r, "\{X\}"'] & \left(\mathrm{Res}_V^\mathcal{T} U\mathcal{O}\right)^{V-\sqcup}
\end{tikzcd}$$

where $\{X\}$ classifies X under Proposition 1.21. We will use the following terminology.

Definition 1.28. Given $\mathcal{P}^\otimes$ a reduced $\mathcal{T}$-operad, $\mathcal{O}^\otimes$ a unital $\mathcal{T}$-operad, and $A\colon \mathcal{P}^\otimes \to \mathcal{O}^\otimes$ an $\mathcal{O}$-algebra in $\mathcal{P}^\otimes$ with underlying $\mathcal{T}$-object $x \in \Gamma^{\mathcal{T}} U\mathcal{O}$, the *reduction of* A is the algebra $\widetilde{A}$ defined as

$$\begin{tikzcd}
\mathcal{P}^\otimes \arrow[dr, dashed, "\widetilde{A}"] \arrow[drr, bend left, "A"] \arrow[ddr, bend right, "!"'] & & \\
& \mathrm{End}_x^{\mathrm{red}}(\mathcal{O}^\otimes) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^\otimes \arrow[d] \\
& \mathrm{Comm}_{\mathcal{T}}^\otimes \arrow[r, "\{x\}"'] & U\mathcal{O}^{\mathcal{T}-\sqcup}
\end{tikzcd}$$

This underlies an adjunction between *pointed* unital $\mathcal{T}$-operads and reduced $\mathcal{T}$-operads [Ste25a]. ◁

Now, the main result we use concerning this is the following.

Proposition 1.29 ([Ste25d])**.** *Suppose $\mathcal{O}^\otimes$ is a unital $\mathcal{T}$-operad such that $U\mathcal{O}$ is a $\mathcal{T}$-space. Then, the maps $\mathrm{End}_X^{\mathrm{red}}(\mathcal{O}^\otimes) \to \mathrm{Res}_V^{\mathcal{T}} \mathcal{O}^\otimes$ assemble to a $\mathcal{T}$-colimit diagram*

$$\underline{\mathrm{colim}}_{X \in U\mathcal{O}} \mathrm{End}_X^{\mathrm{red}}(\mathcal{O}^\otimes) \xrightarrow{\sim} \mathcal{O}^\otimes.$$

Corollary 1.30. *Suppose $\varphi\colon \mathcal{O}^\otimes \overset{\mathrm{BV}}{\otimes} \mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ is a morphism of $\mathcal{T}$-operads inducing equivalences*

$$U\mathcal{O} \times U\mathcal{P} \xrightarrow{\sim} U\mathcal{Q}$$
$$\mathrm{End}_X^{\mathrm{red}}(\mathcal{O}^\otimes) \overset{\mathrm{BV}}{\otimes} \mathrm{End}_Y^{\mathrm{red}}(\mathcal{P}^\otimes) \xrightarrow{\sim} \mathrm{End}_{(X,Y)}^{\mathrm{red}}(\mathcal{Q}^\otimes).$$

Then, φ is an equivalence.

1.2.8. *Remarks on topological categories.* The following proposition may be familiar.

Proposition 1.31 ([HTT, Rem A.3.2.11, Thm 2.2.5.1])**.** *There is a zigzag of Quillen equivalences*

$$\begin{tikzcd}
\mathrm{sSet}_{\mathrm{Joyal}} \arrow[r, bend left, "\mathfrak{C}[-]"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\mathrm{sSet}_{\mathrm{Quillen}}} \arrow[l, bend left, "N"] \arrow[r, bend left, "|\cdot|_*"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\mathrm{Top}_{\mathrm{Quillen}}} \arrow[l, bend left, "\mathrm{Sing}(-)_*"]
\end{tikzcd}$$

between the Joyal model structure on simplicial sets and the Dwyer-Kan model structure on topological(ly enriched) categories with respect to the Quillen (monoidal) model structure on topological spaces.

In particular, the Hammock localization of $\mathrm{Cat}_{\mathrm{Top}}$ at the collection of essentially surjective and homotopy-fully faithful functors presents the ∞-category Cat of small ∞-categories. This allows us license to do *evil*, the first such deed being the following definition.

Definition 1.32. A topological functor $F\colon \mathcal{C} \to \mathcal{D}$ is *point-set monic* if it is injective on objects and for each $(X,Y) \in \mathrm{Ob}(\mathcal{C})$ of hom-topological spaces $\underline{\mathrm{Hom}}(X,Y) \to \underline{\mathrm{Hom}}(FX,FY)$ is injective. ◁

Evil is relatively common, per the following corollary of the axiom of choice.

Lemma 1.33. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is* ***essentially*** *injective and each morphism of hom-topological spaces $\underline{\mathrm{Hom}}(X,Y) \to \underline{\mathrm{Hom}}(FX,FY)$ is injective. Then, there exists a diagram of topological functors*

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "F'"] \arrow[d, "\beta"] & \mathcal{D}' \arrow[d, "\beta'"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

such that F' is point-set monic and β, β' are Dwyer-Kan equivalences.

Proof. Choose $\mathcal{C}', \mathcal{D}'$ to be the full topological categories spanned by skeleta for the underlying categories of $\mathcal{C}$ and $\mathcal{D}$, and note that F' has injective hom functions by cancellation and is injective on objects by skeletality. □

Now, this was designed for the sake of the following proposition.

Proposition 1.34. *Suppose we have a diagram in* $\mathrm{Cat}_{\mathrm{Top},/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})}$ *specifying the solid arrows of*

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r, dashed, "\varphi"] \arrow[d, "\iota"] & \mathcal{P}^{\otimes} \arrow[d, "\iota'"] \\ \mathcal{Q}^{\otimes} \arrow[r, "\psi"] & \mathcal{R}^{\otimes} \end{tikzcd}$$

where ι' is point-set monic and every indicated functor presents a morphism of $\mathcal{T}$-preoperads. Then, there exists at most one functor φ filling in the dashed arrow such that the diagram commutes up to homotopy, in which case φ presents a morphism of $\mathcal{T}$-preoperads. This filler exists if and only if the following conditions are satisfied:

(a) *For all $\mathbf{C} \in \mathcal{O}^{\otimes}$, $\psi\iota(\mathbf{C})$ is in the image of ι'.*

(b) *For all multi-operations $\mu \in \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{C};\mathbf{D})$, the element $\iota\psi(\mu)$ is in the image of ι'.*

In this case, $\varphi(\mathbf{C}) = \iota'^{-1}\psi\iota(\mathbf{C})$ and $\varphi(\mu) = \iota'^{-1}\psi\iota(\mu)$.

Proof. Since ι' is point-set monoic, if φ exists, then it is compatible with inert-cocartesian lifts, i.e. it's a morphism of $\mathcal{T}$-preoperads. Now, note that (a) and (b) are satisfied is φ exists, so instead assume that (a) and (b) are satisfied; then, we define the object function of φ by $\varphi(\mathbf{C}) := \iota'^{-1}\psi\iota(\mathbf{C})$, which is well-defined and unique by assumption, and the morphism-function by $\varphi(\mu) = \iota'^{-1}\psi\iota(\mu)$. The fact that this is functorial follows by the fact that the composite $\iota' \circ \varphi = \psi \circ \iota$ is functorial and ι' is point-set monic. □

Remark 1.35. In the nonequivariant case, [HA, Rmk 5.4.2.14] left implicit the construction of a lift

$$\begin{tikzcd} \mathbb{E}^{\otimes}_{B\mathrm{Top}(k)} \times \mathbb{E}^{\otimes}_{B\mathrm{Top}(k')} \arrow[r, dashed, "\varphi"] \arrow[d, hook] & \mathbb{E}^{\otimes}_{B\mathrm{Top}(k+k')} \arrow[d, hook] \\ B\mathrm{Top}(k)^{\sqcup} \times B\mathrm{Top}(k')^{\sqcup} \arrow[r, "\varpi^*"'] & B\mathrm{Top}(k+k')^{\sqcup} \end{tikzcd}$$

where we've set the notation $\mathbb{E}^{\otimes}_{B\mathrm{Top}(k)} := B\mathrm{Top}(k)^{\otimes}$, the vertical arrows classify the equivalence $U\mathbb{E}_{B\mathrm{Top}(k)} \simeq B\mathrm{Top}(k)$ of [HA, Thm 5.4.1.5], and the bottom arrow is induced by a homeomorphism $\varpi\colon \mathbb{R}^k \oplus \mathbb{R}^{k'} \simeq \mathbb{R}^{k+k'}$. In Section 3.2, we carefully construct point-set monic models for the vertical arrows and observe that such a lift exists on the level of topological functors, which presents a morphism of e-preoperads by Proposition 1.34. ◂

1.2.9. *The $\mathcal{T}$-operadic nerve.* Given $\mathcal{O}$ a $\mathfrak{C}$-colored genuine topological $\mathcal{T}$-operad, we can define a topological category $N^{\otimes}\mathcal{O}$ over $\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ to have objects

$$\mathrm{Ob}\,\pi^{-1}(\{S\}) := \mathfrak{C}^{S},$$

and morphism space lying over $S \xleftarrow{g} R \xrightarrow{f} T$ by

$$\mathrm{Map}_{N^{\otimes}\mathcal{O}}^{S \xleftarrow{g} R \xrightarrow{f} T}(\mathbf{C};\mathbf{D}) := \mathrm{Mul}_{\mathcal{O}}^{f}(g^*\mathbf{C};\mathbf{D}),$$

with identity arrows and composition functions determined by $1_{\bullet}$ and γ.

Proposition 1.36 ([NS22; Ste25b])**.** *$N^{\otimes}\mathcal{O}$ is a $\mathcal{T}$-operad with $\underline{\pi}_0 U N^{\otimes}\mathcal{O} \simeq \mathfrak{C}$ and structure spaces*

$$\mathrm{Mul}_{N^{\otimes}\mathcal{O}}^{f}(\mathbf{C};\mathbf{D}) \simeq \mathrm{Mul}_{\mathcal{O}}^{f}(\mathbf{C};\mathbf{D}).$$

1.3. Weak approximations of $\mathcal{T}$-preoperads, reflective quotients.

1.3.1. *Weak approximations.* We derive the following from [Har] in Appendix A.1.

Proposition 1.37 (c.f. [HA, Thm 2.3.3.22])**.** *If $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a weak approximation of $\mathcal{T}$-preoperads and $\mathcal{C}$ a complete ∞-category, the pullback functor*

$$\mathrm{Mon}_{\mathfrak{P}}(\mathcal{C}) \to \mathrm{Mon}_{\mathfrak{O}}(\mathcal{C})$$

is fully faithful and its essential image consists of those $\mathcal{O}$-monoids $\mathcal{O}^{\otimes} \to \mathcal{C}$ whose restriction admits a factorization

$$U\mathcal{O}^{\otimes} \to U\mathcal{P}^{\otimes} \to \mathcal{C};$$

in particular, every strong approximation is an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence.

We will often check for weak approximations using the following variation of [HA, Cor 2.3.3.16], which we also verify in Appendix A.1.

Corollary 1.38. *Suppose $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a map of $\mathcal{T}$-preoperads over $\mathcal{C}^{\mathcal{T}-\sqcup}$ such that $U\mathcal{P} \to \mathcal{C}$ is a cocartesian fibration whose fibers are spaces. Then, Condition (WA-a) of Definition 1.1 is equivalent to the condition that, for all $O \in \mathcal{O}$ and active $\mathcal{C}^{\mathcal{T}-\sqcup}$-maps $f\colon D \to \pi_{\mathcal{O}} O$, the induced map of spaces*

$$\widetilde{\varphi}\colon B\left(\mathcal{O}^{\mathrm{act}}_{/O} \times_{\mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi_{\mathcal{O}}O}} \{f\}\right) \to \mathcal{P}^{\mathrm{act}}_{/\varphi O} \times_{\mathcal{C}_{/\pi_{\mathcal{O}}O}} \{f\}$$

is an equivalence. In particular, if $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-operad, then the above map has the form

$$\widetilde{\varphi}\colon B\left(\mathcal{O}^{\mathrm{act}}_{/O} \times_{\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\varphi O}} \{f\}\right) \to \mathcal{P}(P;\varphi O)$$

1.3.2. *Reflective quotients.*

Definition 1.39. Given $\mathcal{P}^{\otimes}$ a $\mathcal{T}$-operad and $L\colon U\mathcal{P} \to \mathcal{C}$ a reflective G-subcategory with reflective , we define the corresponding *reflective quotient* by

$$\mathrm{Refl}^{\otimes}_{\mathcal{C}}(\mathcal{P}) := \mathcal{P}^{\otimes} \sqcup_{\mathrm{triv}(U\mathcal{P})} \mathrm{triv}(\mathcal{C}). \qquad \triangleleft$$

The following proposition follows by unwinding definitions.

Proposition 1.40. *Pullback along the structure map $\mathcal{P}^{\otimes} \to \mathrm{Refl}^{\otimes}_{\mathcal{C}}(\mathcal{P})$ yields a filly faithful embedding*

$$\mathrm{Alg}_{\mathrm{Refl}_{\mathcal{C}}(\mathcal{P})}(\mathcal{Q}) \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{Q})$$

with essential image spanned by those $\mathcal{P}$-algebras whose underlying $\mathcal{T}$-funcor $U\mathcal{P} \to U\mathcal{Q}$ factors through L.

Corollary 1.41. *If $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a weak approximation such that $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-operad, then the induced map $\varphi\colon \mathrm{Refl}^{\otimes}_{U\mathcal{O}}\mathcal{P} \to \mathcal{O}^{\otimes}$ is an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence; in particular, for all $\mathcal{T}$-operads $\mathcal{Q}$, the pullback functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{Q}) \longrightarrow \mathrm{Alg}_{\mathcal{P}}(\mathcal{Q})$$

is fully faithful with essential image the $\mathcal{P}$-algebras whose color $\mathcal{T}$-functor factors as $U\mathcal{P} \to U\mathcal{O} \to U\mathcal{Q}$.

Proof. Choosing $\mathcal{Q} := \underline{\mathcal{S}}^{\mathcal{T}-\times}_{\mathcal{T}}$, we acquire a diagram

$$\begin{tikzcd}
\mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \arrow[rr, hook] \arrow[dr, hook] & & \mathrm{Mon}_{\mathcal{P}}(\mathcal{S}) \\
& \mathrm{Mon}_{\mathrm{Refl}_{U\mathcal{O}}(\mathcal{P})}(\mathcal{S}) \arrow[ur, hook] &
\end{tikzcd}$$

so that the pullback functor $\mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Mon}_{\mathrm{Refl}_{U\mathcal{O}}(\mathcal{P})}(\mathcal{S})$ is fully faithful by two-out-of-three. In fact, by Propositions 1.37 and 1.40 we find that it is essentially surjective, so Corollary 1.27 implies that φ is an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence. $\square$

1.4. **The $\mathcal{T}$-preoperadic image.** For the duration of this subsection, let $\mathcal{O}^{\otimes}, \mathcal{Q}^{\otimes}$ be $\mathcal{T}$-preoperads and $\varphi\colon \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ an essentially surjective map of $\mathcal{T}$-preoperads.

Definition 1.42. The *$\mathcal{T}$-preoperadic image of φ* is the wide subcategory $\mathrm{im}\varphi^{\otimes} \subset \mathcal{Q}^{\otimes}$ containing an arrow f if and only if there are some arrows (g) in $\mathcal{O}^{\otimes}$ such that f is homotopic to $\varphi(g)$. $\triangleleft$

Lemma 1.43. $\mathrm{im}\varphi^{\otimes}$ *admits a (unique) $\mathcal{T}$-preoperad structure such that each of the functors*

$$\mathcal{O}^{\otimes} \to \mathrm{im}\varphi^{\otimes} \xrightarrow{\iota} \mathcal{Q}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$$

are $\mathcal{T}$-preoperads maps.

Proof. By construction, the $\mathcal{T}$-preoperad structure must just be the composite $\mathrm{im}\varphi^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$, handling uniqueness. Note that $\mathrm{im}\varphi^{\otimes}$ contains every inert-cocartesian arrow of $\mathcal{Q}^{\otimes}$ by the essential surjectivity assumption; it will suffice to verify that inert-cocartesian arrows of $\mathcal{Q}^{\otimes}$ are ι-cocartesian and apply [HTT, HTT 2.4.1.3.(3)].

Fix some cocartesian arrow $f\colon X \to Y$ in $\mathcal{Q}^{\otimes}$; we are tasked with verifying that the map ψ in the following diagram is an equivalence.

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{im}\varphi^{\otimes}}(X,Z) \arrow[drr, "f^*"] \arrow[dr, "g"] \arrow[ddr, hook] & & \\
& F \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathrm{im}\varphi^{\otimes}}(Y,Z) \arrow[d, hook] \\
& \mathrm{Map}_{\mathcal{Q}^{\otimes}}(X,Z) \arrow[r, "f^*"] & \mathrm{Map}_{\mathcal{Q}^{\otimes}}(Y,Z)
\end{tikzcd}$$

Note that F and $\mathrm{Map}_{\mathrm{im}\varphi^{\otimes}}(X,Y)$ embed as summands of $\mathrm{Map}_{\mathcal{Q}^{\otimes}}(X,Y)$, so it suffices to verify that whenever a $\mathcal{Q}^{\otimes}$-map $h\colon X \to Y$ lies in F, it is in the image of φ. Indeed, such an h admits a factorization $X \xrightarrow{f} Y \xrightarrow{\varphi g} Z$ for some morphism g in $\mathcal{O}^{\otimes}$, and essential surjectivity ensures that $f \sim \varphi\widetilde{f}$ for some $\widetilde{f}$ in $\mathcal{O}^{\otimes}$, so $h \sim \varphi\left(g \circ \widetilde{f}\right)$. such that $\square$

1.5. **Wreath products.**

1.5.1. *What are wreath products?*

Construction 1.44. Given $\mathcal{P}^{\otimes}$ a $\mathcal{T}$-preoperad, in [Ste25d] we used the structure functor on $\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes}$ to construct an alternative structure functor $\rho\colon (\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes})^{\mathcal{T}-\sqcup} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$;

$$\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} := \mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathbb{F}_{\mathcal{T}}),\pi} \left(\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes}\right)^{\mathcal{T}-\sqcup} \xrightarrow{\rho} \mathrm{Span}(\mathbb{F}_{\mathcal{T}}).$$

We verified in [Ste25d] that the localization functor for $\mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes}$ restricts to a "diagonal" functor $\gamma\colon \mathrm{Span}(\mathbb{F}_V) \times \mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes} \to (\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes})^{\mathcal{T}-\sqcup}$ making the following diagram of ∞-categories commute

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[rr, "\pi_{\mathcal{O}}"] \arrow[dr, dashed, "M"] \arrow[ddr, "\mathrm{pr}_1"'] & & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathcal{P}^{\otimes} \arrow[rr, "\pi_{\mathcal{P}}"] \arrow[dr, "\gamma"] & & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \times \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \arrow[dr, "\wedge"] & \\
& \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & (\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes})^{\mathcal{T}-\sqcup} \arrow[rr, "\rho"] \arrow[d, "\pi"] & & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \\
& \mathcal{O}^{\otimes} \arrow[rr, "\pi_{\mathcal{O}}"'] & & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) & &
\end{tikzcd}$$

In particular, we acquire a natural morphism of $\mathcal{T}$-preoperads $M\colon \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$. $\triangleleft$

We defer the proof of following central theorem to Appendix A.2; unfortunately, it is unenlightening.

Theorem 1.45. *If $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are $\mathcal{T}$-operads, then $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ is a Morita equivalence; in particular $L \circ M$ factors through a unique equivalence*

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[r, "M"] \arrow[d, "L"] & \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \arrow[d, "L"] \\
\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \arrow[r, dashed, "\sim"] & L_{\mathrm{Op}_{\mathcal{T}}} \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}
\end{tikzcd}$$

The following proposition is obvious.

Proposition 1.46. *Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be $\mathcal{T}$-operads. Then, the fiber ∞-categories of $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ are given by*

$$\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right)_T \simeq \coprod_{T \to S} \mathcal{O}_S \times \mathcal{P}_T.$$

Moreover, given a multi-arity $g\colon T' \to T$, the multimorphism spaces of $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ over g are

$$\mathrm{Mul}^{g}_{\mathcal{O}\wr\mathcal{P}}(h'\colon T' \to S', \mathbf{C}', \mathbf{D}';\ h\colon T \to S, \mathbf{C}, \mathbf{D}) \simeq \coprod_{\substack{f\colon S' \to S \\ fh = h'g}} \mathrm{Mul}^{f}_{\mathcal{O}}(\mathbf{C}';\mathbf{C}) \times \mathrm{Mul}^{g}_{\mathcal{P}}(\mathbf{D}';\mathbf{D}).$$

1.5.2. *How do we use wreath products?* The wreath product construction is of a nontrivial combinatorial nature; for simplicity's sake, we pre-macerate these combinatorics, yielding recognition results which will provide our interface for working with wreath products going forward. First, we describe how to compute wreath products of $\mathcal{T}$-operads with contractible structure spaces.

Proposition 1.47. *Suppose $\mathcal{O}^\otimes, \mathcal{P}^\otimes, \mathcal{Q}^\otimes$ are topologically enriched categories over $\mathrm{Span}(\mathbb{F}_\mathcal{T})$ which present $\mathcal{T}$-operads and $\mathcal{Q}$ has contractible structure spaces. Then, a $\mathcal{T}$-preoperad map $\mathcal{O}^\otimes \wr \mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ is uniquely specified by a map of coefficient systems of sets*

$$\varphi\colon \left(\coprod_{T\to S} \pi_0 \mathcal{O}_S\right) \times \pi_0 \mathcal{P}_T \to \pi_0 \mathcal{Q}_T$$

such that for all data

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d] & T \arrow[d] \\ S' \arrow[r] & V \end{tikzcd}\ ; \qquad\qquad ((\mathbf{C}'; C), (\mathbf{B}'; \mathbf{B})) \in \mathcal{O}_{S'} \times \mathcal{O}_V \times \mathcal{P}_{T'} \times \mathcal{P}_T.$$

such that $\mathcal{O}(\mathbf{C}'; C) \times \mathrm{Mul}^g_{\mathcal{P}}(\mathbf{B}'; \mathbf{B}) \neq \varnothing$, we have $\mathrm{Mul}^h_{\mathcal{Q}}(\mathbf{B}'; \mathbf{B}) \neq \varnothing$.

Moreover, if $\mathcal{O}^\otimes, \mathcal{P}^\otimes$ also have contractible structure spaces, to verify Condition (WA-a) for the resulting map to the preoperadic image $\widetilde{\varphi}\colon \mathcal{O}^\otimes \wr \mathcal{P}^\otimes \to \mathrm{im}(\varphi)$, it suffices to verify that for all data as above, the induced map of posets

$$\coprod_{S'\to S} \mathcal{O}^{\mathrm{act}}_{/\mathbf{C}} \times_{\mathbb{F}_{\mathcal{T},/T}} \{T'\} \times \mathcal{P}^{\mathrm{act}}_{/\mathbf{B}} \times_{\mathbb{F}_{\mathcal{T},/S}} \{S'\} \longrightarrow \mathcal{Q}^{\mathrm{act}}_{/T} \times_{\mathbb{F}_{\mathcal{T},/T}} \{T'\}$$

is injective.

Proof. First note that φ is a map $\mathrm{Ob}(\mathcal{O}^\otimes \wr \mathcal{P}^\otimes)_T \to \mathcal{Q}_T$; to specify a topological functor $\mathcal{O}^\otimes \wr \mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ over $\mathrm{Span}(\mathbb{F}_\mathcal{T})$ it then suffices to define the data on mapping spaces. Indeed, we determine this canonically via the factorization

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{O}^\otimes \wr \mathcal{P}^\otimes}((\mathbf{C}', \mathbf{B}'), (\mathbf{B}', \mathbf{B})) \arrow[r] & \mathrm{Map}_{\mathcal{Q}^\otimes}(\varphi(\mathbf{C}', \mathbf{B}', \varphi(\mathbf{B}'; \mathbf{B})) \\ \mathrm{Mul}^f_{\mathcal{O}^\otimes \wr \mathcal{P}^\otimes}((\mathbf{C}', \mathbf{B}'); (\mathbf{C}, \mathbf{B})) \arrow[u, hook] \arrow[r, dashed] & \mathrm{Mul}^f_{\mathcal{Q}^\otimes}(\varphi(\mathbf{C}', \mathbf{B}'); \varphi(\mathbf{B}', \mathbf{B})) \arrow[u, hook] \end{tikzcd}$$

There exists an essentially unique filler as in the dashed arrow since whenever the domain is nonempty, the codomain is contractible by assumption. To verify functoriality, we must verify a that a collection of diagrams among products of multimorphism spaces of $\mathcal{Q}^\otimes$ commute; however, since $\mathcal{Q}^\otimes$ has contractible nonempty structure spaces, all terms in such diagrams are either empty or contractible, so all such diagrams commute.

Having proved the first statement, we now verify the remaining condition. Now, fix some $(\mathbf{C}, \mathbf{B}) \in \mathcal{O}^\otimes \wr \mathcal{P}^\otimes$ and some $h \in \mathcal{Q}^{\mathrm{act}}_{/\varphi(\mathbf{C}, \mathbf{B})}$. Let $T' \to T$ be the multi-arity under h. We may form the diagram

$$\begin{tikzcd} F' \arrow[r] \arrow[d, "\psi'"] \arrow[dr, phantom, "\lrcorner"] & \coprod_{S'\to S} \mathcal{O}^{\mathrm{act}}_{/\mathbf{C}} \times_{\mathbb{F}_{\mathcal{T},/T}} \{T'\} \times \mathcal{P}^{\mathrm{act}}_{/\mathbf{B}} \times_{\mathbb{F}_{\mathcal{T},/S}} \{S'\} \arrow[d, "\psi"] & \\ F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^\otimes \wr \mathcal{P}^\otimes)^{\mathrm{act}}_{/(\mathbf{C}, \mathbf{B})} \times_{\mathbb{F}_{\mathcal{T},/T}} \{T'\} \arrow[r] \arrow[d, "\widetilde{\varphi}"] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^\otimes \wr \mathcal{P}^\otimes)^{\mathrm{act}}_{/(\mathbf{C}, \mathbf{B})} \arrow[d] \\ \{h\} \arrow[r] & \mathcal{Q}^{\mathrm{act}}_{/\varphi(\mathbf{C}, \mathbf{B})} \times_{\mathbb{F}_{\mathcal{T},/T}} \{T'\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Q}^{\mathrm{act}}_{/\varphi(\mathbf{C}, \mathbf{B})} \arrow[d] \\ & \{T'\} \arrow[r] & \mathbb{F}_{\mathcal{T},/T} \end{tikzcd}$$

Now, note the following facts:

(1) By assumption, $\mathcal{Q}^{\mathrm{act}}_{/\varphi(\mathbf{C},\mathbf{B})} \times_{\mathbb{F}_{\mathcal{T},/S}} \{T\}$ is a *poset*, as is the contractible category $\{h\}$. In particular, under the assumption that $\mathcal{O}^\otimes$ and $\mathcal{P}^\otimes$ also have contractible nonempty structure spaces, $(\mathcal{O}^\otimes \wr \mathcal{P}^\otimes)^{\mathrm{act}}_{/(\mathbf{C},\mathbf{B})} \times_{\mathbb{F}_{\mathcal{T},/S}} \{T\}$ and $\coprod_{S' \to S} \mathcal{O}^{\mathrm{act}}_{/\mathbf{C}} \times_{\mathbb{F}_{\mathcal{T},/T}} \{T'\} \times \mathcal{P}^{\mathrm{act}}_{/\mathbf{B}} \times_{\mathbb{F}_{\mathcal{T},/S}} \{S'\}$ are posets.
(2) Since the inclusion Poset $\subset$ Cat attains a left adjoint, posets are closed under pullback, so in particular F and F' are presented by the pullback of the associated diagrams *in the category of posets*, i.e. by preimages.
(3) By assumption, $F' \in \{\varnothing, *\}$.
(4) ψ is bijective, so ψ' is also bijective.

By combining these facts, we find that $F \in \{\varnothing, *\}$. Now, instead fix some $h \in \mathrm{im}(\varphi)^{\mathrm{act}}_{/\varphi(\mathbf{C},\mathbf{B})}$ and draw the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^\otimes \wr \mathcal{P}^\otimes)^{\mathrm{act}}_{/(\mathbf{C},\mathbf{B})} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^\otimes \wr \mathcal{P}^\otimes)^{\mathrm{act}}_{/(\mathbf{C},\mathbf{B})} \arrow[d] \\
\{h\} \arrow[r] & \mathrm{im}(\varphi)^{\mathrm{act}}_{/\varphi(\mathbf{C},\mathbf{B})} \arrow[r] & \mathcal{Q}^{\mathrm{act}}_{/\varphi(\mathbf{C},\mathbf{B})}
\end{tikzcd}$$

By definition of $\mathrm{im}(\varphi)$ we know that F is nonempty, so the above argument shows that it consists of a single point, hence it is weakly contractible. □

Now, we cover the case of maps to one-color $\mathcal{T}$-operads.

Corollary 1.48. *Suppose $\mathcal{O}^\otimes, \mathcal{P}^\otimes, \mathcal{Q}^\otimes$ are the topologically enriched nerves of topological genuine $\mathcal{T}$-operads such that $\mathcal{Q}^\otimes$ has one color. A $\mathcal{T}$-preoperad map $\mathcal{O}^\otimes \wr \mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ is uniquely specified by the data of, for each composable arities $\mathrm{Ind}_V^{\mathcal{T}} T \to \mathrm{Ind}_V^{\mathcal{T}} S \to V$ and compatible $\mathcal{P}$ and $\mathcal{O}$ (multi-)profiles $(\mathbf{B}';\mathbf{B})$ and $(\mathcal{C}';C)$, a homotopy class of continuous map*

$$\varphi \colon \mathcal{O}(\mathbf{C}';C) \times \mathrm{Mul}^g_{\mathcal{P}}(\mathbf{B}';\mathbf{B}) \to \mathcal{Q}(T')$$

sending $\varphi 1 = 1$ and such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(\mathbf{C}';C) \times \mathrm{Mul}^g_{\mathcal{P}}(\mathbf{B}';\mathbf{B}) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Res}"] & \mathcal{Q}(T') \arrow[d, "\mathrm{Res}"] \\
\mathcal{O}\left(\mathrm{Res}^V_U \mathbf{C}'; \mathrm{Res}^V_U C\right) \times \mathrm{Mul}_{\mathcal{P}}^{\mathrm{Res}^V_U g}\left(\mathrm{Res}^V_U \mathbf{B}'; \mathrm{Res}^V_U \mathbf{B}\right) \arrow[r, "\varphi"] & \mathcal{Q}\left(\mathrm{Res}^V_U T'\right)
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
\left(\mathcal{O}(\mathbf{C}';\mathbf{C}) \times \mathrm{Mul}^h_{\mathcal{O}}(\mathbf{C}'';\mathbf{C}')\right) \times \left(\mathrm{Mul}^g_{\mathcal{P}}(\mathbf{B}';\mathbf{B}) \times \mathrm{Mul}^f_{\mathcal{P}}(\mathbf{B}'';\mathbf{B}')\right) \arrow[r, "{(\gamma,\gamma)}"] \arrow[d, phantom, "\wr"] & \mathcal{O}(\mathbf{C}'';C) \times \mathrm{Mul}^{f\circ g}_{\mathcal{P}}(\mathbf{B}'';\mathbf{B}) \arrow[dd, "\varphi"] \\
\left(\mathcal{O}(\mathbf{C}';C) \times \mathrm{Mul}^f_{\mathcal{P}}(\mathbf{B}';\mathbf{B})\right) \times \left(\mathrm{Mul}^h_{\mathcal{O}}(\mathbf{C}'';\mathbf{C}') \times \mathrm{Mul}^g_{\mathcal{P}}(\mathbf{B}'';\mathbf{B}')\right) \arrow[d, "{(\varphi,\varphi)}"] & \\
\mathcal{Q}(T') \times \mathrm{Mul}^{f\circ g}_{\mathcal{Q}}(T'';T') \arrow[r, "\gamma"] & \mathcal{Q}(T'')
\end{tikzcd} \tag{3}$$

The associated functor acts on active mapping spaces via φ.

Proof. Since $\mathcal{Q}^\otimes \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ is bijective on objects, there is exactly one map $\mathrm{Ob}(\mathcal{O}^\otimes \wr \mathcal{P}^\otimes) \to \mathrm{Ob}\,\mathcal{Q}^\otimes$ over $\mathrm{Ob}\,\mathrm{Span}(\mathbb{F}_{\mathcal{T}})$: $(\mathbf{C},\mathbf{B}) \mapsto T'$. To construct such a functor which is compatible with cocartesian arrows, we then must determine the data on mapping spaces, uniquely via the factorization

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}^\otimes \wr \mathcal{P}^\otimes}((\mathbf{C}',\mathbf{B}'),(\mathbf{B}',\mathbf{B})) \arrow[r] & \mathrm{Map}_{\mathcal{Q}^\otimes}(T',T) \\
\mathrm{Mul}^f_{\mathcal{O}^\otimes \wr \mathcal{P}^\otimes}((\mathbf{C}',\mathbf{B}');(\mathbf{C},\mathbf{B})) \arrow[r] \arrow[u, hook] \arrow[d] & \mathrm{Mul}^f_{\mathcal{Q}^\otimes}(T',T) \arrow[u, hook] \arrow[d] \\
\mathcal{O}(\mathbf{C}'_K;C_K) \times \mathrm{Mul}^g_{\mathcal{P}}(\mathbf{B}'_K;\mathbf{B}_K) \arrow[r, dashed, "\varphi"] & \mathcal{Q}(T'_K)
\end{tikzcd}$$

We're left with functoriality; this follows by unwinding definitions. □

1.6. **The bookkeeping theorem.** We now prove the main bookkeeping theorem, Theorem 1.2.

Lemma 1.49. *$\alpha_1 \times \alpha_2$ is a weak approximation.*

Proof. Since $\alpha_1 \times \alpha_2 \simeq \alpha_1 \times \mathrm{id} \circ \mathrm{id} \times \alpha_2$, we may assume without loss of generality that $\alpha_2 = \mathrm{id}$, in which case each of the conditions are obvious. □

Proof of Theorem 1.2. In view of Theorem 1.45, applying $\mathrm{Seg}_{(-)}(\mathcal{S})$ to the induced diagram

$$\begin{tikzcd}
\mathcal{O}_1^\otimes \times \mathcal{O}_2^\otimes \arrow[r] \arrow[d] & \mathcal{O}_3^\otimes \arrow[d] \\
\mathcal{P}_1^\otimes \times \mathcal{P}_2^\otimes \arrow[r] & \mathcal{P}_3^\otimes
\end{tikzcd}$$

yields a diagram of fully faithful inclusions

$$\begin{tikzcd}
\mathrm{Seg}_{\mathcal{O}_1}\mathrm{Seg}_{\mathcal{O}_2}(\mathcal{S}) \arrow[r, phantom, "\simeq"] & \mathrm{Seg}_{\mathcal{O}_3}(\mathcal{S}) \\
\mathrm{Seg}_{\mathcal{P}_1}\mathrm{Seg}_{\mathcal{P}_2}(\mathcal{S}) \arrow[u, hook] & \mathrm{Seg}_{\mathcal{P}_3}(\mathcal{S}); \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

It suffices to verify that the images of $(\alpha_1 \times \alpha_2)^*$ and $\varphi_{\mathcal{O}}^* \alpha_3^*$ agree. In fact, they correspond with Segal $\mathcal{O}_1 \times \mathcal{O}_2$-monoids spaces attaining solutions to the inner and outer lifting problem, respectively:

$$\begin{tikzcd}
U\mathcal{O}_1 \times U\mathcal{O}_2 \arrow[rr] \arrow[d] & & \mathcal{S} \\
U\mathcal{P}_1 \times U\mathcal{P}_2 \arrow[r, "U\varphi_{\mathcal{P}}", "\sim"'] \arrow[rru, dashed, bend left] & U\mathcal{P}_3 \arrow[ru, dashed] &
\end{tikzcd}$$

Indeed, $U\varphi_{\mathcal{P}}$ is an equivalence by assumption, so the images agree. □

2. Little disks and prefactorization algebras

Let G be a Lie group and $\mathcal{O}_G^{\text{f.i.}} \subset \mathcal{S}_G$ the non-full subcategory of G-spaces whose objects correspond with homogeneous G-spaces $[G/H]$ for $H \subset G$ a compact closed subgroup and whose morphisms form the subspace of maps $[G/K] \to [G/H]$ homotopic to a quotient along a finite-index subgroup inclusion $K \subset H$ up to conjugation. $\mathcal{O}_G^{\text{f.i.}}$ is an atomic orbital ∞-category whose $\mathcal{O}_G^{\text{f.i.}}$-1-category $\underline{\mathbb{F}}_G$ has values $\left(\underline{\mathbb{F}}_G\right)_H = \mathbb{F}_H$ the finite H-sets. For the remainder of this paper, we refer to $\mathcal{O}_G^{\text{f.i.}}$-equivariant objects as *G-equivariant objects*, and will replace $[G/H]$-decorations with H-decorations.

Warning 2.1. $\mathcal{O}_G^{\text{f.i.}}$-spaces are *not* synonymous with proper equivariant G-spaces if there exist infinite-index compact closed subgroup inclusions $K \subset H$; this only becomes true when all compact closed subgroups of G are discrete, such as when G is discrete. We advise the reader to take care to remember that the associated theories are only proper equivariant *with finite-index restrictions (and transfers).* ◁

We encourage the reader to specialize to the finite case; the additional generality does not introduce additional difficulty, as parameterized higher category theory obviates any group-specific bookkeeping.

2.1. **Fundamentals of little V-disks.** Fix V and W a pair of orthogonal G-representations. This entire subsection will need to be adapted to the multi-representation setting!

2.1.1. *Little V-disks.* Given $H \subset G$ a compact closed subgroup and $S \in \mathbb{F}_H$ a finite H-set, we let $D(V) \subset V$ denote the unit disk in V and we let

$$S \cdot D(V) \coloneqq \coprod_{[H/K] \subset S} \mathrm{Ind}_K^H D\left(\mathrm{Res}_K^G V\right)$$

denote the *S-indexed coproduct* of $D(V)$.

Definition 2.2. Given $f : T \to S$ a morphism of finite H-sets, and $\mathscr{S} \subset \mathrm{Vect}_{\mathbb{R}}^{\omega}$ a subcategory containing V and W, we define the *topological space of H-equivariant $\mathscr{S}$-structured little V-disk embeddings in W over f* to be the topological subspace

$$\mathrm{Emb}_f^{H,\mathscr{S}}(T \cdot D(V), S \cdot D(W)) \subset \mathrm{Map}^H(T \cdot D(V), S \cdot D(W))$$

of H-equivariant maps ι whose induced map $\pi_0^H \iota \colon \pi_0^H(T \cdot D(V)) = T \to S = \pi_0^H(S \cdot D(W))$ is f, which are embeddings of topological H-spaces, and which satisfy the condition that for each orbit $[H/K] \subset T$, the mate $D\left(\mathrm{Res}_K^G V\right) \to D(\mathrm{Res}_K^G W) \simeq \mathrm{Res}_K^H D(\mathrm{Res}_H^G W)$ to the restricted map $\iota_{[H/K]}$ of

$$\begin{tikzcd}
\mathrm{Ind}_K^H D\left(\mathrm{Res}_K^G V\right) \arrow[r, "\iota_{[H/K]}"] \arrow[d, hook] & D\left(\mathrm{Res}_H^G W\right) \arrow[d, hook] \\
T \cdot D(V) \arrow[r, "\iota"] & S \cdot D(W)
\end{tikzcd}$$

is a sum of a constant map and a map lying in $\mathscr{S}$. In the case $S = *_H$ is the terminal H-set, we simply write

$$\mathrm{Emb}^{H,\mathscr{S}}(T \cdot D(V), D(W)) \coloneqq \mathrm{Emb}_!^{H,\mathscr{S}}(T \cdot D(V), *_H \cdot D(W)). \quad \triangleleft$$

Example 2.3.

- In the case that $\mathscr{S} \subset \mathrm{Vect}_{\mathbb{R}}^{\omega,\simeq} \subset \mathrm{Vect}_{\mathbb{R}}^{\omega}$ consists of the subgroup $(0,1]^\times \subset \mathrm{GL}_n(\mathbb{R}^n)$ of short positive scalar multiples of the identity, these are almost the same thing as Guillou-May's *little V-disk maps* [GM17]. The corresponding embeddings are sometimes called "affine embeddings," a piece of terminology we squareley reject; we will instead call these *equidiameter embeddings*.
- In the case that $\mathscr{S} = \mathbb{R}_{>0}^{\times \dim V} \subset \mathrm{GL}(V^e)$ (corresponding with the positive-entry connected diagonal matrices with respect to a fixed basis), we refer to these as *rectilinear embeddings*.
- In the case that $\mathscr{S} = \mathrm{Vect}_{\mathbb{R}}^{\omega}$, we refer to these as *affine embeddings*. $\triangleleft$

For the time being fix $\mathscr{S} \subset \mathscr{S}'$ be a subcategory. First note that structured embeddings behave well with restriction.

Observation 2.4. Let $K \subset H$ be a finite-index closed subgroup inclusion. Restriction factors uniquely:

$$\begin{tikzcd}
\mathrm{Emb}_f^{H,\mathscr{S}}(T \cdot D(V), S \cdot D(V)) \arrow[r, hook] \arrow[d, dashed, "\mathrm{Res}_K^H"] & \mathrm{Emb}_f^{H,\mathscr{S}'}(T \cdot D(V), S \cdot D(V)) \arrow[r, hook] \arrow[d, dashed, "\mathrm{Res}_K^H"] & \mathrm{Map}^H(T \cdot D(V), S \cdot D(V)) \arrow[d, "\mathrm{Res}_K^H"] \\
\mathrm{Emb}_{\mathrm{Res}_K^H f}^{K,\mathscr{S}}(T \cdot D(V), S \cdot D(V)) \arrow[r, hook] & \mathrm{Emb}_{\mathrm{Res}_K^H f}^{K,\mathscr{S}'}(T \cdot D(V), S \cdot D(V)) \arrow[r, hook] & \mathrm{Map}^K(T \cdot D(V), S \cdot D(V))
\end{tikzcd}$$

Indeed uniqueness follows by noting that topological subspace embeddings are monic in Top. $\triangleleft$

Next note that structured embeddings behave well with slicing over orbits.

Observation 2.5. In the case $S = [H/K]$, T is canonically expressed as an induction $T \simeq \mathrm{Ind}_K^H \overline{T}$. Moreover, by unwinding definitions, we see that induction furnishes compatible homeomorphisms

$$\begin{tikzcd}
\mathrm{Emb}_{\mathrm{Ind}_K^H T \to [H/K]}^{H,\mathscr{S}}\left(\mathrm{Ind}_K^H T \cdot D(V), [H/K] \cdot D(V)\right) \arrow[r, phantom, "\simeq"] \arrow[d, hook] & \mathrm{Emb}^{K,\mathscr{S}}(T \cdot D(V), D(V)) \arrow[d, hook] \\
\mathrm{Emb}_{\mathrm{Ind}_K^H T \to [H/K]}^{H,\mathscr{S}'}\left(\mathrm{Ind}_K^H T \cdot D(V), [H/K] \cdot D(V)\right) \arrow[r, phantom, "\simeq"] & \mathrm{Emb}^{K,\mathscr{S}'}(T \cdot D(V), D(V))
\end{tikzcd}$$

$\triangleleft$

Last, we observe that structured embeddings satisfy a Segal condition.

Observation 2.6. Given a summand inclusion $i\colon S' \subset S$, pullback yields a diagram of forgetful maps

$$\begin{tikzcd}
\mathrm{Emb}_f^{H,\mathscr{S}}(T \cdot D(V), S \cdot D(V)) \arrow[r, "i^*"] \arrow[d, hook] & \mathrm{Emb}_{i^*f}^{H,\mathscr{S}}(i^*T \cdot D(V), S' \cdot D(V)). \arrow[d, hook] \\
\mathrm{Emb}_f^{H,\mathscr{S}'}(T \cdot D(V), S \cdot D(V)) \arrow[r, "i^*"] & \mathrm{Emb}_{i^*f}^{H,\mathscr{S}'}(i^*T \cdot D(V), S' \cdot D(V)).
\end{tikzcd}$$

Under this, we may choose the homeomorphism of Observation 2.5 by first restricting as in Observation 2.4 to K then pulling back aong the inclusion of the identity coset

$$*_K = [K/eKe^{-1}] \subset \coprod_{g \in [K \backslash H/K]} [K/gKg^{-1}] \simeq \mathrm{Res}_K^H [H/K].$$

In particular, restriction and pullback along the orbit inclusions $[H/K] \subset S$ yields a chain of homeomorphism

$$\begin{aligned}
\mathrm{Emb}_f^H(T \cdot D(V), S \cdot D(V)) &\simeq \prod_{[H/K] \subset S} \mathrm{Emb}^H_{\mathrm{Ind}_K^H T_K \to [H/K]}\left(\mathrm{Ind}_K^H T_K \cdot D(V), [H/K] \cdot D(V)\right) \\
&\simeq \prod_{[H/K] \subset S} \mathrm{Emb}^K(T_K \cdot D(V), D(V)),
\end{aligned}$$

where we write $\mathrm{Ind}_K^H T_K$ for the canonical induction-expression for $T \times_S [H/K]$, compatibly with affineness. ◁

2.1.2. *Composition.* There are two essential maps on structured embedding spaces; first, composition.

Lemma 2.7. *Composition of continuous maps restricts uniquely to operations*

$$\gamma\colon \mathrm{Emb}_g^{H,\mathcal{S}}(S \cdot D(V), R \cdot D(W)) \times \mathrm{Emb}_f^H(T \cdot D(U), S \cdot D(V)) \longrightarrow \mathrm{Emb}_{g \circ f}^{H,\mathcal{S}}(T \cdot D(U), R \cdot D(W)),$$

compatible with inclusions of structures. Moreover, γ is compatible with restriction, i.e. the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Emb}_g^{H,\mathscr{S}}(S \cdot D(V), R \cdot D(W)) \times \mathrm{Emb}_f^H(T \cdot D(U), S \cdot D(V)) \arrow[r, "\gamma"] \arrow[d, "\mathrm{Res}_K^H"] & \mathrm{Emb}_{g \circ f}^{H,\mathscr{S}}(T \cdot D(U), R \cdot D(W)). \arrow[d, "\mathrm{Res}_K^H"] \\
\mathrm{Emb}_g^{K,\mathscr{S}}(S \cdot D(V), R \cdot D(VW) \times \mathrm{Emb}_f^K(T \cdot D(U), S \cdot D(V)) \arrow[r, "\gamma"] & \mathrm{Emb}_{g \circ f}^{K,\mathscr{S}}(T \cdot D(U), R \cdot D(W)).
\end{tikzcd}$$

Proof. Since π_0^H is functorial, we're tasked with verifying that $\mathscr{S}$-structured embeddings are closed under composition; unwinding definitions, this follows from the fact that $\mathscr{S}$ is closed under composition and the composition of $\mathscr{S}$ with a constant map is a constant map. Restriction-compatibility follows from functoriality of restriction. □

We will use this as an operadic composition map, so we must verify associativity.

Observation 2.8. Given V an orthogonal G-representation, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Map}^H(R \cdot D(V), D(V)) \times \mathrm{Map}^H(S \cdot D(V), R \cdot D(V)) \times \mathrm{Map}^H(T \cdot D(V), S \cdot D(V)) \arrow[r, "\circ"] \arrow[ddd, bend right, "\circ"'] & \mathrm{Map}^H(S \cdot D(V), D(V)) \times \mathrm{Map}^H(T \cdot D(V), S \cdot D(V)) \arrow[ddd, bend left, "\circ"] \\
\mathrm{Emb}_h^{H,\mathscr{S}}(R \cdot D(V), D(V)) \times \mathrm{Emb}_g^{H,\mathscr{S}}(S \cdot D(V), R \cdot D(V)) \times \mathrm{Emb}_f^{H,\mathscr{S}}(T \cdot D(V), S \cdot D(V)) \arrow[u, hook] \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathrm{Emb}_{h \circ g}^{H,\mathscr{S}}(S \cdot D(V), D(V)) \times \mathrm{Emb}_f^{H,\mathscr{S}}(T \cdot D(V), S \cdot D(V)) \arrow[u, hook] \arrow[d, "\gamma"] \\
\mathrm{Emb}_h^{H,\mathscr{S}}(R \cdot D(V), D(V)) \times \mathrm{Emb}_{g \circ f}^{H,\mathscr{S}}(T \cdot D(V), R \cdot D(V)) \arrow[r, "\gamma"] \arrow[d, hook] & \mathrm{Emb}_{h \circ g \circ f}^{H,\mathscr{S}}(T \cdot D(V), D(V)) \arrow[d, hook] \\
\mathrm{Map}^H(R \cdot D(V), D(V)) \times \mathrm{Map}^H(T \cdot D(V), R \cdot D(V)) \arrow[r, "\circ"] & \mathrm{Map}^H(T \cdot D(V), D(V))
\end{tikzcd}$$

Indeed, the commutativity of the inner diagram follows from commutativity of the outer diagram and the top and bottom rectangles, together with the fact that subspace inclusions are monic in Top. ◁

2.1.3. *External products.* Note that direct sums yields a faithful and conservative functor $\oplus\colon \mathrm{Vect}_{\mathbb{R}}^{\omega} \times \mathrm{Vect}_{\mathbb{R}}^{\omega} \to \mathrm{Vect}_{\mathbb{R}}^{\omega}$. Given two structures $\mathscr{S}, \mathscr{S}'$, we define the *product* structure

$$\mathscr{S} \times \mathscr{S}' \to \mathrm{Vect}_{\mathbb{R}}^{\omega} \times \mathrm{Vect}_{\mathbb{R}}^{\omega} \xrightarrow{\oplus} \mathrm{Vect}_{\mathbb{R}}^{\omega}.$$

The relevant closure property is the following.

Definition 2.9. $\mathscr{S}$ is *product-closed* if $\mathscr{S} \times \mathscr{S} \subset \mathscr{S}$. ◁

All of our examples are product-closed. We use this to construct the external product map.

Construction 2.10. Fix V, W a pair of orthogonal G-representations, and once and for all fix a G-equivariant diffeomorphism $\varpi\colon D(V) \times D(W) \simeq D(V \oplus W)$.[4] Fix $\mathscr{S}$ a product-closed structure. We (uniquely) define compatible external product maps

$$\begin{tikzcd}
\mathrm{Emb}_g^{H,\mathscr{S}}(S \cdot D(V), R \cdot D(V)) \times \mathrm{Emb}_f^{H,\mathscr{S}}(T \cdot D(W), S \cdot D(W)) \arrow[r, dashed, "\varphi"] \arrow[d, hook] & \mathrm{Emb}_{g \circ f}^{H,\mathscr{S}}(T \cdot D(V \oplus W), R \cdot D(V \oplus W)) \arrow[d, hook] \\
\mathrm{Map}_g^{H}(S \cdot D(V), R \cdot D(V)) \times \mathrm{Map}_f^{H}(T \cdot D(W), S \cdot D(W)) \arrow[r, "\varphi"] & \mathrm{Map}_{g \circ f}^{H}(T \cdot D(V \oplus W), R \cdot D(V \oplus W))
\end{tikzcd}$$

elementwise by the formula

$$\begin{tikzcd}
T \cdot D(V \oplus W) \arrow[rrr, "\varphi(\tilde{g};\tilde{f})"] \arrow[d, phantom, "\wr"] & & & R \cdot D(V \oplus W) \arrow[d, phantom, "\wr"] \\
D(V) \times (T \cdot D(W)) \arrow[r, "(\mathrm{id},\tilde{g})"] & D(V) \times (S \cdot D(W)) \arrow[r, phantom, "\simeq"] & S \cdot D(V) \times D(W) \arrow[r, "(\tilde{f},\mathrm{id})"] & R \cdot D(V) \times D(W)
\end{tikzcd} \tag{4}$$

◁

We see that φ behaves well with respect to restriction and composition.

Observation 2.11. Eq. (4) is manifestly restriction-stable, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathrm{Emb}_g^{H,\mathscr{S}}(S \cdot D(V), R \cdot D(V)) \times \mathrm{Emb}_f^{H,\mathscr{S}}(T \cdot D(W), S \cdot D(W)) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Res}_K^H"'] & \mathrm{Emb}_{g \circ f}^{H,\mathscr{S}}(T \cdot D(V \oplus W), R \cdot D(V \oplus W)). \arrow[d, "\mathrm{Res}_K^H"] \\
\mathrm{Emb}_{\mathrm{Res}_K^H g}^{K,\mathscr{S}}(S \cdot D(V), R \cdot D(V)) \times \mathrm{Emb}_{\mathrm{Res}_K^H f}^{K,\mathscr{S}}(T \cdot D(W), S \cdot D(W)) \arrow[r, "\varphi"'] & \mathrm{Emb}_{\mathrm{Res}_K^H g \circ f}^{K,\mathscr{S}}(T \cdot D(V \oplus W), R \cdot D(V \oplus W)).
\end{tikzcd}$$

Moreover, it is compatible with composition of structured embeddings, i.e. the top horizontal composite

$$\begin{tikzcd}
P \cdot D(V \oplus W) \arrow[r, "\varphi(\tilde{h};\mathrm{id})"] \arrow[d, leftrightarrow] & T \cdot D(V \oplus W) \arrow[rr, "\varphi(\tilde{g};\tilde{f})"] \arrow[d, leftrightarrow] & & R \cdot D(V \oplus W) \arrow[r, "\varphi(\mathrm{id};\tilde{j})"] \arrow[dd, leftrightarrow] & Q \cdot D(V \oplus W) \arrow[dd, leftrightarrow] \\
D(V) \times (P \cdot D(W)) \arrow[r, "(\mathrm{id},\tilde{h})"] & D(V) \times (T \cdot D(W)) \arrow[r, "(\mathrm{id},\tilde{g})"] & D(V) \times (S \cdot D(W)) \arrow[d, leftrightarrow] & & \\
& & S \cdot D(V) \times D(W) \arrow[r, "(\tilde{f},\mathrm{id})"] & R \cdot D(V) \times D(W) \arrow[r, "(\tilde{j},\mathrm{id})"] & Q \cdot D(V) \times D(W)
\end{tikzcd}$$

clearly agrees with $\varphi\left(\tilde{g} \circ \tilde{h}; \tilde{j} \circ \tilde{f}\right)$. Here, for visual clarity, we've denoted equivalences with two heads and no other decorations Moreover, Eq. (4) is associative, in the sense that we may define an element $\varphi(f;g;h)$

[4] For concreteness, the reader may fix an equivariant diffeomorphism $\varphi_V\colon V \simeq D(V)$ (say, by $\varphi_V(x) = \frac{1}{1+|x|} \cdot x$) and use $\varphi_{V \oplus W} \circ (\varphi_V^{-1}, \varphi_W^{-1})$.

making following diagram commute

$$\begin{tikzcd}
T\cdot D(U\oplus V\oplus W) \arrow[rrr, "\varphi(f;g;h)"] \arrow[d, leftrightarrow] & & & P\cdot D(U\oplus V\oplus W) \arrow[ddd, leftrightarrow] \\
D(U)\times(T\cdot D(V\oplus W)) \arrow[rr, "(\mathrm{id},\varphi(g;h))"] \arrow[dd, leftrightarrow] & & D(U)\times(R\cdot D(V\oplus W)) \arrow[d, leftrightarrow] & \\
& D(U)\times S\cdot D(V)\times D(W) \arrow[r, "(\mathrm{id},g,\mathrm{id})"'] \arrow[d, leftrightarrow] & D(U)\times R\cdot D(V)\times D(W) \arrow[d, leftrightarrow] & \\
D(U)\times D(V)\times(T\cdot D(W)) \arrow[r, "(\mathrm{id},\mathrm{id},h)"] & D(U)\times D(V)\times S\cdot D(W) \arrow[d, leftrightarrow] & R\cdot D(U)\times D(V)\times D(W) \arrow[r, "(f,\mathrm{id},\mathrm{id})"'] & P\cdot D(U)\times D(V)\cdot D(W) \arrow[d, leftrightarrow] \\
& S\cdot D(U\oplus V)\cdot D(W) \arrow[rr, "(\varphi(f;g),\mathrm{id})"'] & & P\cdot D(U\oplus V)\cdot D(W)
\end{tikzcd}$$

We see that "products with the identity self-embedding" yields a map

$$\iota\colon \mathrm{Emb}_f^{H,\mathscr{S}}(S\cdot D(V),R\cdot D(V)) \longrightarrow \mathrm{Emb}_f^{H,\mathscr{S}}(S\cdot D(U\oplus V),R\cdot D(U\oplus V)),$$

which makes the following diagram commute

$$\begin{tikzcd}
\mathrm{Emb}_g^{H,\mathscr{S}}(S\cdot D(V),R\cdot D(V))\times\mathrm{Emb}_f^{H,\mathscr{S}}(T\cdot D(W),S\cdot D(W)) \arrow[r, "\varphi"] \arrow[d, "(\iota,\mathrm{id})"] & \mathrm{Emb}_{g\circ f}^{H,\mathscr{S}}(T\cdot D(V\oplus W),R\cdot D(V\oplus W)) \arrow[d, "\iota"] \\
\mathrm{Emb}_g^{H,\mathscr{S}}(S\cdot D(U\oplus V),R\cdot D(U\oplus V))\times\mathrm{Emb}_f^{H,\mathscr{S}}(T\cdot D(W),S\cdot D(W)) \arrow[r, "\varphi"] & \mathrm{Emb}_{g\circ f}^{H,\mathscr{S}}(T\cdot D(U\oplus V\oplus W),R\cdot D(U\oplus V\oplus W))
\end{tikzcd}$$

Moreover, ι intertwines with restriction and composition in the obvious way. ◃

Lastly, the following property of φ will be key.

Proposition 2.12. *Under the universal property for disjoint unions, φ yields an injective map*

$$\begin{tikzcd}
\coprod_{T\xrightarrow{f}S\xrightarrow{g}R}\mathrm{Emb}_g^{H,\mathscr{S}}(S\cdot D(V),R\cdot D(V))\times\mathrm{Emb}_f^{H,\mathscr{S}}(T\cdot D(W),S\cdot D(W)) \arrow[dr, bend right, "\widetilde{\varphi}"] & \\
& \mathrm{Emb}_{g\circ f}^{H,\mathscr{S}}(T\cdot D(V\oplus W),R\cdot D(V\oplus W)).
\end{tikzcd}$$

Now, note that $\widetilde{\varphi}$ is compatible with sub-structure inclusions, Aff is the universal structure, and right-cancellation of injectives shows that it suffices to prove that $\widetilde{\varphi}$ is injective in the affine case. We have a *center and underlying linear transformation* map

$$(c,t)\colon \mathrm{Emb}_g^{\mathrm{Aff},H}\left([n]\cdot D^{k+k'},D^{k+k'}\right)\longrightarrow \mathrm{Conf}_n(D^{k+k'})\times\mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k+k'}\right)^n.$$

participating in the following diagram, whose dashed arrow we do not yet claim to exist.

$$\begin{tikzcd}
\coprod_{T\to S}\mathrm{Emb}^{\mathrm{Aff},H}(S\cdot D(V),D(V))\times\mathrm{Emb}_f^{\mathrm{Aff},H}(T\cdot D(W),S\cdot D(W)) \arrow[r, "\widetilde{\varphi}"] \arrow[d, hook] & \mathrm{Emb}^{\mathrm{Aff},H}(T\cdot D(V\oplus W),D(V\oplus W)) \arrow[d, hook] \\
\coprod_{f\colon[n]\to[m]}\mathrm{Emb}^{\mathrm{Aff}}\left(m\cdot D^k,D^k\right)\times\mathrm{Emb}_f^{\mathrm{Aff}}\left([n]\cdot D^{k'},[m]\cdot D^{k'}\right) \arrow[r, "\widetilde{\varphi}_e"] \arrow[d, hook, "(c,t)"] & \mathrm{Emb}^{\mathrm{Aff}}\left([n]\cdot D^{k+k'},D^{k+k'}\right) \arrow[d, hook, "(c,t)"] \\
\coprod_{f\colon[n]\to[m]}\mathrm{Conf}_m D^k\times\mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^k\right)^m\times\left(\prod_{\ell\in[m]}\left(\mathrm{Conf}_{f^{-1}(\ell)}D^{k'}\right)\times\mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k'}\right)^{f^{-1}(\ell)}\right) \arrow[r, hook, dashed, "\widetilde{\varphi}_{c,t}"] & \mathrm{Conf}_n(D^{k+k'})\times\mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^K\right)^n
\end{tikzcd} \tag{5}$$

We omit the proof of the following nearly tautological lemma.

Lemma 2.13. *(c,t) is injective.*

The heart of our argument for Proposition 2.12 is the following lemma.

Lemma 2.14. *There exists an injective function* $\widetilde{\varphi}_{c,t}$ *making the above diagram commute.*

Proof. First define the function $\varphi_c \colon \mathrm{Conf}_m D^k \times \prod_{\ell \in [m]} \mathrm{Conf}_{f^{-1}(\ell)} D^{k'} \longrightarrow \mathrm{Conf}_n D^{k+k'}$ by the external product

$$\varphi_c(a; b_1, \dots, b_m)(i) = \varpi\left(a(f(i)), b_{f(i)}(i)\right),$$

where $\varpi \colon D^k \times D^{k'} \xrightarrow{\sim} D^{k+k'}$ is the distinguished diffeomorphism. Next, we define the function

$$\varphi_t \colon \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^k\right)^m \times \prod_{\ell \in [m]} \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k'}\right)^{f^{-1}(\ell)} \to \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k+k'}\right)$$

by

$$\varphi_t(F; G_1, \cdots, G_m)_i = F_i \times G_{f(i)}.$$

This yields a diagram

$$\begin{tikzcd}
\coprod_{f \colon [n] \to [m]} \mathrm{Conf}_m D^k \times \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^k\right)^m \times \left(\prod_{\ell \in [m]} \left(\mathrm{Conf}_{f^{-1}(\ell)} D^{k'}\right) \times \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k'}\right)^{f^{-1}(\ell)}\right) \arrow[drr, "\widetilde{\varphi}_{c,t}"] \arrow[d, phantom, "\wr"] & & \\
\coprod_{f \colon [n] \to [m]} \left(\mathrm{Conf}_m D^k \times \prod_{\ell \in [m]} \mathrm{Conf}_{f^{-1}(\ell)} D^{k'}\right) \times \left(\mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^k\right)^m \times \prod_{\ell \in [m]} \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k'}\right)^{f^{-1}(\ell)}\right) \arrow[rr] & & \mathrm{Conf}_n(D^{k+k'}) \times \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^K\right)^n \\
\left(\mathrm{Conf}_m D^k \times \prod_{\ell \in [m]} \mathrm{Conf}_{f^{-1}(\ell)} D^{k'}\right) \times \left(\mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^k\right)^m \times \prod_{\ell \in [m]} \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k'}\right)^{f^{-1}(\ell)}\right) \arrow[u, hook] \arrow[urr, "\varphi_c \times \varphi_t"'] & &
\end{tikzcd}$$

It follows by unwinding definitions that this makes the above diagram commute. We're left with injectivity, so we'll define a section for $\widetilde{\varphi}_{c,t}$. To begin, given a configuration $\iota \colon [n] \hookrightarrow D^{k+k'}$, we may define the diagram

$$\begin{tikzcd}
f_\iota^{-1}(\ell) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rrr, bend left, "s_{k',\ell}(\iota)"] & {[n]} \arrow[r, "\iota"] \arrow[d, two heads, "f_\iota"] & D^{k+k'} \arrow[r, two heads, "\pi_k"] \arrow[d, two heads, "\pi_k"] & D^{k'} \\
\{\ell\} \arrow[r] & {[m_\iota] \coloneqq \mathrm{im}(\pi_k \iota)} \arrow[r, hook, "s_k(\iota)"'] & D^k &
\end{tikzcd}$$

In particular, we acquire a map $s_c \colon \mathrm{Conf}_n D^{k+k'} \longrightarrow \coprod_{f \colon [n] \to [m]} \mathrm{Conf}_m D^k \times \prod_{\ell \in [m]} \mathrm{Conf}_{f^{-1}(\ell)} D^{k'}$ lying over the assignment $\iota \mapsto f_\iota$. We define the map $s_{c,t}$ with opposite domain and codomain to $\widetilde{\varphi}_{c,t}$ by taking a pair $(\iota, (F_1, \dots, F_n))$ to the configuration data $s_c \iota$ as well as the maps

$$\left(sF_1, \dots, sF_{m_\iota}; (sG_i)_{f^{-1}(1)}, \dots, (sG_i)_{f^{-1}(m_\iota)}\right) \in \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^k\right)^{[m_\iota]} \times \prod_{\ell \in [m_\iota]} \mathrm{End}_{\mathbb{R}}\left(\mathbb{R}^{k'}\right)^{f_\iota^{-1}(\ell)}$$

with data defined by the compositions

$$\begin{tikzcd}
\mathbb{R}^{k'} \arrow[r, hook] \arrow[d, "sG_j"] & \mathbb{R}^{k+k'} \arrow[d, "F_j"] & \mathbb{R}^k \arrow[l, hook'] \arrow[d, "sF_i"] \\
\mathbb{R}^{k'} & \mathbb{R}^{k+k'} \arrow[l, two heads] \arrow[r, two heads] & \mathbb{R}^k
\end{tikzcd} \qquad f_\iota(j) = i$$

Unwinding definitions, we see $s_{c,t}$ is a section of $\widetilde{\varphi}_{c,t}$, so the latter is injective, as desired. □

Proof of Proposition 2.12. Note that the vertical arrows in Eq. (5) are injective by Lemma 2.13; the diagonal composite is then injective by Lemma 2.14. by right-cancellation of injections, this implies that $\widetilde{\varphi}$ is injective. □

2.1.4. *Configurations.* We now verify the following folklore extension of [May72, Thm 4.8].

Lemma 2.15. *Suppose $(0,1]^n \subset \mathscr{S}$ is a substructure such that each space in $\mathscr{S}$ is closed under linear combinations with positive coefficients and $W \subset V$ is a subrepresentation. Then, evaluation at centers*
$$c\colon \mathrm{Emb}^{H,\mathscr{S}}(S \cdot D(W), D(V)) \to \mathrm{Conf}_S^H(V)$$
is an $\mathrm{Aut}_H S$-homotopy equivalence.

Proof. We define the *maximal equidiameter embedding* function $s\colon \mathrm{Conf}_n(V^e) \to \mathrm{Emb}(n \cdot D(W^e), D(V^e))$ by
$$s_{x_1,\dots,x_n}(w) = (x_1,\dots,x_n) + \frac{\min_{i\neq j} d(x_i,x_j)}{2} \cdot w.$$
Now, note that s is continuous and restricts uniquely to an $\mathrm{Aut}_H S$-equivariant map
$$\begin{tikzcd} \mathrm{Conf}_S^H(W) \arrow[r, "s"] \arrow[d, hook] & \mathrm{Emb}^{H,\mathscr{S}}(S \cdot D(V), D(V)) \arrow[d, hook] \\ \mathrm{Conf}_{|S|}(W^e) \arrow[r, "s"] & \mathrm{Emb}^{\mathscr{S}}(|S| \cdot D(V^e), D(V^e)) \end{tikzcd}$$
Moreover, s is a section of c, so for the proposition it suffices to construct an $\mathrm{Aut}_H S$-equivariant deformation retract of $\mathrm{Emb}^{H,\mathscr{S}}(S \cdot D(W), D(V))$ onto $\mathrm{im}\, s$. Indeed, we may do so linearly via the homotopy
$$h_{t,\iota}(w) = t \cdot s_{c(\iota)}(w) + (1-t)\cdot \iota(w).$$
□

We saw in [Ste25c] that $\mathrm{Conf}_S^H(V)$ has an elementwise-contractible basis of *trivializing neighborhoods*
$$\begin{aligned}\mathcal{P}_S^H(V) &:= \left\{ U \subset D(V)^{|S|} \;\middle|\; \exists\, H\text{-equivariant disjoint rectilinear embedding } \iota\colon S\cdot D(V) \to D(V) \text{ s.t. } U = \mathrm{im}(\iota) \right\} \\ &\subset \mathcal{O}\left(\mathrm{Conf}_S^H(V)\right).\end{aligned}$$
We may make an arbitrary choice of rerpresentative for each V, under which such embeddings are necessarily unique. Hypercovering theory allows us to recover the homotopy type of $\mathrm{Conf}_S^H(V)$ from the classifying space of the poset of basis elements under inclusion via the following result.

Proposition 2.16 ([DI04], e.g. as in [Knu18])**.** *If $\mathcal{P}$ is an elementwise contractible basis for a topological space X (considered as a poset), then the canonical map $B\mathcal{P} \to X$ is a weak equivalence.*

In particular, taking centers yields a commutative diagram
$$\begin{tikzcd} B\mathcal{P}_S^H(V) \arrow[r, "\alpha"'] \arrow[rr, bend left, "c'"] & \mathrm{Emb}^{H,\mathscr{S}}(S \cdot D(V), D(V)) \arrow[r, "c"'] & \mathrm{Conf}_S^H(V) \end{tikzcd}$$
such that c and c' are weak equivalences; α is a weak equivalence by two out of three. That is, we've proved the following.

Proposition 2.17. *Under the assumptions of Lemma 2.15, $\alpha\colon B\mathcal{P}_S^H(V) \to \mathrm{Emb}^{H,\mathscr{S}}(S \cdot D(V), D(V))$ is a weak equivalence.*

This will form the heart of our weak approximation, but first we introduce the main topological idea.

2.1.5. *Decomposable little disks.* Let $\iota\colon \mathcal{P}_S^H(V|W) \hookrightarrow \mathcal{P}_S^H(V \oplus W)$ be the embedded sub-poset image of φ.

Proposition 2.18 (G-preoperadic [Dun88, Prop 2.3])**.** *ι induces weak equivalences*
$$B\mathcal{P}_S^H(V|W) \xrightarrow{\sim} B\mathcal{P}_S^H(V \oplus W) \xrightarrow{\sim} \mathbb{E}_{V\oplus W}(S).$$

Proof. Let $\mathcal{P}_S^{H,\mathrm{equi}}(V \oplus W) \subset \mathcal{P}_S^H(V \oplus W)$ be the *equidiameter* embeddings. We consider the intersection (i.e. pullback) diagram
$$\begin{tikzcd} \mathcal{P}_S^{H,\mathrm{equi}}(V|W) \arrow[r, hook, "\iota^{\mathrm{equi}}"] \arrow[d, hook, "i_{V|W}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_S^{H,\mathrm{equi}}(V \oplus W) \arrow[d, hook, "i_{V\oplus W}"] \\ \mathcal{P}_S^H(V|W) \arrow[r, hook, "\iota"] & \mathcal{P}_S^H(V \oplus W) \end{tikzcd}$$

By two out of three, it suffices to verify that ι^{equi}, $i_{V|W}$, and $i_{V\oplus W}$ are B-equivalences; we use the well-known facts that adjoints are B-equivalences.

First note that ι^{equi} has right adjoint taking an equidiameter little disk embedding f to the equidiameter little disk embedding Rf centered on $c(f)$ with diameter

$$\frac{1}{2}\min\left(\max_{\pi_V(x_i)\neq\pi_V(x_j)} d(\pi_V(x_i),\pi_V(x_j)),\quad \max_{i\neq j} d(x_i,x_j)\right)$$

Next note that $i_{V\oplus W}$ has a left adjoint taking a little disk embedding f to the equidiameter little disk embedding Lf centered on $c(f)$ with diameter given by the minimum diameter of the components of f; last, note that L restricts to a left adjoint to $i_{V|W}$. □

2.2. Little V-disk algebras.

2.2.1. *The G-operad $\mathbb{E}_V$.*

Construction 2.19. We define the topological G-symmetric sequence

$$\mathbb{E}^t_{V,\mathscr{S}}(S) := \mathrm{Emb}^{H,\mathscr{S}}(S\cdot D(V), D(V)).$$

with $\mathrm{Aut}_H S$ acting by precomposition and restriction as in Observation 2.4; note that restriction is Borel $\mathrm{Aut}_H S$-equivariant by Lemma 2.7. ◁

Note that the Segal condition of Observation 2.6 characterizes $\mathbb{E}^t_V$-multioperations:

$$\mathrm{Mul}^f_{\mathbb{E}^t_{V,\mathscr{S}}}(T;S) \simeq \mathrm{Emb}^{H,\mathscr{S}}_f(T\cdot D(V), S\cdot D(V)).$$

In particular, the composition map γ yields a composition operation

$$\gamma\colon \mathrm{Mul}^g_{\mathbb{E}^t_V}(S;R)\times \mathrm{Mul}^f_{\mathbb{E}^t_V}(T;S) \longrightarrow \mathrm{Mul}^{g\circ f}_{\mathbb{E}^t_V}(T;R).$$

Proposition 2.20. *$(\mathbb{E}^t_{V,\mathscr{S}}, (\mathrm{id}_{D(V)}), \gamma)$ is a genuine topological G-operad.*

Proof. We're tasked with verifying Conditions (OP-a) to (OP-e). To start, note that Conditions (OP-a) and (OP-c) (i.e. unitality and restriction-stability of units) are obvious. Moreover, Condition (OP-b) is simply Observation 2.8 and Condition (OP-d) is verified in the second statement of Lemma 2.7. Lastly, equivariance of γ follows from the same property for composition of equivariant maps. □

Definition 2.21. The *$\mathscr{S}$-structured little V-disks G-operad* is $\mathbb{E}^{\otimes}_{V,\mathscr{S}} := N^{\otimes}\mathbb{E}^t_V$. The *little V-disks G-operad* is $\mathbb{E}^{\otimes}_V := \mathbb{E}^{\otimes}_{V,\mathrm{Rect}}$. ◁

Remark 2.22. When G is finite, $\mathbb{E}^{\otimes}_{V,\mathrm{equi}}$ is synonymous with the G-operad given the same name in [Hor19]. ◁

Observation 2.23. When $\mathscr{S}$ is product-closed, Observation 2.11 constructs a map $\mathbb{E}^t_{V,\mathscr{S}} \to \mathbb{E}^t_{V\oplus W,\mathscr{S}}$; given a sub-representation inclusion $U \subset V$, choosing an orthogonal complement to U and taking nerves yields a map $\mathbb{E}^{\otimes}_U \to \mathbb{E}^{\otimes}_V$. ◁

2.2.2. *The Dunn map.* The following constructs a *Dunn map* $\varphi_{\mathbb{E}}\colon \mathbb{E}^{\otimes}_{V,\mathscr{S}} \wr \mathbb{E}^{\otimes}_{W,\mathscr{S}} \to \mathbb{E}^{\otimes}_{V\oplus W,\mathscr{S}}$.

Proposition 2.24. *The maps $\varphi\colon \mathbb{E}^t_{V,\mathscr{S}}(S)\times \mathrm{Mul}^{T\to S}_{\mathbb{E}^t_{W,\mathscr{S}}}(T;S) \to \mathbb{E}^t_{V\oplus W,\mathscr{S}}(T)$ of Construction 2.10 satisfy Corollary 1.48.*

Proof. The first and second diagram of Observation 2.11 correspond with Eq. (2) and Eq. (3), respectively. □

Of course, composition yields a diagram of G-preoperads

$$\mathbb{E}^{\otimes}_{V,\mathscr{S}} \times \mathbb{E}^{\otimes}_{W,\mathscr{S}} \to \mathbb{E}^{\otimes}_{V,\mathscr{S}} \wr \mathbb{E}^{\otimes}_{W,\mathscr{S}} \to \mathbb{E}^{\otimes}_{V\oplus W,\mathscr{S}},$$

which corresponds canonically with a map of G-operads $\varphi\colon \mathbb{E}^{\otimes}_{V,\mathscr{S}} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_{W,\mathscr{S}} \to \mathbb{E}^{\otimes}_{V\oplus W,\mathscr{S}}$. The precise form of Theorem A is that φ is an equivalence for $\mathscr{S} = \mathrm{Rect}$ together with the following observation.

Observation 2.25. Observation 2.11 demonstrates that the right square of the following commutes:

$$\begin{tikzcd}
\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \arrow[d] \arrow[rrr, bend left] & \mathbb{E}_U^{\otimes} \times \mathbb{E}_W^{\otimes} \arrow[l] \arrow[r] \arrow[d] & \mathbb{E}_U^{\otimes} \wr \mathbb{E}_W^{\otimes} \arrow[r] \arrow[d] & \mathbb{E}_{U\oplus W}^{\otimes} \arrow[d] \\
\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \arrow[rrr, bend right] & \mathbb{E}_V^{\otimes} \times \mathbb{E}_W^{\otimes} \arrow[l] \arrow[r] & \mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \arrow[r] & \mathbb{E}_{V\oplus W}^{\otimes}
\end{tikzcd}$$

We've already verified that the rest of the inner diagram commutes, so in particular, the outer square commutes; that is, the Dunn map is natural with respect to summand inclusions in each factor. In particular, taking $U = 0$ implies that the composite map

$$\mathbb{E}_W^{\otimes} \simeq \mathbb{E}_0^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \xrightarrow{(\iota,\mathrm{id})} \mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \xrightarrow{\varphi} \mathbb{E}_{V\oplus W}^{\otimes}$$

is ι; we interpret this as showing that "φ extends ι." ◁

2.3. **Additivity for V-disk prefactorization algebras.** Define the G-coefficient system of sets

$$\pi_0 \mathbb{P}_{V,H} := \mathcal{P}_{*_H}^H(V),$$

with evident functoriality. This becomes a G-poset under inclusions, which we denote $\mathbb{P}_{V,\bullet}$. Moreover, we define a $\mathbb{P}_{V,\bullet}$-symmetric sequence by *indexed rectilinear embeddings*

$$\mathbb{P}_V\left((M_K)_S; M\right) = \begin{cases} * & \exists \text{ disjoint factorization } \coprod_K^S M_K \hookrightarrow M \hookrightarrow D(V_{\underline{H}}); \\ \varnothing & \text{otherwise.} \end{cases}$$

By "disjoint factorization," we mean that $\coprod_K^S M_K \hookrightarrow M$ itself is an embedding. Note that the factorization is unique, if it exists.

Proposition 2.26. *There exists an essentially unique $\mathbb{P}_{V,\bullet}$-colored G-0-operad with $\mathbb{P}_{V,\bullet}$-symmetric sequence $\mathbb{P}_V(-;-)$.*

Proof. Since $\mathbb{P}_V(-;-)$ has 0-truncated structure spaces, there is at most one choice of 1_C and of γ, and if they exist, the associativity, unitality, equivariance, and restriction-stability diagrams all commute. Thus it suffices to verify that there *exist* maps $* \to \mathbb{P}_V(N;N)$ and

$$\mathbb{P}_V\left((M_K)_S; N\right) \times \prod_{K\in \mathrm{Orb}(S)} \mathbb{P}_V\left(\left(L_J\right)_{T_K}; M_K\right) \to \mathbb{P}_V\left(\left(L_J\right)_T; N\right)$$

when $\coprod_K^S M \to D(V_{\underline{H}})$ factors through an H-equivariant embedding into N and $\coprod_J^{T_K} T_K \to D(V_{\underline{K}})$ factors through a K-equivariant embedding into M_K. The first is implemented by the identity self-embedding of N and the second by noting that the composite

$$\coprod_J^T L_J \simeq \coprod_K^S \coprod_J^{T_K} L_J \hookrightarrow \coprod_K^S M_K \hookrightarrow N$$

is the desired embedding. □

We define the coefficient system map as the map underlying the $\mathcal{T}$-poset equivalence

$$\pi_0 U\varphi_{\mathbb{P}} := \varphi_{\mathbb{E}}|_H \colon \mathcal{P}_{*_H}^H(V) \times \mathcal{P}_{*_H}^H(W) \to \mathcal{P}_{*_H}^H(V \oplus W).$$

Proposition 2.27. *There is a unique map $\varphi_{\mathbb{P}} \colon \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} \to \mathbb{P}_{V\oplus W}^{\otimes}$ lying over $\pi_0 U\varphi_{\mathbb{P}}$, and the induced map to the $\mathcal{T}$-preoperadic image*

$$\widetilde{\varphi}_{\mathbb{P}} \colon \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} \longrightarrow \mathbb{P}_{V|W}^{\otimes} := \mathrm{im}\left(\varphi_{\mathbb{P}}\right)$$

is an L_{Op_G}-equivalence.

Proof. To verify existence and uniqueness of $\varphi_{\mathbb{P}}$, we must verify the composability condition of Proposition 1.47. Unwinding definitions, this condition stipulates that, given equivariant rectilinear embeddings $g\colon T \cdot D(W) \to S \cdot D(W)$ and $f\colon S \cdot D(V) \to D(V)$, the composite $\varphi(g; f)$ is an equivariant rectilinear embedding, which is obvious from the definition of φ.

Now, as already mentioned the $\mathcal{T}$-poset maps

$$(\mathbb{P}_V \wr \mathbb{P}_W)^{\mathrm{el}} \to \mathbb{P}^{\mathrm{el}}_{V|W} \to \mathbb{P}^{\mathrm{el}}_{V\oplus W}$$

are equivalences, verifying the strong form of Condition (WA-b).

Moreover, by Proposition 1.47, to verify Condition (WA-a), it suffices to verify that the induced map

$$\varphi\colon \coprod_{f\colon T \to S} \left(\mathcal{P}^H_S(V) \times \prod_{[H/K]\subset S} \mathcal{P}^K_{T_K}(W) \right) \longrightarrow \mathcal{P}^H_T(V \oplus W)$$

is injective; this is precisely Proposition 2.12. $\square$

2.4. Locally constant V-disk prefactorization algebras and equivariant Dunn-Lurie additivity.

Construction 2.28. We will define a map $\alpha\colon \mathbb{P}^t_V \to \mathbb{E}^t_V$; in order to do so, given an H-equivariant affine embedding $f\colon S \cdot D(V) \to D(V)$, we're tasked with choosing a point

$$\alpha(v) \in \mathbb{E}^t_V = \mathrm{Emb}^{H,\mathrm{Rect}}(S \cdot D(V), D(V));$$

we choose the tautological one. Restriction-and composition-stability are also tautological, as is the fact that $\alpha \circ \varphi_{\mathbb{E}} \simeq \varphi_{\mathbb{P}} \circ \alpha$. ◁

Proposition 2.29. *The maps $\mathbb{P}^{\otimes}_V \to \mathbb{E}^{\otimes}_V$ and $\mathbb{P}^{\otimes}_{V|W} \to \mathbb{E}^{\otimes}_{V\oplus W}$ are weak approximations; that is, $\mathbb{E}_V$-algebras embed fully faithfully into V-prefactorization algebras as the locally constant ones.*

Proof. For Condition (WA-b), it suffices to note that $\left(U\mathbb{P}_{V|W}\right)_H = (U\mathbb{P}_{V\oplus W})_H$ have a terminal object given by the identity self-embedding of $D(V \oplus W)$. For Condition (WA-a), in light of Corollary 1.38, we're tasked with verifying that the tautological maps

$$\begin{array}{ccccc} B\left(\mathbb{P}^{\mathrm{act}}_{V|W} \times_{\mathbb{F}_{G,/H}} \left\{\mathrm{Ind}^G_H S \to [G/H]\right\}\right) & \hookrightarrow & B\left(\mathbb{P}^{\mathrm{act}}_{V\oplus W} \times_{\mathbb{F}_{G,/H}} \left\{\mathrm{Ind}^G_H S \to [G/H]\right\}\right) & \xrightarrow{\quad\alpha\quad} & \mathbb{E}_V(S) \\ \wr\| & & \wr\| & & \wr\| \\ B\mathcal{P}^H_S(V|W) & \hookrightarrow & B\mathcal{P}^H_S(V \oplus W) & \longrightarrow & \mathrm{Emb}^{\mathrm{Aff},H}(S \cdot D(V\oplus W), D(V \oplus W)) \end{array}$$

are weak equivalences; this is Propositions 2.17 and 2.18. $\square$

We're now ready to conclude homotopical equivariant Dunn additivity.

Proof of Theorem A. As telegraphed in the introduction, it suffices to verify that the square

$$\begin{array}{ccc} \mathbb{P}^{\otimes}_V \wr \mathbb{P}^{\otimes}_W & \xrightarrow{\widetilde{\varphi}_{\mathbb{P}}} & \mathbb{P}^{\otimes}_{V|W} \\ {\scriptstyle \alpha_V \wr \alpha_W}\downarrow & & \downarrow{\scriptstyle \alpha_{V|W}} \\ \mathbb{E}^{\otimes}_V \wr \mathbb{E}^{\otimes}_W & \xrightarrow{\varphi_{\mathbb{E}}} & \mathbb{E}^{\otimes}_{V\oplus W} \end{array}$$

satisfies the conditions of our bookkeeping theorem, Theorem 1.2. Indeed, Condition (BK-a) is Proposition 2.29, Condition (BK-b) is Proposition 2.27, and Condition (BK-c) follows simply by noting that all of the G-operads involved have one color. $\square$

3. Tangential structures

In this section, we study variations of the little disk G-operad where affineness is replaced with linear G-tangential structure. This begins with definitions in Section 3.1, leading to a construction of the additivity (natural) equivalence in Section 3.2. We go on in Section 3.3 to relate this to previous work of Szczesny on this topic, and relate it to the motivating conjecture of Dwyer, Hess, and Knudsen.

3.1. Equivariant framed little disk algebras.

3.1.1. *The Kister-Mazur theorem for orthogonal representations.*

Proposition 3.1. *The inclusions* $\mathrm{Aut}^H(V) \subset \mathrm{Emb}_0^{H,\mathrm{Aff}}(D(V), D(V)) \subset \mathrm{Emb}^{H,\mathrm{Aff}}(D(V), D(V))$ *are homotopy equivalences; in particular,* $\mathrm{Emb}^{H,\mathrm{Aff}}(V,V)$ *is grouplike.*

Proof. As noted in [Hor19], it is easily seen that the proof of the nonequivariant case in [Kup17] preserves the H-equivariant subspaces. Moreover, we may explicitly construct a deformation retract of $\mathrm{Emb}^{H,\mathrm{Aff}}(D(V), D(V))$ onto $\mathrm{Emb}_0^{H,\mathrm{Aff}}(D(V), D(V))$ by

$$h_t(\iota) = (1-t)\iota - t \cdot c(\iota). \qquad \square$$

3.1.2. *The G-operad $\mathbb{E}_V$.*

Notation 3.2. The G-coefficient system of *n-dimensional orthogonal representations* is defined by

$$\underline{\mathbf{Rep}}_G^{O(n)} := \underline{\pi}_0 B_G O(n);$$

in particular, $\left(\underline{\mathbf{Rep}}_G^{O(n)}\right)^H$ is the set of isomorphism classes of n-dimensional orthogonal H-representations and the restriction map $\left(\underline{\mathbf{Rep}}_G^{O(n)}\right)^H \to \left(\underline{\mathbf{Rep}}_G^{O(n)}\right)^K$ is the usual restriction function. ◁

Construction 3.3. We define a $\underline{\mathbf{Rep}}_G^{O(n)}$-symmetric sequence $\mathbb{E}^t_{B_G O(n)}$ by

$$\mathbb{E}^t_{B_G O(n)}((U_K); V) := \mathrm{Emb}_S^{H,\mathrm{lin.isom}}\left(\coprod_K^S U_K, V\right);$$

the restriction action is the evident restriction. ◁

Proposition 3.4. $\left(\mathbb{E}^t_{B_G O(n)}, (\mathrm{id}_\bullet), \gamma\right)$ *is a genuine topological G-operad.*

Proof. Identical to Proposition 2.20. □

Notation 3.5. The *$B_G O(n)$-framed little disks G-operad* is $\mathbb{E}^\otimes_{B_G O(n)} := N^\otimes \mathbb{E}^t_{B_G O(n)}$. ◁

Lemma 3.6. *The equivalence* $\underline{\pi}_0 B_G O(n) \simeq \underline{\pi}_0 U\mathbb{E}_{B_G O(n)}$ *extends to an equivalence of G-spaces*

$$f\colon B_G O(n) \xrightarrow{\sim} U\mathbb{E}_{B_G O(n)}.$$

Proof. Proposition 3.1 yields a filler equivalence for the bottom arrow of the following.

$$\begin{tikzcd} B_G O(n)^H \arrow[r, "f^H"] & U\mathbb{E}_{B_G O(n)} \\ \coprod\limits_{R\colon H \curvearrowright \mathbb{R}^n} O^H(R) \arrow[u, "\wr"] \arrow[r] & \coprod\limits_{R: H \curvearrowright \mathbb{R}^n} \mathrm{Emb}^{H,\mathrm{lin.isom.}}(R,R) \arrow[u, "\wr"] \end{tikzcd}$$

The desired equivalence is classifed by $(f^H)_{H \subset G}$ under Elmendorf's theorem. □

In particular, f is adjunct to a map $\mathbb{E}^\otimes_{B_G O(n)} \to B_G O(n)^{G-\sqcup}$.

Construction 3.7. If $T\colon X \to B_G O(n)$ is a linear G-tangential structure, the *X-framed little disk G-operad* is

$$\mathbb{E}_X^\otimes := L_{\mathrm{Op}_G}\left(\mathbb{E}_{B_G O(n)} \times_{B_G O(n)^{G-\sqcup}} X^{G-\sqcup}\right). \qquad ◁$$

Note that $\mathbb{E}_X^\otimes$ is functorial in $X \in \mathcal{S}_{G,/B_G O(n)}$; in particular, given a point $x \in X$ with stabilizer H, we acquire a map

$$f_x\colon \mathbb{E}^\otimes_{T_x X} \to \mathrm{Res}_H^G \mathbb{E}_X^\otimes.$$

Proposition 3.8. *Given a linear G-tangential structure $T\colon X \to B_G O(n)$ and a point $x \in X$, the reduction of the functoriality map* $f_x\colon \mathbb{E}^\otimes_{T_x X} \to \mathrm{Res}^G_{\mathrm{stab}(x)} \mathbb{E}^\otimes_X$ *is an equivalence*

$$\mathbb{E}^\otimes_{T_x} \xrightarrow{\sim} \mathrm{End}_x^{\mathrm{red}} \mathbb{E}_X^\otimes$$

Proof. Unwinding definitions, we're tasked with proving that the back faces of the following is cartesian.

$$\begin{tikzcd}
\mathrm{Emb}^{\mathrm{Aff}}(S\cdot D(V),D(V)) \arrow[rr] \arrow[dd] \arrow[dr, "\sim"] & & \mathrm{Emb}^{H,\mathrm{Rect}}(S\cdot D(V),D(V)) \arrow[dd] \arrow[dr, "\sim"] & \\
& \mathrm{Conf}^G_S(V) \arrow[rr] \arrow[dd] & & \mathrm{Germ}^G_S(V) \arrow[dd] \\
* \arrow[rr] \arrow[dr, equal] & & \underline{\mathrm{Emb}}^{\mathrm{Rect}}(D(V),D(V))^S \arrow[dr, "\sim"] & \\
& * \arrow[rr] & & O^S(V)
\end{tikzcd}$$

Equivalently, we may prove that the front face is cartesian; this follows from [Hor19, Prop 3.8.7]. □

Corollary 3.9. *If $T\colon X \to B_G O(n)$ is a linear G-tangential structure, then pullback along the functoriality maps $f_x\colon \mathbb{E}^{\otimes}_{T_x} \to \mathrm{Res}^G_{\mathrm{stab}(x)} \mathbb{E}^{\otimes}_X$ yields a natural G-limit presentation*

$$\underline{\mathrm{Alg}}_{\mathbb{E}_X}(\mathcal{C}) \simeq \underline{\lim}_{x\in X} \underline{\mathrm{Alg}}_{\mathbb{E}_{T_x}}(\mathcal{C}).$$

Proof. The disintegration and assembly procedure (as recalled in Proposition 1.29) provides a natural G-colimit presentation

$$\mathbb{E}^{\otimes}_X \simeq \underline{\mathrm{colim}}_{x\in X} \mathbb{E}^{\otimes}_{T_x};$$

since right G-adjoints turn G-colimits into G-limits, this yields a G-limit presentation of G-operads

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_X}(\mathcal{C}) \simeq \underline{\lim}_{x\in X} \underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{T_x}}(\mathcal{C});$$

the result follows by noting that $U\colon \underline{\mathrm{Op}}_G \to \underline{\mathrm{Cat}}_G$ is a right G-adjoint (hence it preserves G-limits) and it takes $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(-)$ naturally to $\underline{\mathrm{Alg}}_{\mathcal{O}}(-)$. □

3.2. Assembly and additivity with equivariant tangential structure.

Proposition 3.10. *There exists a map of G-operads $\widetilde{\varphi}\colon \mathbb{E}^{\otimes}_{B_G O(k)} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_{B_G O(k')} \to \mathbb{E}^{\otimes}_{B_G O(k+k')}$ such that, for all orbits $(x,y)\colon [G/H]\times[G/K] \to B_G O(k)\times B_G O(k')$, the associated diagram commutes*

$$\begin{tikzcd}
\mathbb{E}^{\otimes}_{T_x} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_{T_y} \arrow[r, hook] \arrow[d, "\varphi"] & \mathbb{E}^{\otimes}_{B_G O(k)} \overset{\mathrm{BV}}{\otimes} \mathbb{E}^{\otimes}_{B_G O(k')} \arrow[r, hook] \arrow[d, "\widetilde{\varphi}"] & B_G O(k)^{G-\sqcup} \overset{\mathrm{BV}}{\otimes} B_G O(k')^{G-\sqcup} \arrow[r, phantom, "\simeq"] \arrow[d] & (B_G O(k)\times B_G O(k'))^{G-\sqcup} \arrow[d, "\oplus^{\sqcup}"] \\
\mathbb{E}^{\otimes}_{T_x\oplus T_y} \arrow[r, hook] & \mathbb{E}^{\otimes}_{B_G O(k+k')} \arrow[r, hook] & B_G O(k+k')^{G-\sqcup} \arrow[r, equal] & B_G O(k+k')^{G-\sqcup}
\end{tikzcd}$$

where the middle horizontal arrows classify the equivalences of Lemma 3.6, the top right equivalence is Corollary A.12, and the left horizontal arrows are induced by the functoriality of $\mathbb{E}^{\otimes}_{(-)}$.

Proof. Since all elements of the bottom row are G-operads, it suffices to construct a commutative diagram of topolgoical G-preoperads

$$\begin{tikzcd}
\mathbb{E}^{\otimes}_{T_x} \times \mathbb{E}^{\otimes}_{T_y} \arrow[r, hook] \arrow[d, "\varphi"] & \mathbb{E}^{\otimes}_{B_G O(k)} \times \mathbb{E}^{\otimes}_{B_G O(k')} \arrow[r, hook] \arrow[d, dashed, "\widetilde{\varphi}"] & B_G O(k)^{G-\sqcup} \times B_G O(k')^{G-\sqcup} \arrow[d] \\
\mathbb{E}^{\otimes}_{T_x\oplus T_y} \arrow[r, hook] & \mathbb{E}^{\otimes}_{B_G O(k+k')} \arrow[r, hook] & B_G O(k+k')^{G-\sqcup}
\end{tikzcd}$$

Now, note that the bottom left horizontal arrow has action maps classified by the subspace inclusions

$$\mathrm{Emb}^H\left(\coprod_U^S R_U, R\right) \hookrightarrow \prod_U^S \mathrm{Emb}^H(R_U, R)$$

and is essentially injective; moreover, the top arrow is an external product of two such arrows. In particular, we can replace the desired right square (up to equivalence) with one whose horizontal arrows are point-set monic in the sense of Proposition 1.34 and verify the conditions there to consturct $\widetilde{\varphi}$ such that the right squhare commutes. Since the right horizontal arrows are U-equivalences by definition, condition (a) is immediate. Moreover, condition (b) follows by unwinding definitions, as does commutativity of the outer diagram;commutativity of the left diagram then follows by the fact that point-set monics are monomorphisms in $\mathrm{Cat}_{\mathrm{Top}}$. ☐

Construction 3.11. Given $T\colon X \to B_G O(k)$ and $T'\colon Y \to B_G O(k')$ a pair of G-tangential structures, there is a *Dunn map* of G-preoperads so that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{E}_X^{\otimes} \times \mathbb{E}_Y^{\otimes} \arrow[rrr, hook] \arrow[dr, dashed, "\varphi_{(X,Y)}"] \arrow[dd, "\mathbb{E}_T \times \mathbb{E}_{T'}"'] & & & X^{G-\sqcup} \times Y^{G-\sqcup} \arrow[dr, equal] \arrow[dd, "T^{G-\sqcup} \times T'^{G-\sqcup}"] & \\
& \mathbb{E}_{X \times Y}^{\otimes} \arrow[rrr, hook] \arrow[dd, "\mathbb{E}_{T \times T'}"] \arrow[dr, phantom, "\lrcorner", very near start] & & & (X \times Y)^{G-\sqcup} \arrow[dd, "T \oplus T'^{G-\sqcup}"] \\
\mathbb{E}_{B_G O(k)}^{\otimes} \times \mathbb{E}_{B_G O(k')}^{\otimes} \arrow[rrr, hook] \arrow[dr, "\widetilde{\varphi}"'] & & & B_G O(k)^{G-\sqcup} \times B_G O(k')^{G-\sqcup} \arrow[dr, "\oplus^{\sqcup}"] & \\
& \mathbb{E}_{B_G O(k+k')}^{\otimes} \arrow[rrr, hook] & & & B_G O(k+k')^{G-\sqcup}
\end{tikzcd}$$

In fact, the bottom and right face form a G-functor

$$\begin{aligned}
\underline{\mathcal{S}}_{G,/B_G O(k)} \times \underline{\mathcal{S}}_{G,/B_G O(k')} \longrightarrow & \underline{\mathrm{Fun}}_G\left(\mathrm{Infl}_e^G \Delta^1 \times \Delta^1, \underline{\mathrm{PreOp}}_G\right) \times_{\underline{\mathrm{Fun}}\left(\mathrm{Infl}_e^G \Delta^1, \underline{\mathrm{PreOp}}_G\right)} \underline{\mathrm{Fun}}_G\left(\mathrm{Infl}_e^G \Delta^1 \times \Delta^1, \underline{\mathrm{PreOp}}_G\right) \\
\simeq & \underline{\mathrm{Fun}}_G\left(\mathrm{Infl}_e^G \Delta^1 \times \Delta^1 \sqcup_{\Delta^1} \Delta^1 \times \Delta^1, \underline{\mathrm{PreOp}}_G\right)
\end{aligned}$$

and the formation of pullbacks of the bottom and top cospan extends this to a G-functor

$$\underline{\mathcal{S}}_{G,/B_G \mathrm{Top}(k)} \times \underline{\mathcal{S}}_{G,/B_G \mathrm{Top}(k')} \longrightarrow \underline{\mathrm{Fun}}_G\left(\mathrm{Infl}_e^G \left(\Delta^1\right)^3, \underline{\mathrm{PreOp}}_G\right);$$

evaluation at the dashed arrow yields a commutative diagram of G-functors

$$\begin{tikzcd}
\underline{\mathcal{S}}_{G,/B_G \mathrm{Top}(k) \times B_G O(k')} \arrow[d, "\mathbb{E}_{(-)}"'] & \underline{\mathcal{S}}_{G,/B_G O(k)} \times \underline{\mathcal{S}}_{G,/B_G O(k')} \arrow[l, "\times"'] \arrow[r, "(\mathbb{E}_{(-)}, \mathbb{E}_{(-)})"] \arrow[d, "\varphi"] & \underline{\mathrm{PreOp}}_G \times \underline{\mathrm{PreOp}}_G \arrow[d, "\times"] \\
\underline{\mathrm{PreOp}}_G & \underline{\mathrm{Fun}}_G\left(\mathrm{Infl}_e^G \Delta^1, \underline{\mathrm{PreOp}}_G\right) \arrow[l, "\mathrm{ev}_1"] \arrow[r, "\mathrm{ev}_0"'] & \underline{\mathrm{PreOp}}_G
\end{tikzcd}$$

In other words, we've constructed a natural transformation of G-functors $\mathbb{E}_X^{\otimes} \times \mathbb{E}_Y^{\otimes} \to \mathbb{E}_{X \times Y}^{\otimes}$. ◁

Now, our precise form of Corollary B is the following.

Theorem 3.12. *The map* $\varphi\colon \mathbb{E}_X^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_Y^{\otimes} \to \mathbb{E}_{X \times Y}^{\otimes}$ *is an equivalence.*

Proof. It follows by two-out-of-three that φ always induces an equivalence $U\mathbb{E}_X \times U\mathbb{E}_Y \to U\mathbb{E}_{X \times Y}$. Moreover, for all pairs of orbits $(x,y) \subset X \times Y$, Propositions 3.8 and 3.10 and Construction 3.11 constructs an equivalence of arrows

$$\begin{tikzcd}
\mathbb{E}_{T_x}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{T_y}^{\otimes} \arrow[r, "\varphi"] \arrow[d, phantom, "\wr"] & \mathbb{E}_{T_x \oplus T_y}^{\otimes} \arrow[d, phantom, "\wr"] \\
\mathrm{End}_x^{\mathrm{red}}(\mathbb{E}_X^{\otimes}) \overset{\mathrm{BV}}{\otimes} \mathrm{End}_y^{\mathrm{red}}(\mathbb{E}_Y^{\otimes}) \arrow[r, "\varphi"] & \mathrm{End}_{(x,y)}^{\mathrm{red}}(\mathbb{E}_{X \times Y}^{\otimes})
\end{tikzcd}$$

hence φ induces an equivalence of reduced endomorphism H-operads over pairs of orbtis, and the result follows by Corollary 1.30. ☐

3.3. **Skew little cubes, Szczesny additivity, and equivariant configurations.** Let G be a compact Lie group.

3.3.1. *Homotopy theories for G-operads and Szczesny's additivity theorem.*

Observation 3.13. There are inclusions of G-spaces $B_G \Sigma_n \hookrightarrow \underline{\Sigma}_G$ summing to an equivalence

$$\coprod_{n \in \mathbb{N}} B_G \Sigma_n \xrightarrow{\sim} \underline{\Sigma}_G.$$

In particular, G-symmetric sequences are precisely sequences of genuine G-equivariant principal Σ_n-bundles. ◁

Now, every topological $\Sigma_n \times G$-space has an underlying genuine G-equivariant principal Σ_n-bundle by family-restriction along the inclusion $\mathrm{Tot} B_G \Sigma_n \simeq \mathcal{O}_{G \times \Sigma_n, \Gamma} \subset \mathcal{O}_{G \times \Sigma_n}$; in particular, we get an *underlying topological G-symmetric sequence* functor

$$\mathsf{sseq}\colon \mathsf{Op}(\mathsf{Top}_G) \longrightarrow \mathsf{Fun}\left(\coprod_{n \in \mathbb{N}} BG \times B\Sigma_n, \ \mathsf{Top}\right) \longrightarrow \mathsf{Fun}(\mathrm{Tot}\underline{\Sigma}_G, \mathsf{Top}).$$

Definition 3.14. A morphism $\mathcal{O} \to \mathcal{P}$ of topological G-symmetric sequences is a *weak equivalence* if $\mathcal{O}(S) \to \mathcal{P}(S)$ is a weak equivalence for all closed subgroups $H \subset G$ and finite H-sets $S \in \mathbb{F}_G$. A morphism f in $\mathsf{Op}(\mathsf{Top}_G)$ is a *graph-weak equivalence* if $\mathsf{sseq}(f)$ is a weak equivalence. ◁

Warning 3.15. These are *not* the weak equivalences transferred along

$$\mathsf{Op}\left(\mathsf{Top}_G\right) \to \mathsf{Fun}\left(\coprod_{n \in \mathbb{N}} B\Sigma_n, \mathsf{Top}_G\right)$$

for the projective structure on the codomain with respect to genuine G-equivalences; for that matter, this is fails for $\mathcal{F}$-equivalences for any family $\mathcal{F} \subset \mathcal{O}_G$, such as $\mathcal{F} = BG$. Indeed, for an arbitrary (topological) category $\mathcal{C}$, the Quillen equivalence

$$\mathsf{Fun}\left(\coprod_{n \in \mathbb{N}} B\Sigma_n, \mathsf{Top}^{\mathcal{C}}\right)_{\mathrm{proj.}} \simeq \mathsf{Fun}\left(\coprod_{n \in \mathbb{N}} B\Sigma_n, \mathsf{Top}\right)^{\mathcal{C}}_{\mathrm{proj.}}$$

is compatible with the composition product on each side, furnishing a Quillen equivalence

$$\mathsf{Op}(\mathsf{Top}^{\mathcal{C}})_{\text{transf. proj.}} \simeq \mathsf{Op}(\mathsf{Top})^{\mathcal{C}}_{\mathrm{proj.}}.$$

Now, taking ∞-categories, specializing to $\mathcal{C} = \mathcal{F}$ and applying Elmendorf's theorem [DK84; Elm83], note that $\mathbb{E}_\infty^{\otimes}$ is taken to the constant functor $\mathcal{F} \to \mathsf{Op}$ valued on the terminal object $\mathbb{E}_\infty^{\otimes}$, and hence it is terminal. On the other hand, $\mathbb{E}_\infty^{\otimes} \in \mathsf{Op}_G$ is only terminal if G is nontrivial.

In essence, the transferred weak equivalences on $\mathsf{Op}(\mathsf{Top}^{\mathcal{F}})$ do not respect spaces of transfers, so the corresponding homotopy types can't corepresent the equivariant loop space theory of [GM17; RS00], can't support the (multiplicative) Wirthmüller isomorphisms of [BH21; HHR16; Ste25d], etc. ◁

Construction 3.16 ([BP21, § 4.3] without finiteness)**.** Given $\mathcal{O} \in \mathsf{Op}(\mathsf{Top}_G)$, we may define a genuine topological G-operad $\iota_! \mathcal{O}$ with underlying G-symmetric sequence $\mathsf{sseq}\,\mathcal{O}$, with the same unit maps, and with composition defined by

$$\begin{tikzcd}
\left(\mathcal{O}(|S|) \times \prod\limits_{[H/K] \in S} \mathcal{O}(|T_K|)\right)^{\Gamma_S \times \prod_{[H/K] \subset S} \Gamma_{T_K}} \arrow[d, "\Delta"'] \arrow[r, phantom, "\simeq"] & \mathcal{O}(S) \times \prod\limits_{[H/K] \subset S} \mathcal{O}(T_K) \arrow[r, "\gamma"] & \mathcal{O}(T) \arrow[d, phantom, "\wr"] \\
\left(\mathcal{O}(|S|) \times \prod\limits_{x \in S} \mathcal{O}(|T_x|)\right)^{\Gamma_S \times \prod_{x \in S} \Gamma_{T_x}} \arrow[r, "\gamma^{\Gamma_S \times \prod_{x \in S} \Gamma_{T_x}}"] & \mathcal{O}(|T|)^{\Gamma_S \times \prod_{x \in S} \Gamma_{T_x}} \arrow[r, "\mathrm{Res}"] & \mathcal{O}(|T|)^{\Gamma_T}
\end{tikzcd}$$

We use this only for illustration purposes, so we leave the verification of the conditions for genuine G-operads as an exercise to the reader. ◁

Proposition 3.17 ([Ste25b])**.** *The composite nerve functor $N^{\otimes}\colon \mathsf{Op}(\mathsf{Top}_G) \to s\mathsf{Set}^G_{/\mathrm{Tot}\underline{\mathbb{F}}_{G,*}}$ preserves fibrant objects and preserves and reflects weak equivalences between fibrant objects; in particular, it possesses a* *conservative total right derived functor of ∞-categories*

$$\mathrm{Op}(\mathrm{Top}_G)\left[\Gamma\text{-WEQ}^{-1}\right] \to \mathrm{Op}_G.$$

Szczesny's work then confirms that $\mathbb{E}_V^{\otimes}$ is modelled by Dwyer-Hess-Knudsen.

Theorem 3.18 ([Szc24, Thm 5.6])**.** *Given a dilation representation $V\colon G \to \mathcal{O}(n)$, Dwyer-Hess-Knudsen's skew little cubes operad C_n^G and the little V-disks operad D_V are connected by a zigzag of graph weak equivalences.*

In particular, Szczesny affirmed the following genuine-equivariant version of Dwyer-Hess-Knudsen's skew cubes additivity conjecture [DHK18, Conj 4.18], where $\otimes^{\Gamma}$ is the Top_G-enriched point-set Boardman-Vogt tensor product.

Theorem 3.19 ([Szc24, Thm 1.1])**.** *There is a natural graph weak equivalence $D_V \otimes^{\Gamma} D_W \simeq D_{V\oplus W}$.*

In particular, this and Theorem A construct a family of examples where $\otimes^{\Gamma}$ and $\overset{\mathrm{BV}}{\otimes}$ agree: both satisfy the genuine equivariant skew cubes additivity conjecture. Moreover, in sight of Warning 3.15, we may view Corollary B as constructing *partially* genuine equivariant versions, where the base for Borel equivariance is allowed to be a version of an $\mathbb{E}_1$-group with G-action.

3.3.2. *Corollary B and Dwyer-Hess-Knudsen.* Fix $X \to B_G O(n)$ a linear tangential structure. Let $\underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}$ be Miladinovic's G-symmetric monoidal ∞-category of X-framed G-manifolds and disjoint unions [Mil20] (generalizing Horev [Hor19]). To demonstrate that Corollary B captures an equiavriantization of the aspirations of [DHK18, Conj 4.18], We verify the following corollary, which was implied to be a corollary of [DHK18, Conj 4.18] in the case $G = e$ and $X = BH', Y = BH$.

Corollary 3.20. *There is an equivalence of G-operads*

$$\mathrm{End}_{\mathbb{R}^{n_X}}(\underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup})^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{End}_{\mathbb{R}^{n_Y}}(\underline{\mathrm{Mfld}}_G^{Y-\mathrm{fr},\sqcup})^{\otimes} \simeq \mathrm{End}_{\mathbb{R}^{n_X+n_Y}}(\underline{\mathrm{Mfld}}_G^{X\times Y-\mathrm{fr},\sqcup})^{\otimes}$$

Now, this directly follows from Corollary B and the following variation of [DHK18, Thm 4.14].

Proposition 3.21 (Tangential [Hor19, Prop 3.9.8])**.** *There is an equivalence $\mathrm{End}_{\mathbb{R}^n}\left(\underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}\right) \simeq \mathbb{E}_X^{\otimes}$.*

Proof. It follows by unwinding definitions and applying Proposition 3.1 that

$$U\,\mathrm{End}_{\mathbb{R}_n}\left(\underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}\right) \simeq X.$$

The identity map on X is then mate to a structure map of G-operads

$$\mathrm{End}_{\mathbb{R}_n}\left(\underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}\right) \to X^{G-\sqcup};$$

it follows by the adjunction that the back and bottom face of following diagram commutes, constructing the left front face.

$$\begin{tikzcd}
\mathrm{End}_{\mathbb{R}^n}\underline{\mathrm{Mfld}}_G^{Y-\mathrm{fr},\sqcup} \arrow[d] \arrow[dr, dashed] \arrow[drr] & & \\
\mathrm{End}_{\mathbb{R}^n}\underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup} \arrow[d] \arrow[dr, dashed] \arrow[drr] & \mathbb{E}_Y^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y^{G-\sqcup} \arrow[d] \\
\mathrm{End}_{\mathbb{R}^n}\underline{\mathrm{Mfld}}_G^{\sqcup} \arrow[dr] \arrow[drr] & \mathbb{E}_X^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X^{G-\sqcup} \arrow[d] \\
& \mathbb{E}_{B_G O(n)}^{\otimes} \arrow[r] & B_G O(n)^{G-\sqcup}
\end{tikzcd}$$

In particular, we've constructed a map $\psi\colon \mathrm{End}_{\mathbb{R}^n}\underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup} \to \mathbb{E}_X^{\otimes}$ which is natural in maps of linear G-tangential structures. Specializing to the case that $Y \simeq *$ shows that ψ is a map of X-families of G-operads whose maps on fibers is the usual map $\psi_V\colon \underline{\mathrm{Rep}}_G^{V-\mathrm{fr},\sqcup} \simeq \mathrm{End}_{\mathbb{R}^n}\underline{\mathrm{Mfld}}_G^{V-\mathrm{fr},\sqcup} \to \mathbb{E}_V^{\otimes}$; Disintegration and assembly then reduces the proposition to checking that ψ_V is an equivalence, i.e. in the case that X is contractible. The proof of [Hor19] applies verbatim to prove this case. $\square$

We have a few corollaries to this. One is the simple envelope corollary of Proposition 3.21 and [Hor19].

Corollary 3.22. *The G-symmetric monoidal envelope yields an equivalence*

$$\mathrm{Alg}_{\mathbb{E}_X}(\mathcal{C}) \simeq \mathrm{Fun}_G^{\otimes}\left(\underline{\mathrm{Disk}}_G^{X-\mathrm{fr}}, \mathcal{C}\right).$$

Moreover, Symmetric monoidality of Env_G then yields the following curious corollary about framed disk embeddings, where $\circledast$ is the canonical tensor product of G-commutative monoids [Ste25b], i.e. it is the box product of symmetric monoidal Mackey functors.

Corollary 3.23. *There is an equivalence of G-symmetric monoidal ∞-categories*

$$\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \circledast \underline{\mathrm{Disk}}_G^{Y-\mathrm{fr},\sqcup} \simeq \underline{\mathrm{Disk}}_G^{X\times Y-\mathrm{fr},\sqcup}.$$

To clarify the relationship between our results and the literature, we recover the following a analog of [DHK18, Conj 4.18] in two ways.

Corollary 3.24. *Let $BG \to BO(n)$ and $BH \to BO(m)$ be the tangential structures specified by a pair of orthogonal representations. Then, there is an equivalence of operads*

$$\mathbb{E}_{BG}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{BH}^{\otimes} \simeq \mathbb{E}_{BG\times BH}^{\otimes}.$$

Proof of Corollary 3.24, Lurie style. Apply Corollary B for the trivial group. □

Proof of Corollary 3.24, Szczesny style. Given X be a space considered as an atomic orbital ∞-category, there is a diagram of equivalences

$$\begin{tikzcd}
\mathrm{Op}_X \arrow[d, leftrightarrow, "\text{[Ste24]}"'] & \mathrm{Op}(\mathrm{Top}^X)\left[\text{Ptws-WEQ}^{-1}\right] \arrow[d, leftrightarrow, "3.15"'] & \mathrm{Op}_{/X^{\sqcup}} \\
\mathrm{Op}_{I_\infty(X)} \arrow[r, leftrightarrow, "\text{[Ste25b]}"] & \mathrm{Op}^X \arrow[ur, leftrightarrow, "\text{[Ste25d]}"'] &
\end{tikzcd}$$

Specializing to the case $X = BG$, we find

$$\begin{tikzcd}
\mathrm{Op}_G \arrow[r] & \mathrm{Op}_{BG} \arrow[r, phantom, "\simeq"] & \mathrm{Op}_{/BG^{\sqcup}} \arrow[r] & \mathrm{Op} \\
\mathbb{E}_V^{\otimes} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \mathrm{Bor}_{BG}^G \mathbb{E}_V^{\otimes} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \left(\mathbb{E}_{BG}^{\otimes} \to BG^{\sqcup}\right) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \mathbb{E}_{BG}^{\otimes} \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

Taking inflations, we externalize the tensor product of the corresponding G-operads, acquiring a diagram

$$\begin{tikzcd}
\mathrm{Op}_G \times \mathrm{Op}_H \arrow[r, "\mathrm{Bor}_{BG}^G \times \mathrm{Bor}_{BH}^H"] \arrow[d, hook, "E_{\mathcal{O}_G \vee \mathcal{O}_H}^{G\times H}"'] & \mathrm{Op}_{BG} \times \mathrm{Op}_{BH} \arrow[r, "\mathrm{colim}"] \arrow[d, phantom, "\wr"] & \mathrm{Op} \arrow[d, equal] \\
\mathrm{Op}_{G\times H} \arrow[r, "\mathrm{Bor}_{BG\times BH}^{G\times H}"] & \mathrm{Op}_{BG\times BH} \arrow[r, "\mathrm{colim}"] & \mathrm{Op}
\end{tikzcd}$$

where every arrow intertwines with binary tensor products. Writing F for the bottom horizontal composite (i.e. "underlying semidirect product operad") and applying Theorem A, we acquire a chain of equivalences

$$\begin{aligned}
\mathbb{E}_{BG}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{BH}^{\otimes} &\simeq F\left(\mathbb{E}_{\mathrm{Infl}_G^{G\times H} V}^{\otimes}\right) \overset{\mathrm{BV}}{\otimes} F\left(\mathbb{E}_{\mathrm{Infl}_H^{G\times H} W}^{\otimes}\right) \\
&\simeq F\left(\mathbb{E}_{\mathrm{Infl}_G^{G\times H} V}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{\mathrm{Infl}_H^{G\times H} W}^{\otimes}\right) \\
&\simeq F\left(\mathbb{E}_{\mathrm{Infl}_G^{G\times H} V \oplus \mathrm{Infl}_H^{G\times H} W}^{\otimes}\right) \\
&\simeq \mathbb{E}_{BG\times BH}^{\otimes}.
\end{aligned}$$

□

Appendix A. Deferred proofs

A.1. **Weak approximations and monoids.** We're tasked with proving Proposition 1.37, which we will do by reduction to Harpaz's result, using the comparison to the setting of fibrous $\mathrm{Tot}\underline{\mathbb{F}}_{\mathcal{T},*}$-patterns of [BHS22; Ste25b]. We henceforth assume the reader is familiar with [Har, § 4.2] and [Ste25b, § A]. We begin by recalling the following.

Proposition A.1 ([BHS22], generalized as [Ste25b])**.** *The full subcategory of $\mathrm{Tot}\underline{\mathbb{F}}^{\mathrm{el}}_{\mathcal{T},*,S/} \simeq \mathcal{T} \times_{\mathbb{F}_{\mathcal{T}}} \mathbb{F}_{\mathcal{T},/S}$ consisting of orbit inclusions $U \hookrightarrow S$ is initial.*

Corollary A.2. *If $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-preoperad, then $\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes}$ is a weak ∞-operad; given a map of $\mathcal{T}$-operads $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$, the associated functor $\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathrm{TotTot}_{\mathcal{T}}\mathcal{P}^{\otimes}$ is a functor of weak ∞-operads.*

Lemma A.3. *If $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is weak approximation in the sense of $\mathcal{T}$-preoperads, then $\mathrm{TotTot}_{\mathcal{T}}\varphi$ is a weak approximation of weak ∞-operads.*

Proof. Unwinding definitions, we have a diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \{h\} \arrow[d] & \\
\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\mathrm{act}}_{/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{TotTot}_{\mathcal{T}}\mathcal{P}^{\mathrm{act}}_{/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Tot}\underline{\mathbb{F}}^{\mathrm{act}}_{\mathcal{T},*,/S} \arrow[d] \\
\mathrm{Tot}\mathcal{O}^{\mathrm{act}}_{/S} \arrow[r] & \mathrm{Tot}\mathcal{P}^{\mathrm{act}}_{/S} \arrow[r] & \mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}}_{/S}
\end{tikzcd}$$

□

Proof of Proposition 1.37. We're left with verifying that an $\mathcal{O}$-monoid $\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathcal{S}$ admits a factorization

$$\begin{tikzcd}
\mathrm{TotTot}_{\mathcal{T}}\mathcal{O}^{\mathrm{el}} \arrow[r] \arrow[d, equal] & \mathrm{TotTot}_{\mathcal{T}}\mathcal{P}^{\mathrm{el}} \arrow[r, dashed] \arrow[d, equal] & \mathcal{S} \\
\mathrm{Tot}U\mathcal{O} \arrow[r] & \mathrm{Tot}U\mathcal{P} \arrow[ur, dashed] &
\end{tikzcd}$$

if and only if the associated $\mathcal{T}$-functor $U\mathcal{O} \to \underline{\mathcal{S}}_{\mathcal{T}}$ admits a factorization

$$U\mathcal{O} \longrightarrow U\mathcal{P} \dashrightarrow \underline{\mathcal{S}}_{\mathcal{T}};$$

this follows by noting that the above functors are related by the adjunction $\mathrm{Tot} \dashv \underline{\mathrm{Coeff}}^{\mathcal{T}}$. □

Lemma A.4. *In the situation of Corollary 1.38, the ∞-category $\mathcal{P}^{\mathrm{act}}_{/\varphi O} \times_{\mathcal{C}_{/\pi_{\mathcal{O}} O}} \{f\}$ is a space.*

Proof. The assumptions guarantee that the source functor

$$\mathcal{P}^{\mathrm{act}}_{/\varphi O} \times_{\mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi_{\mathcal{O}} O}} \{f\} \to \mathcal{P}_D$$

has codomain a space; moreover, unwinding definitions, the fibers are always spaces. Together these imply that $\mathcal{P}^{\mathrm{act}}_{/\varphi O} \times_{\mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi_{\mathcal{O}} O}} \{f\}$ is a space. □

Proof of Corollary 1.38. Fixing some $\tilde{f} \in \mathcal{P}^{\mathrm{act}}_{/\varphi\mathcal{O}}$ with underlying $\mathcal{C}^{\mathcal{T}-\sqcup}$-map f, we draw the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{O}^{\mathrm{act}}_{/O} \times_{\mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi O}} \{f\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{O}^{\mathrm{act}}_{/O} \arrow[d] \\
\{\tilde{f}\} \arrow[r] & \mathcal{P}^{\mathrm{act}}_{/\varphi O} \times_{\mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi O}} \{f\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{P}^{\mathrm{act}}_{/\varphi O} \arrow[d] \\
& \{f\} \arrow[r] & \mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi O}
\end{tikzcd}$$

We're tasked with verifying that F is weakly contractible. But we may apply the adjunction between B and $\mathcal{S} \hookrightarrow \mathsf{Cat}$ together with Lemma A.4 to construct a fiber square

$$\begin{tikzcd} BF \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\left(\mathcal{O}^{\mathrm{act}}_{/\mathcal{O}} \times_{\mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi\mathcal{O}}} \{f\}\right) \arrow[d] \\ \left\{\tilde{f}\right\} \arrow[r] & \mathcal{P}^{\mathrm{act}}_{/\varphi\mathcal{O}} \times_{\mathcal{C}^{\mathcal{T}-\sqcup,\mathrm{act}}_{/\pi\mathcal{O}}} \{f\} \end{tikzcd}$$

In particular, weak contractibility of F for all $\left\{\tilde{f}\right\}$ is equivalent to contractibility of the homotopy fibers of $\widetilde{\varphi}$ for all f, which is equivalent to $\widetilde{\varphi}$ being an equivalence for all f, as desired. $\square$

A.2. **Wreath products vs tensor products.** Generalizing [Har, § 4.3], we begin with definitions.

Construction A.5. Define the algebraic pattern $\mathcal{P}^c$ over $\mathsf{Span}(\mathbb{F}_V)$ via the same ∞-category over $\mathsf{Span}(\mathbb{F}_V)$, but with all morphisms inert and all objects elementary. We define the *mixed wreath product* $(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes})^{\mathrm{mix}}$ of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ as the same ∞-category over $\mathsf{Span}(\mathbb{F}_V)$ and inert arrows as $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ and the same elementary objects as $\mathcal{O}^{\otimes} \wr \mathcal{P}^c$, so that

$$\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right)^{\mathrm{el}} = \mathcal{O}^{\mathrm{el}} \times_{\mathbb{F}_{\mathcal{T}}^{\mathrm{op}}} \mathcal{P}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{el}} \times_{\mathbb{F}_{\mathcal{T}}^{\mathrm{op}}} \mathcal{P}^{\mathrm{int}} = \left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right)^{\mathrm{mix-el}} \tag{6}$$

$\triangleleft$

There is a commutative diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[r, "\varphi"] & \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \\ & (\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes})^{\mathrm{mix}} \arrow[u, dashed, "m"] \\ \mathcal{O}^{\otimes} \times \mathcal{P}^{c} \arrow[r, "\varphi^{\mathrm{coarse}}"'] \arrow[uu, dashed, "t"] & \mathcal{O}^{\otimes} \wr \mathcal{P}^{c} \arrow[u, dashed, "c"] \end{tikzcd}$$

whose solid arrows are maps of patterns, and whose dotted arrows denote identity arrows on underlying ∞-categories. We want to verify that the top horizontal arrow is a Morita equivalence, which we will try to deduce from the fact that the bottom is a Morita equivalence, which is the following lemma.

Lemma A.6. *$\varphi^{\mathrm{coarse}}$ is a strong approximation.*

Proof. Since all active arrows in $\mathcal{P}^c$ are equivalences, the inclusion of identities on the $\mathcal{P}^c$ induces an equivalence of arrows

$$\begin{tikzcd} (\mathcal{O}^{\otimes} \times \mathcal{P}^{c})^{\mathrm{act}}_{/(\mathbf{C};\mathbf{D})} \arrow[r] \arrow[d, phantom, "\wr"] & \mathcal{O}^{\otimes} \wr \mathcal{P}^{c}_{/(\mathbf{C};\Delta^S\mathbf{D})} \arrow[d, phantom, "\wr"] \\ \mathcal{O}^{\mathrm{act}}_{/\mathbf{C}} \arrow[r, equal] & \mathcal{O}^{\mathrm{act}}_{/\mathbf{C}} \end{tikzcd}$$

confirming condition (a). Moreover, the condition that $\mathcal{O} \times \mathrm{Tot}\mathcal{P}^{\otimes} \simeq \varphi^{-1}\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right)^{\mathrm{el}} \to \left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right)^{\mathrm{el}}$ is an equivalence follows by unwinding definitions. $\square$

Now, to make comparisons along vertical arrows, we import the following lemma due to Harpaz.

Lemma A.7 ([Har, Lem 4.2.21])**.** *Suppose $(\mathcal{Q}^{\otimes}, \mathcal{Q}^{\mathrm{int}}, \mathcal{Q}^{\mathrm{act}}, \mathcal{Q}^{\mathrm{el}})$ and $(\mathcal{Q}^{\otimes}, \mathcal{Q}^{\mathrm{int}\prime}, \mathcal{Q}^{\mathrm{act}\prime}, \mathcal{Q}^{\mathrm{el}\prime})$ are two weak ∞-operad structures on the same ∞-category such that $\mathcal{Q}^{\mathrm{int}} \subset \mathcal{Q}^{\mathrm{int}\prime}$ and $\mathcal{Q}^{\mathrm{act}\prime} \subset \mathcal{Q}^{\mathrm{act}}$.*

1. *(Unnecessary change of factorization system) Suppose $\mathcal{Q}^{\mathrm{el}} = \mathcal{Q}^{\mathrm{el}\prime}$ and every path $X \xrightarrow{f} Y \xrightarrow{g} Z$ with Z elementary, $g \circ f$ in $\mathcal{O}^{\mathrm{int}}$, and f in $\mathcal{O}^{\mathrm{int}\prime}$, has Y elementary. Then, the identity arrow on $\mathcal{Q}^{\otimes}$ lifts to a Morita equivalence between the two weak ∞-operad structures.*
2. *(Change of elementaries) Suppose $\mathcal{Q}^{\mathrm{int}} = \mathcal{Q}^{\mathrm{int}\prime}$ and $\mathcal{Q}^{\mathrm{act}} = \mathcal{Q}^{\mathrm{act}\prime}$, and additionally $\mathcal{Q}^{\mathrm{el}} \subset \mathcal{Q}^{\mathrm{el}\prime}$. Then, $F\colon \mathcal{Q}^{\otimes} \to \mathcal{C}$ is a $\left(\mathcal{Q}^{\otimes}, \mathcal{Q}^{\mathrm{int}}, \mathcal{Q}^{\mathrm{act}}, \mathcal{Q}^{\mathrm{el}}\right)$-Segal object if and only if it satisfies the following conditions:*

(a) F *is a* $\left(\mathcal{Q}^{\otimes}, \mathcal{Q}^{\mathrm{int}\prime}, \mathcal{Q}^{\mathrm{act}\prime}, \mathcal{Q}^{\mathrm{el}\prime}\right)$*-Segal object, and*
(b) $F|_{\mathcal{Q}^{\mathrm{el}\prime}}$ *is right Kan extended from* $F|_{\mathcal{Q}^{\mathrm{el}}}$.

We separate out the unnecessary change of factorization system first.

Lemma A.8. *c is a Morita equivalence.*

Proof. By Lemma A.7, it suffices to verify that, for all inert arrows $f\colon (\mathbf{C};(\mathbf{D}_U)_S) \to (\mathbf{C}';(\mathbf{D}'_{U'})_{S'})$ in $\mathcal{O}^{\otimes} \wr \mathrm{triv}\mathcal{P}^{\otimes}$ and factorizations $f\colon (\mathbf{C};(\mathbf{D}_U)_S) \xrightarrow{f'} (\mathbf{C}'';(\mathbf{D}''_{U''})_{S''}) \to (\mathbf{C}';(\mathbf{D}'_{U'})_{S'})$ with f' mixed-inert, $(\mathbf{C}'';(\mathbf{D}''_{U''})_{S''})$ is coarse-elementary. Unwinding definitions, f' mixed-inert implies that $\mathbf{C}''$ lies over a transitive V-set, which ensures that $(\mathbf{C}'';(\mathbf{D}''_{U''})_{S''})$ is coarse-elementary, as desired. $\square$

Proof of Theorem 1.45. Lemmas A.6 to A.8 and two-out-of-three construct a commutative diagram of fully faithful functors

$$\begin{tikzcd}
\mathrm{Seg}_{\mathcal{O}}\mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \arrow[d, hook, "\iota_\times"] & \simeq & \mathrm{Seg}_{\mathcal{O}\times\mathcal{P}}(\mathcal{C}) \arrow[d, hook] & & \mathrm{Seg}_{\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes}}(\mathcal{C}) \arrow[ll, hook'] \arrow[d, hook, "\iota_\wr"] \\
\mathrm{Seg}_{\mathcal{O}}\mathrm{Fun}(\mathcal{P},\mathcal{C}) & \simeq & \mathrm{Seg}^{\mathrm{coarse}}_{\mathcal{O}\times\mathcal{P}}(\mathcal{C}) & \simeq \ \mathrm{Seg}^{\mathrm{coarse}}_{\mathcal{O}\wr\mathcal{P}}(\mathcal{C}) \ \simeq & \mathrm{Seg}^{\mathrm{mix}}_{\mathcal{O}\wr\mathcal{P}}(\mathcal{C})
\end{tikzcd}$$

Thus it suffices to verify that, given a Segal $\mathcal{O}^{\otimes}$-object in $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$, the corresponding Segal $(\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes})^{\mathrm{mix}}$-object is a Segal $\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes}$-object, i.e. the image of $\iota_\times$ is contained in the image of $\varphi^{\mathrm{coarse}*}\iota_\wr$. By Lemma A.7, the image of $\varphi^{\mathrm{coarse}}\iota_\wr$ is those coarse Segal objects $F\colon \mathcal{O}^{\otimes}\times\mathcal{P}^{\otimes} \to \mathcal{S}$ such that

(a) $F|_{(\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes})^{\mathrm{mix-el}}}$ is right Kan extended from $F|_{(\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes})^{\mathrm{el}}}$,

and the image of $\iota_\times$ is those F such that

(b) for all $O \in \mathcal{O}^{\otimes}$, the functor $\{O\}\times\mathcal{P}^{\otimes} \to \mathcal{S}$ is a Segal space.

We're tasked with verifying that Conditions (a) and (b) are equivalent. Now, in sight of Eq. (6), this compiles down to verifying that a functor is right Kan extended along a base changed functor $\mathcal{C}\times_{\mathcal{D}}\mathcal{E} \to \mathcal{C}\times_{\mathcal{D}}\mathcal{E}'$ if and only if, for all $C \in \mathcal{C}$, the associated functor is right Kan extended along the functor $\{C\}\times\mathcal{E} \to \{C\}\times\mathcal{E}'$; but this is obvious. $\square$

A.3. **Boardman-Vogt tensor products of commutative operads.** For this subsection, let $\mathcal{T}$ be an atomic orbital ∞-category. We will assume familiarity with the minutiae of [Ste25d], and in particular, we use the following results.

Proposition A.9 ([Ste25d])**.** *There is a unique equivalence* $\mathcal{N}^{\otimes}_{I\vee J\infty} \simeq \mathcal{N}^{\otimes}_{I\infty} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{J\infty}$.

Proposition A.10 ([Ste25d])**.** $\mathcal{N}^{\otimes}_{I\infty} \in \mathrm{Op}_{\mathcal{T}}$ *is an idempotent algebra, and its modules are the $\mathcal{T}$-operads whose underlying I-operads are cocartesian.*

Proposition A.11 ([Ste25d])**.** *Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be almost-unital $\mathcal{T}$-operads.*

(1) *There is an equivalence* $U\left(\mathcal{O}\overset{\mathrm{BV}}{\otimes}\mathcal{P}\right) \simeq U\mathcal{O}\times U\mathcal{P}$.
(2) $A\left(\mathcal{O}\overset{\mathrm{BV}}{\otimes}\mathcal{P}\right) = A\mathcal{O}\vee A\mathcal{P}$.

Corollary A.12. *Let I, J be almost-unital weak indexing systems, and $\mathcal{C}, \mathcal{D}$ a pair of $\mathcal{T}$-∞-categories. Then, there is a unique $I\vee J$-operad equivalence* $\mathcal{C}^{I-\sqcup}\overset{\mathrm{BV}}{\otimes}\mathcal{D}^{J-\sqcup} \simeq (\mathcal{C}\times\mathcal{D})^{I\vee J-\sqcup}$.

Proof. Propositions A.9 and A.10 yield a string of equivalences

$$\begin{aligned}
\mathcal{N}^{\otimes}_{I\vee J}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{I-\sqcup}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{J-\sqcup} &\simeq \mathcal{N}^{\otimes}_{I}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{I-\sqcup}\overset{\mathrm{BV}}{\otimes}\mathcal{N}^{\otimes}_{J\infty}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{J-\sqcup} \\
&\simeq \mathrm{triv}^{\otimes}_{\mathcal{T}}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{I-\sqcup}\overset{\mathrm{BV}}{\otimes}\mathrm{triv}^{\otimes}_{\mathcal{T}}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{J-\sqcup}
\end{aligned}$$

under $\mathrm{triv}^{\otimes}_{\mathcal{T}}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{I-\sqcup}\overset{\mathrm{BV}}{\otimes}\mathrm{triv}^{\otimes}_{\mathcal{T}}\overset{\mathrm{BV}}{\otimes}\mathcal{C}^{J-\sqcup}$; another application of Proposition A.10 implies that $\mathcal{C}^{I-\sqcup}\overset{\mathrm{BV}}{\otimes}\mathcal{D}^{J-\sqcup}$ is has cocartesian underlying $I\vee J$-operad, so Proposition A.11 shows that $\mathcal{C}^{I-\sqcup}\overset{\mathrm{BV}}{\otimes}\mathcal{D}^{J-\sqcup}$ is cocartesian with underlying $\mathcal{T}$-∞-category $\mathcal{C}\times\mathcal{D}$. The result is then an application of the cocartesian rigidity result of [Ste25d, § 1.4]. $\square$

References

[Bar23] Shaul Barkan. *Arity Approximation of ∞-Operads*. 2023. arXiv: 2207.07200 [math.AT] (cit. on p. 5).

[BHS22] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT] (cit. on pp. 4, 5, 9, 34).

[BDGNS16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. *Parametrized higher category theory and higher algebra: A general introduction*. 2016. arXiv: 1608.03654 [math.AT] (cit. on p. 4).

[Ber21] Eric Berry. *Additivity of factorization algebras & the cohomology of real Grassmannians*. Thesis (PhD) – Montana State University. 2021. URL: https://scholarworks.montana.edu/items/af090b2d-18f3-487d-8c6a-6ea015500ba8 (cit. on p. 4).

[BH21] Andrew J. Blumberg and Michael A. Hill. "Equivariant stable categories for incomplete systems of transfers". In: *J. Lond. Math. Soc. (2)* 104.2 (2021), pp. 596–633. ISSN: 0024-6107,1469-7750. DOI: 10.1112/jlms.12441. URL: https://doi.org/10.1112/jlms.12441 (cit. on p. 31).

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Vol. Vol. 347. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1973, pp. x+257 (cit. on p. 2).

[Bon19] Peter Bonventre. "The genuine operadic nerve". In: *Theory Appl. Categ.* 34 (2019), pp. 736–780. ISSN: 1201-561X. URL: https://arxiv.org/abs/1904.01465 (cit. on p. 8).

[BP21] Peter Bonventre and Luís A. Pereira. "Genuine equivariant operads". In: *Adv. Math.* 381 (2021), Paper No. 107502, 133. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2020.107502. URL: https://arxiv.org/abs/1707.02226 (cit. on p. 31).

[CH21] Hongyi Chu and Rune Haugseng. "Homotopy-coherent algebra via Segal conditions". In: *Adv. Math.* 385 (2021), Paper No. 107733, 95. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2021.107733. URL: https://arxiv.org/abs/1907.03977 (cit. on pp. 5, 9).

[DHLSW25] Sanath Devalapurkar, Jeremy Hahn, Tyler Lawson, Andrew Senger, and Dylan Wilson. *Examples of disk algebras*. 2025. arXiv: 2302.11702 [math.AT]. URL: https://arxiv.org/abs/2302.11702 (cit. on p. 3).

[DI04] Daniel Dugger and Daniel C. Isaksen. "Topological hypercovers and A^1-realizations". In: *Math. Z.* 246.4 (2004), pp. 667–689. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-003-0607-y. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/dugger-hypercover.pdf (cit. on p. 24).

[Dun88] Gerald Dunn. "Tensor product of operads and iterated loop spaces". In: *J. Pure Appl. Algebra* 50.3 (1988), pp. 237–258. ISSN: 0022-4049,1873-1376. DOI: 10.1016/0022-4049(88)90103-X. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/Dunn.pdf (cit. on pp. 2, 4, 24).

[DK84] W. G. Dwyer and D. M. Kan. "Singular functors and realization functors". In: *Nederl. Akad. Wetensch. Indag. Math.* 46.2 (1984), pp. 147–153. ISSN: 0019-3577. URL: https://www.sciencedirect.com/science/article/pii/1385725884900167 (cit. on p. 31).

[DHK18] William Dwyer, Kathryn Hess, and Ben Knudsen. *Configuration spaces of products*. 2018. arXiv: 1710.05093 [math.AT]. URL: https://arxiv.org/abs/1710.05093 (cit. on pp. 3, 32, 33).

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy limit functors on model categories and homotopical categories*. Vol. 113. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2004, pp. viii+181. ISBN: 0-8218-3703-6. DOI: 10.1090/surv/113. URL: https://doi.org/10.1090/surv/113 (cit. on p. 5).

[Elm83] A. D. Elmendorf. "Systems of Fixed Point Sets". In: *Transactions of the American Mathematical Society* 277.1 (1983), pp. 275–284. ISSN: 00029947. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/elmendorf-fixed.pdf (visited on 04/22/2023) (cit. on p. 31).

[FV13] Zbigniew Fiedorowicz and Rainer M. Vogt. *An Additivity Theorem for the Interchange of En Structures*. 2013. arXiv: 1102.1311 [math.AT]. URL: https://arxiv.org/abs/1102.1311 (cit. on pp. 2, 5).

[GM11] Bertrand Guillou and J. P. May. *Models of G-spectra as presheaves of spectra*. 2011. arXiv: 1110.3571 [math.AT] (cit. on p. 2).

[GM17] Bertrand J. Guillou and J. Peter May. "Equivariant iterated loop space theory and permutative G-categories". In: *Algebr. Geom. Topol.* 17.6 (2017), pp. 3259–3339. ISSN: 1472-2747. DOI: 10.2140/agt.2017.17.3259. URL: https://arxiv.org/abs/1207.3459 (cit. on pp. 5, 19, 31).

[Har] Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*. URL: https://www.math.univ-paris13.fr/~harpaz/lecture_notes.pdf (cit. on pp. 3–5, 9, 13, 34, 35).

[HHLN23] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten. "Two-variable fibrations, factorisation systems and ∞-categories of spans". In: *Forum Math. Sigma* 11 (2023), Paper No. e111, 70. ISSN: 2050-5094. DOI: 10.1017/fms.2023.107. URL: https://arxiv.org/abs/2011.11042 (cit. on pp. 8, 9).

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. "On the nonexistence of elements of Kervaire invariant one". In: *Ann. of Math. (2)* 184.1 (2016), pp. 1–262. ISSN: 0003-486X. DOI: 10.4007/annals.2016.184.1.1. URL: https://people.math.rochester.edu/faculty/doug/mypapers/Hill_Hopkins_Ravenel.pdf (cit. on p. 31).

[Hil22] Michael A. Hill. "On the algebras over equivariant little disks". In: *J. Pure Appl. Algebra* 226.10 (2022), Paper No. 107052, 21. ISSN: 0022-4049,1873-1376. DOI: 10.1016/j.jpaa.2022.107052. arXiv: 1709.02005 [math.AT] (cit. on pp. 3, 5).

[Hor19] Asaf Horev. *Genuine equivariant factorization homology*. 2019. arXiv: 1910.07226 [math.AT] (cit. on pp. 25, 28, 29, 32, 33).

[Knu18] Ben Knudsen. *Configuration spaces in algebraic topology*. 2018. arXiv: 1803.11165 [math.AT]. URL: https://arxiv.org/abs/1803.11165 (cit. on p. 24).

[Kup17] Alexander Kupers. *Three lectures on topological manifolds*. 2017. URL: https://www.utsc.utoronto.ca/people/kupers/wp-content/uploads/sites/50/2021/01/toplectures.pdf (cit. on p. 28).

[LLP25] Tobias Lenz, Sil Linskens, and Phil Pützstück. *Norms in equivariant homotopy theory*. 2025. arXiv: 2503.02839 [math.AT]. URL: https://arxiv.org/abs/2503.02839 (cit. on p. 10).

[HTT] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. DOI: 10.1515/9781400830558. URL: https://www.math.ias.edu/~lurie/papers/HTT.pdf (cit. on pp. 5, 12, 14).

[HA] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf (cit. on pp. 2, 4, 9, 10, 13, 14).

[May72] J. P. May. *The geometry of iterated loop spaces*. Vol. Vol. 271. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1972, pp. viii+175 (cit. on pp. 2, 24).

[Mil20] Aleksandar Miladinovic. *Equivariant stabilization and factorization homology*. Thesis (PhD) – Université Paris-Nord - Paris XIII. 2020. URL: https://sorbonne-paris-nord.hal.science/tel-03920859/ (cit. on p. 32).

[NS22] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT] (cit. on pp. 4, 7, 9, 10, 13).

[RS00] Colin Rourke and Brian Sanderson. "Equivariant configuration spaces". In: *J. London Math. Soc. (2)* 62.2 (2000), pp. 544–552. ISSN: 0024-6107,1469-7750. DOI: 10.1112/S0024610700001241. URL: https://doi.org/10.1112/S0024610700001241 (cit. on pp. 2, 31).

[SW03] Paolo Salvatore and Nathalie Wahl. "Framed discs operads and Batalin-Vilkovisky algebras". In: *Q. J. Math.* 54.2 (2003), pp. 213–231. ISSN: 0033-5606,1464-3847. DOI: 10.1093/qjmath/54.2.213. URL: https://doi.org/10.1093/qjmath/54.2.213 (cit. on p. 2).

[Sha22] Jay Shah. *Parametrized higher category theory II: Universal constructions*. 2022. arXiv: 2109.11954 [math.CT] (cit. on p. 5).

[Sha23] Jay Shah. "Parametrized higher category theory". In: *Algebr. Geom. Topol.* 23.2 (2023), pp. 509–644. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2023.23.509. URL: https://arxiv.org/pdf/1809.05892.pdf (cit. on p. 5).

[Sta63] James Dillon Stasheff. "Homotopy associativity of H-spaces. I, II". In: *Trans. Amer. Math. Soc.* 108 (1963). **108** (1963), 275-292; ibid, pp. 293–312. ISSN: 0002-9947,1088-6850. DOI:

10.1090/s0002-9947-1963-0158400-5. URL: https://doi.org/10.1090/s0002-9947-1963-0158400-5 (cit. on p. 2).

[Ste24] Natalie Stewart. *Orbital categories and weak indexing systems*. 2024. arXiv: 2409.01377 [math.CT]. URL: https://arxiv.org/abs/2409.01377 (cit. on pp. 4, 33).

[Ste25a] Natalie Stewart. *A homotopically equivariant Eckmann-Hilton argument (forthcoming)*. 2025 (cit. on pp. 4, 12).

[Ste25b] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products*. 2025. arXiv: 2501.02129 [math.CT]. URL: https://arxiv.org/abs/2501.02129 (cit. on pp. 2–5, 7, 9–11, 13, 31, 33, 34).

[Ste25c] Natalie Stewart. *On connectivity of spaces of equivariant configurations (draft)*. 2025. URL: https://nataliesstewart.github.io/files/Conf_draft.pdf (cit. on pp. 3, 4, 24).

[Ste25d] Natalie Stewart. *On tensor products with equivariant commutative operads*. 2025. URL: https://nataliesstewart.github.io/files/ninfty.html (cit. on pp. 2–5, 10–12, 15, 31, 33, 36).

[Szc24] Ben Szczesny. *Equivariant Framed Little Disk Operads are Additive*. 2024. arXiv: 2410.20235 [math.AT]. URL: https://arxiv.org/abs/2410.20235 (cit. on pp. 2, 5, 32).