Homotopy-coherent interchange and equivariant little disk operads

Natalie Stewart

Department of Mathematics
Harvard University

INI Equivariant Homotopy Theory in Context

Queen's University Belfast, April 7 2025

A question you might happen upon

In trace methods for Real algebraic K-theory, THH has a Real analogue:[1]

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) & & \mathrm{Alg}_{\mathbb{E}_\sigma}(\mathrm{Sp}_{C_2}) \\ {\scriptstyle \mathrm{THH}}\downarrow & \xrightarrow{\text{“Reality”}} & \downarrow{\scriptstyle \mathrm{THR}} \\ \mathrm{Sp} & & \mathrm{Sp}_{C_2} \end{array}$$

In this, σ is the sign representation and $\mathbb{E}_\sigma$-algebras are a genuine-equivariant version of rings with anti-involution.

Question (c.f. Dotto-Moi-Patchkoria-Reeh[2] ’17)

What algebraic structure does THR of highly structured C_2-ring spectra have?

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How to construct structure on THH

Observation

THH can be given a symmetric monoidal structure, so we may lift

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{Sp}) \arrow[r, phantom, "\simeq"] \arrow[drr] & & \mathrm{Alg}_{\mathcal{O}}\mathrm{Alg}^{\otimes}_{\mathbb{E}_1}(\mathrm{Sp}) \arrow[r, dashed] \arrow[d, "U"'] & \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \arrow[d, "U"'] \\
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Theorem (Dunn[3] '88, Lurie[4] '09)

$\mathbb{E}_n \simeq \mathbb{E}_{n-1} \otimes \mathbb{E}_1$; hence THH takes $\mathbb{E}_n$-rings to $\mathbb{E}_{n-1}$-rings.

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THR can be given a C_2-symmetric monoidal structure, so we may lift

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Conjecture

$$\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V\oplus W}$$

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For this talk, all terms are defined ∞-categorically.

Theorem (S.[5])

Given V, W orthogonal G-representations, we have

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hence there is an equivalence of ∞-categories

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THR of $\mathbb{E}_{V \oplus \sigma}$-rings has a natural $\mathbb{E}_V$-ring structure.

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How to prove equivariant Dunn additivity

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1. Define “G-operads” as a localizing subcategory of “G-preoperads.”
2. Define G-preoperad models $\times$ and $\wr$ for $\overset{\mathrm{BV}}{\otimes}$, reduce to showing that $\varphi_{\mathbb{E}}$ is an L_{Op_G}-equivalence.
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\mathbb{P}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{P}_W^{\otimes} \arrow[d, "\alpha_V \otimes \alpha_W"'] & \mathbb{P}_V^{\otimes} \times \mathbb{P}_W^{\otimes} \arrow[l, "L_{\mathbb{P}}"'] \arrow[r, "M_{\mathbb{P}}"] \arrow[d, "\alpha_V \times \alpha_W"'] & \mathbb{P}_V^{\otimes} \wr \mathbb{P}_W^{\otimes} \arrow[r, "\widetilde{\varphi}_{\mathbb{P}}"] \arrow[d, "\alpha_V \wr \alpha_W"'] & \mathbb{P}_{V|W}^{\otimes} \arrow[r, hook] \arrow[d, "\alpha_{V|W}"'] & \mathbb{P}_{V\oplus W}^{\otimes} \arrow[d, "\alpha_{V\oplus W}"] \\
\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} & \mathbb{E}_V^{\otimes} \times \mathbb{E}_W^{\otimes} \arrow[l, "L_{\mathbb{E}}"] \arrow[r, "M_{\mathbb{E}}"'] & \mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \arrow[r, "\varphi_{\mathbb{E}}"'] & \mathbb{E}_{V\oplus W}^{\otimes} \arrow[r, equal] & \mathbb{E}_{V\oplus W}^{\otimes}
\end{tikzcd}$$

1. Define "G-operads" as a localizing subcategory of "G-preoperads."
2. Define G-preoperad models $\times$ and $\wr$ for $\overset{\mathrm{BV}}{\otimes}$, reduce to showing that $\varphi_{\mathbb{E}}$ is an L_{Op_G}-equivalence.
3. Define "approximations" α, an approximated "surjection" $\widetilde{\varphi}_{\mathbb{P}}$ onto "decomposable little disks," verify that $\alpha_{V|W}$ is an approximation by lifting Dunn's argument about decomposability.
4. Use ∞-category theory to reduce to showing $\widetilde{\varphi}_{\mathbb{P}}$ is an L_{Op_G}-equivalence.
5. Verify that $\widetilde{\varphi}_{\mathbb{P}}$ by lifting Dunn's argument about uniqueness of decompositions.

A quasi-definition of G-operads

Definition

- A G-**preoperad** is a functor $\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_G)$ with cocartesian lifts over backwards maps.
- A G-**operad** is required to satisfy "Segal conditions."
- An $\mathcal{O}$-**algebra in** $\mathcal{C}^{\otimes}$ is a functor preserving cocartesian arrows :

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{C}^{\otimes} \arrow[dl] \\ & \mathrm{Span}(\mathbb{F}_G) & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Op}_G \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{POp}_G \arrow[l, bend right, "L_{\mathrm{Op}_G}"'] \arrow[r, equal] & \mathrm{Cat}^{\mathrm{Backwards-cocart}}_{\infty, / \mathrm{Span}(\mathbb{F}_G)} \end{tikzcd}$$

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The underlying G-symmetric sequence

Construction

Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G$, $H \subset G$, and $S \in \mathbb{F}_G$, define the **structure space**

$$\begin{tikzcd} \mathcal{O}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mor}\left(\mathcal{O}^{\otimes}\right) \arrow[d] \\ \left\{\mathrm{Ind}_H^G S = \mathrm{Ind}_H^G S \to [G/H]\right\} \arrow[r] & \mathrm{Mor}\left(\mathrm{Span}(\mathbb{F}_G)\right) \end{tikzcd}$$

Proposition (S.[18])

If $\mathcal{O}^{\otimes}$ has "one color" then is it conservatively identified by $\left(\mathcal{O}(S)\right)_{\substack{H \subset G \\ S \in \mathbb{F}_H}}$.

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Example

The **little V-disks G-operad** has S-ary structure space given by **H-equivariant configurations of S in V**:

$$\mathbb{E}_V(S) := \mathrm{Emb}^{H,\mathrm{Affine}}(S \cdot D(V), D(V)) \simeq \mathrm{Conf}_S^H(V).$$

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Given a G-symmetric monoidal category $\mathcal{C}^\otimes : \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$, its unstraightening is a G-operad. Given $X \in \mathcal{C}(G/G)$, there is an **endomorphism G-operad** with

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Example (Horev-Klang-Zou[19] '20)

Let $\underline{\mathcal{S}}_G^{G-\times}$ be the *cartesian structure* on G-spaces. Then, for all $X \in \mathcal{S}_G$, we have $\Omega^V X \in \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{S}_G)$.

Example (Horev-Klang-Zou '20, loc. cit.)

Let $\underline{\mathrm{Sp}}_G^{\otimes}$ be the HHR G-symmetric monoidal structure. If $f \colon \Omega^V X \to \underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}_G)$ is a V-**loop map**, then $\mathrm{Th}(f) \in \mathrm{Alg}_{\mathbb{E}_V}(\mathrm{Sp}_G)$.

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Modelling G-operadic constructions, two ways

Model categorist: **(co)fibrantly replace**, then apply a construction to monoids in G-symmetric sequences.

∞*-categorist*: apply a G-preoperadic construction, then L_{Op_G}**-localize**.

Shared goal: model corepresenting object for **pairings** (aka interchanging algebras, bifunctors, etc.) akin to May.[20]

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The Boardman-Vogt tensor product

Definition

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Op}_G}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_G)^2 \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_G)\right)$$

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ lifts to a "pointwise" G-symmetric monoidal category $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$.

Proposition (S.[21])

$(-) \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes}$ is left adjoint to $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(-)$, so

$$\mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}).$$

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[21] Natalie Stewart. *Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products.* 2025. arXiv: 2501.02129 [math.CT].

The Boardman-Vogt tensor product

Definition

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} := L_{\mathrm{Op}_G}\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathrm{Span}(\mathbb{F}_G)^2 \xrightarrow{\wedge} \mathrm{Span}(\mathbb{F}_G)\right)$$

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ lifts to a “pointwise” G-symmetric monoidal category $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$.

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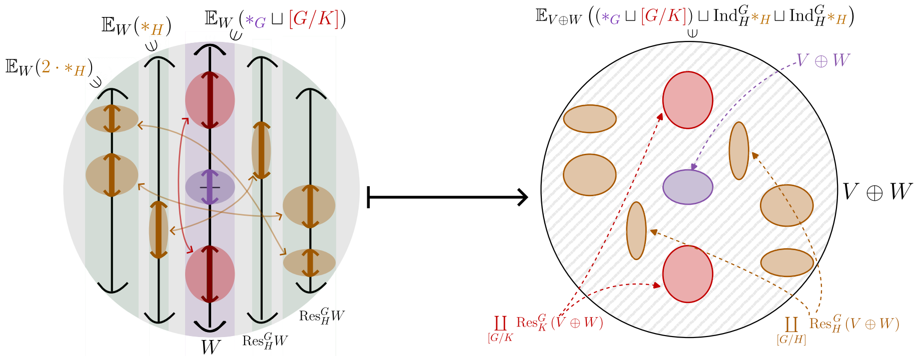
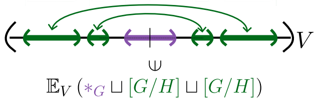
The Dunn map

$$\mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \xrightarrow{\varphi_{\mathbb{E}}} \mathbb{E}_{V \oplus W}^{\otimes}$$





$$\mathrm{Ar}\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right) \times_{\mathrm{ArSpan}(\mathbb{F}_G))} \left\{\mathrm{Ind}_H^G T = \mathrm{Ind}_H^G T \to [G/H]\right\}$$

$$\wr\!\!\mid$$

$$\coprod_{T \to S} \mathcal{O}(S) \times \prod_{[H/K] \subset S} \mathcal{P}\left(T \times_S [H/K]\right)$$

Preoperad models for 1-colored G-operads

Given $\mathcal{P}^{\otimes} \in \mathrm{POp}_G$, We let $\mathcal{P}^{\mathrm{act}}_{/P}$ be the ∞-category of arrows $P' \to P$ projecting to a forward map $T = T \to S$.

Definition

A map of G-preoperads $\alpha : \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ with $\mathcal{O}^{\otimes}$ a “one color” G-operad is a **weak approximation** if

1. The G-∞-category of colors $U\mathcal{P}$ **has a terminal G-object**, and
2. For all $P \in \mathcal{P}^{\otimes}$ and $T \to \pi(P)$, **the map of spaces**

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Proposition (Harpaz[22] '19 + reinterpretation)

If $\alpha: \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a weak approximation, pullback is fully faithful

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}_G) \subset \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}_G)$$

with image the $\mathcal{P}$-monoids whose "color" G-functors $U\mathcal{P} \to \underline{\mathcal{S}}_G$ are constant.

Weak approximations can be made to have many colors; a weak approximation α is a **strong approximation** if $U\mathcal{P} \to U\mathcal{O}$ is an equivalence.

Proposition (S.[23])

$\mathrm{Alg}_{(-)}(\mathcal{S}_G)$ detects L_{Op_G}-equivalences when $U\mathcal{P} \to U\mathcal{O}$ is an equivalence; in particular, **strong approximations are L_{Op_G}-equivalences**.

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The L_{Op_G}-conservativity argument, in short

Proof sketch.

Examine the free $\mathcal{O}$-G-space monad:[24]

$$(T_{\mathcal{O}} X)^H \simeq \coprod_{S \in \mathbb{F}_H} \left(\mathcal{O}(S) \times \left(X^S\right)^H \right)_{h \operatorname{Aut}_H S}.$$

The inclusion $\operatorname{Aut}_H(S) \subset \operatorname{End}_H(S) = \left(S^S\right)^H$ yields natural splitting

$$(T_{\mathcal{O}} S)^H \simeq \mathcal{O}(S) \sqcup \textbf{Junk}.$$

Use monadicity of $\operatorname{Alg}_{\mathcal{O}}(\mathcal{S}_G) \to \mathcal{S}_G$ and conservativity of $\left(\mathcal{O}(S)\right)_{\substack{H \subset G \\ S \in \mathbb{F}_H}}$. □

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Approximating manifolds

Proposition (Dugger-Isaksen[25] '01)

If X is a topological space and $\mathfrak{O} \subset \mathscr{P}(X)$ a basis of contractible open subsets, then we get a weak equivalence

$$B\mathfrak{O} \xrightarrow{\sim} X$$

Corollary

Let $\mathfrak{O}_S^H(V) \subset \mathrm{Conf}_S^H(V)$ be the basis of configurations in affinely $\coprod_S D(V)$-shaped invariant subspaces of $D(V)$. We get a weak equivalence

$$B\mathfrak{O}_S^H(V) \xrightarrow{\sim} \mathbf{Conf}_S^H(V) \simeq \mathbb{E}_V(S).$$

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$$\begin{tikzcd}
 & & & \mathbb{P}^{\otimes}_{V\oplus W} \arrow[d, "\alpha_{V\oplus W}"] \\
\mathbb{E}^{\otimes}_V \wr \mathbb{E}^{\otimes}_W \arrow[r, "\varphi_{\mathbb{E}}"'] & \mathbb{E}^{\otimes}_{V\oplus W} \arrow[r, equal] & & \mathbb{E}^{\otimes}_{V\oplus W}
\end{tikzcd}$$

We define a G-preoperad $\mathbb{P}^{\otimes}_V$ such that $\mathbb{P}^{\mathrm{act}}_{V,/P} \simeq \mathfrak{D}^H_S(V)$, yielding a weak approximation $\alpha_V \colon \mathbb{P}^{\otimes}_V \to \mathbb{E}^{\otimes}_V$. Then, we define a $\mathbb{P}$-Dunn map fitting into the above diagram.

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\mathbb{E}_V^{\otimes} \wr \mathbb{E}_W^{\otimes} \arrow[r, "\varphi_{\mathbb{E}}"'] & \mathbb{E}_{V \oplus W}^{\otimes} \arrow[r, equal] & \mathbb{E}_{V \oplus W}^{\otimes}
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2. A variant of Harpaz's strategy[26] shows that $M_{\mathbb{P}}$ and $M_{\mathbb{E}}$ are L_{Op_G}-equivalences.

[26] Yonatan Harpaz. *Little cubes algebras and factorization homology (course notes)*.

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Stalkwise-linearizable tangential structures

Define $\mathbb{E}^{\otimes}_{B^{\mathrm{lin}}_G \mathbf{Top}(n)}$ to have colors the linearizable G-actions on $\mathbb{R}^n$ and operations the *topological* embeddings.

Given a G-space X with stalkwise-linearizable equivariant $\mathbb{R}^n$-bundle $T_\bullet \colon X \to B^{\mathrm{lin}}_G \mathrm{Top}(n)$, we define the assembly

$$\mathbb{E}^{\otimes}_X := L_{\mathrm{Op}_G}\left(\mathbb{E}^{\otimes}_{B^{\mathrm{lin}}_G \mathrm{Top}(n)} \times_{B^{\mathrm{lin}}_G \mathrm{Top}(n)^{G-\sqcup}} X^{G-\sqcup}\right).$$

Theorem (S.[30])

There is a natural equivariant colimit expression of operads

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[30] Natalie Stewart. *On homotopical additivity of equivariant little disks operads (forthcoming)*. 2025.

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Stalkwise-linearizable tangential structures

Corollary (S.)

Given stalkwise-linearizable equivariant $\mathbb{R}^k$-bundles $X \to B_G^{\mathrm{lin}}\mathrm{Top}(n)$ and $Y \to B_G^{\mathrm{lin}}\mathrm{Top}(m)$, fixing the direct sum structure

$$X \times Y \longrightarrow B_G^{\mathrm{lin}}\mathrm{Top}(n) \times B_G^{\mathrm{lin}}\mathrm{Top}(m) \xrightarrow{\oplus} B_G^{\mathrm{lin}}\mathrm{Top}(n+m),$$

there is an equivalence.

$$\mathbb{E}_X^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_Y^{\otimes} \xrightarrow{\sim} \mathbb{E}_{X\times Y}^{\otimes}.$$

Other cases that are covered by the original argument:

- (Equivariant) swiss cheese
- Stratified little disks

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A curious corollary about algebraic topology

Let $\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \subset \underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}$ be the G-symmetric monoidal full category whose objects are disjoint unions of X-framed G-disks and whose mapping spaces are X-framed equivariant disk embeddings.

Corollary (S., c.f. Dwyer-Hess-Knudsen[31] '19)

There is a G-symmetric monoidal equivalence

$$\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \square \underline{\mathrm{Disk}}_G^{Y-\mathrm{fr},\sqcup} \simeq \underline{\mathrm{Disk}}_G^{X\times Y-\mathrm{fr},\sqcup}.$$

Here, $\square$ is the box product of semi-Mackey functors valued in Cat_∞.

[31] William Dwyer, Kathryn Hess, and Ben Knudsen. "Configuration spaces of products". In: *Trans. Amer. Math. Soc.* 371.4 (2019), pp. 2963–2985.

A curious corollary about algebraic topology

Let $\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \subset \underline{\mathrm{Mfld}}_G^{X-\mathrm{fr},\sqcup}$ be the G-symmetric monoidal full category whose objects are disjoint unions of X-framed G-disks and whose mapping spaces are X-framed equivariant disk embeddings.

Corollary (S., c.f. Dwyer-Hess-Knudsen[31] '19)

There is a G-symmetric monoidal equivalence

$$\underline{\mathrm{Disk}}_G^{X-\mathrm{fr},\sqcup} \boxtimes \underline{\mathrm{Disk}}_G^{Y-\mathrm{fr},\sqcup} \simeq \underline{\mathrm{Disk}}_G^{X\times Y-\mathrm{fr},\sqcup}.$$

Here, $\boxtimes$ is the box product of semi-Mackey functors valued in Cat_∞.

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This presentation was made in Beamer, with figures via tikz-cd and Inkscape, presented via Impressive. The tex is on my website. The title slide is $H^{\star}_{C_2}(*_{C_2}; \underline{\mathbb{Z}})$ [5].