AN ECKMANN-HILTON ARGUMENT IN EQUIVARIANT HIGHER ALGEBRA

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Abstract. Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be k- and ℓ-connected unital G-operads subject to the condition for all S that $\mathcal{O}(S) = \varnothing$ if and only if $\mathcal{P}(S) = \varnothing$. We show that the Boardman-Vogt tensor product $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected; equivalently, $\mathcal{O}\otimes\mathcal{P}$-monoids in any $(k+\ell+3)$-category lift uniquely to incomplete semi-Mackey functors. In particular, under no connectivity assumptions, discrete $\mathcal{O}\otimes\mathcal{P}$-monoids lift uniquely to incomplete semi-Mackey functors, recovering an Eckmann-Hilton argument for "C_p-unital magmas." As a consequence, we show that the smashing localizations on unital G-operads correspond precisely with unital $\mathcal{N}_\infty$-operads, and hence the (finite) poset of unital weak indexing systems.

Along the way we characterize ℓ-connectivity of a unital G-operad $\mathcal{O}^{\otimes}$ equivalently as ℓ-connectivity of $\mathcal{O}$-admissible Wirthmüller maps of $\mathcal{O}$-monoid spaces.

Introduction

The classical *Eckmann-Hilton argument* shows that, given a set with two unital multiplications $(M, *, \cdot)$ satisfying the interchange law

$$(a * b)\cdot(c * d) = (a \cdot c) * (b \cdot d),$$

the unital magmas $(M, *)$ and $(M, \cdot)$ are isomorphic to each other and are commutative monoids. We will study equivariant variations of this result, beginning with a weakening of Dress' Mackey functors [Dre71].

Definition 1. Let $\mathcal{C}$ be a 1-category with finite products and C_p the cyclic group of prime order p. A *C_p-unital magma in $\mathcal{C}$* is a unital magma M^e with a C_p action by unital magma homomorphisms, a unital magma M^{C_p} (with trivial C_p-action), and C_p-equivariant *restriction and transfer* homomorphisms

$$r\colon M^{C_p} \to M^e, \qquad\qquad t\colon M^e \to M^{C_p}$$

subject to the condition that $r \circ t$ is multiplication by p. A *homomorphism* $M \to N$ is a pair of unital magma homomorphisms $F^e\colon M^e \to N^e$ and $F^{C_p}\colon M^{C_p} \to M^e$ such that $F^{C_p} \circ t = t \circ F^e$ and $F^e \circ r = r \circ F^{C_p}$. ◃

Example 2. The $(\lambda+1)$st homotopy coefficient system of a C_p-space attains a functorial C_p-unital magma structure under the evident analog of Lewis' *unstable Mackey structure* [Lew92].[1] ◃

In this article, we prove and vastly generalize the following theorem.

Theorem A. *Suppose (M, M') is a pair of C_p-unital magma structures on the same coefficient system satisfying suitable interchange relations. Then, $M \simeq M'$ and each underlie a semi-Mackey functor; in particular, if the multiplications on M^e and M^{C_p} are invertible, then M and M' are isomorphic Mackey functors.*

To prove this, we embed it in the theory of *algebras over G-operads* in the sense of [NS22]; in particular, we show in Section 3 that C_p-unital magmas are algebras over a particular C_p-operad $\mathbb{A}^{\otimes}_{2,C_p}$ in C_p-coefficient systems valued in $\mathcal{C}$, and spell out the correct interchange relations there.

Crucially, in [Ste25a] we associated to a pair of G-operads $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ a *tensor product* $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$, whose algebras are *interchanging $\mathcal{O}$- and $\mathcal{P}$-algebras*:

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{D}) \simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{D}).$$

In particular, pairs of interchanging C_p-unital magma structures correspond with $\mathbb{A}^{\otimes}_{2,C_p} \overset{\mathrm{BV}}{\otimes} \mathbb{A}^{\otimes}_{2,C_p}$-algebras.

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[1] Explicitly, by V-Mackey functor, we mean a functor $\mathscr{B}_G(V) \to \mathbf{Ab}$ sending disjoint unions to direct sums, where $\mathscr{B}_G(V)$ is Lewis' V-Burnside category; the transfer map $\Sigma^{\lambda+1}_+ *_{C_p} \to \Sigma^{\lambda+1}_+[C_p/e]$ is constructed by the usual $\mathbb{S}_G$-duality construction along an embedding $[C_p/e] \hookrightarrow \lambda$ (see t[Wir75]). λ refers to any nontrivial 2-dimensional C_p-representation, though the same facts are true for the $(\sigma+1)$st homotopy coefficient system when $p = 2$.

Now, G-operads are ∞-categorical gadgets; thankfully, $\mathcal{O}$-algebras in a G-symmetric monoidal n-category are canonically equivalent to algebras over the *homotopy n-operad* $h_n\mathcal{O}^\otimes$, whose structure spaces are the $(n-1)$-truncations of the structure spaces of $\mathcal{O}^\otimes$ [Ste25a]. In particular, if the structure spaces of $\mathcal{O}^\otimes$ are n-connected, then $h_n\mathcal{O}^\otimes$ is canonically equivalent to a (weak) $\mathcal{N}_\infty$-operad in the sense of [BH15; Ste25a], so its algebras in the (cartesian) G-symmetric monoidal n-category of coefficient systems in an n-category $\mathcal{D}$ are precisely incomplete semi-Mackey functors valued in $\mathcal{D}$ [Ste25b].

From this, we identify Theorem A with the statement that $\mathbb{A}_{2,C_p}^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{A}_{2,C_p}^\otimes$ is connected together with the observation that the "arity support" weak indexing category

$$A\mathbb{A}_{2,C_p} := \left\{ T \to S \;\middle|\; \forall\, [G/H] \subset S,\ \mathbb{A}_{2,C_p}(T \times_S [G/H]) \neq \varnothing \right\} \subset \mathbb{F}_{C_p}$$

satisfies $A\mathbb{A}_{2,C_p} = \mathbb{F}_{C_p}$ (so the corresponding incomplete Mackey functors have all transfers). Our main homotopy-coherent lift of Theorem A is the following generalization of [SY19, Thm 1.0.1].

Theorem B. *If $\mathcal{O}^\otimes$ and $\mathcal{P}^\otimes$ are k and ℓ-connected almost essentially unital G-operads with $A\mathcal{O} = A\mathcal{P}$, then $\mathcal{O}^\otimes \otimes \mathcal{P}^\otimes$ is $(k+\ell+2)$-connected.*

For instance, Theorem B, lax G-symmetric monoidality of $\underline{\pi}_0 \colon \underline{\mathrm{Sp}}_G^\otimes \to \underline{\mathrm{Mack}}_G^\square(\mathbf{Ab})$, and the results of [Cha24] together construct a natural $A\mathcal{O}$-Tambara structure on the 0th homotopy groups of $\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{O}$-ring G-spectra;[2] this and a forthcoming equivariant Dunn additivity result will construct a natural AV-Tambara structure on the 0th homotopy Mackey functors of $\mathbb{E}_{2V}$-ring G-spectra.

We may remove the assumption $A\mathcal{O} = A\mathcal{P}$ in Theorem B, but we will need a more refined notion of connectivity. In general, given a weak indexing category I, we say that $\mathcal{O}^\otimes$ is *k-connected at I* if, for all elements of the corresponding weak indexing system

$$T \in \mathbb{F}_{I,H} := \left\{ S \in \mathbb{F}_H \;\middle|\; \mathrm{Ind}_H^G S \to [G/H] \in I \right\},$$

the structure space $\mathcal{O}(T)$ is k-connected.

Given a subgroup $H \subset G$ and a finite H-set $S \in \mathbb{F}_H$, there is a minimal unital H-weak indexing system $\underline{\mathbb{F}}_{I_S} \subset \underline{\mathbb{F}}_H$ containing S, consisting of summands of restrictions of iterated indexed coproducts of S [Ste24]. We say that $\mathcal{O}^\otimes$ is k-connected at S if it's k-connected at I_S. We define the *connectivity function*

$$\mathrm{Conn}_{\mathcal{O}} \colon \coprod_{(H) \subset G} \pi_0 \mathbb{F}_H \to \mathbb{Z} \cup \{\infty\}$$

by the formula $\mathrm{Conn}_{\mathcal{O}}(S) := \min\{k \mid \mathcal{O}^\otimes \text{ is } k\text{-connected at } S\}$. Now, $(\mathbb{Z} \cup \{\infty\})^{\coprod_{(H)\subset G} \pi_0 \mathbb{F}_H}$ forms a commutative monoid under pointwise addition and a poset by pointwise comparison

$$f \le g \iff \forall S,\ f(S) \le g(S).$$

An index-by-index version of Theorem B is the following.

Theorem C. *Given $\mathcal{O}^\otimes, \mathcal{P}^\otimes$ a pair of almost-unital G-operads, the following inequality holds:*

$$\mathrm{Conn}_{\mathcal{O}} + \mathrm{Conn}_{\mathcal{P}} + 2 \le \mathrm{Conn}_{\mathcal{O}\otimes\mathcal{P}}.$$

The key to our strategy for Theorems B and C is the following precise relationship between Wirthmüller map connectivity and connectivity at I, which the author believes to be of independent interest.

Theorem D. *Let $\mathcal{P}^\otimes$ be a G-operad and I an almost essentially unital weak indexing category. Then, the following conditions are equivalent:*

(a) $\mathcal{P}^\otimes$ is ℓ-connected at I.

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[2] To construct this lax symmetric monoidality, first note that $\underline{\mathrm{Sp}}_{G,\ge 0}^\otimes \subset \underline{\mathrm{Sp}}_G^\otimes$ is closed under tensor products, so the localization G-functor $\underline{\mathrm{Sp}}_G \to \underline{\mathrm{Sp}}_{G,\ge 0}^\otimes$ is given a lax G-symmetric monoidal structure by Proposition 36. Moreover, to construct a lax G-symmetric monoidal structure on $\tau_{\le 0} = \pi_0 \colon \underline{\mathrm{Sp}}_{G,\ge 0} \to \underline{\mathrm{Sp}}_G$, in light of [NS22] we need only note that $\otimes$ takes π_0-equivalences to π_0-equivalences and that the resulting structure agrees with the usual one on Mackey functors; the former follows by the same fact applied to geometric fixed points combined with induction up the poset of families using the isotropy separation sequence.

(b) *For all n-toposes $\mathcal{C}$ (with $n \le \infty$), I-admissible H-sets $S \in \mathbb{F}_{I,H}$, and S-indexed tuples of $\mathcal{P}$-monoids $(X_K) \in \prod_{[H/K] \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathrm{Res}^G_K \mathcal{P}}(\mathcal{C})$, the S-indexed $\mathcal{P}$-monoid Wirthmüller map*

$$W_{S,(X_K)} \colon \coprod_K^S X_K \longrightarrow \prod_K^S X_K$$

is ℓ-connected.

(c) *For all I-admissible H-sets $S \in \mathbb{F}_{I,H}$ and S-indexed tuples of $\mathcal{P}$-G-spaces $(X_K) \in \prod_{[H/K] \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathrm{Res}^G_K \mathcal{P}}(\mathcal{S})$, the S-indexed $\mathcal{P}$-G-space Wirthmüller map*

$$W_{S,(X_K)} \colon \coprod_K^S X_K \longrightarrow \prod_K^S X_K$$

is ℓ-connected.

For Theorem D, a morphism $g \colon X \to Y$ in an ∞-category $\mathcal{C}$ is *ℓ-truncated* if, for all $Z \in \mathcal{C}$, the map of spaces $\mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$ is ℓ-truncated, and $f \colon A \to B$ is ℓ-connected if, for all diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

such that g is ℓ-truncated, the space of lifts h is contractible.

Remark 3. In the case that $\mathcal{C}$ is an n-topos for some $0 \le n \le \infty$, the above definitions are equivalent to *ℓ-truncatedness* and *$(\ell-1)$-connectiveness* in the sense of [HTT, Def 6.5.1.10] by [SY19, Lem 4.2.6] and [HTT, Prop 6.5.1.12, Prop 6.5.1.19]. ◁

Additionally, the *S-indexed Wirthmüller map* in a G-∞-category is defined to be the S-indexed semiadditive norm map as in [CLL24; Nar16]; that is, the $[H/K]$-indexed Wirthmüller map $W_{[H/K],X} \colon \mathrm{Ind}_K^H X \to \mathrm{CoInd}_K^H X$ is adjunct to the map

$$X \longrightarrow \mathrm{Res}_K^H \mathrm{CoInd}_K^H X \simeq \prod_{g \in [K \backslash H/K]} \mathrm{CoInd}^H_{H \cap gKg^{-1}} \mathrm{Res}^H_{H \cap gKg^{-1}} X$$

whose projection onto the factor indexed by the identity double coset is the identity and whose other projections are zero. The $\coprod_i [H/K_i]$-indexed Wirthmüller map

$$W_{\coprod_i [H/K_i],(X_i)} \colon \coprod_{K_i}^H X_i \simeq \coprod_i \mathrm{Ind}_{K_i}^H X_i \longrightarrow \prod_i \mathrm{CoInd}_{K_i}^H X_i \simeq \prod_{K_i}^H X_i$$

is classified by the diagonal matrix whose ith entry is $W_{[H/K_i],X_i}$.

Remark 4. In the course of proving Theorem D, we will verify that Condition (b) is further equivalent to the condition that the $\mathrm{Coeff}^H \mathcal{C}$-map underlying $W_{S,(X_K)}$ is pointwise ℓ-connected; moreover, Condition (c) is equivalent to the condition that the underlying H-space map is ℓ-connected, i.e. its associated maps on J-fixed point spaces are surjective on path components with ℓ-connected fiber for each $J \subset H$. ◁

The rest of this paper replaces the orbit category $\mathcal{O}_G$ with an arbitrary atomic orbital ∞-category $\mathcal{T}$; we will prove Theorems B to D in that level of generality. We encourage the reader to either globally specialize to $\mathcal{T} = \mathcal{O}_G$ or familiarize themself with the atomic orbital setting via [Ste25a].

Structural implications. The specialization of Theorem B to infinite tensor powers is the following.

Corollary 5. *Suppose $\mathcal{O}^\otimes$ is an almost-reduced $\mathcal{T}$-operad. Then, the following conditions are equivalent.*

(a) *$\mathcal{O}^\otimes$ is an almost-unital weak $\mathcal{N}_\infty$-operad.*

(b) *($\mathcal{O}^\otimes$-EHA) the unique map $\mathrm{triv}_{\mathcal{T}}^\otimes \to \mathcal{O}^\otimes$ yields an equivalence*

$$\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{BV}{\otimes} \mathrm{triv}_{\mathcal{T}}^\otimes \xrightarrow{\mathrm{id} \otimes \mathrm{can}} \mathcal{O}^\otimes \overset{BV}{\otimes} \mathcal{O}^\otimes.$$

(c) *(abstract $\otimes$-idempotence) there exists an equivalence $\mathcal{O}^\otimes \overset{BV}{\otimes} \mathcal{O}^\otimes \simeq \mathcal{O}^\otimes$.*

Proof. The implication (a) $\implies$ (b) is one of the main results of [Ste25b], and is also implied by Theorem B. The implication (b) $\implies$ (c) is obvious. To see the implication (c) $\implies$ (a), note that Theorem B implies that $\mathcal{O}^{\otimes}$ is ∞-connected, i.e. all of its nonempty structure spaces are contractible. The result follows by the identification of such almost-reduced $\mathcal{T}$-operads with almost-unital weak $\mathcal{N}_\infty$-operads [Ste25a]. $\square$

To see why we may view Condition (b) as an *Eckmann-Hilton argument*, note that it is equivalent to the condition that $\mathcal{O}^{\otimes}$ possesses a unital magma structure in $\mathrm{Op}_{\mathcal{T}}^{\otimes}$ whose multiplication map $\mu\colon \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ is an equivalence; unitality of μ is precisely the condition that the associated diagonal natural transformation

$$\delta\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$$

is split by restriction to either $\mathcal{O}$-algebra structure, and the fact that μ is an equivalence is precisely the condition that δ is a natural equivalence, i.e. pairs of interchanging $\mathcal{O}$-algebra structures agree, and there is one such pair for all $\mathcal{O}$-algebra structures.

On the other hand, Condition (b) is equivalent to the assertion that $\mathcal{O}^{\otimes}$ admits a (unique) structure as an *idempotent algebra* in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni},\otimes}$; taking modules yields a bijective monotone correspondence between these and the smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni},\otimes}$ (see [GGN15, § 3] and [CSY20, § 5.1]).

Corollary 5 classifies smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$; define the full subcategory

$$\mathrm{Op}_{\mathcal{T}}^{I-\mathrm{Wirth}} := \left\{ \mathcal{O}^{\otimes} \;\middle|\; \forall\, S \in \underline{\mathbb{F}}_I,\, \mathcal{C}^{\otimes} \in \mathrm{Cat}_{\mathcal{T}}^{\otimes},\ \bigotimes^{S} \simeq \coprod^{S} \text{ in } \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \right\} \subset \mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}.$$

In [Ste25b] we showed that this is the smashing localization for $\mathcal{N}_{I\infty}^{\otimes}$ in order to compute tensor products of $\mathcal{N}_\infty$-operads. We also showed that idempotent algebras in $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ are almost-reduced, yielding the following.

Corollary E. *The construction $I \mapsto \mathrm{Op}_{\mathcal{T}}^{I-\mathrm{Wirth}}$ yields an isomorphism of posets*

$$\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \xrightarrow{\sim} \left\{ \text{Smashing localizations of } \mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}} \text{ under reverse inclusion} \right\}$$

A striking corollary of this is that there are finitely many smashing localizations on $\mathrm{Op}_{\mathcal{T}}^{a\mathrm{uni}}$ [Ste24].

Consequences in algebraic topology. Let I be an indexing category and Sp_I be the ∞-category presented by Blumberg-Hill's stable model category of I-spectra [BH21]. We say that an I-spectrum E is *connected* if $\underline{\pi}_n(E) \simeq 0$ for all $n \le 0$, i.e. it is the suspension of a connective I-spectrum. We see that *any* loop space theory with arity support I reaches connected I-spectra after infinite iteration.

Corollary 6. *If $\mathcal{O}^{\otimes}$ is a reduced G-operad with $\mathcal{O}(2 \cdot *_G) \neq 0$ and X is a connected G-space with infinitely many interchanging $\mathcal{O}$-algebra structures, then X is the 0th G-space of an essentially unique connected $A\mathcal{O}$-spectrum compatibly with its $\mathcal{O}^{\otimes\infty}$-structure.*

Proof. Note that $\mathcal{O}^{\otimes\infty} := \mathrm{colim}_{n\to\infty} \mathcal{O}^{\otimes n}$ is abstractly $\overset{\mathrm{BV}}{\otimes}$-idempotent, so $\mathcal{O}^{\otimes\infty} \simeq \mathcal{N}_{A\mathcal{O}}^{\otimes}$ by Corollary 5, i.e.

$$\underline{\mathrm{CAlg}}_{A\mathcal{O}}^{\otimes}(\mathcal{C}) \xrightarrow{\sim} \lim_{n\to\infty} \overbrace{\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes} \cdots \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}}^{n\text{-fold}}(\mathcal{C}). \tag{1}$$

Moreover, given a model $\mathcal{P}^{\otimes} \in \mathrm{Op}(\mathrm{sSet}_{BG})$ for $\mathcal{N}_{A\mathcal{O}}^{\otimes}$, [Ste25b] and [Mar24] yield equivalences

$$\mathrm{CAlg}_{A\mathcal{O}}\left(\underline{\mathcal{S}}_{G,\ge 1}^{G-\times}\right) \simeq \mathrm{CMon}_{A\mathcal{O}}(\mathcal{S}_{\ge 1}) \simeq \mathrm{Alg}_{\mathcal{P}}\left(\mathrm{Top}_{G,\ge 1}\right)\left[\mathrm{WEQ}^{-1}\right]$$

over $\mathcal{S}_{G,\ge 1}$, the right hand side denoting the Hammock localization inverting the class of (point-set) $\mathcal{P}$-algebra morphisms whose underlying function of topological G-spaces is a G-weak equivalence.[3] The defining equivalence $\mathrm{Sp}_{A\mathcal{O},\ge 0} \simeq \mathrm{Alg}_{\mathcal{P}}^{\mathrm{grplike}}\left(\mathrm{Top}_G\right)\left[\mathrm{WEQ}^{-1}\right]$ then embeds $\mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Top}_{G,\ge 1}\right)\left[\mathrm{WEQ}^{-1}\right]$ as those $A\mathcal{O}$-spectra whose 0th G-space is connected; it follows by unwinding definitions that this is precisely $\mathrm{Sp}_{A\mathcal{O},\ge 1}$, so Eq. (1) restricts to an equivalence

$$\mathrm{Sp}_{I,\ge 1} \simeq \lim_{n\to\infty} \overbrace{\mathrm{Alg}_{\mathcal{O}} \cdots \underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}}^{n\text{-fold}}(\mathcal{S}_{G,\ge 1})$$

[3] Here, $\mathrm{sSet}_G := \mathrm{sSet}^{BG}$ and $\mathrm{Top}_G := \mathrm{Top}^{BG}$ are the 1-categories of *simplicial sets* and *topological spaces* with G-action.

over $\mathcal{S}_{G,\geq 1}$. $\square$

To construct an infinite loop space theory for I-spectra, one is left with the following question.

Question 7. Given an indexing category I, does there exist a reduced G-operad $\mathcal{O}^{\otimes}$ with $A\mathcal{O} = I$ and a space S^I such that $\mathcal{O}$-monoid structures on a connected G-space X are equivalent to S^I-loop space structures? $\triangleleft$

Remark 8. We chose to specialize to the connected setting for convenience; one could instead assume that there exists some $\mu \in \mathcal{O}(2 \cdot *_G)$ whose action on one of the $\mathcal{O}$-structures on X induces an *invertible* magma structure on the coefficient system $\underline{\pi}_0 X$, in which case the corresponding $A\mathcal{O}$-commutative algebra has an underlying grouplike commutative monoid structure; the variation of Corollary 6 follows *mutatis mutandis*. $\triangleleft$

Additionally, we acquire Ω^V-spectrum structures in a wide variety of circumstances.

Corollary 9. *Fix V an orthogonal G-representation. If $\mathcal{O}^{\otimes}$ is an almost-reduced G-operad with $\mathcal{O}(S) \neq \varnothing$ whenever there exists an embedding $S \hookrightarrow \mathrm{Res}_H^G V$ and X is a connected G-space admitting infinitely many interchanging $\mathcal{O}$-algebra structures, then X admits the structure of a V-infinite loop space.*

Proof. The V-infinite loop space structure corresponds with the $\mathbb{E}_{\infty V}$-structure pulled back along the unique map $\mathbb{E}_{\infty V}^{\otimes} \simeq \mathcal{N}_{AV}^{\otimes} \to \mathcal{N}_{A\mathcal{O}}^{\otimes} \simeq \mathcal{O}^{\otimes\infty}$ under the recognition principle of [GM17; RS00]. $\square$

Sharpness. Theorems B and C are not sharp for all examples. One reason is the discrepancy between unions and joins of weak indexing systems.

Example 10. Given I an almost-unital weak indexing category, let $\mathcal{N}_{I\infty}^{\otimes} \in \mathrm{Op}_G$ be the corresponding weak $\mathcal{N}_\infty$-operad as in [Ste25a]. Unwinding definitions, we find that

$$\mathrm{Conn}_{\mathcal{N}_{I\infty}}(S) = \begin{cases} \infty & S \in \underline{\mathbb{F}}_I \\ -2 & \text{otherwise.} \end{cases}$$

Moreover, we found in [Ste25b] that $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$. This demonstrates a failure of sharpness in Theorem C; indeed, generically, we have

$$\left(\mathrm{Conn}_{\mathcal{N}_{I\infty}} + \mathrm{Conn}_{\mathcal{N}_{J\infty}} + 2\right)^{-1}(\infty) = \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J \subsetneq \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \mathrm{Conn}^{-1}_{\mathcal{N}_{I\infty} \otimes \mathcal{N}_{J\infty}}(\infty). \quad \triangleleft$$

Another issue is topological; in forthcoming work, given V an orthogonal G-representation, we will show that the little V-disks G-operad $\mathbb{E}_V^{\otimes}$ is ℓ-connected at S if and only if the following conditions are satisfied:

(a) For all orbits $[H/K] \subset S$ and intermediate inclusions $K \subset J \subset H$, we have $\dim V^J \geq \dim V^K + \ell + 2$, and
(b) if $\left|S^H\right| \geq 2$, then $\dim V^H \geq \ell + 2$.

Moreover, we will show that $\mathbb{E}_V$ is additive under tensor products, i.e. $\mathbb{E}_V^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_W^{\otimes} \simeq \mathbb{E}_{V\oplus W}^{\otimes}$.

Example 11. Let $G := C_2$, with sign representation σ. Then, we have fixed point dimensions

$$\dim (a + b\sigma)^e = a + b; \qquad \dim (a + b\sigma)^{C_2} = a.$$

In particular, the connectivity function has

$$\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(k *_e) = a + b - 2$$

$$\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(c *_{C_2} + d[C_2/e])) = \begin{cases} a - 2 & d = 0 \\ b - 2 & c < 2 \\ \min(a,b) - 2 & \text{otherwise.} \end{cases}$$

$\mathrm{Conn}_{\mathbb{E}_{a+b\sigma}}(c *_{C_2} + d[C_2/e])$ is as non-additive as is possible in the last case; indeed, the examples $1 + b\sigma$ and $a' + \sigma$ have the same arity-support, but when $a', b > 1$, we have

$$\begin{aligned} \mathrm{Conn}_{1+b\sigma}(2 *_{C_2} + [C_2/e]) + \mathrm{Conn}_{a'+\sigma}(2 *_{C_2} + [C_2/e]) - 2 &= 0 \\ &< \min(a', b) - 1 \\ &= \mathrm{Conn}_{a'+1+(b+1)\sigma}(2 *_{C_2} + [C_2/e]). \end{aligned} \quad \triangleleft$$

Nevertheless, equality is sometimes attained.

Example 12. For all orthogonal G-representations V, it follows from the above description that

$$\mathrm{Conn}_{\mathbb{E}_V \otimes \mathbb{E}_V} = \mathrm{Conn}_{\mathbb{E}_{2V}} = 2\mathrm{Conn}_{\mathbb{E}_V} - 2. \quad \triangleleft$$

The strategy. First, the tautological symmetric monoidal equivalence

$$\mathrm{Op}_{\mathcal{T}}^{\otimes} \simeq \lim_{V \in \mathcal{T}} \mathrm{Op}_V^{\otimes}$$

detects connectivity at an index, so we may assume without loss of generality that $\mathcal{T}$ has a terminal object (and, in particular, it is a 1-category). Second, we have the following.

Lemma 13. *The following theorems imply each other:*

(a) Theorem B in all cases.
(b) Theorem B in the case $A\mathcal{O} \simeq \underline{\mathbb{F}}_{I_W}$ for some finite W-set $S \in \mathbb{F}_W$, where W is the terminal object of $\mathcal{T}$.
(c) Theorem C.

Proof. The implication (a) $\implies$ (b) is obvious. The implication (b) $\implies$ (c) follows by noting that, when $S \in \underline{\mathbb{F}}_{A\mathcal{O}}$, the condition $\mathrm{Conn}_{\mathcal{O}}(S) \geq k$ is precisely the condition that the arity-Borelification $\mathrm{Bor}_{\underline{\mathbb{F}}_S}^{\mathcal{T}} \mathcal{O}^{\otimes}$ is k-connected. The implication (c) $\implies$ (a) follows by monotonicity the function

$$\min_{S \in \underline{\mathbb{F}}_{A\mathcal{O}}} f(S) \colon (\mathbb{Z} \cup \{\infty\})^{\coprod_{V \in \mathcal{T}} \pi_0 \mathbb{F}_V} \to \mathbb{Z} \cup \{\infty\}. \qquad \square$$

We're left with proving Theorem B in the almost-unital case. We will perform a similar reduction to [SY19]; namely, by examining the free $\mathcal{O}$-algebra monad, we reduce this to $(k+1)$-*connectivity of the reduced endomorphism $A\mathcal{O}$-operad in $\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{I-\times}$* in the case $\mathcal{C}$ is the $\mathcal{T}$-∞-category of coefficient systems in a presheaf ∞-topos.

We express the structure space $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C})^{I-\times}\right)(S)$ as the spaces of lifts of $\Delta \colon X^{\sqcup S} \to X$ along the S-indexed Wirthmüller map $W_{X,S} \colon X^{\sqcup S} \to X^{\times S}$, which is directly related to truncatedness of X and connectedness of $W_{X,S}$ [SY19]; hence it suffices to prove Theorem D in the almost-unital case.

We finish by directly relating ℓ-connectivity of $W_{X,S}$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{Mon}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C})$, reducing Theorem D to the fact that $\mathrm{Mon}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C})$ is I-semiadditive when $\mathcal{O}$ is ℓ-connected at I, which we verified in [Ste25b].

Acknowledgements. This article is greatly influenced by the work of Schlank-Yanovski [SY19], which recovers almost all of the results and ideas in this article in the case that G is the trivial group, and has additionally been influential to my thinking in the previous articles [Ste25a; Ste25b]. In general, I'd like to thank my advisor Mike Hopkins for several helpful conversations on this material.

1. I-operads

Throughout this article, we fix $\mathcal{T}$ an atomic orbital ∞-category in the sense of [NS22]; that is, we assume that all retracts in $\mathcal{T}$ are equivalences and that the finite coproduct completion $\mathbb{F}_{\mathcal{T}} := \mathcal{T}^{\sqcup}$ has pullbacks.

We begin in Section 1.1 by recalling the simultaneous generalization and weakening of Blumberg-Hill's G-indexing systems and I-Mackey functors to $\mathcal{T}$-*weak* indexing systems and I-commutative monoids. We go on to Section 1.3 where we recall the relevant background from [NS22; Ste25a; Ste25b] on $\mathcal{T}$-operads, as well as establishing a few foundational results concerning the *doctrinal adjunction* and *reduced endomorphism I-operads.*

1.1. Preliminaries on $\mathcal{T}$-∞-categories and weak indexing systems.

Recall that a $\mathcal{T}$-*coefficient system* is a functor out of $\mathcal{T}^{\mathrm{op}}$:

$$\mathrm{Coeff}^{\mathcal{T}}(\mathcal{C}) := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}).$$

Generalizing Elmendorf's theorem, we define d-truncated $\mathcal{T}$-spaces and $\mathcal{T}$-d-categories as coefficient systems:

$$\mathcal{S}_{\mathcal{T},\leq d} := \mathrm{Coeff}^{\mathcal{T}}(\mathcal{S}_{\leq d}); \qquad \mathrm{Cat}_{\mathcal{T},d} := \mathrm{Coeff}^{\mathcal{T}}(\mathrm{Cat}_d).$$

We write $\mathrm{Cat}_{\mathcal{T}} := \mathrm{Cat}_{\mathcal{T},\infty}$ and $\mathcal{S}_{\mathcal{T}} := \mathcal{S}_{\mathcal{T},\leq\infty}$. Given a $\mathcal{T}$-∞-category $\mathcal{C}$, we write $\mathcal{C}_V$ for the value $\mathcal{C}(V)$ and $\mathrm{Res}_V^W \colon \mathcal{C}_W \to \mathcal{C}_V$ for the functoriality under a map $V \to W$. The ∞-category of $\mathcal{T}$-coefficient systems lifts to a $\mathcal{T}$-∞-category with V-value the $\mathcal{T}_{/V}$-coefficient systems

$$\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})_V := \mathrm{Coeff}^{\mathcal{T}_{/V}}(\mathcal{C});$$

the functoriality is given by restriction. We acquire $\mathcal{T}$-∞-categories $\underline{\mathcal{S}}_{\mathcal{T},\leq d}$ and $\underline{\mathrm{Cat}}_{\mathcal{T},d}$ similarly.

Example 14. We may define a $\mathcal{T}$-∞-category by $\underline{\mathbb{F}}_{\mathcal{T}}$ by values

$$(\underline{\mathbb{F}}_{\mathcal{T}})_V \coloneqq \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}$$

with functoriality given by pullback. We write $\mathbb{F}_V \coloneqq \mathbb{F}_{\mathcal{T},/V}$. Note that this is a $\mathcal{T}$-1-category since $\mathcal{T}_{/V}$ is a 1-category [NS22, Prop 2.5.1]. ◁

Example 15. Given $\mathcal{C}$ an arbitrary n-category, $\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{C})$ is a $\mathcal{T}$-n-category [HTT, Cor 2.3.4.8]. In particular, if $\mathcal{C}$ is an ∞-topos and $\tau_{\leq n-1}\mathcal{C}$ its n-topos of $(n-1)$-truncated objects, then $\underline{\mathrm{Coeff}}^{\mathcal{T}}(\tau_{\leq n-1}\mathcal{C})$ is a $\mathcal{T}$-n-category. ◁

Example 16. The ∞-category of $\mathcal{T}$-∞-categories is Cartesian closed with internal hom characterized by values

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})_V \simeq \mathrm{Fun}_{\mathcal{T}_{/V}}(\mathrm{Res}^{\mathcal{T}}_V\mathcal{C},\mathrm{Res}^{\mathcal{T}}_V\mathcal{D}),$$

where $\mathrm{Res}^{\mathcal{T}}_V \colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}_{/V}}$ is pullback and $\mathrm{Fun}_{\mathcal{T}}(-,-)$ denotes the evident ∞-category of natural transformations [BDGNS16]. By unwinding definitions and applying [HTT, Cor 2.3.4.8], we find that whenever $\mathcal{D}$ is a $\mathcal{T}$-n-category, $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ is a $\mathcal{T}$-n-category. ◁

Example 17. We refer to the adjunction between limits and constant diagrams as the *inflation and fixed point* adjunction

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend left, "\mathrm{Infl}^{\mathcal{T}}_e"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{\mathcal{T}} \arrow[l, bend left, "\Gamma^{\mathcal{T}}"] \end{tikzcd}$$

In the case that $\mathcal{T}$ has a terminal object V, the image of $\mathrm{Infl}^{\mathcal{T}}_e$ consists of the $\mathcal{T}$-∞-categories whose restriction functors Res^V_W are all equivalences. In any case, we may string together natural equivalences

$$\begin{aligned}
\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}^{\mathcal{T}}_e K,\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right)_V &\simeq \mathrm{Fun}_V\left(\mathrm{Infl}^{\mathcal{T}_{/V}}_e K,\underline{\mathrm{Coeff}}^{\mathcal{T}_{/V}}\mathcal{C}\right)\\
&\simeq \mathrm{Fun}\left(K,\mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}},\mathcal{C}\right)\right)\\
&\simeq \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}},\mathrm{Fun}(K,\mathcal{C})\right)\\
&\simeq \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{C}^K\right)_V
\end{aligned}$$

to construct a $\mathcal{T}$-equivalence $\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}^{\mathcal{T}}_e K,\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right) \simeq \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{C}^K\right)$; in particular, choosing $\mathcal{C} = \mathcal{K}$, $\mathcal{T}$-coefficient systems in presheaves of spaces on K can equivalently be realized as $\mathcal{T}$-equivariant presheaves of $\mathcal{T}$-spaces on K with trivial $\mathcal{T}$-equivariant structure. We henceforth write

$$\underline{\mathcal{S}}^K_{\mathcal{T},\leq n} \coloneqq \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{S}^K_{\leq n}\right); \qquad\qquad \underline{\mathcal{S}}^K_{\mathcal{T}} \coloneqq \underline{\mathrm{Coeff}}^{\mathcal{T}}\left(\mathcal{S}^K\right).$$ ◁

Given $V \in \mathcal{T}$ an orbit and $S \in \mathbb{F}_V$ a finite V-set, we write $\varphi_{SV} \colon \mathrm{Ind}^{\mathcal{T}}_V S \to V$ for the corresponding map in $\mathbb{F}_{\mathcal{T}}$, and we write

$$\mathcal{C}_S \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U \simeq \mathrm{Fun}_{\mathcal{T}}\left(\mathrm{Ind}^{\mathcal{T}}_V S,\mathcal{C}\right).$$

Pullback along the structure map φ_{SV} yields an *indexed diagonal* functor

$$\Delta^S \colon \mathcal{C}_V \to \mathcal{C}_S;$$

its values are $\Delta^S X = (\mathrm{Res}^V_U X)_{U \in \mathrm{Orb}(S)}$. The S*-indexed coproduct* (if it exists) is the left adjoint $\coprod^S \colon \mathcal{C}_S \to \mathcal{C}_V$ to Δ^S, and the S*-indexed product* $\prod^S \colon \mathcal{C}_S \to \mathcal{C}_V$ is the right adjoint. These are the ür-examples of *equivariantly indexed operations*, whose combinatorics we control using *weak indexing systems*.

Definition 18. A *one-color weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ which is closed under $\underline{\mathbb{F}}_I$-indexed coproducts and contains $*_V$ for all $V \in \mathcal{T}$. A *one-color weak indexing category* is a pullback-stable wide subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ subject to the condition that $\coprod_i (T_i \to S_i)$ lies in I if and only if each map $T_i \to S_i$ lies in I. ◁

Given I a one-color weak indexing category, we define the I*-admissible* V*-sets* as

$$\underline{\mathbb{F}}_I \coloneqq \left\{S \mid \mathrm{Ind}^{\mathcal{T}}_V S \to V \in I\right\} \subset \underline{\mathbb{F}}_{\mathcal{T}};$$

we verified in [Ste24] that $\underline{\mathbb{F}}_{(-)}$ furnishes an equivalence between one-color weak indexing systems and one-color weak indexing categories, so we safely conflate these notions. For the following example, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is called a $\mathcal{T}$*-family* if, whenever there exists a morphism $V \to W$ with $W \in \mathcal{F}$, we have $V \in \mathcal{F}$.

Example 19. The terminal one-color weak indexing system is $\underline{\mathbb{F}}_{\mathcal{T}}$. We define the following other examples, where $\mathcal{F} \subset \mathcal{T}$ is a fixed $\mathcal{T}$-family:

$$(\underline{\mathbb{F}}_{\mathrm{triv}})_V := \{*_V\}$$
$$\left(\underline{\mathbb{F}}_{0,\mathcal{F}}\right)_V := \begin{cases} \{\varnothing_V, *_V\} & V \in \mathcal{F} \\ \{*_V\} & \text{otherwise.} \end{cases}$$
$$(\underline{\mathbb{F}}_\infty)_V := \{n \cdot *_V \mid n \in \mathbb{N}\}.$$

The corresponding one-color weak indexing categories are denoted $I_{\mathrm{triv}}, I_{0,\mathcal{F}}, I_\infty$. ◁

Construction 20. We write

$$\upsilon(I) := \left\{ V \in \mathcal{T} \mid \varnothing_V \in (\underline{\mathbb{F}}_I)_V \right\} \subset \mathcal{T}.$$

This is a $\mathcal{T}$-family, called the *unit family* of I [Ste24]. ◁

We say that $\underline{\mathbb{F}}_I$ is *almost-unital* if, whenever $\{*_V\} \subsetneq \mathbb{F}_{I,V}$, we have $\varnothing_V \in \mathbb{F}_{I,V}$; that is, $\underline{\mathbb{F}}_I$ is unital over all orbits for which $\underline{\mathbb{F}}_I$ has nontrivial arities. We say $\underline{\mathbb{F}}_I$ is *unital* if $\varnothing_V \in \mathbb{F}_{I,V}$ for all V.

1.2. **Preliminaries on I-commutative monoids and I-symmetric monoidal ∞-categories.** Let I be a one-color weak indexing category. The pair $(\mathbb{F}_{\mathcal{T}}, I)$ is a *span pair* in the sense of [EH23] (i.e. $(\mathbb{F}_{\mathcal{T}}, I, I)$ is an *adequate triple* in the sense of [Bar14]), so it yields a wide subcategory

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \hookrightarrow \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$$

of the effective Burnside ∞-category whose morphisms are given by spans $X \leftarrow R \xrightarrow{f} Y$ with $f \in I$. Given I a one-color weak indexing category and $\mathcal{C}$ an ∞-category, we define the ∞-category of I*-commutative monoids* in $\mathcal{C}$ as

$$\mathrm{CMon}_I(\mathcal{C}) := \mathrm{Fun}^\times(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C}).$$

We define the ∞-category of *small I-symmetric monoidal ∞-categories* as

$$\mathrm{Cat}_I^\otimes := \mathrm{CMon}_I(\mathrm{Cat}).$$

We henceforth ignore size issues and omit the adjective "small." Given an I-symmetric monoidal ∞-category $\mathcal{C}$ and $S \in \mathbb{F}_{I,V}$ an I-admissible V-set, we denote the functoriality of $\mathcal{C}^\otimes$ under the structure map $\mathrm{Ind}_S^{\mathcal{T}} S = \mathrm{Ind}_S^{\mathcal{T}} S \to V$ by

$$\bigotimes^S : \mathcal{C}_S \to \mathcal{C}_V.$$

If I is almost-unital, $S \in \mathbb{F}_{I,V}$ is I-admissible, and $1_U \in \mathcal{C}_U$ is initial whenever it exists, then given an S-indexed tuple $(X_U) \in \mathcal{C}_S$ in an I-symmetric monoidal ∞-category with S-indexed coproducts, we define an *S-indexed tensor Wirthmüller map*

$$W_{S,(X_U)} : \coprod_U^S X_U \longrightarrow \bigotimes_U^S X_U$$

by defining its composite map $\mathrm{Ind}_W^V X_W \hookrightarrow \coprod_U^S X_U \to \bigotimes_U^S X_U$ to be adjunct to the map

$$X_W \simeq X_W \otimes \bigotimes_W^{\mathrm{Res}_W^V S - *_W} 1_U \xrightarrow{(\mathrm{id},\eta)} X_W \otimes \bigotimes^{\mathrm{Res}_W^V S - *_W} X_U \simeq \mathrm{Res}_U^V \bigotimes_U^S X_U;$$

intuitively, on the W'th factor, $W_{S,(X_U)}$ takes x to the simple tensor with x in the W'th place and units elsewhere. Given $J \subset I$, we say that $\mathcal{C}$ is *J-cocartesian* if $W_{S,(X_U)}$ is an equivalence for all $S \in \underline{\mathbb{F}}_I$ and $(X_U) \in \mathcal{C}_S$, and we say that $\mathcal{C}$ is *J-cartesian* if its "vertical opposite"

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \xrightarrow{\mathcal{C}^\otimes} \mathrm{Cat} \xrightarrow{\mathrm{op}} \mathrm{Cat}$$

is a J-cocartesian I-symmetric monoidal ∞-category..

In [Ste25b], given $\mathcal{C}$ a $\mathcal{T}$-∞-category with I-indexed (co)products, we constructed essentially unique *(co)cartesian* I-symmetric monoidal structures on $\mathcal{C}$ and verified that $\mathcal{C}$ is *I-semiadditive* in the sense of [CLL24] if and only if there exists an equivalence $\mathcal{C}^{I-\sqcup} \simeq \mathcal{C}^{I-\times}$, which can be chosen (uniquely) to lie over the identity endofunctor.

1.3. **Preliminaries on I-operads.** In [NS22], an ∞-category $\mathrm{Op}_{\mathcal{T}}$ of *$\mathcal{T}$-operads* was introduced, and in [Ste25a; Ste25b] it was given a symmetric monoidal closed $\mathcal{T}$-∞-category structure $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$. We review the relevant formal properties here; in particular, outside of a small part of the verification of another formal property in Proposition 36, we will only use formal properties of $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$, instead probing its objects via the various functors

$$\begin{tikzcd}
\mathrm{Cat}_{\mathcal{T}}^{\otimes} \arrow[r, hook] & \mathrm{Op}_{\mathcal{T}} \arrow[r, "\mathrm{sseq}"] \arrow[dl, "U"'] \arrow[d, dotted, "\underline{\mathrm{Alg}}_{(-)}(\mathcal{C})"] \arrow[dr, "\underline{\mathrm{Alg}}_{\mathcal{P}}(-)"] & \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}) \\
\mathrm{Cat}_{\mathcal{T}} & \mathrm{Cat}_{\mathcal{T}} & \mathrm{Cat}_{\mathcal{T}}
\end{tikzcd}$$

In this way, this paper can be considered agnostic to the presentation of $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$ and the above functors.

1.3.1. *$\mathcal{T}$-symmetric sequences and I-operads.* Writing $\underline{\Sigma}_{\mathcal{T}}$ for the composite $\mathcal{T}$-∞-category

$$\mathcal{T}^{\mathrm{op}} \xrightarrow{\mathbb{F}_{\mathcal{T}}} \mathrm{Cat} \xrightarrow{(-)^{\simeq}} \mathcal{S} \hookrightarrow \mathrm{Cat}$$

and writing $\mathrm{Tot}\colon \mathrm{Cat}_{\mathcal{T}} \simeq \mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{T}^{\mathrm{op}}} \to \mathrm{Cat}$ for the total category functor, in [Ste25a] we defined a *underlying $\mathcal{T}$-symmetric sequence* functor

$$\mathcal{O}(-)\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}).$$

To characterize this, we need a definition.

Definition 21. We say that an I-operad $\mathcal{O}^{\otimes}$ *has at least one color* if $\mathcal{O}(*_V) \neq \varnothing$ for all $V \in \mathcal{T}$ and *has one color* if $\mathcal{O}(*_V) \simeq *$ for all $V \in \mathcal{T}$, ◁

Proposition 22 ([Ste25a])**.** *The functor $\mathcal{O}(-)\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ has a left adjoint* Fr*; in particular, letting $\mathrm{Fr}_{\mathrm{Op}}(S)$ be the free $\mathcal{T}$-operad on the left Kan extended $\mathcal{T}$-symmetric sequence*

$$\begin{tikzcd}
\{S\} \arrow[r, "*"] \arrow[d, hook] \arrow[dr, phantom, "\Downarrow"] & \mathcal{S} \\
\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \arrow[ur, dashed, "\mathrm{Fr}_{\Sigma,S}(*)"'] &
\end{tikzcd}$$

the adjunctions construct a natural equivalence

$$\mathrm{Alg}_{\mathrm{Fr}_{\mathrm{Op}}(S)}(\mathcal{O}) \simeq \mathcal{O}(S).$$

Moreover, the restricted functor $\mathcal{O}(-)\colon \mathrm{Op}_{\mathcal{T}}^{\mathrm{oc}} \to \mathrm{Fun}(\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ is monadic.

In particular, identifying an object of $\mathrm{Tot}\underline{\Sigma}_{\mathcal{T}}$ with a pair (V, S) where $V \in \mathcal{T}$ and $S \in \mathbb{F}_V$, $\mathcal{T}$-operads are identified conservatively by the functor

$$\mathcal{O} \mapsto \prod_{V,S} \mathcal{O}(S).$$

Intuitively, we view $\mathcal{O}(S)$ as the space of *S-ary operations* $\left(\mathrm{Res}_V^{\mathcal{T}} X\right)^{\otimes S} \to \mathrm{Res}_V^{\mathcal{T}} X$ borne by an $\mathcal{O}$-algebra X. This technology allowed us to define the *arity support* functor

$$A\mathcal{O} := \left\{ T \to S \;\middle|\; \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U) \neq \varnothing \right\} \subset \mathbb{F}_{\mathcal{T}};$$

which we verified in [Ste25a] to be a weak indexing category. In fact, we verified that the essential surjection associated with A possesses a fully faithful right adjoint

$$\begin{tikzcd}
\mathrm{Op}_{\mathcal{T}} \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{wIndexCat}_{\mathcal{T}}; \arrow[l, bend left, "\mathcal{N}^{\otimes}_{(-)\infty}"]
\end{tikzcd} \tag{2}$$

we refer to the $\mathcal{T}$-operad $\mathcal{N}^{\otimes}_{I\infty}$ as the *weak $\mathcal{N}_\infty$-operad associated with I*. Now, we further verified in [Ste25a] that, given a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$, the unique map $\mathcal{O}^{\otimes} \to \mathrm{Comm}^{\otimes}_{\mathcal{T}}$ is a monomorphism if and only if the counit map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{O}}$ is an equivalence; in particular, we acquire an equality of full subcategories

$$\mathrm{Op}_{\mathcal{T},/\mathcal{N}^{\otimes}_{I\infty}} = A^{-1}(\mathrm{wIndexCat}_{\mathcal{T},\leq I}) \;\subset\; \mathrm{Op}_{\mathcal{T}},$$

and a full subcategory of $\mathrm{Op}_{\mathcal{T}}$ has a terminal object if and only if it is of this form. We refer to $\mathrm{Op}_I \coloneqq \mathrm{Op}_{\mathcal{T},/\mathcal{N}^{\otimes}_{I\infty}}$ as the ∞-category of *I-operads*; see [Ste25a] for an intrinsic characterization of Op_I.

Monomorphisms are right-cancellable, so all inclusions $I \subset J$ induce monomorphisms $\iota_I^J\colon \mathcal{N}^{\otimes}_{I\infty} \to \mathcal{N}^{\otimes}_{J\infty}$; in other words, the push-pull adjunction

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "E_I^J = \iota_{I!}^J"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_J \arrow[l, bend left, "\mathrm{Bor}_I^J = \iota_I^{J*}"] \end{tikzcd}$$

witnesses $\mathrm{Op}_I \subset \mathrm{Op}_J$ as a colocalizing subcategory. Moreover, it behaves well with $\overset{\mathrm{BV}}{\otimes}$.

Proposition 23 ([Ste25a])**.** *Suppose $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ have at least one color. Then, there is an equality*

$$A(\mathcal{O} \otimes \mathcal{P}) \simeq A\mathcal{O} \vee A\mathcal{P}.$$

In particular, $\mathrm{Op}_I \subset \mathrm{Op}_{\mathcal{T}}$ is a symmetric monoidal full subcategory.

1.3.2. *I-symmetric monoidal categories and $\mathcal{O}$-algebras.* [NS22] constructed a (non-full) subcategory inclusion

$$\iota\colon \mathrm{Cat}^{\otimes}_I \to \mathrm{Op}_{\mathcal{T}};$$

$\mathcal{T}$-operad maps between I-symmetric monoidal categories are called *lax I-symmetric monoidal functors*, and morphisms in the image of ι are called *I-symmetric monoidal functors*.

Moreover, given $\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, we define *$\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$* to be $\mathcal{T}$-operad maps $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$, which naturally fit into an ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. These have a *pointwise $\mathcal{T}$-operad structure* $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ given by the internal hom in a presentably symmetric monoidal structure on $\mathrm{Op}_{\mathcal{T}}$, whose tensor product we write as $\overset{\mathrm{BV}}{\otimes}$ [Ste25a; Ste25b]. The unit for this symmetric monoidal structure is the $\mathcal{T}$-operad $\mathrm{triv}^{\otimes}_{\mathcal{T}} \coloneqq \mathcal{N}^{\otimes}_{I^{\mathrm{triv}}\infty}$ [Ste25a], i.e. there is a canonical equivalence

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{O}) \simeq \mathcal{O}^{\otimes} \tag{3}$$

Moreover, we verified in [Ste25a] that whenever $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category, $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is as well, and given a $\mathcal{T}$-operad map $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ and an I-symmetric monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, the induced lax I-symmetric monoidal functors

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}); \qquad\qquad \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D})$$

are I-symmetric monoidal. In particular, when $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category and $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are I-operads, there are natural I-symmetric monoidal equivalence

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}\otimes\mathcal{P}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \tag{4}$$

1.3.3. *The underlying $\mathcal{T}$-∞-category.* An I-operad $\mathcal{O}^{\otimes}$ has an underlying $\mathcal{T}$-∞-category $U\mathcal{O}$ [NS22]; indeed, $\mathcal{T}$-operads are equivariantizations of the classical notions of *colored operads*, and $U\mathcal{O}$ the *∞-category of colors*. Moreover, the composite functor $\mathrm{Cat}^{\otimes}_I \to \mathrm{Op}_I \xrightarrow{U} \mathrm{Cat}_{\mathcal{T}}$ is the usual *underlying $\mathcal{T}$-∞-category* functor.

U behaves well with respect to $\underline{\mathrm{Alg}}^{\otimes}$; indeed, we verified in [Ste25a] that the underlying $\mathcal{T}$-∞-category has values

$$U\left(\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})\right)_V \simeq \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{O}}\left(\mathrm{Res}^{\mathcal{T}}_V \mathcal{C}\right),$$

where $\mathrm{Res}^{\mathcal{T}}_V\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_V$ is a restriction functor, and furthermore

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \Gamma^{\mathcal{T}} U \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}).$$

It was observed in [NS22] that the composite functor $\mathrm{Op}_{I^{\mathrm{triv}}} \subset \mathrm{Op}_{\mathcal{T}} \xrightarrow{U} \mathrm{Cat}_{\mathcal{T}}$ is an equivalence, and that U factors as $\mathrm{Op}_{\mathcal{T}} \xrightarrow{\mathrm{Bor}_{I^\infty}^{\mathrm{triv}}} \mathrm{Op}_{I^{\mathrm{triv}}} \simeq \mathrm{Cat}_{\mathcal{T}}$. We write $\mathrm{triv}(-)^{\otimes}$ for the composite functor

$$\mathrm{triv}(-)^{\otimes}\colon \mathrm{Cat}_{\mathcal{T}} \xrightarrow{U^{-1}} \mathrm{Op}_{I^\infty} \hookrightarrow \mathrm{Op}_{\mathcal{T}};$$

unwinding definitions, we find that there is a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathrm{triv}(\mathcal{C})}(\mathcal{O}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, U\mathcal{O});$$

that is, $\mathrm{triv}(\mathcal{C})$ algebras are simply $\mathcal{C}$-indexed diagrams of objects.

1.3.4. *$\mathcal{T}$-operadic inflation and fixed points.* In [Ste25a] we constructed an equivalence

$$\varphi\colon \mathrm{Op}_{I_\infty} \xrightarrow{\sim} \mathrm{Coeff}^{\mathcal{T}}\mathrm{Op}$$

exhibiting natural equivalences $\varphi\mathcal{O}_V(n) \simeq \mathcal{O}(n \cdot *_V)$. Limits and constant diagrams yields an *inflation and fixed point* adjunction

$$\begin{tikzcd} \mathrm{Op} \arrow[r, bend left, "\mathrm{Infl}_e^{\mathcal{T}}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{I_\infty} \arrow[l, bend left, "\Gamma^{\mathcal{T}}"] \arrow[r, bend left, "E_{I_\infty}^{\mathcal{T}}"] \arrow[r, phantom, "\perp"] & \mathrm{Op}_{\mathcal{T}}; \arrow[l, bend left, "\mathrm{Bor}_{I_\infty}^{\mathcal{T}}"] \end{tikzcd}$$

we refer to the composite adjunction $\mathrm{Op} \rightleftarrows \mathrm{Op}_{\mathcal{T}}$ also as $\mathrm{Infl}_e^{\mathcal{T}} \dashv \Gamma^{\mathcal{T}}$. For instance we have

$$\mathrm{Alg}_{\mathrm{Infl}(\mathcal{O})}(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}}\left(\Gamma^{\mathcal{T}}\mathcal{P}\right); \tag{5}$$

moreover, we can identify the image of $\mathrm{Infl}_e^{\mathcal{T}}$ easily: they are the I_∞-operads $\mathcal{O}^{\otimes}$ whose underlying $\mathcal{T}$-∞-category is inflated and whose restriction maps

$$\mathcal{O}(C; D) \to \mathcal{O}(\mathrm{Res}_U^V C; \mathrm{Res}_U^V D)$$

are all equivalences.

Example 24. The above description yields a natural equivalence $\mathrm{Infl}_e^{\mathcal{T}}(\mathrm{triv}(\mathcal{C})^{\otimes}) \simeq \mathrm{triv}(\mathrm{Infl}_e^{\mathcal{T}}\mathcal{C})^{\otimes}$. ◁

Example 25. The $\mathcal{T}$-operads $\mathbb{E}_0^{\otimes} \coloneqq \mathcal{N}_{I_{0,\mathcal{T}}}^{\otimes}$ and $\mathbb{E}_\infty^{\otimes} \coloneqq \mathcal{N}_{I_\infty}^{\otimes}$ are inflated from operads of the same names; in particular, unwinding definitions, we may identify $\mathbb{E}_0$-algebras by the formula

$$\underline{\mathrm{Alg}}_{\mathbb{E}_0}(\mathcal{C})_V \simeq \mathcal{C}_{V, 1_V/}.$$

If 1_V is terminal for all $V \in \mathcal{T}$, then this is the $\mathcal{T}$-category of pointed objects $\mathcal{C}_*$. ◁

1.3.5. *Unital I-operads.* Assume that I is an almost unital weak indexing category. In [Ste25b] we introduced the following gamut of definitions, each of which will be useful.

Definition 26. We say that an I-operad $\mathcal{O}^{\otimes}$

- is *almost unital* if it has at least one color and whenever there exists some $S \in \mathbb{F}_V$ such that $\mathcal{O}(S) \neq \varnothing$, we have $\mathcal{O}(\varnothing_V) \simeq *$,
- is *unital* if it has at least one color and $\mathcal{O}(\varnothing_V) \simeq \mathcal{N}_{I\infty}(\varnothing_V)$ for all $V \in \mathcal{T}$, and
- is *almost reduced* if it is almost unital and has one color, and
- is *reduced* if it is unital and has one color. ◁

A $\mathcal{T}$-operad is almost unital if and only if it's a unital I-operad for *some* almost-unital weak indexing category I. For this reason, we'll usually focus on either unital I-operads or almost-unital $\mathcal{T}$-operads. It will be important to keep the I-symmetric monoidal case in mind.

Example 27. We verified in [Ste25b] that an I-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a unital I-operad if and only if, for all $V \in v(I)$, the unit object $1_V \in \mathcal{C}_V$ is initial. ◁

Write $\mathbb{E}_{0,v(I)}^{\otimes} \coloneqq \mathcal{N}_{I_{0,v(I)}}^{\otimes}$. We will largely use the following result of [Ste25b] to access unital I-operads.

Proposition 28 ([Ste25b])**.** *The full subcategory* $\mathrm{Op}_I^{\mathrm{uni}} \subset \mathrm{Op}_I$ *of unital* I*-operads is both a localizing and colocalizing subcategory, i.e. the inclusion participates in a double adjunction*

$$\begin{tikzcd} \mathrm{Op}_I \arrow[r, bend left, "(-)\overset{BV}{\otimes}\mathbb{E}_{0,v(I)}^{\otimes}"] \arrow[r, bend right, "\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(-)"'] & \mathrm{Op}_I^{\mathrm{uni}}. \arrow[l, hook'] \end{tikzcd}$$

In particular, if $\mathcal{O}^{\otimes}$ *and* $\mathcal{C}^{\otimes}$ *are unital, then there are natural equivalences*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}\otimes\mathbb{E}_{0,v(I)}}(\mathcal{C});$$
$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D}) \simeq \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_{0,v(I)}}(\mathcal{D}).$$

We accomplished this in part by recognizing an equality of full subcategories $\mathrm{Op}_I^{\mathrm{uni}} = \mathrm{Op}_I^{I_{0,v(I)}-\mathrm{Wirth}}$; that is, an I-operad is unital if and only if its I-symmetric monoidal ∞-categories of algebras have V-units which are initial for each $V \in v(I)$, which is true if and only if they are unital by Example 27. Moreover, since the $\overset{\mathrm{BV}}{\otimes}$-unit $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ is initial among one color I-operads, this yields the following easy corollary.

Corollary 29. $\mathbb{E}_{0,v(I)}^{\otimes}$ *is initial among reduced* I*-operads.*

$\mathrm{Op}_I^{\mathrm{red}}$ has initial unit object; interestingly, it has *absorptive* terminal object.

Proposition 30 ([Ste25b])**.** *If* $\mathcal{O}^{\otimes}$ *is a unital* I*-operad, then the map* $\mathbb{E}_{0,v(I)}^{\otimes} \to \mathcal{O}^{\otimes}$ *induces a (unique) equivalence*

$$\mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes} \overset{BV}{\otimes} \mathbb{E}_{0,v(I)}^{\otimes} \xrightarrow{\sim} \mathcal{N}_{I\infty}^{\otimes} \overset{BV}{\otimes} \mathcal{O}^{\otimes}.$$

1.3.6. *Cartesian and cocartesian* I*-symmetric monoidal* ∞*-categories.* In [Ste25b], given $\mathcal{C}$ a $\mathcal{T}$-∞-category with I-indexed (co)products, we defined *cocartesian* and *cartesian* I-symmetric monoidal ∞-categories $\mathcal{C}^{I-\sqcup}$ and $\mathcal{C}^{I-\times}$, which are determined by the properties that their I-indexed tensor products are canonically equivalent to indexed (co)products. We gave algebras in cartesian I-symmetric monoidal ∞-categories an explicit presentation generalizing the $\mathcal{O}$*-monoids* of [HA] (as $\mathcal{T}$-functors satisfying "Segal conditions") which we will not mention explicitly here; as a relic of this, we will simply use the notation

(6) $$\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{D}) := \underline{\mathrm{Alg}}_{\mathcal{O}}\left(\mathcal{D}^{I-\times}\right); \qquad \mathrm{Mon}_{\mathcal{O}}(\mathcal{D}) := \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{D}^{I-\times}\right).$$

The associated I-symmetric monoidal structure is cartesian [Ste25b]. When $\mathcal{C}$ is an ∞-category, we will write

(7) $$\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) := \underline{\mathrm{Mon}}_{\mathcal{O}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right); \qquad \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) := \mathrm{Mon}_{\mathcal{O}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C}\right).$$

instead we will use their monadic presentation, which goes as follows.

Proposition 31 ([Ste25a])**.** *Suppose* $\mathcal{C}$ *is a presentable and cartesian closed* ∞*-category. Then, the monad* $T_{\mathcal{O}}$ *associated with the monadic functor* $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ *has fixed points*

$$(T_{\mathcal{O}}X)^W \simeq \coprod_{S \in \mathbb{F}_{I,W}} \left(\mathrm{Fr}_{\mathcal{C}}\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} X^U\right)_{h\mathrm{Aut}_W(S)},$$

where $\mathrm{Fr}_{\mathcal{C}}\colon \mathcal{S} \to \mathcal{C}$ *is the unique left adjoint sending* $*$ *to the terminal object of* $\mathcal{C}$*.*

Moreover, in the case that $\mathcal{O}^{\otimes}$ is unital, we characterized cocartesian algebras simply as diagrams

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\left(\mathcal{C}^{I-\sqcup}\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O}, \mathcal{C})^{I-\sqcup};$$

in fact, $\mathcal{C}^{I-\sqcup}$ still exists as an I-operad with the above algebras in when $\mathcal{C}$ is not assumed to have I-indexed coproducts. In particular, in the unital case, we acquire a double adjunction

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{T}} \arrow[r, bend left, "\mathrm{triv}(-)^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}"] \arrow[r, bend right, "(-)^{I-\sqcup}"'] & \mathrm{Op}_I^{\mathrm{uni}} \arrow[l, "U" description] \end{tikzcd}. \tag{8}$$

Example 32. In [Ste25b] we gave a general formula for $\mathcal{C}^{I-\sqcup}$, but the mapping-in property makes it easy enough to determine this in the case that $\mathcal{C}$: there is an equivalence

$$\mathrm{Alg}_{\mathcal{O}}\left(*_{\mathcal{T}}^{I-\sqcup}\right) \simeq * \simeq \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{N}_{I\infty}^{\otimes}\right),$$

natural in the unital I-operad $\mathcal{O}^{\otimes}$, constructing an equivalence $\mathcal{N}_{I\infty}^{\otimes} \simeq *_{\mathcal{T}}^{I-\sqcup}$ by Yoneda's lemma. ◁

1.3.7. *I-d-operads.* In [Ste25a], we defined the full subcategory $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ of *$\mathcal{T}$-d-operads* to be those such that $\mathcal{O}(S)$ is a $(d-1)$-truncated space for all $S \in \underline{\mathbb{F}}_{A\mathcal{O}}$, and verified the following.

Proposition 33 ([Ste25a])**.** *Fix $d \geq -1$ and $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$.*

(1) The inclusion $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ has a left adjoint $h_d\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Op}_{\mathcal{T},d}$ satisfying

$$h_d\mathcal{O}(S) \simeq \tau_{\leq d-1}\mathcal{O}(S).$$

(2) The unit of the h_0-localization adjunction is the map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}}^{\otimes}$; in particular, $\mathcal{N}_{(-)\infty}^{\otimes}$ factors through an equivalence

$$\mathrm{wIndexCat}_{\mathcal{T}} \simeq \mathrm{Op}_{\mathcal{T},0}.$$

(3) When $\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-d-operad, there is a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{h_d\mathcal{O}}^{\otimes}(\mathcal{P}),$$

and each are $\mathcal{T}$-d-operads.

(4) An I-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-d-operad if and only if $U\mathcal{C}$ is a $\mathcal{T}$-d-category.

We call $h_d\mathcal{O}^{\otimes}$ the *homotopy d-operad of $\mathcal{O}^{\otimes}$*.

1.3.8. *$\mathcal{O}$-algebras in I-symmetric monoidal 1-categories.* Fix $\mathcal{C}^{\otimes}$ an I-symmetric monoidal 1-category; in light of Proposition 33, to characterize $\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$, we may equivalently characterise $h_1\mathcal{O}$-algebras in $\mathcal{C}$, so assume $\mathcal{O}^{\otimes}$ is an I-1-operad, i.e. its structure spaces are sets.

We gave a simple combinatorial model for I-1-operads in [Ste25a], which we will not relitigate here, instead focusing only on algebras. Given a $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$, we defined the *unreduced endomorphism I-operad* $\mathrm{End}_X(\mathcal{C})$ as a one-colored I-1-operad with structure sets

$$\mathrm{End}_X(\mathcal{C})(S) \simeq \mathrm{Hom}_{\mathcal{C}_V}\left(X_{\underline{V}}^{\otimes S}, X_{\underline{V}}\right),$$

where $X_{\underline{V}} \in \mathcal{C}_V$ is the V-object underlying X. 1-categorical algebras take a familiar form.

Proposition 34 ([Ste25a])**.** *Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_{I,1}^{\mathrm{oc}}$, $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a 1-category whose objects are pairs $(X \in \Gamma^{\mathcal{T}}\mathcal{C}, \varphi\colon \mathcal{O} \to \mathrm{End}_X(\mathcal{C}))$ and whose morphisms are $\Gamma^{\mathcal{T}}\mathcal{C}$-maps $f\colon X \to Y$ such that the corresponding diagram commutes*

$$\begin{tikzcd} & \mathrm{End}_X(\mathcal{C}) \arrow[d, "\mathrm{End}_f"] \\ \mathcal{O}^{\otimes} \arrow[ur] \arrow[dr] & \\ & \mathrm{End}_Y(\mathcal{C}) \end{tikzcd}$$

Moreover, we may exploit this to explicitly describe interchange.

Corollary 35 ([Ste25a])**.** *Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes} \in \mathrm{op}_{I,1}^{\mathrm{oc}}$, an $\mathcal{O} \overset{BV}{\otimes} \mathcal{P}$-algebra structure on X is precisely a pair of $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures such that, for all $\mu \in \mathcal{O}(S)$, the corresponding $\mathcal{C}$-map $X_{\underline{V}}^{\otimes S} \to X_{\underline{V}}$ is a morphism of $\mathcal{P}$-algebras; a morphism of $\mathcal{O} \overset{BV}{\otimes} \mathcal{P}$-algebras is a $\Gamma^{\mathcal{T}}\mathcal{C}$-map which is separately an $\mathcal{O}$-algebra and $\mathcal{P}$-algebra morphism.*

1.4. **The doctrinal adjunction.** The following proposition will play a crucial role in constructing I-symmetric monoidal left adjoints. We temporarily assume that the reader is familiar with [Ste25a, § 2].

Proposition 36 (Doctrinal adjunction)**.** *Suppose $L^\otimes \colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is an I-symmetric monoidal functor whose underlying $\mathcal{T}$-functor L admits a right adjoint R. Then, R lifts to a canonical lax I-symmetric monoidal right adjoint $R^\otimes \vdash L^\otimes$. Moreover, for any $\mathcal{T}$-operad $\mathcal{O}^\otimes$ the postcomposition lax I-symmetric monoidal functors partake in a lax I-symmetric monoidal adjunction*

$$L_*^\otimes \colon \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C}) \rightleftarrows \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{D}) \colon R_*^\otimes$$

such that $L^{\otimes_*}$ is I-symmetric monoidal. If $R^\otimes$ is symmetric monoidal then $R_*^\otimes$ is symmetric monoidal; if R is also fully faithful, then $R_*^\otimes$ is fully faithful.

Proof. Applying [HA, Prop 7.3.2.6] to the fibrations on opposite categories, we acquire a right adjoint $R^\otimes \vdash L^\otimes$ relative to $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$. Moreover, an identical argument to [HA, Cor 7.3.2.7] shows that $R^\otimes$ preserves cocartesian lifts for inert morphisms. The lax I-symmetric monoidal functors $L_*^\otimes$ and $R_*^\otimes$ are then constructed in [Ste25a], where postcomposition along an I-symmetric monoidal functor is verified to be I-symmetric monoidal; in particular, $L_*^\otimes$ is always I-symmetric monoidal and $R_*^\otimes$ is I-symmetric monoidal whenever $R^\otimes$ is.

Note that postcomposition along the unit and counit data for $L^\otimes \dashv R^\otimes$ yield unit and counit data for $L_*^\otimes$ and $R_*^\otimes$ in any case. When $R^\otimes, L^\otimes$ are symmetric monoidal and R is fully faithful, the counit $\varepsilon \colon L^\otimes R^\otimes \mathcal{C}^\otimes \to \mathcal{C}^\otimes$ is an I-symmetric monoidal functor whose underlying $\mathcal{T}$-functor is an equivalence, so ε is an I-symmetric monoidal equivalence; in particular, this implies that the counit of $L_*^\otimes \dashv R_*^\otimes$ is an equivalence, so $R_*^\otimes$ is fully faithful. □

1.5. **Recognizing h_{n+1}-equivalences.** Theorem D recognizes morphisms of $\mathcal{T}$-operads which become equivalences after applying h_{n+1}, so we now spell out some of its antecedents.

Proposition 37. *Let $\varphi \colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ be a morphism of $\mathcal{T}$-operads. The following are equivalent:*

(a) for all $S \in \underline{\mathbb{F}}_{A\mathcal{O}} \cup \underline{\mathbb{F}}_{A\mathcal{P}}$, the map of spaces

$$\varphi(S) \colon \mathcal{O}(S) \to \mathcal{P}(S)$$

is an n-equivalence;

(b) φ is an h_{n+1}-equivalence;

(c) for all $\mathcal{T}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}$, the pullback $\mathcal{T}$-symmetric monoidal functor

$$\underline{\mathrm{Alg}}_{\mathcal{P}}^\otimes(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}^\otimes(\mathcal{C})$$

is an equivalence;

(d) the pullback functor

$$\mathrm{Mon}_{\mathcal{P}}(\mathcal{S}_{\le n}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{S}_{\le n})$$

is an equivalence; and

(e) for all ∞-categories K, the pullback map of spaces

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{S}_{\le n}^K\right)^{\simeq} \to \mathrm{Mon}_{\mathcal{O}}\left(\mathcal{S}_{\le n}^K\right)^{\simeq}$$

is an equivalence.

To prove this, we apply the following lemma.

Lemma 38. *Given a $\mathcal{T}$-operad $\mathcal{P}^\otimes$ and a pair of ∞-categories $\mathcal{D}, K$ such that $\mathcal{D}$ admits finite products, there is an equivalence*

$$\underline{\mathrm{Mon}}_{\mathcal{P}}\left(\mathcal{D}^K\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{D})\right),$$

natural in functors of K, product-preserving functors of $\mathcal{D}$, and $\mathcal{T}$-operad maps of $\mathcal{P}$; in particular, taking $\mathcal{T}$-fixed points yields a natural equivalence of categories

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{D}^K\right) \simeq \mathrm{Mon}_{\mathcal{P}}(\mathcal{D})^K.$$

Proof. We construct a chain of equivalences

$$\begin{aligned}
\underline{\mathrm{Mon}}_{\mathcal{P}}\left(\mathcal{D}^K\right) &\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D}^K)^{\mathcal{T}-\times}) && \text{Eqs. (6) and (7)}\\
&\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\right)^{\mathcal{T}-\times} && \text{Example 17}\\
&\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}(\mathrm{Infl}_e^{\mathcal{T}} K)}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) && \text{Eq. (3)}\\
&\simeq \underline{\mathrm{Alg}}_{\mathcal{P}}\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{Infl}_e^{\mathcal{T}}\mathrm{triv}(K)}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) && \text{Example 24}\\
&\simeq \underline{\mathrm{Alg}}_{\mathrm{Infl}_e^{\mathcal{T}}\mathrm{triv}(K)}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}^{\mathcal{T}-\times}\right) && \text{Eq. (4)}\\
&\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Alg}}_{\mathcal{P}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}, \mathcal{D}^{\mathcal{T}-\times}\right)\right) && \text{Eq. (5)}\\
&\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}} K, \underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{D})\right) && \text{Eqs. (6) and (7)}
\end{aligned}$$

The remaining equivalence follows by noting that $\Gamma^{\mathcal{T}}\mathrm{Infl}_e^{\mathcal{T}}\mathcal{C} \simeq \mathcal{C}$, naturally in $\mathcal{C}$. □

Proof of Proposition 37. A generalization of the equivalence between Conditions (a) to (d) was proved in [Ste25a], and Condition (c) clearly implies Condition (e). Moreover, fixing $\mathcal{D} = \mathcal{S}_{\leq n}$ and taking cores of Lemma 38 yields a natural equivalence

$$\mathrm{Mon}_{\mathcal{P}}\left(\mathcal{S}_{\leq n}^K\right)^{\simeq} \simeq \mathrm{Map}_{\mathrm{Cat}}(K, \mathrm{Mon}_{\mathcal{P}}(\mathcal{S}_{\leq n}))$$

so Condition (e) and Yoneda's lemma together imply Condition (d). □

We say that $\mathcal{O}^{\otimes}$ is *n-connected* if the unique map $\mathcal{O}^{\otimes} \to \mathcal{N}^{\otimes}_{A\mathcal{P}}$ is an h_{n+1}-equivalence. In [Ste25b] we acquired the following additional characterizations for n-connected $\mathcal{T}$-operads:

Proposition 39. *Suppose $\mathcal{O}^{\otimes}$ is an almost-unital $\mathcal{T}$-operad. Then, the following conditions are equivalent:*

- *(b') $\mathcal{O}^{\otimes}$ is n-connected.*
- *(f') For all $A\mathcal{O}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}^{\otimes}$, the $A\mathcal{O}$-symmetric monoidal $(n+1)$-category $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is cocartesian.*
- *(g') The $\mathcal{T}$-$(n+1)$-category $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S}_{\leq n})$ is $A\mathcal{O}$-semiadditive.*

1.6. **The reduced endomorphism I-operad as a right adjoint.** In [Ste25b], we introduced the *reduced endomorphism I-operad* of a $\mathcal{T}$-operad for the purpose of lifting the disintegration and assembly process of [HA]. In this section, we gain explicit computational control over reduced endomorphism I-operads of unital I-symmetric monoidal ∞-categories.

Proposition 40. *The inclusion $\mathrm{Op}_I^{\mathrm{red}} \simeq \mathrm{Op}^{\mathrm{red}}_{I,\mathbb{E}_{0,v(I)}/} \hookrightarrow \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}_{0,v(I)}/}$ has a right adjoint computed by the pullback*

$$\begin{tikzcd}
\mathrm{End}_X^{I,\mathrm{red}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{O}^{\otimes} \arrow[d, "\eta"] \\
\mathcal{N}^{\otimes}_{I\infty} \arrow[r, "\{X\}"'] & \mathcal{O}^{\mathcal{T}-\sqcup}
\end{tikzcd} \tag{9}$$

In the case that $\mathcal{C}^{\otimes}$ is a unital I-symmetric monoidal ∞-category and $X \in \mathcal{C}_V$ is a V-object, mapping in from the free unital I-operad $\mathrm{Fr}_{\mathrm{Op}}(S) \overset{BV}{\otimes} \mathbb{E}_{0,v(I)}$ on an operation in arity $S \in \mathbb{F}_{I,V}$ yields a pullback

$$\begin{tikzcd}
\mathrm{End}_X^{I,\mathrm{red}}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right) \arrow[d, "W^*_{S,X}"] \\
\{\nabla\} \arrow[r] & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\sqcup S}, X\right)
\end{tikzcd}$$

i.e. $\mathrm{End}_X^{I,\mathrm{red}}(S)$ *is equivalent to the space of lifts along the following dashed arrow in* $\mathcal{C}_V$

$$\begin{tikzcd} X^{\sqcup S} \arrow[r, "\nabla"] \arrow[d, "W_{S,X}"'] & X \arrow[d, "!"] \\ X^{\otimes S} \arrow[r, "!"'] \arrow[ur, dashed] & * \end{tikzcd}$$

Proof. We will apply the general reduction procedure of [SY19, Prop 2.1.5], applied to the *sliced* adjunction

$$U_*\colon \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}^{\otimes}_{0,v(I)}/} \rightleftarrows \mathrm{Cat}_{\mathcal{T},*}\colon \eta^*(-)^{I-\sqcup},$$

whose right adjoint is $(-)^{I-\sqcup}$ together with the precomposed structure map

$$\mathbb{E}^{\otimes}_{0,v(I)} \xrightarrow{\eta} \mathcal{N}^{\otimes}_{I\infty} \simeq *^{I-\sqcup}_{\mathcal{T}} \to \mathcal{C}^{I-\sqcup}.$$

Indeed, $\mathrm{Cat}_{\mathcal{T},*}$ admits an initial object $*_{\mathcal{T}} \simeq U\mathbb{E}_{0,v(I)}$, and $\mathrm{Op}^{\otimes}_{I,\mathbb{E}^{\otimes}_{0,v(I)}/}$ admits all limits, which are preserved by U since it is a right adjoint by Eq. (8). Moreover, $\mathbb{E}_{0,v(I)} \in \mathrm{Op}^{\mathrm{red}}_I$ is initial by Corollary 29, there is a unique equivalence $\mathcal{N}^{\otimes}_{I\infty} \simeq *^{I-\sqcup}_{\mathcal{T}}$ by Eq. (2) and Example 32, and $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}_{0,v(I)}/}$ corresponds with a reduced I-operad if and only if $U\mathcal{O}^{\otimes} \in \mathrm{Cat}_{\mathcal{T},*}$ is initial, so the first claim follows by [SY19, Prop 2.1.5].

To acquire the second pullback square, one need only note that the natural equivalences

$$\mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{C}^{\otimes}\right) \simeq \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right),$$
$$\mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{N}^{\otimes}_{I\infty}\right) \simeq *$$

follow by Propositions 22 and 28. What remains is to verify that the right vertical arrow is $W^*_{S,X}$ and the bottom arrow includes the fold map ∇; both facts were verified in [Ste25b]. □

In fact, [SY19, Prop 4.2.8] introduced a result on connectivity of such spaces of lifts, immediately yielding the following corollary.

Corollary 41. *If $X \in \mathcal{C}_V$ is a $(k+\ell+2)$-truncated object and the Wirthmüller map $W_{S,X}\colon X^{\sqcup S} \to X^{\otimes S}$ is ℓ-connected, then the space $\mathrm{End}_X^{I,\mathrm{red}}(\mathcal{C})(S)$ is k-truncated.*

In general, reduction is an incarnation of the *disintegration and assembly* procedure of [HA; Ste25b]; given a reduced I-operad $\mathcal{P}^{\otimes}$ and a V-object $X \in \mathcal{O}_V$, applying $\mathcal{P}$-algebras to Eq. (9) yields a pullback

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})_V \arrow[d, "U"] \\ \{X\} \arrow[r, hook] & U\mathcal{O}_V \end{tikzcd} \tag{10}$$

In the case that $U\mathcal{O}$ is a $\mathcal{T}$-space, U is a automatically cocartesian fibration, so $\mathcal{O}$-algebras are $U\mathcal{O}$-indexed diagrams of $\mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})$-algebras. Unfortunately, this is far from our case; the best we can do is take cores of the above pullback square, resulting in the following proposition.

Proposition 42. *Suppose $\mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ is a morphism of I-operads inducing an equivalence of spaces*

$$\varphi_X^{*,\simeq}\colon \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{Q}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})^{\simeq} \longrightarrow \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})^{\simeq}$$

for all $V \in \mathcal{T}$ and $X \in U\mathcal{O}_V$. Then, the induced map of $\mathcal{T}$-spaces

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^{\simeq} \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^{\simeq}$$

is an equivalence; in particular, passing to $\mathcal{T}$-fixed points, the induced map of spaces

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{O})^{\simeq} \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{O})^{\simeq}$$

is an equivalence.

Proof. Taking cores of Eq. (10), we find that that $\varphi_X^{*,\simeq}$ is the induced map on the homotopy fiber over X of the following map of $\mathcal{T}$-spaces over $U\mathcal{O}$:

$$\begin{tikzcd} \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^{\simeq} \arrow[rr, "\varphi^{*,\simeq}"] \arrow[dr] & & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^{\simeq} \arrow[dl] \\ & U\mathcal{O} & \end{tikzcd}$$

$\varphi^{*,\simeq}$ is an equivalence if and only if its V-fixed points are an equivalence for all $V \in \mathcal{T}$, and the homotopy fibers of $\varphi^{*,\simeq,V}$ are contractible by the above argument, so $\varphi^{*,\simeq,V}$ is an equivalence for all V. Hence $\varphi^{*,\simeq}$ is an equivalence, proving the proposition. □

2. Connectivity and Wirthmüller maps

In this section, we verify Theorem D; in particular, we will acquire the following technical corollary.

Corollary 43. *If $\mathcal{P}^{\otimes}$ is ℓ-connected at I, then for all $(k+\ell+2)$-toposes $\mathcal{C}$, the reduced endomorphism I-operad $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{I-\times}\right)$ is an I-$(k+1)$-operad.*

Proof. Since $\mathcal{C}$ is a $(k+\ell+2)$-category, X is $(k+\ell+2)$-truncated, and Theorem D implies that $W_{X,S}$ is ℓ-connected, so the result follows from Corollary 41. □

Before moving on, we show how this yields the atomic orbital generalization of Theorem B.

Proof of Theorem B. By passing to restrictions and Borelifications, we assume that $\mathcal{O}, \mathcal{P}$ are almost reduced. By Proposition 37, we're tasked with verifying that, for all presheaf $(k+\ell+2)$-toposes $\mathcal{C}$, the map of spaces

$$\mathrm{Mon}_{\mathcal{O}}\mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq} \to \mathrm{CMon}_{A\mathcal{O}}(\mathcal{C})^{\simeq}.$$

is an equivalence; since $\mathcal{N}_{A\mathcal{O}\infty}^{\otimes} \simeq \mathcal{P}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ by Proposition 30, we may equivalently require that the map

$$\mathrm{Mon}_{\mathcal{O}}\mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq} \to \mathrm{CMon}_{A\mathcal{O}}\mathrm{Mon}_{\mathcal{P}}(\mathcal{C})^{\simeq}$$

is an equivalence. In particular, by Propositions 37 and 42, it suffices to prove that $\mathrm{End}_X\left(\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{C})^{A\mathcal{O}-\times}\right)$ is an $A\mathcal{O}$-$(k+1)$-operad, which is Corollary 43. □

2.1. Connectivity of algebras can be detected in the value topos.

Fix $\mathcal{C}$ an n-topos for some $n \le \infty$.

Lemma 44. *A map $f\colon C \to D$ in $\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ is ℓ-connected if and only if, for all $V \in \mathcal{T}^{\mathrm{op}}$, the fixed point map $C^V \to D^V$ is ℓ-connected.*

Proof. Per Remark 3, it is equivalent to prove that ℓ-connectiveness of a morphism in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ is measured elementwise. Indeed, since (co)limits in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ are computed elementwise, effective epimorphisms and diagonals are as well. The former proves the statement for (-2)-connectiveness, and the latter together with the diagonal presentation of [HTT, Prop 6.5.1.18] shows that the statement for $(\ell-1)$-connectiveness implies the statement for ℓ-connectiveness, so the lemma follows by induction. □

Proposition 45. *Given a map $f\colon X \to Y$ in $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$, if the underlying map Uf in $\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ is ℓ-connected, then f is ℓ-connected.*

Proof. In view of [SY19, Lem 4.4.1], it suffices to verify that the monad $T_{\mathcal{O}}\colon \mathrm{Coeff}^{\mathcal{T}}\mathcal{C} \to \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ preserves ℓ-connected morphisms; by Lemma 44, it suffices to verify that whenever each $\mathcal{C}$-diagram $X^V \to Y^V$ is ℓ-connected, each induced map $T_{\mathcal{O}}X^W \to T_{\mathcal{O}}X^W$ is ℓ-connected. But by Proposition 31, it suffices to note that ℓ-connected morphisms in an ∞-topos are closed under cartesian products and colimits [HTT, Cor 6.5.1.13, Prop 5.2.8.6]. □

For instance, U preserves the terminal object and is conservative, so it also reflects the property of being terminal; applying Proposition 45 in the case $Y = *$ shows that U reflects n-connectivity of objects.

Remark 46. Since U is a right adjoint, it preserves n-truncatedness and n-truncated objects. ◁

Warning 47. Proposition 45 is delicate for a few reasons.

(1) 1f $\mathcal{O}$ is not n-connected, then the free $\mathcal{O}$-algebra monad $T_{\mathcal{O}}\colon \mathcal{C}_V \to \mathcal{C}_V$ may itself may fail to preserve n-connected objects; indeed, we have $T_{\mathcal{O}} *_V \simeq \coprod_{S \in \mathbb{F}_V} \mathrm{Fr}_{\mathcal{C}} \mathcal{O}(S)_{h\mathrm{Aut}_V S}$, which is often not much more highly connected than the individual spaces $\mathcal{O}(S)_{h\mathrm{Aut}_V S}$.
(2) U does not generally *preserve* ℓ-connectivity of objects or morphisms for instance, given an $\ell \geq (k+1)$-connected space X, the equivalence $\Omega^k\colon \mathcal{S}_{*,\geq k+1} \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S}_{\geq 1})$ exhibits Ω^k as an ℓ-connected $\mathbb{E}_k$-algebra such that $U\Omega^n$ is only in general $(\ell-k)$-connected.
(3) For a similar reason, U does not usually reflect ℓ-truncatedness of morphisms or objects. ◁

2.2. **The proof of Theorem D.** We now begin to reduce Theorem D to the case $n \leq \ell+1$ with the following.

Lemma 48. *The truncation functor $\tau_{\leq \ell}\colon \mathcal{C} \to \tau_{\leq \ell}\mathcal{C}$ extends to a $\mathcal{T}$-functor*

$$\tau_{\mathcal{O}}\colon \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C})$$

satisfying $\tau_{\mathcal{O}} W_{S,X} = W_{S,\tau_{\mathcal{O}} X}$. Moreover, the inclusion $\iota\colon \tau_{\leq \ell}\mathcal{C} \to \mathcal{C}$ extends to a fully faithful $\mathcal{T}$-functor

$$\iota_{\mathcal{O}}\colon \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C}) \hookrightarrow \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C})$$

such that $\tau_{\mathcal{O}} W_{S,\iota_{\mathcal{O}} X} = W_{S,X}$.

Proof. Since $\tau_{\leq \ell}$ is product-preserving [HTT, Lem 6.5.1.2], $\tau_{\leq \ell}\colon \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C} \to \underline{\mathrm{Coeff}}^{\mathcal{T}} \tau_{\leq \ell \mathcal{C}}$ is a $\mathcal{T}$-symmetric monoidal left adjoint for the cartesian structure [Ste25b]; everything other than the equalities involving $W_{S,X}$ then follows straightforwardly from Proposition 36.

In particular, $\tau_{\mathcal{O}}$ is a $\mathcal{T}$-functor which preserves indexed products and coproducts; this implies that $\tau_{\mathcal{O}} W_{S,X} = W_{S,\tau_{\mathcal{O}} X}$. The remaining equality follows from fully faithfulness by noting that

$$\tau_{\mathcal{O}} W_{S,\iota_{\mathcal{O}} X} = W_{S,\tau_{\mathcal{O}} \iota_{\mathcal{O}} X} = W_{S,X}. \qquad \square$$

We say that a map $f\colon X \to Y$ in an n-topos is an *ℓ-equivalence* if it is a $\tau_{\leq \ell}$-equivalence; if f admits a section, this is equivalent to f being ℓ-connected (see [SY19, Prop 4.3.5] or note that this follows by splitting the long exact sequence in homotopy). We apply this by equivariantizing [SY19, Lem 5.1.1].

Lemma 49. *If $\mathcal{C}^{I-\times}$ is a Cartesian I-symmetric monoidal ∞-category and $S \in \underline{\mathbb{F}}_I$, then the image of the $\mathcal{O}$-algebra Wirthmüller map $W_{X,S}\colon \coprod_U^S X_U \to \prod_U^S X_U$ under $U\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_V \to \mathcal{C}_V$ admits a section.*

Given (Y_U) an S-tuple and $U \in \mathrm{Orb}(S)$ a distinguished orbit, choose the distinguished fixed point α whose induction is the following

$$\begin{tikzcd}
U \arrow[dr, dashed, "\alpha"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \mathrm{Ind}_U^{\mathcal{T}} \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

(see [Ste24] for the fact that this is indeed a summand inclusion). Let

$$\beta\colon Y_U \to \mathrm{Res}_U^V \mathrm{CoInd}_U^V Y_U \simeq \prod_W^{\mathrm{Res}_U^V \mathrm{Ind}_U^V *_U} \mathrm{CoInd}_W^U \mathrm{Res}_W^U Y_U$$

be the map whose corresponding map $\mathrm{Res}_W^U Y_U \to \mathrm{Res}_W^U Y_U$ is 0 when $W \neq \alpha$ and the identity otherwise. Let

$$\iota_U\colon Y_U \to \mathrm{Res}_U^V \prod_W^S Y_U \simeq \mathrm{Res}_U^V \mathrm{CoInd}_U^V Y_U \times \mathrm{Res}_U^V \prod_W^{S-U} Y_W$$

be the map corresponding with β on the first factor and 0 on the other. Let $i_U\colon Y_U \to \mathrm{Res}_U^V \coprod_{U'}^S Y_{U'}$ be adjunct to the inclusion $\mathrm{Ind}_U^V Y_U \hookrightarrow \coprod_{U'}^S Y_{U'}$.

Proof of Lemma 49. Fix some operation $\mu \in \mathcal{O}(S)$. We will verify that the following diagram commutes. Then, $\mu\sigma_1 f$ will be the desired section for $W_{X,S}$.

$$\begin{tikzcd}
\prod_U^S\left(\mathrm{Res}_U^V \coprod_U^V X_U\right) \arrow[r, "\sim"'] & \left(\coprod_U^S X_U\right)^{\times S} \arrow[r, "\mu"] \arrow[d, "h=\left(W_{\mathrm{Res}_U^V X, \mathrm{Res}_U^V S}\right)_{U\in \mathrm{Orb}(S)}"] & \coprod_U^S X_U \arrow[d, "W_{X,S}"] \\
& \left(\prod_U^S X_U\right)^{\times S} \arrow[r, "\mu"] \arrow[d, "\sigma_2"'] & \prod_U^S X_U \arrow[d, equal] \\
\prod_U^S X_U \arrow[uu, "f=(i_U)_{U\in \mathrm{Orb}(S)}"] \arrow[r, "g=(\iota_U)_{U\in\mathrm{Orb}(S)}"] \arrow[rr, equal, bend right] & \prod_U^S X_U^{\times \mathrm{Res}_U^V S} \arrow[r, "\mu"] & \prod_U^S X_U
\end{tikzcd}$$

Note that the top right square is commutative by the fact that $W_{S,X}$ is an $\mathcal{O}$-algebra morphism and the bottom right follows by unwinding the definition of μ.

Now, note that $\mu \circ g$ is the external product of a collection of endomorphisms $X_U \xrightarrow{\iota_U} X_U^{\times \mathrm{Res}_U^V S} \xrightarrow{\mu} X_U$; unwinding definitions, ι_U is the inclusion of a unit on all but one factor:

$$\begin{tikzcd}
X_U \arrow[r, "\iota_U"] \arrow[d, equal] & X_U^{\times \mathrm{Res}_U^V S} \arrow[r, "\mu"] \arrow[d, equal] & X_U \\
X_U \times \prod_W^{\mathrm{Res}_U^V S - \{\alpha\}} 1_W \arrow[r, "(\mathrm{id},\eta)"] & X_U \times \prod_W^{\mathrm{Res}_U^V S - \{\alpha\}} X_W \arrow[ur] &
\end{tikzcd}$$

in particular, $\mu \circ \iota_U$ is homotopic to the identity, so $\mu \circ g$ is homotopic to the identity, and the bottom triangle commutes.

To characterize the composite morphism of the left rectangle, we may equivalently characterize the composite map $\pi_U \sigma_2 h \sigma_1 f \colon \prod_U^S X_U \to \mathrm{CoInd}_U^V X_U^{\times \mathrm{Res}_U^V S}$; in fact, under the expression $X_U^{\times \mathrm{Res}_U^V S} \simeq \prod_W^{\mathrm{Res}_U^V S} \mathrm{Res}_W^U X_U$, it suffices to characterize the composite morphism $\prod_U^S X_U \to \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$ and verify that it is homotopic to the relevant projection of g for each W, U.

In particular, relevant projection of g is the composite morphism

$$\prod_U^S X_U \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\delta_{U,W}} \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$$

where $\delta_{U,W}$ is a Kronecker delta

$$\delta_{U,W} = \begin{cases} \mathrm{id} & U = W; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, note that the projection $\pi_U \sigma_2 h \sigma_1 \colon \prod_U^S X_U \to X_U^{\times \mathrm{Res}_U^V S}$ itself factors as

$$\prod_U^S \left(\mathrm{Res}_U^V \coprod_U^V X_U\right) \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\widetilde{f}_U} \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U,$$

so we're tasked with verifying that $\widetilde{f}_U$ is homotopic to $\delta_{U,W}$. Indeed, this follows by examining the following diagram:

$$\begin{tikzcd}
\prod\limits_U^S X_U \arrow[r, "f"] \arrow[d, twoheadrightarrow] & \prod\limits_U^S\left(\mathrm{Res}_U^V \coprod\limits_U^V X_U\right) \arrow[r, phantom, "\simeq"] \arrow[d, twoheadrightarrow] & \left(\coprod\limits_U^S X_U\right)^{\times S} \arrow[r, "h"] & \left(\prod\limits_U^S X_U\right)^{\times S} \arrow[r, phantom, "\simeq"] & \prod\limits_U^S X_U^{\times \mathrm{Res}_U^V S} \arrow[d, twoheadrightarrow] \\
\mathrm{CoInd}_U^V X_U \arrow[r, "\mathrm{CoInd}_U^V i_U"] \arrow[drrr, bend right, "\delta_{U,W}"] & \mathrm{CoInd}_U^V \mathrm{Res}_U^V \coprod\limits_U^V X_U \arrow[rr, "\mathrm{CoInd}_U^V W"] & & \mathrm{CoInd}_U^V \mathrm{Res}_U^V \prod\limits_U^V X_U \arrow[r, phantom, "\simeq"] \arrow[d, twoheadrightarrow] & X_U^{\times \mathrm{Res}_U^V S} \\
& & & \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U &
\end{tikzcd}$$

□

Proof of Theorem D. Assume $\mathcal{O}^\otimes$ is ℓ-connected at I, i.e. Condition (a). We study the behavior of $W_{S,X}$ under the following diagram:

$$\begin{tikzcd}
\underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C}) \arrow[r, "\iota_{\mathcal{O}}"] \arrow[d, "U_{\leq \ell}"] & \underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "L_{\mathcal{O}}"] \arrow[d, "U"] & \underline{\mathrm{Mon}}_{\mathcal{O}}(\tau_{\leq \ell}\mathcal{C}) \arrow[d, "U_{\leq \ell}"] \\
\underline{\mathrm{Coeff}}^{\mathcal{T}}\, \tau_{\leq \ell}\mathcal{C} \arrow[r, "\iota"] & \underline{\mathrm{Coeff}}^{\mathcal{T}}\, \mathcal{C} \arrow[r, "L"] & \underline{\mathrm{Coeff}}^{\mathcal{T}}\, \tau_{\leq \ell}\mathcal{C}
\end{tikzcd}$$

In particular, by Proposition 37 and Lemma 48, $L_{\mathcal{O}} W_{S,X} = W_{S,L_{\mathcal{O}}X}$ is an equivalence, so $U_{\leq \ell} L_{\mathcal{O}} W_{S,X} = LUW_{S,X}$ is an equivalence, i.e. $UW_{S,X}$ is an ℓ-equivalence. In turn, by Lemma 49 this implies that $UW_{S,X}$ is ℓ-connected, so Proposition 45 implies that $W_{S,X}$ is ℓ-connected, i.e. Condition (b).

The implication Condition (b) $\implies$ Condition (c) is immediate, so assume Condition (c), i.e. fix the case $\mathcal{C} := \mathcal{S}$ and assume that that $W_{S,X}$ is ℓ-connected for all $X \in \mathrm{Alg}_{\mathcal{O}}\mathcal{S}$ and $S \in \underline{\mathbb{F}}_I$. We may invert the above argument: this time, we find that $UW_{S,\iota_{\mathcal{O}}Y}$ is an ℓ-equivalence for all $Y \in \mathrm{Alg}_{\mathcal{O}}\mathcal{S}_{\leq \ell}$, so $LUW_{S,Y} = U_{\leq \ell} L_{\mathcal{O}} W_{S,\iota_{\mathcal{O}}Y} = U_{\leq \ell} W_{S,Y}$ is an equivalence. By conservativity of $U_{\leq \ell}$, this implies that $W_{S,Y}$ is an equivalence, so $\mathcal{O}^\otimes$ is ℓ-connected at I by Proposition 37, proving Condition (a). □

3. The C_p-operads $\mathbb{A}_{2,C_p}^\otimes$ and $\mathbb{A}_{2,C_p}^\otimes \overset{\mathrm{BV}}{\otimes} \mathbb{A}_{2,C_p}^\otimes$

For the rest of this article, we specialize to $\mathcal{T} = \mathcal{O}_{C_p}$, where C_p is the group of prime order p, and $\mathcal{C}$ is a 1-category. As in Proposition 22, let $\mathrm{Fr}_\Sigma(S)$ denote the free C_p-symmetric sequence on an operation in arity S. Now, the pointwise formula for left Kan extensions yields equivalences

$$\mathrm{Fr}_{\Sigma, p\cdot *_{C_p}}(*)(p \cdot *_e) \simeq \Sigma_p;$$
$$\mathrm{Fr}_{\Sigma, [C_p/e]}(*)(p \cdot *_e) \simeq \Sigma_p.$$

We define the C_p-symmetric sequence of sets F_{2,C_p} as the coequalizer

$$F_{2,C_p} := \mathrm{CoEq}\left(\Sigma_p[p \cdot *_e] \rightrightarrows \left(\mathrm{Fr}_{\Sigma,[C_p/e]}(*) \sqcup \mathrm{Fr}_{\Sigma, p\cdot *_{C_p}}(*)\right)\right),$$

where $\Sigma_p[p \cdot *_e]$ is the C_p-symmetric sequence defined by

$$\Sigma_p[p \cdot *_e](S) := \begin{cases} \Sigma_p & S = p \cdot *_e; \\ \varnothing & \text{otherwise.} \end{cases}$$

and the two arrows are the inclusions of $\Sigma_p[p \cdot *_e$. We define the unital C_p-operad $\mathbb{A}_{2,C_p}^\otimes$ by the Boardman-Vogt tensor product

$$\mathbb{A}_{2,C_p}^\otimes := \mathbb{E}_0^\otimes \overset{\mathrm{BV}}{\otimes} \mathrm{Fr}_{\mathrm{Op}}\left(F_{2,C_p}\right).$$

As promised, we verify that $\mathbb{A}_{2,C_p}$-monoids are the same as C_p-unital magmas.

Proposition 50. *There is an equivalence between $\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C})$ and C_p-unital magmas in $\mathcal{C}$.*

Proof. By Example 25 and Proposition 28 we have

$$\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})}\underline{\mathrm{Mon}}^{\otimes}_{\mathbb{E}_0}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})}\mathcal{C}_*.$$

Moreover, by Proposition 34, the data of an $\mathbb{A}_{2,C_p}$-monoid structure on $X \in \mathrm{Coeff}^{C_p}\mathcal{C}$ is equivalently viewed as a map $\eta\colon *_{C_p} \to X$ (which we identify with an element $\widetilde{X} \in \mathrm{Coeff}^{C_p}\mathcal{C}_*$) and an element of

$$\begin{aligned}\mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})}(\mathrm{End}_{\widetilde{X}}(\mathcal{C}_*))^{\simeq} &\simeq \mathrm{Hom}_{\mathrm{Fun}\left(\mathrm{Tot}\Sigma_{C_p},\mathcal{S}\right)}\left(F_{2,C_p},\mathrm{End}_{\widetilde{X}}(\mathcal{C}_*)\right)\\ &\simeq \mathrm{Hom}_{\mathrm{Coeff}^{C_p}\mathcal{C}_*}\left(\widetilde{X}^p,\widetilde{X}\right)\times_{\mathrm{Hom}_{\mathcal{C}_*}\left((\widetilde{X}^e)^p,\widetilde{X}^e\right)}\mathrm{Hom}_{\mathrm{Coeff}^{C_p}\mathcal{C}_*}\left(\mathrm{CoInd}_e^{C_p}\widetilde{X}^e,\widetilde{X}\right).\end{aligned}$$

We're left with interpreting this concretely: by a standard argument, $\mathrm{Hom}_{\mathrm{Coeff}^{C_p}\mathcal{C}_*}(\widetilde{X}^p,\widetilde{X})$ corresponds bijectively with the set of unital magma structures on X with unit η, and this corresponds bijectively with the pairs of unital magma structures on X^{C_p} and X^e with unit maps η^{C_p} and η^e such that the restriction map is a homomorphism. Under this bijection, the forgetful map $\mathrm{Hom}_{\mathrm{Coeff}^{C_p}\mathcal{C}_*}\left(\widetilde{X}^p,\widetilde{X}\right) \to \mathrm{Hom}_{\mathcal{C}_*}\left(\left(\widetilde{X}^e\right)^p,\widetilde{X}\right)$ simply forgets the data of X^{C_p} and the restriction.

Similarly, since C_p-coefficient coinduction is presented by the coefficient system $X^p \xleftarrow{\Delta} X$ with permutation action, $\mathrm{Hom}_{\mathrm{Coeff}^{C_p}\mathcal{C}^*}\left(\mathrm{CoInd}_e^{C_p}\widetilde{X}^e,\widetilde{X}\right)$ corresponds bijectively with the set of unital C_p-equivariant transfers $t\colon X^e \to X^{C_p}$ and unital magma structures on X^e with unit η^e satisfying the condition that the following diagram commutes.

$$\begin{tikzcd} X^e \arrow[r, "t"] \arrow[d, "\Delta"] & X^{C_p} \arrow[d, "r"] \\ (X^e)^p \arrow[r, "*"] & X^e \end{tikzcd}$$

Once again, the forgetful map restricts to the unital magama structure on η^e; thus the fiber product corresponds exactly with G-unital magma structures on X with units η^e and η^{C_p}.

Now, what we've described is a bijective assignment of *sets* $\mathrm{Ob}\,\mathrm{Mon}_{\mathbb{A}_{2,C_p}}(\mathcal{C}) \to \mathrm{Ob}\,\mathrm{Magma}^{\mathrm{uni}}_{C_p}(\mathcal{C})$ over $\mathrm{Ob}\,\mathcal{C}$. To conclude, it suffices to prove that a $\mathrm{Coeff}^{C_p}\mathcal{C}$ morphism between a pair of C_p-unital magmas is a C_p-unital magma homomorphism if and only if it's an $\mathbb{A}_{2,C_p}$-algebra homomorphism.

To prove this, note that an $\mathbb{A}_{2,C_p}$-monoid morphism is equivalently a $\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})$-monoid morphism of pointed objects, i.e. a pair of maps $F^e\colon M^e \to N^e$ and $F^{C_p}\colon M^{C_p} \to N^{C_p}$ which are compatible with units, satisfying $F^{C_p} \circ t = t \circ F^e$ and $F^e \circ r = r \circ F^{C_p}$ together with p-degree additivity

$$\begin{tikzcd} \left(M^{C_p}\right)^p \arrow[r] \arrow[d] & \left(N^{C_p}\right)^p \arrow[d] & (M^e)^p \arrow[r] \arrow[d] & (M^e)^p \arrow[d] \\ M^{C_p} \arrow[r] & N^{C_p} & M^e \arrow[r] & N^e \end{tikzcd}$$

It suffices to note that a map between the pointed sets underlying unital magmas is a homomorphism if and only if it intertwines with $n$ary addition for *some* $n \geq 2$; indeed, one can simply identify binary addition with n-ary addition whose first $(n-2)$-factors are the unit. □

We now spell out the interchange relations explicitly.

Proposition 51. *There is an equivalence between* $\mathrm{Mon}_{\mathbb{A}_{2,C_p}\otimes\mathbb{A}_{2,C_p}}(\mathcal{C})$ *and pairs of* G*-unital magma structures* $(M,*,\bullet,t_*,t_\bullet)$ *in* $\mathcal{C}$ *satisfying the interchange relations* $1_* = 1_\bullet$ *and*

$$\begin{tikzcd} (X^p)^p \arrow[r, "(\bullet)"] \arrow[d, "(*)"'] & X^p \arrow[d, "*"'] \\ X^p \arrow[r, "\bullet"] & X \end{tikzcd} \qquad \begin{tikzcd} X^{C_p} \arrow[d, "r"'] & X^e \arrow[l, "t_\bullet"'] \arrow[r, "t_*"] \arrow[d, "\Delta"'] & X^{C_p} \arrow[d, "r"] \\ X^e & (X^e)^p \arrow[l, "*"] \arrow[r, "\bullet"'] & X^e \end{tikzcd} \qquad \begin{tikzcd} (X^e)^p \arrow[r, "(t_\bullet)"] \arrow[d, "*"'] & \left(X^{C_p}\right)^p \arrow[d, "*"'] \\ X^e \arrow[r, "t_\bullet"'] & X^{C_p} \end{tikzcd} \qquad \begin{tikzcd} (X^e)^p \arrow[r, "(t_*)"] \arrow[d, "\bullet"'] & \left(X^{C_p}\right)^p \arrow[d, "\bullet"'] \\ X^e \arrow[r, "t_*"'] & X^{C_p} \end{tikzcd}$$

Proof. Example 25 and Proposition 28 yields an equivalence.

$$\mathrm{Mon}_{\mathbb{A}_{2,C_p}^{\otimes 2}}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Fr}_{\mathrm{Op}}(F_{2,C_p})^{\otimes 2}}(\mathcal{C}_*).$$

This is characterized explicitly by Corollary 35 and Proposition 50; it suffices to note that the specified interchange relations correspond precisely with the conditions that $t_\bullet$ and $\bullet$ are C_p-unital magma homomorphisms. □

We conclude the following form of Theorem A.

Corollary 52. *Given $\mathcal{C}$ a 1-category, the forgetful functor*

$$\begin{aligned}\mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{C_p}), \mathcal{C}) \longrightarrow & \mathrm{Mon}_{\mathbb{A}_{2,C_p} \otimes \mathbb{A}_{2,C_p}}(\mathcal{C}) \\ & \simeq \left\{ \textit{Interchanging pairs of } C_p\textit{-unital magmas in } \mathcal{C} \right\}\end{aligned}$$

is an equivalence of categories.

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