A directory of higher G-algebra

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Contents

The purpose of the following document is to serve as a directory to a large swath of the current knowledge concerning *genuine equivariantly homotopy coherent algebra*. The reader should interpret this as inherently limited to those papers which the contributors have happened to find, which at the present is just the author.

Remark. Throughout this text, we embrace a linguistic dichotomy regarding homotopy-coherent algebra: that which is founded in 1-categories with auxiliary structure (homotopical categories, model categories, etc.) is called *brave new algebra* (following Waldhausen's coinage), and that which is founded in ∞-categories is called *higher algebra* (following Lurie's coinage). Seeing as this text is not called "A directory of brave new G-algebra," we restrict our focus almost exclusively to equivariant higher algebra. $\triangleleft$

Any errors belong to the author. If any errata or missing references are found, please direct them to her, or ask her for edit permissions.

This is a work in progress!

Part I

HA index

Chapter 1

Stable ∞-categories

The major reference for this section is [Nar16].

1.1 Foundations

- HA.1.1.2.14 follows for the fibers of $\mathcal{T}$-categories by noting that $\mathcal{T}$-stability implies fiberwise stability [Nar16, Def 7.1].

1.2 Stable ∞-categories and homological algebra

I don't know of G-references for any of this section.

1.3 Homological algebra and derived categories

I don't know of G-references for any of this section.

1.4 Spectra and stabilization

- HA.1.4.1.10: I don't know of a reference for this.
- HA.1.4.2.14: I don't know of a reference for this.
- HA.1.4.2.17: this follows for $\mathcal{T}$ by [Nar16, Lem 7.2].
- HA.1.4.2.23: this follows for $\mathcal{T}$ by [Nar16, Thm 7.4].
- HA.1.4.2.27: this follows for $\mathcal{T}$ under the additional assumption that the category is $\mathcal{T}$-semiadditive by [NS22, Def 7.1].
- Section 1.4.3: I don't know of any references on equivariant t-structures.
- HA.1.4.4.2 may follow for $\mathcal{T}$ from the presentable-idempotents equivalence of [Hil24, Thm 6.5.4] combined with a version of the characterization of $\mathcal{T}$-stable Ind-completions of [Hil24, Prop 5.2.3]. In general, results from [Hil24, § 6] should help.
- HA 1.4.4.4-1.4.4.6 should probably follow by adding $\mathcal{T}$ to the HA proof using the $\mathcal{T}$-AFT of [Hil24, Thm 6.2.1]; it is worth noting that the $\mathcal{T}$-category of $\mathcal{T}$-spectra is probably presentable by a combination of [NS22, Thm 7.4] and a version of [Hil24, Lem 6.7.4] for linear functors.

Chapter 2

∞-operads

The major references for this section are [CH21; CH23; Hau23; NS22].

2.1 Foundations

Section 2.1 is relitigated for $\mathcal{T}$ in [NS22, § 2], and modernized in [BHS22; CH21; CH23].

2.2 Constructions of ∞-operads

- HA.2.2.1 for $\mathcal{T}$ is [NS22, § 2.9]
- HA.2.2.2 will appear in upcoming work of mine concerning the BV indexed tensor product.
- I don't know of a reference for HA.2.2.3.
- HA.2.2.4 for $\mathcal{T}$ is [NS22, § 2.8] and it is modernized in [BHS22].
- HA.2.2.5 appears in some guise in [Bar18]; I plan to cover this for $\mathcal{T}$ in upcoming work concerning BV tensor products of $\mathcal{N}_\infty$ and $\mathbb{E}_V$ G-operads. Nevertheless, the associated *internal hom* was constructed in [NS22, § 5.3], and a proof that it is left adjoint to the "obvious" definition of binary G-BV tensors appears in some lecture notes of mine.
- HA.2.2.6 for $\mathcal{T}$ is [NS22, § 3].

2.3 Disintegration and assembly

Approximation of ∞-operads is modernized in [BHS22, § 5], but otherwise I don't know references for HA.2.3.

2.4 Products and coproducts

1. HA.2.4.1: the construction of cartesian Symmetric monoidal structures for $\mathcal{T}$ appears as [NS22, Ex 2.4.1], but I don' know of an existing reference for HA 2.4.1.8-2.4.1.9; this will appear for I-operads in my upcoming work on $\mathcal{N}_\infty$-operads, where I is an indexing system.
2. I don't know of a reference for HA.2.4.2.5, but this is not hard to show in general.
3. HA.2.4.3.9 and its consequences will appear in upcoming work of ine about $\mathcal{N}_\infty$-operads.
4. I don't know of any reference for HA.2.4.4.

Chapter 3

Algebras and modules of ∞-operads

3.1 Free algebras

- HA.3.1.1-2 seems to be developed for $\mathcal{T}$ in [NS22, § 4].
- HA.3.1.3 seems to be developed for $\mathcal{T}$ as [NS22, Thm 4.3.4]; we will develop the practical computation of the monad associated with algebras in a distributive G-symmetric monoidal category in upcoming work in $\mathcal{N}_\infty$ operads (c.f. [SY19, Lem 2.4.2])

3.2 Limits and colimits of algebras

- HA 3.2.1 is [NS22, § 5.2].
- Much of HA 3.2.2-3.2.3 may be summarized by [NS22, Cor 5.1.5].
- HA 3.2.4.7 is kind of developed for I as [NS22, Cor 5.3.8], so long as "is cocartesian" is interpreted as "is a cocartesian structure;" we prove that this identifies the I-symmetric monoidal structure in upcoming work on $\mathcal{N}_\infty$-operads using an I-lift of HA 2.4.1.8-2.4.1.9.

3.3 Modules over ∞-operads

Upcoming!

3.4 General features of module ∞-categories

Upcoming!

Chapter 4

Associative algebras and their modules

(to be filled later, but it's mostly missing)

Chapter 5

Little cubes and factorizable sheaves

5.1 Definitions and basic properties

- I know of neither a reference nor a precise statement for HA.5.1.1.1.
- HA.5.1.1.4 is lifted to $\mathbb{E}_V$ as the main theorem of [Ste24], and HA.5.1.1.5 immediately follows.
- HA.5.1.1.7 is lifted to $\mathbb{E}_V$ in [Ste24].
- I know of no (homotopical) reference for HA.5.1.2.1, and it is likely to be quite difficult; I expect to resolve this in the spectral setting in upcoming work.
- I know of no reference for HA.5.1.3-HA.5.1.5

5.2 Bar constructions and Koszul duality

(to be filled later, but it's mostly missing)

5.3 Centers and centralizers

(to be filled later, but it's mostly missing)

5.4 Little cubes and manifold topology

(to be filled in later, see [Hor19])

5.5 Topological Chiral homology

(to be filled in later, see [Hor19])

Chapter 6

The calculus of functors

(to be filled later, but it's mostly missing)

Chapter 7

Algebra in the stable homotopy category

(to be filled in later, but it's mostly upcoming)

Bibliography

[BHS22] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. 2022. arXiv: 2208.07183 [math.CT] (cit. on p. 6).

[Bar18] Clark Barwick. "From operator categories to higher operads". In: *Geom. Topol.* 22.4 (2018), pp. 1893–1959. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2018.22.1893. URL: https://doi.org/10.2140/gt.2018.22.1893 (cit. on p. 6).

[CH21] Hongyi Chu and Rune Haugseng. "Homotopy-coherent algebra via Segal conditions". In: *Adv. Math.* 385 (2021), Paper No. 107733, 95. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2021.107733. URL: https://arxiv.org/abs/1907.03977 (cit. on p. 6).

[CH23] Hongyi Chu and Rune Haugseng. *Enriched homotopy-coherent structures*. 2023. arXiv: 2308.11502 [math.CT] (cit. on p. 6).

[Hau23] Rune Haugseng. *An allegendly somewhat friendly introduction to ∞-operads*. 2023. URL: https://folk.ntnu.no/runegha/iopd.pdf (cit. on p. 6).

[HH16] Michael A. Hill and Michael J. Hopkins. *Equivariant symmetric monoidal structures*. 2016. arXiv: 1610.03114 [math.AT] (cit. on p. 15).

[Hil24] Kaif Hilman. *Parametrised Presentability over Orbital Categories*. 2024. arXiv: 2202.02594 [math.AT] (cit. on p. 5).

[Hor19] Asaf Horev. *Genuine equivariant factorization homology*. 2019. arXiv: 1910.07226 [math.AT] (cit. on p. 9).

[Nar16] Denis Nardin. *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category*. 2016. arXiv: 1608.07704 [math.AT] (cit. on p. 5).

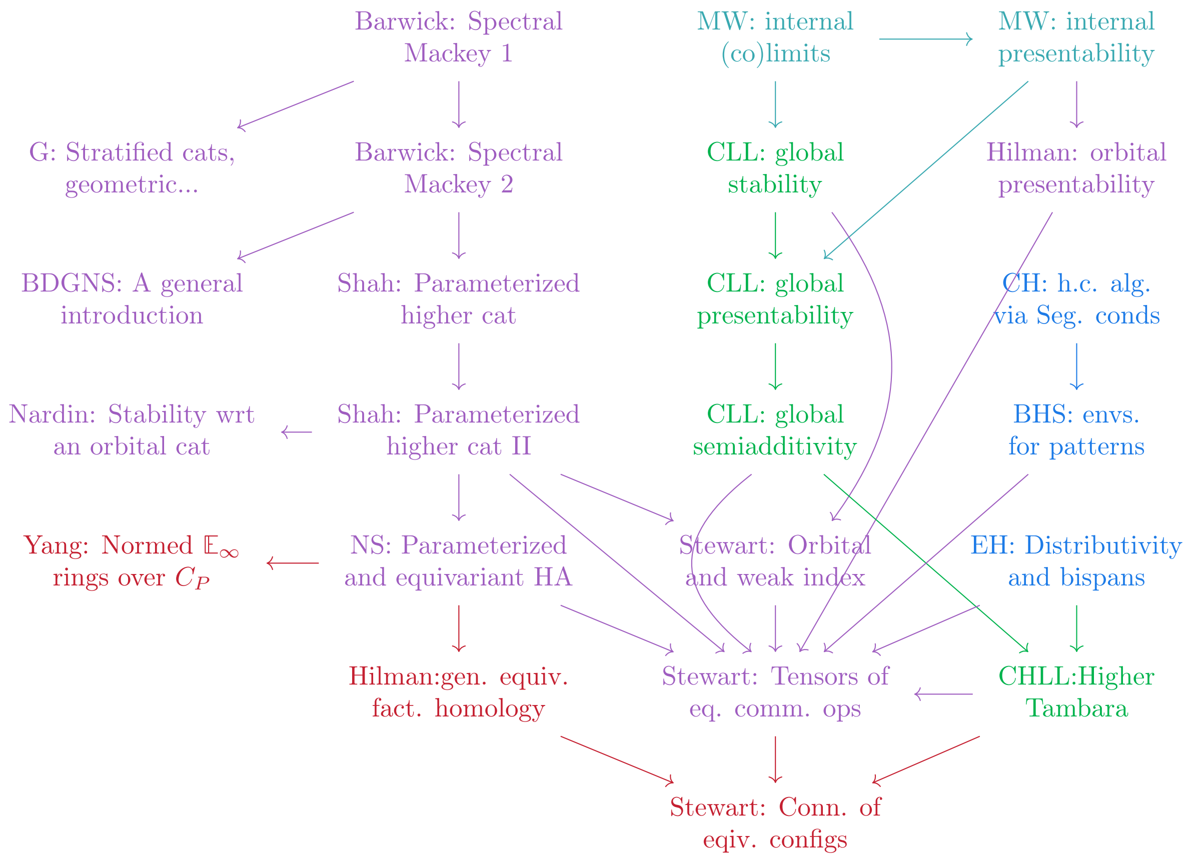
[NS22] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. arXiv: 2203.00072 [math.AT] (cit. on pp. 5–7).

[SY19] Tomer M. Schlank and Lior Yanovski. "The ∞-categorical Eckmann-Hilton argument". In: *Algebr. Geom. Topol.* 19.6 (2019), pp. 3119–3170. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2019.19.3119. URL: https://doi.org/10.2140/agt.2019.19.3119 (cit. on p. 7).

[Ste24] Natalie Stewart. *On connectivity of spaces of equivariant configurations (draft)*. 2024. URL: https://nataliesstewart.github.io/files/Conf_draft.pdf (cit. on p. 9).

Chapter 8

A graph of dependencies in the literature



Part II

The work of Barwick-Dotto-Glasman-Nardin-Shah

The ∞-categorical perspective on equivariant homotopy perhaps first began in the two papers of Barwick and Barwick-Glasman-Shah on spectral Mackey functors (i.e. G-spectra, when G is finite). This is where we first see the *effective burnside category* come clearly into view in ∞-category theory, defined functorially for arbitrary *disjunctive triples*, and with compatibility with certain fibrations via the *unfurling construction*.

Following this, the writing of book was declared by Barwick-Dotto-Glasman-Nardin-Shah, fleshing out Hill's program (c.f. [HH16]) for equivariant algebra in the ∞-categorical perspective, to be distributed first as a sequence of 9 exposés. In retrospect, around half of these were finished, and it doesn't appear that the members are still publishing papers are still publishing in stable equivariant homotopy.

Nevertheless, several immensefully useful papers were developed; of particular note are Glasman's paper on isotropy separation over (epi)orbital categories, Shah's two papers thoroughly covering equivariant higher category theory, Nardin's paper on equivariant stability, and Nardin-Shah's paper covering some rudiments of equivariant operads.

SPECTRAL MACKEY FUNCTORS AND EQUIVARIANT ALGEBRAIC K-THEORY (I)

CLARK BARWICK

For my dear friend Dan Kan.

Abstract. *Spectral Mackey functors* are homotopy-coherent versions of ordinary Mackey functors as defined by Dress. We show that they can be described as excisive functors on a suitable ∞-category, and we use this to show that universal examples of these objects are given by algebraic K-theory.

More importantly, we introduce the *unfurling* of certain families of Waldhausen ∞-categories bound together with suitable adjoint pairs of functors; this construction completely solves the homotopy coherence problem that arises when one wishes to study the algebraic K-theory of such objects as spectral Mackey functors.

Finally, we employ this technology to lay the foundations of equivariant stable homotopy theory for profinite groups; the lack of such foundations has been a serious impediment to progress on the conjectures of Gunnar Carlsson. We also study fully functorial versions of A-theory, upside-down A-theory, and the algebraic K-theory of derived stacks.

Contents

0. Summary

This paper lays the foundations of what might be called *axiomatic derived representation theory*. Inspired by Bert Guillou and Peter May [11, 12, 13] and Dmitry Kaledin [18], we construct here a very general homotopy theory of *spectral Mackey functors* — families of spectra equipped with operations that mirror the restriction and induction operations found in ordinary Mackey functors. Our theory of spectral Mackey functors accounts for all of the compositions of these operations, their homotopies, their homotopies between homotopies, etc.

The basic input is an ∞-category C with two subcategories, $C_{\dagger} \subset C$, whose maps we call *ingressive*, and $C^{\dagger} \subset C$, whose maps we call *egressive*. We require that ingressive and egressive maps are stable under pullback, and we require that C admit finite coproducts that act effectively as disjoint unions (Df. 5.2). We call this a *disjunctive triple* $(C, C_{\dagger}, C^{\dagger})$.

Example. The ordinary category of finite continuous G-sets for a profinite group G defines a disjunctive triple in which every morphism is both ingressive and egressive.

Example. The ordinary category of varieties over a field defines a disjunctive triple in which every morphism is egressive but only flat and proper maps are ingressive.

Example. The ∞-category of spaces defines a disjunctive triple in which every morphism is ingressive but only morphisms with finite (homotopy) fibers are egressive.

The names "ingressive" and "egressive" are meant to suggest functorialities: a spectral Mackey functor M on C should consist of a covariant functor

$$M_{\star} \colon C_{\dagger} \longrightarrow \mathbf{Sp}$$

and a contravariant functor

$$M^{\star} \colon (C^{\dagger})^{op} \longrightarrow \mathbf{Sp},$$

each valued in the ∞-category $\mathbf{Sp}$ of spectra. These functors will be required to carry coproducts to wedges of spectra. These two functors will agree on objects, so that given a map $f \colon X \longrightarrow Y$, we obtain a *pullback* map

$$f^{\star} \colon M(Y) = M^{\star}(Y) \longrightarrow M^{\star}(X) = M(X)$$

and a *pushforward* map

$$f_{\star} \colon M(X) = M_{\star}(X) \longrightarrow M_{\star}(Y) = M(Y).$$

Furthermore, the pullback and pushforward maps are required to satisfy a *base change condition*; namely, for any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

(with abusively named morphisms), we require that

$$g^{\star} f_{\star} \simeq f_{\star} g^{\star}.$$

But this description won't quite do as a definition of Mackey functors. After all, the base change condition is no condition at all: the homotopies $g^{\star} f_{\star} \simeq f_{\star} g^{\star}$

are additional *data*, and these data have to satisfy additional coherences, which themselves are homotopies that in turn have to satisfy further coherences, etc., etc., ad infinitum. To encode all this data efficiently, we define the *effective Burnside ∞-category* $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ (Df. 5.7). The objects of this ∞-category are the objects of C, and a morphism from X to Y in $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ is a span

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & & Y, \end{array}$$

in C in which $U \twoheadrightarrow X$ is egressive and $U \rightarrowtail Y$ is ingressive. Composition is then defined by forming pullbacks. Of course, pullbacks are only unique up to a contractible choice, so composition in $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ is only defined up to a contractible choice. This is no cause for concern, however, as this is exactly the sort of thing ∞-categories were designed to handle gracefully. In particular, even when C is an ordinary category, $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ typically won't be.

Now the ∞-category $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ has direct sums (Pr. 4.3 and 5.8), which are given by the coproduct in C, and a *spectral Mackey functor* on $(C, C_\dagger, C^\dagger)$ is a functor

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Sp}$$

that carries this direct sum to the wedge of spectra (Df. 6.1). When C is the category of finite G-sets for some finite group G, a spectral Mackey functor on C is tantamount to a genuine G-equivariant spectrum. (This is a theorem of Guillou and May [12].) If we replace the ∞-category of spectra in this discussion with an ordinary abelian category A, then we recover the usual notion of Mackey functors for G with values in A in the sense of Dress [4]. So the homotopy groups of a spectral Mackey functor form an ordinary Mackey functor in abelian groups, just as one would expect. (We will actually formulate our definition in terms of general target additive ∞-categories.)

If C is the ordinary category of finite G-sets for some finite group G, the homotopy category $hA^{\mathit{eff}}(C)$ of the effective Burnside ∞-category is not quite what one would typically call the Burnside category. Rather, the Burnside category is obtained by forming the group completion of the Hom sets — the "local group completion" — of $A^{\mathit{eff}}(C)$. (This is the significance of the word "effective" here; it's meant as a loose pun on the phrase "effective divisor.") Since our target ∞-categories will be additive and thus already locally group complete, however, the local group completion of $A^{\mathit{eff}}(C)$ is a layer of complication we can do without.

The main result of this paper is the discovery that there is a deep connection between algebraic K-theory and the homotopy theory of spectral Mackey functors. We show that spectral Mackey functors can be described as excisive functors from a certain "derived Burnside ∞-category" (Lm. 7.3). This identification allows us to use the Goodwillie differential [10] to construct *Mackey stabilizations* of functors from $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ to the ∞-category **Kan** of spaces (Pr. 7.4). It turns out that this use of the Goodwillie differential can in important cases be related (§9) to the use of the Goodwillie differential in our characterization of algebraic K-theory [2]. This little observation permits us to express *representable* Mackey functors $\mathbf{S}^X$ as *equivariant algebraic K-theory spectra.* The spectral Mackey functor $\mathbf{S}$ represented by the terminal object 1 is called the *Burnside Mackey functor* (Df. 8.1); it is the analog of the sphere spectrum in this context. (In a sequel to this work, we will show that in fact it is the unit for the natural symmetric monoidal structure on Mackey functors.) Our formula for the Mackey stabilization gives us a K-theoretic interpretation of this Mackey functor (§13).

More importantly, we solve the central homotopy coherence problem of equivariant algebraic K-theory: we prove that the algebraic K-theories of a family of Waldhausen ∞-categories [2] connected by suitable adjoint pairs of functors together define a spectral Mackey functor. This we do via an operation we call *unfurling* (Df. 11.3). The resulting structure provides a complete accounting for the functorialities enjoyed by the algebraic K-theory of such a family of Waldhausen ∞-categories.

The structure of a spectral Mackey functor is a very rich one, and it may be difficult to appreciate its utility from the abstract formalism alone. Therefore, in a sequence of appendices, we have the pleasure of studying our examples in some detail. Readers may find it useful to flip back and forth between bits of the body of the paper and these examples.

(A) The full subcategory of a coherent n-topos in the sense of Lurie ($1 \leq n \leq \infty$) spanned by the coherent objects is a disjunctive triple in which every morphism is both ingressive and egressive (Ex. A.2). If, moreover, every coherent object can be written as a coproduct of finitely many *connected objects* (Df. A.7), then our K-theoretic description of the Burnside Mackey functor $\mathbf{S}$ gives us a formula:

$$\mathbf{S}(1) \simeq \bigvee_X \Sigma^\infty_+ B\,\mathrm{Aut}(X),$$

where the wedge is taken over all equivalence classes of connected objects X, and $\mathrm{Aut}(X)$ denotes the space of auto-equivalences of X (Th. A.9). This is a very general form of the *Segal–tom Dieck splitting*.

(B) Suppose G a profinite group. Then the (nerve of the) ordinary category of finite continuous G-sets (i.e., finite sets with an action of G whose stabilizers are all open) is an example of the kind above. In particular, we may speak of spectral Mackey functors for G. When G is finite, it follows from work of Bert Guillou and Peter May [12] that the homotopy theory of spectral Mackey functors is equivalent to the homotopy theory of G-equivariant spectra in the sense of Lewis–May–Steinberger [24], Mandell–May [32], and Hill–Hopkins–Ravenel [14, 16, 15]. For general profinite groups, we believe that our ∞-category of spectral Mackey functors for G is the first definition of the homotopy theory of G-equivariant spectra. (The lack of foundations for such a subject has been a serious impediment to real progress on the conjectures of Gunnar Carlsson.) The generalized Segal–tom Dieck splitting (Th. A.9) gives a formula for the G-fixed points of the G-equivariant sphere spectrum:

$$\mathbf{S}^G_G \simeq \bigvee_H \Sigma^\infty_+ B(N_G H/H),$$

where the wedge is taken over conjugacy classes of open subgroups $H \leq G$, and $N_G H$ denotes the normalizer of H in G (Pr. B.8).

(C) For any space X, there are two kinds of conditions one might impose on a retractive spaces $X \longrightarrow X' \longrightarrow X$ (Nt. C.9): one could demand that X' is a retract of a finite CW complex (or, alternately, a finite CW complex itself); alternately, one could demand that the homotopy fibers of $X' \longrightarrow X$ are retracts of finite CW complexes (or, alternately, finite CW complexes). The algebraic K-theories of these ∞-categories are spectra denoted $\mathbf{A}(X)$ and $\mathbf{V}(X)$. The former is covariantly functorial and the latter is contravariantly functorial. But each possesses additional functorialities: if $X \longrightarrow Y$ is a map whose homotopy fibers are retracts of finite CW complexes, then we obtain umkehr maps $\mathbf{A}(Y) \longrightarrow \mathbf{A}(X)$ and $\mathbf{V}(X) \longrightarrow \mathbf{V}(Y)$. We can assemble these functorialities together to get Mackey functors $\mathbf{A}$ (respectively, $\mathbf{V}$) for the disjunctive triple given by the ∞-category of spaces in which every map is ingressive (resp., egressive), and only those maps whose homotopy fibers are retracts of finite CW complexes are egressive (resp., ingressive) (Nt. C.11). This provides a host of interesting *assembly maps* (C.12)

$$\mathbf{A}^{\oplus}(X,Y) \wedge \mathbf{A}(X) \longrightarrow \mathbf{A}(Y) \quad \text{and} \quad \mathbf{A}^{\oplus}(X,Y) \wedge \mathbf{V}(Y) \longrightarrow \mathbf{V}(X),$$

where $\mathbf{A}^{\oplus}(X,Y)$ is the group completion of the E_∞ space of diagrams

$$X \longleftarrow U \longrightarrow Y,$$

where the homotopy fibers of $U \longrightarrow X$ are finite CW complexes, and the E_∞ structure is given by coproduct. When $X = *$, the maps $\mathbf{S} \longrightarrow \mathbf{A}(*) \longrightarrow \mathbf{S}$ can be composed with these assembly maps to obtain, for any retract U of a finite CW complex, maps

$$\Sigma^\infty_+ \operatorname{Map}(U,Y) \longrightarrow \mathbf{A}(Y) \quad \text{and} \quad \Sigma^\infty_+ \operatorname{Map}(U,Y) \longrightarrow D\mathbf{V}(Y).$$

Dually, when $Y = *$, we obtain, for any space V, maps

$$\Sigma^\infty_+ \operatorname{Map}_{\mathrm{rc}}(X,V) \longrightarrow D\mathbf{A}(X) \quad \text{and} \quad \Sigma^\infty_+ \operatorname{Map}_{\mathrm{rc}}(X,V) \longrightarrow \mathbf{V}(X),$$

where $\operatorname{Map}_{\mathrm{rc}}$ denotes the space of maps whose homotopy fibers are retracts of finite CW complexes. Special cases of these maps have been constructed by Waldhausen, Malkiewich, and others. Moreover, it turns out that all of this holds when the ∞-category of spaces is replaced with any compactly generated ∞-topos whose terminal object is compact.

(D) In [2, §12], we defined the algebraic K-theory of derived stacks, and we observed there that it was contravariantly functorial in morphisms of (nonconnective) spectral Deligne–Mumford stacks. Here we push this further by including the covariant functoriality of the algebraic K-theory of spectral Deligne–Mumford stacks in certain and its compatibility with the contravariant functoriality. More precisely, we construct two disjunctive triples of derived stacks. The first is the ∞-category of spectral Deligne–Mumford stacks in which we declare that every morphism is egressive, and a morphism is ingressive if and only if it is strongly proper, of finite Tor-amplitude, and locally of finite presentation (D.17). The second is the ∞-category of all flat sheaves (of spaces) on the ∞-category of connective E_∞ rings in which we declare that every morphism is egressive, and a morphism is ingressive if and only if it is quasi-affine representable and *perfect* in the sense that the pushforward of the structure sheaf is perfect (Pr. D.20). We prove that algebraic K-theory is a spectral Mackey functor for each of these disjunctive triples (Nt. D.19 and Nt. D.22). This can be thought of as a very general and very structured form of proper base change for K-theory. It also ensures the existence of interesting *assembly maps*

$$\mathbf{A}^{\oplus}_{\mathrm{DM}}(X,Y) \wedge \mathbf{K}(X) \longrightarrow \mathbf{K}(Y) \quad (\text{respectively, } \mathbf{A}^{\oplus}_{\mathrm{Shv}}(X,Y) \wedge \mathbf{K}(X) \longrightarrow \mathbf{K}(Y)\ ),$$

where $\mathbf{A}^{\oplus}_{\mathrm{DM}}(X, Y)$ (resp., $\mathbf{A}^{\oplus}_{\mathrm{Shv}}(X, Y)$) is the group completion of the E_∞ space of diagrams

$$X \longleftarrow U \longrightarrow Y$$

of spectral Deligne–Mumford stacks (resp., of flat sheaves), where $U \longrightarrow Y$ is strongly proper, of finite Tor-amplitude, and locally of finite presentation (resp., quasi-affine representable and perfect), and the E_∞ structure is given by coproduct. When $X = \operatorname{Spec} \mathbf{S}$, the maps $\mathbf{S} \longrightarrow \mathbf{K}(\mathbf{S}) \longrightarrow \mathbf{S}$ can be composed with these assembly maps to obtain, for any spectral Deligne–Mumford stack (resp., any flat sheaf) U, maps

$$\Sigma^\infty_+ \operatorname{Map}_{\mathrm{pr}}(U, Y) \longrightarrow \mathbf{K}(Y) \quad \text{and} \quad \Sigma^\infty_+ \operatorname{Map}_{\mathrm{perf}}(U, Y) \longrightarrow \mathbf{K}(Y),$$

where $\operatorname{Map}_{\mathrm{pr}}(U, Y)$ (resp., $\operatorname{Map}_{\mathrm{perf}}(U, Y)$) denotes the space of morphisms $U \longrightarrow Y$ that are strongly proper, of finite Tor-amplitude, and locally of finite presentation (resp., quasi-affine representable and perfect). Dually, when $Y = *$, we obtain, for any spectral Deligne–Mumford stack (resp., any flat sheaf) V that is strongly proper, of finite Tor-amplitude, and locally of finite presentation (resp., quasi-affine representable and perfect) over $\operatorname{Spec} \mathbf{S}$, maps

$$\Sigma^\infty_+ \operatorname{Map}(V, X) \longrightarrow D\mathbf{K}(X) \quad \text{and} \quad \Sigma^\infty_+ \operatorname{Map}(V, X) \longrightarrow D\mathbf{K}(X).$$

Restricting this to étale covers of a fixed (nice) scheme X with a geometric point $x \in X(\Omega)$, we obtain the *Galois equivariant K-theory spectrum* (Ex. D.24)

$$K_{\pi_1^{\text{ét}}(X,x)}(X) \colon B^{fin}_{\pi_1^{\text{ét}}(X,x)} \longrightarrow \mathbf{Sp}.$$

In a very precise manner, this object encodes the failure of K-theory to satisfy Galois descent; this is the subject of a series of conjectures of Gunnar Carlsson [3], which can now be formulated thanks to the foundations we provide here. We intend to study this object in great detail in future work.

Related work. It is probably safe to say that others have anticipated a fully ∞-categorical (and thus "model-independent") construction of equivariant stable homotopy theory, but, as always, the devil is in the details. We believe this is the first ∞-categorical approach to equivariant stable homotopy theory, and it is the first to provide a complete construction of equivariant algebraic K-theory of families of Waldhausen ∞-categories of the kind described above. There are, however, a number of precursors to this paper.

In a brilliant series of papers, Bert Guillou and Peter May construct [11, 12, 13] a homotopy theory of spectral Mackey functors, and they show that for a finite group G, the homotopy theory of spectral Mackey functors for finite G is equivalent to the homotopy theory of genuine G-spectra. Our homotopy theory of spectral Mackey functors for finite G-sets is easily seen to be equivalent to theirs.

Dmitry Kaledin has also developed [18] a theory of "derived Mackey functors" (again for finite groups G). Our work here is a generalization of Kaledin's: the homotopy category of Mackey functors valued in the derived ∞-category of abelian groups is naturally equivalent to his derived category of Mackey functors.

Aderemi Kuku, extending work of Andreas Dress [5], has worked for many years on the foundations of equivariant higher algebraic K-theory as a construction that yields ordinary abelian group-valued Mackey functors (partly joint with Dress) [6, 7, 19, 20, 21, 22, 23]. For Mackey functors for finite groups, our work can be understood as lifting the target of Kuku's constructions to the ∞-category of spectrum-valued Mackey functors.

Work of Shimakawa [34, 35, 36, 37, 38] shows that the K-theory of permutative categories with a suitable action of a finite group can be given the structure of a genuine G-spectrum. The work here amounts to the generalization of this result to the context of Waldhausen K-theory for ∞-categories with suitable G-actions.

Mona Merling has an alternate approach to constructing G-spectra from Waldhausen categories with a suitable action of a finite group G. It is possible that her approach and the one given here are suitably equivalent; however, it seems that the two approaches differ significantly in the details, and Merling's appears to be adapted to the technology of equivariant homotopy theory as developed by Lewis–May–Steinberger [24], Mandell–May [32], and Hill–Hopkins–Ravenel [14, 16, 15].

Acknowledgments. I have had very helpful conversations with David Ayala and Haynes Miller about the contents of this paper and its sequels.

1. Preliminaries on ∞-categories

In general, we use the terminology from [25, 2, 1]. We review some of the relevant notation here.

1.1. **Notation.** In order to deal gracefully with size issues, we'll use Grothendieck universes in this paper. In particular, we fix, once and for all, three uncountable, strongly inaccessible cardinals $\kappa_0 < \kappa_1 < \kappa_2$ and the corresponding universes $\mathbf{V}_{\kappa_0} \in \mathbf{V}_{\kappa_1} \in \mathbf{V}_{\kappa_2}$. Now a set, simplicial set, category, etc., will be said to be ***small*** if it is contained in the universe $\mathbf{V}_{\kappa_0}$; it will be said to be ***large*** if it is contained in the universe $\mathbf{V}_{\kappa_1}$; and it will be said to be ***huge*** if it is contained in the universe $\mathbf{V}_{\kappa_2}$. We will say that a set, simplicial set, category, etc., is ***essentially small*** if it is equivalent (in the appropriate sense) to a small one.

1.2**.** The model of ∞-categories we will employ is Joyal's model of *quasicategories*, which we will here call ***∞-categories***. We refer systematically to [25] for details about this model of higher categories.

1.3. **Notation.** A ***simplicial category*** — that is, a category enriched in the category of simplicial sets — will frequently be denoted with a superscript $(-)^{\Delta}$.

Suppose $\mathbf{C}^{\Delta}$ a simplicial category. Then we write $(\mathbf{C}^{\Delta})_0$ for the ordinary category given by taking the 0-simplices of the Mor spaces. That is, $(\mathbf{C}^{\Delta})_0$ is the category whose objects are the objects of $\mathbf{C}$, and whose morphisms are given by

$$(\mathbf{C}^{\Delta})_0(x,y) := \mathbf{C}^{\Delta}(x,y)_0.$$

If the Mor spaces of $\mathbf{C}^{\Delta}$ are all fibrant, then we will often write

$$\mathbf{C} \quad \text{for the simplicial nerve} \quad N(\mathbf{C}^{\Delta})$$

[25, Df. 1.1.5.5], which is an ∞-category [25, Pr. 1.1.5.10].

1.4. **Notation.** For any ∞-category A, there exists a simplicial subset $\iota A \subset A$, which is the largest Kan simplicial subset of A [25, 1.2.5.3]. We shall call this space the ***interior ∞-groupoid of*** A. The assignment $A \longmapsto \iota A$ defines a right adjoint ι to the inclusion functor u from Kan simplicial sets to ∞-categories.

1.5. **Notation.** The large simplicial category $\mathbf{Kan}^{\Delta}$ is the category of small Kan simplicial sets, with the usual notion of mapping space. The large simplicial category $\mathbf{Cat}_{\infty}^{\Delta}$ is defined in the following manner [25, Df. 3.0.0.1]. The objects of $\mathbf{Cat}_{\infty}^{\Delta}$ are small ∞-categories, and for any two ∞-categories A and B, the morphism space

$$\mathbf{Cat}_{\infty}^{\Delta}(A,B) := \iota\,\mathrm{Fun}(A,B)$$

is the interior ∞-groupoid of the ∞-category $\mathrm{Fun}(A,B)$.

Similarly, for any strongly inaccessible cardinal τ, we may define the locally τ-small simplicial category $\mathbf{Kan}(\tau)^{\Delta}$ of τ-small simplicial sets and the locally τ-small simplicial category $\mathbf{Cat}_{\infty}(\tau)^{\Delta}$ of τ-small ∞-categories.

2. The twisted arrow ∞-category

We have elsewhere [] spoken of the twisted arrow ∞-category of an ∞-category. Let us recall the basic facts here.

2.1. **Proposition.** *The following are equivalent for a functor $\theta\colon \Delta \longrightarrow \Delta$.*

(2.1.1) The functor $\theta^{op}\colon N\Delta^{op} \longrightarrow N\Delta^{op}$ is cofinal in the sense of Joyal [25, Df. 4.1.1.1].

(2.1.2) The induced endofunctor $\theta^{\star}\colon s\mathbf{Set} \longrightarrow s\mathbf{Set}$ on the ordinary category of simplicial sets (so that $(\theta^{\star}X)_n = X_{\theta(n)}$) carries every standard simplex Δ^m to a weakly contractible simplicial set.

(2.1.3) The induced endofunctor $\theta^{\star}\colon s\mathbf{Set} \longrightarrow s\mathbf{Set}$ on the ordinary category of simplicial sets is a left Quillen functor for the usual Quillen model structure.

Proof. By Joyal's variant of Quillen's Theorem A [25, Th. 4.1.3.1], the functor θ^{op} is cofinal just in case, for any integer $m \geq 0$, the nerve $N(\theta/\mathbf{m})$ is weakly contractible. The category $(\theta/\mathbf{m})$ is clearly equivalent to the category of simplices of $\theta^{\star}(\Delta^m)$, whose nerve is weakly equivalent to $\theta^{\star}(\Delta^m)$. This proves the equivalence of the first two conditions.

It is clear that for any functor $\theta\colon \Delta \longrightarrow \Delta$, the induced functor $\theta^{\star}\colon s\mathbf{Set} \longrightarrow s\mathbf{Set}$ preserves monomorphisms. Hence $\theta^{\star}$ is left Quillen just in case it preserves weak equivalences. Hence if $\theta^{\star}$ is left Quillen, then it carries the map $\Delta^n \xrightarrow{\sim} \Delta^0$ to an equivalence $\theta^{\star}\Delta^n \xrightarrow{\sim} \theta^{\star}\Delta^0 \cong \Delta^0$, and, conversely, if $\theta^{op}\colon N\Delta^{op} \longrightarrow N\Delta^{op}$ is cofinal, then for any weak equivalence $X \xrightarrow{\sim} Y$, the induced map $\theta^{\star}X \longrightarrow \theta^{\star}Y$ factors as

$$\begin{aligned} \theta^{\star}X \simeq \operatorname{hocolim}_n X_{\theta(n)} &\simeq \operatorname{hocolim}_n P_N \\ &\simeq X \\ &\xrightarrow{\sim} Y \\ &\simeq \operatorname{hocolim}_n Y_n \\ &\simeq \operatorname{hocolim}_n Y_{\theta(n)} \simeq \theta^{\star}Y, \end{aligned}$$

which is a weak equivalence. This proves the equivalence of the third condition with the first two. □

2.2. One may call any functor $\theta\colon \Delta \longrightarrow \Delta$ satisfying the equivalent conditions above a ***combinatorial subdivision***. Work of Katerina Velcheva shows that in fact combinatorial subdivisions can be classified: they are all iterated joins of id and op. The example in which we are interested, the join $op \star \mathrm{id}$, is originally due to Segal.

2.3. **Notation.** Denote by $\epsilon\colon \Delta \longrightarrow \Delta$ the combinatorial subdivision

$$[\mathbf{n}] \longmapsto [\mathbf{n}]^{op} \star [\mathbf{n}] \cong [\mathbf{2n+1}].$$

Including $[\mathbf{n}]$ into either factor of the join $[\mathbf{n}]^{op} \star [\mathbf{n}]$ (either contravariantly or covariantly) defines two natural transformations $op \longrightarrow \epsilon$ and $\mathrm{id} \longrightarrow \epsilon$. Precomposition with ϵ induces an endofunctor $\epsilon^{\star}$ on the ordinary category of simplicial sets, together with natural transformations $\epsilon^{\star} \longrightarrow op$ and $\epsilon^{\star} \longrightarrow \mathrm{id}$.

For any simplicial set X, the ***edgewise subdivision*** of X is the simplicial set

$$\widetilde{\mathscr{O}}(X) := \epsilon^{\star} X.$$

That is, $\widetilde{\mathscr{O}}(X)$ is given by the formula

$$\widetilde{\mathscr{O}}(X)_n = \mathrm{Mor}(\Delta^{n,op} \star \Delta^n, X) \cong X_{2n+1}.$$

The two natural transformations described above give rise to a morphism

$$\widetilde{\mathscr{O}}(X) \longrightarrow X^{op} \times X,$$

functorial in X.

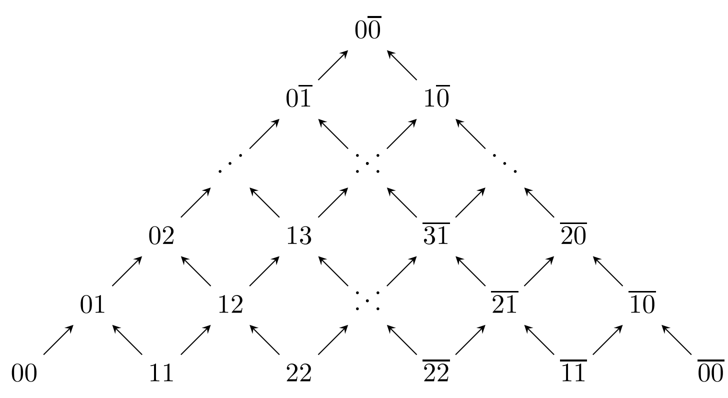
2.4. For any simplicial set X, the vertices of $\widetilde{\mathscr{O}}(X)$ are edges of X; an edge of $\widetilde{\mathscr{O}}(X)$ from $u \longrightarrow v$ to $x \longrightarrow y$ can be viewed as a commutative diagram (up to chosen homotopy)

$$\begin{array}{ccc} u & \longleftarrow & x \\ \downarrow & & \downarrow \\ v & \longrightarrow & y \end{array}$$

When X is the nerve of an ordinary category C, $\widetilde{\mathscr{O}}(X)$ is isomorphic to the nerve of the twisted arrow category of C in the sense of [8]. When X is an ∞-category, we are therefore inclined to call $\widetilde{\mathscr{O}}(X)$ the ***twisted arrow ∞-category*** of X. This terminology is justified by the following.

2.5. **Proposition** (Lurie, [29, Pr. 4.2.3])**.** *If X is an ∞-category, then the functor $\widetilde{\mathscr{O}}(X) \longrightarrow X^{op} \times X$ is a left fibration; in particular, $\widetilde{\mathscr{O}}(X)$ is an ∞-category.*

2.6. **Example.** To illustrate, for any object $\mathbf{p} \in \Delta$, the ∞-category $\widetilde{\mathscr{O}}(\Delta^p)$ is the nerve of the category



(Here we write $\overline{n}$ for $p - n$.)

3. The effective Burnside ∞-category

We now employ the edgewise subdivision to define a quasicategorical variant of the Burnside category. The essence of the idea was explored in our work on the ∞-categorical Q construction.

3.1. **Notation.** For any ∞-category C, denote by $\mathbf{R}_*(C)\colon \Delta^{op} \longrightarrow s\mathbf{Set}$ the functor given by the assignment

$$[\mathbf{n}] \longmapsto \iota\,\mathrm{Fun}(\widetilde{\mathscr{O}}(\Delta^n)^{op}, C).$$

3.2. **Proposition.** *The functor $\mathbf{R}_*\colon \mathbf{Cat}_\infty^0 \longrightarrow \mathrm{Fun}(\Delta^{op}, s\mathbf{Set})$ carries every quasicategory to a Reedy fibrant simplicial space, and it preserves weak equivalences.*

Proof. We first show that for any ∞-category C, the simplicial space $\mathbf{R}_*(C)$ is Reedy fibrant. This is the condition that for any monomorphism $K \hookrightarrow L$, the map

$$\iota\,\mathrm{Fun}(\widetilde{\mathscr{O}}(L)^{op}, C) \longrightarrow \iota\,\mathrm{Fun}(\widetilde{\mathscr{O}}(K)^{op}, C)$$

is a Kan fibration of simplicial sets. This follows immediately from Pr. 2.1 and [25, Lm. 3.1.3.6]. To see that $\mathbf{R}_*$ preserves weak equivalences, we note that since $\mathrm{Fun}(\widetilde{\mathscr{O}}(\Delta^n)^{op}, -)$ preserves weak equivalences, so does $\mathbf{R}_n$. □

3.3. **Definition.** Suppose C an ∞-category. For any integer $n \geq 0$, let us say that a functor $X\colon \widetilde{\mathscr{O}}(\Delta^n)^{op} \longrightarrow C$ is ***cartesian*** if, for any integers $0 \leq i \leq k \leq \ell \leq j \leq n$, the square

$$\begin{array}{ccc} X_{ij} & \longrightarrow & X_{kj} \\ \downarrow & & \downarrow \\ X_{i\ell} & \longrightarrow & X_{k\ell} \end{array}$$

is a pullback.

Write $\mathbf{A}_*^{\mathit{eff}}(C) \subset \mathbf{R}_*(C)$ for the subfunctor in which $\mathbf{A}_n^{\mathit{eff}}(C)$ is the full simplicial subset of $\mathbf{R}_n(C)$ spanned by the cartesian functors

$$X\colon \widetilde{\mathscr{O}}(\Delta^n)^{op} \longrightarrow C.$$

Note that since any functor that is equivalent to an cartesian functor is itself cartesian, the simplicial set $\mathbf{A}_n^{\mathit{eff}}(C)$ is a union of connected components of $\mathbf{R}_n(C)$.

3.4. **Proposition.** *For any ∞-category C that admits all pullbacks, the simplicial space $\mathbf{A}_*^{\mathit{eff}}(C)$ is a complete Segal space.*

Proof. The Reedy fibrancy of $\mathbf{A}_*^{\mathit{eff}}(C)$ follows easily from the Reedy fibrancy of $\mathbf{R}_*(C)$.

To see that $\mathbf{A}_*^{\mathit{eff}}(C)$ is a Segal space, it is necessary to show that for any integer $n \geq 1$, the Segal map

$$\mathbf{A}_n^{\mathit{eff}}(C) \longrightarrow \mathbf{A}_1^{\mathit{eff}}(C) \times_{\mathbf{A}_0^{\mathit{eff}}(C)} \cdots \times_{\mathbf{A}_0^{\mathit{eff}}(C)} \mathbf{A}_1^{\mathit{eff}}(C)$$

is an equivalence. Let L_n denote the ordinary category

$$00 \longleftarrow 01 \longrightarrow 11 \longleftarrow 12 \longrightarrow \cdots (n-1)(n-1) \longleftarrow (n-1)n \longrightarrow nn.$$

The target of the Segal map can then be identified with the maximal Kan complex contained in the ∞-category

$$\mathrm{Fun}(NL_n, C).$$

The Segal map is therefore an equivalence by the uniqueness of limits in ∞-categories [25, Pr. 1.2.12.9].

Finally, to check that $\mathbf{A}_*^{eff}(C)$ is complete, let E be the nerve of the contractible ordinary groupoid with two objects; then completeness is equivalent to the assertion that the Rezk map

$$\mathbf{A}_0^{eff}(C) \longrightarrow \lim_{[\mathbf{n}]\in(\Delta/E)^{op}} \mathbf{A}_n^{eff}(C)$$

is a weak equivalence. The source of this map can be identified with ιC; its target can be identified with the full simplicial subset of

$$\iota \operatorname{Fun}(\widetilde{\mathscr{O}}(E)^{op}, C)$$

spanned by those functors $X\colon \widetilde{\mathscr{O}}(E)^{op} \longrightarrow C$ such that for any simplex $\Delta^n \longrightarrow E$, the induced functor $\widetilde{\mathscr{O}}(\Delta^n)^{op} \longrightarrow C$ is ambigressive. Note that the twisted arrow category of the contractible ordinary groupoid with two objects is the contractible ordinary groupoid with four objects. Consequently, the image of any functor $X\colon \widetilde{\mathscr{O}}(E)^{op} \longrightarrow C$ is contained in ιC. Thus the target of the Rezk map can be identified with $\iota \operatorname{Fun}(\widetilde{\mathscr{O}}(E)^{op}, C)$ itself, and the Rezk map is an equivalence. □

3.5. It is now clear that $\mathbf{A}_*^{eff}$ defines a functor of ∞-categories

$$\mathbf{A}_*^{eff}\colon \mathbf{Cat}_\infty^{\mathrm{lex}} \longrightarrow \mathbf{CSS},$$

where $\mathbf{Cat}_\infty^{\mathrm{lex}} \subset \mathbf{Cat}_\infty$ is the subcategory consisting of ∞-categories with all finite limits and left exact functors between them, and where $\mathbf{CSS} \subset \operatorname{Fun}(\Delta^{op}, \mathbf{Kan})$ is the full subcategory spanned by complete Segal spaces.

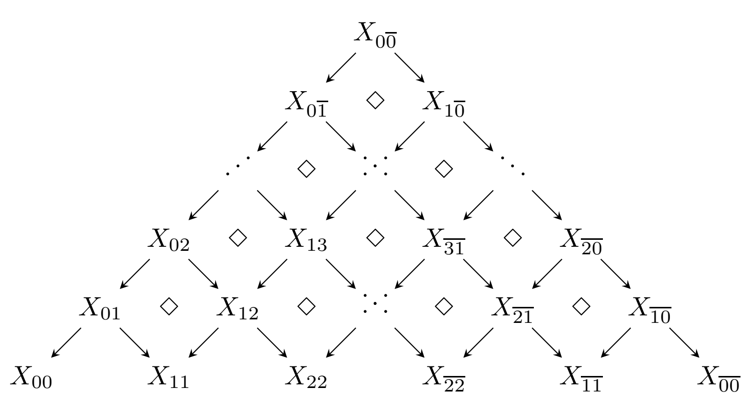
Joyal and Tierney show that the functor that carries a simplicial space $\mathbf{X}$ to the simplicial set whose n-simplices are the vertices of $\mathbf{X}_n$ induces an equivalence of ∞-categories $\mathbf{CSS} \longrightarrow \mathbf{Cat}_\infty$. This leads us to the following definition.

3.6. **Definition.** For any ∞-category C that admits all pullbacks, denote by $A^{eff}(C)$ the ∞-category whose n-simplices are vertices of $\mathbf{A}_n^{eff}(C)$, i.e., cartesian functors $\widetilde{\mathscr{O}}(\Delta^n)^{op} \longrightarrow C$. We may call this the ***effective Burnside*** ∞***-category of*** C.

This defines a functor of ∞-categories

$$A^{eff}\colon \mathbf{Cat}_\infty^{\mathrm{lex}} \longrightarrow \mathbf{Cat}_\infty.$$

3.7. For any ∞-category C that admits all pullbacks, an n-simplex of $A^{eff}(C)$ is a diagram



of C in which every square is a pullback. Here we write $\overline{n}$ for $p-n$.

Another way of describing $A^{\mathit{eff}}(C)$ is as follows. The objects of $A^{\mathit{eff}}(C)$ are precisely those of C. Between objects X and Y, the space of maps is given by

$$\mathrm{Map}_{A^{\mathit{eff}}(C)}(X,Y) \simeq \iota C_{/\{X,Y\}},$$

where $\{X,Y\}$ denotes the diagram $\{x,y\} \longrightarrow C$ from the discrete simplicial set $\{x,y\}$ to C that carries x to X and y to Y. Composition

$$\iota C_{/\{X,Y\}} \times \iota C_{/\{Y,Z\}} \longrightarrow \iota C_{/\{X,Z\}}$$

is defined, up to coherent homotopy, by pullback $- \times_Y -$.

3.8**.** Note that the traditional Burnside category is distinct from $A^{\mathit{eff}}(C)$ in two ways. First, when forming the traditional Burnside category, one begins by studying *isomorphism classes* of spans between objects. This is to ensure that one obtains a nice set of maps for which the pullback construction is sensible. In our effective Burnside ∞-category, we do not pass to isomorphism classes. Rather, we are content to take the entire *space* of spans between objects as our mapping space. We again use pullback to define composition, and we lose no sleep over the fact that pullbacks are only defined up to coherent equivalence, since composition in any ∞-category is only required to be defined up to coherent equivalence in the first place. The ***ordinary effective Burnside category*** of an ordinary category C may be identified with the homotopy category $hA^{\mathit{eff}}(NC)$ of $A^{\mathit{eff}}(NC)$.

Second, the ordinary Burnside category is usually defined as the "local group completion" of this ordinary effective Burnside category. Then Mackey functors are then defined as additive functors from this Burnside category to, say, the category of abelian groups. This is overkill: if the target is already group complete, one knows already what additive functors from the Burnside category will be in terms of the category before group completion. The group completion is a relatively minor procedure for ordinary categories, but for ∞-categories, group completion is serious business. Indeed, if $\mathbf{F}$ is the ordinary category of finite sets, then when one forms the local group completion $A(N\mathbf{F})$ of $A^{\mathit{eff}}(N\mathbf{F})$, the space of endomorphisms on the one point set becomes

$$\mathrm{Map}_{A(N\mathbf{F})}(*,*) \simeq QS^0,$$

by Barratt–Priddy–Quillen. To avoid such complications, we happily stick with the effective Burnside ∞-category.

3.9. **Notation.** The two natural transformations $\varepsilon^\star \longrightarrow \mathit{op}$ and $\varepsilon^\star \longrightarrow \mathrm{id}$ induce two natural transformations

$$(\cdot)_\star \colon \mathrm{id} \longrightarrow A^{\mathit{eff}} \quad \text{and} \quad (\cdot)^\star \colon \mathit{op} \longrightarrow A^{\mathit{eff}}.$$

For any morphism $f \colon U \longrightarrow V$ of an ∞-category C that admits all pullbacks, one thus obtains morphisms

$$f_\star \colon U \longrightarrow V \quad \text{and} \quad f^\star \colon V \longrightarrow U.$$

For any pullback square

$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[j]{} & Y, \end{array}$$

one obtains a homotopy

$$g^{\star} \circ j_{\star} \simeq i_{\star} \circ f^{\star} \colon V \longrightarrow X.$$

3.10. **Notation.** Additionally, there is a self-duality equivalence

$$\widetilde{\mathscr{O}}^{op} \xrightarrow{\sim} \widetilde{\mathscr{O}},$$

whence we have a natural equivalence

$$D \colon A^{\mathit{eff},op} \xrightarrow{\sim} A^{\mathit{eff}}$$

such that the diagram

$$\begin{array}{ccccc} & & A^{\mathit{eff},op} & & \\ & \overset{(\cdot)^{\star,op}}{\nearrow} & & \overset{(\cdot)_{\star}^{op}}{\nwarrow} & \\ \mathrm{id} & & \downarrow D & & op \\ & \underset{(\cdot)_{\star}}{\searrow} & & \underset{(\cdot)^{\star}}{\swarrow} & \\ & & A^{\mathit{eff}} & & \end{array}$$

commutes.

4. Disjunctive ∞-categories

An ∞-category with all finite limits is disjunctive if the coproduct acts effectively as a disjoint union. The effective Burnside ∞-category $A^{\mathit{eff}}(C)$ of a disjunctive ∞-category C has the peculiar property that the initial object of C becomes a zero object in $A^{\mathit{eff}}(C)$, and the coproduct in C becomes both the coproduct and the product in $A^{\mathit{eff}}(C)$. This permits us to regard the effective Burnside ∞-category as somewhat "algebraic" in nature.

4.1. **Definition.** Suppose C is an ∞-category. Then C is said to ***admit direct sums*** if the following conditions hold.

(4.1.1) The ∞-category C is pointed.
(4.1.2) The ∞-category C has all finite products and coproducts.
(4.1.3) For any finite set I and any I-tuple $(X_i)_{i \in I}$ of objects of C, the map

$$\coprod X_I \longrightarrow \prod X_I$$

in hC — given by the maps $\phi_{ij} \colon X_i \longrightarrow X_j$, where ϕ_{ij} is zero unless $i = j$, in which case it is the identity — is an isomorphism.

If C admits finite direct sums, then for any finite set I and any I-tuple $(X_i)_{i \in I}$ of objects of C, we denote by $\bigoplus X_I$ the product (or, equivalently, the coproduct) of the X_i.

If C admits direct sums, then C will be said to be ***additive*** if its homotopy category hC is additive. Denote by $\mathbf{Cat}_{\infty}^{add} \subset \mathbf{Cat}_{\infty}(\kappa_1)$ the subcategory consisting of locally small additive ∞-categories and functors between them that preserve direct sums.

We are mostly interested in ∞-categories C such that the ∞-category $A^{\mathit{eff}}(C)$ admit direct sums. To ensure this, we introduce the following class of ∞-categories.

4.2. **Definition.** An ∞-category will be called ***disjunctive*** if it admits all finite limits and finite coproducts and if, in addition, finite coproducts are disjoint and universal [25, §6.1.1, (ii) and (iii)].

Equivalently, an ∞-category C that admits all finite limits and all finite coproducts is disjunctive just in case, for any finite set I and any collection $\{X_i\}_{i\in I}$ of objects of C, the natural functor

$$\prod_{i\in I} C_{/X_i} \longrightarrow C_{/\coprod_{i\in I} X_i}$$

given by the coproduct is an equivalence of ∞-categories. Its inverse is given informally by the assignment

$$U \longmapsto (U \times_{\coprod_{j\in I} X_j} X_i)_{i\in I}$$

Let us denote by $\mathbf{Cat}_\infty^{disj} \subset \mathbf{Cat}_\infty$ the subcategory whose objects are (small) disjunctive ∞-categories and whose morphisms are those functors that preserve pullbacks and finite coproducts.

In ordinary category theory, it may be more customary to refer to categories with the properties described above with the portmanteau "lextensive." We won't be doing that.

4.3. **Proposition.** *If C is a disjunctive ∞-category, then the ∞-category $A^{eff}(C)$ admits direct sums.*

Proof. We show that the natural functor $(\cdot)_\star \colon C \longrightarrow A^{eff}(C)$ preserves coproducts. The result will then follow from the self-duality $D\colon A^{eff}(C)^{op} \simeq A^{eff}(C)$. Unwinding the definitions, the claim that $(\cdot)_\star$ preserves coproducts amounts to the following claim: for any object Y of C, the space

$$\iota C_{/\{\varnothing,Y\}}$$

is contractible (which follows directly from [25, Lm. 6.1.3.6]), and for any objects X and X' of C, the map

$$\iota C_{/\{X\sqcup Y,Z\}} \longrightarrow \iota C_{/\{X,Z\}} \times \iota C_{/\{Y,Z\}}$$

given informally by the assignment

$$W \longmapsto (W \times_{X\sqcup Y} X, W \times_{X\sqcup Y} Y)$$

is an equivalence. We claim that the map

$$\iota C_{/\{X,Z\}} \times \iota C_{/\{Y,Z\}} \longrightarrow \iota C_{/\{X\sqcup Y,Z\}}$$

given informally by the assignment

$$(U,V) \longmapsto U \sqcup V$$

is a homotopy inverse. Indeed, the statement that

$$W \simeq (W \times_{X\sqcup Y} X) \sqcup (W \times_{X\sqcup Y} Y)$$

follows from the universality of finite coproducts, and the statement that

$$U \simeq (U \sqcup V) \times_{X\sqcup Y} X \quad \text{and} \quad V \simeq (U \sqcup V) \times_{X\sqcup Y} Y$$

follows from the identifications

$$X \simeq X \times_{X\sqcup Y} X, \qquad \varnothing \simeq X \times_{X\sqcup Y} Y \quad \text{and} \quad Y \simeq Y \times_{X\sqcup Y} Y,$$

all of which follow easily from the disjointness and universality of coproducts. □

Note that we do not quite use the full strength of disjunctivity here. It would have been enough to assume only that C admits pullbacks and finite coproducts and that finite coproducts are disjoint and universal. However, we will use the further condition that C admits a terminal object and hence all finite products when we study the Burnside Mackey functor.

5. Disjunctive triples

In a little while we will define Mackey functors on a disjunctive ∞-category C (valued in spectra, say) as direct-sum preserving functors M from the effective Burnside category $A^{eff}(C)$. This means that for any object X of C, we'll have an associated spectrum $M(X)$, and for any morphism $X \longrightarrow Y$ of C, we'll have both a morphism $M(X) \longrightarrow M(Y)$ and a morphism $M(Y) \longrightarrow M(X)$. So a Mackey functor will splice together a covariant functor and a contravariant functor.

However, it is not always reasonable to expect both covariance and contravariance for all morphisms simultaneously. Rather, one may wish instead to specify classes of morphisms in which one has covariance and contravariance. This leads to the notion of a disjunctive triple.

5.1. Recall [2, Df. 1.11] that a ***pair*** of ∞-categories $(C, C_\dagger)$ consists of an ∞-category C and a subcategory $C_\dagger$ [25, §1.2.11] that contains all the equivalences.

A ***triple*** of ∞-categories is an ∞-category equipped with two pair structures. That is, a triple $(C, C_\dagger, C^\dagger)$ consists of an ∞-category C and two subcategories

$$C_\dagger, C^\dagger \subset C,$$

each of which contains all the equivalences. We call morphisms of $C_\dagger$ ***ingressive*** and morphisms of $C^\dagger$ ***egressive***.

5.2. **Definition.** A triple $(C, C_\dagger, C^\dagger)$ of ∞-categories is said to be ***adequate*** if the following conditions obtain.

(5.2.1) For any ingressive morphism $Y \rightarrowtail X$ and any egressive morphism $X' \twoheadrightarrow X$, there exists a pullback square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \rightarrowtail & X. \end{array}$$

(5.2.2) In any pullback square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow[f]{} & X. \end{array}$$

if f is ingressive (respectively, egressive), then so is f'.

We will say that an adequate triple $(C, C_\dagger, C^\dagger)$ is a ***disjunctive triple*** if the following further conditions obtain.

(5.2.3) The ∞-category C admits finite coproducts.

(5.2.4) The class of ingressive morphisms and the class of egressive morphisms are each ***compatible with coproducts*** in the following sense. First, any morphism from an initial object is both ingressive and egressive. Second, for any objects $X, Y, Z \in C$, a morphism $X \sqcup Y \longrightarrow Z$ is ingressive (respectively, egressive) just in case both the restrictions $X \longrightarrow Z$ and $Y \longrightarrow Z$ are so.

(5.2.5) Suppose I and J finite sets. Suppose that for any pair $(i, j) \in I \times J$, we are given a pullback square

$$\begin{array}{ccc} X'_{ij} & \rightarrowtail & Y'_j \\ \downarrow & & \downarrow \\ X_i & \rightarrowtail & Y \end{array}$$

a pullback square in which $X_i \rightarrowtail Y$ is ingressive and $Y'_j \twoheadrightarrow Y$ is egressive. Then the resulting square

$$\begin{array}{ccc} \coprod_{(i,j)\in I\times J} X'_{ij} & \rightarrowtail & \coprod_{j\in J} Y'_j \\ \downarrow & & \downarrow \\ \coprod_{i\in I} X_i & \rightarrowtail & Y \end{array}$$

is also a pullback square.

If $(C, C_\dagger, C^\dagger)$ is a disjunctive triple, then we shall call a pullback square

$$\begin{array}{ccc} Y' & \overset{j}{\rightarrowtail} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \underset{i}{\rightarrowtail} & X \end{array}$$

of C ***disjunctive*** if i (and hence also j) is ingressive and p (and hence also p') is egressive.

Now a ***functor of disjunctive triples*** is a functor of triples

$$f\colon (C, C_\dagger, C^\dagger) \longrightarrow (D, D_\dagger, D^\dagger)$$

(i.e., a functor $C \longrightarrow D$ that carries ingressive morphisms to ingressive morphisms and egressive morphisms to egressive morphisms) such that f preserves finite coproducts and ambigressive pullbacks.

5.3. **Notation.** Let us write $\mathbf{Trip}_\infty^{disj}$ for the subcategory of the ∞-category $\mathbf{Trip}_\infty$ of small triples of ∞-categories whose objects are disjunctive triples and whose morphisms are functors of disjunctive triples.

5.4. **Example.** Of course every disjunctive ∞-category C admits its ***maximal triple structure*** (C, C, C). Hence anything said of disjunctive triples specializes to disjunctive ∞-categories.

5.5. **Definition.** Suppose $(C, C_\dagger, C^\dagger)$ a triple of ∞-categories. For any integer $n \geq 0$, let us say that a functor $X\colon \widetilde{\mathscr{O}}(\Delta^n)^{op} \longrightarrow C$ is ***ambigressive cartesian*** (relative to the triple structure) if, for any integers $0 \leq i \leq k \leq \ell \leq j \leq n$, the square

$$\begin{array}{ccc} X_{ij} & \twoheadrightarrow & X_{kj} \\ \downarrow & & \downarrow \\ X_{i\ell} & \rightarrowtail & X_{k\ell} \end{array}$$

is a pullback in which the morphisms $X_{ij} \rightarrowtail X_{kj}$ and $X_{i\ell} \rightarrowtail X_{k\ell}$ are ingressive, and the morphisms $X_{ij} \twoheadrightarrow X_{i\ell}$ and $X_{kj} \twoheadrightarrow X_{k\ell}$ are egressive.

Recall the functor $\mathbf{R}_*(C)\colon \Delta^{op} \longrightarrow \mathbf{Kan}$ from Nt. 3.1. Write $\mathbf{A}^{\mathit{eff}}_*(C, C_\dagger, C^\dagger) \subset \mathbf{R}_*(C)$ for the subfunctor in which $\mathbf{A}^{\mathit{eff}}_n(C, C_\dagger, C^\dagger)$ is the full simplicial subset of $\mathbf{R}_n(C)$ spanned by the cartesian functors $X\colon \widetilde{\mathscr{O}}(\Delta^n)^{op} \longrightarrow C$. Note that since any functor that is equivalent to an cartesian functor is itself cartesian, the simplicial set $\mathbf{A}^{\mathit{eff}}_n(C, C_\dagger, C^\dagger)$ is a union of connected components of $\mathbf{R}_n(C)$.

The proof of the following is virtually identical to that of Pr. 3.4.

5.6. **Proposition.** *For any adequate triple of ∞-categories $(C, C_\dagger, C^\dagger)$, the simplicial space $\mathbf{A}^{\mathit{eff}}_*(C, C_\dagger, C^\dagger)$ is a complete Segal space.*

5.7. **Definition.** For any adequate triple of ∞-categories $(C, C_\dagger, C^\dagger)$, denote by $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ the ∞-category whose n-simplices are vertices of $\mathbf{A}^{\mathit{eff}}_n(C, C_\dagger, C^\dagger)$, i.e., ambigressive cartesian functors $\widetilde{\mathscr{O}}(\Delta^n)^{op} \longrightarrow C$. We may call this the ***effective Burnside ∞-category of*** $(C, C_\dagger, C^\dagger)$.

5.8**.** Suppose $(C, C_\dagger, C^\dagger)$ a locally small adequate triple of ∞-categories $(C, C_\dagger, C^\dagger)$. Here's an alternate way to go about defining $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$. Write $\mathscr{P}(C) := \mathrm{Fun}(C^{op}, \mathbf{Kan})$ for the usual ∞-category of presheaves of spaces. We may describe (an equivalent version of)

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \subset A^{\mathit{eff}}(\mathscr{P}(C))$$

as the subcategory whose objects are those functors that are representable, in which a morphism $F \longrightarrow G$ of $A^{\mathit{eff}}(\mathscr{P}(C))$ exhibited as a diagram

$$\begin{array}{ccccc} & & H & & \\ & \swarrow & & \searrow & \\ F & & & & G, \end{array}$$

lies in $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ just in case the morphism of C representing the morphism $H \longrightarrow F$ of $\mathscr{P}(C)$ is egressive and the morphism of C representing the morphism $H \longrightarrow G$ of $\mathscr{P}(C)$ is ingressive. Since the pullback of an ingressive (respectively, egressive) morphism along an egressive (resp., ingressive) morphism is ingressive (resp., egressive) of C, and since the Yoneda embedding preserves all limits that exist in C, it follows that $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ is indeed a subcategory.

Note that for any disjunctive triple $(C, C_\dagger, C^\dagger)$, the subcategory

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \subset A^{\mathit{eff}}(\mathscr{P}(C))$$

is closed under direct sums.

5.9. **Notation.** Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple. The two natural transformations

$$(\cdot)_\star\colon \mathrm{id} \longrightarrow A^{\mathit{eff}} \quad \text{and} \quad (\cdot)^\star\colon \mathit{op} \longrightarrow A^{\mathit{eff}}$$

restrict to yield functors

$$(\cdot)_\star\colon C_\dagger \longrightarrow A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \quad \text{and} \quad (\cdot)^\star\colon C^{\dagger,op} \longrightarrow A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$$

Consequently, for any ingressive morphism $f\colon U \rightarrowtail V$ of C, one obtains a morphism $f_\star\colon U \longrightarrow V$, and for any egressive morphism $f\colon U \twoheadrightarrow V$ of C, one obtains a morphism $f^\star\colon V \longrightarrow U$. Additionally, for any pullback square

$$\begin{array}{ccc} U & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \underset{j}{\rightarrowtail} & Y \end{array}$$

in which i, j are ingressive and f, g are egressive, one obtains a homotopy

$$g^\star \circ j_\star \simeq i_\star \circ f^\star\colon V \longrightarrow X.$$

5.10. **Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. Then the self-duality equivalence

$$\widetilde{\mathscr{O}}^{op} \xrightarrow{\sim} \widetilde{\mathscr{O}},$$

induces the natural equivalence

$$D\colon A^{\mathit{eff}}(C, C^\dagger, C_\dagger)^{op} \xrightarrow{\sim} A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$$

such that the diagram

$$\begin{array}{ccccc} & & A^{\mathit{eff}}(C, C^\dagger, C_\dagger)^{op} & & \\ & \nearrow^{(\cdot)^{\star,op}} & & \nwarrow^{(\cdot)_\star^{op}} & \\ C_\dagger & & \downarrow{\scriptstyle D} & & C^{\dagger,op} \\ & \searrow_{(\cdot)_\star} & & \swarrow_{(\cdot)^\star} & \\ & & A^{\mathit{eff}}(C, C_\dagger, C^\dagger) & & \end{array}$$

commutes.

5.11. A functor of disjunctive triples $f\colon (C, C_\dagger, C^\dagger) \longrightarrow (D, D_\dagger, D^\dagger)$ induces a functor $A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow A^{\mathit{eff}}(D, D_\dagger, D^\dagger)$ that preserves direct sums.

6. Mackey functors

6.1. **Definition.** Suppose E an additive ∞-category, and suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. Then a ***Mackey functor on*** $(C, C_\dagger, C^\dagger)$ valued in E is a functor

$$M\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow E$$

that preserves direct sums. If C itself is a disjunctive ∞-category, then a ***Mackey functor on*** C is nothing more than a Mackey functor on the maximal triple (C, C, C).

6.2. **Example.** When E is the nerve of an ordinary additive category, a Mackey functor $M\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow E$ factors in an essentially unique fashion through the homotopy category $hA^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ and then its local group completion (obtained by taking the Grothendieck group of the Hom sets). Hence the notion of Mackey functor described here subsumes the one defined by Dress [4].

Note that some authors define "ordinary" Mackey functors as functors on the local group completion of the *opposite* category $hA^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{op}$. This is just a matter of convention, as the duality functor provides the equivalence $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{op} \simeq A^{\mathit{eff}}(C, C^\dagger, C_\dagger)$.

6.3. **Notation.** Suppose E an additive ∞-category, and suppose $(C, C_\dagger, C^\dagger)$ a small disjunctive triple. Then we denote by

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; E) \subset \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$$

the full subcategory spanned by the Mackey functors. This is covariantly functorial for additive functors in E. For any functor $f\colon (C, C_\dagger, C^\dagger) \longrightarrow (D, D_\dagger, D^\dagger)$ of disjunctive triples, we have an induced functor

$$f^\star\colon \mathbf{Mack}(D, D_\dagger, D^\dagger, E) \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger, E).$$

These functors fit together to yield a functor

$$\mathbf{Mack}\colon \mathbf{Trip}_\infty^{disj,op} \times \mathbf{Cat}_\infty^{add} \longrightarrow \mathbf{Cat}_\infty(\kappa_1).$$

If C is a disjunctive ∞-category, then $\mathbf{Mack}(C, E) = \mathbf{Mack}(C, C, C; E)$.

In fact, let's see that the functor **Mack** is valued in $\mathbf{Cat}_\infty^{add}$.

6.4. **Proposition.** *For any disjunctive triple $(C, C_\dagger, C^\dagger)$ and for any additive ∞-category E, the ∞-category $\mathbf{Mack}(C, C_\dagger, C^\dagger; E)$ is additive.*

Proof. It is easy to see that $\mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$ is additive. The full subcategory

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; E) \subset \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$$

is closed under finite direct sums by noting that the constant functor at a zero object clearly preserves direct sums, and for any Mackey functors M and N, the functor $M \oplus N$ carries zero objects to zero objects, and

$$\begin{aligned}(M \oplus N)(X \oplus Y) &= M(X \oplus Y) \oplus N(X \oplus Y)\\ &\simeq M(X) \oplus M(Y) \oplus N(X) \oplus N(Y)\\ &\simeq (M \oplus N)(X) \oplus (M \oplus N)(Y)\end{aligned}$$

for any objects $X, Y \in A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$. □

Perhaps surprisingly, Mackey functors are closed under all limits and colimits.

6.5. **Proposition.** *For any disjunctive triple $(C, C_\dagger, C^\dagger)$ and for any additive ∞-category E that admits all limits (respectively, all colimits), the full subcategory $\mathbf{Mack}(C, C_\dagger, C^\dagger; E) \subset \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$ is closed under limits (resp., under colimits).*

Proof. We will prove the statement about colimits. The statement about limits will then follow from consideration of the equivalence

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; E)^{op} \simeq \mathbf{Mack}(C, C^\dagger, C_\dagger; E^{op}).$$

We have already seen that $\mathbf{Mack}(C, C_\dagger, C^\dagger; E) \subset \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$ is closed under finite coproducts; it therefore remains to show that if Λ is a sifted ∞-category and if $M\colon \Lambda \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; E)$ is a diagram of Mackey functors, then the colimit $M_\infty = \mathrm{colim}_{\alpha\in\Lambda} M_\alpha$ in $\mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$ is again a Mackey functor. For this, suppose $X, Y \in A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$, and observe that

$$M_\infty(X \oplus Y) \simeq \mathrm{colim}_{\alpha\in\Lambda} M_\alpha(X \oplus Y) \simeq \mathrm{colim}_{\alpha\in\Lambda} M_\alpha(X) \oplus M_\alpha(Y).$$

Now since Λ is sifted, we further have

$$\mathrm{colim}_{\alpha\in\Lambda} M_\alpha(X) \oplus M_\alpha(Y) \simeq \mathrm{colim}_{\alpha,\beta\in\Lambda} M_\alpha(X) \oplus M_\beta(Y).$$

Since

$$\mathrm{colim}_{\alpha,\beta\in\Lambda} M_\alpha(X) \oplus M_\beta(Y) \simeq \mathrm{colim}_{\alpha\in\Lambda} M_\alpha(X) \oplus \mathrm{colim}_{\beta\in\Lambda} M_\beta(Y),$$

we obtain

$$M_\infty(X \oplus Y) \simeq M_\infty(X) \oplus M_\infty(Y). \qquad \square$$

6.5.1. **Corollary.** *For any disjunctive triple $(C, C_\dagger, C^\dagger)$ and for any additive ∞-category E that admits all limits (respectively, all colimits), limits (resp., colimits) in $\mathbf{Mack}(C, C_\dagger, C^\dagger; E)$ are computed objectwise.*

6.6. **Example.** For any disjunctive triple $(C, C_\dagger, C^\dagger)$, the effective Burnside ∞-category admits a local group completion, which is a universal target for Mackey functors. This is the ***Burnside ∞-category***. More precisely, there exists an additive category $A(C, C_\dagger, C^\dagger)$ and a Mackey functor

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow A(C, C_\dagger, C^\dagger)$$

with the following universal property. For any additive ∞-category E, the functor

$$\mathrm{Fun}^{add}(A(C, C_\dagger, C^\dagger), E) \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; E)$$

is an equivalence, where $\mathrm{Fun}^{add}(A(C, C_\dagger, C^\dagger), E) \subset \mathrm{Fun}(A(C, C_\dagger, C^\dagger), E)$ is the full subcategory of additive functors.

This follows from the fact that the functor

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; -)\colon \mathbf{Cat}_\infty^{add} \longrightarrow \mathbf{Cat}_\infty^{add}$$

preserves all limits, which in turn follows from the fact that limits in $\mathbf{Cat}_\infty^{add}$ are computed in the ∞-category $\mathbf{Cat}_\infty$. We leave the details to the reader, as we will not need the Burnside ∞-category itself in this paper.

Mackey functors valued in an additive ∞-category E will inherit duality functors. To illustrate, we focus particularly on the case of Mackey functors valued in finite spectra.

6.7. Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. Then the equivalence

$$D\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \xrightarrow{\sim} A^{\mathit{eff}}(C, C^\dagger, C_\dagger)^{op}$$

induces an equivalence

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; E) \simeq \mathbf{Mack}(C, C^\dagger, C_\dagger; E^{op})^{op}$$

for any additive ∞-category E.

6.8. **Definition.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. The (Spanier–Whitehead) duality functor $\mathbf{Sp}^\omega \xrightarrow{\sim} \mathbf{Sp}^{\omega,op}$ for finite spectra can be composed with the equivalence above to yield an equivalence

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}^\omega)^{op} \simeq \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}^{\omega,op})^{op} \simeq \mathbf{Mack}(C, C^\dagger, C_\dagger; \mathbf{Sp}^\omega).$$

The image of a Mackey functor

$$M\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Sp}^\omega$$

under this equivalence is the ***dual Mackey functor***

$$M^\vee\colon A^{\mathit{eff}}(C, C^\dagger, C_\dagger) \longrightarrow \mathbf{Sp}^\omega.$$

We conclude this subsection by remarking on the potential *covariant* dependence of $\mathbf{Mack}(C, C_\dagger, C^\dagger; E)$ on C.

6.9. **Notation.** Suppose E a presentable additive ∞-category, suppose $(C, C_\dagger, C^\dagger)$ and $(D, D_\dagger, D^\dagger)$ disjunctive ∞-categories, and suppose

$$f\colon (C, C_\dagger, C^\dagger) \longrightarrow (D, D_\dagger, D^\dagger)$$

a functor of disjunctive triples. Then the induced functor

$$f^\star\colon \mathbf{Mack}(D, D_\dagger, D^\dagger; E) \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; E)$$

preserves limits (since they are computed objectwise), whence it follows from the Adjoint Functor Theorem [25] that it admits a left adjoint $f_!$. These left adjoints fit together to yield a functor

$$\mathbf{Mack}\colon \mathbf{Trip}_\infty^{disj} \times \mathbf{Pr}^{\mathbf{L},add} \longrightarrow \mathbf{Pr}^{\mathbf{L},add},$$

where, in the notation of [25, Df. 5.5.3.1],

$$\mathbf{Pr}^{\mathbf{L},add} := \mathbf{Pr}^{\mathbf{L}} \cap \mathbf{Cat}_\infty^{add},$$

the ∞-category of presentable additive ∞-categories, whose morphisms are left adjoints.

7. Mackey stabilization

7.1. **Definition.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, suppose E a presentable ∞-category in which filtered colimits are left exact [25, Df. 7.3.4.2], and suppose

$$f\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow E \quad \text{and} \quad F\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Sp}(E)$$

two functors. Then a natural transformation

$$\eta\colon f \longrightarrow \Omega^\infty \circ F$$

will be said to exhibit F as the ***Mackey stabilization*** of f if F is a Mackey functor, and if, for any Mackey functor $M\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Sp}(E)$, the map

$$\mathrm{Map}_{\mathbf{Mack}(C,C_\dagger,C^\dagger;\mathbf{Sp}(E))}(F, M) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(A^{\mathit{eff}}(C,C_\dagger,C^\dagger),E)}(f, \Omega^\infty \circ M)$$

induced by η is an equivalence.

We shall now show that Mackey stabilization exist and are computable by means of a Goodwillie derivative. In particular, we will show that the functor

$$\Omega^\infty \circ -\colon \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)) \longrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$$

admits a left adjoint.

7.2. **Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. Then we write

$$\mathrm{D}A(C, C_\dagger, C^\dagger) := \mathscr{P}_\Sigma(A^{\mathit{eff}}(C, C_\dagger, C^\dagger))$$

for the nonabelian derived ∞-category of $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ [25]. This ∞-category admits all colimits, and it comes equipped with a fully faithful functor

$$j\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \hookrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger).$$

These are uniquely characterized by *either* of the following conditions.

(7.2.1) For any ∞-category E that admits all colimits, the restriction functor
$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E) \longrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$$
induced by j is fully faithful, and its essential image is spanned by those functors $A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow E$ that preserve finite coproducts.

(7.2.2) For any ∞-category E with all sifted colimits, the restriction functor
$$\mathrm{Fun}_{\mathscr{G}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E) \longrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$$
induced by j is an equivalence, where $\mathscr{G}$ denotes the class of small sifted simplicial sets.

The following is now immediate from [31].

7.3. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose E a presentable ∞-category in which filtered colimits are left exact. Then the inclusion functor*
$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \hookrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger)$$
and the 0-th space functor $\Omega^\infty \colon \mathbf{Sp}(E) \longrightarrow E$ induce equivalences
$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{L}}(\mathrm{D}A(C, C_\dagger, C^\dagger), \mathbf{Sp}(E)) & \xrightarrow{\sim} & \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)) \\ \| & & \\ \mathrm{Exc}_{\mathscr{F}}(\mathrm{D}A(C, C_\dagger, C^\dagger), \mathbf{Sp}(E)) & \xrightarrow{\sim} & \mathrm{Exc}_{\mathscr{F}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E) \end{array}$$
where $\mathrm{Exc}_{\mathscr{F}}$ denotes the ∞-category of (1-)excisive functor that preserve all filtered colimits.

Now we are well positioned to obtain Mackey stabilizations.

7.4. **Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose E a presentable ∞-category in which filtered colimits are left exact. Then any functor $f \colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow E$ admits a Mackey stabilization. In particular, the functor*
$$\Omega^\infty \circ - \colon \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)) \longrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$$
admits a left adjoint.

Proof. Compose the equivalences of the previous lemma with the 1-excisive approximation functor $P_1 \colon \mathrm{Fun}_{\mathscr{F}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E) \longrightarrow \mathrm{Exc}_{\mathscr{F}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E)$, which is left adjoint to the inclusion. Employing the equivalences of the previous lemma, a left adjoint to the inclusion functor
$$\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)) \simeq \mathrm{Exc}_{\mathscr{F}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E) \hookrightarrow \mathrm{Fun}_{\mathscr{F}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E).$$
This left adjoint may now be composed with the inclusion
$$\mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E) \simeq \mathrm{Fun}_{\mathscr{G}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E) \hookrightarrow \mathrm{Fun}_{\mathscr{F}}(\mathrm{D}A(C, C_\dagger, C^\dagger), E)$$
to obtain the desired Mackey stabilization functor. □

Happily, Tom Goodwillie has provided us with a formula for the 1-excisive approximation [31, Cnstr. 7.1.1.27]. Hence for any functor $f \colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow E$, we obtain a formula for its Mackey stabilization as a 1-excisive functor $\Omega^\infty \circ F \colon \mathrm{D}A(C, C_\dagger, C^\dagger) \longrightarrow E$:
$$\Omega^\infty \circ F \simeq \operatorname*{colim}_{n \geq 0} \Omega^n \circ \overline{f} \circ \Sigma^n_{\mathrm{D}A(C, C_\dagger, C^\dagger)},$$

where $\overline{f}\colon \mathrm{D}A(C, C_{\dagger}, C^{\dagger}) \longrightarrow E$ is the essentially unique functor that preserves sifted colimits such that

$$\overline{f}|_{A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})} = f.$$

Consequently, we have the task of studying the suspension functor $\Sigma_{\mathrm{D}A(C,C_{\dagger},C^{\dagger})}$ on $\mathrm{D}A(C, C_{\dagger}, C^{\dagger})$. In particular, we are interested in its values on objects of the effective Burnside ∞-category. For any object $X \in A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})$, we have a simplicial object $B_*(0, X, 0)$ given by

$$\mathbf{n} \longmapsto X^n,$$

whose geometric realization in $\mathrm{D}A(C, C_{\dagger}, C^{\dagger})$ is $\Sigma_{\mathrm{D}A(C,C_{\dagger},C^{\dagger})}X$. Since $\overline{f}$ preserves geometric realizations, we find

$$\Omega^{\infty}F(X) \simeq \operatorname*{colim}_{n\geq 0} \Omega^n \left(\operatorname*{colim}_{(\mathbf{k}_1,\dots,\mathbf{k}_n)\in(N\Delta^{op})^n} f\left(X^{k_1+\cdots+k_n}\right)\right).$$

In one important class of cases, it follows immediately from Segal's delooping machine that passage to the colimit is unnecessary.

7.5. **Proposition.** *Suppose $(C, C_{\dagger}, C^{\dagger})$ a disjunctive triple, and suppose E an ∞-topos. If $f\colon A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger}) \longrightarrow E$ is a functor that preserves finite products, the Mackey stabilization F of f is defined by the formula*

$$\Omega^{\infty}F(X) \simeq \Omega\big|B_*(*, f(X), *)\big|_{N\Delta^{op}},$$

where $B_*(*, f(X), *)$ is the simplicial object $\mathbf{k} \longmapsto f(X)^k$, and $|\cdot|_{N\Delta^{op}}$ denotes geometric realization.

8. Representable spectral Mackey functors and assembly morphisms

The Mackey stabilization is useful for constructing universal examples of Mackey functors.

8.1. **Definition.** Suppose $(C, C_{\dagger}, C^{\dagger})$ a disjunctive triple, and suppose X and object of C. Then the Mackey stabilization of the functor $A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger}) \longrightarrow \mathbf{Kan}$ corepresented by X will be denoted

$$\mathbf{S}^X_{(C,C_{\dagger},C^{\dagger})}\colon A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger}) \longrightarrow \mathbf{Sp}.$$

(We will drop the subscript and write $\mathbf{S}^X$ when the chosen disjunctive triple is clear from the context.) This is the ***Mackey functor corepresented by*** X

We will call the Mackey functor corepresented by the terminal object 1 the ***Burnside Mackey functor***. In this case, we drop the superscript and write simply $\mathbf{S}_{(C,C_{\dagger},C^{\dagger})}$.

The following is now an immediate consequence of the universal property of the Mackey stabilization.

8.2. **Proposition.** *Suppose $(C, C_{\dagger}, C^{\dagger})$ a disjunctive triple, and suppose X and object of C. Then the corepresentable Mackey functor has the universal property that for any Mackey functor $M\colon A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger}) \longrightarrow \mathbf{Sp}$, there is an identification*

$$\mathrm{Map}_{\mathbf{Mack}(C,C_{\dagger},C^{\dagger};\mathbf{Sp})}(\mathbf{S}^X, M) \simeq \Omega^{\infty}M(X),$$

functorial in M.

8.3. **Definition.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, suppose X and object of C, and suppose $M\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Sp}$ a Mackey functor. Then the identity functor $M(X) \longrightarrow M(X)$ defines a morphism of Mackey functors

$$\mathbf{S}^X \longrightarrow F(M(X), M),$$

where the target is the composite

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \xrightarrow{M} \mathbf{Sp} \xrightarrow{F(M(X),-)} \mathbf{Sp}.$$

For any object $Y \in C$, we call the corresponding morphism

$$\alpha\colon \mathbf{S}^X(Y) \wedge M(X) \longrightarrow M(Y)$$

the ***assembly morphism*** for M, X, and Y.

9. Mackey stabilization via algebraic K-theory

Let us discuss a key circumstance in which we can express the Mackey stabilization (Df. 7.1) of a functor in terms of the additivization presented in [2, Df. 7.9]: we are interested in the situation in which a functor is given by composing a Mackey functor valued in Waldhausen ∞-categories with a suitable theory in the sense of [2, Df. 7.1].

9.1. **Definition.** Suppose $\phi\colon \mathbf{Wald}_\infty \longrightarrow \mathscr{E}$ a pre-additive theory [2, Df. 7.11]. Then we will say that a Waldhausen ∞-category $\mathscr{C}$ is ***ϕ-split*** if, for any integer $m \geq 0$, the functors $\mathscr{F}_m(\mathscr{C}) \longrightarrow \mathscr{C}$ and $\mathscr{F}_m(\mathscr{C}) \longrightarrow \mathscr{S}_m(\mathscr{C})$ induce an equivalence

$$\phi(\mathscr{F}_m(\mathscr{C})) \xrightarrow{\sim} \phi(\mathscr{C}) \times \phi(\mathscr{S}_m(\mathscr{C})).$$

9.2. **Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose*

$$\mathscr{X}\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Wald}_\infty$$

a Mackey functor valued in the ∞-category of Waldhausen ∞-categories. Suppose, additionally, that $\mathscr{E}$ is an ∞-topos and $\phi\colon \mathbf{Wald}_\infty \longrightarrow \mathscr{E}$ a pre-additive theory such that for any object $s \in C$, the Waldhausen ∞-category is ϕ-split. Then the Mackey stabilization of the composite $\phi \circ \mathscr{X}$ is given by

$$s \longmapsto \mathbf{D}\phi(\mathscr{X}(s)),$$

where $\mathbf{D}\phi\colon \mathbf{Wald}_\infty \longrightarrow \mathbf{Sp}(\mathscr{E})$ denote the canonical lift of the additivization of [2, Cor. 7.6.1].

Proof. Let us extend $\phi \circ \mathscr{X}$ to a functor $\mathrm{D}A(C, C_\dagger, C^\dagger) \longrightarrow \mathscr{E}$ and compute the 1-excisive approximation. The Mackey stabilization of $\phi \circ \mathscr{X}$ is then the spectrum-valued lift of the 1-excisive approximation of the composite

$$\mathrm{D}A(C, C_\dagger, C^\dagger) \xrightarrow{\mathbf{L}\mathscr{X}} \mathrm{D}(\mathbf{Wald}_\infty) \xrightarrow{\Phi} \mathscr{E},$$

where Φ is the left derived functor of ϕ [2, Df. 4.14], and $\mathbf{L}\mathscr{X}$ is the essentially unique colimit-preserving functor whose restriction to $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ is $\mathscr{X}$. Then one has (by, for example, [31, Rk. 7.1.1.30])

$$P_1(\Phi \circ \mathbf{L}\mathscr{X}) \simeq P_1(\Phi) \circ \mathbf{L}\mathscr{X};$$

hence the Mackey stabilization of $\phi \circ \mathscr{X}$ is given by the spectrum-valued lift of the functor

$$s \longmapsto \operatorname{colim}_m \Omega^m \Phi \Sigma^m \mathscr{X}(s).$$

Here Σ denotes the suspension in $\mathrm{D}(\mathbf{Wald}_\infty)$. It can be computed by means of a bar construction:

$$\Sigma\mathscr{Y} \simeq |B_*(0, \mathscr{Y}, 0)|_{N\Delta^{op}},$$

where $B_n(0, \mathscr{Y}, 0) \simeq \mathscr{Y}^n$. Consequently, one may compute $\Phi \circ \Sigma$ as a geometric realization:

$$\Phi\Sigma(\mathscr{Y}) \simeq |B_*(*, \Phi(\mathscr{Y}), *)|_{N\Delta^{op}}.$$

Now let us assume further that for any object $s \in S$, the Waldhausen ∞-category $\mathscr{X}(s)$ is ϕ-split. In this case, one has

$$|B_*(*, \phi(\mathscr{X}(s)), *)|_{N\Delta^{op}} \simeq \Phi(\mathscr{S}\mathscr{X}(s)),$$

whence $\Omega\Phi\Sigma(\mathscr{X}(s))$ is the additivization [2, Df. 7.9] of ϕ applied to $\mathscr{X}(s)$. It follows that the colimit stabilizes, and the result follows. □

9.2.1. **Corollary.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose*

$$\mathscr{X}\colon A^{\mathrm{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Wald}_\infty$$

a Mackey functor valued in the ∞-category of Waldhausen ∞-categories. If each Waldhausen ∞-category $\mathscr{X}(s)$ is ι-split, then the Mackey stabilization of $\iota \circ \mathscr{X}$ is given by $\mathbf{K} \circ \mathscr{X}$, where $\mathbf{K}$ denotes connective algebraic K-theory.

10. Waldhausen bicartesian fibrations

We have already seen that there is a close relationship between algebraic K-theory and spectral Mackey functors. The inputs required there were Mackey functors valued in Waldhausen ∞-categories. Unfortunately, in nature, these Mackey functors tend not to appear with all of their coherences splayed out. Instead, the most interesting examples are found *furled* — as fibrations that exhibit both covariant functoriality and contravariant functoriality along with a compatibility between the two in certain situations. We call these fibrations *Waldhausen bicartesian fibrations*. In this section, we define this notion, and in the next, we show how to *unfurl* these fibrations to extract the desired Mackey functors valued in Waldhausen ∞-categories.

10.1**.** Suppose $p\colon X \longrightarrow S$ a cartesian and cocartesian fibration. Then for any morphism $f\colon s \longrightarrow t$ of S, one has and adjoint pair of functors

$$f_!\colon X_s \rightleftarrows X_t\colon f^\star.$$

For any square

$$\begin{array}{ccc} s & \xrightarrow{i} & s' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ t & \xrightarrow[j]{} & t', \end{array} \tag{10.1.1}$$

the unit $\eta\colon \mathrm{id} \longrightarrow j^\star \circ j_!$ induces a natural transformation

$$q^\star \xrightarrow{q^\star(\eta)} q^\star \circ j^\star \circ j_! \simeq i^\star \circ q'^\star \circ j_!,$$

which is adjoint to a natural transformation

$$i_! \circ q^\star \longrightarrow q'^\star \circ j_!, \tag{10.1.2}$$

which we call the ***base change natural transformation***. Equivalently, we may construct this natural transformation by using the counit $\epsilon\colon q_! \circ q^\star \longrightarrow \mathrm{id}$ to define

$$q'_! \circ i_! \circ q^\star \simeq j_! \circ f_! \circ q^\star \xrightarrow{j_!(\epsilon)} j_!;$$

its right adjoint is then the base change natural transformation (10.1.2).

When the base change natural transformation is an equivalence, then one says that the square

$$\begin{array}{ccc} X_s & \xrightarrow{i_!} & X_{s'} \\ {\scriptstyle q_!}\downarrow & & \downarrow{\scriptstyle q'_!} \\ X_t & \xrightarrow[j_!]{} & X_{t'}, \end{array} \qquad \text{(respectively, the square} \qquad \begin{array}{ccc} X_{t'} & \xrightarrow{q'^\star} & X_{s'} \\ {\scriptstyle j^\star}\downarrow & & \downarrow{\scriptstyle i^\star} \\ X_t & \xrightarrow[q^\star]{} & X_s, \end{array} \quad)$$

is ***right adjointable*** (resp., ***left adjointable***) [25]. Apparently this is sometimes called the *Beck–Chevalley condition*.

If we only assume p an inner fibration, we can make sense of the base change natural transformation (10.1.2) in the presence of a small amount of extra information. Of course one needs to know that the functors $i_!$, $j_!$, $q^\star$, and $q'^\star$ all exist, and in order to construct (10.1.2), it is enough to assume *either* that the functors $q_!$ and $q'_!$ exist *or* that the functors $i^\star$ and $j^\star$. That is, if $\sigma\colon \Delta^1 \times \Delta^1 \longrightarrow S$ is given by the square (10.1.1), then it suffices to assume only *one* of the following.

(10.1.3) The functor

$$X \times_S (\Delta^1 \times \Delta^1) \longrightarrow \Delta^1 \times \Delta^1$$

is a cocartesian fibration, and the functor

$$X \times_S (\Delta^1 \times \partial\Delta^1) \longrightarrow \Delta^1 \times \partial\Delta^1$$

is a cartesian fibration.

(10.1.4) The functor

$$X \times_S (\Delta^1 \times \Delta^1) \longrightarrow \Delta^1 \times \Delta^1$$

is a cartesian fibration, and the functor

$$X \times_S (\partial\Delta^1 \times \Delta^1) \longrightarrow \partial\Delta^1 \times \Delta^1$$

is a cocartesian fibration.

We will apply this idea in the case that (10.1.1) is a square of a disjunctive triple in which the vertical morphisms q, q' of are egressive and the horizontal morphisms i, j are ingressive.

10.2. **Definition.** A triple $(C, C_\dagger, C^\dagger)$ is said to be ***left complete*** if $C_\dagger \subset C^\dagger$ and ***right complete*** if $C^\dagger \subset C_\dagger$.

In the examples of disjunctive triples of greatest interest to us, it is often the case that either every map is egressive or every map is ingressive. In particular, most of the examples of interest to us are either left complete or right complete.

Our equivariant K-theory will take as input assignments of Waldhausen ∞-categories to objects of disjunctive triples $(C, C_\dagger, C^\dagger)$ that are covariant in ingressive morphisms and contravariant in egressive morphisms. We will insist that for morphisms that are both ingressive and egressive, the resulting functors are adjoint. Finally, we will assume that for pullback squares of egressive morphisms along ingressive morphisms, the base change natural transformation is an equivalence. But for the base change natural transformation to make sense, we must assume that $(C, C_\dagger, C^\dagger)$ is either left or right complete. This leads us to the following.

10.3. **Definition.** Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple that is either left complete or right complete. An inner fibration $p\colon X \longrightarrow C$ is said to be ***adequate over the triple*** $(C, C_\dagger, C^\dagger)$ if the following conditions obtain.

(10.3.1) For any ingressive morphism $f\colon s \rightarrowtail t$ and any object $x \in X_s$, there exists a p-cocartesian edge $x \longrightarrow y$ covering f. In particular, the functor

$$p_\dagger\colon X \times_C C_\dagger \longrightarrow C_\dagger$$

is a cocartesian fibration.

(10.3.2) For any egressive morphism $f\colon s \twoheadrightarrow t$ and any object $y \in X_t$, there exists a p-cartesian edge $x \longrightarrow y$ covering f. In particular, the functor

$$p^\dagger\colon X \times_C C^\dagger \longrightarrow C^\dagger$$

is a cartesian fibration.

(10.3.3) For any ambigressive pullback square

$$\begin{array}{ccc} s & \overset{i}{\rightarrowtail} & s' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ t & \underset{j}{\rightarrowtail} & t', \end{array}$$

the base change natural transformation

$$i_! \circ q^\star \longrightarrow q'^\star \circ j_!$$

is an equivalence.

Suppose now that $(C, C_\dagger, C^\dagger)$ is a disjunctive triple that is either left complete or right complete. A ***Waldhausen bicartesian fibration over the triple*** $(C, C_\dagger, C^\dagger)$

$$q\colon \mathscr{X} \longrightarrow C$$

is a functor of pairs $\mathscr{X} \longrightarrow C^\flat$ that enjoys the following properties.

(10.3.4) The underlying functor $q\colon \mathscr{X} \longrightarrow C$ is an adequate inner fibration over the triple $(C, C_\dagger, C^\dagger)$.

(10.3.5) For any ingressive morphism $\eta\colon s \rightarrowtail t$, the induced functor $\eta_!\colon \mathscr{X}_s \longrightarrow \mathscr{X}_t$ carries cofibrations to cofibrations, and it is an exact functor

$$(\mathscr{X}_s, \mathscr{X}_s \times_{\mathscr{X}} \mathscr{X}_\dagger) \longrightarrow (\mathscr{X}_t, \mathscr{X}_t \times_{\mathscr{X}} \mathscr{X}_\dagger)$$

of Waldhausen ∞-categories.

(10.3.6) For any egressive morphism $\eta\colon s \twoheadrightarrow t$, the induced functor $\eta^\star\colon \mathscr{X}_t \longrightarrow \mathscr{X}_s$ carries cofibrations to cofibrations, and it is an exact functor

$$(\mathscr{X}_t, \mathscr{X}_t \times_{\mathscr{X}} \mathscr{X}_\dagger) \longrightarrow (\mathscr{X}_s, \mathscr{X}_s \times_{\mathscr{X}} \mathscr{X}_\dagger)$$

of Waldhausen ∞-categories.

(10.3.7) For any finite set I and any collection $\{s_i \mid i \in I\}$ of objects of C indexed by the elements of I with coproduct s, the functors

$$j_i^\star\colon \mathscr{X}_s \longrightarrow \mathscr{X}_{s_i}$$

induced by the inclusions $j_i\colon s_i \hookrightarrow s$ together exhibit $\mathscr{X}_s$ as the direct sum $\bigoplus_{i \in I} \mathscr{X}_{s_i}$.

The following lemma is just an unwinding of the relevant definitions, but it will come in handy later.

10.4. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple that is either left or right complete. If $p\colon X \longrightarrow C$ is an inner fibration satisfying conditions (10.3.1) and (10.3.2), then condition (10.3.3) is equivalent to the condition that for any square*

$$\begin{array}{ccc} x & \overset{\eta}{\rightarrowtail} & x' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ y & \underset{\theta}{\rightarrowtail} & y' \end{array}$$

of $\mathscr{X}$ that covers an ambigressive pullback square

$$\begin{array}{ccc} s & \overset{i}{\rightarrowtail} & s' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ t & \underset{j}{\rightarrowtail} & t' \end{array}$$

of C, if α and α' are p-cartesian and θ is p-cocartesian, then η is also p-cocartesian.

11. Unfurling

Here is the central construction of this paper. A Waldhausen bicartesian fibration over a left or right complete disjunctive triple $(C, C_\dagger, C^\dagger)$ has all the elements that we might look for in a Mackey functor valued in Waldhausen ∞-categories: there's a covariant functor $C_\dagger \longrightarrow \mathbf{Wald}_\infty$ and a contravariant functor $(C^\dagger)^{op} \longrightarrow \mathbf{Wald}_\infty$, and the two are glued via base change equivalences. Unfortunately, these data are not displayed in a fashion that makes it easy to spot the functoriality in the effective Burnside ∞-category $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$. In order to extract something that is visibly functorial in the effective Burnside ∞-category, we must perform an operation, which we call *unfurling*. When we unfurl a Waldhausen bicartesian fibration $\mathscr{X} \longrightarrow C$ for the triple $(C, C_\dagger, C^\dagger)$, we end up with a Waldhausen cocartesian fibration $\Upsilon(\mathscr{X}/(C, C_\dagger, C^\dagger)) \longrightarrow A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$, which we may then straighten into a Mackey functor on $(C, C_\dagger, C^\dagger)$ valued in Waldhausen ∞-categories.

11.1. **Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a triple, and suppose $p\colon X \longrightarrow C$ an inner fibration. Denote by $X_\dagger \subset X \times_C C_\dagger$ (respectively, $X^\dagger \subset X \times_C C^\dagger$) the subcategory containing all the objects whose morphisms are p-cocartesian (resp., p-cartesian).

11.2. **Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple (Df. 5.2), and suppose $p\colon X \longrightarrow C$ an adequate inner fibration over $(C, C_\dagger, C^\dagger)$ (Df. 10.3). Then the triples $(X, X \times_C C_\dagger, X^\dagger)$ and $(X, X_\dagger, X \times_C C^\dagger)$ are adequate as well.*

Proof. We show that $(X, X \times_C C_\dagger, X^\dagger)$ is adequate; the case of $(X, X_\dagger, X \times_C C^\dagger)$ is dual. Suppose $\sigma\colon \Delta^1 \times \Delta^1 \longrightarrow C$ an ambigressive pullback

$$\begin{array}{ccc} s' & \rightarrowtail & t' \\ \downarrow & & \downarrow \\ s & \rightarrowtail & t, \end{array}$$

and suppose $x \rightarrowtail y$ a morphism covering $s \rightarrowtail t$ and $y' \twoheadrightarrow y$ a p-cartesian edge covering $t' \longrightarrow t$. Then there exists a p-cartesian edge $x' \twoheadrightarrow x$ covering $s' \longrightarrow s$, and after filling in the inner horn $x' \longrightarrow x \longrightarrow y$ and the outer horn $x' \longrightarrow y \longleftarrow y'$ (using the p-cartesianness of $y' \twoheadrightarrow y$), we obtain a pullback square

$$\begin{array}{ccc} x' & \rightarrowtail & y' \\ \downarrow & & \downarrow \\ x & \rightarrowtail & y \end{array}$$

covering σ. ☐

11.3. **Definition.** Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple, and suppose $p\colon X \longrightarrow C$ an adequate inner fibration over $(C, C_\dagger, C^\dagger)$. Then the ***unfurling*** of p is the ∞-category

$$\Upsilon(X/(C, C_\dagger, C^\dagger)) := A^{eff}(X, X \times_C C_\dagger, X^\dagger).$$

Composition with p defines a natural map

$$\Upsilon(p)\colon \Upsilon(X/(C, C_\dagger, C^\dagger)) \longrightarrow A^{eff}(C, C_\dagger, C^\dagger).$$

We'll prove the following brace of lemmas in the next section.

11.4. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple, and suppose $p\colon X \longrightarrow C$ an adequate inner fibration over $(C, C_\dagger, C^\dagger)$. Then*

$$\Upsilon(p)\colon \Upsilon(X/(C, C_\dagger, C^\dagger)) \longrightarrow A^{eff}(C, C_\dagger, C^\dagger)$$

is an inner fibration.

11.5. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple, and suppose $p\colon X \longrightarrow C$ an adequate inner fibration over $(C, C_\dagger, C^\dagger)$. An edge $f : y \longrightarrow z$ of $\Upsilon(X/(C, C_\dagger, C^\dagger))$ is $\Upsilon(p)$-cocartesian if it is represented as a span*

$$\begin{array}{ccccc} & & u & & \\ & \overset{\phi}{\swarrow} & & \overset{\psi}{\searrow} & \\ y & & & & z, \end{array}$$

in which ϕ is p-cartesian over an egressive morphism and ψ is p-cocartesian over an ingressive morphism.

The following is now immediate.

11.6. **Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ an adequate triple that is either left or right complete, and suppose*

$$p\colon \mathscr{X} \longrightarrow C$$

an adequate inner fibration over $(C, C_\dagger, C^\dagger)$. Then the unfurling $\Upsilon(p)$ is a cocartesian fibration.

More particularly, for any disjunctive triple $(C, C_\dagger, C^\dagger)$ that is either left or right complete (Df. 10.2), and for any Waldhausen bicartesian fibration $p\colon \mathscr{X} \longrightarrow C$ over $(C, C_\dagger, C^\dagger)$ (Df. 10.3), it follows that $\Upsilon(p)$ is a cocartesian fibration. Moreover, for any object s of C, consider the functor $\mathscr{X}_s \longrightarrow \Upsilon(\mathscr{X}/(C, C_\dagger, C^\dagger))_s$ induced by the natural transformation $\varepsilon^\star \longrightarrow op$. This functor is the identity on objects, and it is easy to see that it is fully faithful. Now for any $s \longrightarrow t$ of $A^{\mathrm{eff}}(C, C_\dagger, C^\dagger)$ represented as a span

$$\begin{array}{ccc} & u & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ s & & t, \end{array}$$

in C, the induced functor

$$\mathscr{X}_s \simeq \Upsilon(\mathscr{X}/(C, C_\dagger, C^\dagger))_s \longrightarrow \Upsilon(\mathscr{X}/(C, C_\dagger, C^\dagger))_t \simeq \mathscr{X}_t$$

is equivalent to $g_! \circ f^\star$. In particular, it is exact, whence we have the following.

11.7. **Proposition.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a disjunctive triple that is either left or right complete, and suppose*

$$p\colon \mathscr{X} \longrightarrow C$$

a Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$*. Then the unfurling* $\Upsilon(p)$ *is a Waldhausen cocartesian fibration.*

Now condition (10.3.2) immediately implies the main feature of unfurlings.

11.8. **Theorem.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a disjunctive triple, and suppose*

$$p\colon \mathscr{X} \longrightarrow C$$

a Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$*. Then a functor*

$$\mathscr{M}_p\colon A^{\mathrm{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Wald}_\infty$$

that classifies the unfurling $\Upsilon(p)$ *is a Mackey functor.*

One may compose a functor classifying the unfurling of a Waldhausen bicartesian fibration with the delooping $\mathbf{Wald}_\infty \longrightarrow \mathbf{Sp}$ of any additive theory to obtain the following.

11.8.1. **Corollary.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a disjunctive triple, and suppose*

$$p\colon \mathscr{X} \longrightarrow C$$

a Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$*. Suppose*

$$\mathscr{M}_p\colon A^{\mathrm{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Wald}_\infty$$

a functor that classifies the unfurling $\Upsilon(p)$*. Then for any additive theory*

$$F\colon \mathbf{Wald}_\infty \longrightarrow \mathscr{E}$$

in the sense of [2, Df. 7.1]*, the composition*

$$\mathbf{F} \circ \mathscr{M}_p\colon A^{\mathrm{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Sp}(\mathscr{E}),$$

where $\mathbf{F}\colon \mathbf{Wald}_\infty \longrightarrow \mathbf{Sp}(\mathscr{E})$ *is the canonical delooping of* F [2, Cor. 7.6.1]*, is a Mackey functor.*

In particular, we see that the algebraic K-theory of a Waldhausen bicartesian fibration naturally organizes itself into a Mackey functor valued in spectra.

12. Horn filling in effective Burnside ∞-categories

In both Lm. 11.4 and Lm. 11.5, we are interested in filling horns in effective Burnside ∞-categories. These correspond to extensions along inclusions of the form $\widetilde{\mathscr{O}}(\Lambda^m_k)^{op} \hookrightarrow \widetilde{\mathscr{O}}(\Delta^m)^{op}$ that enjoy certain properties. In this section, we construct a filtration that provides a general strategy for constructing the desired extensions, and we use this to prove Lms. 11.4 and 11.5.

The reader uninterested in such nitty-gritty may be forgiven for skipping this section; however, one must acknowledge that this is where the rubber meets the road. Ultimately, it is the combinatorics of simplices that allow us to solve homotopy-coherence problems.

12.1. **Notation.** In this section, let $(C, C_\dagger, C^\dagger)$ and $(D, D_\dagger, D^\dagger)$ denote two adequate triples, and let $p\colon (C, C_\dagger, C^\dagger) \longrightarrow (D, D_\dagger, D^\dagger)$ be a functor of triples that carries ambigressive pullbacks to ambigressive pullbacks. Write $p_\dagger$ for the restriction $C_\dagger \longrightarrow D_\dagger$, and write $p^\dagger$ for the restriction $C^\dagger \longrightarrow D^\dagger$.

Here is what we will prove.

12.2. **Theorem.** *Assume that the underlying functor $C \longrightarrow D$ is an inner fibration. Then the induced functor*

$$A^{eff}(p)\colon A^{eff}(C, C_\dagger, C^\dagger) \longrightarrow A^{eff}(D, D_\dagger, D^\dagger)$$

is an inner fibration.

Furthermore, assume the following.

(12.2.1) For any ingressive morphism $g\colon s \rightarrowtail t$ of D and any object $x \in C_s$, there exists an ingressive morphism $f\colon x \rightarrowtail y$ of C covering g that is both p-cocartesian and $p_\dagger$-cocartesian.

(12.2.2) Suppose σ a commutative square

$$\begin{array}{ccc} x' & \overset{f'}{\rightarrowtail} & y' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ x & \underset{f}{\rightarrowtail} & y, \end{array}$$

of C such that the square $f(\sigma)$ is an ambigressive pullback in D, the morphism f' is ingressive, the morphism ϕ is egressive, and the morphism f is p-cocartesian. Then f' is p-cocartesian if and only if the square is an ambigressive pullback (and in particular ψ is egressive).

Then an edge $f\colon y \longrightarrow z$ of $A^{eff}(C, C_\dagger, C^\dagger)$ is $A^{eff}(p)$-cocartesian if it is represented as a span

$$y \xleftarrow{\ \phi\ } u \xrightarrow{\ \psi\ } z,$$

in which ϕ is egressive and p-cartesian and ψ is ingressive and p-cocartesian.

The proof will occupy the entirety of this section. This implies both Lm. 11.4 and Lm. 11.5 as special cases. (Note that (12.2.2) is an immediate consequence of the Beck–Chevvaley condition (10.3.3).) This result may also be used to give an alternative argument that $A^{eff}(C, C_\dagger, C^\dagger)$ is an ∞-category.

12.3. **Notation.** Suppose $m \geq 2$ and suppose $0 \leq k < m$ (note that we are excluding the case $k = m$), and suppose we are given a commutative square

$$\begin{array}{ccc} \Lambda^m_k & \longrightarrow & A^{eff}(C, C_\dagger, C^\dagger). \\ \big\downarrow & & \big\downarrow{\scriptstyle A^{eff}(p)} \\ \Delta^m & \longrightarrow & A^{eff}(D, D_\dagger, D^\dagger), \end{array}$$

and we seek a lift $\Delta^m \longrightarrow A^{eff}(C, C_\dagger, C^\dagger)$. This corresponds to a (solid) commutative square

$$\begin{array}{ccc} \widetilde{\mathscr{O}}(\Lambda^m_k)^{op} & \xrightarrow{g} & C \\ \big\downarrow & \overset{\overline{g}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ \widetilde{\mathscr{O}}(\Delta^m)^{op} & \xrightarrow[h]{} & D \end{array}$$

of simplicial sets in which h is ambigressive cartesian and, for every $i \neq k$, the restriction

$$\widetilde{\mathscr{O}}(\Delta^{\{0,\dots,\hat{\imath},\dots,m\}})^{op} \subset \widetilde{\mathscr{O}}(\Lambda^m_k)^{op} \xrightarrow{g} C$$

is ambigressive cartesian. Our objective then becomes to construct a (dotted) lift

$$\overline{g} \colon \widetilde{\mathscr{O}}(\Delta^m)^{op} \longrightarrow C$$

that is ambigressive cartesian.

12.4. **Definition.** Let us call a m-simplex

$$i_0 j_0 \longrightarrow i_1 j_1 \longrightarrow \cdots \longrightarrow i_m j_m$$

of $\widetilde{\mathscr{O}}(\Delta^m)^{op}$ ***completely factored*** if $i_r - i_{r-1} + j_{r-1} - j_r = 1$ for each $1 \leq r \leq m$. Note that any complete m-simplex is in particular nondegenerate, and $i_0 = 0$ and $j_0 = m$.

12.5**.** Completely factored m-simplices are essentially the same as walks on the poset $\widetilde{\mathscr{O}}([\mathbf{m}])^{op}$ that begin at the point $0m$ and end at a point of the form pp. We may therefore parametrize the completely factored m-simplices as follows. For each integer $0 \leq N \leq 2^m - 1$, let $\sigma(N)$ be the unique completely factored m-simplex

$$i_0 j_0 \longrightarrow i_1 j_1 \longrightarrow \cdots \longrightarrow i_m j_m$$

such that if d_r is the r-th binary digit of N (read left to right, so that $N = \sum_{s=1}^{m} 2^{m-s} d_s$), then

$$d_r = \begin{cases} 0 & \text{if } i_{r-1} = i_r; \\ 1 & \text{if } j_{r-1} = j_r. \end{cases}$$

We order these simplices accordingly. Hence $\sigma(N)$ is the m-simplex

$$i_0 j_0 \longrightarrow i_1 j_1 \longrightarrow \cdots \longrightarrow i_m j_m$$

with

$$i_r = \sum_{s=1}^{r} d_s \quad \text{and} \quad j_r = (m - r) + \sum_{s=1}^{r} d_s.$$

Fig. 1 shows the completely factored 5-simplex $\sigma(01101) = \sigma(13) \subset \widetilde{\mathscr{O}}(\Delta^m)^{op}$.

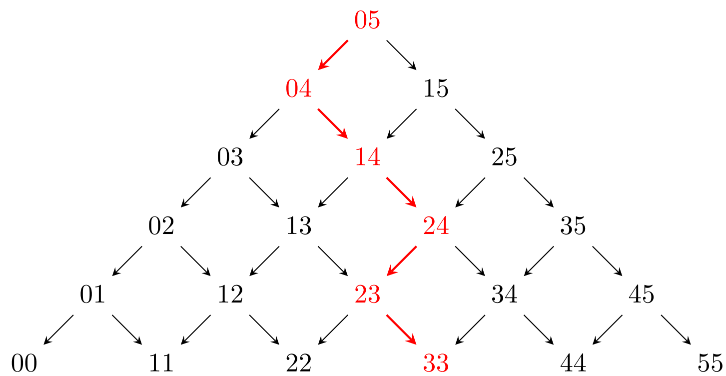


Figure 1. The completely factored 5-simplex $\sigma(01101) = \sigma(13) \subset \widetilde{\mathscr{O}}(\Delta^5)^{op}$, drawn in red as a walk from 05 to 33. The juts of this 5-simplex are 3 and 5. There are no crossings away from 1; if $k \neq 1$, the only crossing away from 1 is 2.

Now for any integer $0 \leq N \leq 2^m$, write

$$P_N(k) := \widetilde{\mathscr{O}}(\Lambda^m_k)^{op} \cup \bigcup_{0 \leq K < N} \sigma(K);$$

this provides a filtration

$$\widetilde{\mathscr{O}}(\Lambda^m_k)^{op} = P_0(k) \subset \cdots \subset P_{2^m}(k) = \widetilde{\mathscr{O}}(\Delta^m)^{op}.$$

Our aim is to find conditions that permit us to extend g along this filtration.

We proceed to analyze the intersections $\sigma(N) \cap P_N(k)$ as a subset of $\sigma(N) \cong \Delta^m$. We will find that each $\sigma(N) \cap P_N(k)$ is in fact a union of faces of Δ^m. There are two kinds of faces that will appear in this intersection. To describe these, let's introduce some simplifying notation.

12.6. **Notation.** For any nonempty totally ordered finite set T and any element $j \in T$, write $\Delta^{\hat{\jmath}}$ for the face

$$\Delta^{T-\{j\}} \subset \Delta^T.$$

More generally, for any ordered subsets $S \subset T$, write $\Lambda^T_S \subset \Delta^T$ for the union of all the faces (i.e., $\#T$-simplices) of Δ^T that contain the simplex Δ^S. In other words, let

$$\Lambda^T_S := \bigcup_{j \notin S} \Delta^{\hat{\jmath}}.$$

When $T = \{0, \ldots, m\}$, we just write Λ^m_S for Λ^T_S.

In this notation, we have

$$\sigma(N) \cap P_N(k) = \left(\bigcup_{0 \leq K < N} \sigma(N) \cap \sigma(K) \right) \cup \left(\bigcup_{j \neq k} \sigma(N) \cap \widetilde{\mathscr{O}}(\Delta^{\hat{\jmath}})^{op} \right),$$

and our claim is that there is a set $E(N,k) \subset \{0, \ldots, m\}$ such that

$$\sigma(N) \cap P_N(k) = \Lambda^m_{E(N,k)}.$$

We proceed to construct this set now.

12.7. **Definition.** Suppose N an integer such that $0 \leq N \leq 2^m - 1$, written as $N = \sum_{s=1}^{m} 2^{m-s} d_s$. A ***jut*** of the completely factored m-simplex $\sigma(N)$ is an integer $z \in \{1, \dots, m\}$ such that

- $d_z = 1$, and
- either $d_{z+1} = 0$ or $z = m$.

Denote by $Z(N) \subset \{1, \dots, m\}$ the set of juts of $\sigma(N)$.

For any jut z, write

$$K_z = \sum_{s=1}^{m} 2^{m-s} d_{z,s},$$

where

$$d_{z,s} = \begin{cases} d_s & \text{if } s \notin \{z, z+1\}; \\ 0 & \text{if } s = z; \\ 1 & \text{if } s = z+1. \end{cases}$$

12.8. **Lemma.** *Suppose N an integer such that $0 \leq N \leq 2^m - 1$, written as $N = \sum_{s=1}^{m} 2^{m-s} d_s$. Then*

$$\bigcup_{0 \leq K < N} \sigma(N) \cap \sigma(K) = \bigcup_{z \in Z(N)} \Delta^{\hat{z}}.$$

Proof. It is easy to see that in the poset of simplicial subsets of $\sigma(N)$ of the form $\sigma(N) \cap \sigma(K)$, the maximal elements consist of those subsets of the form $\sigma(N) \cap \sigma(K_z)$, where z in a jut of $\sigma(N)$. Of course $\Delta^{\hat{z}} = \sigma(N) \cap \sigma(K_z)$. □

12.9. **Definition.** Suppose N an integer such that $0 \leq N \leq 2^m - 1$ written as $N = \sum_{s=1}^{m} 2^{m-s} d_s$. A ***crossing*** of $\sigma(N)$ away from k is an integer $x \in \{0, \dots, m-1\}$ such that one of the following holds:

- $x = 0$, $d_1 = 0$;
- $x = 0$, $d_1 = 1$, and $k \neq 0$;
- $x > 0$, $d_x = d_{x+1} = 1$, and $i_x \neq k$; or
- $x > 0$, $d_x = d_{x+1} = 0$ and $j_x \neq k$.

Denote by $X(N, k) \subset \{0, \dots, m-1\}$ the set of crossings away from k.

The crossings away from k are now all we need to complete our computation of the intersections $\sigma(N) \cap P_N(k)$.

12.10. **Proposition.** *Suppose N an integer such that $0 \leq N \leq 2^m - 1$, written as $N = \sum_{s=1}^{m} 2^{m-s} d_s$. Then*

$$\sigma(N) \cap P_N(k) = \bigcup_{y \in Z(N) \cup X(N,k)} \Delta^{\hat{y}}.$$

Proof. For any crossing x of $\sigma(N)$ away from k, it is clear that the corresponding face is given by

$$\Delta^{\hat{x}} = \begin{cases} \sigma(N) \cap \widetilde{\mathscr{O}}(\Delta^{\hat{\imath}_x})^{op} & \text{if } d_x = 1; \\ \sigma(N) \cap \widetilde{\mathscr{O}}(\Delta^{\hat{\jmath}_x})^{op} & \text{if } d_x = 0. \end{cases}$$

Now for any $j \in \{0, \dots, m\}$, if the set

$$\{r \in \{0, \dots, m\} \mid i_r = j \text{ or } j_r = j\}$$

contains more than one element, then it contains a jut z, and consequently,

$$\sigma(N) \cap \widetilde{\mathscr{O}}(\Delta^{\hat{j}})^{op} \subset \Delta^{\hat{z}} = \sigma(N) \cap \sigma(K_z). \qquad \square$$

12.11. **Warning.** Note that this doesn't quite work if $k = m$ (which we expressly excluded): in this case it is just not true that $\sigma(N) \cap P_N(k)$ is a union of faces. For example, in the completely factored 5-simplex $\sigma(13)$ depicted in Fig. 1, if $k = 5$, then the simplicial subset $\sigma(13) \cap P_{13}(5) \subset \sigma(13)$ is the union

$$\Delta^{\hat{3}} \cup \Delta^{\hat{5}} \cup \Delta^{\{2,3,4,5\}}.$$

Let us reformulate what we have shown.

12.12. **Definition.** Suppose N an integer such that $0 \leq N \leq 2^m - 1$ written as $N = \sum_{s=1}^{m} 2^{m-s} d_s$. Let us call an integer $s \in \{0, \dots, m\}$ an ***essential vertex*** of $\sigma(N)$ for k if it is neither a jut nor a crossing away from k. Denote by

$$E(N, k) := \{0, \dots, m\} - (Z(N) \cup X(N, k))$$

the ordered set of essential vertices of $\sigma(N)$ for k.

We have thus shown that we may write

$$\sigma(N) \cap P_N(k) = \Lambda^m_{E(N,k)} \subset \Delta^m \cong \sigma(N).$$

Now we want to extend g along each inclusion $P_{N-1}(k) \hookrightarrow P_N(k)$, which we now write as pushout

$$\begin{array}{ccc} \Lambda^m_{E(N,k)} & \longrightarrow & \Delta^m \\ \downarrow & & \downarrow \\ P_N(k) & \longrightarrow & P_{N+1}(k). \end{array}$$

For this, we need to determine just what sort of inclusions these ore. For example, if $\Lambda^m_{E(N,k)} \hookrightarrow \Delta^m$ is inner anodyne, one has the desired extension simply because p is an inner fibration. Let's determine precisely when this does the job.

12.13. **Lemma.** *Suppose $S \subset \{0, \dots, m\}$ a nonempty ordered subset. Then the inclusion $\Lambda^m_S \hookrightarrow \Delta^m$ is inner anodyne if the following condition holds.*

(∗) *there exists elements $a < s < b$ of $\{0, \dots, m\}$ such that $s \in S$, but $a, b \notin S$.*

Proof. The claim is trivial if either $m = 2$ or S has cardinality 1. For $m \geq 3$ and $\#S \geq 2$, assume that the result holds both for all smaller values of m and for subsets S of smaller cardinality.

Choose an element $s \in S$ as follows: if $0 \in S$, let $s = 0$; otherwise, if $m \in S$, let $s = m$; otherwise choose $s \in S$ arbitrarily. Then the subset

$$S - \{s\} \subset \{0, \dots, \widehat{s}, \dots, m\}$$

satisfies condition (∗) for $m - 1$; hence the inclusion

$$\Lambda^m_S \cap \Delta^{\hat{s}} = \Lambda^{\{0,\dots,\widehat{s},\dots,m\}}_{S-\{s\}} \hookrightarrow \Delta^{\{0,\dots,\widehat{s},\dots,m\}}$$

is inner anodyne by the inductive hypothesis. The pushout of this edge along the inclusion

$$\Lambda^m_S \cap \Delta^{\hat{s}} \hookrightarrow \Lambda^m_S$$

is the inclusion

$$\Lambda^m_S \hookrightarrow \Lambda^m_{S-\{s\}},$$

which is thus inner anodyne. Now our claim follows from the observation that the subset $S-\{s\} \subset \{0, \ldots, m\}$ also satisfies condition $(*)$, whence the inclusion

$$\Lambda^m_{S-\{s\}} \hookrightarrow \Delta^m$$

is also inner anodyne by the inductive hypothesis. □

Suppose N an integer such that $0 \leq N \leq 2^m - 1$ written as $N = \sum_{s=1}^m 2^{m-s} d_s$. We have shown that if $E(N,k)$ satisfies condition $(*)$, then $P_N(k) \hookrightarrow P_{N+1}(k)$ is inner anodyne. If however $E(N,k)$ fails condition $(*)$, let's refer to $\sigma(N)$ as an ***exceptional case***.

Indeed, such an m-simplex is quite exceptional: it cannot contain more than one jut, since there must be an essential vertex between any two juts. Consequently, for any $t \in \{0, \ldots, m\}$, if

$$d_s = \begin{cases} 1 & \text{if } s \leq t; \\ 0 & \text{if } s > t, \end{cases}$$

then let's write $N_t := \sum_{s=1}^m 2^{m-s} d_s$. Then $\sigma(N_t)$ is the walk

$$0m \longrightarrow 1m \longrightarrow \cdots \longrightarrow tm \longrightarrow t(m-1) \cdots \longrightarrow tt.$$

One thus sees that any exceptional case is of the form $\sigma(N_t)$.

Furthermore, the simplex $\sigma(N_t)$ begins with a crossing if either $k \neq 0$ or $N_t < 2^{m-1}$, so $0 \in E(N_t, k)$ if and only if both $k = 0$ and $N_t \geq 2^{m-1}$. Dually, $\sigma(N_t)$ ends in a jut precisely when N_t is odd, so $m \in E(k)$ if and only if N_t is even. Now a quick analysis of the location of the crossings away from k yields the following classification of all the exceptional cases.

12.14. **Proposition.** *When $k \neq 0$, there are only two exceptional cases:*

(12.14.1) $t = k-1$, *in which case* $E(N_{k-1}, k) = \{m-1, m\}$.
(12.14.2) $t = k$, *in which case* $E(N_k, k) = \{m\}$.

When $k = 0$, there are $t+1$ exceptional cases:

(12.14.3) $t = 0$, *in which case* $E(0,0) = \{m\}$.
(12.14.4) $0 < t < m$, *in which case* $E(N_t, 0) = \{0, m\}$.
(12.14.5) $t = m$, *in which case* $E(2^m - 1, 0) = \{0\}$.

To illustrate, in Fig. 2, the two exceptional cases in $\widetilde{\mathscr{O}}(\Delta^5)^{op}$ for $k = 3$ are depicted.

We can now begin the proof of Th. 12.2. Here is the first bit.

12.15. **Lemma.** *Suppose $0 < k < m$. Then there exists a (dotted) lift*

$$\begin{array}{ccc} \widetilde{\mathscr{O}}(\Lambda^m_k)^{op} & \xrightarrow{g} & C \\ \downarrow & \overset{\overline{g}}{\nearrow} & \downarrow p \\ \widetilde{\mathscr{O}}(\Delta^m)^{op} & \xrightarrow[h]{} & D \end{array}$$

Proof. Let's handle the case $m = 2$ separately. Note that $\widetilde{\mathscr{O}}(\Delta^2)^{op} = (\widetilde{\mathscr{O}}(\Delta^2_1)^{op})^{\triangleleft}$, so we may form the desired lift $\overline{g}$ simply by forming the p-limit in the sense of [25, Df. 4.3.1.1].

For $m \geq 3$, of course we will proceed by induction on the filtration

$$\widetilde{\mathscr{O}}(\Lambda^m_k)^{op} = P_0(k) \subset \cdots \subset P_{2^m}(k) = \widetilde{\mathscr{O}}(\Delta^m)^{op}.$$

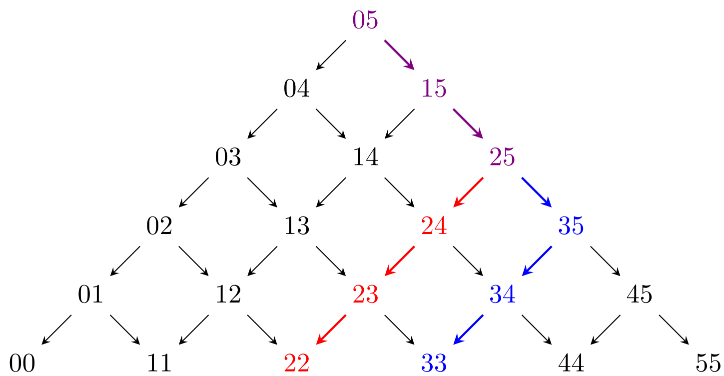


FIGURE 2. The two exceptional cases $\sigma(N_2)$ and $\sigma(N_3)$ for $k = 3$ in $\widetilde{\mathscr{O}}(\Delta^5)^{op}$, drawn in red and blue (respectively) as walks from 05 to 22 and from 05 to 33.

Given a lift

$$\begin{array}{ccc} \widetilde{\mathscr{O}}(\Lambda^m_k)^{op} & \xrightarrow{g} & C \\ \downarrow & \nearrow_{g_N} & \downarrow p \\ P_N(k) & \xrightarrow[h|P_N(k)]{} & D, \end{array}$$

we seek a (dotted) lift

$$\begin{array}{ccc} P_N(k) & \xrightarrow{g_N} & C \\ \downarrow & \nearrow_{g_{N+1}} & \downarrow p \\ P_{N+1}(k) & \xrightarrow[h|P_{N+1}(k)]{} & D. \end{array}$$

The only catch will be that we must choose the extensions to the exceptional m-simplices and some of their neighbors carefully.

To begin, for $0 \leq N < N_{k-1}$, we use the right lifting property with respect to the inner anodyne inclusions $\sigma(N) \cap P_N(k) \hookrightarrow \sigma(N)$ to obtain the desired lift

$$g_{N_{k-1}} \colon P_{N_{k-1}}(k) \longrightarrow C.$$

Now let us call a completely factored m-simplex $\sigma(N)$ ***special*** just in case the corresponding integer $N = \sum_{s=1}^{m} 2^{m-s} d_s$ has the property that $d_s = 1$ for every $1 \leq s \leq k-1$, and no more than one of $d_k, \ldots, d_m$ is equal to 1. Let $R(k)$ be the collection of those N such that $\sigma(N)$ is special. Note that the exceptional m-simplex $\sigma(N_{k-1})$ of (12.14.1) is the first special m-simplex, and the exceptional m-simplex $\sigma(N_k)$ of (12.14.2) is the last special m-simplex. Also observe that for any $N \in R(k)$, one has

$$\sigma(N) \cap P_N = (\sigma(N) \cap P_{N_{k-1}}) \cup \bigcup_{K \in R(k),\ K<N} (\sigma(N) \cap \sigma(K)).$$

Now we have a functor

$$f_k \colon \Delta^1 \times \Delta^{m-1} \longrightarrow \widetilde{\mathscr{O}}(\Delta^m)^{op}$$

that is determined on its value on objects:

$$f_k(u,v) := (\max\{0, v-k+1\}, \max\{m-v-u, k-u\}).$$

The functor f_k restricts to a functor

$$(\Delta^1 \times \Lambda^{m-1}_{m-1}) \cup^{(\Delta^{\{1\}} \times \Lambda^{m-1}_{m-1})} (\Delta^{\{1\}} \times \Delta^{m-1}) \longrightarrow P_{N_{k-1}}(k).$$

Note that (dotted) lifts in the diagram

$$\begin{array}{ccc} (\Delta^1 \times \Lambda^{m-1}_{m-1}) \cup^{(\Delta^{\{1\}} \times \Lambda^{m-1}_{m-1})} (\Delta^{\{1\}} \times \Delta^{m-1}) & \xrightarrow{g_{N_{k-1}}} & C \\ \downarrow & \overset{g'}{\nearrow} & \downarrow p \\ \Delta^1 \times \Delta^{m-1} & \xrightarrow[h|(\Delta^1 \times \Delta^{m-1})]{} & D \end{array}$$

are in bijection with (dotted) lifts in the adjoint diagram

$$\begin{array}{ccc} \Lambda^{m-1}_{m-1} & \longrightarrow & \mathrm{Fun}(\Delta^1, C) \\ \downarrow & \nearrow & \downarrow (t,p) \\ \Delta^{m-1} & \xrightarrow[\eta]{} & \mathrm{Fun}(\Delta^{\{1\}}, C) \times_{\mathrm{Fun}(\Delta^{\{1\}}, D)} \mathrm{Fun}(\Delta^1, D). \end{array}$$

Such a lift exists by [25, Lm. 6.1.1.1]. Consequently, a lift g' exists, and it specifies a family of maps

$$g'_N \colon \sigma(N) \longrightarrow C$$

for $N \in R(k)$ with $N > N_{k-1}$.

Now to obtain a (dotted) lift

$$\begin{array}{ccc} P_{N_{k-1}}(k) & \xrightarrow{g_{N_{k-1}}} & C \\ \downarrow & \overset{g_{N_{k-1}+1}}{\nearrow} & \downarrow p \\ P_{N_{k-1}+1}(k) & \xrightarrow[h|P_{N_{k-1}+1}(k)]{} & D, \end{array}$$

we must extend along

$$\sigma(N_{k-1}) \cap P_{N_{k-1}} \cong \Lambda^m_{\{m-1,m\}} \hookrightarrow \Delta^m \cong \sigma(N_{k-1}),$$

which we factor as the composite

$$\Lambda^m_{\{m-1,m\}} \hookrightarrow \Lambda^m_{\{m-1,m\}} \cup^{\Lambda^{\{0,\dots,m-1\}}_{m-1}} \Delta^{\{0,\dots,m-1\}} \cong \Lambda^m_{m-1} \hookrightarrow \Delta^m.$$

We extend across the first inclusion by using the restriction of the lift g' we have constructed, and then we extend across the inner horn $\Lambda^m_{m-1} \hookrightarrow \Delta^m$ using the fact that p is an inner fibration.

Now for any $N > N_{k-1}$, extend across P_{N+1} as follows: if $N \in R(k)$, extend using the chosen map g'_N, which since

$$\sigma(N) \cap P_N = (\sigma(N) \cap P_{N_{k-1}}) \cup \bigcup_{K \in R(k),\ K<N} (\sigma(N) \cap \sigma(K)),$$

is compatible with the map $P_N \longrightarrow C$ constructed so far. If $N \notin R(k)$, simply extend using the fact that $\sigma(N) \cap P_N \hookrightarrow \sigma(N)$ is an inner horn inclusion. At the end of this procedure, the desired extension

$$\overline{g} \colon P_{2^m} = \widetilde{\mathscr{O}}(\Delta^m)^{op} \longrightarrow C$$

is constructed, and it is ambigressive cartesian by construction. □

Now let us complete the proof of the theorem.

12.16. **Lemma.** *Suppose p satisfies conditions (12.2.1-3) and that $k = 0$. If the morphism $g(01) \twoheadrightarrow g(00)$ is p-cartesian and the morphism $g(01) \rightarrowtail g(11)$ is p-cocartesian, then there is an ambigressive cartesian (dotted) lift*

$$\begin{array}{ccc} \widetilde{\mathscr{O}}(\Lambda^m_k)^{op} & \xrightarrow{g} & C \\ \downarrow & \overset{\overline{g}}{\nearrow} & \downarrow p \\ \widetilde{\mathscr{O}}(\Delta^m)^{op} & \xrightarrow[h]{} & D \end{array}$$

Proof. Again let's treat the case $m = 2$ separately. In this case, since the morphism $g(0,1) \twoheadrightarrow g(0,0)$ is p-cartesian, we obtain a 2-simplex

$$\begin{array}{ccccc} & & g(0,1) & & \\ & \nearrow & & \searrow & \\ g(0,2) & & \twoheadrightarrow & & g(0,0). \end{array}$$

Now after filling inner horns, we choose a p-cocartesian edge $g(0,2) \rightarrowtail g(1,2)$ lying over $h(0,2) \rightarrowtail h(1,2)$, and then by filling the corresponding outer horns, we obtain a diagram

$$\begin{array}{ccccc} & & g(0,2) & & \\ & \swarrow & & \searrow & \\ g(0,1) & & & & g(0,2) \\ \swarrow \quad \searrow & & & & \swarrow \quad \searrow \\ g(0,0) \qquad & & g(1,1) & & \qquad g(2,2) \end{array}$$

It follows from conditions (12.2.1-2) that the morphisms are ingressive or egressive as marked and that the square is ambigressive cartesian.

For $m \geq 3$, we will once again proceed by induction on the filtration

$$\widetilde{\mathscr{O}}(\Lambda^m_k)^{op} = P_0(k) \subset \cdots \subset P_{2^m}(k) = \widetilde{\mathscr{O}}(\Delta^m)^{op}.$$

Given a lift

$$\begin{array}{ccc} \widetilde{\mathscr{O}}(\Lambda^m_k)^{op} & \xrightarrow{g} & C \\ \downarrow & \overset{g_N}{\nearrow} & \downarrow p \\ P_N(k) & \xrightarrow[h|P_N(k)]{} & D, \end{array}$$

we seek a (dotted) lift

$$\begin{array}{ccc} P_N(k) & \xrightarrow{g_N} & C \\ \downarrow & \overset{g_{N+1}}{\nearrow} & \downarrow p \\ P_{N+1}(k) & \xrightarrow[h|P_{N+1}(k)]{} & D. \end{array}$$

Once again, one really only has to tiptoe around the exceptional m-simplices.

To begin, we may easily extend g along the inclusion

$$\sigma(0) \cap P_0 \cong \Lambda^m_m \hookrightarrow \Delta^m \cong \sigma(0)$$

(the exceptional m-simplex of type (12.14.3)), since the edge $g|\Delta^{\{m-1,m\}}$ is p-cartesian.

Now for $0 < N < 2^m - 1$, we have two options for the inclusion

$$\sigma(N) \cap P_N \cong \Lambda^m_{E(N,0)} \hookrightarrow \Delta^m \cong \sigma(N):$$

either it is inner anodyne, in which case it is easy to extend along it, using the fact that p is an inner fibration, or else N is exceptional of type (12.14.4), and hence $N = N_t$ for some integer $0 < t < m$, and $E(N_t, 0) = \{0, m\}$.

To extend along the inclusion

$$\sigma(N_t) \cap P_{N_t} \cong \Lambda^m_{\{0,m\}} \hookrightarrow \Delta^m \cong \sigma(N_t),$$

we factor it as the composite

$$\Lambda^m_{\{0,m\}} \hookrightarrow \Lambda^m_{\{0,m\}} \cup^{\Lambda_0^{\{0,\dots,m-1\}}} \Delta^{\{0,\dots,m-1\}} \cong \Lambda^m_0 \hookrightarrow \Delta^m.$$

Extensions along each of these inclusions exists simply because the edge $g|\Delta^{\{0,1\}}$ is p-cocartesian.

At the end of this procedure, we are left with an extension $P_{2^m-1} \longrightarrow C$. To extend over $\sigma(2^m - 1)$ (the exceptional m-simplex of type (12.14.5)), it suffices just to note that, by assumption, $g|\Delta^{\{0,1\}}$ is p-cocartesian, so one may extend over the inclusion

$$\sigma(2^m - 1) \cap P_{2^m-1} \cong \Lambda^m_0 \hookrightarrow \Delta^m \cong \sigma(2^m - 1).$$

The result is the desired extension

$$\overline{g}\colon P_{2^m} = \widetilde{\mathscr{O}}(\Delta^m)^{op} \longrightarrow C,$$

which is ambigressive cartesian by construction. □

The proof of Th. 12.2 is complete.

13. The Burnside Waldhausen bicartesian fibration

Perhaps the most important Waldhausen bicartesian fibration is the one whose algebraic K-theory will be the spectral Burnside Mackey functor $\mathbf{S}_{(C,C_\dagger,C^\dagger)}$. To describe it, we need some preparatory material.

13.1. **Notation.** In this subsection, let us fix a disjunctive triple $(C, C_\dagger, C^\dagger)$.

13.2. **Definition.** Let us say a morphism $X \longrightarrow U$ of C is a ***summand inclusion*** if there exists a morphism $X' \longrightarrow U$ of C that, together with $X \longrightarrow U$, exhibits U as the coproduct $X \sqcup X'$.

Now if $i\colon X \hookrightarrow U$ is a summand inclusion, a ***complement*** of i is a summand inclusion $i'\colon X' \hookrightarrow U$ such that any square

$$\begin{array}{ccc} \varnothing & \hookrightarrow & X \\ \downarrow & & \downarrow i \\ X' & \underset{i'}{\hookrightarrow} & U \end{array}$$

in which $\varnothing$ is initial in C is a pushout square.

13.3. The compatibility of ingressive and egressive morphisms with coproducts implies that summand inclusions are necessarily ingressive and egressive. Note also that the pullback of a summand inclusion along a morphism that is both ingressive and egressive is again a summand inclusion. Furthermore, the pushout of a summand inclusion $i\colon X \hookrightarrow U$ along any map $f\colon X \longrightarrow Y$ exists and is again a summand inclusion. Finally, a complement for i is a complement for the pushout $j\colon Y \hookrightarrow V$ of i along f.

The following lemma will allow us to formulate definitions using complements, as long as we do not use any non-homotopy-invariant constructions.

13.4. **Lemma.** *Suppose $i\colon X \hookrightarrow U$ a summand inclusion. If $\mathrm{Compl}(i) \subset C_{/U}$ denotes the full subcategory spanned by the complements of i, then the Kan complex $\iota\mathrm{Compl}(i)$ is contractible.*

Proof. We show that the diagonal map $\iota\mathrm{Compl}(i) \longrightarrow \iota\mathrm{Compl}(i) \times \iota\mathrm{Compl}(i)$ is a weak equivalence. To this end, we observe that since C admits all ambigressive pullbacks, it follows that the full subcategory $\mathrm{Sum}(U) \subset C_{/U}$ spanned by the summand inclusions admits all finite products. Consequently, the diagonal functor $\mathrm{Sum}(U) \longrightarrow \mathrm{Sum}(U) \times \mathrm{Sum}(U)$ admits a right adjoint, which is given informally by the assignment

$$(X'_1 \hookrightarrow U,\ X'_2 \hookrightarrow U) \longmapsto (X'_1 \times_U X'_2 \hookrightarrow U).$$

Our claim is that this right adjoint restricts to a quasi-inverse

$$\iota\mathrm{Compl}(i) \times \iota\mathrm{Compl}(i) \longrightarrow \iota\mathrm{Compl}(i)$$

of the diagonal. For this, we must show that if

$$i'_1\colon X'_1 \hookrightarrow U \quad \text{and} \quad i'_2\colon X'_2 \hookrightarrow U$$

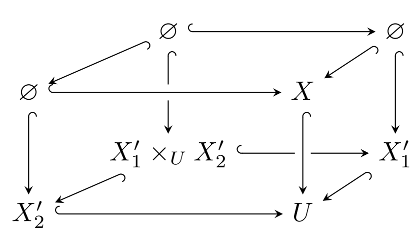
are complements of i, then the projection maps

$$X'_1 \times_U X'_2 \longrightarrow X'_1 \quad \text{and} \quad X'_1 \times_U X'_2 \longrightarrow X'_2$$

are equivalences, and the morphism $i'_{12}\colon X'_1 \times_U X'_2 \hookrightarrow U$ is a complement of i. Indeed, the universality of coproducts implies that the projection $X'_1 \times_U X'_2 \longrightarrow X'_1$ factors as

$$X'_1 \times_U X'_2 \xrightarrow{\sim} (X'_1 \times_U X'_2) \sqcup (X'_1 \times_U X) \simeq X'_1 \times_U (X'_2 \sqcup X) \simeq X'_1 \times_U U \simeq X'_1$$

(and similarly for the projection $X'_1 \times_U X'_2 \longrightarrow X'_2$). Now in the cube



every face is a pullback, and all faces but the top and bottom squares are pushouts, whence i'_{12} is a complement of i. □

13.5. **Definition.** We consider the fibration

$$p\colon \mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, C) \cong \mathrm{Fun}(\Delta^2, C) \times_{\mathrm{Fun}(\Delta^{\{0,2\}},C)} C \longrightarrow C.$$

We may think of the objects of the ∞-category $\mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, C)$ as retract diagrams

$$S_0 \longrightarrow S_1 \longrightarrow S_0;$$

the functor p is given by the assignment

$$[S_0 \longrightarrow S_1 \longrightarrow S_0] \longmapsto S_0.$$

We therefore denote by $C_{S_0/\ /S_0}$ the fiber of p over an object $S_0 \in C$.

We consider the full subcategory $\mathscr{R}(C) \subset \mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, C)$ spanned by those objects S such that the morphism $S_0 \longrightarrow S_1$ is a summand inclusion. We endow $\mathscr{R}(C)$ with the structure of a pair in the following manner. A morphism $T \longrightarrow S$ will be declared ingressive just in case $T_0 \longrightarrow S_0$ is an equivalence, and $T_1 \longrightarrow S_1$ is a summand inclusion.

Now let $\mathscr{R}(C, C_\dagger, C^\dagger) \subset \mathscr{R}(C)$ be the full subcategory spanned by those objects $S\colon \Delta^2/\Delta^{\{0,2\}} \longrightarrow C$ such that for any complement $S_0' \hookrightarrow S_1$ of the summand inclusion $S_0 \hookrightarrow S_1$,

(13.5.1) the essentially unique morphism $S_0' \longrightarrow *$ to the terminal object of C is egressive, and

(13.5.2) the composite $S_0' \longrightarrow S_1 \longrightarrow S_0$ is ingressive.

We endow $\mathscr{R}(C, C_\dagger, C^\dagger)$ with the pair structure induced by $\mathscr{R}(C)$. We will abuse notation by denoting the restriction of the functor $p\colon \mathscr{R}(C) \longrightarrow C$ to the subcategory $\mathscr{R}(C, C_\dagger, C^\dagger) \subset \mathscr{R}(C)$ again by p.

We will now show that p is a Waldhausen bicartesian fibration. This claim follows from the following sequence of observations.

13.6**.** For any object S_0 of C, the fiber $\mathscr{R}(C)_{S_0}$ can be identified with the full subcategory of $C_{S_0/\ /S_0}$ spanned by those objects U such that $S \hookrightarrow U$ is a summand inclusion. A morphism $S_1 \longrightarrow S_1'$ of this ∞-category is ingressive just in case it is a summand inclusion. It is an easy consequence of the existence of finite coproducts and the compatibility of the triple structure with these coproducts that the full subcategory $\mathscr{R}(C, C_\dagger, C^\dagger)_{S_0} \subset \mathscr{R}(C)_{S_0}$ is a Waldhausen ∞-category.

13.7**.** For any ingressive morphism $f\colon S_0 \rightarrowtail T_0$ and for any object $S \in \mathscr{R}(C, C_\dagger, C^\dagger)$ over S_0, there exists a pushout diagram

$$\begin{array}{ccc} S_0 & \rightarrowtail & T_0 \\ \downarrow & & \downarrow \\ S_1 & \rightarrowtail & T_1 \\ \downarrow & & \downarrow \\ S_0 & \rightarrowtail & T_0, \end{array}$$

hence a p-cocartesian edge covering f. The compatibility of the triple structure with coproducts ensures that this defines a functor

$$f_!\colon \mathscr{R}(C, C_\dagger, C^\dagger)_{S_0} \longrightarrow \mathscr{R}(C, C_\dagger, C^\dagger)_{T_0}.$$

The functors $f_!$ are exact because they are all left adjoints, which preserve any colimits that exist.

13.8. Dually, for any egressive morphism $f\colon S_0 \twoheadrightarrow T_0$ and for any object $T \in \mathscr{R}(C, C_\dagger, C^\dagger)$ over T_0, there exists a pullback diagram

$$\begin{array}{ccc} S_0 & \twoheadrightarrow & T_0 \\ \downarrow & & \downarrow \\ S_1 & \twoheadrightarrow & T_1 \\ \downarrow & & \downarrow \\ S_0 & \twoheadrightarrow & T_0. \end{array}$$

We claim that the functor $f^\star\colon \mathscr{R}(C)_{T_0} \longrightarrow \mathscr{R}(C)_{S_0}$ is given informally by the assignment $T_1 \longmapsto T_1 \times_{T_0} S_0$. This follows from the fact that for any cofibration $T_0' \rightarrowtail T_0$ such that the morphism $T_0' \twoheadrightarrow *$ is egressive, the pullback $T_0' \times_{T_0} S_0 \rightarrowtail S_0$ is a cofibration and the morphism $T_0' \times_{T_0} S_0 \longrightarrow *$ is egressive. This follows from the fact that the pullback of an egressive map is egressive, and the pullback of an ingressive map along an egressive map is ingressive. The universality of finite coproducts in C ensures that the functors $f^\star$ preserve finite coproducts; in particular the functors $f^\star$ preserve summand inclusions and pushouts along summand inclusions.

13.9. Now suppose I a finite set, and suppose $\{X_i \mid i \in I\}$ a collection of objects of C indexed by the elements of I with coproduct X. We claim that the functor

$$C_{/X} \xrightarrow{\sim} \prod_{i\in I} C_{/X_i}$$

induced by the inclusions $X_i \hookrightarrow X$ induce an equivalence

$$\mathscr{R}(C, C_\dagger, C^\dagger)_X \xrightarrow{\sim} \prod_{i\in I} \mathscr{R}(C, C_\dagger, C^\dagger)_{X_i}.$$

One need only note that both this functor, which is given by pullbacks along summand inclusions (which are egressive), and its left adjoint, which is given by coproduct, preserve the desired subcategories and restrict to adjoint equivalences.

13.10. Finally, the base change condition for $\mathscr{R}(C, C_\dagger, C^\dagger)$ states that for any ambigressive pullback square

$$\begin{array}{ccc} S_0 & \overset{i}{\rightarrowtail} & S_0' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ T_0 & \underset{j}{\rightarrowtail} & T_0', \end{array}$$

of C and for any object T_1' of C over T_0, the base change morphism

$$((T_0 \sqcup T_1') \times_{T_0} S_0) \cup^{S_0} S_0' \longrightarrow ((T_0 \sqcup T_1') \cup^{T_0} T_0') \times_{T_0'} S_0'$$

is an equivalence. This follows from the identifications

$$\begin{aligned}((T_0 \sqcup T_1') \times_{T_0} S_0) \cup^{S_0} S_0' &\simeq (S_0 \sqcup (T_1' \times_{T_0} S_0)) \cup^{S_0} S_0' \\ &\simeq S_0' \sqcup (T_1' \times_{T_0} S_0) \\ &\simeq S_0' \sqcup (T_1' \times_{T_0'} S_0') \\ &\simeq (T_0' \sqcup T_1') \times_{T_0'} S_0' \\ &\simeq ((T_0 \sqcup T_1') \cup^{T_0} T_0') \times_{T_0'} S_0',\end{aligned}$$

which all follow from the universality of finite coproducts in C and the equivalence $S_0 \simeq T_0 \times_{T_0'} S_0'$.

We thus conclude the following.

13.11. **Theorem.** *For any disjunctive triple $(C, C_\dagger, C^\dagger)$ that is either left or right complete, the functor*

$$p\colon \mathscr{R}(C, C_\dagger, C^\dagger) \longrightarrow C$$

is a Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$.

Now let's unfurl this Waldhausen bicartesian fibration to obtain a Waldhausen cocartesian fibration

$$\Upsilon(p)\colon \Upsilon(\mathscr{R}(C, C_\dagger, C^\dagger)) \longrightarrow A^{eff}(C, C_\dagger, C^\dagger),$$

whence we obtain a Mackey functor

$$\mathscr{M}_p\colon A^{eff}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Wald}_\infty.$$

Note that the assignment

$$[1 \twoheadleftarrow U \rightarrowtail X] \longmapsto [X \hookrightarrow X \sqcup U \longrightarrow X]$$

defines a functor

$$A^{eff}(C, C_\dagger, C^\dagger)_{1/} \longrightarrow \iota_{A^{eff}(C,C_\dagger,C^\dagger)}\mathscr{R}(C, C_\dagger, C^\dagger)$$

of left fibrations over $A^{eff}(C, C_\dagger, C^\dagger)$, and it follows from Lm. 13.4 that it is a fiberwise equivalence. Consequently, we deduce that the functor $\iota \circ \mathscr{M}_p$ is naturally equivalent to the functor represented by the terminal object 1.

Now, almost by definition, the Waldhausen ∞-category $\mathscr{M}_p(S) \simeq \mathscr{R}(C, C_\dagger, C^\dagger)_S$ is ι-split. We therefore conclude the following.

13.12. **Theorem.** *For any disjunctive triple $(C, C_\dagger, C^\dagger)$ that is either left or right complete, the functor*

$$\mathbf{K} \circ \mathscr{M}_p\colon A^{eff}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Sp}$$

is the Burnside Mackey functor $\mathbf{S}_{(C,C_\dagger,C^\dagger)}$ — i.e., the Mackey functor represented by the terminal object $1 \in C$ (Df. 8.1).

Example A. Coherent n-topoi and the Segal–tom Dieck splitting

The similarities between the axioms for a disjunctive ∞-category and the Giraud axioms for n-topoi [25, Th. 6.1.0.6(3)] suggest a deep relationship between the two notions. Here, we briefly describe a way for higher topoi to give rise to a disjunctive ∞-category.

A.1. **Example.** Any subcategory of an n-topos ($1 \leq n \leq \infty$) that is stable under finite limits and finite coproducts is obviously disjunctive. All our examples in this paper are ultimately of this kind. In fact, one can show that every disjunctive ∞-category arises (possibly after a change of universe) in this manner. (If τ is a strongly inaccessible uncountable cardinal, then any τ-small disjunctive ∞-category C can be embedded in the full subcategory $\mathscr{X} \subset \mathrm{Fun}(C^{op}, \mathbf{Kan}(\tau))$ spanned by the functors that preserve products. This is an accessible localization of the ∞-category $\mathrm{Fun}(C^{op}, \mathbf{Kan}(\tau))$, and one can show that the localization functor $\mathrm{Fun}(C^{op}, \mathbf{Kan}(\tau)) \longrightarrow \mathscr{X}$ is left exact, whence $\mathscr{X}$ is an ∞-topos.)

A.2. **Example.** More particularly, the full subcategory $\mathscr{X}^{coh}$ of a coherent ∞-topos $\mathscr{X}$ spanned by the coherent objects [27, Df. 3.12] is thus a disjunctive ∞-category. Furthermore, for any natural number n, the full subcategory $\tau_{\leq n}\mathscr{X}^{coh}$ spanned by the n-truncated objects is a full subcategory of an $(n+1)$-topos that is closed under finite coproducts and finite limits; hence it too is a disjunctive ∞-category.

For $1 \leq n < \infty$, let us say that an object U of an n-topos $\mathscr{X}$ is ***coherent*** if for any n-localic ∞-topos $\mathscr{Y}$ and any equivalence $\phi\colon \mathscr{X} \xrightarrow{\sim} \tau_{\leq n-1}\mathscr{Y}$, the object $\phi(U)$ is coherent. If $\mathscr{Y}$ is coherent, then we will say that $\mathscr{X}$ is ***coherent***, and the full subcategory $\mathscr{X}^{coh} \subset \mathscr{X}$ by the coherent objects is a disjunctive ∞-category.

A.3. **Example.** In particular, the ∞-category $\mathbf{Kan}^{coh}$ of Kan simplicial sets all of whose homotopy groups are finite is a disjunctive ∞-category, and for any $n \geq 0$, the truncation $\tau_{\leq n}\mathbf{Kan}^{coh}$ (whose objects may be called ***finite*** n***-groupoids***) is a disjunctive ∞-category.

The effective Burnside ∞-category $A^{\mathit{eff}}(\tau_{\leq n}\mathbf{Kan}^{coh})$ is an $(n+1)$-category in the sense of [25, Df. 2.3.4.1]. The homotopy category of $A^{\mathit{eff}}(\tau_{\leq 1}\mathbf{Kan}^{coh}) = A^{\mathit{eff}}(\mathbf{F})$ (where $\mathbf{F}$ is the nerve of the ordinary category of finite sets) is the ordinary effective Burnside category for the trivial group.

The following proposition, whose proof we leave to the reader, can be summarized by saying that the ∞-category $A^{\mathit{eff}}(\mathbf{F})$ as the free ∞-category with direct sums generated by a single object (the terminal object 1 of $\mathbf{F}$).

A.4. **Proposition.** *For any ∞-category E that admits direct sums, evaluation at the terminal object* 1 *induces an equivalence of ∞-categories*

$$\mathbf{Mack}(\mathbf{F}, E) \xrightarrow{\sim} E.$$

Let us turn now to a general version of the Segal–tom Dieck splitting theorem.

A.5. **Notation.** To this end, note that if $\mathscr{X}$ is a coherent n-topos ($1 \leq n \leq \infty$), then there is a functor $\delta\colon \mathbf{F} \longrightarrow \mathscr{X}^{coh}$ informally given by the assignment

$$S \longmapsto S \otimes 1 \simeq \coprod_{s \in S} 1.$$

See [25, §4.4.4].

A.6. **Definition.** A coherent n-topos $\mathscr{X}$ will be said to be ***locally connected*** if the functor δ admits a left adjoint π.

The idea here is of course that for any coherent object X, the unit morphism $X \longrightarrow \delta\pi X$ will decompose X into finitely many summands, each of which will be "connected," in a unique manner. Indeed, for any object X, one may exhibit X as a canonical coproduct

$$X \simeq \coprod_{\alpha \in \pi X} (X \times_{\delta\pi X} \delta\{\alpha\}).$$

These objects $X \times_{\delta\pi X} \delta\{\alpha\}$ are now connected in the following sense.

A.7. **Definition.** Suppose $\mathscr{X}$ a coherent n-topos. A coherent object X of $\mathscr{X}$ will be said to be ***connected*** if the functor $\mathscr{X}^{coh} \longrightarrow \mathbf{Kan}$ it corepresents preserves finite coproducts. Equivalently, X is connected just in case πX is a one-point set. Denote by $\mathscr{X}^{conn} \subset \mathscr{X}^{coh}$ the full subcategory spanned by the connected objects.

A.8. **Warning.** It is not necessarily the case that the terminal object $*$ of a locally connected coherent n-topos $\mathscr{X}$ is connected. If it is, then $\mathscr{X}$ is said to be ***connected***, and in this case δ is fully faithful.

It turns out that the spectral Burnside ring of a locally connected coherent n-topos can be identified with a suspension spectrum.

A.9. **Theorem** (Segal–tom Dieck, [39])**.** *Suppose $\mathscr{X}$ a locally connected coherent n-topos. Then there is a natural equivalence*

$$\mathbf{S}_{\mathscr{X}^{coh}}(1) \simeq \Sigma^{\infty}_{+} \iota\mathscr{X}^{conn} \simeq \bigvee_{X} \Sigma^{\infty}_{+} B\,\mathrm{Aut}(X),$$

where the wedge is taken over all equivalence classes of connected objects X, and $\mathrm{Aut}(X)$ *denotes the space of auto-equivalences of X.*

Proof. It follows from Th. 13.12 that $\mathbf{S}_{\mathscr{X}^{coh}}(1)$ can be identified with the group completion of the object $\iota\mathscr{X}^{coh} \in \mathrm{CAlg}(\mathbf{Kan})$, where the commutative algebra structure is given by coproduct. It therefore suffices to show that $\iota\mathscr{X}^{coh}$ is the free commutative algebra generated by the space $\iota\mathscr{X}^{conn}$ in the sense of [31, Ex. 3.1.3.12].

For this, note that the (homotopy) fiber of the map $\iota(\pi)\colon \iota\mathscr{X}^{coh} \longrightarrow \iota N\mathbf{F}$ over a finite set I of cardinality n may be described as the space of pairs (X, f) consisting of an object $X \in \iota\mathscr{X}^{coh}$ and an isomorphism $\pi(X) \xrightarrow{\sim} I$, which can in turn be identified with the product $(\iota\mathscr{X}^{conn})^n$. We therefore obtain an identification

$$\mathrm{Sym}^n(\iota\mathscr{X}^{conn}) \simeq B\Sigma_n \times^h_{\iota N\mathbf{F}} \iota\mathscr{X}^{coh}.$$

Since one has $\iota N\mathbf{F} \simeq \coprod_{n \geq 0} B\Sigma_n$, we find that

$$\iota\mathscr{X}^{coh} \simeq \coprod_{n \geq 0} \mathrm{Sym}^n(\iota\mathscr{X}^{conn}).$$

The proof is thus complete by [31, Ex. 3.1.3.11]. □

Example B. Equivariant spectra for a profinite group

Certain topological groups give a subexample of Ex. A.

B.1. **Definition.** A topological group is ***coherent*** if for every open subgroup H of G, there exist only finitely many subsets of the form HgH for $g \in G$.

B.2. Suppose G a coherent topological group. Suppose B_G the classifying 1-topos of G; that is, B_G is the nerve of the ordinary category of sets equipped with a continuous action of G. Then B_G is the nerve of a coherent topos [17, §D3.4], and the full subcategory B_G^{fin} spanned by the coherent objects is a disjunctive ∞-category.

B.3. A pro-discrete group G is coherent just in case it is profinite. In this case, the coherent objects of B_G are simply those continuous G-sets with only finitely many orbits. Hence the full subcategory $B_G^{\mathit{fin}} \subset B_G$ spanned by finite G-sets with open stabilizers is a disjunctive ∞-category. The effective Burnside ∞-category of B_G^{fin} will be denoted, abusively, $A^{\mathit{eff}}(G)$.

B.4. For any finite group G, the ∞-category $A^{\mathit{eff}}(G)$ is in fact a 2-category whose homotopy category $hA^{\mathit{eff}}(G)$ is the ordinary effective Burnside category for G; the ordinary Burnside category for G is the local group completion (obtained by forming the Grothendieck group of each of the Hom-sets under direct sum).

B.5. **Notation.** When $C = NB_G^{\mathit{fin}}$ for some profinite group G and E is some additive ∞-category, we will write

$$\mathbf{Mack}_G(E) := \mathbf{Mack}(NB_G^{\mathit{fin}}, E).$$

B.6. **Example.** When G is finite and $E = N\mathbf{Ab}$ is the nerve of the ordinary category of abelian groups, the ∞-category $\mathrm{Fun}(A^{\mathit{eff}}(NB_G^{\mathit{fin}}), N\mathbf{Ab})$ is naturally equivalent to the nerve of the ordinary category of functors $\mathrm{Fun}(hA^{\mathit{eff}}(NB_G^{\mathit{fin}}), \mathbf{Ab})$. Using the fact that the homotopy category $hA^{\mathit{eff}}(NB_G^{\mathit{fin}})$ is the ordinary effective Burnside category (3.8), we conclude that the full subcategory $\mathbf{Mack}_G(N\mathbf{Ab})$ is equivalent to the full subcategory of $\mathrm{Fun}(hA^{\mathit{eff}}(NB_G^{\mathit{fin}}), \mathbf{Ab})$ spanned by Mackey functors in the classical sense.

When G is finite and $E = \mathbf{Sp}$ is the ∞-category of spectra, work of Guillou and May [12] show that this ∞-category is equivalent to the underlying ∞-category of the relative category of genuine G-spectra in the sense of Lewis–May–Steinberger [24], Mandell–May [32], and Hill–Hopkins–Ravenel [14, 16, 15]. For general profinite groups G, we call $\mathbf{Mack}_G(\mathbf{Sp})$ the ***∞-category of G-equivariant spectra***.

B.7. Suppose G a profinite group, and suppose H a closed normal subgroup of G. Then the natural functor $\varphi\colon G \longrightarrow G/H$ induces a morphism of topoi

$$\varphi^\star\colon B_{G/H} \rightleftarrows B_G\colon \varphi_\star.$$

Both functors preserve coherent objects, finite coproducts and pullbacks. Hence we obtain functors

$$\varphi^\star\colon NB_{G/H}^{\mathit{fin}} \longrightarrow NB_G^{\mathit{fin}} \quad \text{and} \quad \varphi_\star\colon NB_G^{\mathit{fin}} \longrightarrow NB_{G/H}^{\mathit{fin}}.$$

For any presentable additive ∞-category E, we obtain adjunctions

$$\varphi^\star_!\colon \mathbf{Mack}_{G/H}(E) \rightleftarrows \mathbf{Mack}_G(E)\colon \varphi^{\star\star}$$

and

$$\varphi_{\star!}\colon \mathbf{Mack}_G(E) \rightleftarrows \mathbf{Mack}_{G/H}(E)\colon \varphi_\star^\star.$$

We write $\Psi^H := \varphi^{\star\star}$ and $\Phi^H := \varphi_{\star!}$. When G is finite, one may use the equivalence of Guillou–May [12] to interpret Ψ^H and Φ^H as functors on the ∞-categories of G- and G/H-equivariant spectra. One may show that these functors agree (up to equivalence) with the *Lewis–May fixed points* Ψ^H and the *geometric fixed points* Φ^H constructed by Mandell–May [32].

For any profinite group G, the 1-topos NB_G^{fin} is locally connected (connected, in fact). This fact permits us to use our Segal–tom Dieck Theorem A.9 to compute the value of the Burnside spectral Mackey functor

$$\mathbf{S}_G := \mathbf{S}_{NB_G^{fin}}$$

on the terminal G-set $[G/G]$. Indeed, the connected objects of NB_G^{fin} are precisely the finite G-sets with open stabilizers that are transitive. Up to equivalence, these are classified by conjugacy classes of open subgroups of G. The space of autoequivalences of the transitive G-set $[G/H]$ is equivalent to the quotient $N_G H/H$, whence we obtain the traditional Segal–tom Dieck splitting [39], now for profinite groups:

B.8. **Proposition.** *For any profinite group G, one has*

$$\mathbf{S}_G([G/G]) \simeq \bigvee_H \Sigma^\infty_+ B(N_G H/H),$$

where the wedge is indexed by conjugacy classes of open subgroups $H \leq G$.

Example C. A-theory and upside-down A-theory of ∞-topoi

In this section, we introduce two dual disjunctive triple structures on an ∞-topos, where the ingressive or egressive morphisms are defined by means of a finiteness condition. We use these structures to construct both the A-theory and the V-theory of ∞-topoi together with all their functorialities.

To begin, if we consider an ∞-topos as a disjunctive ∞-category, then there is a natural Waldhausen bicartesian fibration that lies over it. Let us investigate this now.

The following is an easy analogue of [25, Lm. 6.1.1.1].

C.1. **Lemma.** *If C is an ∞-category that admits both pullbacks and pushouts, then the functor*

$$\mathrm{Fun}(\Delta^2, C) \longrightarrow \mathrm{Fun}(\Delta^{\{0,2\}}, C)$$

is a both a cartesian fibration and a cocartesian fibration.

C.2. **Notation.** For the remainder of this section, suppose $\mathscr{X}$ an ∞-topos. We consider the fibration

$$p\colon \mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathscr{X}) \cong \mathrm{Fun}(\Delta^2, \mathscr{X}) \times_{\mathrm{Fun}(\Delta^{\{0,2\}}, \mathscr{X})} \mathscr{X} \longrightarrow \mathscr{X},$$

which is both cartesian and cocartesian. We may think of the objects of the ∞-category $\mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathscr{X})$ as retract diagrams

$$X \longrightarrow X' \longrightarrow X;$$

the functor p is given by the assignment

$$[X \longrightarrow X' \longrightarrow X] \longmapsto X.$$

We therefore denote $\mathscr{X}_{X/\ /X}$ the fiber of p over an object X of $\mathscr{X}$. For any morphism $f\colon X \longrightarrow Y$, the corresponding functors

$$f_!\colon \mathscr{X}_{X/\ /X} \longrightarrow \mathscr{X}_{Y/\ /Y} \quad \text{and} \quad f^\star\colon \mathscr{X}_{Y/\ /Y} \longrightarrow \mathscr{X}_{X/\ /X}$$

are given informally by the assignments $X' \longmapsto Y \cup^X X'$ and $Y' \longmapsto Y' \times_Y X$.

C.3. **Proposition.** *Endow the ∞-category $\mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathscr{X})$ with the pair structure in which a morphism f is ingressive just in case $p(f)$ is an equivalence. Then $p\colon \mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathscr{X}) \longrightarrow \mathscr{X}$ is a Waldhausen bicartesian fibration.*

Proof. The only nontrivial points are the following. For any morphism $f\colon X \longrightarrow Y$, the functor $f^\star$ preserves pushout squares by the universality of colimits in $\mathscr{X}$. The admissibility of p also follows from the universality of colimits in $\mathscr{X}$. Finally, if I is a finite set and $\{X_i\}_{i\in I}$ a family of objects of $\mathscr{X}$ with coproduct X, then pullback along the various inclusions $X_i \hookrightarrow X$ defines an equivalence of ∞-categories

$$\mathscr{X}_{X/\ /X} \xrightarrow{\sim} \prod_{i\in I} \mathscr{X}_{X_i/\ /X_i}.$$ □

This Waldhausen bicartesian fibration is not so interesting from the point of view of algebraic K-theory, as each ∞-category $\mathscr{X}_{X/\ /X}$ has vanishing K-theory. To make it more interesting, we must restrict attention to objects with a finiteness condition. We thus turn to the study of finiteness conditions on objects of the ∞-topos $\mathscr{X}$. We begin with the following result.

C.4. **Proposition.** *Suppose $X \in \mathscr{X}$ an object. An object $X' \in \mathscr{X}_{/X}$ is compact if and only if it is compact as an object of $\mathscr{X}$.*

Proof. It is easy to see that if X' is compact in $\mathscr{X}$, then it is compact in $\mathscr{X}_{/X}$. Conversely, the forgetful functor $\mathscr{X}_{/X} \longrightarrow \mathscr{X}$ admits a right adjoint that, thanks to the universality of colimits, preserves colimits. It follows from [25, Pr. 5.5.7.2(1)] that if X' is compact as an object of $\mathscr{X}_{/X}$, then it is compact as an object of $\mathscr{X}$. □

The following is now immediate.

C.4.1. **Corollary.** *If the ∞-topos $\mathscr{X}$ is compactly generated, then for any object $X \in \mathscr{X}$, the ∞-topos $\mathscr{X}_{/X}$ is compactly generated as well.*

In the ∞-category of spaces, those maps whose fibers are (homotopy) retracts of finite CW complexes play a special role. In a more general ∞-topos, this role is played by the *relatively compact* morphisms, which we define now.

C.5. **Definition** (Lurie, [25, Df. 6.1.6.4]). A morphism $Y \longrightarrow X$ of $\mathscr{X}$ is said to be ***relatively compact*** just in case, for any compact object S of $\mathscr{X}$ and any morphism $\eta\colon S \longrightarrow X$ and any pullback square

$$\begin{array}{ccc} Y_\eta & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & \xrightarrow[\eta]{} & X, \end{array}$$

the object Y_η is compact.

C.6. **Proposition.** *Suppose $\mathscr{G} \subset \mathscr{X}$ a full subcategory whose objects are compact that generates $\mathscr{X}$ under colimits (so that $\mathscr{X}$ is compactly generated). Then a morphism $Y \longrightarrow X$ of $\mathscr{X}$ is relatively compact just in case, for any object $T \in \mathscr{G}$ and* *any morphism* $\xi\colon T \longrightarrow X$ *and any pullback square*

$$\begin{array}{ccc} Y_\xi & \longrightarrow & Y \\ \downarrow & & \downarrow \\ T & \underset{\xi}{\longrightarrow} & X, \end{array}$$

the object Y_ξ *is compact.*

Proof. To prove this claim, it suffices to show that if

(C.6.1) $S \in \mathscr{X}^\omega$,
(C.6.2) $Z \in \mathscr{X}$, and
(C.6.3) $Z \longrightarrow S$ is a morphism such that for any pullback square

$$\begin{array}{ccc} Z_\xi & \longrightarrow & Z \\ \downarrow & & \downarrow \\ T & \underset{\xi}{\longrightarrow} & S, \end{array}$$

of $\mathscr{X}$ in which $T \in \mathscr{G}$, the object Z_ξ is compact,

then the object Z is also compact. To see this, we first argue that $\mathscr{X}^\omega$ is generated under finite colimits and retracts by $\mathscr{G}$; indeed, if $\mathscr{X}^f \subset \mathscr{X}^\omega$ denotes the full subcategory generated by $\mathscr{G}$ under finite colimits, then [25, Pr. 5.3.5.11] implies that the colimit preserving functor $\mathrm{Ind}(\mathscr{X}^f) \longrightarrow \mathrm{Ind}(\mathscr{X}^\omega) = \mathscr{X}$ corresponding to the inclusion $\mathscr{X}^f \hookrightarrow \mathscr{X}^\omega$ is an equivalence, whence $\mathscr{X}^\omega$ is the idempotent completion of $\mathscr{X}^f$ thanks to [25, Pr. 5.5.7.8]. We therefore may write S as a retract of a finite colimit of objects of $\mathscr{G}$, and employing the universality of colimits, we obtain a presentation of Z as a retract of a finite colimit of compact objects. □

C.7. **Example.** In particular, we deduce that the ∞-category **Kan** admits a disjunctive triple structure in which every morphism is ingressive, and a morphism is egressive just in case its fibers are compact (or, in the parlance of [9, p. 3], *homotopy finitely dominated*).

C.8. **Proposition.** *If* $\mathscr{X}_{\mathrm{rc}} \subset \mathscr{X}$ *denotes the subcategory of relatively compact morphisms, then the triples* $(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}})$ *and* $(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})$ *are disjunctive.*

Proof. It is obvious that relatively compact morphisms are closed under pullback. The universality of colimits and the fact that compact objects of $\mathscr{X}$ are closed under finite coproducts implies that the class of relatively compact morphisms is compatible with coproducts. The other axioms all follow directly from the universality of colimits and the disjointness of finite coproducts in $\mathscr{X}$. □

C.9. **Notation.** Now let us restrict the Waldhausen bicartesian fibration

$$p\colon \mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathscr{X}) \longrightarrow \mathscr{X}.$$

Let us write $\mathscr{I}(\mathscr{X})$ for the full subcategory of $\mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathscr{X})$ spanned by those retract diagrams

$$X \longrightarrow X' \longrightarrow X$$

such that the object $X' \in \mathscr{X}_{X/\ /X}$ is compact. Dually, let us write $\mathscr{J}(\mathscr{X})$ for the full subcategory of $\mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, \mathscr{X})$ spanned by those retract diagrams

$$X \longrightarrow X' \longrightarrow X$$

such that the morphism $X' \longrightarrow X$ is relatively compact.

C.10. **Theorem.** *Assume that $\mathscr{X}$ is compactly generated, and assume that the terminal object $1 \in \mathscr{X}$ is compact. The functor p restricts to Waldhausen bicartesian fibrations*

$$p_{\mathscr{I}(\mathscr{X})} \colon \mathscr{I}(\mathscr{X}) \longrightarrow \mathscr{X}$$

over $(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}})$ and

$$p_{\mathscr{J}(\mathscr{X})} \colon \mathscr{J}(\mathscr{X}) \longrightarrow \mathscr{X}$$

over $(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})$.

Proof. The only points left to be shown are the following for a morphism $f \colon X \longrightarrow Y$ of $\mathscr{X}$.

(A) If $X' \in \mathscr{X}_{X/\ /X}$ is compact, then $X' \cup^X Y \in \mathscr{X}_{Y/\ /Y}$ is also compact.
(B) If $Y' \longrightarrow Y$ is relatively compact, then the pullback $Y' \times_Y X \longrightarrow X$ is also relatively compact.
(C) If f is relatively compact and $Y' \in \mathscr{X}_{Y/\ /Y}$ is compact, then $Y' \times_Y X \in \mathscr{X}_{X/\ /X}$ is also compact.
(D) If f is relatively compact and $X' \longrightarrow X$ is relatively compact, then the pushout $X' \cup^X Y \longrightarrow Y$ is also relatively compact.

To prove (1), note that the functor $\mathscr{X}_{X/\ /X} \longrightarrow \mathscr{X}_{Y/\ /Y}$ given by the assignment $X' \longmapsto X' \cup^X Y$ can be identified with the tensor product

$$u \otimes \mathrm{id} \colon \mathscr{X}_{/X} \otimes \mathbf{Kan}_* \longrightarrow \mathscr{X}_{/Y} \otimes \mathbf{Kan}_*$$

of presentable ∞-categories (in the sense of [31, §6.3.1]) of the forgetful functor $u \colon \mathscr{X}_{/X} \longrightarrow \mathscr{X}_{/Y}$ with the identity functor [31, Pr. 6.3.2.11]. By Pr. C.4 and Cor. C.4.1, u is an ω-good functor between compactly generated ∞-categories in the sense of [31, Nt. 6.3.7.8]. Hence by [31, Lm. 6.3.7.11], $u \otimes \mathrm{id}$ is ω-good as well. In particular, it preserves compact objects.

Assertion (2) is clear.

To prove (3), note that the functor $v \colon \mathscr{X}_{/Y} \longrightarrow \mathscr{X}_{/X}$ given by the assignment $Y' \longmapsto Y' \times_Y X$ preserves colimits by the universality of colimits in $\mathscr{X}$, and it preserves compact objects thanks to Pr. C.4; we thus conclude that v is ω-good. Once again we may identify the functor $\mathscr{X}_{Y/\ /Y} \longrightarrow \mathscr{X}_{X/\ /X}$ given by the assignment $Y' \longmapsto Y' \times_Y X$ with $v \otimes \mathrm{id}$, and once again we may appeal to [31, Lm. 6.3.7.11] to conclude that $v \otimes \mathrm{id}$ is ω-good.

Finally, to prove (4), assume that $S \in \mathscr{X}$ is a compact object and $S \longrightarrow Y$ a morphism. The universality of colimits implies that

$$(X' \cup^X Y) \times_Y S \simeq (X' \times_Y S) \cup^{(X \times_Y S)} S.$$

Since both $X' \longrightarrow X$ and $X \longrightarrow Y$ are relatively compact, it follows that both $X' \times_Y S$ and $X \times_Y S$ are compact. Since compact objects are closed under finite colimits, it follows that $(X' \cup^X Y) \times_Y S$ is compact as well, and this completes the proof that $X' \cup^X Y \longrightarrow Y$ is relatively compact. □

C.10.1. **Corollary.** *Assume that $\mathscr{X}$ is compactly generated, and assume that the terminal object $1 \in \mathscr{X}$ is compact. The unfurlings*

$$\Upsilon(p_{\mathscr{I}(\mathscr{X})}) \longrightarrow A^{\mathrm{eff}}(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}}) \quad \textit{and} \quad \Upsilon(p_{\mathscr{J}(\mathscr{X})}) \longrightarrow A^{\mathrm{eff}}(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})$$

classify Mackey functors

$$\mathscr{M}_{\mathscr{I}(\mathscr{X})} \colon A^{\mathrm{eff}}(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}}) \longrightarrow \mathbf{Wald}_\infty$$

and

$$\mathscr{M}_{\mathscr{J}(\mathscr{X})} \colon A^{\mathrm{eff}}(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X}) \longrightarrow \mathbf{Wald}_\infty.$$

C.11. **Notation.** Assume that $\mathscr{X}$ is compactly generated, and assume that the terminal object $1 \in \mathscr{X}$ is compact. Write $\mathbf{A}$ and $\mathbf{\forall}$ for the composite Mackey functors

$$\mathbf{A}_{\mathscr{X}} := \mathbf{K} \circ \mathscr{M}_{\mathscr{I}(\mathscr{X})} \quad \text{and} \quad \mathbf{\forall}_{\mathscr{X}} := \mathbf{K} \circ \mathscr{M}_{\mathscr{J}(\mathscr{X})}.$$

For any object $X \in \mathscr{X}$, the spectrum $\mathbf{A}_{\mathscr{X}}(X)$ is the algebraic K-theory of the full subcategory $\mathscr{I}(\mathscr{X})_X \subset \mathscr{X}_{X/\ /X}$ spanned by the compact objects, and $\mathbf{\forall}_{\mathscr{X}}(X)$ is the algebraic K-theory of the full subcategory $\mathscr{J}(\mathscr{X})_X \subset \mathscr{X}_{X/\ /X}$ spanned by those retract diagrams $X \longrightarrow X' \longrightarrow X$ such that $X' \longrightarrow X$ is relatively compact.

C.12**.** Assume that $\mathscr{X}$ is compactly generated, and assume that the terminal object $1 \in \mathscr{X}$ is compact. We have equivalences

$$\mathrm{Map}_{\mathbf{Mack}(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}}; \mathbf{Sp})}(\mathbf{S}_{(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}})}, \mathbf{A}_{\mathscr{X}}) \simeq \mathbf{A}_{\mathscr{X}}(1)$$

and

$$\mathrm{Map}_{\mathbf{Mack}(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X}; \mathbf{Sp})}(\mathbf{S}_{(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})}, \mathbf{\forall}_{\mathscr{X}}) \simeq \mathbf{\forall}_{\mathscr{X}}(1),$$

where 1 denotes the terminal object of $\mathscr{X}$. The object $1 \sqcup 1 \in \mathscr{X}_{1/\ /1}$ lies in both $\mathscr{I}(\mathscr{X})_1$ and $\mathscr{J}(\mathscr{X})_1$, whence its classes in the corresponding K-theories specify morphisms of Mackey functors

$$\mathbf{S}_{(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}})} \longrightarrow \mathbf{A}_{\mathscr{X}} \quad \text{and} \quad \mathbf{S}_{(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})} \longrightarrow \mathbf{\forall}_{\mathscr{X}}.$$

We thus obtain, for any object $X \in \mathscr{X}$, assembly morphisms

$$\mathbf{S}_{(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}})}(X) \longrightarrow \mathbf{A}_{\mathscr{X}}(X) \quad \text{and} \quad \mathbf{S}_{(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})}(X) \longrightarrow \mathbf{\forall}_{\mathscr{X}}(X).$$

We in turn obtain, for any three objects $U, V, X \in \mathscr{X}$ such that $U \longrightarrow 1$ is relatively compact, assembly morphisms

$$\Sigma^\infty_+ \mathrm{Map}_{\mathscr{X}}(U, X) \longrightarrow \mathbf{A}_{\mathscr{X}}(X) \quad \text{and} \quad \Sigma^\infty_+ \mathrm{Map}_{\mathscr{X}_{\mathrm{rc}}}(V, X) \longrightarrow \mathbf{\forall}_{\mathscr{X}}(X).$$

As in Df. 8.3, we obtain, for any object $X \in \mathscr{X}$, morphisms of Mackey functors

$$\mathbf{S}^X_{(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}})} \longrightarrow F(\mathbf{A}_{\mathscr{X}}(X), \mathbf{A}_{\mathscr{X}}) \quad \text{and} \quad \mathbf{S}^X_{(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})} \longrightarrow F(\mathbf{\forall}_{\mathscr{X}}(X), \mathbf{\forall}_{\mathscr{X}}),$$

which induce, for any object $Y \in \mathscr{X}$, assembly morphisms

$$\mathbf{S}^X_{(\mathscr{X}, \mathscr{X}, \mathscr{X}_{\mathrm{rc}})}(Y) \wedge \mathbf{A}_{\mathscr{X}}(X) \longrightarrow \mathbf{A}_{\mathscr{X}}(Y)$$

and

$$\mathbf{S}^X_{(\mathscr{X}, \mathscr{X}_{\mathrm{rc}}, \mathscr{X})}(Y) \wedge \mathbf{\forall}_{\mathscr{X}}(X) \longrightarrow \mathbf{\forall}_{\mathscr{X}}(Y),$$

We in turn obtain, for any objects $U, V \in \mathscr{X}$ such that $U \longrightarrow 1$ is relatively compact, assembly morphisms

$$\Sigma^\infty_+ \mathrm{Map}_{\mathscr{X}}(U, X) \longrightarrow F(\mathbf{\forall}_{\mathscr{X}}(X), \mathbf{\forall}_{\mathscr{X}}(1))$$

and

$$\Sigma^\infty_+ \mathrm{Map}_{\mathscr{X}_{\mathrm{rc}}}(V, X) \longrightarrow F(\mathbf{A}_{\mathscr{X}}(X), \mathbf{A}_{\mathscr{X}}(1)).$$

C.13. **Example.** If $\mathscr{X} = \mathbf{Kan}$, then the functors $\mathbf{A}$ and $\mathbf{V}$ are fully functorial version of A-theory and V-theory as considered by Waldhausen in [40, §2.1] (modulo the small point that here we deal with finitely dominated spaces in place of finite spaces). The assembly morphisms above described above are

$$\Sigma^\infty_+ \operatorname{Map}(U, X) \longrightarrow \mathbf{A}(X) \quad \text{and} \quad \Sigma^\infty_+ \operatorname{Map}_{\mathrm{rc}}(V, X) \longrightarrow \mathbf{V}(X),$$

any three objects $U, V, X \in \mathscr{X}$ such that $U \longrightarrow *$ is relatively compact, where $\operatorname{Map}_{\mathrm{rc}}(V, X) \subset \operatorname{Map}(V, X)$ is the union of the connected components corresponding to maps $V \longrightarrow X$ with finitely dominated (homotopy) fibers. When $U = *$, the first of these morphisms is the usual assembly morphism $\Sigma^\infty_+ X \longrightarrow \mathbf{A}(X)$. Dually, we also have co-assembly morphisms

$$\Sigma^\infty_+ \operatorname{Map}(U, X) \longrightarrow F(\mathbf{V}(X), \mathbf{A}(*))$$

and

$$\Sigma^\infty_+ \operatorname{Map}_{\mathrm{rc}}(V, X) \longrightarrow F(\mathbf{A}(X), \mathbf{A}(*)).$$

(Observe that $\mathbf{A}(*) \simeq \mathbf{V}(*)$.) When $U = *$, the first of these morphisms seems to have been studied independently by Cary Malkiewich. When composed with the morphism induced by the trace

$$\mathbf{A}(*) \simeq \mathbf{K}(\mathbf{S}) \longrightarrow \mathrm{THH}(\mathbf{S}) \simeq \mathbf{S},$$

we obtain morphisms

$$\Sigma^\infty_+ \operatorname{Map}(U, X) \longrightarrow D\mathbf{V}(X) \quad \text{and} \quad \Sigma^\infty_+ \operatorname{Map}_{\mathrm{rc}}(V, X) \longrightarrow D\mathbf{A}(X).$$

Example D. Algebraic K-theory of derived stacks

We turn to a more geometric setting. Over suitable schemes, complexes of quasicoherent sheaves admit both a pushforward and pullback functor. The pullback carries perfect complexes to perfect complexes, but the pushforward does not, unless some heavy constraints are placed on the morphisms of schemes involved. In this subsection, we discuss such restrictions for a very general class of derived stacks.

There are two classes of examples in which we will be interested: *(i)* spectral Deligne–Mumford stacks and *(ii)* arbitrary sheaves of spaces for the flat site. In each case we will construct algebraic K-theory as a spectral Mackey functor relative to certain disjunctive triple structures.

D.1**.** For simplicity, in this section we work *absolutely*, i.e., over the sphere spectrum. Nothing here uses that fact in a nontrivial way, and all the results of this section can obviously be adapted to work over more general bases.

D.2. **Notation.** We consider the ∞-category $\mathbf{CAlg}^{\mathrm{cn}}$ of connective E_∞ rings and the huge ∞-topos

$$\mathbf{Shv}_{\mathrm{flat}} \subset \operatorname{Fun}(\mathbf{CAlg}^{\mathrm{cn}}, \mathbf{Kan}(\kappa_1))$$

of large sheaves on $\mathbf{CAlg}^{\mathrm{cn},op}$ for the flat topology [27, Pr. 5.4].

Let us begin by identifying two sources of disjunctive triples within the ∞-category $\mathbf{Shv}_{\mathrm{flat}}$. Since our aim is to study categories of modules that are contravariantly functorial in all morphisms but only covariantly functorial in certain classes of morphisms, these disjunctive triples will have the property that every morphism will be egressive, but the ingressives will be heavily restricted.

D.3. **Notation.** The full subcategory $\mathbf{DM} \subset \mathbf{Shv}_{\mathrm{flat}}$ spanned by those sheaves that are representable by spectral Deligne–Mumford stacks is closed under finite coproducts and pullbacks, whence it is a disjunctive ∞-category.

For any class $\mathscr{P}$ of morphisms of $\mathbf{DM}$ that is stable under pullback and compatible with coproducts in the sense of (5.2.4), one obtains a disjunctive triple

$$(\mathbf{DM}, \mathbf{DM}_{\mathscr{P}}, \mathbf{DM}),$$

where $\mathbf{DM}_{\mathscr{P}} \subset \mathbf{DM}$ is the subcategory that contains all the objects, in which the morphisms lie in $\mathscr{P}$.

D.4. **Notation.** Alternately, one may opt to keep all the objects of $\mathbf{Shv}_{\mathrm{flat}}$. Then for any class $\mathscr{P}$ of morphisms of $\mathbf{Shv}_{\mathrm{flat}}$ that is stable under pullback and compatible with coproducts in the sense of (5.2.4), one obtains a disjunctive triple

$$(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}}),$$

where $\mathbf{Shv}_{\mathrm{flat},\mathscr{P}} \subset \mathbf{Shv}_{\mathrm{flat}}$ is the subcategory that contains all the objects, in which the morphisms lie in $\mathscr{P}$.

Our ultimate interest will be in the study of perfect modules over these sorts of objects, but let us first consider the larger ∞-category of quasicoherent modules.

D.5. **Notation.** We let

$$\begin{array}{ccc} \mathbf{Mod} & \longrightarrow & \mathbf{QCoh} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{CAlg}^{\mathrm{cn}} & \hookrightarrow & \mathbf{Shv}^{op}_{\mathrm{flat}} \end{array}$$

be a pullback square in which q is the cocartesian fibration of [31, Df. 4.4.1.1], and p is a cocartesian fibration classified by the right Kan extension of the functor that classifies q. The objects of $\mathbf{QCoh}$ can be thought of as pairs (X, M) consisting of a sheaf $X : \mathbf{CAlg}^{\mathrm{cn}} \longrightarrow \mathbf{Kan}(\kappa_1)$ for the flat topology and a quasicoherent module M over X.

D.6**.** Let us note that by [28, Pr. 2.7.17(1)], the fibers of p are all stable ∞-categories. Furthermore, since the functor $\mathbf{Shv}^{op}_{\mathrm{flat}} \longrightarrow \mathbf{Cat}_\infty(\kappa_1)$ that classifies p preserves limits [28, Pr. 2.7.14], it follows that for any finite set I and any collection $\{X_i \mid i \in I\}$ of sheaves, the natural functor

$$\mathbf{QCoh}_{\coprod_{i\in I} X_i} \longrightarrow \prod_{i \in I} \mathbf{QCoh}_{X_i}$$

is an equivalence.

D.7. **Warning.** Note that p is not a cartesian fibration. In general, the functor

$$f^\star : \mathbf{QCoh}_Y \longrightarrow \mathbf{QCoh}_X$$

induced by a natural transformation $f : X \longrightarrow Y$ will preserve all small colimits [28, Pr. 2.7.17(2)], but the ∞-categories may not be presentable unless one knows that X and Y are in some sense "small."

This smallness is ensured if, for example, X and Y are represented by spectral Deligne–Mumford stacks ([28, Prs. 2.3.13 and 2.7.18]). We therefore conclude the following.

D.8. **Lemma.** *The pulled back functor*

$$\mathbf{QCoh} \times_{\mathbf{Shv}^{op}_{\mathrm{flat}}} \mathbf{DM}^{op} \longrightarrow \mathbf{DM}^{op}$$

is both a cocartesian fibration and a cartesian fibration.

D.9. **Notation.** In the ∞-category $\mathbf{DM}$, consider the class $\mathscr{RS}$ of ***relatively scalloped*** morphisms in the sense of [28, Df. 2.5.10]. Then since relatively scalloped morphisms are stable under pullback and compatible with coproducts in the sense of (5.2.4), we obtain a disjunctive triple

$$(\mathbf{DM}, \mathbf{DM}_{\mathscr{RS}}, \mathbf{DM}).$$

D.10. **Proposition.** *The functor*

$$\mathbf{QCoh}^{op} \times_{\mathbf{Shv}_{\mathrm{flat}}} \mathbf{DM} \longrightarrow \mathbf{DM}$$

is a Waldhausen bicartesian fibration for the left complete disjunctive triple

$$(\mathbf{DM}, \mathbf{DM}_{\mathscr{RS}}, \mathbf{DM}).$$

Proof. By [28, Pr. 2.5.14], the relevant base change functors are all equivalences. □

D.11. **Notation.** In the bigger ∞-category of all flat sheaves, we consider the class $\mathscr{QA}$ in $\mathbf{Shv}_{\mathrm{flat}}$ of ***quasi-affine representable*** morphisms — that is, those morphisms $X \longrightarrow Y$ such that for any connective E_∞ ring R and any R-point $\eta\colon \operatorname{Spec} R \longrightarrow Y$, the pullback $X \times_Y \operatorname{Spec} R$ is representable by a spectral Deligne–Mumford stack X_η, and the map $X_\eta \longrightarrow \operatorname{Spec} R$ is quasi-affine [28, Df. 3.1.24]. Then since quasi-affine representable morphisms are stable under pullback and compatible with coproducts in the sense of (5.2.4), we have a disjunctive triple

$$(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{QA}}, \mathbf{Shv}_{\mathrm{flat}}).$$

The following is immediate from [28, Pr. 3.2.5].

D.12. **Lemma.** *The pulled back functor*

$$\mathbf{QCoh} \times_{\mathbf{Shv}^{op}_{\mathrm{flat}}} \mathbf{Shv}^{op}_{\mathrm{flat},\mathscr{QA}} \longrightarrow \mathbf{Shv}^{op}_{\mathrm{flat},\mathscr{QA}}$$

is both a cocartesian fibration and a cartesian fibration.

Furthermore, we deduce from [28, Cor. 3.2.6(2)] that the relevant base change functors are all equivalences, whence we conclude the following.

D.13. **Proposition.** *The functor*

$$p\colon \mathbf{QCoh}^{op} \longrightarrow \mathbf{Shv}_{\mathrm{flat}}$$

is a Waldhausen bicartesian fibration for the left complete disjunctive triple

$$(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{QA}}, \mathbf{Shv}_{\mathrm{flat}}).$$

D.14. **Notation.** Now denote by $\mathbf{Perf} \subset \mathbf{QCoh}$ the full subcategory spanned by those pairs (X, M) in which M is a perfect quasicoherent module on X — i.e., a strongly dualizable object of the symmetric monoidal ∞-category $\mathbf{QCoh}_X$. Endow $\mathbf{Perf}^{op}$ with its fiberwise maximal pair structure, so that

$$\mathbf{Perf}^{\dagger} := \mathbf{Perf} \times_{\mathbf{Shv}^{op}_{\mathrm{flat}}} \iota\mathbf{Shv}^{op}_{\mathrm{flat}}.$$

Note that the restricted functor

$$\mathbf{Perf}^{op} \longrightarrow \mathbf{Shv}_{\mathrm{flat}}$$

remains a Waldhausen cartesian fibration, since pullbacks are symmetric monoidal.

We now wish to shrink our classes of ingressives on $\mathbf{DM}$ and $\mathbf{Shv}_{\mathrm{flat}}$ to ensure that perfect modules define a supported Waldhausen bicartesian fibration. For this, it's enough to find a class of morphisms such that the pushforward preserves perfection.

D.15. **Definition.** Suppose $f\colon X \longrightarrow Y$ a morphism of $\mathbf{Shv}_{\mathrm{flat}}$ such that the functor $f^{\star}\colon \mathbf{QCoh}_Y \longrightarrow \mathbf{QCoh}_X$ admits a right adjoint $f_{\star}$. Call f ***perfect*** if $f_{\star}\colon \mathbf{QCoh}_X \longrightarrow \mathbf{QCoh}_Y$ carries perfect objects [28, Df. 2.7.21] to perfect objects.

Between spectral Deligne–Mumford stacks, Lurie identifies an important class of perfect morphisms.

D.16. **Proposition** (Lurie, [30, Pr. 3.3.20])**.** *If a morphism $f\colon X \longrightarrow Y$ of spectral Deligne–Mumford stacks is strongly proper* [30, Df. 3.1.1]*, of finite Tor-amplitude* [30, Df. 3.3.9]*, and locally almost of finite presentation* [26, Df. 8.16]*, then it is perfect.*

D.17. **Notation.** Denote by $\mathscr{FP}$ the class of strongly proper morphisms of finite Tor-amplitude [30, Df. 3.3.9] and locally almost of finite presentation. To see that the class $\mathscr{FP}$ is stable under pullbacks, combine [30, Rk. 3.1.5 and Pr. 3.3.16] and [26, Pr. 8.25]. It is easy to see that this class is compatible with coproducts in the sense of (5.2.4).

D.18. **Proposition.** *The functor*

$$p_{\mathbf{DM}}\colon \mathbf{Perf}^{op} \times_{\mathbf{Shv}_{\mathrm{flat}}} \mathbf{DM} \longrightarrow \mathbf{DM}$$

is a Waldhausen bicartesian fibration for the left complete disjunctive triple

$$(\mathbf{DM}, \mathbf{DM}_{\mathscr{FP}}, \mathbf{DM}).$$

D.18.1. **Corollary.** *The unfurling*

$$\Upsilon(p_{\mathbf{DM}}) \longrightarrow A^{\mathit{eff}}(\mathbf{DM}, \mathbf{DM}_{\mathscr{FP}}, \mathbf{DM})$$

is classified by a Mackey functor

$$\mathscr{M}_{\mathbf{DM}}\colon A^{\mathit{eff}}(\mathbf{DM}, \mathbf{DM}_{\mathscr{FP}}, \mathbf{DM}) \longrightarrow \mathbf{Wald}_{\infty}.$$

D.19. **Notation.** As a result, we obtain a Mackey functor

$$\mathbf{K} \circ \mathscr{M}_{\mathbf{DM}}\colon A^{\mathit{eff}}(\mathbf{DM}, \mathbf{DM}_{\mathscr{FP}}, \mathbf{DM}) \longrightarrow \mathbf{Sp}.$$

The algebraic K-theory of any Deligne–Mumford stack is an E_∞ ring spectrum, whence it has a canonical K-theory class, given by the unit. Correspondingly, we obtain a morphism of Mackey functors

$$\mathbf{S}_{(\mathbf{DM}, \mathbf{DM}_{\mathscr{FP}}, \mathbf{DM})} \longrightarrow \mathbf{K} \circ \mathscr{M}_{\mathbf{DM}}.$$

We thus obtain, for any Deligne–Mumford stacks U and X, an assembly morphism

$$\Sigma^{\infty}_{+} \operatorname{Mor}_{\mathscr{FP}}(U, X) \longrightarrow \mathbf{K}(X),$$

where $\operatorname{Mor}_{\mathscr{FP}}$ is the space of strongly proper morphisms of finite Tor-amplitude and locally almost of finite presentation.

As in Df. 8.3, we obtain a morphism of spectral Mackey functors

$$\mathbf{S}^{X}_{(\mathbf{DM}, \mathbf{DM}_{\mathscr{FP}}, \mathbf{DM})} \longrightarrow F(\mathbf{K}(X), \mathbf{K} \circ \mathscr{M}_{\mathbf{DM}}),$$

which induces assembly morphisms

$$\mathbf{S}^{X}_{(\mathbf{DM}, \mathbf{DM}_{\mathscr{FP}}, \mathbf{DM})}(Y) \wedge \mathbf{K}(X) \longrightarrow \mathbf{K}(Y).$$

We in turn obtain, for any spectral Deligne–Mumford stack U such that the morphism $U \longrightarrow \mathrm{Spec}^{\text{ét}}(\mathbf{S})$ is strongly proper, of finite Tor-amplitude, and locally almost of finite presentation, an assembly morphism

$$\Sigma^\infty_+ \operatorname{Map}(U, X) \longrightarrow F(\mathbf{K}(X), \mathbf{K}(\mathbf{S})).$$

When composed with the morphism induced by the trace

$$\mathbf{K}(\mathbf{S}) \longrightarrow \mathrm{THH}(\mathbf{S}) \simeq \mathbf{S},$$

we obtain a morphism

$$\Sigma^\infty_+ \operatorname{Map}(U, X) \longrightarrow D\mathbf{K}(X).$$

Now among the more general class of flat sheaves, let us characterize perfect morphisms that are also quasi-affine representable.

D.20. **Proposition.** *The following are equivalent for a quasi-affine representable morphism $f \colon X \longrightarrow Y$ of $\mathbf{Shv}_{\mathrm{flat}}$.*

(D.20.1) The morphism f is perfect.

(D.20.2) The quasicoherent module $f_\star \mathscr{O}_X$ is perfect.

Proof. That the first condition implies the second is obvious. To prove the converse, let us first reduce to the case where Y is affine. Indeed, for any quasicoherent module M over X, the quasicoherent module $f_\star M$ is perfect just in case, for any point $\eta \colon \operatorname{Spec} R \longrightarrow Y$, the R-module $\eta^\star f_\star M$ is perfect. Now consider the pullback

$$\begin{array}{ccc} X_\eta & \xrightarrow{f'} & \operatorname{Spec} R \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\eta} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

One now has $\eta^\star f_\star M \simeq f'_\star \varepsilon^\star M$ [28, Cor. 3.2.6(2)], and if M is perfect, so is $\varepsilon^\star M$.

So we now assume that $Y = \operatorname{Spec} R$. Now let $A := f_\star \mathscr{O}_X$, an R-module. In light of [28, Pr. 3.2.5], we obtain an equivalence $\mathbf{QCoh}_X \simeq \mathbf{Mod}_A$, under which the functor $f_\star$ can be identified with the forgetful functor

$$U \colon \mathbf{Mod}_A \longrightarrow \mathbf{Mod}_R.$$

Now $U^{-1}(\mathbf{Perf}_R)$ is a stable subcategory that is closed under retracts, and by assumption it contains A. Hence $\mathbf{Perf}_A \subset U^{-1}(\mathbf{Perf}_R)$. ☐

D.20.1. **Corollary.** *The collection $\mathscr{Q}\mathscr{P}$ of quasi-affine representable and perfect morphisms of $\mathbf{Shv}_{\mathrm{flat}}$ is stable under pullback.*

Proof. Suppose

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

a pullback square in $\mathbf{Shv}_{\mathrm{flat}}$, and suppose f quasi-affine representable and perfect. Then the quasicoherent module

$$f'_\star \mathscr{O}_{X'} \simeq f'_\star \alpha^\star \mathscr{O}_X \simeq \beta^\star f_\star \mathscr{O}_X$$

is perfect as well. ☐

It is a straightforward matter to see that $\mathscr{Q}\mathscr{P}$ is compatible with coproducts in the sense of (5.2.4). We thus conclude the following.

D.21. **Proposition.** *The functor*

$$\mathbf{Perf}^{op} \longrightarrow \mathbf{Shv}_{\mathrm{flat}}$$

is a Waldhausen bicartesian fibration for the left complete disjunctive triple

$$(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{Q}\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}}).$$

D.21.1. **Corollary.** *The unfurling*

$$\Upsilon(p_{\mathbf{Shv}_{\mathrm{flat}}}) \longrightarrow A^{\mathit{eff}}(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{Q}\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}})$$

is classified by a Mackey functor

$$\mathscr{M}_{\mathbf{Shv}_{\mathrm{flat}}}\colon A^{\mathit{eff}}(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{Q}\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}}) \longrightarrow \mathbf{Wald}_{\infty}.$$

D.22. **Notation.** As a result, we obtain a Mackey functor

$$\mathbf{K}\circ\mathscr{M}_{\mathbf{Shv}_{\mathrm{flat}}}\colon A^{\mathit{eff}}(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{Q}\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}}) \longrightarrow \mathbf{Sp}.$$

The algebraic K-theory of any flat sheaf is an E_∞ ring spectrum, whence it has a canonical K-theory class, given by the unit. Correspondingly, we obtain a morphism of Mackey functors

$$\mathbf{S}_{(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{Q}\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}})} \longrightarrow \mathbf{K}\circ\mathscr{M}_{\mathbf{Shv}_{\mathrm{flat}}}.$$

We thus obtain, for any flat sheaves U and X, an assembly morphism

$$\Sigma^\infty_+ \mathrm{Mor}_{\mathscr{Q}\mathscr{P}}(U,X) \longrightarrow \mathbf{K}(X),$$

where $\mathrm{Mor}_{\mathscr{Q}\mathscr{P}}$ is the space of quasi-affine representable and perfect morphisms.

Dually, as in Df. 8.3, we obtain a morphism of Mackey functors

$$\mathbf{S}^X_{(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{Q}\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}})} \longrightarrow F(\mathbf{K}(X), \mathbf{K}\circ\mathscr{M}_{\mathbf{Shv}_{\mathrm{flat}}}),$$

which induces assembly morphisms

$$\mathbf{S}^X_{(\mathbf{Shv}_{\mathrm{flat}}, \mathbf{Shv}_{\mathrm{flat},\mathscr{Q}\mathscr{P}}, \mathbf{Shv}_{\mathrm{flat}})}(Y)\wedge\mathbf{K}(X) \longrightarrow \mathbf{K}(Y)$$

for any flat sheaf Y. We in turn obtain, for any flat sheaf U such that $U \longrightarrow \mathrm{Spec}^{\mathrm{f}}(\mathbf{S})$ is quasiaffine representable and perfect, an assembly morphism

$$\Sigma^\infty_+ \mathrm{Map}(U,X) \longrightarrow F(\mathbf{K}(X), \mathbf{K}(\mathbf{S})).$$

When composed with the morphism induced by the trace

$$\mathbf{A}(*) \simeq \mathbf{K}(\mathbf{S}) \longrightarrow \mathrm{THH}(\mathbf{S}) \simeq \mathbf{S},$$

we obtain a morphism

$$\Sigma^\infty_+ \mathrm{Map}(U,X) \longrightarrow D\mathbf{K}(X).$$

Finally, let us restrict Ex. D to relate it to Ex. B.

D.23. **Notation.** Suppose X a spectral Deligne–Mumford stack. We denote by $\mathbf{F\acute{E}t}(X)$ the subcategory of $\mathbf{DM}_{/X}$ whose objects are finite [30, Df. 3.2.4] and étale morphisms $Y \longrightarrow X$ and whose morphisms are finite and étale morphisms over X. We will abuse notation and write $A^{\mathit{eff}}(X)$ for the effective Burnside ∞-category of $\mathbf{F\acute{E}t}(X)$.

D.24. **Example.** Suppose X a connected, noetherian scheme, suppose x a geometric point of X. Then if $\pi_1^{\text{ét}}(X,x)$ is the étale fundamental group of X, then by Grothendieck's Galois duality [33, Exp. V, §7], the ∞-category $\mathbf{F\acute{E}t}(X)$ is canonically equivalent to the ∞-category $NB^{fin}_{\pi_1^{\text{ét}}(X,x)}$ of Ex. B, whence we have a canonical equivalence

$$A^{eff}(\pi_1^{\text{ét}}(X,x)) \simeq A^{eff}(X).$$

(We can, of course, relax the condition that X be connected by passing to the étale fundamental groupoid.)

Restricting the Mackey functor $\mathbf{K} \circ \mathscr{M}_{\mathbf{DM}}$ of Nt. D.19 to

$$A^{eff}(X) \subset A^{eff}(\mathbf{DM}, \mathbf{DM}_{\mathscr{F}\mathscr{P}}, \mathbf{DM}),$$

we obtain a $\pi_1^{\text{ét}}(X,x)$-equivariant K-theory spectrum

$$\mathbf{K}_{\pi_1^{\text{ét}}(X,x)}(X) \colon A^{eff}(\pi_1^{\text{ét}}(X,x)) \longrightarrow \mathbf{Sp},$$

whose value on the $\pi_1^{\text{ét}}(X,x)$-set $[\pi_1^{\text{ét}}(X,x)/H]$ (with $H \leq \pi_1^{\text{ét}}(X,x)$ open) is the algebraic K-theory of the étale cover $X' \longrightarrow X$ corresponding to H. Corresponding to the unit of $\mathbf{K}(X)$, we obtain a morphism of Mackey functors

$$\mathbf{S}_{\pi_1^{\text{ét}}(X,x)} \longrightarrow \mathbf{K}_{\pi_1^{\text{ét}}(X,x)}(X).$$

The Segal–tom Dieck splitting thus provides a collection of maps

$$\Sigma^{\infty}_{+} B(N_{\pi_1^{\text{ét}}(X,x)}H/H) \longrightarrow \mathbf{K}(X),$$

one for each conjugacy class of open subgroups $H \leq \pi_1^{\text{ét}}(X,x)$.

References

1. C. Barwick, *On exact ∞-categories and the theorem of the heart*, Preprint `arXiv:1212.5232`, January 2013.
2. ———, *On the algebraic K-theory of higher categories.*, Preprint `arXiv:1204.3607`, April 2013.
3. Gunnar Carlsson, *Derived representation theory and the algebraic k-theory of fields*, Preprint from the web page of the author, 2003.
4. A. W. M. Dress, *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*, Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971, With the aid of lecture notes, taken by Manfred Küchler. MR 0360771 (50 #13218)
5. ———, *Contributions to the theory of induced representations*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 183–240. Lecture Notes in Math., Vol. 342. MR 0384917 (52 #5787)
6. A. W. M. Dress and A. O. Kuku, *The Cartan map for equivariant higher algebraic K-groups*, Comm. Algebra **9** (1981), no. 7, 727–746. MR 609220 (82i:18014)
7. ———, *A convenient setting for equivariant higher algebraic K-theory*, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., vol. 966, Springer, Berlin-New York, 1982, pp. 59–68. MR 689366 (84k:18007)
8. W. G. Dwyer and D. M. Kan, *Function complexes for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 2, 139–147. MR 705421 (85e:55038)
9. W. G. Dwyer, M. Weiss, and B. Williams, *A parametrized index theorem for the algebraic K-theory Euler class*, Acta Math. **190** (2003), no. 1, 1–104. MR 1982793 (2004d:19004)
10. T. G. Goodwillie, *Calculus. III. Taylor series*, Geom. Topol. **7** (2003), 645–711 (electronic). MR 2026544 (2005e:55015)
11. B. Guillou and J. P. May, *Enriched model categories and diagram categories*, Preprint `arXiv:1110.3567v3`, July 2013.
12. ———, *Models of G-spectra as presheaves of spectra*, Preprint `arXiv:1110.3571v3`, July 2013.

13. ______, *Permutative G-categories in equivariant infinite loop space theory*, Preprint `arXiv:1207.3459v2`, March 2014.
14. M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *The Arf-Kervaire invariant problem in algebraic topology: introduction*, Current developments in mathematics, 2009, Int. Press, Somerville, MA, 2010, pp. 23–57. MR 2757358
15. ______, *On the non-existence of elements of Kervaire invariant one*, Preprint `arXiv:0908.3724v2`, November 2010.
16. ______, *The Arf-Kervaire problem in algebraic topology: sketch of the proof*, Current developments in mathematics, 2010, Int. Press, Somerville, MA, 2011, pp. 1–43. MR 2906370
17. P. T. Johnstone, *Sketches of an elephant: a topos theory compendium. Vol. 2*, Oxford Logic Guides, vol. 44, The Clarendon Press, Oxford University Press, Oxford, 2002. MR 2063092 (2005g:18007)
18. D. Kaledin, *Derived Mackey functors*, Mosc. Math. J. **11** (2011), no. 4, 723–803, 822. MR 2918295
19. A. O. Kuku, *Equivariant K-theory and the cohomology of profinite groups*, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math., vol. 1046, Springer, Berlin, 1984, pp. 235–244. MR 750684 (86h:18005)
20. ______, *Equivariant higher K-theory for compact Lie group actions*, Beiträge Algebra Geom. **41** (2000), no. 1, 141–150. MR 1745585 (2001h:57044)
21. ______, *K-theory and representation theory*, Contemporary developments in algebraic K-theory, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 259–355 (electronic). MR 2175639 (2007e:19001)
22. ______, *Equivariant higher algebraic K-theory for Waldhausen categories*, Beiträge Algebra Geom. **47** (2006), no. 2, 583–601. MR 2310576 (2008c:19002)
23. ______, *Representation theory and higher algebraic K-theory*, Pure and Applied Mathematics (Boca Raton), vol. 287, Chapman & Hall/CRC, Boca Raton, FL, 2007. MR 2259035 (2007k:19001)
24. L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986, With contributions by J. E. McClure. MR 866482 (88e:55002)
25. J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)
26. ______, *Derived algebraic geometry IX. Closed immersions*, Preprint from the web page of the author, June 2011.
27. ______, *Derived algebraic geometry VII. Spectral schemes*, Preprint from the web page of the author, June 2011.
28. ______, *Derived algebraic geometry VIII. Quasi-coherent sheaves and Tannaka duality theorems*, Preprint from the web page of the author, May 2011.
29. ______, *Derived algebraic geometry X. Formal moduli problems*, Preprint from the web page of the author, September 2011.
30. ______, *Derived algebraic geometry XII. Proper morphisms, completions, and the Grothendieck existence theorem*, Preprint from the web page of the author, November 2011.
31. ______, *Higher algebra*, Preprint from the web page of the author, August 2012.
32. M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108. MR 1922205 (2003i:55012)
33. *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois Marie 1960–61 (SGA 1). Dirigé par A. Grothendieck. Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin, 1960–61. MR 50 #7129
34. K. Shimakawa, *A note on Γ_G-spaces*, Osaka J. Math. **28** (1991), no. 2, 223–228. MR 1132161 (92j:55015)
35. ______, *Mackey structures on equivariant algebraic K-theory*, K-Theory **5** (1991/92), no. 4, 355–371. MR 1162447 (93h:19003)
36. ______, *Mackey systems of monoidal categories and G-spectra*, K-Theory **5** (1991/92), no. 4, 373–394. MR 1162448 (94b:55016)
37. ______, *Note on the equivariant K-theory spectrum*, Publ. Res. Inst. Math. Sci. **29** (1993), no. 3, 449–453. MR 1233745 (94k:55020)

38. ———, *An E_∞ ring G-spectrum representing the equivariant algebraic K-theory of a bipermutative G-category*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 179, 367–395. MR 1295582 (95m:19003)
39. T. tom Dieck, *Orbittypen und äquivariante Homologie. II*, Arch. Math. (Basel) **26** (1975), no. 6, 650–662. MR 0436177 (55 #9127)
40. F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR 86m:18011

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SPECTRAL MACKEY FUNCTORS AND EQUIVARIANT ALGEBRAIC K-THEORY (II)

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Abstract. We study the "higher algebra" of spectral Mackey functors, which the first named author introduced in Part I of this paper. In particular, armed with our new theory of symmetric promonoidal ∞-categories and a suitable generalization of the second named author's Day convolution, we endow the ∞-category of Mackey functors with a well-behaved symmetric monoidal structure. This makes it possible to speak of *spectral Green functors* for any operad O. We also answer a question of Mathew, proving that the algebraic K-theory of group actions is lax symmetric monoidal. We also show that the algebraic K-theory of derived stacks provides an example. Finally, we give a very short, new proof of the equivariant Barratt–Priddy–Quillen theorem, which states that the algebraic K-theory of the category of finite G-sets is simply the G-equivariant sphere spectrum.

Contents

0. Summary

This paper is part of an effort to give a complete description of the structures available on the algebraic K-theory of varieties and schemes (and even of various derived stacks) with all their concomitant functorialities and homotopy coherences.

So suppose X a scheme (quasicompact and quasiseparated). The derived tensor product $\otimes^{\mathbf{L}}$ on perfect complexes on X defines a symmetric monoidal structure on the derived category D_X^{perf} of perfect complexes on X. With a little more effort, one can lift this structure to a symmetric monoidal structure on the stable ∞-category of perfect complexes on X. This suffices to get a product on algebraic K-theory

$$\otimes\colon K(X) \wedge K(X) \longrightarrow K(X)$$

that is associative and commutative up to coherent homotopy. Thus, $K(X)$ has not only the structure of a connective spectrum, but also the structure of a *connective E_∞ ring spectrum*. This is an exceedingly rich structure: not only do the homotopy groups $K_*(X)$ form a graded commutative ring, but these homotopy groups also support (in a functorial way) a tremendous amount of structure involving intricate higher homotopy operations called *Toda brackets*. Still more information (in the form of *Dyer-Lashof operations*) can be found on the $\mathbf{F}_p$-cohomology of $K(X)$.

Now for any morphism $f\colon Y \longrightarrow X$ of schemes, the derived functor

$$\mathbf{L}f^\star\colon D_X^{qcoh} \longrightarrow D_Y^{qcoh}$$

on the category of complexes with quasicoherent cohomology preserves perfect complexes, and the resulting functor $\mathbf{L}f^\star\colon D_X^{perf} \longrightarrow D_Y^{perf}$ induces a morphism

$$f^\star\colon K(X) \longrightarrow K(Y)$$

on the algebraic K-theory. The functor $\mathbf{L}f^\star$ is compatible with the derived tensor product, in the sense that for any perfect complexes E and F on X, there is a canonical isomorphism

$$\mathbf{L}f^\star(E \otimes^{\mathbf{L}} F) \simeq (\mathbf{L}f^\star E) \otimes^{\mathbf{L}} (\mathbf{L}f^\star F).$$

Again this can be lifted to the level of stable ∞-categories, whence the induced morphism $f^\star$ on K-theory turns out to be a morphism of connective E_∞ ring spectra. This implies that the induced homomorphism on homotopy groups

$$f^\star\colon K_*(X) \longrightarrow K_*(Y)$$

is a homomorphism of graded commutative rings, and it must respect all the higher homotopy operations on $K_*(X)$ as well.

Furthermore, one can fit all the functors $\mathbf{L}f^\star$ together to get a presheaf $U \rightsquigarrow D_U^{perf}$ on the big site of all schemes. This can even be viewed as a presheaf of stable ∞-categories, which suffices to give us a presheaf of connective spectra $U \rightsquigarrow K(U)$. Since the morphisms $f^\star$ are morphisms of connective E_∞ ring spectra, we can regard this as presheaf of E_∞ ring spectra.

If one wanted, one might "externalize" the product on K-theory in the following manner. For any two schemes X and Y over a base scheme S, one may define an external tensor product

$$\boxtimes^{\mathbf{L}}\colon D_X^{perf} \times D_X^{perf} \longrightarrow D_{X\times_S Y}^{perf}$$

by the assignment $(E, F) \rightsquigarrow (\mathbf{L}\,\mathrm{pr}_1^\star E) \otimes^{\mathbf{L}} (\mathbf{L}\,\mathrm{pr}_2^\star F)$. Note that we have natural equivalences

$$(\mathbf{L}f^\star E) \boxtimes^{\mathbf{L}} (\mathbf{L}g^\star F) \simeq \mathbf{L}(f \times g)^\star(E \boxtimes^{\mathbf{L}} F)$$

If we lift this to the level of stable ∞-categories, this gives rise to an external pairing

$$\boxtimes\colon K(X) \wedge K(Y) \longrightarrow K(X \times_S Y),$$

which is functorial (contravariantly) in X and Y. The E_∞ product on $K(X)$ can now be obtained by pulling back this external pairing along the diagonal map:

$$K(X) \wedge K(X) \longrightarrow K(X \times_S X) \longrightarrow K(X).$$

A morphism of schemes $f\colon Y \longrightarrow X$ may induce morphisms in the *covariant* direction as well. The pushforward $\mathbf{R}f_\star\colon D^{qcoh}_Y \longrightarrow D^{qcoh}_X$ generally will not preserve perfect complexes. If, however, f is flat and proper, then for any perfect complex E, the complex $\mathbf{R}f_\star E$ *is* perfect. Thus in this case $\mathbf{R}f_\star$ restricts to a functor $\mathbf{R}f_\star\colon D^{perf}_Y \longrightarrow D^{perf}_X$, and after lifting this to the stable ∞-categories, we find an induced morphism

$$f_\star\colon K(Y) \longrightarrow K(X)$$

on the algebraic K-theory. One thus obtains a covariant functor $U \rightsquigarrow K(U)$, but only with respect to flat and proper morphisms. Observe, however, that since the functors $\mathbf{R}f_\star$ do not commute with the derived tensor product, this functor is *not* valued in ring spectra.

Nevertheless, if $f\colon Y \longrightarrow X$ is proper and flat, we do have an algebraic structure preserved by $\mathbf{R}f_\star$. Observe that one may regard $K(Y)$ as a module over the E_∞ ring spectrum $K(X)$ via $f^\star$. For any perfect complexes E on Y and F on X, one has a canonical equivalence

$$(\mathbf{R}f_\star E) \otimes^{\mathbf{L}} F \simeq \mathbf{R}f_\star(E \otimes^{\mathbf{L}} \mathbf{L}f^\star F)$$

of perfect complexes; this is the usual projection formula [8, Exp. III, Pr. 3.7]. At the level of K-theory, this translates to the observation that the morphism

$$f_\star\colon K(Y) \longrightarrow K(X)$$

is a morphism of connective $K(X)$-modules. The induced map on homotopy groups

$$f_\star\colon K_*(Y) \longrightarrow K_*(X)$$

is therefore a homomorphism of $K_*(X)$-modules.

Note that the *external* tensor product $\boxtimes^{\mathbf{L}}$ is actually perfectly compatible with the pushforwards, in the sense that one has natural equivalences

$$(\mathbf{R}f_\star E) \boxtimes^{\mathbf{L}} (\mathbf{R}g_\star F) \simeq \mathbf{R}(f \times g)_\star(E \boxtimes^{\mathbf{L}} F),$$

so on K-theory the external product $\boxtimes\colon K(X) \wedge K(Y) \longrightarrow K(X \times_S Y)$ is functorial (covariantly) in X and Y.

Last, but certainly not least, there is a compatibility between the morphisms $f^\star$ and the morphisms $g_\star$, which results from the base change theorem for complexes [8, Exp. IV, Pr. 3.1.0]. Suppose that

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$

is a pullback square of schemes in which the horizontal maps g are flat and proper. Then the canonical morphism

$$\mathbf{L}f^\star \mathbf{R}g_\star \longrightarrow \mathbf{R}g_\star \mathbf{L}f^\star$$

is an objectwise equivalence of functors $D^{perf}_{X'} \longrightarrow D^{perf}_Y$. This translates to the condition that there is a canonical homotopy

$$f^\star g_\star \simeq g_\star f^\star\colon K(X') \longrightarrow K(Y)$$

of morphisms of $K(X)$-modules. In fact, this compatibility between the pullbacks and the pushforwards, combined with the compatibility between $f_\star$ and the external tensor product, allows us to *deduce* the projection formula.

Let us summarize the structure we've found on the assignment $U \rightsquigarrow K(U)$:

- For every scheme X, we have an E_∞ ring spectrum $K(X)$. Moreover, for any two schemes X and Y over a base S, one has an external pairing
$$\boxtimes\colon K(X) \wedge K(Y) \longrightarrow K(X \times_S Y).$$
- For every morphism $f\colon Y \longrightarrow X$, we have a pullback morphism
$$f^\star\colon K(X) \longrightarrow K(Y),$$
which is compatible with the external pairings and thus also with the E_∞ product.
- For every flat and proper morphism $f\colon Y \longrightarrow X$, we have a pushforward morphism
$$f_\star\colon K(Y) \longrightarrow K(X),$$
which is compatible with the external pairings and thus (in light of the next condition) also with the $K(X)$-module structure.
- For any pullback square
$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$
in which the horizontal maps g are flat and proper, we have a canonical homotopy
$$f^\star g_\star \simeq g_\star f^\star\colon K(X') \longrightarrow K(Y).$$
of morphisms of $K(X)$-modules.

In this paper, we will demonstrate that these structures, along with all of their homotopy coherences, are neatly packaged in a *spectral Green functor* on the category of schemes.

This structure is the origin of both the $\mathrm{Gal}(E/F)$-equivariant E_∞ ring spectrum structure on the algebraic K-theory of a Galois extension $E \supset F$ and the cyclotomic structure on the p-typical curves on a smooth $\mathbf{F}_p$-scheme. For the former, see 9.7, and for the latter, see the forthcoming paper [7].

In order to describe all the structure we see here, we study the "higher algebra" (in the sense of Lurie's book [19], for example) of spectral Mackey functors, which we introduced in Part I of this paper [4]. The ∞-category of spectral Mackey functors turns out to admit all the same well-behaved structures as the ∞-category of spectra itself. In particular, the ∞-category of Mackey functors admits a well-behaved symmetric monoidal structure. This, combined with Saul Glasman's convolution for ∞-categories [11], makes it possible to speak of E_1 algebras, E_∞ algebras, or indeed O-algebras for any operad O in this context; these are called *O-Green functors*.

We use this framework to provide a very simple answer to a question posed to us by Akhil Mathew, in which we demonstrate that the functor that assigns to any ∞-category with an action of a finite group G its equivariant algebraic K-theory is lax symmetric monoidal. We also show that the algebraic K-theory of derived stacks with its transfer maps as described above offers an example of an E_∞ Green functor. We also use this theory to give a new proof of the equivariant Barratt–Priddy–Quillen theorem, which states that the algebraic K-theory of the category of finite G-sets is simply the G-equivariant sphere spectrum. (In fact, we will generalize this result dramatically.)

Warning. Let us emphasize that E_∞-Green functors for a finite group G are *not* equivalent to algebras in G-equivariant spectra structured by the equivariant linear isometries operad on a complete G-universe. To describe the latter in line with the discussion here – and to find such structures on algebraic K-theory spectra – it is necessary to develop elements of the theory of G-∞-categories. This we do in the forthcoming joint paper [5].

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1. ∞-ANTI-OPERADS AND SYMMETRIC PROMONOIDAL ∞-CATEGORIES

One of the many complications that arises when one combines an ∞-category and its opposite in the way we have in our construction of the effective Burnside ∞-category is that our constructions are extremely intolerant of asymmetries in basic definitions. This complication rears its head the moment we want to contemplate the symmetric monoidal structure on the Burnside ∞-category. In effect, the description of a symmetric monoidal ∞-categories given in [19, Ch. 4] forces one to specify the data of maps *out of* various tensor products in a suitably compatible fashion. Thus symmetric monoidal categories are there identified as certain ∞-*operads*. But since we are also working with opposites of symmetric monoidal ∞-categories, we will come face-to-face with circumstances in which we must identify the data of maps *into* various tensor products in a suitably compatible fashion. We will call the resulting opposites of ∞-operads ∞-*anti-operads*.[1] Awkward as this may seem, it cannot be avoided.

1.1. **Notation.** Let $\Lambda(\mathbf{F})$ denote the following ordinary category. The objects will be finite sets, and a morphism $J \longrightarrow I$ will be a map $J \longrightarrow I_+$; one composes $\psi\colon K \longrightarrow J_+$ with $\phi\colon J \longrightarrow I_+$ by forming the composite

$$K \xrightarrow{\psi} J_+ \xrightarrow{\phi_+} I_{++} \xrightarrow{\mu} I_+,$$

where $\mu\colon I_{++} \longrightarrow I_+$ is the map that simply identifies the two added points. (Of course $\Lambda(\mathbf{F})$ is equivalent to the category $\mathbf{F}_*$ of pointed finite sets, but we prefer to think of the objects of $\Lambda(\mathbf{F})$ as unpointed. This is the natural perspective on this category from the theory of operator categories [1].)

1.2. **Definition.** (1.2.1) An ***∞-anti-operad*** is an inner fibration

$$p\colon O_\otimes \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$$

whose opposite

$$p^{op}\colon (O_\otimes)^{op} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

is an ∞-operad.

[1] We do not know a standard name for this structure. In a previous verion of this paper, CB called these "cooperads," but this conflicts with better-known terminology.

(1.2.2) If $p\colon O_{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$ is an ∞-anti-operad, then an edge of $O_{\otimes}$ will be said to be ***inert*** if it is cartesian over an edge of $\mathrm{N}\Lambda(\mathbf{F})^{op}$ that corresponds to an inert map in $\Lambda(\mathbf{F})$, that is, a map $\phi\colon J \longrightarrow I_{+}$ such that the induced map $\phi^{-1}(I) \longrightarrow I$ is a bijection [19, Df. 2.1.1.8], [1, Df. 8.1].

(1.2.3) A cartesian fibration

$$q\colon X_{\otimes} \longrightarrow O_{\otimes}$$

will be said to ***exhibit*** $X_{\otimes}$ ***as an*** $O_{\otimes}$***-monoidal*** ∞***-category*** just in case the cocartesian fibration

$$q^{op}\colon (X_{\otimes})^{op} \longrightarrow (O_{\otimes})^{op}$$

exhibits $(X_{\otimes})^{op}$ as an $(O_{\otimes})^{op}$-monoidal ∞-category in the sense of [19, Df. 2.1.2.13]. When $O_{\otimes} = \mathrm{N}\Lambda(\mathbf{F})^{op}$, we will say that q ***exhibits*** $X_{\otimes}$ ***as a symmetric monoidal*** ∞***-category***.

(1.2.4) A ***morphism*** $f\colon O_{\otimes} \longrightarrow P_{\otimes}$ ***of*** ∞***-anti-operads*** is a morphism over $\mathrm{N}\Lambda(\mathbf{F})^{op}$ that carries inert edges to inert edges. If $O_{\otimes}$ and $P_{\otimes}$ are symmetric monoidal ∞-categories, then f is a ***symmetric monoidal functor*** if it carries all cartesian edges to cartesian edges.

1.3. **Example.** Suppose C an ∞-category. We define the ***cartesian*** ∞***-anti-operad*** as

$$p\colon C_{\times} := ((C^{op})^{\sqcup})^{op} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op},$$

where the notation $(\cdot)^{\sqcup}$ refers to the cocartesian ∞-operad [19, Cnstr. 2.4.3.1]. If C is an ∞-category that admits all products, then the functor p exhibits $C_{\times}$ as a symmetric monoidal ∞-category [19, Rk. 2.4.3.4].

An object (I, X) of $C_{\times}$ consists of a finite set I and a family $\{X_i \mid i \in I\}$; a morphism $(\phi, \omega)\colon (I, X) \longrightarrow (J, Y)$ of $C_{\times}$ consists of a map of finite sets $\phi\colon J \longrightarrow I_{+}$ and a family of morphisms

$$\left\{\omega_j\colon X_{\phi(j)} \longrightarrow Y_j \;\middle|\; j \in \phi^{-1}(I)\right\}$$

of C. If C admits finite products, then the morphisms ω_j determine and are determined by a family of morphisms

$$\left\{ \omega_{J_i}\colon X_i \longrightarrow \prod_{j \in J_i} Y_j \;\middle|\; i \in I \right\};$$

here J_i denotes the fiber $\phi^{-1}(i)$.

Observe that the cartesian ∞-anti-operad is significantly simpler to define than the cartesian ∞-operad. Note also that $(\Delta^0)_{\times} = \mathrm{N}\Lambda(\mathbf{F})^{op}$.

It is extremely useful to note that the condition that an ∞-operad $C^{\otimes}$ be a symmetric monoidal ∞-category can be broken into two conditions:

(1) The first of these is *corepresentability* [19, Df. 6.2.4.3]; this is the condition that the functors $\mathrm{Map}^{\xi_I}_{C^{\otimes}}(x_I, -)\colon C \longrightarrow \mathbf{Top}$ be corepresentable, where ξ_I is the unique active map $I \longrightarrow *$ in $\Lambda(\mathbf{F})$. A compact expression of this is simply to say (as Lurie does) that the inner fibration $C^{\otimes} \longrightarrow N\Lambda(\mathbf{F})$ is locally cocartesian.

(2) The second condition is *symmetric promonoidality*. This can be expressed in a number of ways. One may say that for any active map $\phi\colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$, for

any object $x_J \in C^{\otimes}_J$, and for any object $z \in C$, the natural map

$$\int^{y_I \in C^{\otimes}_I} \mathrm{Map}^{\xi_I}_{C^{\otimes}}(y_I, z) \times \mathrm{Map}^{\phi}_{C^{\otimes}}(x_J, y_I) \longrightarrow \mathrm{Map}^{\xi_J}_{C^{\otimes}}(x_J, z)$$

is an equivalence; this is an operadic version of the condition expressed in [19, Ex. 6.2.4.9]. Equivalently, $C^{\otimes}$ is a symmetric promonoidal ∞-category if it represents a commutative algebra object in the ∞-category of ∞-categories and profunctors. In light of [19, §B.3], we make the following definition.

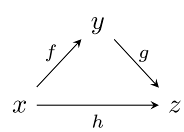
1.4. **Definition.** We will say that an ∞-operad $C^{\otimes}$ is ***symmetric promonoidal*** if the structure map $C^{\otimes} \longrightarrow N\Lambda(\mathbf{F})$ is a flat inner fibration [19, Df. B.3.8]. Similarly, we will say that an ∞-anti-operad $C_{\otimes}$ is ***symmetric promonoidal*** if the structure map $C_{\otimes} \longrightarrow N\Lambda(\mathbf{F})^{op}$ is a flat inner fibration.

Our claim now is that the conjunction of these two conditions are equivalent to the condition that $C^{\otimes}$ be a symmetric monoidal ∞-category. That is, we claim that a symmetric monoidal ∞-category is *precisely* a corepresentable symmetric promonoidal ∞-category. This follows immediately from the following.

1.5. **Proposition.** *The following are equivalent for an inner fibration $p\colon X \longrightarrow S$.*

(1.5.1) The inner fibration p is flat and locally cocartesian.
(1.5.2) The inner fibration p is cocartesian.

Proof. The second condition implies the first by [19, Ex. B.3.11]. Let us show that the first condition implies the second. By [16, Pr. 2.4.2.8], it suffices to consider the case in which $S = \Delta^2$, and to show that for any section of p given by a commutative triangle



in which f and g are locally p-cocartesian, the edge h is locally p-cocartesian as well.

In this case, by [16, Cor. 3.3.1.2], we can find a cocartesian fibration $q\colon Y \longrightarrow \Delta^2$ along with an equivalence

$$\phi\colon X \times_{\Delta^2} \Lambda^2_1 \xrightarrow{\sim} Y \times_{\Delta^2} \Lambda^2_1$$

of cocartesian fibrations over Λ^2_1. Now since p is flat, the inclusion $X \times_{\Delta^2} \Lambda^2_1 \hookrightarrow X$ is a categorical equivalence over Δ^2. Consequently, we may lift to obtain a map $\psi\colon X \longrightarrow Y$ over Δ^2 extending ϕ. This map is a categorical equivalence since both p and q are flat.

Now $\psi(f) = \phi(f)$ and $\psi(g) = \phi(g)$ are q-cocartesian, whence so is $\psi(h)$. The stability of relative colimits under categorical equivalences [16, Pr. 4.3.1.6], in light of [16, Ex. 4.3.1.4], now implies that h is p-cocartesian. □

One reason to treasure symmetric promonoidal structures is the fact that, as we shall now prove, they are precisely the structure needed on an ∞-category C in order for $\mathrm{Fun}(C, D)$ to admit a *Day convolution* symmetric monoidal structure.[2]

[2]We would like to acknowledge that Dylan Wilson has independently made this observation.

To explain, suppose first $C^{\otimes}$ a small symmetric monoidal ∞-category, and suppose $D^{\otimes}$ a symmetric monoidal ∞-category such that D admits all colimits, and the tensor product preserves colimits separately in each variable. In [11], Glasman constructs a symmetric monoidal structure on the functor ∞-category $\mathrm{Fun}(C, D)$ which is the natural ∞-categorical generalization of Day's convolution product. As in Day's construction, the convolution $F \otimes G$ of two functors $F, G\colon C \longrightarrow D$ in Glasman's symmetric monoidal structure is given by the left Kan extension of the composite

$$C \times C \xrightarrow{(F,G)} D \times D \xrightarrow{\otimes} D$$

along the tensor product $\otimes\colon C \times C \longrightarrow C$.

In particular, for any finite set I, and for any I-tuple $\{F_i\}_{i \in I}$ of functors $C \longrightarrow D$, the value of the tensor product is given by the coend

$$\left(\bigotimes_{i \in I} F_i\right)(x) \simeq \int^{u_I \in C_I^{\otimes}} \mathrm{Map}^{\xi_I}_{C^{\otimes}}(u_I, x) \otimes \bigotimes_{i \in I} F_i(u_i).$$

Equivalently, the Day convolution on $\mathrm{Fun}(C, D)$ is the essentially unique symmetric monoidal structure that enjoys the following criteria:

- ▶ The tensor product
$$- \otimes -\colon \mathrm{Fun}(C, D) \times \mathrm{Fun}(C, D) \longrightarrow \mathrm{Fun}(C, D)$$
preserves colimits separately in each variable.
- ▶ The functor given by the composite
$$C^{op} \times D \xrightarrow{j \times \mathrm{id}} \mathrm{Fun}(C, \mathbf{Kan}) \times D \xrightarrow{m} \mathrm{Fun}(C, D)$$
is symmetric monoidal, where j denotes the Yoneda embedding, and m is the functor corresponding to the composition
$$\mathrm{Fun}(C, \mathbf{Kan}) \longrightarrow \mathrm{Fun}(D \times C, D \times \mathbf{Kan}) \longrightarrow \mathrm{Fun}(D \times C, D)$$
in which the first functor is the obvious one, and the functor $D \times \mathbf{Kan} \longrightarrow D$ is the tensor functor $(X, K) \rightsquigarrow X \otimes K$ of [16, §4.4.4].

Conveniently, we can extend Glasman's Day convolution to situations in which $C^{\otimes}$ is only symmetric promonoidal.

1.6. **Proposition.** *For any symmetric promonoidal ∞-category $C^{\otimes}$ and any symmetric monoidal ∞-category $D^{\otimes}$ such that D admits all colimits and $\otimes\colon D \times D \longrightarrow D$ preserves colimits separately in each variable,* $\mathrm{Fun}(C, D)$ *admits a symmetric monoidal structure such that the E_∞-algebras therein are morphisms of ∞-operads $C^{\otimes} \longrightarrow D^{\otimes}$.*

Proof. The results of the first two sections of [11] hold when $C^{\otimes}$ is symmetric promonoidal with only one change: in the proof of [11, Lm. 2.3], the reference to [16, Pr. 3.3.1.3] should be replaced with a reference to [19, Pr. B.3.14]. Consequently, our claim follows from [11, Prs. 2.11 and 2.12]. □

1.7. Once again, for any finite set I, and for any I-tuple $\{F_i\}_{i \in I}$ of functors $C \longrightarrow D$, the value of the tensor product is given by the coend

$$\left(\bigotimes_{i \in I} F_i\right)(x) \simeq \int^{u_I \in C_I^{\otimes}} \mathrm{Map}^{\xi_I}_{C^{\otimes}}(u_I, x) \otimes \bigotimes_{i \in I} F_i(u_i).$$

2. The symmetric promonoidal structure on the effective Burnside ∞-category

Suppose C a disjunctive ∞-category. The product on C does not induce the product on the effective Burnside ∞-category $A^{eff}(C)$. (Indeed, recall that the effective Burnside ∞-category admits direct sums, and these direct sums are induced by the *coproduct* in C.) However, a product on C (if it exists) *does* induce a symmetric monoidal structure on $A^{eff}(C)$. The construction of the previous example is just what we need to describe this structure, and it will work for a broad class of disjunctive triples – which we call *cartesian* – as well.

It turns out to be convenient to consider situations in which C does not actually have products. In this case, the effective Burnside ∞-category $A^{eff}(C)$ admits not a symmetric monoidal structure, but only a symmetric promonoidal structure, which suffices to get the Day convolution on ∞-categories of Mackey functors.

2.1. **Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. We now define a triple structure $(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)$ on $C_\times$ in the following manner. A morphism

$$(\phi, \omega)\colon (I, X) \longrightarrow (J, Y)$$

of $C_\times$ will be ingressive just in case ϕ is a bijection, and each morphism

$$\omega_j\colon X_{\phi(j)} \longrightarrow Y_j$$

is ingressive. The morphism (ϕ, ω) will be egressive just in case each morphism

$$\omega_j\colon X_{\phi(j)} \longrightarrow Y_j$$

is egressive (with no condition on ϕ).

It is a trivial matter to verify the following.

2.2. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. Then the triple*

$$(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)$$

is adequate in the sense of [4, Df. 5.2].

In particular, for any left complete disjunctive triple $(C, C_\dagger, C^\dagger)$, one may consider the effective Burnside ∞-category

$$A^{eff}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger).$$

2.3. **Example.** Note in particular that

$$((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger) \simeq (\mathrm{N}\Lambda(\mathbf{F})^{op}, \iota\mathrm{N}\Lambda(\mathbf{F})^{op}, \mathrm{N}\Lambda(\mathbf{F})^{op}),$$

whence one proves easily that the inclusion

$$\mathrm{N}\Lambda(\mathbf{F}) \simeq (((\Delta^0)_\times)^\dagger)^{op} \hookrightarrow A^{eff}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)$$

is an equivalence.

We'll use the following pair of results. They follow the same basic pattern as [4, Lms. 11.4 and 11.5]; in particular, they too follow immediately from the first author's "omnibus theorem" [4, Th. 12.2].

2.4. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. Then the natural functor*

$$A^{eff}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger) \longrightarrow A^{eff}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)$$

is an inner fibration.

2.5. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. Then for any object Y of $C_\times$ lying over an object $J \in (\Delta^0)_\times$ and any inert morphism $\phi\colon I \longrightarrow J$ of $N\Lambda(\mathbf{F})$, there exists a cocartesian edge $Y \longrightarrow X$ for the inner fibration*

$$A^{\mathit{eff}}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger) \longrightarrow A^{\mathit{eff}}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)$$

lying over the image of ϕ under the equivalence of Ex. 2.3.

Now we can go about defining the symmetric promonoidal structure on the effective Burnside ∞-category of a disjunctive triple.

2.6. **Notation.** For any disjunctive triple $(C, C_\dagger, C^\dagger)$, we define $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes$ as the pullback

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes := A^{\mathit{eff}}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger) \times_{A^{\mathit{eff}}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)} \mathrm{N}\Lambda(\mathbf{F}),$$

equipped with its canonical projection to $\mathrm{N}\Lambda(\mathbf{F})$. Note that because the inclusion

$$\mathrm{N}\Lambda(\mathbf{F}) \hookrightarrow A^{\mathit{eff}}((\Delta^0)_\times, (\Delta^0)_{\times,\dagger}, (\Delta^0)_\times^\dagger)$$

is an equivalence, it follows that the projection functor

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow A^{\mathit{eff}}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)$$

is actually an equivalence.

2.7. **Remark.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. The objects of the total ∞-category $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes$ are pairs (I, X_I) consisting of a finite set I and an I-tuple $X_I = (X_i)_{i\in I}$ of objects of C. A morphism

$$(J, Y_J) \longrightarrow (I, X_I)$$

of $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes$ can be thought of as a morphism $\phi\colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$ and a collection of diagrams

$$\left\{ \begin{array}{ccc|c} & U_{\phi(j)} & & \\ \swarrow & & \searrow & j \in \phi^{-1}(I) \\ Y_j & & X_{\phi(j)}, & \end{array} \right\}$$

such that for any $j \in J$, the morphism $U_{\phi(j)} \rightarrowtail X_{\phi(j)}$ is ingressive, and the morphism

$$U_{\phi(j)} \twoheadrightarrow Y_j$$

is egressive.

Composition is then defined by pullback; that is, a 2-simplex

$$(K, Z_K) \longrightarrow (J, Y_J) \longrightarrow (I, X_I)$$

consists of morphisms $\psi\colon K \longrightarrow J$ and $\phi\colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$ along with a collection of diagrams

$$\left\{ \begin{array}{ccccc|c} & & W_{\phi(\psi(k))} & & & \\ & \swarrow & & \searrow & & \\ & V_{\psi(k)} & & U_{\phi(\psi(k))} & & k \in (\phi\psi)^{-1}(I) \\ \swarrow & & \searrow \quad \swarrow & & \searrow & \\ Z_k & & Y_{\psi(k)} & & X_{\phi(\psi(k))} & \end{array} \right\}$$

in which the square in the middle exhibits each W_i (for $i \in I$) as the iterated fiber product over U_i of the set of objects $\{V_j \times_{Y_j} U_i \mid j \in J_i\}$. (Note that the left completeness is used to show that this iterated fiber product exists.)

In particular, $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_{\{1\}}$ may be identified with the effective Burnside ∞-category $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ itself, and for any finite set I, the inert morphisms $\chi_i \colon I \longrightarrow \{i\}_+$ together induce an equivalence

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_{I} \xrightarrow{\sim} \prod_{i \in I} A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_{\{i\}}.$$

For the proofs of the next few results it is convenient to introduce a bit of notation.

2.8. **Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a triple, suppose A and B are two sets, and suppose $S \colon A \sqcup B \longrightarrow C$ a functor. Then let

$$C'_{/\{S_x \ ; \ S_y\}_{x \in A, y \in B}} \subseteq C_{/\{S_z\}_{z \in A \sqcup B}}$$

denote the full subcategory spanned by those objects such that the morphisms to the objects S_x are egressive and the morphisms to the objects S_y are ingressive. In particular, note that

$$\mathrm{Map}_{A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}}((J, Y_J), (*, X)) \simeq \iota C'_{/\{Y_j \ ; \ X\}_{j \in J}}.$$

We have almost proven the following.

2.9. **Proposition.** *For any left complete disjunctive triple* $(C, C_\dagger, C^\dagger)$*, the inner fibration*

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

is an ∞*-operad.*

Proof. Following Rk. 2.7, it only remains to show that given an edge $\alpha \colon I \longrightarrow J$ in $\mathrm{N}\Lambda(\mathbf{F})$ and objects $(I, X), (J, Y)$ in $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}$, the cocartesian edges

$$\begin{array}{ccc} & (*, Y_j) & \\ \swarrow\!\!= & & \searrow \\ (J, Y) & & (*, Y_j), \end{array}$$

over the inert edges $\rho^j \colon J \longrightarrow *$ induce an equivalence

$$\mathrm{Map}^{\alpha}_{A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}}((I, X), (J, Y)) \longrightarrow \prod_{j \in J} \mathrm{Map}^{\rho^j \circ \alpha}_{A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}}((I, X), (*, Y_j)).$$

But this is indeed true, since the map identifies the left-hand side as

$$\prod_{j \in J} \iota C'_{/\{X_i \ ; \ Y_j\}_{i \in \alpha^{-1}(j)}}.$$ □

We now show that the ∞-operad $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}$ is symmetric promonoidal.

2.10. **Proposition.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a left complete disjunctive triple. Then the* ∞*-operad*

$$p \colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

is symmetric promonoidal; that is, p *is a flat inner fibration.*

Proof. Suppose $\sigma\colon \Delta^2 \longrightarrow N\Lambda(\mathbf{F})$ a 2-simplex given by a diagram

$$\begin{array}{ccc} & J & \\ {}^{\alpha}\nearrow & & \searrow^{\beta} \\ I & \xrightarrow[\gamma]{} & K \end{array}$$

a 2-simplex of $N\Lambda(\mathbf{F})$. Suppose

$$\begin{array}{ccc} & (K,W) & \\ \swarrow & & \searrow \\ (I,X) & & (K,Z), \end{array}$$

an edge $\widetilde{\gamma}$ of

$$\Sigma := A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{N\Lambda(\mathbf{F}),\sigma} \Delta^2$$

lifting γ. Set

$$E := \Sigma_{(I,X)/\ /(K,Z)} \times_{N\Lambda(\mathbf{F})} \{J\}$$

be the ∞-category of factorizations of $\widetilde{\gamma}$ through Σ_J. Observe that an n-simplex of E is a cartesian functor $\widetilde{\mathscr{O}}(\Delta^{n+2})^{op} \longrightarrow (C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})$ satisfying certain conditions.

We aim to show that E is weakly contractible. To this end, we will identify a full subcategory $E' \subset E$ whose inclusion functor admits a right adjoint such that E' contains a terminal object.

To begin, let us define a functor $\epsilon\colon E \times \Delta^1 \longrightarrow E$ extending the projection $E \times \{1\} \xrightarrow{\sim} E$ as follows: given non-negative integers $k \leq n$, let

$$f_{n,k}\colon \widetilde{\mathscr{O}}(\Delta^{n+3}) \longrightarrow \widetilde{\mathscr{O}}(\Delta^{n+2})$$

be the unique functor which on objects is given by

$$f_{n,k}(ij) := \begin{cases} 0j & \text{if } i \leq k+1 \text{ and } j \leq k+1; \\ 0(j-1) & \text{if } i \leq k+1 \text{ and } j > k+1; \\ (i-1)(j-1) & \text{if } i > k+1. \end{cases}$$

Then for every n-simplex $\sigma\colon \Delta^n \longrightarrow E$ corresponding to a functor

$$\overline{\sigma}\colon \widetilde{\mathscr{O}}(\Delta^{n+2})^{op} \longrightarrow C_{\times},$$

define $\epsilon(\sigma)\colon \Delta^n \times \Delta^1 \longrightarrow E$ to be the unique functor which sends the nondegenerate $(n+1)$-simplex

$$(0,0) \longrightarrow \cdots \longrightarrow (0,k) \longrightarrow (1,k) \longrightarrow \cdots \longrightarrow (1,n)$$

to the $(n+1)$-simplex $\Delta^{n+1} \longrightarrow E$ corresponding to the functor

$$\overline{\sigma} \circ f_{nk}^{op}\colon \widetilde{\mathscr{O}}(\Delta^{n+3})^{op} \longrightarrow C_{\times}.$$

It is easy (albeit tedious) to verify that the functors $\epsilon(\sigma)$ assemble to yield a unique functor ϵ. Now set

$$R := \epsilon|_{(E\times\{0\})}.$$

Given an object $\tau \in E$ displayed as a 2-simplex

$$\begin{array}{ccccc} & & (K,W) & & \\ & \swarrow & & \searrow & \\ & (J,Y_{01}) & & (K,Y_{12}) & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ (I,X) & & (J,Y) & & (K,Z) \end{array}$$

of Σ, the edge $\epsilon_\tau\colon R(\tau) \longrightarrow \tau$ to be

$$\begin{array}{ccccccc} & & & (K,W) & & & \\ & & \swarrow & & /\!\!/ & & \\ & & (J,Y_{01}) & & (K,W) & & \\ & /\!\!/ & & \backslash\!\!\backslash \quad \swarrow & & \searrow & \\ & (J,Y_{01}) & & (J,Y_{01}) & & (K,Y_{12}) & \\ \swarrow & & \backslash\!\!\backslash \quad /\!\!/ & & \searrow \quad \swarrow & & \searrow \\ (I,X) & & (J,Y_{01}) & & (J,Y) & & (K,Z) \end{array}$$

From this, it is apparent that the essential image E' of R is the full subcategory spanned by those $\tau \in E$ such that the morphism $(J,Y_{01}) \longrightarrow (J,Y)$ is an equivalence, and by the dual of [16, Pr. 5.2.7.4], R is a colocalization functor.

We now define $(J,\overline{W}) \in C_\times$ by

$$\overline{W}_j := \begin{cases} W_{\beta(j)} & \text{if } \beta(j) \neq *; \\ \varnothing & \text{if } \beta(j) = *. \end{cases}$$

There is an obvious factorization of $(K,W) \longrightarrow (I,X)$ through $(J,\overline{W})$, and we define an object $\omega \in E'$ as

$$\begin{array}{ccccc} & & (K,W) & & \\ & \swarrow & & /\!\!/ & \\ & (J,\overline{W}) & & (K,W) & \\ \swarrow & & \backslash\!\!\backslash \quad \swarrow & & \searrow \\ (I,X) & & (J,\overline{W}) & & (K,Z) \end{array}$$

We now claim that ω is terminal in E'. Let $\tau \in E'$ be any object displayed as a 2-simplex

$$\begin{array}{ccccc} & & (K,W) & & \\ & \swarrow & & \searrow & \\ & (J,Y_{01}) & & (K,Y_{12}) & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ (I,X) & & (J,Y) & & (K,Z) \end{array}$$

of Σ. We have a homotopy pullback square

$$\begin{array}{ccc}
\mathrm{Map}_E(\tau,\omega) & \longrightarrow & \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), d_2(\omega)) \\
\big\downarrow & & \big\downarrow{\scriptstyle \omega_*} \\
\Delta^0 & \underset{\tau}{\longrightarrow} & \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), \widetilde{\gamma})
\end{array}$$

and the terms on the right-hand side are in turn given as homotopy pullbacks

$$\begin{array}{ccc}
\mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), d_2(\omega)) & \longrightarrow & \mathrm{Map}_{\Sigma}((J,Y),(J,\overline{W})) \\
\big\downarrow & & \big\downarrow{\scriptstyle d_2(\tau)^*} \\
\Delta^0 & \underset{d_2(\omega)}{\longrightarrow} & \mathrm{Map}_{\Sigma}((I,X),(J,\overline{W})),
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), \widetilde{\gamma}) & \longrightarrow & \mathrm{Map}_{\Sigma}((J,Y),(K,Z)) \\
\big\downarrow & & \big\downarrow{\scriptstyle d_2(\tau)^*} \\
\Delta^0 & \underset{\widetilde{\gamma}}{\longrightarrow} & \mathrm{Map}_{\Sigma}((I,X),(K,Z)).
\end{array}$$

In light of the equivalence $(J, Y_{01}) \xrightarrow{\sim} (J, Y)$, we obtain equivalences

$$\mathrm{Map}_{\Sigma}((J,Y),(J,\overline{W})) \simeq \prod_{j\in J} \iota C'_{/\{(Y_{01})_j\ ;\ \overline{W}_j\}};$$
$$\mathrm{Map}_{\Sigma}((I,X),(J,\overline{W})) \simeq \prod_{j\in J} \iota C'_{/\{X_i\ ;\ \overline{W}_j\}_{i\in\alpha^{-1}(j)}}.$$

Under these equivalences the map $d_2(\tau)^*$ is given by $\prod_{j\in J}\phi_j$ where

$$\phi_j \colon \iota C'_{/\{(Y_{01})_j\ ;\ \overline{W}_j\}} \longrightarrow \iota C'_{/\{X_i\ ;\ \overline{W}_j\}_{i\in\alpha^{-1}(j)}}$$

is defined by postcomposition by the maps $(Y_{01})_j \longrightarrow X_i$ (with $i \in \alpha^{-1}(j)$). As a corollary of Cor. 2.11.1 below, we may factor the square in question into two homotopy pullback squares:

$$\begin{array}{ccccc}
\mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), d_2(\omega)) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_{\mathrm{id}}}((J,\overline{W}),(J,Y_{01})) & \longrightarrow & \prod_{j\in J}\iota C'_{/\{(Y_{01})_j\ ;\ \overline{W}_j\}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\Delta^0 & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_{\alpha}}((J,\overline{W}),(I,X)) & \longrightarrow & \prod_{j\in J}\iota C'_{/\{X_i\ ;\ \overline{W}_j\}_{i\in\alpha^{-1}(j)}}.
\end{array}$$

Similarly, we factor the second square into two homotopy pullback squares:

$$\begin{array}{ccccc}
\mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), \widetilde{\gamma}) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_{\beta}}((K,W),(J,Y_{01})) & \longrightarrow & \prod_{k\in K}\iota C'_{/\{(Y_{01})_j\ ;\ Z_k\}_{j\in\beta^{-1}(k)}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\Delta^0 & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_{\gamma}}((K,W),(I,X)) & \longrightarrow & \prod_{k\in K}\iota C'_{/\{X_i\ ;\ Z_k\}_{i\in\gamma^{-1}(k)}}
\end{array}$$

The map ω_* is then seen to be equivalent to the induced map between the fibers of the horizontal maps in the following commutative square:

$$\begin{array}{ccc} \mathrm{Map}_{(C_\times)^\dagger_{\mathrm{id}}}((J,\overline{W}),(J,Y_{01})) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_\alpha}((J,\overline{W}),(I,X)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{(C_\times)^\dagger_\beta}((K,W),(J,Y_{01})) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_\gamma}((K,W),(I,X)). \end{array}$$

The left vertical map is the equivalence

$$\prod_{j\in\beta^{-1}(K)} \mathrm{Map}_{C^\dagger}(W_{\beta(j)},(Y_{01})_j) \xrightarrow{\sim} \prod_{k\in K}\prod_{j\in\beta^{-1}(k)} \mathrm{Map}_{C^\dagger}(W_k,(Y_{01})_j),$$

and the right vertical map is the equivalence

$$\prod_{j\in\beta^{-1}(K)}\prod_{i\in\alpha^{-1}(j)} \mathrm{Map}_{C^\dagger}(W_{\beta(j)},X_i) \xrightarrow{\sim} \prod_{k\in K}\prod_{i\in\gamma^{-1}(k)} \mathrm{Map}_{C^\dagger}(W_k,X_i),$$

so the square is in fact a homotopy pullback square and ω_* is an equivalence. Hence the mapping space $\mathrm{Map}_E(\tau,\omega)$ is contractible and ω is a terminal object of E'. This proves that E is weakly contractible. □

We digress briefly to give the following proposition, which is useful for studying the interaction of the over and undercategory functors with homotopy colimit diagrams.

2.11. **Proposition.** *Suppose C an ∞-category, and let $s\mathbf{Set}_{/C}$ be endowed with the model structure created by the forgetful functor to $s\mathbf{Set}$ equipped with the Joyal model structure. Then we have a Quillen adjunction*

$$C_{(-)/}\colon s\mathbf{Set}_{/C} \rightleftarrows (s\mathbf{Set}_{/C})^{op} : C_{/(-)}$$

between the over and undercategory functors.

Proof. The displayed functors are indeed adjoint to each other, since for objects $\phi\colon X \longrightarrow C$ and $\psi\colon Y \longrightarrow C$ we have natural isomorphisms

$$\mathrm{Hom}_{/C}(X, C_{\psi/}) \cong \mathrm{Hom}_{(X\sqcup Y)/}(X\star Y, C) \cong \mathrm{Hom}_{/C}(Y, C_{/\phi}).$$

To check that this adjunction is a Quillen adjunction, we check that $C_{(-)/}$ preserves cofibrations and trivial cofibrations. Let $\tau\colon \phi \longrightarrow \phi'$ be a map in $s\mathbf{Set}_{/C}$, and let $f = d_2(\tau)\colon X \longrightarrow X'$. If f is a monomorphism, by [16, 2.1.2.1] we have that $C_{\phi'/} \longrightarrow C_{\phi/}$ is a left fibration, hence by [16, 2.4.6.5] a categorical fibration. If f is a monomorphism and a categorical equivalence, by [16, 4.1.1.9] and [16, 4.1.1.1(4)] f is right anodyne, hence by [16, 2.1.2.5] $C_{\phi'/} \longrightarrow C_{\phi/}$ is a trivial Kan fibration. □

2.11.1. **Corollary.** *Let C be an ∞-category and suppose given a morphism f : $x \longrightarrow y$ in C and a diagram*

$$\begin{array}{ccccc} K & \overset{\phi}{\hookrightarrow} & L & \xrightarrow{p} & C \\ \downarrow & & \downarrow & \nearrow_{p'} & \\ K\sqcup\Delta^0 & \overset{\phi'}{\hookrightarrow} & L\sqcup\Delta^0 & & \end{array}$$

of simplicial sets where $\phi' = \phi \sqcup \mathrm{id}$ *and* $p'|_{\Delta^0}$ *selects* y*. Then we have a homotopy pullback square of* ∞*-categories*

$$\begin{array}{ccc} \{x\} \times_C C_{/p} & \xrightarrow{F} & C_{/p'} \\ \downarrow & & \downarrow \\ \{x\} \times_C C_{/p\circ\phi} & \xrightarrow{G} & C_{/p'\circ\phi'} \end{array}$$

where the vertical functors are given by change of diagram and the horizontal functors are to be defined.

Proof. Define the functor F as follows: the datum of an n-simplex $\Delta^n \longrightarrow \{x\} \times_C C_{/p}$ consists of a map $\alpha\colon \Delta^n \star L \longrightarrow C$ which restricts to p on L and to the constant map to x on Δ^n, and we use this to define $\Delta^n \star (L \sqcup \Delta^0) \longrightarrow C$ to be the unique map which restricts to α on $\Delta^n \star L$ and to

$$\Delta^n \star \Delta^0 \longrightarrow \Delta^1 \xrightarrow{f} C$$

on $\Delta^n \star \Delta^0$; this gives the n-simplex of $C_{/p'}$. The definition of G is analogous. The square in question then fits into a rectangle

$$\begin{array}{ccccc} \{x\} \times_C C_{/p} & \xrightarrow{F} & C_{/p'} & \longrightarrow & C_{/p} \\ \downarrow & & \downarrow & & \downarrow \\ \{x\} \times_C C_{/p\circ\phi} & \xrightarrow{G} & C_{/p'\circ\phi'} & \longrightarrow & C_{/p\circ\phi} \end{array}$$

where the long horizontal functors are given as the inclusion of the fiber over x and the functors in the righthand square are given by change of diagram. By Prp. 2.11 and left properness of the Joyal model structure, the righthand square is a homotopy pullback square. The vertical functor $C_{/p} \longrightarrow C_{/p\circ\phi}$ is a right fibration, so the outermost square is a homotopy pullback square. The conclusion follows. □

If we want the symmetric promonoidal ∞-category

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

to be symmetric monoidal, we need a nontrivial condition on our disjunctive triple.

2.12. **Definition.** A disjunctive triple $(C, C_\dagger, C^\dagger)$ will be said to be ***cartesian*** just in case it enjoys the following properties

(2.12.1) It is left complete.

(2.12.2) The underlying ∞-category C admits finite products.

(2.12.3) For any object $X \in C$, the product functor

$$X \times -\colon C \longrightarrow C$$

preserves finite coproducts; that is, for any finite set I and any collection $\{U_i \mid i \in I\}$ of objects of C, the natural map

$$\coprod_{i\in I}(X \times U_i) \longrightarrow X \times \left(\coprod_{i\in I} U_i\right)$$

is an equivalence.

(2.12.4) A morphism $X \longrightarrow \prod_{j\in J} Y_j$ is egressive just in case each of the components $X \longrightarrow Y_j$ is so.

2.13. **Example.** Note that a disjunctive ∞-category C that admits a teminal object, when equipped with the maximal triple structure (in which every morphism is both ingressive and egressive) is always cartesian. More generally, any disjunctive triple that contains a terminal object 1 with the property that every morphism $X \longrightarrow 1$ is ingressive and egressive is cartesian.

2.14. **Proposition.** *If $(C, C_\dagger, C^\dagger)$ is a cartesian disjunctive triple, then the symmetric promonoidal ∞-category*

$$p\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

is symmetric monoidal; that is, p is a cocartesian fibration.

Proof. Since p is flat, by Pr. 1.5 it suffices to verify that p is a locally cocartesian fibration. Since p is an ∞-operad, by the dual of [16, Lm. 2.4.2.7] we reduce to checking that for any active edge $\alpha\colon I \longrightarrow J$ and any object (I, X) over I, there exists a locally p-cocartesian edge $\widetilde{\alpha}$ covering α. For each $j \in J$, let $\widetilde{X}_j = \prod_{i\in\alpha^{-1}(j)} X_i$, and define $\widetilde{\alpha}$ to be

$$\begin{array}{ccc} & (J, \widetilde{X}) & \\ \swarrow & & \| \\ (I, X) & & (J, \widetilde{X}), \end{array}$$

where the morphism $(J, \widetilde{X}) \longrightarrow (I, X)$ is defined using the various projection maps $\widetilde{X}_{\alpha(i)} \longrightarrow X_i$. Then $\widetilde{\alpha}$ is a locally p-cocartesian edge if for all $(J, Y) \in A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_J$, the induced map

$$\widetilde{\alpha}^*\colon \mathrm{Map}_{A^{\mathit{eff}}(C,C_\dagger,C^\dagger)^{\otimes}_J}((J, \widetilde{X}), (J, Y)) \longrightarrow \mathrm{Map}_{A^{\mathit{eff}}(C,C_\dagger,C^\dagger)^{\otimes}_\alpha}((I, X), (J, Y))$$

is an equivalence. This map is in turn equivalent to the map

$$\prod_{j\in J} \phi_j\colon \prod_{j\in J} \iota C'_{/\{\prod_{i\in\alpha^{-1}(j)} X_i\ ;\ Y_j\}} \longrightarrow \prod_{j\in J} \iota C'_{/\{X_i\ ;\ Y_j\}_{i\in\alpha^{-1}(j)}}$$

where ϕ_j is induced by postcomposition by the projection maps $\prod_{i\in\alpha^{-1}(j)} X_i \longrightarrow X_i$. Since $(C, C_\dagger, C^\dagger)$ is a cartesian disjunctive triple, we have that the functor

$$(C^\dagger)_{/\prod_{i\in\alpha^{-1}(j)} X_i} \longrightarrow (C^\dagger)_{/(X_i, i\in\alpha^{-1}(j))}$$

is an equivalence. Hence in light of Prp. 2.11 we have a homotopy pullback square

$$\begin{array}{ccc} \prod_{j\in J} \iota C'_{/\{\prod_{i\in\alpha^{-1}(j)} X_i\ ;\ Y_j\}} & \xrightarrow{\phi_j} & \prod_{j\in J} \iota C'_{/\{X_i\ ;\ Y_j\}_{i\in\alpha^{-1}(j)}} \\ \downarrow & & \downarrow \\ (C^\dagger)_{/\prod_{i\in\alpha^{-1}(j)} X_i} & \longrightarrow & (C^\dagger)_{/\{X_i\}_{i\in\alpha^{-1}(j)}} \end{array}$$

where the horizontal maps are equivalences. We deduce that the map $\widetilde{\alpha}^*$ is an equivalence, as desired. ☐

In light of Lm. 2.5 and Rk. 2.7, we obtain the following.

2.15. **Theorem.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. Then the functor*

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

exhibits $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}$ as a symmetric promonoidal ∞-category, the underlying ∞-category of which is the effective Burnside ∞-category $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$. Furthermore, if $(C, C_\dagger, C^\dagger)$ is cartesian, then $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}$ is symmetric monoidal.

2.16. **Notation.** When $(C, C_\dagger, C^\dagger)$ is a *right* complete disjunctive triple, we may employ duality and write

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)_{\otimes} := (A^{\mathit{eff}}(C, C^\dagger, C_\dagger)^{\otimes})^{op}.$$

The functor $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)_{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$ is then a symmetric promonoidal structure on the Burnside ∞-category $A^{\mathit{eff}}(C, C^\dagger, C_\dagger)^{op} \simeq A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$.

2.17. Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple. Note that the formula

$$\coprod_{i \in I}(X \times U_i) \simeq X \times \left(\coprod_{i \in I} U_i\right)$$

implies immediately that the tensor product functor

$$\otimes \colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \times A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$$

preserves direct sums separately in each variable.

More generally, suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple, suppose I a finite set, and suppose $\{x_i\}_{i \in I}$ a collection of objects of C, which we view, by the standard abuse, as an object of $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_{I}$. Consider the 1-simplex $\xi_I \colon \Delta^1 \longrightarrow N\Lambda(\mathbf{F})$, and denote by $h^{\{x_i\}_{i \in I}}$ the restriction of the functor

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes} \times_{N\Lambda(\mathbf{F})} \Delta^1 \longrightarrow \mathbf{Kan}$$

corepresented by $\{x_i\}_{i \in I}$ to $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$. Informally, this is the functor

$$\mathrm{Map}^{\xi_I}_{C^{\otimes}}(\{x_i\}_{i \in I}, -).$$

Suppose $j \in I$, and suppose $\{y_k \longrightarrow x_j\}_{k \in K}$ a family of morphisms that together exhibit x_j as the coproduct $\coprod_{k \in K} y_k$. For each $i \in I$ and $k \in K$, write

$$x'_{i,k} := \begin{cases} y_k & \text{if } i = j; \\ x_i & \text{if } i \neq j. \end{cases}$$

Then the natural map

$$h^{\{x_i\}_{i \in I}} \longrightarrow \prod_{k \in K} h^{\{x'_{i,k}\}_{i \in I}}$$

is an equivalence.

2.18. For any disjunctive ∞-category C that admits a terminal object, the duality functor

$$D \colon A^{\mathit{eff}}(C)^{op} \xrightarrow{\sim} A^{\mathit{eff}}(C)$$

of [4, Nt. 3.10] provides duals for the symmetric monoidal ∞-category $A^{\mathit{eff}}(C)^{\otimes}$ [17, Df. 2.3.5]. More precisely, for any object X of $A^{\mathit{eff}}(C)$, there exists an evaluation morphism $X \otimes DX \longrightarrow 1$ given by the diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\Delta}\swarrow & & \searrow{\scriptstyle !} \\ X \times X & & 1, \end{array}$$

and, dually, there exists a coevaluation morphism $1 \longrightarrow DX \otimes X$ given by the diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle !}\swarrow & & \searrow{\scriptstyle\Delta} \\ 1 & & X \times X. \end{array}$$

Since the square

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta \times \mathrm{id}} \\ X \times X & \xrightarrow[\mathrm{id} \times \Delta]{} & X \times X \times X \end{array}$$

is a pullback, it follows that the composite

$$X \longrightarrow X \otimes DX \otimes X \longrightarrow X$$

in $A^{eff}(C)$ is homotopic to the identity. We conclude that $A^{eff}(C)^{\otimes}$ is a symmetric monoidal ∞-category with duals.

2.19. If $(C, C_{\dagger}, C^{\dagger})$ is a cartesian disjunctive triple, then in general it is not quite the case that the symmetric monoidal ∞-category $A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes}$ admits duals. We have an evaluation morphism $X \otimes DX \longrightarrow 1$ in $A^{eff}(C, C_{\dagger}, C^{\dagger})$ just in case the diagonal $\Delta\colon X \longrightarrow X \times X$ of C is egressive, and the morphism $!\colon X \longrightarrow 1$ is ingressive. We have a coevaluation morphism $1 \longrightarrow DX \otimes X$ in $A^{eff}(C, C_{\dagger}, C^{\dagger})$ just in case Δ is ingressive and ! is egressive.

2.20. If $(C, C_{\dagger}, C^{\dagger})$ and $(D, D_{\dagger}, D^{\dagger})$ are left complete disjunctive triples, then it is easy to see that a functor of disjunctive triples

$$f\colon (C, C_{\dagger}, C^{\dagger}) \longrightarrow (D, D_{\dagger}, D^{\dagger})$$

induces a functor of adequate triples

$$(C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger}) \longrightarrow (D_{\times}, (D_{\times})_{\dagger}, (D_{\times})^{\dagger})$$

and thus a morphism of ∞-operads

$$A^{eff}(f)^{\otimes}\colon A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \longrightarrow A^{eff}(D, D_{\dagger}, D^{\dagger})^{\otimes}.$$

If, furthermore, $(C, C_{\dagger}, C^{\dagger})$ and $(D, D_{\dagger}, D^{\dagger})$ are cartesian and f preserves finite products, then $A^{eff}(f)^{\otimes}$ is of course a symmetric monoidal functor.

3. Green functors

Andreas Dress [10] defined Green functors as Mackey functors equipped with certain pairings. Gaunce Lewis [14] noticed that these pairings made them commutative monoids for the Day convolution tensor product on the category of Mackey functors. By an old observation of Brian Day [9, Ex. 3.2.2], these are precisely the lax symmetric monoidal additive functors on the effective Burnside category. Thanks to recent work of Saul Glasman [11], this characterization of monoids for the Day convolution holds in the ∞-categorical context as well.

3.1. **Definition.** We shall say that a symmetric monoidal ∞-category $E^{\otimes}$ is ***additive*** if the underlying ∞-category E is additive, and the tensor product functor $\otimes\colon E \times E \longrightarrow E$ preserves direct sums separately in each variable.

3.2. **Definition.** (3.2.1) Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple and $E^{\otimes}$ an additive symmetric monoidal ∞-category. Then a ***commutative Green functor*** is a morphism of ∞-operads

$$A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \longrightarrow E^{\otimes}$$

such that the underlying functor $A^{eff}(C, C_{\dagger}, C^{\dagger}) \longrightarrow E$ preserves direct sums.

(3.2.2) More generally, if $O^{\otimes}$ is an ∞-operad, then an $O^{\otimes}$***-Green functor*** is a morphism of ∞-operads

$$A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes} \longrightarrow E^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes}$$

over $O^{\otimes}$ such that for any object X of the underlying ∞-category O, the functor

$$A^{eff}(C, C_{\dagger}, C^{\dagger}) \simeq (A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes})_X \longrightarrow (E^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes})_X \simeq E$$

preserves direct sums.

(3.2.3) Similarly, for any perfect operator category Φ, we may define a Φ***-Green functor*** as a morphism

$$A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} \mathrm{N}\Lambda(\Phi) \longrightarrow E^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} \mathrm{N}\Lambda(\Phi)$$

of ∞-operads over Φ such that the underlying functor $A^{eff}(C, C_{\dagger}, C^{\dagger}) \longrightarrow E$ preserves direct sums.

3.3. **Notation.** Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose $E^{\otimes}$ an additive symmetric monoidal ∞-category. For any ∞-operad $O^{\otimes}$, let us write, employing the notation of [19, Df. 2.1.3.1]

$$\mathbf{Green}_{O^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) \subset \mathbf{Alg}_{A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes} \ /O^{\otimes}}(E^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes})$$

for the full subcategory spanned by the $O^{\otimes}$-Green functors.

3.4. **Example.** We define *modules* over an *associative Green functor* in this way. Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose $E^{\otimes}$ an additive symmetric monoidal ∞-category. Then we may consider the ∞-operad of [19, Df. 4.2.7], which we will denote $\mathrm{LM}^{\otimes}$. The inclusion $\mathrm{Ass}^{\otimes} \hookrightarrow \mathrm{LM}^{\otimes}$ induces a functor

$$\mathbf{Green}_{\mathrm{LM}^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) \longrightarrow \mathbf{Green}_{\mathrm{Ass}^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}).$$

An object A of the target may be called an ***associative Green functor***, and an object of the fiber of this functor over A may be called a ***left*** A***-module***. We write

$$\mathbf{Mod}^{\ell}_{A}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) := \mathbf{Green}_{\mathrm{LM}^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) \times_{\mathbf{Green}_{\mathrm{Ass}^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes})} \{A\}$$

for the ∞-category of left A-modules. When A is a commutative Green functor, we will drop the superscript ℓ.

The convolution of two Mackey functors will not in general be a Mackey functor, but it can replaced with one by employing a localization (which we might as well call Mackeyification). To prove that convolution followed by Mackeyification defines a symmetric monoidal structure on the ∞-category of Mackey functors, it is necessary to show that Mackeyification is *compatible* with the convolution symmetric monoidal structure in the sense of Lurie [19, Df. 2.2.1.6, Ex. 2.2.1.7].

The following is immediate from [4, Pr. 6.5].

3.5. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose E a presentable additive ∞-category. Then the ∞-category $\mathbf{Mack}(C, C_\dagger, C^\dagger; E)$ is an accessible localization of the ∞-category $\mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$.*

3.6. **Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive ∞-category, and suppose E a presentable additive ∞-category. Then write M for the left adjoint to the fully faithful inclusion

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; E) \hookrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E).$$

3.7. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive ∞-category, and suppose $E^\otimes$ a presentable symmetric monoidal additive ∞-category. Then the left adjoint M constructed above is compatible in the sense of* [19, Df. 2.2.1.6] *with Glasman's Day convolution symmetric monoidal structure on* $\mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$.

Proof. For any collection of objects $\{s_i \mid i \in I\}$ of C, let

$$h^{\{s_i\}}\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Kan}$$

be as in 2.17, and for any object $x \in E$, let

$$-\otimes x\colon \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), \mathbf{Kan}) \longrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$$

be the composition with the tensor product $-\otimes x\colon \mathbf{Kan} \longrightarrow E$ with spaces [16, §4.]. Thus objects of the form $h^{\{s_i\}} \otimes x$ generate the ∞-category $\mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$ under colimits. It is easy to see that for any functors $f, g\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Kan}$ and any object $x \in E$, the map

$$(f \times g) \otimes x \longrightarrow (f \otimes x) \oplus (g \otimes x)$$

is an M-equivalence; furthermore, the class of M-equivalences is the strongly saturated class generated by the canonical morphisms

$$h^{s\oplus t} \otimes x \longrightarrow (h^s \otimes x) \oplus (h^t \otimes x).$$

This tensor product and the Day convolution are compatible in the sense that there are natural equivalences

$$(h^s \otimes x) \otimes (h^t \otimes y) \simeq h^{\{s,t\}} \otimes (x \otimes y),$$

whence one obtains natural M-equivalences

$$\begin{aligned}
((h^s \otimes x) \oplus (h^t \otimes x)) \otimes (h^u \otimes y) &\simeq ((h^s \otimes x) \otimes (h^u \otimes y)) \oplus ((h^t \otimes x) \otimes (h^u \otimes y)) \\
&\simeq (h^{\{s,u\}} \otimes x \otimes y) \oplus (h^{\{t,u\}} \otimes x \otimes y) \\
&\longrightarrow (h^{\{s,u\}} \times h^{\{t,u\}}) \otimes x \otimes y \\
&\simeq h^{\{s\oplus t,u\}} \otimes x \otimes y \\
&\simeq h^{s\oplus t} \otimes x \otimes h^u \otimes y.
\end{aligned}$$

It follows that for any M-equivalence $X \longrightarrow Y$ and any object Z of the ∞-category $\mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$, the morphism

$$X \otimes Z \longrightarrow Y \otimes Z$$

is an M-equivalence. □

3.8. In particular, if $(C, C_\dagger, C^\dagger)$ is a left complete disjunctive triple, and if $E^\otimes$ a presentable symmetric monoidal additive ∞-category, we obtain a symmetric monoidal ∞-category $\mathbf{Mack}(C, C_\dagger, C^\dagger; E)^\otimes$, and, in light of [11], for any ∞-operad $O^\otimes$, one obtains an equivalence

$$\mathbf{Alg}_{O^\otimes}(\mathbf{Mack}(C, C_\dagger, C^\dagger; E)^\otimes) \simeq \mathbf{Green}_{O^\otimes}(C, C_\dagger, C^\dagger; E).$$

4. Green stabilization

Now let us address the issue of multiplicative structures on the Mackey stabilization, as constructed in [4, §7]. In particular, we aim to show that if E is an ∞-topos, then the Mackey stabilization of a morphism of operads

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow E^\times$$

naturally admits the structure of a Green functor

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Sp}(E)^\wedge.$$

4.1. **Definition.** Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple, suppose E an ∞-topos, and suppose

$$f\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow E^\times \quad \text{and} \quad F\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Sp}(E)^\otimes$$

morphisms of ∞-operads. Then a morphism of $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes$-algebras

$$\eta\colon f \longrightarrow \Omega^\infty \circ F$$

will be said to exhibit F as the ***Green stabilization*** of f if F is a Green functor, and if, for any Green functor $R\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Sp}(E)^\otimes$, the map

$$\mathrm{Map}_{\mathbf{Green}_{E_\infty}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)^\otimes)}(F, R) \longrightarrow \mathrm{Map}_{\mathbf{Alg}_{A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes}(E^\times)}(f, \Omega^\infty \circ R)$$

induced by η is an equivalence.

The following result is essentially the same as [2, Pr. 2.1].

4.2. **Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple. There exists a symmetric monoidal ∞-category $\mathrm{D}A(C, C_\dagger, C^\dagger)^\otimes$ and a fully faithful symmetric monoidal functor*

$$j^\otimes\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \hookrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger)^\otimes$$

with the following properties.

(4.2.1) The ∞-category $\mathrm{D}A(C, C_\dagger, C^\dagger)$ underlies $\mathrm{D}A(C, C_\dagger, C^\dagger)^\otimes$, and the underlying functor of $j^\otimes$ is the inclusion

$$j\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \hookrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger)$$

of [4, Nt. 7.2].

(4.2.2) *For any symmetric monoidal ∞-category $E^{\otimes}$ whose underlying ∞-category admits all sifted colimits such that the tensor product preserves sifted colimits separately in each variable, the induced functor*

$$\mathbf{Alg}_{\mathrm{D}A(C,C_{\dagger},C^{\dagger})^{\otimes}}(E^{\otimes}) \longrightarrow \mathbf{Alg}_{A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})^{\otimes}}(E^{\otimes})$$

exhibits an equivalence from the full subcategory spanned by those morphisms of ∞-operads A whose underlying functor $A\colon \mathrm{D}A(C,C_{\dagger},C^{\dagger}) \longrightarrow E$ preserves sifted colimits to the full subcategory spanned by those morphisms of ∞-operads B whose underlying functor $B\colon A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger}) \longrightarrow E$ preserves filtered colimits.

(4.2.3) *The tensor product functor*

$$\otimes\colon \mathrm{D}A(C,C_{\dagger},C^{\dagger}) \times \mathrm{D}A(C,C_{\dagger},C^{\dagger}) \longrightarrow \mathrm{D}A(C,C_{\dagger},C^{\dagger})$$

preserves all colimits separately in each variable.

Proof. The only part that is not a consequence of [19, Pr. 4.8.1.10 and Var. 4.8.1.11] is the assertion that the tensor product functor

$$\otimes\colon \mathrm{D}A(C,C_{\dagger},C^{\dagger}) \times \mathrm{D}A(C,C_{\dagger},C^{\dagger}) \longrightarrow \mathrm{D}A(C,C_{\dagger},C^{\dagger})$$

preserves direct sums separately in each variable. This assertion holds for objects of the effective Burnside category $A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})$ thanks to the universality of coproducts in C; the general case follows by exhibiting any object of $\mathrm{D}A(C,C_{\dagger},C^{\dagger})$ as a colimit of a sifted diagram of objects of $A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})$ and using the fact that both the tensor product and the direct sum commute with sifted colimits. ☐

In light of [2, Pr. 3.5] and [19, Pr. 6.2.4.14 and Th. 6.2.6.2], we now have the following.

4.3. **Proposition.** *Suppose $(C,C_{\dagger},C^{\dagger})$ a disjunctive triple, suppose E an ∞-topos, and suppose*

$$f\colon A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})^{\otimes} \longrightarrow E^{\times}$$

a morphism of ∞-operads. Then a Green stabilization of f exists. In particular, the functor

$$\Omega^{\infty}\circ -\colon \mathbf{Green}(C,C_{\dagger},C^{\dagger};\mathbf{Sp}(E)^{\otimes}) \longrightarrow \mathbf{Alg}_{A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})^{\otimes}}(E^{\times})$$

admits a left adjoint that covers the left adjoint of the functor

$$\Omega^{\infty}\circ -\colon \mathbf{Mack}(C,C_{\dagger},C^{\dagger};\mathbf{Sp}(E)) \longrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger}),E).$$

4.4. **Example.** Suppose $(C,C_{\dagger},C^{\dagger})$ a cartesian disjunctive triple. Then the functor

$$A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger}) \longrightarrow \mathbf{Kan}$$

corepresented by the terminal object 1 of C is the unit for the Day convolution symmetric monoidal structure of Glasman, and hence it is an E_{∞} algebra in an essentially unique fashion. Thus we can consider its Green stabilization

$$^{\otimes} = {}^{\otimes}_{(C,C_{\dagger},C^{\dagger})}\colon A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})^{\otimes} \longrightarrow \mathbf{Sp}^{\wedge},$$

whose underlying Mackey functor is the Burnside Mackey functor $_{(C,C_{\dagger},C^{\dagger})}$ of [4]. We call $^{\otimes}$ the ***Burnside Green functor***.

In a similar vein, we immediately have the following:

4.5. **Proposition.** *For any cartesian disjunctive triple $(C, C_\dagger, C^\dagger)$, the functor*

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{op} \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})$$

given by the assignment $X \rightsquigarrow {}^{X}$ is naturally symmetric monoidal. That is, for any two objects $X, Y \in C$, one has a canonical equivalence

$${}^{X} \otimes {}^{Y} \simeq {}^{X \times Y}$$

4.5.1. **Corollary.** *Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple. For any spectral Mackey functor M thereon, write $F(M, -)$ for the right adjoint to the functor*

$$- \otimes M \colon \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}) \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}).$$

Then for any object $X \in C$, the Mackey functor $F({}^{X}, M)$ is given by the assignment

$$Y \rightsquigarrow M(X \times Y).$$

The following is now immediate.

4.6. **Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple. The Burnside Mackey functor ${}_{(C, C_\dagger, C^\dagger)}$ is the unit in the symmetric monoidal ∞-category $\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})^{\otimes}$. Consequently, the Burnside Green functor ${}^{\otimes}_{(C, C_\dagger, C^\dagger)}$ is the initial object in the ∞-category $\mathbf{Green}_{\mathrm{N}\Lambda(\mathbf{F})}(C, C_\dagger, C^\dagger; \mathbf{Sp}^{\otimes})$, and the forgetful functor*

$$\mathbf{Mod}_{\otimes}(C, C_\dagger, C^\dagger; \mathbf{Sp}^{\otimes}) \xrightarrow{\sim} \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})$$

is an equivalence.

5. Duality

In this section, suppose C a disjunctive ∞-category that admits a terminal object. Since the functor $X \rightsquigarrow {}^{X}$ is symmetric monoidal, it follows immediately that every representable Mackey functor ^{X} is strongly dualizable, and

$$({}^{X})^{\vee} \simeq {}^{DX}$$

5.1. **Notation.** For any associative spectral Green functor R and for any object $X \in C$, denote by R^{X} the left R-module $R \otimes {}^{X}$, and denote by ${}^{X}R$ the right R-module ${}^{X} \otimes R$.

Of course for any left (respectively, right) R-module M, one has

$$\mathrm{Map}(R^{X}, M) \simeq \Omega^{\infty} M(X) \qquad (\text{resp.,} \quad \mathrm{Map}({}^{X}R, M) \simeq \Omega^{\infty} M(X) \quad).$$

5.2. **Definition.** For any associative spectral Green functor R on C, denote by $\mathbf{Perf}^{\ell}_{R}$ the smallest stable subcategory of the ∞-category $\mathbf{Mod}^{\ell}_{R}$ that contains the left R-modules R^{X} (for $X \in C$) and is closed under retracts. Similarly, denote by $\mathbf{Perf}^{r}_{R}$ the smallest stable subcategory of the ∞-category $\mathbf{Mod}^{r}_{R}$ that contains the right R-modules ${}^{X}R$ (for $X \in C$) and is closed under retracts.

The objects of $\mathbf{Perf}^{\ell}_{R}$ (respectively, $\mathbf{Perf}^{r}_{R}$) will be called ***perfect*** left (resp., right) modules over R.

Now we obtain the following, which is a straightforward analogue of [19, Pr. 7.2.5.2].

5.3. **Proposition.** *For any associative spectral Green functor R, a left R-module is compact just in case it is perfect.*

Proof. For any $X \in C$, the functor corepresented by R^X is the assignment $M \rightsquigarrow \Omega^\infty M(X)$, which preserves filtered colimits. Hence R^X is compact, and thus any perfect left R-module is compact.

Conversely, there is a fully faithful, colimit-preserving functor

$$F\colon \mathrm{Ind}(\mathbf{Perf}^\ell_R) \hookrightarrow \mathbf{Mod}_R$$

induced by the inclusion $\mathbf{Perf}^\ell_R \hookrightarrow \mathbf{Mod}^\ell_R$. If this is not essentially surjective, there exists a nonzero left R-module M such that for every R-module N in the essential image of F, the group $[N, M]$ vanishes. In particular, for any integer n and any object $X \in C$,

$$\pi_n M(X) \cong [R^X[n], M] \cong 0,$$

whence $M \simeq 0$. □

The proof of the following is word-for-word identical to that of [19, Pr. 7.2.5.4].

5.4. **Proposition.** *For any associative spectral Green functor R on C, a left R-module M is perfect just in case there exists a right R-module $M^\vee$ that is dual to M in the sense that the functor*

$$\mathrm{Map}(, M^\vee \otimes_R -)\colon \mathbf{Mod}^\ell_R \longrightarrow \mathbf{Kan}$$

is the functor that M corepresents.

5.5. **Example.** Note that, in particular, for any object $X \in C$, one has

$$(R^X)^\vee \simeq {}^{DX}R.$$

6. The Künneth spectral sequence

Let us note that the Künneth spectral sequence works in the Mackey functor context more or less exactly as in the ordinary ∞-category of spectra. To this end, let us first discuss t-structures on ∞-categories of spectral Mackey functors.

6.1. **Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose A a stable ∞-category equipped with a t-structure $(A_{\geq 0}, A_{\leq 0})$. Then the two subcategories*

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; A)_{\geq 0} := \mathbf{Mack}(C, C_\dagger, C^\dagger; A_{\geq 0})$$

and

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; A)_{\leq 0} := \mathbf{Mack}(C, C_\dagger, C^\dagger; A_{\leq 0})$$

define a t-structure on $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$.

Proof. Consider the functor $L\colon \mathbf{Mack}(C, C_\dagger, C^\dagger; A) \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ given by composition with $\tau_{\leq -1}$; it is clear that L is a localization functor. Furthermore, the essential image of L is the ∞-category $\mathbf{Mack}(C, C_\dagger, C^\dagger; A_{\leq -1})$, which is closed under extensions, since $A_{\leq -1}$ is. Now we apply [19, Pr. 1.2.1.16]. □

6.2**.** Note that if A a stable ∞-category equipped with a t-structure $(A_{\geq 0}, A_{\leq 0})$, then for any disjunctive triple $(C, C_\dagger, C^\dagger)$, the heart of the induced t-structure on $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ is given by

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; A)^\heartsuit \simeq \mathbf{Mack}(C, C_\dagger, C^\dagger; A^\heartsuit).$$

Furthermore, it is clear that many properties of the t-structure on A are inherited by the induced t-structure $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$: in particular, one verifies easily that the t-structure on $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ is left bounded, right bounded, left complete, right complete, compatible with sequential colimits, compatible with filtered colimits, or accessible if the t-structure on A is so.

6.3. **Example.** For any disjunctive triple $(C, C_\dagger, C^\dagger)$, the ∞-category of spectral Mackey functors $\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})$ admits an accessible t-structure that is both left and right complete whose heart is the abelian category $\mathbf{Mack}(C, C_\dagger, C^\dagger; N\mathbf{Ab})$. Observe that the corepresentable functors $\tau_{\leq 0}{}^X$ are projective objects in the heart, and thus the heart has enough projectives.

In particular, if G is a profinite group and if C is the disjunctive ∞-category of finite G-sets, then the ∞-category $\mathbf{Mack}_G$ of spectral Mackey functors for G admits an accessible t-structure that is both left and right complete, in which the heart $\mathbf{Mack}_G^\heartsuit$ is the nerve of the usual abelian category of Mackey functors for G.

6.4. **Construction.** Suppose A a stable ∞-category equipped with a t-structure. Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose $X \colon N\mathbf{Z} \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ a filtered Mackey functor with colimit $X(+\infty)$. Then we have the spectral sequence

$$E_r^{p,q} := \operatorname{im}\left[\pi_{p+q}\left(\frac{X(p)}{X(p-r)}\right) \longrightarrow \pi_{p+q}\left(\frac{X(p+r-1)}{X(p-1)}\right)\right]$$

associated with X [19, Df. 1.2.2.9].

Note that this is a spectral sequence of $A^\heartsuit$-valued Mackey functors. Since limits and colimits of Mackey functors are defined objectwise, it follows that for any object $U \in A^{eff}(C, C_\dagger, C^\dagger)$, the value $E_r^{p,q}(U)$ is the spectral sequence (in $A^\heartsuit$) associated with the filtered object $X(U) \colon N\mathbf{Z} \longrightarrow A$.

6.5**.** In the setting of Cnstr. 6.4, assume that A admits all sequential colimits and that the t-structure is compatible with these colimits. If $X(n) \simeq 0$ for $n \ll 0$, then the associated spectral sequence converges to a filtration on $\pi_{p+q}(X(+\infty))$ [19, 1.2.2.14]. That is:

- For any p and q, there exists $r \gg 0$ such that the differential
$$d_r \colon E_r^{p,q} \longrightarrow E_r^{p-r,q+r-1}$$
vanishes.
- For any p and q, there exist a discrete, exhaustive filtration
$$\cdots \subset F_{p+q}^{-1} \subset F_{p+q}^{0} \subset F_{p+q}^{1} \subset \cdots \subset \pi_{p+q}X(+\infty)$$
and an isomorphism $E_\infty^{p,q} \cong F_{p+q}^p / F_{p+q}^{p-1}$.

In more general circumstances, one can obtain a kind of "local convergence." Suppose again that A admits all sequential colimits, and that the t-structure is compatible with these colimits. Now suppose that for every object $U \in A^{eff}(C, C_\dagger, C^\dagger)$, there exists $n \ll 0$ such that $X(n)(U) \simeq 0$. Then for every object $U \in A^{eff}(C, C_\dagger, C^\dagger)$, the spectral sequence $E_r^{p,q}(U)$ converges to $\pi_{p+q}(X(+\infty)(U))$. In finitary cases (e.g., when C is the disjunctive ∞-category of finite G-sets for a finite group G), there is no difference between the local convergence and the global convergence.

Better convergence results can be obtained when the filtered Mackey functor is the skeletal filtration of a simplicial connective object Y_* [19, Pr. 1.2.4.5]. In this case, we do not need to assume that the t-structure on A is compatible with sequential colimits, the associated spectral sequence is a first-quadrant spectral sequence, and it converges to a length $p + q$ filtration on $\pi_{p+q}|Y_*|$.

Now, to construct the Künneth spectral sequence for Mackey functors, we can follow very closely the arguments of Lurie [19, §7.2.1].

6.6. **Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple. Then the collection of corepresentable Mackey functors $\{^X \mid X \in A^{eff}(C, C_\dagger, C^\dagger)\}$ is a set of compact projective generators for $\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}_{\geq 0})$ in the sense of* [16, Dfn. 5.5.2.3].

Proof. The corepresentable functors provide a set of compact projective generators for the ∞-category $\mathrm{Fun}^\times(A^{eff}(C, C_\dagger, C^\dagger), \mathbf{Kan})$ because this category is precisely $P_\Sigma(A^{eff}(C, C_\dagger, C^\dagger)^{op})$. The functor

$$\Omega^\infty \circ -\colon \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}_{\geq 0}) \longrightarrow \mathrm{Fun}^\times(A^{eff}(C, C_\dagger, C^\dagger), \mathbf{Kan})$$

preserves sifted colimits and is conservative, since $\Omega^\infty\colon \mathbf{Sp}_{\geq 0} \longrightarrow \mathbf{Kan}$ preserves sifted colimits by [19, 1.4.3.9] and is conservative, and the inclusion of both sides into all functors preserves sifted colimits (we use that $\mathbf{Kan}$ is cartesian closed). We conclude by applying [19, 4.7.4.18]. □

To set up the spectral sequence we need to impose the hypotheses of strong dualizability on the ^{X}. Because of this, we now work in the generality of C a disjunctive ∞-category which admits a terminal object.

Suppose

$$R\colon A^{eff}(C)^\otimes \times_{N\Lambda(\mathbf{F})} \mathrm{Ass}^\otimes \longrightarrow \mathbf{Sp}^\wedge \times_{N\Lambda(\mathbf{F})} \mathrm{Ass}^\otimes$$

an associative Green functor, suppose M a right R-module, and suppose N a left R-module. There is a comparison map

$$\mathrm{Tor}_0^{\pi_* R}(\pi_* M, \pi_* N) \longrightarrow \pi_*(M \otimes_R N)$$

constructed as follows: given $x \in \pi_m M(U)$ and $y \in \pi_n N(V)$, choose representatives $\Sigma^m(^U R) \longrightarrow M$ and $\Sigma^n(R^V) \longrightarrow N$ and take their smash product to obtain a map

$$\Sigma^{m+n}(^{U\times V}) \longrightarrow \Sigma^{m+n}(^{U\times V}) \otimes R \simeq \Sigma^m(^U R) \otimes_R \Sigma^n(R^V) \longrightarrow M \otimes_R N$$

and thus an element $x \otimes y \in \pi_{m+n}(M \otimes_R N)(U \times V)$; this is suitably natural so that it descends to a map out of the Day convolution tensor product $\pi_* M \otimes_{\pi_* R} \pi_* N$ to $\pi_*(M \otimes_R N)$. This map is not usually an isomorphism. Instead, we construct a spectral sequence that converges to $\pi_*(M \otimes_R N)$, in which this map appears as an edge homomorphism.

Let S denote the class of left R-modules of the form $\Sigma^n R^X$ for $n \in \mathbf{Z}$ and $X \in C$. By [19, Pr. 7.2.1.4], there exists an S-free S-hypercovering $P_\bullet \longrightarrow N$ in the (presentable) stable ∞-category $\mathbf{Mod}_R^\ell$.

6.7. **Lemma.** *For any S-hypercovering $P_\bullet \longrightarrow N$, we have that $|P_\bullet| \simeq N$.*

Proof. Let $S_{\geq n}$ be the subset of S on $\Sigma^m \circ R^X$ for $m \geq n$. From our S-hypercovering $P_\bullet \longrightarrow N$, we obtain $S_{\geq n}$-hypercoverings $\tau_{\geq n} P_\bullet \longrightarrow \tau_{\geq n} N$ for every $n \in \mathbf{Z}$. Since the $\Sigma^n S^X$, $X \in C$ constitute a set of projective generators for $\mathbf{Mack}(C; \mathbf{Sp}_{\geq n})$ by Lm. 6.6, we have that $|\tau_{\geq n} P_\bullet| \simeq \tau_{\geq n} N$ by the hypercompleteness of $\mathbf{Kan}$. By the right completeness of the t-structure, we deduce that $|P_\bullet| \simeq N$. □

By passing to the skeletal filtration of $M \otimes_R |P_\bullet|$, we obtain a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 1}$ that converges to $\pi_{p+q}(M \otimes_R N)$. The complex $(E_1^{*,q}, d_1)$ is the normalized chain complex $N_*(\pi_q(M \otimes_R P_\bullet))$.

To proceed, we need to prove the following analogue of [19, Pr. 7.2.1.17].

6.8. **Lemma.** *If P is a direct sum of objects in S, then the map*

$$\mathrm{Tor}_0^{\pi_* R}(\pi_* M, \pi_* P) \longrightarrow \pi_*(M \otimes_R P)$$

is an isomorphism.

Proof. Both sides commute with direct sums and shifts, so we reduce to the case of $P = R^X$. We claim first that for any spectral Mackey functor E,

$$\pi_* E \otimes \tau_{\leq 0}{}^X \cong \pi_*(E \otimes {}^X).$$

Since $\tau_{\leq 0}{}^Y$ corepresents evaluation at Y for **Ab**-valued Mackey functors, and $\tau_{\leq 0}{}^X$ has dual $\tau_{\leq 0}{}^{DX}$, we have $(\pi_* E \otimes \tau_{\leq 0}{}^X)(Y) \cong (\pi_* E)(Y \times DX)$. Similarly, corepresentability and strong dualizability on the level of the **Sp**-valued Mackey functors implies that $\pi_*(E \otimes {}^X)(Y) \cong (\pi_* E)(Y \times DX)$, so we conclude. Now we apply this claim both for M and R to see that

$$\begin{aligned} \pi_* M \otimes_{\pi_* R} \pi_*(R^X) &\cong \pi_* M \otimes_{\pi_* R} (\pi_* R \otimes \tau_{\leq 0}{}^X) \\ &\cong \pi_* M \otimes \tau_{\leq 0}{}^X \\ &\cong \pi_*(M \otimes {}^X) \\ &\cong \pi_*(M \otimes_R R^X). \end{aligned}$$

We leave the identification of the specified map with this isomorphism to the reader. □

We thus obtain an isomorphism

$$\mathrm{Tor}_0^{\pi_* R}(\pi_* M, \pi_* P_\bullet) \cong \pi_*(M \otimes_R P_\bullet).$$

As $P_\bullet$ is an S-free S-hypercovering of N, $N_*(\pi_* P_\bullet)$ is a resolution of $\pi_* N$ by projective $\pi_* R$-modules. It follows that the E_2 page is given by

$$E_2^{p,q} \cong \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q.$$

As in [19, Cor. 7.2.1.23], we have an immediate corollary.

6.8.1. **Corollary.** *Suppose C, R, M, and N as above. Suppose that R, M, and N are all connective. Then $M \otimes_R N$ is connective, and one has an isomorphism of ordinary Mackey functors*

$$\pi_0(M \otimes_R N) \cong \pi_0 M \otimes_{\pi_0 R} \pi_0 N.$$

6.9. **Example.** If C is the category of finite G-sets for G a finite group, then our Künneth spectral sequence recovers that of Lewis and Mandell in [15]. We refer the reader there to a more extensive discussion of this spectral sequence in that particular case.

7. Symmetric monoidal Waldhausen bicartesian fibrations

In [3], we define an $O^\otimes$-monoidal Waldhausen ∞-category for any ∞-operad $O^\otimes$ as an $O^\otimes$-algebra in the symmetric monoidal ∞-category $\mathbf{Wald}_\infty^\otimes$. We give two equivalent fibrational formulations of this notion.

7.1. **Definition.** Suppose $O^\otimes$ an ∞-operad. An ***$O^\otimes$-monoidal Waldhausen ∞-category*** consists of a pair cocartesian fibration [3, Df. 3.8]

$$p^\otimes \colon \mathbf{X}^\otimes \longrightarrow O^\otimes$$

such that the following conditions obtain.

(7.1.1) The composite

$$\mathbf{X}^{\otimes} \longrightarrow O^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

exhibits $\mathbf{X}^{\otimes}$ as an ∞-operad.

(7.1.2) The fiber $p\colon \mathbf{X} \longrightarrow O$ over $* \in \mathrm{N}\Lambda(\mathbf{F})$ is a Waldhausen cocartesian fibration.

(7.1.3) For any finite set I and any choice of inert morphisms $\{\rho_i\colon s \longrightarrow s_i\}_{i\in I}$ covering the inert morphisms $I \longrightarrow \{i\}$, an edge η of $\mathbf{X}^{\otimes}_s$ is ingressive if and only if, for every $i \in I$, the edge $\rho_{i,!}(\eta)$ of $\mathbf{X}_{s_i}$ is ingressive.

(7.1.4) For any finite set I, any morphism $\mu\colon s \longrightarrow t$ of $O^{\otimes}$ covering the unique active morphism $I \longrightarrow \{\xi\}$, and any choice of inert morphisms $\{s \longrightarrow s_i \mid i \in I\}$ covering the inert morphisms $I \longrightarrow \{i\}$, the functor of pairs

$$\mu_!\colon \prod_{i\in I} \mathbf{X}_{s_i} \simeq \mathbf{X}^{\otimes}_s \longrightarrow \mathbf{X}_t$$

is exact separately in each variable [2].

Dually, suppose $O_{\otimes}$ an ∞-anti-operad. Then a ***$O_{\otimes}$-monoidal Waldhausen ∞-category*** is a pair cartesian fibration

$$p_{\otimes}\colon \mathbf{X}_{\otimes} \longrightarrow O_{\otimes}$$

such that the following conditions obtain.

(7.1.5) The composition

$$\mathbf{X}_{\otimes} \longrightarrow O_{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$$

exhibits $\mathbf{X}_{\otimes}$ as an ∞-anti-operad.

(7.1.6) The fiber $p\colon \mathbf{X} \longrightarrow O$ over $* \in \mathrm{N}\Lambda(\mathbf{F})^{op}$ is a Waldhausen cartesian fibration.

(7.1.7) For any finite set I and any choice of inert morphisms $\{\pi_i\colon s \longrightarrow s_i\}_{i\in I}$ covering the inert morphisms $I \longrightarrow \{i\}$, an edge η of $\mathbf{X}^{\otimes}_s$ is ingressive if and only if, for every $i \in I$, the edge $\pi_i^{\star}(\eta)$ of $\mathbf{X}_{s_i}$ is ingressive.

(7.1.8) For any finite set I, any morphism $\mu\colon t \longrightarrow s$ of $O_{\otimes}$ covering the opposite of the unique active morphism $I \longrightarrow \{\xi\}$, and any choice of inert morphisms $\{s_i \longrightarrow s\}_{i\in I}$ covering the inert morphisms $I \longrightarrow \{i\}$, the functor of pairs

$$\mu^{\star}\colon \prod_{i\in I} \mathbf{X}_{s_i} \simeq \mathbf{X}_{\otimes,s} \longrightarrow \mathbf{X}_t$$

is exact separately in each variable.

Employing [19, Ex. 2.4.2.4 and Pr. 2.4.2.5] and [2, Lm 1.4], one deduces the following.

7.2. **Proposition.** *Suppose $O^{\otimes}$ (respectively, $O_{\otimes}$) an ∞-operad (resp., an ∞-anti-operad). Then the functor*

$$O^{\otimes} \longrightarrow \mathbf{Cat}_{\infty} \qquad \textit{(resp., the functor} \quad (O_{\otimes})^{op} \longrightarrow \mathbf{Cat}_{\infty} \quad \textit{)}$$

classifying an $O^{\otimes}$-monoidal Waldhausen ∞-category (resp., an $O_{\otimes}$-monoidal Waldhausen ∞-category) factors through an essentially unique morphism of ∞-operads

$$O^{\otimes} \longrightarrow \mathbf{Wald}^{\otimes}_{\infty} \qquad \textit{(resp., the functor} \quad (O_{\otimes})^{op} \longrightarrow \mathbf{Wald}^{\otimes}_{\infty} \quad \textit{)}$$

7.3. **Definition.** Now suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple. A ***symmetric monoidal Waldhausen bicartesian fibration***

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

over $(C, C_{\dagger}, C^{\dagger})$ is a functor of pairs $\mathbf{X}_{\boxtimes} \longrightarrow (C_{\times})^{\flat}$ with the following properties.

(7.3.1) The underlying functor $p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$ is an inner fibration.
(7.3.2) For any egressive morphism $(\phi,\omega) : (I,X) \twoheadrightarrow (J,Y)$ of $C_{\times}$ (in the sense of Nt. 2.1) and for any object Q of the fiber $(\mathbf{X}_{\boxtimes})_{(J,Y)}$, there exists a $p_{\boxtimes}$-cartesian morphism $P \longrightarrow Q$ covering (ϕ,ω).
(7.3.3) The composition

$$\mathbf{X}_{\boxtimes} \longrightarrow C_{\times} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$$

exhibits $\mathbf{X}_{\boxtimes}$ as an ∞-anti-operad.
(7.3.4) The fiber $p\colon \mathbf{X} \longrightarrow C$ over $* \in \mathrm{N}\Lambda(\mathbf{F})^{op}$ is a Waldhausen bicartesian fibration $\mathbf{X} \longrightarrow C$ over $(C, C_{\dagger}, C^{\dagger})$.

7.4. This is a lot of data, so let's unpack it a bit.

First, a symmetric monoidal Waldhausen bicartesian fibration

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

over $(C, C_{\dagger}, C^{\dagger})$ admits an underlying Waldhausen bicartesian fibration $p\colon \mathbf{X} \longrightarrow C$ over $(C, C_{\dagger}, C^{\dagger})$. This provides, for any object $S \in C$, a Waldhausen ∞-category $\mathbf{X}_S$, and for any morphism $\phi\colon S \longrightarrow T$ of C, it provides an exact "pushforward" functor $\phi_!\colon \mathbf{X}_S \longrightarrow \mathbf{X}_T$ whenever ϕ is ingressive and an exact "pullback" functor $\phi^{\star}\colon \mathbf{X}_T \longrightarrow \mathbf{X}_S$ whenever ϕ is egressive. These are compatible with composition, and when ϕ is ingressive and (therefore) egressive, these two are adjoint.

There's more structure here: for any finite set I and any I-tuple $(S_i)_{i\in I}$ of objects of C with product S, consider the cartesian edge

$$(\{\xi\}, S) \longrightarrow (I, S_I)$$

of $C_{\times}$ lying over the morphism $\{\xi\} \longrightarrow I$ of $\Lambda(\mathbf{F})^{op}$ corresponding to the unique active morphism $I \longrightarrow \{\xi\}$ of $\Lambda(\mathbf{F})$; it is of course egressive in $\mathbf{X}_{\boxtimes}$. Hence there is a functor

$$\boxtimes_{i\in I}\colon \prod_{i\in I} \mathbf{X}_{S_i} \longrightarrow \mathbf{X}_S,$$

exact separately in each variable. If $(\phi_i\colon S_i \longrightarrow T_i)_{i\in I}$ is an I-tuple of morphisms of C with product $\phi\colon S \longrightarrow T$ then the square

$$\begin{array}{ccc} \prod_{i\in I} \mathbf{X}_{T_i} & \xrightarrow{\boxtimes_{i\in I}} & \mathbf{X}_T \\ {\scriptstyle \prod_{i\in I}\phi_i^{\star}}\downarrow & & \downarrow{\scriptstyle \phi^{\star}} \\ \prod_{i\in I} \mathbf{X}_{S_i} & \xrightarrow[\boxtimes_{i\in I}]{} & \mathbf{X}_S \end{array}$$

commutes.

When $(C, C_{\dagger}, C^{\dagger})$ is cartesian, this structure endows each fiber $\mathbf{X}_S$ with a symmetric monoidal structure: indeed, for any finite set I, we may define

$$\bigotimes_{i\in I} := \Delta^* \circ \boxtimes_{i\in I},$$

where $\Delta\colon S \longrightarrow S^I$ is the diagonal. One sees easily that the commutativity of the square above implies that any functor $\phi^{\star}$ induced by a morphism $\phi\colon S \longrightarrow T$ is symmetric monoidal in a natural way. Furthermore, a simple argument demonstrates that the external product $\boxtimes_{i\in I}$ can be recovered from the symmetric monoidal structures along with the pullback functors; for example, $X \boxtimes Y \simeq \mathrm{pr}_1^{\star} X \otimes \mathrm{pr}_2^{\star} Y$.

Now it follows from [19, Cor. 7.3.2.7] that if $\phi\colon S \longrightarrow T$ is both ingressive and egressive in C, then $\phi_!$ extends to a lax symmetric monoidal functor $\mathbf{X}_S^{\otimes} \longrightarrow \mathbf{X}_T^{\otimes}$.

7.5. **Lemma.** *Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose*

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

a symmetric monoidal Waldhausen bicartesian fibration over $(C, C_{\dagger}, C^{\dagger})$. Then the inner fibration

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

is an adequate inner fibration [4, Df. 10.3] *for the triple $(C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})$ (Nt. 2.1).*

Proof. The only condition of adequate inner fibrations that isn't explicitly part of the definition above is the assertion that for any ingressive morphism (ϕ, ω) : $(I, X) \rightarrowtail (J, Y)$ of $C_{\times}$ and for any object P of the fiber $(\mathbf{X}_{\boxtimes})_{(I,X)}$, there exists a $p_{\boxtimes}$-cocartesian morphism $P \longrightarrow Q$ covering (ϕ, ω).

So suppose that $(\phi, \omega) : (I, X) \rightarrowtail (J, Y)$ is ingressive — i.e., that $\phi\colon J \longrightarrow I$ is a bijection and each morphism $\omega_{\phi^{-1}(i)}\colon X_i \longrightarrow Y_{\phi^{-1}(i)}$ is ingressive —, and suppose that P is an object of $\mathbf{X}_{\boxtimes}$ that lies over (I, X). Then under the equivalence

$$(\mathbf{X}_{\boxtimes})_I \simeq \prod_{i \in I} \mathbf{X}_{\{i\}},$$

the object P corresponds to a family $(P_i)_{i \in I}$ of objects such that P_i lies over X_i for any $i \in I$. For each $i \in I$, select a p-cocartesian edge $P_i \longrightarrow Q_{\phi^{-1}(i)}$ covering $\omega_{\phi^{-1}(i)}$. Now there is an essentially unique morphism $P \longrightarrow Q$ covering (ϕ, ω) that corresponds under the equivalence above to the edges $P_i \longrightarrow Q_{\phi^{-1}(i)}$, and it is easy to see that it is $p_{\boxtimes}$-cocartesian. ☐

If $(C, C_{\dagger}, C^{\dagger})$ is a left complete disjunctive triple, and if $p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$ a symmetric monoidal Waldhausen bicartesian fibration for $(C, C_{\dagger}, C^{\dagger})$, then our goal is now to equip the unfurling of $\mathbf{X}$ with the structure of a $A^{\mathit{eff}}(C)^{\otimes}$-monoidal Waldhausen structure. It will then follow that the corresponding Mackey functor is in fact a commutative Green functor.

7.6. **Construction.** Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

a symmetric monoidal Waldhausen bicartesian fibration over $(C, C_{\dagger}, C^{\dagger})$. Then we define $\Upsilon(\mathbf{X}/(C, C_{\dagger}, C^{\dagger}))^{\otimes}$ as the pullback

$$\Upsilon(\mathbf{X}_{\boxtimes}/(C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})) \times_{A^{\mathit{eff}}(C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})} A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})^{\otimes}.$$

The inner fibration [4, Lm. 11.4]

$$\Upsilon(\mathbf{X}_{\boxtimes}/(C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})) \longrightarrow A^{\mathit{eff}}(C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})$$

pulls back to an inner fibration

$$\Upsilon(p)^{\otimes}\colon \Upsilon(\mathbf{X}/(C, C_{\dagger}, C^{\dagger}))^{\otimes} \longrightarrow A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})^{\otimes}.$$

We call this the ***unfurling*** of the symmetric monoidal Waldhausen bicartesian fibration $p_{\boxtimes}$.

7.7. Suppose, for simplicity, that $(C, C_\dagger, C^\dagger)$ is cartesian. Unwinding the definitions, one sees that the objects of $\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes$ are precisely the objects of $\mathbf{X}_\boxtimes$. These, in turn, can be thought of as triples (I, S_I, P_{S_I}) consiting of a finite set I, an I-tuple $S_I := (S_i)_{i\in I}$, and an object P_{S_I} of the fiber

$$(\mathbf{X}_\otimes)_{S_I} \simeq \prod_{i\in I} \mathbf{X}_{S_i},$$

which corresponds to an I-tuple $(P_{S_i})_{i\in I}$ of objects of the various Waldhausen ∞-categories $\mathbf{X}_{S_i}$. Now a morphism $(J, T_J, Q_{T_J}) \longrightarrow (I, S_I, P_{S_I})$ of the unfurling $\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes$ can be thought of as the following data:

(7.7.1) a morphism $\phi\colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$;

(7.7.2) a collection of diagrams

$$\left\{ \begin{array}{ccc} & U_{\phi(j)} & \\ {\scriptstyle\tau_j}\swarrow & & \searrow{\scriptstyle\sigma_{\phi(j)}} \\ T_j & & S_{\phi(j)}, \end{array} \;\middle|\; j \in \phi^{-1}(I) \right\}$$

of C such that for any $j \in \phi^{-1}(I)$, the morphism $\sigma_j\colon U_{\phi(j)} \rightarrowtail S_{\phi(j)}$ is ingressive, and the morphism $\tau_j\colon U_{\phi(j)} \twoheadrightarrow T_j$ is egressive; and

(7.7.3) a collection of morphisms

$$\left\{ \sigma_{\phi(j),!}\tau^\star_{J_i}\left(\boxtimes_{j\in J_i} Q_{T_j}\right) \longrightarrow P_{S_i} \;\middle|\; i \in I \right\}$$

in the various ∞-categories $\mathbf{X}_{S_i}$, where τ_{J_i} is the edge $(\{i\}, U_i) \longrightarrow (J_i, T_{J_i})$ corresponding to the tuple $(\tau_j)_{j\in J_i}$.

7.8. **Theorem.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple, and suppose*

$$p_\boxtimes\colon \mathbf{X}_\boxtimes \longrightarrow C_\times$$

a symmetric monoidal Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$. Then the functor $\Upsilon(p)^\otimes$ exhibits the ∞-category $\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes$ as a $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes$-monoidal Waldhausen ∞-category.

Proof. We first observe that, in light of [4, Pr. 11.6] and Lm. 7.5, the functor $\Upsilon(p)^\otimes$ is a cocartesian fibration. Let us check that the composite cocartesian fibration

$$\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes \longrightarrow A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

exhibits $\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes$ as a symmetric monoidal ∞-category.

To this end, it suffices to show that for any finite set I and any I-tuple $S_I := (S_i)_{i\in I}$ of objects of C, the functor

$$\prod_{i\in I}\chi_{i,!}\colon (\mathbf{X}_\boxtimes)_{S_I} \simeq \Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes_{S_I} \longrightarrow \prod_{i\in I}\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))_{S_i} \simeq \prod_{i\in I}\mathbf{X}_{S_i}$$

induced by the cocartesian edges covering the inert maps $\chi_i\colon I \longrightarrow \{i\}_+$ is an equivalence. But this morphism can be identified with

$$\prod_{i\in I}\left(\mathrm{id}_! \circ \mathrm{id}^\star \circ \boxtimes_{i\in\{i\}}\right) \colon \prod_{i\in I}\mathbf{X}_{S_i} \longrightarrow \prod_{i\in I}\mathbf{X}_{S_i},$$

which is homotopic to the identity.

Now for any finite set J, a morphism $T \longrightarrow S$ of $A^{eff}(C, C_\dagger, C^\dagger)^\otimes$ covering the unique active morphism $J \longrightarrow \{\xi\}$ is represented by a collection of spans

$$\left\{ \begin{array}{ccc} & U & \\ {}^{\phi_j}\swarrow & & \searrow^{\psi} \\ T_j & & S. \end{array} \;\middle|\; j \in J \right\}$$

The tensor product functor can therefore be written as

$$\psi_! \circ \phi_J^\star \circ \boxtimes_{j\in J} \colon \prod_{j\in J} \mathbf{X}_{T_j} \simeq \mathbf{X}_T \longrightarrow \mathbf{X}_S,$$

which is exact separately in each variable. □

In light of Pr. 7.2, we have the following.

7.8.1. **Corollary.** *Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple that is either left complete or right complete, and suppose $p_\boxtimes \colon \mathbf{X}_\boxtimes \longrightarrow C_\times$ a symmetric monoidal Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$. Then the cocartesian fibration $\Upsilon(p)^\otimes$ is classified by a Green functor*

$$\mathbf{M}_p^\otimes \colon A^{eff}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Wald}_\infty^\otimes.$$

8. Equivariant algebraic K-theory of group actions

In this section, we answer a question of Akhil Mathew. Namely, for any Waldhausen ∞-category C with an action of a finite group G, can one form an equivariant algebraic K-theory spectrum $K_G(C)$ whose H-fixed point spectrum is the algebraic K-theory of the homotopy fixed point ∞-category C^{hH}? Furthermore, can one do this in a lax symmetric monoidal fashion, so that if C is an algebra in Waldhausen ∞-categories over an ∞-operad $O^\otimes$, then $K_G(C)$ is an algebra over $O^\otimes$ in $\mathbf{Mack}(\mathbf{F}_G; \mathbf{Sp})$? The answer to both of these questions is yes, and our framework makes it an almost trivial matter to see how.

8.1. **Construction.** Suppose G a finite group. Let denote by $\mathbf{F}_G^{free} \subset \mathbf{F}_G$ the full subcategory spanned by those finite G-sets upon which G acts freely. Observe that $\mathbf{F}_G^{free}$ is the finite-coproduct completion of BG; that is, it is the free ∞-category with finite coproducts generated by BG. Consequently, $A^{eff}(\mathbf{F}_G^{free})$ is the free semiadditive ∞-category generated by BG; that is, evaluation at G/e defines an equivalence

$$\mathbf{Mack}(\mathbf{F}_G^{free}; A) \xrightarrow{\sim} \mathrm{Fun}(BG, A).$$

At the same time, the subcategory $\mathbf{F}_G^{free} \subset \mathbf{F}_G$ is clearly closed under coproducts, and since $\mathbf{F}_G^{free}$ is a sieve in $\mathbf{F}_G$, it follows that it is stable under pullbacks and binary products as well. Consequently, we obtain a fully faithful inclusion

$$A^{eff}(\mathbf{F}_G^{free}) \hookrightarrow A^{eff}(\mathbf{F}_G).$$

We thus obtain, for any semiadditive ∞-category A, a corresponding restriction functor

$$\mathbf{Mack}(\mathbf{F}_G; A) \longrightarrow \mathbf{Mack}(\mathbf{F}_G^{free}; A).$$

If A is an addition presentable, then the restriction functor admits a right adjoint

$$B_G \colon \mathrm{Fun}(BG, A) \longrightarrow \mathbf{Mack}(\mathbf{F}_G; A),$$

given by right Kan extension. We shall call this the *Borel functor*, since it assigns to any "naïve" G-object the corresponding *Borel-equivariant* object.

Applying this when $A = \mathbf{Wald}_\infty$ and apply algebraic K-theory, we obtain the *algebraic K-theory of group actions*:

$$\mathbf{K} \circ B_G \colon \mathrm{Fun}(BG, \mathbf{Wald}_\infty) \longrightarrow \mathbf{Mack}(\mathbf{F}_G; \mathbf{Sp}).$$

8.2. **Proposition.** *The algebraic K-theory of group actions extends naturally to a lax symmetric monoidal functor*

$$\mathbf{K}^{\otimes} \circ B_G^{\otimes} \colon \mathrm{Fun}(BG, \mathbf{Wald}_\infty)^{\otimes} \longrightarrow \mathbf{Mack}(\mathbf{F}_G; \mathbf{Sp})^{\otimes}.$$

for the objectwise symmetric monoidal structure relative to the symmetric monoidal structure on $\mathbf{Wald}_\infty$ [2] *and the additivized Day convolution on spectral Mackey functors.*

Proof. Since $\mathbf{K}^{\otimes}$ is lax symmetric monoidal [2], it suffices to show that for any presentable semiadditive symmetric monoidal ∞-category $E^{\otimes}$, the Borel functor B_G extends to a symmetric monoidal functor

$$B_G^{\otimes} \colon \mathrm{Fun}(BG, E)^{\otimes} \simeq \mathbf{Mack}(\mathbf{F}_G^{\mathit{free}}; A)^{\otimes} \longrightarrow \mathbf{Mack}(\mathbf{F}_G; E)^{\otimes}.$$

This will follow directly from [19], once one knows that the restriction functor

$$\mathbf{Mack}(\mathbf{F}_G; E) \longrightarrow \mathrm{Fun}(BG, E)$$

extends to a symmetric monoidal functor

$$\mathbf{Mack}(\mathbf{F}_G; E)^{\otimes} \longrightarrow \mathbf{Mack}(\mathbf{F}_G^{\mathit{free}}; A)^{\otimes} \simeq \mathrm{Fun}(BG, E)^{\otimes}.$$

For this, observe that since $\mathbf{F}_G^{\mathit{free}} \subset \mathbf{F}_G$ is stable under binary products, the inclusion

$$A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}}) \hookrightarrow A^{\mathit{eff}}(\mathbf{F}_G)$$

extends to a symmetric monoidal functor

$$A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})^{\otimes} \hookrightarrow A^{\mathit{eff}}(\mathbf{F}_G)^{\otimes}.$$

It thus suffices to note that for any free finite G-set V, the subcategory

$$(A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}}) \times A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})) \times_{A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})} A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})_{/V} \subset (A^{eff}(\mathbf{F}_G) \times A^{\mathit{eff}}(\mathbf{F}_G)) \times_{A^{\mathit{eff}}(\mathbf{F}_G)} A^{\mathit{eff}}(\mathbf{F}_G)_{/V}$$

is cofinal. □

9. Equivariant algebraic K-theory of derived stacks

In this section, we construct two symmetric monoidal Waldhausen bicartesian fibrations that extend the following two Waldhausen bicartesian fibrations introduced in [4, §D]:

- the Waldhausen bicartesian fibration

  $$\mathbf{Perf}^{op} \times_{\mathbf{Shv}_{\mathit{flat}}} \mathbf{DM} \longrightarrow \mathbf{DM}$$

  for the left complete disjunctive triple $(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$ of spectral Deligne–Mumford stacks, in which the ingressive morphisms are strongly proper morphisms of finite Tor-amplitude, and all morphisms are egressive [4, Pr. D.18], and

- the Waldhausen bicartesian fibration

$$\mathbf{Perf}^{op} \longrightarrow \mathbf{Shv}_{flat}$$

for the left complete disjunctive triple $(\mathbf{Shv}_{flat}, \mathbf{Shv}_{flat,\mathbf{QP}}, \mathbf{Shv}_{flat})$ of flat sheaves in which the ingressive morphisms are the quasi-affine representable and perfect morphisms, and all morphisms are egressive [4, Pr. D.21].

These will give algebraic K-theory the structure of a commutative Green functor for these two triples.

9.1. To begin, we let

$$\begin{array}{ccc} \mathbf{Mod}^{\otimes} & \longrightarrow & \mathbf{QCoh}^{\otimes} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{CAlg}^{cn} \times N\Lambda(\mathbf{F}) & \hookrightarrow & \mathbf{Shv}^{op}_{flat} \times N\Lambda(\mathbf{F}) \end{array}$$

be a pullback square in which q is the cocartesian fibration of [19, Th. 4.5.3.1], and p is a cocartesian fibration classified by the right Kan extension of the functor that classifies q. The objects of $\mathbf{QCoh}^{\otimes}$ can be thought of as triples (X, I, M_I) consisting of a sheaf $X\colon \mathbf{CAlg}^{cn} \longrightarrow \mathbf{Kan}(\kappa_1)$ for the flat topology, a finite set I, and an I-tuple $M_I = \{M_i\}_{i\in I}$ of quasicoherent modules M over X.

9.2. We may now pass to the cocartesian ∞-operads to obtain a cocartesian fibration of ∞-operads

$$p^{\sqcup}\colon (\mathbf{QCoh}^{\otimes})^{\sqcup} \longrightarrow (\mathbf{Shv}^{op}_{flat} \times N\Lambda(\mathbf{F}))^{\sqcup} \simeq (\mathbf{Shv}_{flat,\times})^{op} \times_{N\Lambda(\mathbf{F})} N\Lambda(\mathbf{F})^{\sqcup}.$$

Now $N\Lambda(\mathbf{F})^{\sqcup} \longrightarrow N\Lambda(\mathbf{F})$ admits a section that carries any finite set I to the pair $(I, *_I)$, where $*_I = \{*\}_{i\in I}$. Let us pull back $p^{\sqcup}$ along this section to obtain a cocartesian fibration of ∞-operads

$$p^{\boxtimes}\colon \mathbf{QCoh}^{\boxtimes} := (\mathbf{QCoh}^{\otimes})^{\sqcup} \times_{N\Lambda(\mathbf{F})^{\sqcup}} N\Lambda(\mathbf{F}) \longrightarrow (\mathbf{Shv}_{flat,\times})^{op}.$$

9.3. Passing to opposites, we obtain a functor

$$(\mathbf{QCoh}^{op})_{\boxtimes} := (\mathbf{QCoh}^{\boxtimes})^{op} \longrightarrow \mathbf{Shv}_{flat,\times}$$

which

- restricts to a symmetric monoidal Waldhausen bicartesian fibration

$$(\mathbf{QCoh}^{op})_{\boxtimes} \times_{\mathbf{Shv}_{flat,\times}} \mathbf{DM}_{\times} \longrightarrow \mathbf{DM}_{\times}$$

that extends the Waldhausen bicartesian fibration of [4, Pr. D.10] for the disjunctive triple of spectral Deligne–Mumford stacks, in which the ingressive morphisms are relatively scalloped, and all morphisms are egressive, and

- gives a symmetric monoidal Waldhausen bicartesian fibration

$$(\mathbf{QCoh}^{op})_{\boxtimes} \longrightarrow \mathbf{Shv}_{flat,\times}$$

that extends the Waldhausen bicartesian fibration of [4, Pr. D.13] for the disjunctive triple of flat sheaves, in which the ingressive morphisms are quasi-affine representable, and all morphisms are egressive.

9.4. At last, restricting to perfect modules, we obtain the desired symmetric monoidal Waldhausen bicartesian fibrations

$$(\mathbf{Perf}^{op})_{\boxtimes} \times_{(\mathbf{Shv}_{flat})_{\times}} \mathbf{DM}_{\times} \longrightarrow \mathbf{DM}_{\times}$$

for $(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$ and

$$(\mathbf{Perf}^{op})_{\boxtimes} \longrightarrow (\mathbf{Shv}_{flat})_{\times}$$

for $(\mathbf{Shv}_{flat}, \mathbf{Shv}_{flat,\mathbf{QP}}, \mathbf{Shv}_{flat})$.

Now, passing to the unfurling, we obtain the following pair of results.

9.5. **Proposition.** *The Mackey functor*

$$\mathbf{M}_{\mathbf{DM}} \colon A^{eff}(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM}) \longrightarrow \mathbf{Wald}_{\infty}$$

of [4, Cor. D.18.1] *admits a natural structure of a commutative Green functor* $\mathbf{M}^{\otimes}_{\mathbf{DM}}$. *In particular, the algebraic K-theory of spectral Deligne–Mumford stacks is naturally a commutative spectral Green functor for* $(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$.

9.6. **Proposition.** *The Mackey functor*

$$\mathbf{M}_{\mathbf{Shv}_{flat}} \colon A^{eff}(\mathbf{Shv}_{flat}, \mathbf{Shv}_{flat,\mathbf{QP}}, \mathbf{Shv}_{flat}) \longrightarrow \mathbf{Wald}_{\infty}$$

of [4, Cor. D.21.1] *admits a natural structure of a commutative Green functor* $\mathbf{M}^{\otimes}_{\mathbf{Shv}_{flat}}$. *In particular, the algebraic K-theory of flat sheaves is naturally a commutative spectral Green functor for* $(\mathbf{Shv}_{flat}, \mathbf{Shv}_{flat,\mathbf{QP}}, \mathbf{Shv}_{flat})$.

9.7. **Construction.** Suppose X a spectral Deligne–Mumford stack. As in [4, Nt. D.23], we denote by $\mathbf{F\acute{E}t}(X)$ the subcategory of $\mathbf{DM}_{/X}$ whose objects are finite [18, Df. 3.2.4] and étale morphisms $Y \longrightarrow X$ and whose morphisms are finite and étale morphisms over X. Observe that the fiber product $- \times_X -$ endows $\mathbf{F\acute{E}t}(X)$ with the structure of a cartesian disjunctive ∞-category. We will abuse notation and write $A^{eff}(X)^{\otimes}$ for the symmetric monoidal effective Burnside ∞-category of $\mathbf{F\acute{E}t}(X)$.

Now the inclusion

$$(\mathbf{F\acute{E}t}(X), \mathbf{F\acute{E}t}(X), \mathbf{F\acute{E}t}(X)) \hookrightarrow (\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$$

is clearly a morphism of cartesian disjunctive triples, whence one can restrict the commutative Green functor $\mathbf{M}^{\otimes}_{\mathbf{DM}}$ above along the morphism of ∞-operads

$$A^{eff}(X)^{\otimes} \longrightarrow A^{eff}(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})^{\otimes}$$

to a commutative Green functor

$$\mathbf{M}_X \colon A^{eff}(X)^{\otimes} \longrightarrow \mathbf{Wald}^{\otimes}_{\infty}.$$

Now if X is (say) a connected, noetherian scheme, then a choice of geometric point x of X gives rise to an equivalence

$$A^{eff}(\pi_1^{\acute{e}t}(X,x))^{\otimes} \simeq A^{eff}(X)^{\otimes}.$$

Applying algebraic K-theory, we obtain a commutative spectral Green functor for the étale fundamental group:

$$\mathbf{K}^{\otimes}_{\pi_1^{\acute{e}t}(X,x)}(X) \colon A^{eff}(\pi_1^{\acute{e}t}(X,x))^{\otimes} \longrightarrow \mathbf{Sp}^{\otimes}.$$

This commutative Green functor deserves the handle *Galois-equivariant algebraic K-theory*.

10. An equivariant Barratt–Priddy–Quillen Theorem

10.1. **Notation.** In this section, suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple.

10.2. **Recollection.** Recall [4, Df. 13.5] that $\mathbf{R}(C) \subset \mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, C)$ is the full subcategory spanned by those retract diagrams

$$S_0 \longrightarrow S_1 \longrightarrow S_0;$$

such that the morphism $S_0 \longrightarrow S_1$ is a summand inclusion. We endow $\mathbf{R}(C)$ with the structure of a pair in the following manner. A morphism $T \longrightarrow S$ will be declared ingressive just in case $T_0 \longrightarrow S_0$ is an equivalence, and $T_1 \longrightarrow S_1$ is a summand inclusion. Write p for the functor $\mathbf{R}(C) \longrightarrow C$ given by evaluation at the vertex $0 = 2$:

$$[S_0 \longrightarrow S_1 \longrightarrow S_0] \rightsquigarrow S_0.$$

Recall also that $\mathbf{R}(C, C_\dagger, C^\dagger) \subset \mathbf{R}(C)$ is the full subcategory spanned by those objects

$$S \colon \Delta^2/\Delta^{\{0,2\}} \longrightarrow C$$

such that for any complement $S_0' \hookrightarrow S_1$ of the summand inclusion $S_0 \hookrightarrow S_1$,

(10.2.1) the essentially unique morphism $S_0' \longrightarrow 1$ to the terminal object of C is egressive, and

(10.2.2) the composite $S_0' \longrightarrow S_1 \longrightarrow S_0$ is ingressive.

We endow $\mathbf{R}(C, C_\dagger, C^\dagger)$ with the pair structure induced by $\mathbf{R}(C)$. We will abuse notation by denoting the restriction of the functor $p \colon \mathbf{R}(C) \longrightarrow C$ to the subcategory $\mathbf{R}(C, C_\dagger, C^\dagger) \subset \mathbf{R}(C)$ again by p.

We proved in [4, Th. 13.11] that p is a Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$.

10.3. **Construction.** Recall that an object of the ∞-category $\mathbf{R}(C, C_\dagger, C^\dagger)_\times$ can be described as pairs (I, X) consisting of a finite set I and a collection $X = \{X_i \mid i \in I\}$ of objects of $\mathbf{R}(C, C_\dagger, C^\dagger)$ indexed by the elements of I. Accordingly, a morphism $(I, X) \longrightarrow (J, Y)$ of $\mathbf{R}(C, C_\dagger, C^\dagger)_\times$ can be described as a map $J \longrightarrow I_+$ of finite sets and a collection

$$\left\{ X_i \longrightarrow \prod_{j \in J_i} Y_j \;\middle|\; i \in I \right\}$$

of morphisms of $\mathbf{R}(C, C_\dagger, C^\dagger)$.

We now define a subcategory $\mathbf{R}(C, C_\dagger, C^\dagger)_\boxtimes \subset \mathbf{R}(C, C_\dagger, C^\dagger)_\times$ that contains all the objects. A morphism $(I, X) \longrightarrow (J, Y)$ of $\mathbf{R}(C, C_\dagger, C^\dagger)_\times$ is a morphism of $\mathbf{R}(C, C_\dagger, C^\dagger)_\boxtimes$ if and only if, for every $i \in I$, every nonempty proper subset $K_i \subset J_i$, and every choice of a complement $Y_{j,0}' \hookrightarrow Y_{j,1}$ of the summand inclusion $Y_{j,0} \hookrightarrow Y_{j,1}$, the square

$$\begin{array}{ccc} \varnothing & \longrightarrow & X_{i,1} \\ \downarrow & & \downarrow \\ \prod_{j \in K_i} Y_{j,0} \times \prod_{j \in J_i \setminus K_i} Y_{j,0}' & \longrightarrow & \prod_{j \in J_i} Y_{j,1}, \end{array}$$

in which $\varnothing$ is initial and the bottom morphism is the obvious summand inclusion, is a pullback.

Let us endow this ∞-category with a pair structure in the following manner. We declare a morphism $(I,X)\longrightarrow(J,Y)$ of $\mathbf{R}(C,C_\dagger,C^\dagger)_\boxtimes$ to be ingressive just in case the map $J\longrightarrow I_+$ represents an isomorphism in $\Lambda(\mathbf{F})$, and, for every $i\in I$, the map $X_i\longrightarrow Y_{\phi(i)}$ of $\mathbf{R}(C,C_\dagger,C^\dagger)$ is ingressive.

The following is now immediate.

10.4. **Proposition.** *The functor*

$$p_\boxtimes\colon \mathbf{R}(C,C_\dagger,C^\dagger)_\boxtimes\longrightarrow C_\times$$

given by evaluation at $0=2$ *in* $\Delta^2/\Delta^{\{0,2\}}$ *exhibits* $\mathbf{R}(C,C_\dagger,C^\dagger)$ *as a symmetric monoidal Waldhausen bicartesian fibration over* $(C,C_\dagger,C^\dagger)$.

10.5. **Construction.** Now we may the unfurling construction of [4, §11] to the symmetric monoidal Waldhausen bicartesian fibration $p_\boxtimes$ to obtain an $A^{eff}(C,C_\dagger,C^\dagger)^\otimes$-monoidal Waldhausen ∞-category (in the sense of [2])

$$\Upsilon(p)^\otimes\colon \Upsilon(\mathbf{R}(C,C_\dagger,C^\dagger)/(C,C_\dagger,C^\dagger))^\otimes\longrightarrow A^{eff}(C,C_\dagger,C^\dagger)^\otimes.$$

As we've demonstrated, $\Upsilon(p)^\otimes$ is classified by an E_∞ Green functor

$$\mathbf{M}_p^\otimes\colon A^{eff}(C,C_\dagger,C^\dagger)^\otimes\longrightarrow \mathbf{Wald}_\infty^\otimes$$

whose underlying functor is the Mackey functor

$$\mathbf{M}_p\colon A^{eff}(C,C_\dagger,C^\dagger)\longrightarrow \mathbf{Wald}_\infty$$

corresponding to the unfurling of the Waldhausen bicartesian fibration

$$\mathbf{R}(C,C_\dagger,C^\dagger)\longrightarrow C$$

over $(C,C_\dagger,C^\dagger)$.

In [2], we demonstrated that algebraic K-theory lifts in a natural fashion to a morphism of ∞-operads, whence we may contemplate the commutative Green functor

$$\mathbf{K}^\otimes\circ\mathbf{M}_p^\otimes\colon A^{eff}(C,C_\dagger,C^\dagger)^\otimes\longrightarrow \mathbf{Sp}^\otimes.$$

Observe that by [4, Th. 13.12], the underlying Mackey functor

$$_{(C,C_\dagger,C^\dagger)} := \mathbf{K}\circ\mathbf{M}_p$$

of $\mathbf{K}^\otimes\circ\mathbf{M}_p^\otimes$ is the spectral Burnside Mackey functor for $(C,C_\dagger,C^\dagger)$, as defined in [4, Df. 8.1]. In particular, it is unit for the symmetric monoidal ∞-category $\mathbf{Mack}(C,C_\dagger,C^\dagger;\mathbf{Sp})$, which of course admits an essentially unique E_∞ structure. Consequently, we deduce the following.

10.6. **Theorem** (Equivariant Barratt–Priddy–Quillen)**.** *The Green functor* $\mathbf{K}^\otimes\circ\mathbf{M}_p^\otimes$ *is the spectral Burnside Green functor* $_{(C,C_\dagger,C^\dagger)}$.

Of course, this result directly implies the original Barratt–Priddy–Quillen Theorem, which states that the algebraic K-theory of the ordinary Waldhausen category $\mathbf{F}_*$ of pointed finite sets (in which the cofibrations are the monomorphisms) is the sphere spectrum . Furthermore, the essentially unique E_∞ structure on is induced by the smash product of pointed finite sets.

11. A brief epilogue about the Theorems of Guillou–May

Suppose G a finite group. Write $\mathbf{OrthSp}_G$ for the underlying ∞-category of the relative category of orthogonal G-spectra. The Equivariant Barratt–Priddy–Quillen Theorem of Guillou–May [12] provides a similar description in $\mathbf{OrthSp}_G$ of certain mapping spectra. Note that this is not *a priori* related to Th. 10.6 when $C = \mathbf{F}_G$. Nevertheless, a suitable comparison theorem (which of course Guillou–May provide in [13]) offers an implication.

On the other hand, the proof of our result here, combined with work from our forthcoming book [6], will allow us to reprove, using entirely different methods, the comparison result of Guillou–May. Indeed, if we can extend the functor $\Sigma^\infty_+ \colon \mathbf{F}_G \longrightarrow \mathbf{OrthSp}_G$ to a suitable functor $A^{\mathit{eff}}(\mathbf{F}_G) \longrightarrow \mathbf{OrthSp}_G$, then the Equivariant Barratt–Priddy–Quillen Theorem above and the Schwede–Shipley theorem [20] together will imply the result of Guillou–May [13] providing the equivalence

$$\mathbf{Sp}^G \simeq \mathbf{OrthSp}_G.$$

It is, however, difficult to construct the desired functor $A^{\mathit{eff}}(\mathbf{F}_G) \longrightarrow \mathbf{OrthSp}_G$ directly, as this involves nontrivial homotopy coherence problems. However, in the language of G-equivariant ∞-category theory, which we develop in the forthcoming [6] provides a universal property for the G-equivariant effective Burnside ∞-category. This will provide us with the desired functor, and we will easily deduce the desired equivalence as a corollary.

References

1. C. Barwick, *From operator categories to topological operads*, Preprint `arXiv:1302.5756`, February 2013.
2. ———, *Multiplicative structures on algebraic K-theory*, Preprint `arXiv:1304.4867`., April 2013.
3. ———, *On the algebraic K-theory of higher categories.*, Preprint `arXiv:1204.3607`, April 2013.
4. ———, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Preprint `arXiv:1404.0108`, April 2014.
5. C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah, *Parametrized higher algebra*, To appear.
6. ———, *Parametrized higher category theory*, To appear.
7. C. Barwick and S. Glasman, *Cyclonic spectra, cyclotomic spectra, and a conjecture of Kaledin*, Preprint `arXiv:1602.02163`.
8. P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois Marie 1966–67 (SGA 6). Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud, J. P. Serre. Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin, 1966–67. MR 50 #7133
9. B. Day, *Construction of biclosed categories*, Ph.D. thesis, University of New South Wales, September 1970.
10. A. W. M. Dress, *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*, Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971, With the aid of lecture notes, taken by Manfred Küchler. MR 0360771 (50 #13218)
11. S. Glasman, *Day convolution for ∞-categories*, Preprint, `arXiv:1308.4940`, August 2013.
12. B. Guillou and J. P. May, *Models of G-spectra as presheaves of spectra*, Preprint `arXiv:1110.3571v3`, July 2013.
13. ———, *Permutative G-categories in equivariant infinite loop space theory*, Preprint `arXiv:1207.3459v2`, March 2014.
14. L. G. Lewis, Jr., *The theory of Green functors*, Unpublished manuscript.

15. L. Gaunce Lewis and Michael A. Mandell, *Equivariant universal coefficient and künneth spectral sequences*, Proceedings of the London Mathematical Society **92** (2006), no. 2, 505–544.
16. J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)
17. ______, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. MR 2555928 (2010k:57064)
18. ______, *Derived algebraic geometry XII. Proper morphisms, completions, and the Grothendieck existence theorem*, Preprint from the web page of the author, November 2011.
19. ______, *Higher algebra*, Preprint from the web page of the author, September 2014.
20. Stefan Schwede and Brooke Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), no. 1, 103–153. MR 2003g:55034

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PARAMETRIZED HIGHER CATEGORY THEORY AND HIGHER ALGEBRA: A GENERAL INTRODUCTION

CLARK BARWICK, EMANUELE DOTTO, SAUL GLASMAN, DENIS NARDIN, AND JAY SHAH

Let k denote a field, and let $E \supseteq k$ be a finite Galois extension thereof with Galois group G. The algebraic K-groups $K_n(k)$ and $K_n(E)$, as defined by Quillen, together exhibit some interesting structure. Since these groups are defined in terms of the categories of finite-dimensional vector spaces (along with their additive structure), the forgetful functor $\mathbf{Vect}(E) \longrightarrow \mathbf{Vect}(k)$ and the functor $\mathbf{Vect}(k) \longrightarrow \mathbf{Vect}(E)$ given by $X \rightsquigarrow X \otimes_k E$ give rise to homomorphisms

$$V\colon K_n(E) \longrightarrow K_n(k) \quad \text{and} \quad F\colon K_n(k) \longrightarrow K_n(E).$$

Ordinary Galois theory shows that the composite functor $\mathbf{Vect}(E) \longrightarrow \mathbf{Vect}(E)$ given by $Y \rightsquigarrow Y \otimes_k E$ can be described as the direct sum

$$\bigoplus_{g \in G} g\colon \mathbf{Vect}(E) \longrightarrow \mathbf{Vect}(E),$$

where G acts in the obvious manner. Accordingly, we have an action of G on $K_n(E)$ for which both V and F are equivariant, and a formula

$$FV = \sum_{g \in G} g.$$

Note that the equivariance of V implies that it factors through the orbits $K_n(E)_G$, and the equivariance of F implies that it factors through the fixed points $K_n(E)^G$, but these maps do not typically identify $K_n(k)$ with either the orbits or the fixed points. The data of $K_n(k)$ is an added piece of structure; that is, $K_n(k)$ cannot in general be recovered from $K_n(E)$ as a G-module.

But the problem is even deeper than this. Even if one considers all the K-groups together as a single entity (by thinking of these groups as the homotopy groups of a space or spectrum), one can construct a descent spectral sequence

$$E^2_{p,q} = H^{-p}(G, K_q(E)),$$

but this will not, as a rule, converge to the groups $K_{p+q}(k)$. In other words, the space or spectrum $K(k)$ is not the homotopy fixed point space/spectrum of the action of G on the space/spectrum $K(E)$. Consequently, even knowing the homotopy type $K(E)$ with its action of G is insufficient to recover the groups $K_n(k)$. This is the *descent problem in algebraic K-theory.*

There is, of course, no need to consider the K-theories of E and k in isolation. One can also include the information of the K-groups of all the various subextensions $E \supseteq L \supseteq k$. In other words, for any subgroup $H \leq G$, one can contemplate the K-groups $K_n(E^H)$ of the fixed field E^H. These abelian groups each have conjugation homomorphisms

$$c_g\colon K_n(E^H) \longrightarrow K_n(E^{gHg^{-1}})$$

for any $g \in G$. Additionally, for subgroups $K, L \leq H \leq G$, one again has the forgetful functor $\mathbf{Vect}(E^K) \longrightarrow \mathbf{Vect}(E^H)$ and the functor $\mathbf{Vect}(E^H) \longrightarrow \mathbf{Vect}(E^L)$ given by $Y \rightsquigarrow Y \otimes_{E^H} E^L$, so again one has homomorphisms

$$V_K^H \colon K_n(E^K) \longrightarrow K_n(E^H) \quad \text{and} \quad F_L^H \colon K_n(E^H) \longrightarrow K_n(E^L).$$

Again, a small amount of Galois theory reveals that these two homomorphisms compose in the following manner:

$$F_L^H V_K^H = \sum_{x \in L \backslash G / K} V^L_{L \cap (xKx^{-1})} c_x F^K_{(x^{-1}Lx) \cap K} \colon K_n(E^K) \longrightarrow K_n(E^L).$$

And again, of course, the groups $K_n(E^H)$ cannot be recovered from the G-module $K_n(E)$ or the homotopy type $K(E)$ with its action of G.

Combined, this structure on the assignment $H \rightsquigarrow K_n(E^H)$ makes up what is called a *Mackey functor* for G. As we see, this is strictly more structure than a G-module. Similarly, the assignment $H \rightsquigarrow K(E^H)$ is a *spectral Mackey functor* for G in the sense of the first author [2]. This is strictly more structure than a spectrum with a G-action. We call this object the *G-equivariant K-theory of E over k*.

In this monograph, we tease out the kind of structure on the categories $\mathbf{Vect}(E^H)$ that provides their K-theory with the structure of a spectral Mackey functor for G. As a first approximation, we note that, because the category of subextensions of E is equivalent to the category of transitive G-sets, the functors $Y \rightsquigarrow Y \otimes_{E^H} E^L$ together define what we call a *G-category* – a diagram of categories indexed on the opposite of the orbit category $\mathbf{O}_G$ of G. Let us write $\underline{\mathbf{Vect}}_{E \supseteq k}$ for this G-category.

Of course, the G-category $\underline{\mathbf{Vect}}_{E \supseteq k}$ is relatively simple: after all, if one thinks of the action of G on $\mathbf{Vect}(E)$, then $\mathbf{Vect}(E^H)$ is the category of E-vector spaces equipped with a semilinear action of H. In other words, $\mathbf{Vect}(E^H)$ is simply the homotopy fixed point category for the action of H on $\mathbf{Vect}(E)$. So we might at first contemplate $\mathbf{Vect}(E)$ with its G-action. However, the adjoints to the functors in this G-category – the forgetful functors – contain extra information that compels us to contemplate entire G-category structure.

For example, the forgetful functor $\mathbf{Vect}(E) \longrightarrow \mathbf{Vect}(k)$ is a kind of generalized *product* of vector spaces: we regard it as *indexed*, not over a mere set, but over the G-set G/e. To see why this is appropriate, first note that by the normal basis theorem, if Y is an E-vector space with basis $\{v_i\}_{1 \leq i \leq n}$, then there is an element $\theta \in E$ such that Y has basis $\{g\theta v_i\}_{1 \leq i \leq n, g \in G}$ over k. But without choosing this element, we would still be entitled to write

$$\prod_{\alpha \in G/e} Y$$

for this k-vector space. In the same manner, the presence of all the other right adjoints $\mathbf{Vect}(E^H) \longrightarrow \mathbf{Vect}(E^K)$ in this diagram of categories can be regarded as the existence of various *indexed products*

$$\prod_{\alpha \in K/H} Z$$

on this G-category. At the same time, since our field extensions are separable, these right adjoints are all also *left* adjoints, and so we even think of this as endowing our G-category with *indexed direct sums*

$$\bigoplus_{\alpha\in K/H} Z.$$

The point here is that the transfer structure on the equivariant algebraic K-groups arises from the additional structure of indexed products or coproducts on the G-category $\underline{\mathbf{Vect}}_{E\supseteq k}$. And this example refects a general principle: to get the full structure of a Mackey functor on equivariant algebraic K-theory of E over k, one must work not only with the diagram of categories indexed by $\mathbf{O}_G^{op}$, but also the G-direct sums thereupon.

The G-category $\underline{\mathbf{Vect}}_{E\supseteq k}$ also carries a sophisticated multiplicative structure. Of course, the tensor product over k provides an *external product*

$$\mathbf{Vect}(E^K)\times\mathbf{Vect}(E^L)\longrightarrow\mathbf{Proj}^{fg}(E^K\otimes_k E^L)\simeq\prod_{x\in L\backslash G/K}\mathbf{Vect}(E^{(x^{-1}Lx)\cap K}).$$

In [9], we demonstrated that the external products provide the equivariant algebraic K-groups with the structure of a graded Green functor, and, even better, they provide the equivariant algebraic K-theory spectra with the structure of a spectral Green functor.

However, there is a still richer multiplicative structure, whose impact on equivariant K-theory is studied here for the first time. Just as the usual norm of an element of E is automatically Galois-invariant, we see that for any finite-dimensional E-vector space V, the tensor power $V^{\otimes G}$ comes with canonical descent data. We call the resulting k-vector space $N_E^k(V)$ the *multiplicative norm* from E to k. Quite simply, $N_E^k(V)$ is the k-vector space (of dimension $(\dim V)^{\#G}$) such that the set $\mathrm{Hom}_k(N_E^k(V),W)$ is in bijection with the set of *norm forms* $V^{\times G}\longrightarrow W\otimes_k E$ *for* E/k – i.e., k-multilinear maps

$$\Phi\colon V^{\times G}\longrightarrow W\otimes_k E$$

such that for any element $(v_h)_{h\in G}\in V^{\times G}$, any element $g\in G$, and any element $\lambda\in E$,

$$\Phi((v'_h)_{h\in G})=(g\lambda)\Phi((v_h)_{h\in G}),$$

where

$$v'_h=\begin{cases}\lambda v_g & \text{if } h=g;\\ v_h & \text{if } h\neq g,\end{cases}$$

and

$$g\Phi((v_h)_{h\in G})=\Phi((v_{gh})_{h\in G}.$$

So, $N_E^k(V)$ is the dual of the k-vector space of norm forms $V^{\times G}\longrightarrow E$ for E/k. In particular, when $k=\mathbf{R}$ and $E=\mathbf{C}$, then $N_{\mathbf{C}}^{\mathbf{R}}(V)$ is precisely the dual space of the $\mathbf{R}$-vector space of hermitian forms on V.

More generally, there are multiplicative norms for any subgroups $K\leq L\leq G$. Together with the external products, these multiplicative norms furnish $\underline{\mathbf{Vect}}_{E\supseteq k}$ with a *G-symmetric monoidal structure*. In effect, this provides tensor products indexed over any finite G-set $U=\coprod_{i\in I}(G/H_i)$, which amount to functors

$$\bigotimes_{u\in U}\colon\prod_{i\in I}\mathbf{Vect}(E^{H_i})\longrightarrow\mathbf{Vect}(k),$$

which are suitably associative and commutative.

This additional structure on $\underline{\mathbf{Vect}}_{E\supseteq k}$ descends to an analogous structure on the equivariant algebraic K-theory of E over k. These provide the equivariant algebraic K-theory of E over k with the full structure of a *G-E_∞-algebra*.

Hill's program

To tell this story, we pursue here the general theory of *G-∞-categories*. But we are by no means the first to contemplate this possibility.

In their landmark solution of the Kervaire Invariant Problem [23], Mike Hill, Mike Hopkins, and Doug Ravenel developed a perspective on equivariant stable homotopy thery that centered on the study of indexed products, indexed coproducts, and indexed symmetric monoidal structures (incorporating their multiplicative norms). They argued that these structures were fundamental to the basic structure of equivariant stable homotopy theory.

In 2012, Hill presented (partly jointly with Hopkins) a sketch of a program to rewire huge swaths of higher category theory in order to embed these structures into the very fabric of the homotopy theory of homotopy theories. Hill sought a theory of G-∞-categories and G-functors, along with a concomitant theory of internal homs, G-limits, G-colimits, G-Kan extensions, etc. He furthermore conjectured that, equipped with this technology, one could prove the following, which is an analogue of the universal property of the ∞-category **Top** of spaces.

Theorem A. *The G-∞-category $\underline{\mathbf{Top}}_G$ of G-spaces – whose value on an orbit G/H is the ∞-category of H-spaces – is freely generated under G-colimits by the contractible G-space; that is, for any G-∞-category D with all G-colimits, evaluation on the generator defines an equivalence of G-∞-categories*

$$\underline{\mathrm{Fun}}_G^L\left(\underline{\mathbf{Top}}_G, D\right) \xrightarrow{\sim} D.$$

Here $\underline{\mathrm{Fun}}_G^L$ is the G-∞-category of G-colimit-preserving functors.

In this text, we develop all this machinery, and this is the first main theorem.

Recall that one may speak of semiadditive ∞-categories, in which finite products and finite coproducts exist and coincide. In the same manner, Hill expected that one may speak of G-semiadditive ∞-categories, in which finite G-products and finite G-coproducts exist and coincide. Furthermore, the effective Burnside ∞-category $A^{\mathit{eff}}(\mathbf{F})$ of finite sets is equivalent to the ∞-category of finitely generated free E_∞-spaces, whence it is the free semiadditive ∞-category on one generator. Accordingly, in equivariant higher category theory, we have the following.

Theorem B. *The G-∞-category $\underline{A}^{\mathit{eff}}(G)$ – whose value on G/H is the effective Burnside ∞-category of finite H-sets – is equivalent to the G-∞-category of finitely generated free G-E_∞-spaces. In other words, it is the free G-semiadditive G-∞-category on one generator; that is, for any G-semiadditive G-∞-category A, evaluation on the generator defines an equivalence of G-∞-categories*

$$\underline{\mathrm{Fun}}_G^{\oplus}\left(\underline{A}^{\mathit{eff}}(G), A\right) \xrightarrow{\sim} A.$$

Here $\underline{\mathrm{Fun}}_G^{\oplus}$ is the G-∞-category of G-coproduct-preserving functors.

As suggested by work of Andrew Blumberg [14], the G-stability of a G-∞-category can be defined as ordinary stability along with G-semiadditivity. Consequently, the two previous theorems, with some effort, together provide the following, also conjectured by Hill:

Theorem C. *The G-∞-category $\underline{\mathbf{Sp}}^G$ of G-spectra – whose value on an orbit G/H is the ∞-category $\mathbf{Sp}^H$ of genuine H-spectra – is the free G-stable G-∞-category with all G-colimit on one generator; that is, for any G-stable G-∞-category E, evaluation on the generator defines an equivalence of G-∞-categories*

$$\underline{\mathrm{Fun}}_G^L\left(\underline{\mathbf{Sp}}^G, A\right) \xrightarrow{\sim} A.$$

Going further, Hill also expected that the multiplicative norms of Hill–Hopkins–Ravenel would be part of a new type of structure – a *G-symmetric monoidal G-∞-category.* In effect, a G-symmetric monoidal G-∞-category is a G-∞-category C along with tensor product functors over finite G-sets. In particular, one has a functor

$$N^G\colon C(G/e) \longrightarrow C(G/G),$$

which is exactly the desired multiplicative norm.

Work of Hill and Hopkins [22] has already laid out the idea of G-symmetric monoidal ordinary categories, but incorporating homotopy coherence into this sort of structure is a taller order. The situation is roughly analogous to the situation with the smash product in model categories of spectra: there are genuine obstructions to making a G-symmetric monoidal structure maximally compatible with a model category of genuine G-spectra. However, when we pass to the world of ∞-categories as in [24], the situation becomes much cleaner: not only can one give an explicit, homotopy invariant construction of the smash product on the ∞-category $\mathbf{Sp}$ of spectra, but this smash product enjoys a universal property that characterizes it up to a contractible space of choices.

We bring exactly this kind of conceptual clarity (and technical power) to the study of homotopy coherent G-commutative structures in this text. We define the notions of *G-∞-operad* and *G-symmetric monoidal G-∞-category.* We find that G-products define G-symmetric monoidal structures on the G-∞-category $\underline{\mathbf{Cat}}_{\infty,G}$ of G-∞-categories and the G-∞-category $\underline{\mathbf{Top}}_G$ of G-spaces. The G-commutative algebra objects of $\underline{\mathbf{Cat}}_{\infty,G}$ are precisely the G-symmetric monoidal ∞-categories, and the G-commutative algebra objects of $\underline{\mathbf{Cat}}_{\infty,G}$ are precisely the G-E_∞-spaces.

Similarly, there is a G-subcategory $\underline{\mathbf{Pr}}_G^L \subset \underline{\mathbf{Cat}}_{\infty,G}$ of G-presentable G-∞-categories and G-left adjoints. This too has a G-symmetric monoidal structure, but it is not given by G-products; rather, the G-commutative algebra objects of $\underline{\mathbf{Pr}}_G^L$ are precisely the G-symmetric monoidal ∞-categories that are presentable and in which the tensor product preserves G-colimits separately in each variable.

Theorem D. *The G-∞-category $\underline{\mathbf{Top}}_G$ is the unit in the G-symmetric monoidal G-∞-category $\underline{\mathbf{Pr}}_G^L$. In particular, it admits an essentially unique G-symmetric monoidal structure in which the tensor product preserves G-colimits separately in each variable, which is given by the G-products.*

Even further, the full G-subcategory $\underline{\mathbf{Pr}}_{G,st}^L \subset \underline{\mathbf{Pr}}_G^L$ spanned by the G-stable G-presentable G-∞-categories inherits the G-symmetric monoidal structure, and we have the following.

Theorem E. *The G-∞-category $\underline{\mathbf{Sp}}^G$ is the unit object in the G-symmetric monoidal G-∞-category $\underline{\mathbf{Pr}}^L_{G,st}$. In particular, it admits an essentially unique G-symmetric monoidal structure in which the tensor product preserves G-colimits separately in each variable.*

In particular, this provides a universal description of the Hill–Hopkins–Ravenel multiplicative norm. With some work, this even provides a universal characterization of an *individual* norm functor.

Theorem F. *The norm functor $N^G\colon \mathbf{Sp} \longrightarrow \mathbf{Sp}^G$ is the inital object of the ∞-category*

$$\mathrm{Fun}^{\otimes}(\mathbf{Sp}, \mathbf{Sp}^G) \times_{\mathrm{Fun}^{\otimes}(\mathbf{Top}, \mathbf{Sp}^G)} \mathrm{Fun}^{\otimes}(\mathbf{Top}, \mathbf{Sp}^G)_{\Sigma^{G,\infty}_{+} \circ \Pi_G /},$$

where $\Pi_G\colon \mathbf{Top} \longrightarrow \mathbf{Top}^G$ is the G-product, and $\mathrm{Fun}^{\otimes}$ denotes the ∞-category of symmetric monoidal functors.

In this text, we completely realize Hill's vision, and we prove Theorems A–F.

Taking the G out of Genuine

Formally, one may now note that the orbit category $\mathbf{O}_G$ of the group G plays a much more significant role in these results than does G itself. In particular, although the ∞-category $\mathbf{Sp}^G$ can be obtained by taking the ∞-category $\mathbf{Top}_G$ and inverting the representation spheres, our viewpoint regards the role of representation spheres as incidental.

One is thus led to ask whether one might untether equivariant homotopy theory from dependence upon a group. (We thank Haynes Miller for the pun of the section heading.) That is, first, do Theorems A–F hold more generally? And, second, is there any value in proving them in greater generality? The answer to both questions turns out to be *yes*.

Indeed, when one examines the proofs of the results above, one finds that the unstable results continue to hold when $\mathbf{O}_G$ is replaced with any base ∞-category T. The stable results require only very mild conditions on T; in effect, one requires the analogue of the Mackey decomposition theorem in T ("T is *orbital*") and a condition that no nontrivial retracts exist ("T is *atomic*"). We can even extend this further, and define an *incompleteness class* R on the orbital ∞-category T; in effect, this serves to place limits on the classes of transfers that exist in the corresponding ∞-category of spectra.

As it happens, there are many examples that make this generality worthwhile. Here are a few.

1. As a mild extension of the example $\mathbf{O}_G$, consider a family $\mathscr{F}$ of subgroups of G such that if $K \leq L$ lie in $\mathscr{F}$, then any subgroup $H \leq G$ that is conjugate to a subgroup H' such that $K \leq H' \leq L$ also lies in $\mathscr{F}$. Then the full subcategory $\mathbf{O}_{G,\mathscr{F}} \subseteq \mathbf{O}_G$ is also an atomic orbital category. Such categories (along with various inclusions of "closed" and "open" subcategories) appear naturally when we contemplate the isotropy separation sequence in equivariant stable homotopy theory.
2. Following Blumberg and Hill [15], any incomplete G-universe U gives rise to an incompleteness class R_U on $\mathbf{O}_G$, and this permits us to model G-spectra relative to U.

3. Furthermore, one can also work with orbit categories of profinite groups (where the stabilizers are required to be open) and locally finite groups (where the stabilizers are required to be finite). This provides extensions of equivariant stable and unstable homotopy theory to these contexts.
4. Any ∞-groupoid (= Kan complex) X is atomic orbital. The corresponding ∞-category of X*-spaces* is equivalent to the ∞-category of functors $X \longrightarrow \mathbf{Top}$; likewise, the ∞-category of X*-spectra* is equivalent to the ∞-category of functors $X \longrightarrow \mathbf{Sp}$. In other words, X-spaces are local systems of spaces over X, and X-spectra are local systems of spectra over X. Consequently, this example actually recovers parametrized homotopy theory as studied by Peter May and Johann Sigurdsson [25]; in fact, this example was the inspiration for our title.
5. Combining the previous example with the ur-example, for any G-space X, one can construct the *total orbital* ∞*-category* $\underline{\mathrm{X}}$. One sees that $\underline{X}$-spaces are local systems of G-spaces over X, and $\underline{X}$-spectra are local systems of G-spectra over X.
6. The cyclonic orbit 2-category $\mathbf{O}_{©}$, whose objects are $\mathbf{Q}/\mathbf{Z}$-sets with finite stabilizers, whose 1-morphisms are equivariant maps, and whose 2-morphisms are certain intertwiners, is an orbital ∞-category [8]. The corresponding homotopy theory of B-spectra is the homotopy theory of S^1-equivariant spectra relative to the family of finite subgroups. This is precisely the sort of equivariance that one sees on topological Hochschild homology [6]. To construct the homotopy category of *cyclotomic* spectra, one forms the fixed points of this homotopy theory relative to the action of the monoid of *open immersions* from $\mathbf{O}_{©}$ into itself.
7. Generalizing the previous example are the *multi-cyclonic orbit* 2*-categories* which control torus-equivariance and *multi-cyclotomic* structures, which appear naturally on higher forms of topological Hochschild homology [5].
8. The 2-category Γ of finite connected groupoids and covering maps is atomic orbital. The corresponding homotopy theory of Γ-spectra is a variant of Stefan Schwede's global equivariant homotopy theory [27]. To get *exactly* Schwede's global equivariant homotopy theory (for finite groups) in our framework requires a larger orbital ∞-category of finite connected groupoids equipped with an incompletness class.
9. The category $\mathbf{Surj}_{\leq n}$ of finite sets of cardinality $\leq n$ and surjective maps is an atomic orbital category. This one is extremely strange, however, as it doesn't have much at all to do with any groups. Nevertheless, the third author shows in [21] that the corresponding homotopy theory of $\mathbf{F}_{\leq n}$-spectra is equivalent to the homotopy theory of n-excisive functors $\mathbf{Sp} \longrightarrow \mathbf{Sp}$, generalizing Tom Goodwillie's classification of homogeneous functors. Indeed, it is the inclusion of $\mathbf{Surj}_{\leq n-1}$ into $\mathbf{Surj}_{\leq n}$, combined with the complmentary inclusion of $B\Sigma_n$ into $\mathbf{Surj}_{\leq n}$, that together reconstruct the recollement of n-excisive functors by $(n-1)$-excisive functors and n-homogeneous functors.
10. The ∞-categories $\mathbf{Surj}(\mathbf{R})_{\leq n}$ and $\mathbf{Surj}(\mathbf{C})_{\leq n}$ obtained from the topological categories of finite-dimensional inner product spaces (over $\mathbf{R}$ and $\mathbf{C}$, respectively) of dimension $\leq n$ and orthogonal projections are atomic orbital as well. Just as stable homotopy theory parametrized on the orbital categories $\mathbf{Surj}_{\leq n}$ "controls" the Goodwillie tower, so the stable homotopy theory parametrized on the

orbital categories $\mathbf{Surj}(\mathbf{R})_{\leq n}$ "controls" the Weiss orthogonal calculus [29]. Likewise, stable homotopy theory parametrized on the orbital categories $\mathbf{Surj}(\mathbf{C})_{\leq n}$ "controls" the unitary calculus. We hope to return to this point in future work.

11. Our framework also covers and extends a setting previously defined in work of Bill Dwyer and Dan Kan, Emanuel Dror Farjoun, and Boris Chorny and Bill Dwyer. In [19], Farjoun builds on work of [20] and defines a model structure on the category of diagrams of spaces indexed on a small category I, called the I-equivariant model structure, which depends on the "I-orbits": the diagrams $I \longrightarrow \mathbf{Top}$ whose strict (= 1-categorical) colimit is equal to a point. In particular if $I = G$ is a group these are precisely the G-orbits, and the resulting homotopy theory is the fixed-points model structure on G-spaces. Moreover Farjoun's construction admits an Elmendorf–McClure theorem, in the sense that the I-equivariant model structure is Quillen-equivalent to a presheaf category (on the orbit category when the orbits are either small or complete). This result was proved in different levels of generality in [20] and [17], and in full generality in the more recent [16]. The category of I-orbits $\mathbf{O}_I$ is an atomic orbital category, and by the above mentioned Elmendorf–McClure theorem, Farjoun's I-equivariant model category is equivalent to our homotopy theory of $\mathbf{O}_I$-spaces. Our construction exhibits the I-equivariant homotopy theory as a fiber of a full-fledged $\mathbf{O}_I$-category, thus enabling one to exploit the full theory of $\mathbf{O}_I$-equivariant limits and colimits.

Such a wealth of examples compels us to prove Theorems A–F in the generality atomic orbital ∞-categories, and, where possible, we develop elements of the theory in even greater generality.

Plan

This text consists of nine Exposés:

I. We introduce the basic elements of the theory of *parametrized ∞-categories* and functors between them. Following the lessons of [13], these notions are defined as suitable fibrations of ∞-categories and functors between them. We give as many examples as we are able at this stage. Simple operations, such as the formation of opposites and the formation of functor ∞-categories, become slightly more involved in the parametrized setting, but we explain precisely how to perform these constructions. All of these constructions can be performed explicitly, without resorting to such acts of desperation as straightening. The key results of this Exposé are: (1) a universal characterization of the T-∞-category of T-objects in any ∞-category, (2) the existence of an internal Hom for T-∞-categories, and (3) a parametrized Yoneda lemma. [3]

II. We dive deep into the fundamentals of parametrized ∞-category theory in the second Exposé. In particular, we construct parametrized versions of join and slice, and use these to define *parametrized colimits and limits* as well as *parametrized left and right Kan extensions*. At the heart of this is the difficult but technically powerful result that, just as one may decompose colimits into coproducts and geometric realizations in ∞-category theory, similarly one may decompose parametrized colimits into parametrized coproducts and geometric realizations in the ordinary sense. This has the effect of elevating parametrized coproducts and products to a special status within the theory. Theorem A is proved (and generalized) here. [28]

III. We next introduce *orbital ∞-categories*, along with a host of examples. There are actually different sorts of functor between orbital ∞-categories, and we taxonomize these according to certain algebro-geometric intuitions. For any orbital ∞-category T, we have a corresponding ∞-category $\mathbf{Sp}^T$ (even T-∞-category) of *T-spectra*, which under our algebro-geometric analogy corresponds roughly to an ∞-category of "quasicoherent sheaves on T." The different sorts of functors between orbital ∞-categories induce suitable functors between the ∞-categories of T-spectra, and these behave as the names suggest. Perhaps most importantly, *closed immersions* of orbital ∞-categories admit *open complements*, and these two functors induce a *recollement* of the corresponding ∞-category of spectra; this is how one obtains the *isotropy separation sequence* and generalizations thereof. With a little care, we are able to extend all this to the context of an orbital ∞-category equipped with an incompleteness class. [7]

IV. In the fourth Exposé, we define *semiadditive parametrized ∞-categories*, and we prove Theorem B. Then we use the work of Exposé II to show that parametrized stability can be expressed as ordinary stability combined with parametrized semiadditivity. This now makes it possible to prove Theorem C. [26]

V. Next, we introduce the notion of *parametrized Waldhausen ∞-categories*. We show that the algebraic K-theory of a Waldhausen T-∞-category naturally carries the structure of a T-spectrum. [1]

VI. From here, we move toward the algebraic structures in parametrized higher category theory. We introduce the notions of *T-∞-operad* and *T-symmetric monoidal ∞-category* for an orbital ∞-category T, and we offer up numerous examples. Perhaps most importantly, parametrized ∞-categories with all T-coproducts (or, dually, T-products) inherit canonical T-symmetric monoidal structures. [4]

VII. In the seventh Exposé, we prove that when T is an atomic orbital ∞-category, the T-∞-category $\underline{\mathbf{Pr}}_T^L$ of T-presentable T-∞-categories admits a T-symmetric monoidal structure analogous to the symmetric monoidal structure on presentable ∞-categories. Theorem A then implies, more or less directly, Theorem D. Moreover, T-stable T-presentable T-∞-categories form a symmetric monoidal localization of $\underline{\mathbf{Pr}}_T^L$, and the localization is given by tensoring with $\underline{\mathbf{Sp}}^T$. Theorem E follows immediately. Moreover, one deduces a different universal property of $\underline{\mathbf{Sp}}^T$, which is that it is, in effect, the result of inverting the analogues of the permutation representation spheres in the T-symmetric monoidal T-∞-category $\underline{\mathbf{Top}}_G$. From this, we are able to deduce the universal property of the norm (Theorem F). [12]

VIII. In the penultimate Exposé, we introduce the T-∞-category $\underline{\mathbf{Mod}}(A)$ of modules over a T-E_∞-algebra A (for an atomic orbital ∞-category T). We show that it is T-symmetric monoidal, and we describe how it transforms in both A and T. [11]

IX. Finally, we return to the subject of equivariant algebraic K-theory, where we show that the equivariant algebraic K-theory of a T-symmetric monoidal Waldhausen T-∞-category admits the natural structure of a T-E_∞ ring spectrum. This applies not only in the field case of the beginning of this introduction, but also to those forms of equivariant algebraic K-theory that arise in the work of Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel [18] as well as the nascent subject of equivariant (derived) algebraic geometry. [10]

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References

1. C. Barwick, *Algebraic K-theory of parametrized ∞-categories*, To appear.
2. ______, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Preprint arXiv:1404.0108, April 2014.
3. C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah, *Elements of parametrized higher category theory*, To appear.
4. ______, *Parametrized ∞-operads and parametrized symmetric monoidal ∞-categories*, To appear.
5. C. Barwick and S. Glasman, *Multicyclonic spectra, multicyclotomic spectra, and higher topological hochschild homology*, To appear.
6. ______, *Noncommutative syntomic realization*, To appear.
7. ______, *Orbital ∞-categories*, To appear.
8. ______, *Cyclonic spectra, cyclotomic spectra, and a conjecture of Kaledin*, Preprint arXiv:1602.02163, February 2016.
9. C. Barwick, S. Glasman, and J. Shah, *Spectral Mackey functors and equivariant algebraic K-theory (II)*, Preprint arXiv:1505.03098, May 2015.
10. C. Barwick and D. Nardin, *Norms in algebraic K-theory*, To appear.
11. C. Barwick, D. Nardin, and J. Shah, *Module theory for parametrized E_∞ ring spectra*, To appear.
12. ______, *Parametrized presentable ∞-categories and distributive functors*, To appear.
13. C. Barwick and J. Shah, *Fibrations in ∞-category theory*, Preprint arXiv:1607.04343, July 2016.
14. A. J. Blumberg, *Continuous functors as a model for the equivariant stable homotopy category*, Algebr. Geom. Topol. **6** (2006), 2257–2295. MR 2286026 (2008a:55006)
15. A. J. Blumberg and M. A. Hill, *G-symmetric monoidal categories of modules over equivariant commutative ring spectra*, Preprint arXiv:1511.07363, November 2015.
16. B. Chorny, *Homotopy theory of relative simplicial presheaves*, Israel J. Math. **205** (2015), no. 1, 471–484. MR 3314596
17. B. Chorny and W. G. Dwyer, *Homotopy theory of small diagrams over large categories*, Forum Math. **21** (2009), no. 2, 167–179. MR 2503302
18. D. Clausen, A. Mathew, N. Naumann, and J. Noel, *Descent in algebraic K-theory and a conjecture of Ausoni–Rognes*, Preprint arXiv:1606.03328, June 2016.
19. E. Dror Farjoun, *Homotopy theories for diagrams of spaces*, Proc. Amer. Math. Soc. **101** (1987), no. 1, 181–189. MR 897092
20. W. G. Dwyer and D. M. Kan, *Singular functors and realization functors*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no. 2, 147–153. MR 749528
21. S. Glasman, *Goodwillie calculus and Mackey functors*, To appear.

22. M. A. Hill and M. J. Hopkins, *Equivariant symmetric monoidal structures*, Preprint from the web page of the first author.
23. M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262. MR 3505179
24. J. Lurie, *Higher algebra*, Preprint from the web page of the author, August 2012.
25. J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs, vol. 132, American Mathematical Society, Providence, RI, 2006. MR 2271789
26. D. Nardin, *Stability with respect to an orbital* ∞*-category*, To appear.
27. S. Schwede, *Global homotopy theory*, Preprint from the web page of the author.
28. J. Shah, *Parametrized homotopy limits and colimits*, To appear.
29. Michael Weiss, *Orthogonal calculus*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 3743–3796. MR 1321590

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Stratified categories, geometric fixed points and a generalized Arone-Ching theorem

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Abstract

We develop a theory of Mackey functors on epiorbital categories which simultaneously generalizes the theory of genuine G-spectra for a finite group G and the theory of n-excisive functors on the category of spectra. Using a new theory of stratifications of a stable ∞-category along a finite poset, we prove a simultaneous generalization of two reconstruction theorems: one by Abram and Kriz on recovering G-spectra from structure on their geometric fixed point spectra for abelian G, and one by Arone and Ching that recovers an n-excisive functor from structure on its derivatives. We deduce a strong splitting theorem for $K(n)$-local G-spectra and reprove a theorem of Kuhn on the $K(n)$-local splitting of Taylor towers.

1 Introduction

Equivariant stable homotopy theory has become notorious for its profusion of fixed point functors. The most superficially arcane of these, the *geometric fixed points* first defined in [LMS86, §II.9], also have the best formal properties. Let G be a finite group and H a subgroup; let $\mathbf{Sp}^G$ be the ∞-category of genuine G-spectra. Then the geometric H-fixed point spectrum, which we will regard as a functor

$$\Phi^H : \mathbf{Sp}^G \to \mathbf{Sp},$$

is uniquely determined by the following properties:

1. Φ^H commutes with all homotopy colimits, and
2. Φ^H is compatible with the suspension spectrum functor in that the diagram of functors

$$\begin{array}{ccc} \mathbf{Top}^G & \xrightarrow{\Sigma^\infty_+} & \mathbf{Sp}^G \\ \downarrow{\scriptstyle (-)^H} & & \downarrow{\scriptstyle \Phi^H} \\ \mathbf{Top} & \xrightarrow{\Sigma^\infty_+} & \mathbf{Sp} \end{array}$$

commutes.

The following additional pleasant properties follow:

3. Φ^H has a natural enhancement to a symmetric monoidal functor.

4. (Geometric fixed point Whitehead theorem) If E is a G-spectrum such that $\Phi^H E$ is contractible for *all* subgroups H of G, then E itself is contractible.

This last property strongly suggests that it might be possible to present G-spectra as diagrams of their geometric fixed point spectra, much as the spectral Mackey functor approach of [GM11] and [Bar14] employs the genuine fixed point spectra E^H. Such a presentation would obviously be desirable, since there are many spectra - for example, those arising as norms in the sense of [HHR09, §B.5] - whose geometric fixed points are much more accessible than their genuine fixed points. Some subtlety turns out to be required in this. For example, already for $G = C_2$, it's easy to see that the spaces of natural transformations in both directions between Φ^G and the underlying spectrum, or identity fixed point, functor $\Phi^{\{e\}}$ are contractible. Nevertheless, it can be done, and in this paper, we give a construction that accomplishes this and significantly more, drawing together work of Abram and Kriz [AK13] and Arone and Ching [AC15], which itself generalizes previous work of Bauer and McCarthy [BM03]. We stress that this presentation gives a completely new model of G-spectra in which the fundamental data are geometric fixed point spectra, their homotopical actions by groups derived from G, and gluing data relating these. To motivate this generalization, we'll draw attention to an analogy between equivariant stable homotopy theory and functor calculus, which was first pointed out to us by Mike Hopkins. For $G = C_2$ and a G-spectrum E, we have a natural cofiber sequence, the *norm cofibration sequence*

$$E_{hG} \to E^G \to \Phi^G E.$$

On the other hand, suppose that $F : \mathbf{Sp} \to \mathbf{Sp}$ is a reduced 2-excisive functor in the sense of Goodwillie [Goo91]. Then the Taylor tower of F is simply a cofiber sequence of functors

$$D_2 F \to F \to P_1 F$$

where $D_2 F$ is the 2-homogeneous part of F and $P_1 F$ is the 1-excisive approximation. Fixing a spectrum X, let's specialize further to the case where E is the indexed smash product $X^{\wedge C_2}$ and F is the functor

$$W : \mathbf{Sp} \to \mathbf{Sp}, \quad W(T) = (T^{\wedge C_2})^{C_2},$$

evaluated on X. Then the norm cofibration sequence and the Taylor tower become equivalent cofiber sequences

$$(X^{\wedge C_2})_{hC_2} \to (X^{\wedge C_2})^{C_2} \to X.$$

This equivalence can be made into the basis for an equivalence of ∞-categories

$$\mathbf{Sp}^{C_2} \simeq \mathrm{Fun}^{\text{2-exc}}(\mathbf{Sp}, \mathbf{Sp})$$

between $\mathbf{Sp}^{C_2}$ and the category of reduced 2-excisive functors $\mathbf{Sp} \to \mathbf{Sp}$ under which the identity fixed points correspond to the second derivative and the geometric C_2-fixed points correspond to the first derivative.

This coincidence suggests the existence of a systematic analogy between Goodwillie derivatives and geometric fixed points. This paper develops a common context for the Goodwillie calculus of functors between stable ∞-categories and equivariant stable homotopy theory - that of *Mackey functors on epiorbital categories* - which makes this analogy precise. In brief, n-excisive functors are governed by the category $\mathcal{F}_s^{\leq n}$ of finite sets of cardinality at most n and surjective maps in precisely the same way as G-spectra are governed by the category $\mathcal{O}_G$ of transitive G-sets, and the visible equivalence of categories between $\mathcal{F}_s^{\leq 2}$ and $\mathcal{O}_{C_2}$ accounts for the equivalence between $\mathbf{Sp}^{C_2}$ and $\mathrm{Fun}^{\text{2-exc}}(\mathbf{Sp}, \mathbf{Sp})$.

A beautiful presentation of n-excisive functors on spectra via structure on their derivatives has been constructed by Arone and Ching in [AC15], and we extend their result to our more general context, where it provides the desired presentation of the ∞-category of G-spectra. Along the way, we develop a formalism of *stratified stable ∞-categories* that encompasses, on the one hand, the category of Mackey functors on a epiorbital category, and on the other hand, monoidal stable ∞-categories equipped with a family of homological localizations, such as the category of p-local spectra with its chromatic filtration. Our theorem will be a special case of a general reconstruction theorem for objects of stratified stable ∞-categories. In the context of families of homological localizations, similar results have been obtained by Antolín-Camarena and Barthel [ACB14]. In upcoming work, we plan to use this presentation to give an explicit and homotopy-invariant description of the Hill-Hopkins-Ravenel norm.

The structure of the paper is as follows. In Section 2, we develop the theory of epiorbital categories and their Mackey functors, culminating in the statement and proof of our version, Theorem 2.38, of the Arone-Ching comonadicity theorem [AC15, Theorem 3.13]. In Section 3, we define stratified stable ∞-categories (Definition 3.5) and give several examples, then state and prove the classification of objects of a stratified stable ∞-category (Theorem 3.21), in effect giving a description of the category of coalgebras for the comonad of Section 2. We unpack the implications of our theorem for C_p-spectra in detail in Examples 3.29, obtaining the classical description of C_p-spectra via the Tate fracture square. Finally, in the short Section 4, we prove a strong and general splitting theorem reminiscent of the tom Dieck splitting, Theorem 4.2, for Mackey functors valued in $K(n)$-local spectra, recovering a result of Kuhn [Kuh04] on functor calculus. We believe the G-spectrum case of this theorem to be new.

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2 Epiorbital categories and Mackey functors

Our Arone-Ching theorem will be applicable to a certain class of "generalized equivariant homotopy theories" that encompasses the usual theory of genuine G-spectra and the theory of n-excisive functors on spectra. Such a theory springs from a "epiorbital category" with properties similar to the category of orbits for a finite group G.

Definition 2.1. A *epiorbital category* or EOC is an essentially finite category $\mathcal{M}$ satisfying the following condtions:

- Every morphism in $\mathcal{M}$ is an epimorphism.
- $\mathcal{M}$ admits pushouts and coequalizers; equivalently, $\mathcal{M}$ admits colimits over finite connected diagrams.

Epiorbital categories have a strong directionality.

Definition 2.2. It follows immediately from Yoneda's lemma that any endomorphism in a epiorbital category is an isomorphism, and so the set of isomorphism classes of objects of $\mathcal{M}$ carries a natural partial order wherein $[X] \geq [Y]$ if and only if the morphism set $\mathcal{M}(X, Y)$ is nonempty. We'll call this poset $\mathcal{P}_{\mathcal{M}}$.

Example 2.3. Let G be a finite group. Then the *orbit category* $\mathcal{O}_G$, which is defined as the category of sets with transitive G-action, is a EOC.

Example 2.4. If G is a finite group and H a subgroup, then the full subcategory

$$\mathcal{O}_{G/H} \subseteq \mathcal{O}_G$$

spanned by those orbits on which H acts trivially is again a EOC. There's no clash of notation here, for if H happens to be normal, then $\mathcal{O}_{G/H}$ is clearly just the orbit category of the group G/H.

Example 2.5. Let $\mathcal{F}_s$ be the category of finite sets and surjective maps and let $\mathcal{F}_s^{\leq n}$ be the full subcategory of $\mathcal{F}_s$ spanned by the sets of cardinality at most n. Then $\mathcal{F}_s^{\leq n}$ is a EOC.

The following properties of epiorbital categories will be useful:

Lemma 2.6. Let $\mathcal{M}$ be a EOC, let $\mathcal{K}$ be any finite category and let $\rho : \mathcal{K} \to \mathcal{M}$ be a functor. Then the overcategory $\mathcal{M}_{/\rho}$ is a EOC.

Proof. Immediate. □

Lemma 2.7. Let $\mathcal{M}$ be a EOC. Then any connected component of $\mathcal{M}$ has a final object.

Proof. We'll show that any object of $\mathcal{M}$ whose isomorphism class is minimal with respect to the natural partial order is final in its connected component. Indeed, let T be such an object and let X an object in its connected component. Then there's a zigzag of morphisms

$$T \overset{f_1}{\leftarrow} Y_1 \overset{g_1}{\rightarrow} Y_2 \overset{f_2}{\leftarrow} \cdots \overset{f_n}{\leftarrow} X.$$

By taking iterated pushouts, we can inductively replace this zigzag with a diagram

$$T \overset{f}{\rightarrow} Y \overset{g}{\leftarrow} X.$$

Moreover, by the choice of T, f must be an isomorphism, so there is a morphism from X to T. Now suppose we have a pair of morphisms

$$f, g : X \rightrightarrows T.$$

Then h, the coequalizer of f and g, is a morphism with source T, and therefore an isomorphism. By composing with h^{-1}, we see that $f = g$. □

If $\mathcal{C}$ is any category, we'll denote by $\mathcal{C}^{\amalg}$ the closure of $\mathcal{C}$ under formal finite coproducts. By definition, $\mathcal{C}^{\amalg}$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$ spanned by the finite coproducts of representable functors. This is an operation we'll frequently want to perform on EOCs.

Example 2.8. $\mathcal{O}_G^{\amalg}$ is equivalent to $\mathcal{F}^G$, the category of all finite G-sets, since any finite G-set decomposes uniquely into orbits.

One very relevant property of $\mathcal{F}^G$ is that it's meaningful to take Mackey functors over it: $\mathcal{F}^G$ is *disjunctive* in the sense of [Bar14], so its effective Burnside ∞-category $A^{eff}(\mathcal{F}^G)$ can be formed and admits direct sums. We'll quickly recall these ideas.

Definition 2.9. Let $\mathbf{C}$ be an ∞-category which admits finite products and coproducts and a zero object. We'll say that $\mathbf{C}$ *admits direct sums*, or is *semiadditive*, if for every pair of objects $X, Y \in \mathbf{C}$, the natural map

$$X \amalg Y \to X \times Y$$

provided by the zero object is an equivalence.

Remark 2.10. It's not hard to see that each mapping space of a semiadditive ∞-category naturally carries the structure of a commutative monoid space; the term *additive* ∞-category is traditionally reserved for categories whose mapping spaces are grouplike.

Definition 2.11. An ∞-category $\mathbf{C}$ is called *disjunctive* if

- $\mathbf{C}$ admits pullbacks and finite coproducts,

- for each finite set I and for each I-tuple $(X_i)_{i\in I}$ of objects of $\mathbf{C}$, the natural functor

$$\prod_i \mathbf{C}_{/X_i} \to \mathbf{C}_{/\coprod_i X_i}$$

  is an equivalence of categories.

Let $\mathbf{C}$ be an ∞-category which admits pullbacks. Then one can construct [Bar14, 3.6] an ∞-category $A^{eff}(\mathbf{C})$, called the *effective Burnside category of* $\mathbf{C}$, whose objects are those of $\mathbf{C}$, whose morphisms are spans

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

in $\mathbf{C}$, and where composition is performed by forming pullbacks.

Proposition 2.12. [Bar14, 4.3] If $\mathbf{C}$ is disjunctive, then $A^{eff}(\mathbf{C})$ is semiadditive.

$\square$

Example 2.13. A functor from $A^{eff}(\mathcal{F}^G)$ to $\mathbf{Ab}$ which preserves direct sums is precisely a Mackey functor in the sense of [tD73].

The content of the next lemma is that it's meaningful to talk about Mackey functors over arbitrary epiorbital categories:

Lemma 2.14. Let $\mathcal{M}$ be a epiorbital category. Then $\mathcal{M}^{\amalg}$ is disjunctive.

Proof. The condition that

$$\prod_i \mathcal{C}_{/X_i} \to \mathcal{C}_{/\coprod_i X_i}$$

is an equivalence is satisfied for any $\mathcal{C}$ of the form $\mathcal{D}^{\amalg}$, so we only need to show that $\mathcal{M}^{\amalg}$ admits pullbacks.

Let $\rho : \Lambda^2_2 \to \mathcal{M}^{\amalg}$ be a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in $\mathcal{M}^{\amalg}$. If any of X, Y or Z are empty then the pullback exists and is empty, so let's assume all are nonempty.

If Z decomposes nontrivially as a coproduct $Z_1 \coprod Z_2$, we get a decomposition of diagrams $\rho = \rho_1 \coprod \rho_2$, and if each ρ_i admits a limit W_i, then $W_1 \coprod W_2$ is a limit of ρ. Thus it suffices to assume Z is representable.

On the other hand, if X decomposes nontrivially as $X_1 \coprod X_2$, and if

$$\begin{array}{ccc} W_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z, \end{array} \qquad \begin{array}{ccc} W_2 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

are pullback diagrams, then

$$\begin{array}{ccc} W_1 \coprod W_2 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a pullback diagram. After carrying out the same argument for Y, it's enough to assume that X, Y and Z are all representable - in other words, that ρ may be lifted to a diagram $\widetilde{\rho} : \Lambda^2_2 \to \mathcal{M}$.

But it now follows from Lemma 2.6 and Lemma 2.7 that ρ admits a limit, since to give a limit of ρ in $\mathcal{M}^{\amalg}$ is, tautologically, to give a final object in each connected component of the EOC $\mathcal{M}_{/\widetilde{\rho}}$. □

Definition 2.15. There is a more general notion of *orbital ∞-category* which features centrally in the upcoming work [BDG+16]. An orbital ∞-category is simply any ∞-category $\mathcal{M}$ for which $\mathcal{M}^{\amalg}$ admits pullbacks, from which it follows that $\mathcal{M}^{\amalg}$ is disjunctive. By Lemma 2.14, epiorbital categories are orbital. Epiorbital categories are the focus of this paper, but some results will be stated for general orbital ∞-categories.

Definition 2.16. If $\mathcal{M}$ is an orbital ∞-category, we'll write $A^{eff}(\mathcal{M})$ for the effective Burnside category $A^{eff}(\mathcal{M}^{\amalg})$; we don't expect this notation to cause confusion. $A^{eff}(\mathcal{M})$ is semiadditive, and if $\mathbf{C}$ is a semiadditive presentable ∞-category, then we'll denote by $\mathbf{Mack}(\mathcal{M}, \mathbf{C})$ the category of $\mathbf{C}$-valued Mackey functors on $\mathcal{M}$: the category of additive (i.e. direct-sum-preserving) functors from $\mathcal{M}$ to $\mathbf{C}$. If $\mathbf{C} = \mathbf{Sp}$, then we'll usually omit $\mathbf{C}$ and refer to the category simply as $\mathbf{Mack}(\mathcal{M})$.

Example 2.17. When $\mathcal{M}$ is the orbit category $\mathcal{O}_G$, $\mathbf{Mack}(\mathcal{M})$ is the category of *spectral Mackey functors* for G [Bar14], which is a model for the homotopy theory of genuine G-spectra.

Example 2.18. The natural target of both the genuine fixed point functor $(-)^H$ and the geometric fixed point functor Φ^H (of which more anon) on $\mathbf{Mack}(\mathcal{O}_G)$ is $\mathbf{Mack}(\mathcal{O}_{G/H})$ (Example 2.4), even when H is not normal in G.

Theorem 2.19. When $\mathcal{M}$ is the category $\mathcal{F}_s^{\leq n}$ of Example 2.5, $\mathbf{Mack}(\mathcal{M}, \mathbf{C})$ is equivalent to the category of (reduced, filtered-colimit-preserving) n-excisive functors from $\mathbf{Sp}$ to $\mathbf{C}$. This equivalence has the property that if $F : \mathbf{Sp} \to \mathbf{C}$ is n-excisive, if E is the corresponding Mackey functor and if S is a set, then

$E(S)$ is equivalent to the S-indexed cross-effect of F evaluated on an S-indexed set of spheres. In particular, if S has n-elements, then $E(S)$ is equivalent to the nth derivative $\mathbb{D}_n F$ as spectra with Σ_n-action.

This equivalence is the subject of the separate paper [Gla16].

The following is a significant technical lemma that provides control over the values of many universally defined Mackey functors, including the fixed points of the free genuine equivariant G-spectrum on a spectrum with G-action.

Lemma 2.20. Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are semiadditive ∞-categories and $\phi : \mathbf{A} \to \mathbf{B}$, $F : \mathbf{A} \to \mathbf{C}$ are additive functors. Suppose the left Kan extension $\phi_! F : \mathbf{B} \to \mathbf{C}$ exists. Then $\phi_! F$ is additive.

Proof. We must verify that $\phi_! F$ preserves zero objects and direct sums of pairs of objects. The first is obvious, so let X, Y be objects of $\mathbf{B}$. Then

$$\phi_! F(X \oplus Y) \simeq \operatorname*{colim}_{(\phi(Z) \to X \oplus Y) \in \mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/X \oplus Y}} F(Z).$$

Let

$$a : (\mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/X}) \times (\mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/Y}) \to \mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/X \oplus Y}$$

be the functor with

$$a(\phi(Z_1) \to X, \phi(Z_2) \to Y) = (\phi(Z_1 \oplus Z_2) \simeq \phi(Z_1) \oplus \phi(Z_2) \to X \oplus Y).$$

Then we claim that a is cofinal. Thus we must verify that for each object $k : \phi(Z) \to X \oplus Y$ of $\mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/X \oplus Y}$, the overcategory

$$\mathbf{O} := (\mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/X}) \times (\mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/Y}) \times_{\mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/X \oplus Y}} (\mathbf{A} \times_{\mathbf{B}} \mathbf{B}_{/X \oplus Y})_{/k}$$

is weakly contractible. Indeed, we claim that $\mathbf{O}$ has an initial object. An object of $\mathbf{O}$ is a pair (Z_1, Z_2) of objects of $\mathbf{A}$ together with a morphism $\delta : Z \to Z_1 \oplus Z_2$ and a commutative diagram

$$\begin{array}{ccc} \phi(Z) & \xrightarrow{k} & X \oplus Y \\ \downarrow{\scriptstyle \phi(\delta)} & \nearrow & \\ \phi(Z_1 \oplus Z_2). & & \end{array}$$

Then the initial object of $\mathbf{O}$ is evidently the diagonal map $\Delta : Z \to Z \oplus Z$ together with the commutative diagram

$$\begin{array}{ccc} \phi(Z) & \xrightarrow{k} & X \oplus Y \\ \downarrow{\scriptstyle \phi(\Delta)} & \nearrow_{k_X \oplus k_Y} & \\ \phi(Z \oplus Z). & & \end{array}$$

Now there is a commutative diagram

$$\begin{array}{ccc} (\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X})\times(\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/Y}) & \xrightarrow{(F\oplus F)} & \mathbf{C} \\ \downarrow a & \nearrow F & \\ \mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X\oplus Y}, & & \end{array}$$

allowing us, by our cofinality result, to rewrite

$$\phi_! F(X\oplus Y)\simeq \underset{(\phi(Z_1)\to X,\phi(Z_2)\to Y)\in(\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X})\times(\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/Y})}{\operatorname{colim}} F(Z_1)\oplus F(Z_2).$$

Now for any pair of functors $b_1, b_2 : K \to \mathbf{C}$, we have the commutation of colimits

$$\operatorname{colim}\,(b_1\oplus b_2)\simeq \operatorname{colim}\,(b_1)\oplus\operatorname{colim}\,(b_2).$$

In particular, if b_2 is the constant functor valued at some object P, then

$$\operatorname{colim}\,(b_1\oplus b_2)\simeq \operatorname{colim}\,(b_1)\oplus(P\otimes K),$$

and if K is weakly contractible, then

$$\operatorname{colim}\,(b_1\oplus b_2)\simeq \operatorname{colim}\,(b_1)\oplus P.$$

Note that $\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X}$ is weakly contractible for every $X\in\mathbf{B}$: indeed, the essentially unique object

$$(\mathbf{0}_A, \phi(\mathbf{0}_A)\simeq \mathbf{0}_B\to X)$$

is an initial object of $\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X}$. Hence

$$\begin{aligned} \phi_! F(X\oplus Y) &\simeq \underset{(\phi(Z_1)\to X,\phi(Z_2)\to Y)\in(\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X})\times(\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/Y})}{\operatorname{colim}} F(Z_1)\oplus F(Z_2) \\ &\simeq \underset{(\phi(Z_1)\to X)\in\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X}}{\operatorname{colim}}\ \underset{(\phi(Z_2)\to Y)\in\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/Y}}{\operatorname{colim}} F(Z_1)\oplus F(Z_2) \\ &\simeq \underset{(\phi(Z_1)\to X)\in\mathbf{A}\times_{\mathbf{B}}\mathbf{B}_{/X}}{\operatorname{colim}} F(Z_1)\oplus\phi_! F(Y) \\ &\simeq \phi_! F(X)\oplus\phi_! F(Y). \end{aligned}$$

□

If $\mathbf{I}$ and $\mathbf{J}$ are disjunctive ∞-categories and $F : \mathbf{I}\to\mathbf{J}$ is a functor which preserves pullbacks and finite coproducts, then F clearly induces an additive functor

$$A^{eff}(F) : A^{eff}(\mathbf{I})\to A^{eff}(\mathbf{J}).$$

So if $\mathbf{C}$ admits limits and colimits, we get functors

$$A^{eff}(F)_!, A^{eff}(F)_* : \mathbf{Mack}(\mathbf{I},\mathbf{C})\to\mathbf{Mack}(\mathbf{J},\mathbf{C}).$$

If $\mathcal{M}$ is an orbital ∞-category, we'll call a full subcategory $\mathcal{N}$ of $\mathcal{M}$ *downwardly-closed* if whenever $X \in \mathcal{N}$ and $Y \in \mathcal{M}$ with $\mathrm{Map}(X, Y)$ nonempty, we also have $Y \in \mathcal{N}$. Equivalently, $\mathcal{N} = \phi^{-1}(\{1\})$ for some functor $\phi : \mathcal{M} \to \Delta^1$. Upwardly-closed subcategories are defined dually.

There are plenty of examples of these: if $\mathcal{M}$ is epiorbital, then for any $X \in \mathcal{M}$, the full subcategory of $\mathcal{M}_{\leq X} \subseteq \mathcal{M}$ spanned by those objects Y with $[Y] \leq [X]$ is downwardly-closed. Note also that any downwardly-closed subcategory of an orbital ∞-category is itself orbital, and any downwardly-closed subcategory of an EOC is itself a EOC.

For the following few lemmas, we'll let $\mathcal{M}$ be orbital, let $\mathcal{N}$ be a downwardly-closed subcategory of $\mathcal{M}$ and let $\mathcal{T}$ be its upwardly-closed complement.

Lemma 2.21. The inclusion $i_{\mathcal{N}} : \mathcal{N}_{\amalg} \to \mathcal{M}_{\amalg}$ admits a canonical retraction $j_{\mathcal{N}}$ which is right adjoint to $i_{\mathcal{N}}$.

Proof. By definition, $\mathcal{M}_{\amalg}$ is the full subcategory of $\mathrm{Fun}(\mathcal{M}^{\mathrm{op}}, \mathbf{Top})$ spanned by coproducts of representables, and $i_{\mathcal{N}}$ is given by left Kan extension. But since the value of the functor represented by an object of $\mathcal{T}$ on any object of $\mathcal{N}$ is empty, the restriction $\mathrm{Fun}(\mathcal{M}^{\mathrm{op}}, \mathcal{S}et)$ to $\mathrm{Fun}(\mathcal{N}^{\mathrm{op}}, \mathcal{S}et)$ preserves coproducts of representables. This is the desired retraction. (Note that orbitality of $\mathcal{M}$ is not required for this lemma to hold.) $\square$

We should think of $j_{\mathcal{N}}$ as "formally set all objects of $\mathcal{T}$ to $\emptyset$". We note that $j_{\mathcal{N}}$ preserves coproducts.

Definition 2.22. We define the *geometric value at* $\mathcal{N}$ functor

$$\Phi^{\mathcal{N}} : \mathbf{Mack}(\mathcal{M}, \mathbf{C}) \to \mathbf{Mack}(\mathcal{N}, \mathbf{C})$$

by the left Kan extension $(A^{eff}(j_{\mathcal{N}}))_!$.

The right adjoint of $\Phi^{\mathcal{N}}$, the *extension by zero from* $\mathcal{N}$ functor, will be denoted $\Xi^{\mathcal{N}}$. For an object X of $\mathcal{M}$, we'll denote $\Phi^{\mathcal{M}_{\leq X}}$ by Φ^X. If E is an object of $\mathbf{Mack}(\mathcal{M}, \mathbf{C})$, then we'll write $E^{\Phi X}$ for the value of $\Phi^X E$ on X, the "geometric fixed point spectrum at X".

Example 2.23. Let G be a finite group and let $H \leq G$ be a subgroup. If $\mathcal{M} = \mathcal{O}_G$, then $\Phi^{G/H}$ is the classical functor of geometric fixed points [Bar14, Example B.6]. Here the usual notation would be Φ^H, not $\Phi^{G/H}$, and we apologize for the clash.

We have an equivalence

$$(\mathcal{O}_G)_{\leq G/H} \simeq \mathcal{O}_{G/H}$$

and so $\Phi^{G/H}$ naturally takes values in $\mathbf{Mack}(\mathcal{O}_{G/H})$, as previously claimed.

Example 2.24. If $\mathcal{M} = \mathcal{F}_s^{\leq n}$, and $k < n$, then $\Xi^{\mathcal{F}_s^{\leq k}}$ is the functor which regards a k-excisive functor as an n-excisive functor, and its left adjoint $\Phi^{\mathcal{F}_s^{\leq k}}$ is the k-excisive approximation functor.

Definition 2.25. Using the notation of Lemma 2.21, let $A^{eff}(\mathcal{T})$ be the effective Burnside category of $(\mathcal{T})^{\amalg}$, or equivalently, the full subcategory of $A^{eff}(\mathcal{M})$ spanned by the objects of $(\mathcal{T})^{\amalg}$. Let $A^{eff}(i_{\mathcal{T}})$ be the inclusion of this full subcategory.

Remark 2.26. Let $\mathcal{G}$ be a groupoid. We can form the effective Burnside category $A^{eff}(\mathcal{G})$, since any commutative square in a groupoid is a pullback square. Moreover, there's a natural equivalence of ∞-categories

$$c_{\mathcal{G}} : A^{eff}(\mathcal{G}) \xrightarrow{\sim} \mathcal{G}$$

which maps the span

$$x \xleftarrow{g} y \xrightarrow{h} z$$

to the morphism $hg^{-1} : x \to z$. We'll sometimes implicitly invoke this equivalence.

If X is an object of $\mathcal{M}$, let $\mathcal{G}_X$ be the groupoid spanned by the isomorphism class of X. Form the effective Burnside category $A^{eff}(\mathcal{G}_X^{\amalg})$ and let $i_X : A^{eff}(\mathcal{G}_X) \to A^{eff}(\mathcal{M}_{\leq X})$ be the inclusion. It follows from the directionality of $\mathcal{M}_{\leq X}$ that i_X is fully faithful.

Since there's no room for meaningful transfer maps, we might guess that a Mackey functor on a group $\mathcal{G}$ contains no more information than an object with $\mathcal{G}$-action. This is indeed the case, but the proof is technical and so we defer the bulk of it to Appendix A.

Theorem 2.27. Let $\mathcal{G}$ be a groupoid. Then $A^{eff}(\mathcal{G}^{\amalg})$ is the free semiadditive ∞-category on $\mathcal{G}$: for any semiadditive ∞-category $\mathbf{C}$, the natural inclusion induces an equivalence of categories

$$\mathrm{Fun}^{\oplus}(A^{eff}(\mathcal{G}^{\amalg}), \mathbf{C}) \to \mathrm{Fun}(\mathcal{G}, \mathbf{C}).$$

We'll use this equivalence implicitly from now on.

The class of Mackey functors left or right Kan extended from groupoids is an interesting one. For instance, let G be a finite group and let X be a spectrum with G-action, which by Theorem 2.27 we may regard as an object of $\mathbf{Mack}(\mathcal{G}_{G/e})$. Denote by $i : \mathcal{G}_{G/e}^{\amalg} \hookrightarrow \mathcal{F}_G$ the inclusion. Then the left Kan extension of X along $A^{eff}(i)$ is the *free genuine G-spectrum on X*, often denoted by

$$EG_+ \wedge X.$$

Similarly, the right Kan extension is the *cofree genuine G-spectrum on X*, also known as

$$F(EG_+, X).$$

We know from other models of G-spectra that for any subgroup $H \leq G$,

$$(EG_+ \wedge X)^H \simeq X_{hH},\ F(EG_+, X)^H \simeq X^{hH}.$$

It would be desirable, however, to have a proof of these facts internal to our framework. The following lemma is a more general version of this result:

Lemma 2.28. Suppose $\mathcal{M}$ is an orbital ∞-category and $i : \mathcal{T} \hookrightarrow \mathcal{M}$ is the inclusion of an upwardly closed subcategory. Let $\mathbf{C}$ be a semiadditive ∞-category with all colimits, and let $B \in \mathbf{Mack}(\mathcal{T}, \mathbf{C})$ be a Mackey functor. Then for any $Y \in \mathcal{M}$,

$$i_! B(Y) \simeq \operatorname*{colim}_{\mathcal{M}^{\amalg}_{/Y} \times_{\mathcal{M}^{\amalg}} \mathcal{T}^{\amalg}} B.$$

Proof. For now, we content ourselves with a sketched proof and leave the details as an exercise. This result will be assumed in some examples in Section 3 but will not feature integrally in the results of the paper.

For the sake of avoiding the ambiguity that can precipitate from the use of overcategory notation, we clarify our notation. Write

$$\mathcal{T}^{\amalg}_{/Y} = \mathcal{M}^{\amalg}_{/Y} \times_{\mathcal{M}^{\amalg}} \mathcal{T}^{\amalg}$$

and

$$(A^{eff}(\mathcal{T}))_{/Y} = (A^{eff}(\mathcal{M}))_{/Y} \times_{A^{eff}(\mathcal{M})} A^{eff}(\mathcal{T}).$$

Then we have

$$i_! B(Y) \simeq \operatorname*{colim}_{(A^{eff}(\mathcal{T}))_{/Y}} B.$$

The objects of $(A^{eff}(\mathcal{T}))_{/Y}$ are of the form

$$\begin{array}{ccc} & T' & \\ {\scriptstyle f}\swarrow & & \searrow \\ T & & Y. \end{array}$$

We can define a functor $V : \mathcal{T}^{\amalg}_{/Y} \to (A^{eff}(\mathcal{T}))_{/Y}$ by

$$V(T \to Y) = \left(\begin{array}{ccc} & T & \\ /\!\!/ & & \searrow \\ T & & Y \end{array} \right).$$

The essential image of V comprises those diagrams for which $f : T' \to T$ is an equivalence, and it can be shown that V is homotopic to the inclusion of the full subcategory $(\overline{A^{eff}(\mathcal{M})})_{/Y}$ spanned by such diagrams. Moreover, the inclusion of $(\overline{A^{eff}(\mathcal{M})})_{/Y}$ has a left adjoint ϕ with

$$\phi \left(\begin{array}{ccc} & T' & \\ \swarrow & & \searrow \\ T & & Y \end{array} \right) \simeq \left(\begin{array}{ccc} & T' & \\ /\!\!/ & & \searrow \\ T' & & Y \end{array} \right).$$

Thus this inclusion is cofinal, which yields the result. $\square$

Corollary 2.29. Suppose $\mathcal{M}$ is an epiorbital category and $X \in \mathcal{M}$ is a maximal object. Let

$$i_X : A^{eff}(\mathcal{G}_X) \to A^{eff}(\mathcal{M})$$

be the inclusion, let $\mathbf{C}$ be a semiadditive ∞-category with all colimits, and let $B \in \mathbf{Mack}(\mathcal{G}_X, \mathbf{C})$ be a Mackey functor, which by Theorem 2.27 is uniquely determined by $B(X)$ regarded as an object of $\mathbf{C}$ with $\mathrm{Aut}(X)$-action. Then for any $Y \in \mathcal{M}$,

$$(i_X)_! B(Y) \simeq (B(X) \times \mathrm{Hom}_{\mathcal{M}}(X, Y))_{h\mathrm{Aut}(X)}.$$

In particular, if Y is a final object,

$$(i_X)_! B(Y) \simeq B(X)_{h\mathrm{Aut}(X)}.$$

□

Definition 2.30. Let $\mathcal{M}$ be an EOC and $X \in \mathcal{M}$ an object. The *Taylor coefficient at* X functor is the functor $D^X : \mathbf{Mack}(\mathcal{M}, \mathbf{C}) \to \mathrm{Fun}(\mathcal{G}_X, \mathbf{C})$ given by $i_X^* \circ \Phi^X$.

Definition 2.31. Let $\mathcal{M}^\sim$ be the maximal subgroupoid of $\mathcal{M}$ and define the *Taylor sequence* functor

$$D : \mathbf{Mack}(\mathcal{M}, \mathbf{C}) \to \mathrm{Fun}(\mathcal{M}^\sim, \mathbf{C})$$

by

$$\bigvee_{[X]} D^X.$$

We'll now enforce the hypothesis that $\mathbf{C}$ is stable for the remainder of the section. This allows us to state the following important proposition, which is a generalization of the "norm cofibration sequence" from equivariant stable homotopy theory:

Theorem 2.32. Let $\mathcal{N}$ be a downwardly closed subcategory of an orbital ∞-category $\mathcal{M}$ and $\mathcal{T}$ its upwardly closed complement. Denote the restriction $A^{eff}(i_{\mathcal{T}})^*$ and the left Kan extension $A^{eff}(i_{\mathcal{T}})_!$ respectively by $\Pi^{\mathcal{T}}$ and $\Gamma^{\mathcal{T}}$. There's a cofiber sequence of functors $\mathbf{Mack}(\mathcal{M}, \mathbf{C}) \to \mathbf{Mack}(\mathcal{M}, \mathbf{C})$

$$\Gamma^{\mathcal{T}} \Pi^{\mathcal{T}} \xrightarrow{\epsilon} \mathrm{Id} \xrightarrow{\eta} \Xi^{\mathcal{N}} \Phi^{\mathcal{N}},$$

where ϵ and η are the counit and unit of their respective adjunctions.

Theorem 2.32 has the following important equivalent form, whose theme is that a Mackey functor contains no secret data not detected by its values:

Corollary 2.33. $\Xi^{\mathcal{N}}$ is fully faithful, and its essential image is the category $\mathbf{Mack}_{\mathcal{N}}(\mathcal{M}, \mathbf{C})$ of Mackey functors on $\mathcal{M}$ supported on $\mathcal{N}$.

Proof. If $M \in \mathbf{Mack}_{\mathcal{N}}(\mathcal{M}, \mathbf{C})$, then evidently $\Pi^{\mathcal{T}} M$ is zero, and so the cofiber sequence of Theorem 2.32 collapses to an equivalence

$$M \simeq \Xi^{\mathcal{N}} \Phi^{\mathcal{N}} M,$$

which establishes the essential image of $\Xi^{\mathcal{N}}$.

Since $\Xi^{\mathcal{N}}$ is visibly conservative, it follows for $N \in \mathbf{Mack}(\mathcal{N}, \mathbf{C})$, setting $M = \Xi^{\mathcal{N}} N$, that

$$N \simeq \Phi^{\mathcal{N}} \Xi^{\mathcal{N}} N,$$

and the full faithfulness of $\Xi^{\mathcal{N}}$ follows by adjunction. □

Remark 2.34. Observe that Corollary 2.33 also implies Theorem 2.32. Indeed, it follows abstractly that the functor

$$M \mapsto \mathrm{cof}(\epsilon_M)$$

is the localization into $\mathbf{Mack}_{\mathcal{N}}(\mathcal{M}, \mathbf{C})$. On the other hand, assuming that $\Xi^{\mathcal{N}}$ is fully faithful, $\Xi^{\mathcal{N}} \Phi^{\mathcal{N}}$ is the localization into its essential image. By Corollary 2.33, these two localizations coincide, yielding 2.32.

Definition 2.35. Let $\mathcal{D}A(\mathcal{M})$ denote the nonabelian derived category of $A^{eff}(\mathcal{M})$: the category of product-preserving functors $A^{eff}(\mathcal{M}) \to \mathbf{Top}$ [Lur09, Definition 5.5.8.8]. Equivalently, $\mathcal{D}A(\mathcal{M})$ is the category of Mackey functors on $\mathcal{M}$ valued in the category **CMon** of commutative monoid spaces (see, for instance, [Gla16, Remark 2.7]).

Let $\mathbf{C}$ and $\mathbf{D}$ be presentable ∞-categories. Recall that the category of coproduct-preserving functors from $\mathbf{C}$ to $\mathbf{D}$ is denoted $\mathrm{Fun}^{\amalg}(\mathbf{C}, \mathbf{D})$, and that the category of functors from $\mathbf{C}$ to $\mathbf{D}$ that preserve all colimits is denoted $\mathrm{Fun}^{L}(\mathbf{C}, \mathbf{D})$. Now for presentable $\mathbf{C}$, by [Lur09, Proposition 5.5.8.15], there's an equivalence of categories

$$\mathrm{Fun}^{L}(\mathcal{D}A(\mathcal{M}), \mathbf{C}) \simeq \mathrm{Fun}^{\amalg}(A^{eff}(\mathcal{M}), \mathbf{C}).$$

If $\mathbf{C}$ is, in addition, semiadditive, this can be written as an equivalence

$$\mathrm{Fun}^{L}(\mathcal{D}A(\mathcal{M}), \mathbf{C}) \simeq \mathbf{Mack}(\mathcal{M}, \mathbf{C}). \qquad (*)$$

Lemma 2.36. Theorem 2.32 holds when $\mathbf{C} = \mathbf{CMon}$.

Before proving Lemma 2.36, let's deduce Theorem 2.32 from Lemma 2.36. We know

$$\Phi^{\mathcal{N}} : \mathcal{D}A(\mathcal{M}) \to \mathcal{D}A(\mathcal{N})$$

is a localization. Under the equivalence of $(*)$, $\Xi^{\mathcal{N}}$ corresponds to

$$(\Phi^{\mathcal{N}})^* : \mathrm{Fun}^{L}(\mathcal{D}A(\mathcal{N}), \mathbf{C}) \to \mathrm{Fun}^{L}(\mathcal{D}A(\mathcal{M}), \mathbf{C}),$$

and by the universal property of a localization, $(\Phi^{\mathcal{N}})^*$ is fully faithful, and objects of its essential image are functors which map $\Phi^{\mathcal{N}}$-equivalences to equivalences.

Now here's where we use the stability of $\mathbf{C}$: since a morphism in a stable ∞-category is an equivalence if and only if its fiber is zero, the objects of the essential image of $(\Phi^{\mathcal{N}})^*$ are equivalently those functors which map objects in the essential image of

$$\Gamma^{\mathcal{T}} : \mathcal{D}A(\mathcal{T}) \to \mathcal{D}A(\mathcal{M})$$

to $0 \in \mathbf{C}$. But since the equivalence $(*)$ is given by the Yoneda embedding, and since the diagram

$$\begin{array}{ccc} A^{eff}(\mathcal{T}) & \xrightarrow{\sim} & \mathcal{D}A(\mathcal{T}) \\ {\scriptstyle A^{eff}(i_{\mathcal{T}})}\downarrow & & \downarrow{\scriptstyle \Gamma^{\mathcal{T}}} \\ A^{eff}(\mathcal{M}) & \xrightarrow{\sim} & \mathcal{D}A(\mathcal{M}) \end{array}$$

commutes, these correspond under $(*)$ to the objects of $\mathbf{Mack}_{\mathcal{N}}(\mathcal{M}, \mathbf{C})$. This proves Corollary 2.33, and therefore Theorem 2.32.

Proof of Lemma 2.36. The proof of this lynchpin lemma is very technical, and so we've relegated it to Appendix B. It's not required reading for those who don't care to learn to make a very specific kind of sausage, but we note that it's our main point of contact with the combinatorics of the effective Burnside category. □

Let $\mathcal{M}$ be a EOC. For each $X \in \mathcal{M}$, we may define functors

$$R^X = \Xi^X \circ (i_X)_*$$

and

$$L^X = \Xi^X \circ (i_X)_!,$$

where $(i_X)_!$ and $(i_X)_*$ are left and right Kan extension respectively. Observe that $(i_X)_!$ and $(i_X)_*$ are fully faithful, since they're Kan extensions along a fully faithful functor. Since we've already seen that Ξ^X is fully faithful, we conclude that both L^X and R^X are fully faithful. Moreover, R^X is right adjoint to the Taylor coefficient functor D^X.

Similarly, we can define

$$R = \bigvee_{[X]} R^X$$

and

$$L = \bigvee_{[X]} L^X$$

and R is right adjoint to the Taylor sequence functor D.

Proposition 2.37. *L is a section of D; that is, $D \circ L$ is homotopic to the identity.*

Proof. We'll use induction on the number of isomorphism classes of objects in $\mathcal{M}$. If $\mathcal{M}$ is a groupoid, both L and D are already the identity. In general, let X be a maximal object of $\mathcal{M}$. It's clear that $D \circ L$ is homotopic to the identity when restricted to $\mathcal{G}_X$.

Then the morphism

$$(i_X)_!(i_X)^* L \to L$$

is equivalent to the summand inclusion

$$L^X \to \bigvee_{[Y]} L^Y$$

and therefore by Theorem 2.32, we have a cofiber sequence

$$L^X \to L \to \Phi^{\mathcal{M}_{<X}} L$$

which shows that the left square in the diagram

$$\begin{array}{ccccc} \mathrm{Fun}(\mathcal{M}^{\sim}, \mathbf{C}) & \xrightarrow{L} & \mathbf{Mack}(\mathcal{M}, \mathbf{C}) & \xrightarrow{D} & \mathrm{Fun}(\mathcal{M}^{\sim}, \mathbf{C}) \\ \downarrow & & \downarrow{\scriptstyle \Phi^{\mathcal{M}_{<X}}} & & \downarrow \\ \mathrm{Fun}(\mathcal{M}^{\sim}_{<X}, \mathbf{C}) & \xrightarrow{L} & \mathbf{Mack}(\mathcal{M}_{<X}, \mathbf{C}) & \xrightarrow{D} & \mathrm{Fun}(\mathcal{M}^{\sim}_{<X}, \mathbf{C}) \end{array}$$

commutes. The right square commutes by construction, and the bottom composite is homotopic to the identity by the induction hypothesis. By circumnavigating the diagram, we conclude that $D \circ L$ is homotopic to the identity when restricted to $\mathcal{G}_Y$ for any $Y \neq X$. □

The following is our Arone-Ching theorem in its general form.

Theorem 2.38. Let $\mathcal{M}$ be an epiorbital category. Then the adjunction (D, R) is comonadic.

Proof. (D, R) is comonadic if and only if the natural transformation

$$t : \mathrm{Id} \to \mathbf{Tot}(\mathrm{Cobar}(R, DR, D))$$

is an equvalence. We'll closely follow Arone and Ching's proof in [AC15]. This involves showing, for each downwardly-closed subcategory $\mathcal{N}$ of $\mathcal{M}$, that the natural map

$$t^{\mathcal{N}} : \Xi^{\mathcal{N}} \Phi^{\mathcal{N}} \to \mathbf{Tot}(\Xi^{\mathcal{N}} \Phi^{\mathcal{N}} \mathrm{Cobar}(R, DR, D))$$

is an equivalence, by induction on the number of isomorphism classes of objects in $\mathcal{N}$. Since $\Xi^{\mathcal{M}} \Phi^{\mathcal{M}}$ is the identity functor, this will give the result.

So assume that $t^{\mathcal{P}}$ is an equivalence for all $\mathcal{P}$ with at most k isomorphism classes of objects, and suppose $\mathcal{N}$ has $k+1$ isomorphism classes of objects. Let X be a maximal object of $\mathcal{N}$ and let $\mathcal{N}' = \mathcal{N} \setminus \mathcal{G}_X$ be the result of removing the isomorphism class of X. The cofiber sequence of Theorem 2.32 gives a cofiber sequence of functors

$$L^X D^X \to \Xi^{\mathcal{N}} \Phi^{\mathcal{N}} \to \Xi^{\mathcal{N}'} \Phi^{\mathcal{N}'},$$

which in turn gives a map of cofiber sequences

$$\begin{array}{ccc}
L^X D^X & \xrightarrow{t^X} & \mathbf{Tot}(L^X D^X \mathrm{Cobar}(R, DR, D)) \\
\downarrow & & \downarrow \\
\Xi^{\mathcal{N}} \Phi^{\mathcal{N}} & \xrightarrow{t^{\mathcal{N}}} & \mathbf{Tot}(\Xi^{\mathcal{N}} \Phi^{\mathcal{N}} \mathrm{Cobar}(R, DR, D)) \\
\downarrow & & \downarrow \\
\Xi^{\mathcal{N}'} \Phi^{\mathcal{N}'} & \xrightarrow{t^{\mathcal{N}'}} & \mathbf{Tot}(\Xi^{\mathcal{N}'} \Phi^{\mathcal{N}'} \mathrm{Cobar}(R, DR, D)).
\end{array}$$

By the induction hypothesis, $t^{\mathcal{N}'}$ is an equivalence, so it'll suffice to show that t^X is an equivalence. This also starts the induction, since $t^{\mathcal{N}} = t^X$ if $\mathcal{N} = \mathcal{G}_X$ is a connected groupoid. But now we observe that

$$D^X = ev_X D$$

and so

$$\begin{aligned}
\mathbf{Tot}(L^X D^X \mathrm{Cobar}(R, DR, D)) &\simeq \mathbf{Tot}(L^X ev_X D \mathrm{Cobar}(R, DR, D)) \\
&\simeq \mathbf{Tot}(L^X ev_X \mathrm{Cobar}(DR, DR, D)) \\
&\simeq L^X ev_X D \\
&\simeq L^X D^X
\end{aligned}$$

by the usual extra codegeneracy argument. This completes the proof. □

The next section aims to characterize the comonad DR.

3 Categories stratified along a poset

The categories $\mathbf{Mack}(\mathcal{M})$ for $\mathcal{M}$ a EOC, along with many other categories occuring in nature, share a significant structural property: any object of $\mathbf{Mack}(\mathcal{M})$ can be torn open by a series of fracture squares. More precisely, suppose that $\mathcal{N}$ is a downwardly-closed subcategory, $\mathcal{T}$ is its upwardly-closed complement and $X \in \mathbf{Mack}(\mathcal{M})$. Then we'll see that there's a pullback square

$$\begin{array}{ccc}
X & \longrightarrow & (i_{\mathcal{T}})_*(i_{\mathcal{T}})^* X \\
\downarrow & & \downarrow \\
\Phi^{\mathcal{N}} X & \longrightarrow & \Phi^{\mathcal{N}}(i_{\mathcal{T}})_*(i_{\mathcal{T}})^* X
\end{array}$$

To build a theory of how X might be recovered from such data, it'll be helpful to widen our scope. First it's important to advertise a potential point of significant notational confusion.

Warning 3.1. When we regard a poset as a category in this paper, we will use the *opposite* of the usual convention that there is a morphism from x to y if $x \leq y$. For us, the space of morphisms from x to y will be contractible if $x \geq y$ and empty otherwise. We adopt this strange convention in order to preserve intuitions about size of objects in our chief motivating examples of posets: the posets of isomorphism classes of objects of the EOCs $\mathcal{O}_G$ and $\mathcal{F}_s^{\leq n}$.

Definition 3.2. Let $\mathcal{P}$ be a poset. An *interval* in $\mathcal{P}$ is a subset $I \subseteq \mathcal{P}$ such that whenever $x, y \in I$ and $x < z < y$, we have $z \in I$. If $\mathcal{P}$ is any poset, then we denote by $\mathcal{I}_{\mathcal{P}}$ be the poset of intervals in $\mathcal{P}$ ordered by inclusion. If I and J are a pair of intervals, we'll write $I \prec J$ if $I \cap J = \emptyset$ and there is no pair $(i \in I, j \in J)$ with $i > j$.

Note that the relation $\prec$ is *not* a partial order: for example, if $p, q \in \mathcal{P}$ are incomparable, then both $\{p\} \prec \{q\}$ and $\{q\} \prec \{p\}$.

Definition 3.3. Suppose $\mathbf{C}$ is a stable ∞-category. Let $\mathcal{E}_{\mathbf{C}}$ be the poset of stable reflective subcategories of $\mathbf{C}$, ordered by inclusion; equivalently, $\mathcal{E}_{\mathbf{C}}$ is the opposite of the poset of exact localizations of $\mathbf{C}$. Let $\mathbf{P}$ be a poset. Then a *pre-stratification of* $\mathbf{C}$ *along* $\mathcal{P}$ is a map of posets

$$\mathfrak{S} : \mathcal{I}_{\mathcal{P}} \to \mathcal{E}_{\mathbf{C}}.$$

Before we give the criteria that will qualify a pre-stratification as a stratification, it'll be useful to record an elementary fact about localizations.

Lemma 3.4. Given two localizations $\mathcal{L}_1$ and $\mathcal{L}_2$ on a stable ∞-category $\mathbf{C}$ such that $\mathcal{L}_1\mathcal{L}_2 = 0$, the following conditions are equivalent:

(1) The natural diagram

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & \mathcal{L}_1 \\ \downarrow & & \downarrow \\ \mathcal{L}_2 & \longrightarrow & \mathcal{L}_2\mathcal{L}_1 \end{array}$$

is a pullback square.

(2) The containment $\mathcal{L}_2\mathbf{C} \subseteq \ker(\mathcal{L}_1)$ is an equality.

(3) $\mathcal{L}_1$ and $\mathcal{L}_2$ are jointly conservative.

Proof. The implication (1) $\Rightarrow$ (3) is obvious. We'll prove (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are jointly conservative; then if some

$$X \in \ker(\mathcal{L}_1) \setminus \mathcal{L}_2\mathbf{C},$$

the localization map $X \to \mathcal{L}_2 X$ is a non-equivalence which becomes an equivalence after applying either $\mathcal{L}_1$ or $\mathcal{L}_2$, establishing (2). If we now denote by $\mathcal{C}$ the fiber of id $\to \mathcal{L}_1$, then

$$\mathrm{im}\ \mathcal{C} \subseteq \mathcal{L}_2\mathbf{C}$$

(in fact, $\mathcal{C}$ is the coreflection into $\mathcal{L}_2\mathbf{C}$). Now taking horizontal fibers in the square diagram gives the morphism

$$\mathcal{C} \to \mathcal{L}_2\mathcal{C}$$

which is an equivalence, establishing (1). □

Definition 3.5. In the notation of Definition 3.3, let

$$\mathcal{L}_I : \mathbf{C} \to \mathbf{C}$$

be the localization functor corresponding to $\mathfrak{S}(I)$. Then we call $\mathfrak{S}$ a *stratification of* $\mathbf{C}$ *along* $\mathcal{P}$ if the following conditions hold:

(1) $\mathfrak{S}(\mathcal{P}) = \mathbf{C}$,

(2) if $I_2 \prec I_1$, then $\mathcal{L}_{I_1}\mathcal{L}_{I_2} = 0$,

(3) and if $I = I_1 \coprod I_2$, then $\mathcal{L}_{I_1}$ and $\mathcal{L}_{I_2}$, viewed as localization functors on $\mathfrak{S}(I)$, satisfy the equivalent conditions of Lemma 3.4.

Remark 3.6. Since $\emptyset \prec \emptyset$, axiom (2) implies that $\mathcal{S}(\emptyset) = \{0\}$.

Remark 3.7. It follows from the latter two axioms that if $I = I_1 \coprod I_2$ and $I_2 \prec I_1$, then $\mathfrak{S}(I)$ is a *recollement* of $\mathfrak{S}(I_2)$ and $\mathfrak{S}(I_1)$ in the sense of [Lur12, Definition A.8.1].

Remark 3.8. For most of this section we'll assume that $\mathcal{P}$ is finite, but our main result generalizes easily to certain infinite posets (Definition 3.30). n will usually denote the cardinality of $\mathcal{P}$. We'll also assume that $\mathcal{P}$ is connected; it's easy to see that a category stratified along a disconnected poset decomposes naturally as a direct sum of categories stratified along the connected components.

Next we'll see some examples of stratifications. Suppose $\mathbf{C}$ is a presentable symmetric monoidal stable ∞-category whose tensor product preserves colimits in each variable, so that we can talk about the homological localization with respect to an object $E \in \mathbf{C}$ [Bou79]:

Recollection 3.9. An object $F \in \mathbf{C}$ is called *E-acyclic* if $E \otimes F$ is zero. An object $G \in \mathbf{C}$ is called *E-local* if $\mathrm{Map}(F, G)$ is contractible for any E-acyclic F. The E-local objects of $\mathbf{C}$ form a reflective subcategory $\mathbf{L}_E$, with associated localization functor $\mathcal{L}_E$.

Let $\mathcal{P}$ be a poset and suppose we have an object K_p for each $p \in \mathcal{P}$ such that any K_p-local object is K_q-acyclic unless $p \geq q$. We can define a pre-stratification $\mathfrak{S}_{K_\bullet}$ of $\mathbf{C}$ along $\mathcal{P}$ by assigning to I the category of objects which are local with respect to the object $\bigvee_{i \in I} K_i$.

Proposition 3.10. $\mathfrak{S}_{K_\bullet}$ is a stratification of $\mathfrak{S}_{K_\bullet}(\mathcal{P}) = \mathbf{L}_{\bigvee_{p \in \mathcal{P}} K_p}$.

Proof. The proof will consist of two lemmas, and will use induction on the cardinality of $\mathcal{P}$.

Lemma 3.11. Suppose E and F are such that any E-local object is F-acyclic. Then the square of functors

$$\begin{array}{ccc} \mathcal{L}_{E\vee F} & \xrightarrow{f} & \mathcal{L}_F \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} \\ \mathcal{L}_E & \xrightarrow{i} & \mathcal{L}_E\mathcal{L}_F \end{array}$$

is a pullback square.

Proof. This fact is folklore, and cases of it go back to Bousfield and further. The proof, which we now give, is simple.

g and h are both E-localizations, and therefore E-equivalences, and so the total fiber of the square is E-acyclic. On the other hand, f is an F-equivalence, and so is i because its source and target are both F-acyclic. Thus the total fiber of the square is F-acyclic, and so $E \vee F$-acyclic. But everything in the square is $E \vee F$-local, so the total fiber must also be $E \vee F$-local, and hence zero. □

Lemma 3.12. Suppose $\mathcal{I} \subseteq \mathcal{P}$ is a proper interval and $p \in \mathcal{P}$ is such that $\mathcal{I} \prec \{p\}$. Then any object $E \in \mathfrak{S}_{K_\bullet}(\mathcal{I})$ is K_p-acyclic.

Proof. The proof will use Theorem 3.21 (spoilers). By the induction hypothesis, $\mathfrak{S}_{K_\bullet}$ restricts to a stratification of $\mathfrak{S}_{K_\bullet}(\mathcal{I})$ along $\mathcal{I}$. Then the proof of Theorem 3.21 expresses E as a finite limit of objects which are K_i-local for some $i \in \mathcal{I}$, and thus K_p-acyclic. Therefore E is K_p-acyclic. □

□

Remark 3.13. For this class of stratifications, in the case where $\mathcal{P}$ is totally ordered, a theorem similar to Theorem 3.21 has appeared previously in [ACB14].

We'll now give a pair of quick applications of Proposition 3.10.

Example 3.14. Fix a prime p. Then the Morava K-theory spectra $K(0), \cdots, K(n)$ give rise to a stratification of the category of $\bigvee_{i=0}^n K(i)$-local spectra along $(D^n)^{\mathrm{op}}$. (Recall that a spectrum is $\bigvee_{i=0}^n K(i)$-local if and only if it's local with respect to the Morava E-theory E_n.) The pullback squares in this stratification include the famous *chromatic fracture squares*

$$\begin{array}{ccc} \mathcal{L}_{E_n} & \longrightarrow & \mathcal{L}_{K_n} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{L}_{E_{n-1}} & \longrightarrow & \mathcal{L}_{E_{n-1}}\mathcal{L}_{K_n} \end{array}$$

which are the subject of Hopkins' chromatic splitting conjecture [Hov93].

Example 3.15. Let X be a scheme; let $(U_i)_{0\leq i\leq n}$ be locally closed subschemes of X such $U_0 = X$, U_1 is an open subscheme of X, and for $i \geq 2$, U_i is an open subscheme of $U_{i-1} \setminus U_{i-2}$. We say that the U_i form a *stratification* of X.

Let $\mathbf{QC}(X)$ be the stable ∞-category of quasicoherent complexes on $\mathbf{X}$. Then the structure sheaves $\mathcal{O}_{U_i} \in \mathbf{QC}(X)$, $1 \leq i \leq n$, satisfy the hypotheses of Proposition 3.10 and so form a stratification of $\mathcal{L}_{\bigoplus_{i=1}^n \mathcal{O}_{U_i}}\mathbf{QC}(X)$ along Δ^{n-1}. In the case where

$$\bigcup_{i=1}^n U_i = X,$$

this is a stratification of $\mathbf{QC}(X)$ itself.

Our other main source of examples of stratifications comes from the theory developed in Section 2. We'll be able to say something about orbital ∞-categories and substantially more about epiorbital categories.

Let $\mathcal{M}$ be an orbital ∞-category, let $\mathcal{N}$ be a downwardly-closed subcategory of $\mathcal{M}$, and let $\mathcal{T}$ be its upwardly-closed complement. If $\mathbf{C}$ is a stable ∞-category with all limits and colimits, let $\mathbf{Mack}(\mathcal{M};\mathbf{C})$ be the category of $\mathbf{C}$-valued Mackey functors on $\mathcal{M}$ (Definition 2.16). We define a pre-stratification $\mathfrak{S}_{\mathcal{M}}$ of $\mathbf{Mack}(\mathcal{M},\mathbf{C})$ along Δ^1 as follows:

- $\mathfrak{S}_{\mathcal{M}}(\Delta^1) = \mathbf{Mack}(\mathcal{M},\mathbf{C})$,
- $\mathfrak{S}_{\mathcal{M}}(\{1\})$ is the category $\mathbf{Mack}^{\mathcal{T}}(\mathcal{M},\mathbf{C})$ of Mackey functors in the essential image of the right Kan extension from $\mathbf{Mack}(\mathcal{T},\mathbf{C})$,
- $\mathfrak{S}_{\mathcal{M}}(\{0\})$ is the category $\mathbf{Mack}_{\mathcal{N}}(\mathcal{M},\mathbf{C})$ of Mackey functors supported on $\mathcal{N}$ (see Corollary 2.33).

Proposition 3.16. $\mathfrak{S}_{\mathcal{M}}$ is a stratification.

Proof. We must show that the square

$$\begin{array}{ccc} \mathrm{id}_{\mathbf{Mack}(\mathcal{M},\mathbf{C})} & \longrightarrow & \mathcal{L}_1 \\ \downarrow & & \downarrow \\ \mathcal{L}_0 & \longrightarrow & \mathcal{L}_0\mathcal{L}_1 \end{array}$$

is a pullback square of endofunctors. But by taking vertical fibers and applying Theorem 2.32, we're reduced to showing that the natural map

$$\Gamma^{\mathcal{T}}\Pi^{\mathcal{T}} \to \Gamma^{\mathcal{T}}\Pi^{\mathcal{T}}\mathcal{L}_1$$

is an equivalence, which is obvious. □

Now suppose $\mathcal{M}$ is epiorbital, and let $\mathcal{P}_{\mathcal{M}}$ be the poset of isomorphism classes in $\mathcal{M}$ (Definition 2.2).

Definition 3.17. We define a pre-stratification $\mathfrak{S}_{\mathcal{M}}$ on $\mathbf{Mack}(\mathcal{M};\mathbf{C})$ as follows. If $I \subseteq \mathcal{P}_{\mathcal{M}}$ is any interval, let $\mathcal{I}$ be the corresponding full subcategory of $\mathcal{M}$. If I is downwardly-closed, then we define

$$\mathfrak{S}_{\mathcal{M}}(I) = \mathbf{Mack}_{\mathcal{I}}(\mathcal{M};\mathbf{C}) \qquad \text{(Corollary 2.33)}$$

If $J \subseteq \mathcal{P}_{\mathcal{M}}$ is upwardly closed, then let $\mathbf{Mack}^{\mathcal{J}}(\mathcal{M},\mathbf{C})$ be the essential image of the right Kan extension

$$A^{eff}(i_{\mathcal{J}})_* : \mathbf{Mack}(\mathcal{J},\mathbf{C}) \to \mathbf{Mack}(\mathcal{M},\mathbf{C}).$$

We define

$$\mathfrak{S}_{\mathcal{M}}(J) = \mathbf{Mack}^{\mathcal{J}}(\mathcal{M},\mathbf{C}).$$

If $I \subseteq \mathcal{P}_{\mathcal{M}}$ is any interval, then we can write

$$I = I_+ \cap I_-$$

where I_+ is the smallest upwardly-closed set containing I and I_- is, likewise, the smallest downwardly-closed set containing I. Then we define

$$\mathfrak{S}_{\mathcal{M}}(I) = \mathbf{Mack}^{\mathcal{I}_+}(\mathcal{M},\mathbf{C}) \cap \mathbf{Mack}_{\mathcal{I}_-}(\mathcal{M},\mathbf{C}).$$

Proposition 3.18. $\mathfrak{S}_{\mathcal{M}}$ is a stratification of $\mathbf{Mack}(\mathcal{M},\mathbf{C})$ along $\mathcal{P}_{\mathcal{M}}$.

Proof. Clearly

$$\mathfrak{S}_{\mathcal{M}}(\mathcal{P}_{\mathcal{M}}) = \mathbf{Mack}(\mathcal{M},\mathbf{C}).$$

We must verify (3) in Definition 3.5. Let $\mathcal{I},\mathcal{I}_1,\mathcal{I}_2$, be the full subcategories of $\mathcal{M}$ corresponding respectively to I, I_1, I_2, and let $\mathcal{D}$ be the smallest downwardly-closed subcategory of $\mathcal{M}$ containing $\mathcal{I}$. Then by passing to $\mathbf{Mack}_{\mathcal{D}}(\mathcal{M},\mathbf{C})$ if necessarily, we may assume that I_1 is upwardly-closed.

Assume for a moment that $I = \mathcal{P}_{\mathcal{M}}$. Then, as in the proof of Proposition 3.16, we conclude by taking fibers vertically and invoking Theorem 2.32. In general, we note that

$$I_2 = I \cap (\mathcal{P}_{\mathcal{M}} \setminus I_1).$$

We know that the natural diagram

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & \mathcal{L}_{I_1} \\ \downarrow & & \downarrow \\ \mathcal{L}_{\mathcal{P}_{\mathcal{M}}\setminus I_1} & \longrightarrow & \mathcal{L}_{\mathcal{P}_{\mathcal{M}}\setminus I_1}\mathcal{L}_{I_1} \end{array}$$

is a pullback square, and applying $\mathcal{L}_I$ to the entire square gives the result. □

We'll now start setting up for our main result on the reconstruction of objects in a stratified category from their atomic localizations.

Definition 3.19. Let $\mathcal{P}$ be a finite poset. Define $\mathbb{P}(\mathcal{P})$ to be the poset of nonempty subsets $T = \{i_1, i_2, \cdots, i_k\}$ of $\mathcal{P}$, ordered by reverse inclusion.

Definition 3.20. Let $(\mathbf{C}, \mathfrak{S})$ be a stable ∞-category stratified along $\mathcal{P}$. We define an ∞-category $\mathbf{C}^{\mathfrak{S}}$ as the full subcategory of $\mathrm{Fun}(\mathbb{P}(\mathcal{P}), \mathbf{C})$ spanned by those functors

$$F : \mathbb{P}(\mathcal{P}) \to \mathbf{C}$$

such that

- for each $T \in \mathbb{P}(\mathcal{P})$ and for each minimal element $t \in T$, $F(T)$ is in $\mathfrak{S}(\{t\})$;
- if e is an edge of $\mathbb{P}(\mathcal{P})$ of the form $T \to T \cup \{p\}$ where $\{p\} \prec T$, then $F(e)$ exhibits $F(T \cup \{p\})$ as the $\mathfrak{S}(\{p\})$-localization of $F(T)$.

The following, describing how objects of a stratified category can be assembled using higher fracture squares, is the main theorem of this section.

Theorem 3.21. There's an equivalence of categories

$$\mathfrak{d} : \mathbf{C} \to \mathbf{C}^{\mathfrak{S}}.$$

This equivalence will be constructed as an explicit zigzag in the course of the proof.

The first step is to realize that we don't have enough posets, and define some more posets.

Definition 3.22. Let $\mathbb{P}'(\mathcal{P})$ be the set whose elements are nonempty sets $\{I_1, \cdots, I_k\}$ of nonempty, disjoint intervals in $\mathcal{P}$ such that for each pair of indices i, j, either $I_i \prec I_j$ or $I_j \prec I_i$. We'll put a partial order on $\mathbb{P}'(\mathcal{P})$ by letting

$$\{I_1, \cdots, I_k\} \geq \{J_1, \cdots, J_l\}$$

if there exist distinct indices $(i_1, \cdots, i_k)$ such that $J_{i_j} \subseteq I_j$ for each j.

If $T \in \mathbb{P}'(\mathcal{P})$ and $I \in T$, then we'll call I *$\prec$-minimal* if for each $J \in T$ with $J \neq I$, $I \prec J$. This doesn't necessarily imply that there is no $J \in T$ such that $J \prec I$. However, since $\prec$ is transitive, each $T \in \mathbb{P}'(\mathcal{P})$ has at least one $\prec$-minimal element.

Here's a way of "coordinatizing" a poset.

Definition 3.23. A *threading* of a finite poset $\mathcal{P}$ is a filtration

$$\emptyset = \mathcal{P}_{\leq 0} \subseteq \mathcal{P}_{\leq 1} \subseteq \cdots \mathcal{P}_{\leq n} = \mathcal{P}$$

such that for each i, $\mathcal{P}_{\leq i}$ is downwardly closed, and

$$|\mathcal{P}_{\leq i}| = i.$$

We'll fix, once and for all, a threading on $\mathcal{P}$.

Definition 3.24. Let $\mathbb{P}^m(\mathcal{P})$ be the subset of $\mathbb{P}'(\mathcal{P})$ containing those sets

$$T = \{I_1, \cdots, I_k\}$$

such that for each I_i, either $|I_i| = 1$ or

$$I_i = \mathcal{P}_{\leq i} \text{ for some } i \leq m.$$

In particular, at most one of the I_i may have cardinality > 1, and if this occurs then I_i must be a $\prec$-minimal element of T.

There's an obvious isomorphism $\mathbb{P}^1(\mathcal{P}) \cong \mathbb{P}(\mathcal{P})$.

Definition 3.25. Let $\mathbf{C}_m^{\mathfrak{S}}$ be the full subcategory of $\mathrm{Fun}(\mathbb{P}^m(\mathcal{P}))$ spanned by those functors

$$F : \mathbb{P}^m(\mathcal{P}) \to \mathbf{C}$$

such that

- for each $T = \{I_1, \cdots, I_k\} \in \mathbb{P}^m(\mathcal{P})$ and for each I_i that is $\prec$-minimal in T, $F(T) \in \mathfrak{S}(I_i)$;
- if $e : T_1 \to T_2$ is an edge of $\mathbb{P}^m(\mathcal{P})$ of the form

$$\{I_1, I_2, \cdots, I_k\} \to \{I_1, I_2, \cdots, J_k\}$$

  with I_k $\prec$-minimal in T_1 and $J_k \subseteq I_k$, then $F(e)$ exhibits $F(T_2)$ as the $\mathfrak{S}(J_k)$-localization of $F(T_1)$;
- if $e : T_1 \to T_2$ is an edge of $\mathbb{P}^m(\mathcal{P})$ of the form

$$\{I_1, I_2, \cdots, I_k\} \to \{I_1, I_2, \cdots, I_k, I_{k+1}\}$$

  with I_{k+1} $\prec$-minimal, then $F(e)$ exhibits $F(T_2)$ as the $\mathfrak{S}(I_{k+1})$-localization of $F(T_1)$.

Proposition 3.26. If $\mathcal{P}$ has cardinality at most m, then the functor

$$e_{\mathcal{P}} : \mathbf{C}_m^{\mathfrak{S}} \to \mathbf{C}$$

given by evaluation at $\{\mathcal{P}\}$ is an equivalence of categories.

Proof. We'll prove this by induction on m. If $m = 1$, there's nothing to do. If $m = 2$, then $\mathcal{P} \cong \Delta^1$ and we'll label its elements 0 and 1, with $0 > 1$ (this is unfortunately necessitated by our conventions). Then $\mathbf{C}_m^{\mathfrak{S}}$ is the category of squares of the form

$$\begin{array}{ccc} E & \rightarrowtail & E_0 \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & E_{10} \end{array}$$

in which $E_0 \in \mathfrak{S}(0)$, $E_1, E_{10} \in \mathfrak{S}(1)$ and the tailed arrows are localizations. By the stratification axioms, all such squares are cartesian. Then the fact that $e_{\Delta^1} : \mathbf{C}_m^{\mathfrak{S}} \to \mathbf{C}$ is an equivalence is discussed, in almost exactly these terms, in the proof of [Lur12, Proposition A.8.11].

In general, let $x \in \mathcal{P}$ be the unique element of $\mathcal{P} \backslash \mathcal{P}_{\leq n-1}$; then x is a maximal element. For convenience, we'll write $\mathcal{Q}$ for $\mathcal{P}_{\leq n-1}$. Observe that we have an isomorphism of posets

$$h : \Delta^1 \times ((\mathbb{P}^m(\mathcal{Q}))^{\triangleleft}) \to \mathbb{P}^m(\mathcal{P})$$

given by

$$\begin{aligned} h(0, c) &= \{\mathcal{P}\}, \\ h(1, c) &= \{\{x\}\}, \\ h(0, T) &= T, \\ h(1, T) &= \{\{x\}\} \cup T, \end{aligned}$$

where c is the cone point.

Now $\mathbf{C}$ admits a stratification $\mathfrak{S}'$ along Δ^1 wherein

$$\mathfrak{S}'(1) = \mathfrak{S}(\{x\}),\ \mathfrak{S}'(0) = \mathfrak{S}(\mathcal{Q}),$$

so that

$$\phi_{\Delta^1} : \mathbf{C}_m^{\mathfrak{S}'} \to \mathbf{C}$$

is an equivalence. On the other hand, $\mathfrak{S}(\mathcal{Q})$ obviously inherits a stratification $\mathfrak{S}''$ along $\mathcal{Q}$, and the functor

$$\phi_{\mathcal{Q}} : \mathfrak{S}(\mathcal{Q})_m^{\mathfrak{S}''} \to \mathfrak{S}(\mathcal{Q})$$

is an equivalence.

Here's what we deduce by combining these two equivalences. Let

$$\mathcal{K} = \mathbb{P}^m(\Delta^1) \amalg_{\Delta^1} (\mathbb{P}^m(\mathcal{Q}) \times \Delta^1)$$

where we've glued the edge $\{0\} \to \{0,1\}$ of $\mathbb{P}^m(\Delta^1)$ to the edge $\{\mathcal{Q}\} \times \Delta^1$ of $\mathbb{P}^m(\mathcal{Q}) \times \Delta^1$. Let $\overline{\mathbf{C}}$ be the full subcategory $\mathrm{Fun}(\mathcal{K}, \mathbf{C})$ spanned by those F for which

$$F|_{\mathbb{P}^m(\Delta^1)} \in \mathbf{C}_m^{\mathfrak{S}'}$$

and

$$F|_{\mathbb{P}^m(\mathcal{Q}) \times \{0\}}, F|_{\mathbb{P}^m(\mathcal{Q}) \times \{1\}} \in \mathfrak{S}(\mathcal{Q})_m^{\mathfrak{S}''}.$$

Then evaluation on $\{\Delta^1\} \in \mathbb{P}^m(\Delta^1)$ induces an equivalence of categories

$$e_{\Delta^1} : \overline{\mathbf{C}} \xrightarrow{\sim} \mathbf{C}.$$

But

$$\mathbb{P}^m(\Delta^1) \cong \Delta^1 \times \Delta^1,$$

and so

$$\mathcal{K} \cong (\Delta^1 \amalg_{\{1\}} \mathbb{P}^m(\mathcal{Q})) \times \Delta^1.$$

For any simplicial set S, the inclusion

$$\Delta^1 \amalg_{\{1\}} S \hookrightarrow S^{\triangleleft}$$

is inner anodyne, and so we get an inner anodyne composite

$$\mathcal{K} \hookrightarrow \Delta^1 \times ((\mathbb{P}^m(\mathcal{Q}))^{\triangleleft}) \xrightarrow{h} \mathbb{P}^m(\mathcal{P}),$$

giving an equivalence of categories $\mathrm{Fun}(\mathbb{P}^m(\mathcal{P}), \mathbf{C}) \simeq \mathrm{Fun}(\mathcal{K}, \mathbf{C})$, which restricts to an equivalence of categories $\mathbf{C}_m^{\mathfrak{S}} \to \overline{\mathbf{C}}$. Composing with e_{Δ^1} completes the proof. □

Proposition 3.27. For any $\mathcal{P}$ and any $\mathbf{C}$ stratified along $\mathcal{P}$, the restriction functor

$$r_m : \mathbf{C}_m^{\mathfrak{S}} \to \mathbf{C}_{m-1}^{\mathfrak{S}}$$

is an equivalence.

Proof. Let

$$\kappa_m : \mathbf{C}_{m-1}^{\mathfrak{S}} \to \mathrm{Fun}(\mathbb{P}^m(\mathcal{P}), \mathbf{C})$$

be the right Kan extension functor. We claim that $\mathbf{C}_m^{\mathfrak{S}}$ is equal to the essential image of κ_m.

Indeed, let q be the unique element of $\mathcal{P}_{\leq m} \setminus \mathcal{P}_{\leq m-1}$. Suppose

$$T \in \mathbb{P}^m(\mathcal{P}) \setminus \mathbb{P}^{m-1}(\mathcal{P}).$$

Then T is of the form

$$\{\{p_1\}, \cdots, \{p_k\}, \mathcal{P}^{\leq m}\}.$$

Let $\alpha : \Lambda_2^2 \to \mathbb{P}^{m-1}(\mathcal{P})$ be the functor with

$$\begin{aligned}
\alpha_T(0) &= \{\{p_1\}, \cdots, \{p_k\}, \{q\}\}, \\
\alpha_T(1) &= \{\{p_1\}, \cdots, \{p_k\}, \mathcal{P}_{\leq m-1}\}, \\
\alpha_T(2) &= \{\{p_1\}, \cdots, \{p_k\}, \{q\}, \mathcal{P}_{\leq m-1}\}.
\end{aligned}$$

Then α is coinitial in $\mathbb{P}^{m-1}(\mathcal{P})_{T/}$. Moreover, $F : \mathbb{P}^m(\mathcal{P}) \to \mathbf{C}$ is an object of $\mathbf{C}_m^{\mathfrak{S}}$ if and only if

- $F|_{\mathbb{P}^{m-1}(\mathcal{P})} \in \mathbf{C}_{m-1}^{\mathfrak{S}}$, and
- for each $T \in \mathbb{P}^m(\mathcal{P}) \setminus \mathbb{P}^{m-1}(\mathcal{P})$, $F(T) \in \mathfrak{S}(\mathcal{P}_{\leq m})$ and the maps

  $$F(T) \to F(\alpha_T(0)), F(T) \to \mathcal{F}(\alpha_T(1))$$

  are localizations.

But by the stratification axiom, the latter condition is equivalent to the condition that the square

$$\begin{array}{ccc} F(T) & \longrightarrow & F(\alpha_T(0)) \\ \downarrow & & \downarrow \\ F(\alpha_T(1)) & \longrightarrow & F(\alpha_T(2)) \end{array}$$

be a pullback. This completes the proof. □

If $n = |\mathcal{P}|$, we now have equivalences of categories $\mathbf{C}_n^{\mathfrak{S}} \xrightarrow{\sim} \mathbf{C}$ (Proposition 3.26) and $\mathbf{C}_n^{\mathfrak{S}} \xrightarrow{\sim} \mathbf{C}_1^{\mathfrak{S}} \cong \mathbf{C}^{\mathfrak{S}}$ (inductively, using Proposition 3.27). This consitutes a proof of Theorem 3.21.

Example 3.28. When $\mathbf{C}$ is the category $\mathbf{Sp}^G$ for a finite abelian group G, we recover the statement of [AK13, Theorem 3], though in substantially different language.

Example 3.29. Suppose p is a prime and $\mathcal{M} = \mathcal{O}_{C_p}$, so that $\mathcal{P} = \Delta^1$. Then Theorem 3.21 states, after unwinding the definition, that an object E of of $\mathbf{Mack}(\mathcal{M}) \simeq \mathbf{Sp}^{C_p}$ is given by the following data:

- A spectrum with C_p-action
$$E_1 \in \mathfrak{S}_{\mathcal{M}}(\{\{1\}\}) \simeq \mathrm{Fun}(BC_p, \mathbf{Sp}),$$
the underlying spectrum of E;
- a spectrum
$$E_0 \in \mathfrak{S}_{\mathcal{M}}(\{\{0\}\}) \simeq \mathbf{Sp},$$
the C_p-geometric fixed point spectrum of E;
- and a map
$$E_0 \to \mathcal{L}_{\{0\}} E_1 \simeq E_1^{tC_p}$$
where $(-)^{tG}$ is the Tate spectrum, defined by the cofiber sequence
$$(-)_{hG} \to (-)^{hG} \to (-)^{tG}.$$

The epiorbital category $\mathcal{F}_s^{\leq 2}$ is visibly equivalent to $\mathcal{O}_{C_2}$, so an object

$$F \in \mathbf{Mack}(\mathcal{F}_s^{\leq 2}) \simeq \mathrm{Fun}^{2-exc}(\mathbf{Sp}, \mathbf{Sp})$$

is given by the same data as an object of $\mathbf{Sp}^{C_2}$, but in this case, E_1 and E_0 are interpreted as the second and first derivatives of F, respectively. This classification of 2-excisive functors was first carried out in [AC15, §5].

We'll close this section by saying a few words about what happens for infinite posets. Let $\mathcal{P}$ be an infinite poset equipped with a system of finite subposets

$$\emptyset = \mathcal{P}_{\leq 0} \subseteq \mathcal{P}_{\leq 1} \subseteq \cdots \subseteq \mathcal{P}_{\leq n} \subseteq \cdots \subseteq \mathcal{P}$$

which is a threading in the sense that for each i, $\mathcal{P}_{\leq i}$ is downwardly closed and has cardinality i, and

$$\bigcup_n \mathcal{P}_{\leq n} = \mathcal{P}.$$

Definition 3.30. Suppose

$$0 = \mathbf{C}_0 \subseteq \mathbf{C}_1 \subseteq \cdots \subseteq \mathbf{C}_n \subseteq \cdots$$

is a sequence of stable ∞-categories such that $\mathbf{C}_{n-1}$ is a reflective stable subcategory of $\mathbf{C}_n$ for all n. Suppose we have, for each n, a stratification $\mathfrak{S}_n$ of $\mathbf{C}_n$ along $\mathcal{P}_{\leq n}$, and that all of these are compatible in the sense that

$$\mathbf{C}_{n-1} = \mathfrak{S}_n(\mathcal{P}_{\leq n-1})$$

and $\mathfrak{S}_{n-1}$ is the induced stratification. Let

$$\mathbf{C}_\infty := \lim_n \mathbf{C}_n$$

be the limit over the localization maps. We call this data a *pro-stratification* of $\mathbf{C}_\infty$ along $\mathcal{P}$.

Then it follows from the proof of Theorem 3.21 that the diagram

$$\begin{tikzcd} \mathbf{C}_n \arrow[d, "\mathbf{L}_{\mathcal{P}_{\leq n-1}}"'] & (\mathbf{C}_n)_n^{\mathfrak{S}_n} \arrow[l, "\sim"', "e_{\mathcal{P}_{\leq n}}"] \arrow[r, "\sim"] \arrow[d] & \mathbf{C}_n^{\mathfrak{S}_n} \arrow[d] \\ \mathbf{C}_{n-1} & (\mathbf{C}_{n-1})_{n-1}^{\mathfrak{S}_{n-1}} \arrow[l, "\sim"', "e_{\mathcal{P}_{\leq n-1}}"] \arrow[r, "\sim"] & \mathbf{C}_{n-1}^{\mathfrak{S}_{n-1}} \end{tikzcd}$$

commutes up to homotopy for every n. Taking the limit as $n \to \infty$ gives an equivalence

$$\mathbf{C}_\infty \to \lim_n \mathbf{C}_n^{\mathfrak{S}_n} =: \mathbf{C}_\infty^{\mathfrak{S}}.$$

The limit $\mathbf{C}_\infty^{\mathfrak{S}}$ can be described explicitly as follows. Let

$$\mathbb{P}^\infty = \operatorname*{colim}_n \mathbb{P}(\mathcal{P}_n).$$

and

$$\mathbf{C}^\infty := \operatorname*{colim}_n \mathbf{C}_n$$

(which differs from $\mathbf{C}_\infty$ in that we have taken the colimit over the inclusions rather than the limit over the localizations). Then $\mathbf{C}_\infty^{\mathfrak{S}}$ is the full subcategory of $\mathrm{Fun}(\mathbb{P}^\infty, \mathbf{C}^\infty)$ spanned by those functors F for which

$$F|_{\mathbb{P}(\mathcal{P}_n)} \in \mathbf{C}_n^{\mathfrak{S}_n}.$$

Thus we have a description of $\mathbf{C}_\infty$ in terms of (infinite) diagrams of maximally local objects.

Example 3.31. With $\mathcal{P}$ as above, let $\mathbf{C}$ be a symmetric monoidal presentable stable ∞-category and let $(K_p)_{p\in\mathcal{P}}$ be a collection of objects of $\mathbf{C}$ such that any K_p-local object is K_q-acyclic unless $p \geq q$. Then letting

$$\mathbf{C}_n = \mathbf{L}_{\bigvee_{p\in\mathcal{P}_{\leq n}} K_p}\mathbf{C}$$

gives a pro-stratification of

$$\mathbf{C}_\infty = \mathbf{L}_{\bigvee_{p\in\mathcal{P}} K_p}\mathbf{C}$$

along $\mathcal{P}$. In the case where $\mathcal{P} = \mathbb{N}^{\mathrm{op}}$ and K_n is the Morava K-theory $K(n)$, we have expressed the category of harmonic spectra in terms of diagrams of $K(n)$-local spectra.

Example 3.32. Let

$$\emptyset = \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots \subseteq \mathcal{M}_n \subseteq \cdots \subseteq \mathcal{M} = \operatorname*{colim}_n \mathcal{M}_n$$

be a sequence of inclusions of EOCs giving rise to the threading

$$\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \cdots \subseteq \mathcal{P}_n \subseteq \cdots$$

on posets of isomorphism classes. Then $\mathcal{M}^{\amalg}$ is disjunctive and we may speak of the category $\mathbf{Mack}(\mathcal{M}, \mathbf{C})$ of additive functors from $A^{eff}(\mathcal{M}^{\amalg})$ into some stable target category $\mathbf{C}$. Letting

$$\mathbf{C}_n = \mathbf{Mack}(\mathcal{M}_n, \mathbf{C})$$

gives a pro-stratification of $\mathbf{Mack}(\mathcal{M}, \mathbf{C})$ along $\mathcal{P}$.

Example 3.32 has a couple of interesting special cases:

Example 3.33. Let $\mathcal{M}_n = \mathcal{F}_s^{\leq n}$. Then $\mathcal{M}$ is the category $\mathcal{F}_s$ of all finite sets and surjective maps, and $\mathbf{Mack}(\mathcal{M}, \mathbf{C})$ is equivalent to the category of functors $F : \mathbf{Sp} \to \mathbf{C}$ which are *weakly analytic* in the sense that

$$F \simeq \lim_n P_n F.$$

Example 3.34. Let G be a profinite group and let $\mathcal{M}$ be the category of finite G-orbits. Then any threading of the poset $\mathcal{P}$ of isomorphism classes in $\mathcal{M}$ gives a pro-stratification of $\mathbf{Mack}(\mathcal{M}, \mathbf{C})$, which should be thought of a kind of category of genuine G-objects in which only the fixed points under cofinite subgroups are salient.

It's worth noting that given a cofinal system of finite quotients $(H_m)_{m\in\mathbb{N}}$ of G, we could choose our threading so that for each m, there is some n_m such that

$$\mathcal{M}_{n_m} = \mathcal{O}_{H_m}.$$

Thus

$$\mathbf{Mack}(\mathcal{M}, \mathbf{C}) \simeq \lim_{G\twoheadrightarrow H, H \text{ finite}} \mathbf{Mack}(\mathcal{O}_H, \mathbf{C}).$$

4 $K(n)$-local theory

In this brief section, we'll see that symmetry properties which emerge when one works locally with respect to the Morava K-theories $K(n)$ cause large chunks of this theory to collapse. We'll reprove a result of Kuhn on the $K(n)$-local splitting of Taylor towers, and give a new tom Dieck-like splitting result for $K(n)$-local G-spectra.

The following "chromatic blueshift" theorem is a consequence of the results of [GS96] and [HS96]; it appears in roughly this form in [HL13], and as we shall see, [Kuh04] is also highly relevant.

Theorem 4.1. Let G be a finite group and let E be a $K(n)$-local spectrum with G-action. Then the transfer map

$$N : E_{hG} \to E^{hG}$$

is a $K(n)$-local equivalence. Thus the Tate spectrum E^{tG} is $K(n)$-acyclic.

Theorem 4.2. If $\mathcal{M}$ is a epiorbital category and $\mathbf{C}$ is a stable ∞-category such that all Tate spectra are zero - for instance, the category of $K(n)$-local spectra - then the comonad DR of Theorem 2.38 is the identity comonad, and so the Taylor sequence functor

$$D : \mathbf{Mack}(\mathcal{M}, \mathbf{C}) \to \mathrm{Fun}(\mathcal{M}^{\sim}, \mathbf{C})$$

is an equivalence.

Having got this far, the proof is fairly simple.

Proof. Let X be an object of $\mathcal{M}$ and let i_X once again denote the full inclusion $A^{eff}(\mathcal{G}_X) \hookrightarrow A^{eff}(\mathcal{M}_{\leq X})$, where $\mathcal{G}_X$ is the full subcategory of $\mathcal{M}$ spanned by X. Let $E \in \mathrm{Fun}(\mathcal{G}_X, \mathbf{C})$. Then for any $Y \in \mathcal{M}_{\leq X}$, the natural map

$$(i_X)_!(E)(Y) \to (i_X)_*(E)(Y)$$

takes the form

$$\bigoplus_{f \in \mathrm{Map}_{\mathcal{M}}(X,Y)/\text{isomorphism}} E_{h\mathrm{Aut}_f(X)} \to \bigoplus_{f \in \mathrm{Map}_{\mathcal{M}}(X,Y)/\text{isomorphism}} E^{h\mathrm{Aut}_f(X)}$$

and is thus an equivalence, by our hypothesis. We deduce that for each $X \in \mathcal{M}$,

$$L^X \simeq R^X,$$

and so

$$L \simeq R.$$

But $DL \simeq \mathrm{id}$ (Proposition 2.37) and so $DR \simeq \mathrm{id}$. This completes the proof. $\square$

Corollary 4.3. Any $K(n)$-local G-spectrum E (by which we mean a Mackey functor valued in the $K(n)$-local category) satisfies a very strong tom Dieck splitting property: we have an equivalence

$$E \simeq \bigvee_{H \leq G/\text{conjugacy}} L^{G/H} E^{\Phi H}.$$

In particular, for each $H \leq G$, we have a canonical decomposition

$$E^H \simeq \bigvee_{(K \leq H)/\text{conjugacy in } G} \left((E^{\Phi K})_{hW(H,K)}\right)$$

where $W(H, K)$ is the relative Weyl group, defined as

$$W(H, K) := (N_G(K) \cap H)/K.$$

Corollary 4.4 (Kuhn)**.** Let $F : \mathbf{Sp} \to \mathbf{Sp}$ be an m-excisive functor taking values in $K(n)$-local spectra. Then the Taylor tower for F splits: we have an equivalence

$$F(X) \simeq \bigvee_{i=0}^{m} \mathbb{D}_i F(X).$$

A The free semiadditive ∞-category on a group

This appendix is devoted to proving Theorem 2.27, which we restate here (with slightly different notation) for convenience:

Theorem A.1. Let G be an (ordinary) groupoid. Then $A^{eff}(G^{\amalg})$ is the free semiadditive ∞-category on G: for any semiadditive ∞-category $\mathbf{C}$, the natural inclusion induces an equivalence of categories

$$\mathrm{Fun}^{\oplus}(A^{eff}(G^{\amalg}), \mathbf{C}) \to \mathrm{Fun}(G, \mathbf{C}).$$

First, we note that we may assume G is connected. Indeed, having proved this, the general case will follow from the fact that if $(\mathcal{M}_i)_{i \in I}$ is an I-indexed family of orbital categories, then

$$A^{eff}\left(\left(\coprod_I \mathcal{M}_i\right)^{\amalg}\right) \simeq \bigoplus_I A^{eff}(\mathcal{M}_i^{\amalg}).$$

We will further assume that our connected groupoid has only one object, and denote the corresponding group, too, by G.

Now let's get some notation out of the way. Let $\mathcal{F}_*$ be the category of finite pointed sets. If $S \in \mathcal{F}_*$, denote by S^o the finite set $S \setminus \{*\}$. If $s \in S^o$, denote by $\chi_s : S \to \{s\}_+$ the characteristic map at s:

$$\chi_s(t) = \begin{cases} s & t = s \\ * & \text{otherwise.} \end{cases}$$

Definition A.2. [Lur12, Remark 2.4.2.2] Let $\mathbf{C}$ be an ∞-category which admits finite products. Recall that by definition, the category $\mathbf{CMon}(\mathbf{C})$ of *commutative monoids* in $\mathbf{C}$ is the full subcategory of $\mathrm{Fun}(\mathcal{F}_*, \mathbf{C})$ spanned by those functors F satisfying the Segal condition: for each $S \in \mathcal{F}_*$, the edges $F(S) \to F(\{s\}_+)$ determine an equivalence

$$F(S) \simeq \prod_{s \in S^o} F(\{s\}_+).$$

We'll abbreviate $\mathbf{CMon}(\mathbf{Top})$ to $\mathbf{CMon}$.

Now let's begin the proof. First we note that $G^{\amalg}$ is equivalent to the category $\mathcal{F}r^{\mathcal{G}}$ of finite sets with *free* G-action.

Definition A.3. Let $\mathcal{L}(G)$ be the Lawvere theory of commutative monoids with G-action: the full subcategory of $\mathrm{Fun}(G, \mathbf{CMon})$, which is equivalent to $\mathbf{CMon}(\mathrm{Fun}(G, \mathbf{Top}))$, spanned by the the essential image of $\mathcal{F} \subseteq \mathbf{Top}$ under the left adjoint of the forgetful functor $\mathrm{Fun}(G, \mathbf{CMon}) \to \mathbf{Top}$.

Theorem A.4. There is an equivalence of categories between $A^{eff}(\mathcal{F}r^G)$ and $\mathcal{L}(G)$.

Proof. First let's construct the functor. $\mathrm{Fun}(G, \mathbf{CMon})$ is a certain full subcategory of $\mathrm{Fun}(G \times \mathcal{F}_*, \mathbf{Top})$, so we can do this by constructing a functor

$$A^{eff}(\mathcal{F}r^G) \times G \times \mathcal{F}_* \to \mathbf{Top}$$

adjointing over, and checking it makes sense on objects.

We'll do the construction in two stages. First, note that we have a functor

$$A^{eff}(\mathcal{F}r^G) \times A^{eff}(\mathcal{F}) \to A^{eff}(\mathcal{F}r^G)$$

simply by taking objectwise products of staircase diagrams. We also have an inclusion $i : \mathcal{F}_* \to A^{eff}(\mathcal{F})$ as follows: an n-simplex of $\mathcal{F}_*$, given by a chain of pointed maps

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

maps to the staircase diagram $(A_{ij})_{0 \le i \le j \le n}$ with

$$\begin{cases} A_{ii} = X_i^o & i = j \\ A_{ij} = (f_j f_{j-1} \cdots f_{i+1})^{-1} X_j^o & i \neq j. \end{cases}$$

Functoriality is easily checked, as is the fact that all squares which ought to be pullbacks are pullbacks. Composing these two and multiplying by G, we get a map

$$\mu : A^{eff}(\mathcal{F}r^G) \times G \times \mathcal{F}_* \to A^{eff}(\mathcal{F}r^G) \times G.$$

Next, we'll define a functor $A^{eff}(\mathcal{F}r^G) \times G \to \mathbf{Top}$ by defining a left fibration

$$\kappa : A^{eff}(\mathcal{F}r^G)_+ \ltimes G \to A^{eff}(\mathcal{F}r^G) \times G.$$

Here $A^{eff}(\mathcal{F}r^G)_+ \ltimes G$ is itself the total space of a cocartesian fibration over G. A vertex of $A^{eff}(\mathcal{F}r^G)_+ \ltimes G$ is a free G-set U together with a finite set S and a map of sets $S \to U$. An edge of $A^{eff}(\mathcal{F}r^G)_+ \ltimes G$ with source $a_1 : S_1 \to U_1$ and target $a_2 : S_2 \to U_2$ is an element $g \in G$ together with a diagram

$$\begin{tikzcd} S_1 \arrow[d, "g\circ a_1"] & S_2 \arrow[l] \arrow[d] \arrow[dd, bend left, "a_2"] \arrow[dl, phantom, "\lrcorner"] \\ U_1 & W \arrow[l] \arrow[d] \\ & U_2 \end{tikzcd}$$

with higher simplices defined analogously. We define κ to be the map that forgets S. To prove that κ is a left fibration, define $A^{eff}(\mathcal{F}r^G)_+$ to be the fiber of $A^{eff}(\mathcal{F}r^G)_+ \ltimes G$ over the vertex of G. κ restricts to a map

$$A^{eff}(\mathcal{F}r^G)_+ \to A^{eff}(\mathcal{F}r^G)$$

which is actually isomorphic to the target map

$$t : A^{eff}(\mathcal{F}r^G)_{G/} \to A^{eff}(\mathcal{F}r^G)$$

which is definitely a left fibration. Together with the fact that the preimage under κ of an edge of G is an equivalence, this implies that κ itself is a left fibration.

Let K be a functor $A^{eff}(\mathcal{F}r^G) \times G \to \mathbf{Top}$ that classifies κ. We now have a well-defined functor

$$\sigma : A^{eff}(\mathcal{F}r^G) \to \mathrm{Fun}(G \times \mathcal{F}_*, \mathbf{Top})$$

defined by composing μ with K and then taking adjoints. For each free finite G-set U, we must show that the functor

$$\mathcal{F}_* \times \{U\} \xrightarrow{\mu} A^{eff}(\mathcal{F}r^G) \to \mathrm{Fun}(G, \mathbf{Top})$$

is a commutative monoid. Unwinding the definitions shows that this is a consequence of the fact that K preserves products. Moreover, one identifies $\sigma(U)$ with

$$(\Sigma)^{\times U},$$

where $\Sigma = \coprod_{n \geq 0} \Sigma_n$ and G acts by permutation on the factors. This is equivalent to the free commutative monoid in $\mathrm{Fun}(G, \mathbf{Top})$ on the set U/G. So σ factors through a functor

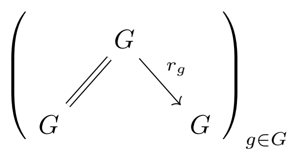
$$\alpha : A^{eff}(\mathcal{F}r^G) \to \mathcal{L}(G).$$

From here, showing that α is an equivalence is the easy part. Essential surjectivity is obvious. For full faithfulness, it suffices to show that α induces an equivalence

$$\alpha_0 : \mathrm{Map}_{A^{eff}(\mathcal{F}r^G)}(G, G) \to \mathrm{Map}_{\mathcal{L}(G)}(\Sigma^{\times G}, \Sigma^{\times G})$$

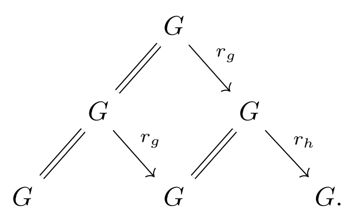
since all of the other relevant maps are products of some copies of this one.

Since G is a commutative monoid in $A^{eff}(\mathcal{F}r^G)$ and $\Sigma^{\times G}$ is a commutative monoid in $\mathcal{L}(G)$, α_0 underlies a map of commutative monoids, both of which are easily seen to be equivalent as commutative monoids to $\Sigma^{\times G}$. Thus it's enough to check that α_0 takes a set of free generators to a set of free generators. On the left, we may take this set to be



where r_g is right multiplication by g. On the other hand, we may take our set of generators on the right to be those automorphisms of $\Sigma^{\times G}$ induced by right multiplication by elements of g. Tracing through the definitions a final time, we see that α_0 maps the one set of generators to the other. This completes the proof. ☐

We have the functor $G \to A^{eff}(\mathcal{F}r^G)$ that takes, for example, a 2-simplex (g, h) to the diagram

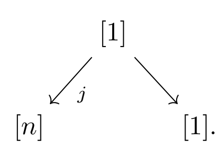


We have the composite

$$i_G : G \times \mathcal{F}_* \to A^{eff}(\mathcal{F}r^G) \times A^{eff}(\mathcal{F}) \to A^{eff}(\mathcal{F}r^G).$$

Proposition A.5. i_G is the universal commutative monoid with G-action, which is to say the initial functor satisfying the Segal condition from $G \times \mathcal{F}_*$ to a category with finite products.

Proof. First we must show that i_G is indeed a commutative monoid. G doesn't make any difference here; we just need to show that $i : \mathcal{F}_* \to A^{eff}(\mathcal{F})$ satisfies the Segal condition. This follows from the fact that the image of the inert map $\chi_j : \langle n \rangle \to \langle 1 \rangle$ is the span



Now let $U(G)$ be the universal category supporting a commutative monoid with G-action. Since $U(G)$ is the initial category under $G \times \mathcal{F}_*$ that takes certain diagrams to limit diagrams, Proposition 5.3.6.2 of [Lur09] gives a prescription for building it as the opposite of a full subcategory of a certain localization $S^{-1}\mathbf{Psh}((G\times\mathcal{F}_*)^{\mathrm{op}})$ of the presheaf category $\mathbf{Psh}((G\times\mathcal{F}_*)^{\mathrm{op}})$. Since we're in this business, let's let $Y : (G\times\mathcal{F}_*)^{\mathrm{op}} \to \mathbf{Psh}((G\times\mathcal{F}_*)^{\mathrm{op}})$ denote the Yoneda embedding.

In this case, the localization S is generated by the morphisms

$$\coprod_{\langle n\rangle^o} Y(\langle 1\rangle) \to Y(\langle n\rangle)$$

given by precomposition with the inert maps, and so localization is "Segalification" and the local objects are exactly the commutative monoid spaces with G-action. Thus $U(G)^{\mathrm{op}}$ is the full subcategory of $\mathrm{Fun}(\mathcal{F}_*\times G,\mathbf{Top})$ spanned by the Segalifications of

$$\coprod_{\langle n\rangle^o} Y(\langle 1\rangle)$$

as n varies. But $Y(\langle 1\rangle)$ is, by definition, left Kan extended along the inclusion

$$\langle 1\rangle : * \to \mathcal{F}_*\times G$$

and so its Segalification is the free commutative monoid with G-action on one generator. Since Segalification preserves coproducts, the other objects follow.

Now we've given an equivalence between $U(G)^{\mathrm{op}}$ and $\mathcal{L}(G)$, and therefore $A^{eff}(\mathcal{F}r^G)$. We know that this category is canonically self-opposite, so we might as well forget the op on $U(G)$. Let's show that this equivalence comes from i_G.

Since $A^{eff}(\mathcal{F}r^G)$ is semiadditive, specifying a commutative monoid with G-action $BG\times\mathcal{F}_*\to A^{eff}(\mathcal{F}r^G)$ is equivalent to specifying it on $G\times\langle 1\rangle$ (see Corollary 2.4.3.10 of [Lur12]). Both i_G and the universal commutative monoid in $U(G)$ take $G\times\langle 1\rangle$ to the G-set G with its right action on itself. This completes the proof. ◻

Corollary A.6. Let $\mathbf{C}$ be an ∞-category with finite products. Then pullback along i_G gives an equivalence

$$\mathrm{Fun}^{\times}(A^{eff}(\mathcal{F}r^G),\mathbf{C}) \to \mathbf{CMon}(\mathrm{Fun}(G,\mathbf{C})).$$

Proof. This is just a restatement of A.5. ◻

We deduce Theorem 2.27 as the special case of A.6 where $\mathbf{C}$ is semiadditive.

B The proof of Lemma 2.36

Lemma B.1. Let $\mathbf{C}$ be an ∞-category, $z : \mathbf{D}\to\mathbf{C}$ the inclusion of a full subcategory and $X\in\mathbf{C}$ an object. Let $\star$ denote the join of simplicial sets, and let

$$\begin{aligned}i_0 : \Delta^0 &\to \Delta^0 \star (\Delta^n \times \Delta^1)\\ i_1 : \Delta^n \times \{0\} &\to \Delta^0 \star (\Delta^n \times \Delta^1)\\ i_2 : \Delta^n \times \{1\} &\to \Delta^0 \star (\Delta^n \times \Delta^1)\end{aligned}$$

be the natural inclusions. We define a simplicial set $\mathbf{C}_{X/\mathbf{D}/}$ whose n-simplices are maps $\alpha : \Delta^0 \star (\Delta^n \times \Delta^1) \to \mathbf{C}$ with

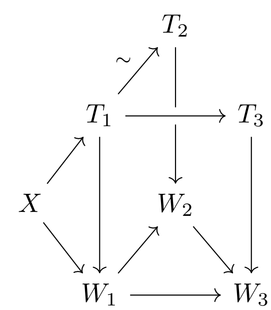
$$\alpha \circ i_0 = X, \quad \alpha \circ i_1 \in \mathrm{Fun}(\Delta^n, \mathbf{D}).$$

Then the map $p_2 : \mathbf{C}_{X/\mathbf{D}/} \to \mathbf{C}$ coming from precomposition with i_2 is cocartesian, and its cocartesian edges are those α for which the image of $\alpha \circ i_1$ is an equivalence. Moreover, the inclusion

$$\lambda : \mathbf{C}_{X/} \times_{\mathbf{C}} \mathbf{D} \hookrightarrow \mathbf{C}_{X/\mathbf{D}/}$$

formed by precomposition with the collapse map $\Delta^0 \star (\Delta^n \times \Delta^1) \to \Delta^0 \star \Delta^n$ is coinitial, and therefore left anodyne [Lur09, Proposition 4.1.1.3].

Proof. First we show that any edge α for which $\alpha \circ i_1$ is an equivalence is cocartesian. This is the claim that any commutative diagram of the form

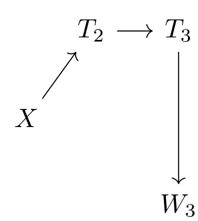


with $T_1 \to T_2$ an equivalence can be completed to a diagram from $\Delta^0 \star (\Delta^2 \times \Delta^1)$, which is clear by inspection. Since there are plenty of these edges, p_2 is cocartesian.

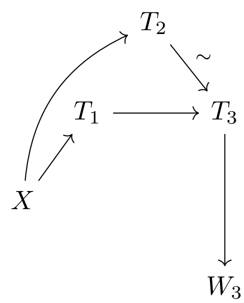
Now we tackle the coinitiality claim. In fact, we'll show that λ admits a right adjoint, which suffices. Let $\Lambda \to \Delta^1$ be the cocartesian fibration classified by λ; an n-simplex of Λ is a map $\tau : \Delta^n \to \Delta^1$ together with a map

$$\alpha : \left(\Delta^0 \star \left\{(i, j) \in \Delta^n \times \Delta^1 \,|\, j = 0 \text{ or } i \in \tau^{-1}(1)\right\}\right) \to \mathbf{C}$$

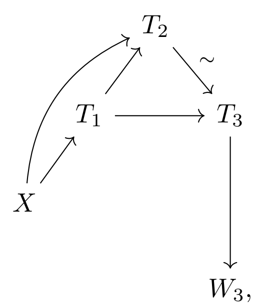
We wish to show that α is also cartesian. In fact, we claim that an edge



of Λ over the nondegenerate edge of Δ^1 is cartesian if $T_2 \to T_3$ is an equivalence. This is the claim that any commuting diagram of the form



can be extended to a commuting diagram of the form



which, again, is clear. □

We list some formal consequences of Lemma B.1.

Corollary B.2. Let $\mathbf{C}^{\mathbf{Top}}_{X/\mathbf{D}/}$ be a fibrant replacement for $\mathbf{C}_{X/\mathbf{D}/}$ in the covariant model structure over $\mathbf{C}$, so that $\mathbf{C}^{\mathbf{Top}}_{X/\mathbf{D}/} \to \mathbf{C}$ is a left fibration and for each object $W \in \mathbf{C}$, the map

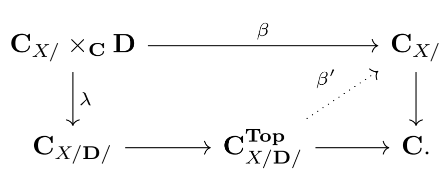
$$(\mathbf{C}^{\mathbf{Top}}_{X/\mathbf{D}/})_W \to (\mathbf{C}_{X/\mathbf{D}/})_W$$

is a Kan-Quillen weak equivalence. Then the functor classified by $\mathbf{C}^{\mathbf{Top}}_{X/\mathbf{D}/}$ is equivalent to the restriction and left Kan extension $z_! z^* \mathrm{Map}(X, -)$ of the functor corepresented by X.

Now let $\beta : \mathbf{C}_{X/\mathbf{D}/} \to \mathbf{C}_{X/}$ be given on n-simplices by precomposition with

$$i_0 \star i_2 : \Delta^0 \star \Delta^n \to \Delta^0 \star (\Delta^n \times \Delta^1).$$

Since $\mathbf{C}_{X/}$ is a left fibration, we have a commutative diagram



It follows that the counit map $c_X : z_! z^* \mathrm{Map}(X, -) \to \mathrm{Map}(X, -)$ is given, after unstraightening, by β'.

Lemma B.3. Let $W \in \mathbf{C}$ and let $f : X \to W$ be a morphism in $\mathbf{C}$. Then the homotopy fiber of

$$c_{X,W} : (z_! z^* \mathrm{Map}(X, -))_W \to \mathrm{Map}(X, W)$$

over f is given, up to weak equivalence, by the simplicial set $\mho_{W,f}$ whose n-simplices are maps $Z : \Delta^{n+2} \to \mathbf{C}$ such that

- $Z|_{\Delta^{\{0,n+2\}}} = f$, and
- for each i with $0 < i < n+2$, $Z(i) \in \mathbf{D}$.

Proof. We know that the Joyal model structure is self-enriched, by using [Lur09, Corollary 2.2.5.4] to deduce that the pushout-product of a trivial cofibration with a cofibration is a trivial cofibration, and it follows that if $K \to L$ is any cofibration of simplicial sets and $\mathbf{E}$ is a quasicategory, then $\mathrm{Fun}(L, \mathbf{E}) \to \mathrm{Fun}(K, \mathbf{E})$ is a categorical fibration. Since β is formed from such a fibration by pullback, β is also a categorical fibration.

The value of c_X on W is given, up to weak equivalence, by the map

$$\beta_W : (\mathbf{C}_{X/\mathbf{D}/})_W \to (\mathbf{C}_{X/})_W = \mathrm{Hom}^L(X, W).$$

Since the target of β_W is a Kan complex and β_W is a categorical fibration, it is a cocartesian fibration [Lur09, Proposition 3.3.18], and since a fibrant replacement for β_W in the covariant model structure over $\mathrm{Hom}^L(X, W)$ is automatically a Kan fibration, the fibers of β_W are its homotopy fibers.

By definition, the fiber $\beta_{W,f}$ over $f \in \mathrm{Hom}^L(X, W)$ is the simplicial set whose n-simplices are maps $Z' : (\Delta^0 \star (\Delta^n \times \Delta^1))/(\Delta^n \times \{1\}) \to \mathbf{C}$ such that

- $Z' \circ i_1 \subseteq \mathbf{D}$, and
- the $(n+1)$-simplex $Z' \circ (i_0 \star i_2)$ is the image of f under the rightmost degeneracy; that is, it is the totally degenerate n-simplex of $\mathbf{C}_{X/}$ at f.

In other words, an n-simplex of $\beta_{W,f}$ is a map

$$Z'' : \Delta^0 \star ((\Delta^n \times \Delta^1)/(\Delta^n \times \{1\}))$$

with $Z''(\Delta^n \times \{0\}) \in \mathbf{D}_n$ and $Z''(\Delta^0 \star ((\Delta^n \times \{1\})/(\Delta^n \times \{1\})) = f$. From here, the proof that $\beta_{W,f} \simeq \mho_{W,f}$ is a minor variant of the proof of [Lur09, Proposition 4.2.1.5]. ∎

We now immerse ourselves in the notation of Lemma 2.36, which we restate here for convenience.

Lemma B.4. Let $\mathcal{M}$ be an epiorbital category, $\mathcal{N}$ a downwardly closed subcategory of $\mathcal{M}$ and $\mathcal{T}$ its upwardly closed complement. Denote the restriction $A^{eff}(i_{\mathcal{T}})^*$ and the left Kan extension $A^{eff}(i_{\mathcal{T}})_!$ respectively by $\Pi^{\mathcal{T}}$ and $\Gamma^{\mathcal{T}}$, and similarly denote $A^{eff}(j_{\mathcal{N}})^*$ and $A^{eff}(j_{\mathcal{N}})_!$ by $\Xi^{\mathcal{N}}$ and $\Phi^{\mathcal{N}}$ respectively. Then there's a cofiber sequence of functors $\mathbf{Mack}(\mathcal{M}, \mathbf{CMon}) \to \mathbf{Mack}(\mathcal{M}, \mathbf{CMon})$

$$\Gamma^{\mathcal{T}}\Pi^{\mathcal{T}} \xrightarrow{\epsilon} \mathrm{Id} \xrightarrow{\eta} \Xi^{\mathcal{N}}\Phi^{\mathcal{N}},$$

where ϵ and η are the counit and unit of their respective adjunctions.

Proof. Since all of the functors in this sequence are colimit-preserving, it suffices to check that its value on each corepresentable Mackey functor is a cofiber sequence. Let $X \in A^{eff}(\mathcal{M})$ be an object and let $f : X \leftarrow Y \to W$ be a morphism in $A^{eff}(\mathcal{M})$. We'll analyze the fiber $\mho_{W,f}$ of Lemma B.3.

For $\mathbf{C}$ an ∞-category, let $\widetilde{\mathcal{O}}_{\mathbf{C}}$ be the twisted arrow category of $\mathbf{C}$ (see [Bar14, §2]) and let $\widetilde{\mathcal{O}}^{\mathbf{C}} := \widetilde{\mathcal{O}}_{\mathbf{C}}^{\mathrm{op}}$ be its opposite. Then an n-simplex of $\mho_{W,f}$ is by definition a functor

$$\varrho : \widetilde{\mathcal{O}}^{\Delta^{n+2}} \to \mathcal{M}^{\amalg}$$

such the span

$$\varrho(0,0) \leftarrow \varrho(0,n+2) \to \varrho(n+2,n+2)$$

coincides with f and $\varrho(i,i) \in \mathcal{T}^{\amalg}$ for all i with $0 < i < n+2$. By the upward closedness of $\mathcal{T}$, the latter condition implices that $\varrho(i,j) \in \mathcal{T}^{\amalg}$ for all $(i,j) \neq (0,0), (n+2,n+2)$, and in particular $\mho_{W,f}$ is empty unless $Y \in \mathcal{T}^{\amalg}$. We claim that if $Y \in \mathcal{T}^{\amalg}$, then $\mho_{W,f}$ is contractible. In fact, let $\Gamma_{W,f}$ be the subsimplicial set of $\mho_{W,f}$ whose n-simplices are those which factor through the morphism

$$\gamma : \widetilde{\mathcal{O}}^{\Delta^{n+2}} \to \widetilde{\mathcal{O}}^{\Delta^{n+2}} \qquad \gamma(i,j) = \begin{cases} (0,0) & \text{if } (i,j) = (0,0) \\ (n+2,n+2) & \text{if } (i,j) = (n+2,n+2) \\ (0,n+2) & \text{otherwise.} \end{cases}$$

If $Y \in \mathcal{T}^{\amalg}$, then clearly $\Gamma_{W,f} \cong *$, and we claim that $\Gamma_{W,f}$ is a simplicial deformation retract of $\mho_{W,f}$. We'll do this in two stages as follows. For each integer k, let $\gamma_k^L : \widetilde{\mathcal{O}}^{\Delta^{n+2}} \to \widetilde{\mathcal{O}}^{\Delta^{n+2}}$ be defined by

$$\gamma_k^L(i,j) = (\min(i-k,0), j)$$

and dually, define $\gamma_k^R : \widetilde{\mathcal{O}}^{\Delta^{n+2}} \to \widetilde{\mathcal{O}}^{\Delta^{n+2}}$ by

$$\gamma_k^R(i,j) = (i, \max(j+k, n+2)).$$

Let $\Gamma_{W,f}^L$ be the subsimplicial set of $\mho_{W,f}$ whose n-simplices factor through γ_{n+1}^L, and define $\Gamma_{W,f}^R$ similarly; we will show that each of $\Gamma_{W,f}^L$ and $\Gamma_{W,f}^R$ is a simplicial deformation retract of $\mho_{W,f}$, and since

$$\Gamma_{W,f} = \Gamma_{W,f}^L \cap \Gamma_{W,f}^R,$$

this will complete the proof of the claim. We will prove the result for $\Gamma^R_{W,f}$; the result for $\Gamma^L_{W,f}$ is, of course, entirely dual.

For each n, l with $0 \leq l \leq n+1$, let $\tau_{n,l} : \Delta^n \to \Delta^1$ be the unique map with

$$\tau_{n,l}^{-1}(0) = [0, \cdots, n-l],$$

where we interpret $[0, -1]$ as the empty interval. Then we define a map

$$\Theta : \mho_{W,f} \times \Delta^1 \to \mho_{W,f}$$

on n-simplices by

$$(\varrho, \tau_{n,l}) \mapsto \varrho \circ \gamma_l^R.$$

Then Θ_1 is a retraction onto $\Gamma^R_{W,f}$, so we have the required simplicial homotopy.

What we have proved so far is that

$$\epsilon_{c_X} : \Gamma^{\mathcal{T}} \Pi^{\mathcal{T}} c_X \to c_X,$$

after evaluation on an object W, is homotopic to the inclusion of the connected components of $\mathrm{Map}(X, W)$ comprising those maps $f : X \leftarrow Y \to W$ with $Y \in A^{eff}(\mathcal{T})$. What's left is easy: $\eta_{c_X}(W)$ is the natural map

$$\mathrm{Map}_{A^{eff}(\mathcal{M})}(X, W) \to \mathrm{Map}_{A^{eff}(\mathcal{N})}(j(X), j(W)),$$

which is just the projection away from the image of $\epsilon_{c_X}(W)$. This concludes the proof of Lemma 2.36. □

References

[AC15] Gregory Arone and Michael Ching. A classification of Taylor towers of functors of spaces and spectra. *Advances in Mathematics*, 272:471–552, 2015.

[ACB14] Omar Antolín-Camarena and Tobias Barthel. Chromatic fracture cubes. *arXiv preprint arXiv:1410.7271*, 2014.

[AK13] William Abram and Igor Kriz. The equivariant complex cobordism ring of a finite abelian group. *preprint*, 2013.

[Bar14] Clark Barwick. Spectral Mackey functors and equivariant algebraic K-theory (I). *arXiv preprint arXiv:1404.0108*, 2014.

[BDG+16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. Parametrized higher categories and parametrized higher algebra. In progress, 2016.

[BM03] Kristine Bauer and Randy McCarthy. On vanishing Tate cohomology and decompositions in Goodwillie calculus. Preprint, 2003.

[Bou79] Aldridge K Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[Gla16] Saul Glasman. Goodwillie calculus and mackey functors. *arXiv preprint arXiv:1610.03127*, 2016.

[GM11] Bertrand Guillou and JP May. Models of G-spectra as presheaves of spectra. *arXiv preprint arXiv:1110.3571*, 2011.

[Goo91] Thomas G Goodwillie. Calculus II: analytic functors. *K-theory*, 5(4):295–332, 1991.

[GS96] J PC Greenlees and Hal Sadofsky. The Tate spectrum of v_n-periodic complex oriented theories. *Mathematische Zeitschrift*, 222(3):391–405, 1996.

[HHR09] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. On the non-existence of elements of Kervaire invariant one. *http://arxiv.org/abs/0908.3724*, 2009.

[HL13] Michael Hopkins and Jacob Lurie. Ambidexterity in $K(n)$-local stable homotopy theory. 2013.

[Hov93] Mark Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *Proceedings of the Cech Centennial Conference on Homotopy Theory*, 1993.

[HS96] Mark Hovey and Hal Sadofsky. Tate cohomology lowers chromatic Bousfield classes. *Proceedings of the American Mathematical Society*, 124(11):3579–3585, 1996.

[Kuh04] Nicholas J Kuhn. Tate cohomology and periodic localization of polynomial functors. *Inventiones mathematicae*, 157(2):345–370, 2004.

[LMS86] L Gaunce Lewis, J Peter May, and Mark Steinberger. *Equivariant stable homotopy theory*. Springer, 1986.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[Lur12] Jacob Lurie. *Higher Algebra*. 2012.

[tD73] Tammo tom Dieck. Equivariant homology and Mackey functors. *Mathematische Annalen*, 206(1):67–78, 1973.

PARAMETRIZED HIGHER CATEGORY THEORY

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Abstract. We develop foundations for the theory of ∞-categories parametrized by a base ∞-category. Our main contribution is a theory of indexed homotopy limits and colimits, which specializes to a theory of G-colimits for G a finite group when the base is chosen to be the orbit category of G. We apply this theory to show that the G-∞-category of G-spaces is freely generated under G-colimits by the contractible G-space, thereby affirming a conjecture of Mike Hill.

Contents

1. Introduction

Motivation from equivariant homotopy theory: This paper lays foundations for a theory of ∞-categories parametrized by a base ∞-category S. Our interest in this project originates in attempting to locate the core homotopy theories of interest in equivariant homotopy theory - those of G-spaces and G-spectra - within the appropriate ∞-categorical framework. To explain, let G be a finite group and let us review the definitions of the ∞-categories of G-spaces and G-spectra, with a view towards endowing them with universal properties.

Consider a category $\mathbf{Top}_G$ of (nice) topological spaces equipped with G-action, with morphisms given by the G-equivariant continuous maps. There are various homotopy theories that derive from this category, depending on the class of weak equivalences that one chooses to invert. At one end, we can invert the class $\mathcal{W}_1$ of G-equivariant maps which induce a weak homotopy equivalence of underlying topological spaces, forgetting the G-action. If we let $\mathbf{Spc}$ denote the ∞-category of spaces (i.e., ∞-groupoids), then inverting $\mathcal{W}_1$ obtains the *∞-category of spaces with G-action*

$$\mathbf{Top}_G[\mathcal{W}_1^{-1}] \simeq \mathrm{Fun}(BG, \mathbf{Spc}).$$

For many purposes, $\mathrm{Fun}(BG, \mathbf{Spc})$ is the homotopy theory that one wishes to contemplate, but here we instead highlight its main deficiency. Namely, passing to this homotopy theory blurs the distinction between homotopy fixed points and actual fixed points, in that the functor $\mathbf{Top}_G \longrightarrow \mathrm{Fun}(BG, \mathbf{Spc})$ forgets the homotopy types of the various spaces X^H for H a nontrivial subgroup of G. Because many arguments in equivariant homotopy theory involve comparing X^H with the homotopy fixed points X^{hH}, we want to retain this data. To this end, we can instead let $\mathcal{W}$ be the class of G-equivariant maps which induce an equivalence on H-fixed points for every subgroup H of G. Let $\mathbf{Spc}_G := \mathbf{Top}_G[\mathcal{W}^{-1}]$; this is the *$\infty$-category of G-spaces.*

Like with $\mathbf{Top}_G[\mathcal{W}_1^{-1}]$, we would like a description of $\mathbf{Spc}_G$ which eliminates any reference to topological spaces with G-action, for the purpose of comprehending its universal property. Elmendorf's theorem grants such a description: we have

$$\mathbf{Spc}_G \simeq \mathrm{Fun}(\mathbf{O}_G^{op}, \mathbf{Spc}),$$

where $\mathbf{O}_G$ is the category of orbits of the group G. Thus, as an ∞-category, $\mathbf{Spc}_G$ is the *free cocompletion* of $\mathbf{O}_G$.

It is a more subtle matter to define the homotopy theory of G-spectra. There are at least three possibilities:

(1) The ∞-category of *Borel G-spectra*, i.e. spectra with G-action: This is
$$\mathbf{Sp}^{hG} := \mathrm{Fun}(BG, \mathbf{Sp}),$$
which is the stabilization of $\mathrm{Fun}(BG, \mathbf{Spc})$.

(2) The ∞-category of *'naive' G-spectra*, i.e. spectral presheaves on $\mathbf{O}_G$: This is
$$\mathbf{Sp}_G := \mathrm{Fun}(\mathbf{O}_G^{op}, \mathbf{Sp}),$$
which is the stabilization of $\mathbf{Spc}_G$.[1]

(3) The ∞-category of *'genuine' G-spectra*, i.e. spectral Mackey functors on the category $\mathbf{F}_G$ of finite G-sets: Let $A^{eff}(\mathbf{F}_G)$ be the effective Burnside $(2,1)$-category of G, given by taking as objects finite G-sets, as morphisms spans of finite G-sets, and as 2-morphisms isomorphisms between spans. Then, the ∞-category of genuine G-spectra is defined to be
$$\mathbf{Sp}^G := \mathrm{Fun}^{\oplus}(A^{eff}(\mathbf{F}_G), \mathbf{Sp}),$$
the ∞-category of direct-sum preserving functors from $A^{eff}(\mathbf{F}_G)$ to $\mathbf{Sp}$.[2]

The third possibility incorporates essential examples of cohomology theories for G-spaces, such as equivariant K-theory, because G-spectra in this sense possess transfers along maps of finite G-sets, encoded by the covariant maps in $A^{eff}(\mathbf{F}_G)$. It is thus what homotopy theorists customarily mean by G-spectra. However, from a categorical perspective it is a more mysterious object than the ∞-category of naive G-spectra, since it is *not* the stabilization of G-spaces. We are led to ask:

Question: What is the universal property of $\mathbf{Sp}^G$? More precisely, we have an adjunction
$$\Sigma^\infty_+ : \mathbf{Spc}_G \rightleftarrows \mathbf{Sp}^G : \Omega^\infty$$
with the right adjoint given by taking $\Omega^\infty : \mathbf{Sp} \longrightarrow \mathbf{Spc}$ objectwise and restricting along the evident map $\mathbf{O}_G^{op} \longrightarrow A^{eff}(\mathbf{F}_G)$, and we would like a universal property for Σ^∞_+ or Ω^∞.

Put another way, what is the categorical procedure which manufactures $\mathbf{Sp}^G$ from $\mathbf{Spc}_G$?

The key idea is that for this procedure of 'G-stabilization' one needs to enforce 'G-additivity' over and above the usual additivity satisfied by a stable ∞-category: that is, one wants the coincidence of coproducts and products indexed not just by finite sets but by finite sets with G-action. Reflecting upon the possible homotopical meaning of such a G-(co)product, we see that for a transitive G-set G/H, $\coprod_{G/H}$ and $\prod_{G/H}$ should be interpreted to mean the left and right adjoints to the restriction functor $\mathbf{Sp}^G \longrightarrow \mathbf{Sp}^H$, i.e. the induction and coinduction functors, and G-additivity then becomes the Wirthmüller isomorphism. In particular, we see that G-additivity is not a property that $\mathbf{Sp}^G$ can be said to enjoy in isolation, but rather one satisfied by the *presheaf* $\underline{\mathbf{Sp}}^G$ of ∞-categories indexed by $\mathbf{O}_G$; here, for every G-orbit U, a choice of basepoint specifying an isomorphism $U \cong G/H$ yields an equivalence $\underline{\mathbf{Sp}}^G(U) \simeq \mathbf{Sp}^H$, and the functoriality in maps of orbits is that of conjugation and

[1]The usage of a subscript G to indicate presheaves on $\mathbf{O}_G$ (whether valued in spaces or spectra) is consistent with our later notation for the S-category of S-objects in an arbitrary ∞-category – see Construction 3.9.

[2]This is not the definition which first appeared in the literature for G-spectra, but it is equivalent to e.g. the homotopy theory of orthogonal G-spectra by the pioneering work of Guillou-May [7]. For an ∞-categorical treatment, see [1].

restriction (in particular, recording the residual actions of the Weyl groups on $\mathbf{Sp}^H$). Correspondingly, we must rephrase our question so as to inquire after the universal property of the *morphism of* $\mathbf{O}_G$*-presheaves*

$$\underline{\Sigma}^\infty_+ : \underline{\mathbf{Spc}}_G \longrightarrow \underline{\mathbf{Sp}}^G,$$

where $\underline{\Sigma}^\infty_+$ is objectwise given by genuine H-suspension ranging over all subgroups $H \leq G$.

We now pause to observe that for the purpose of this analysis the group G is of secondary importance as compared to its associated category of orbits $\mathbf{O}_G$. Indeed, we focused on G-additivity as the distinguishing feature of genuine vs. naive G-spectra, as opposed to the invertibility of representation spheres, in order to evade representation theoretic aspects of equivariant stable homotopy theory. In order to frame our situation in its proper generality, let us now dispense with the group G and replace $\mathbf{O}_G$ by an arbitrary ∞-category T. Call a presheaf of ∞-categories on T a *T-category.* The *T-category of T-spaces* $\underline{\mathbf{Spc}}_T$ is given by the functor $T^{op} \longrightarrow \mathbf{Cat}_\infty$, $t \mapsto \mathrm{Fun}((T^{/t})^{op}, \mathbf{Spc})$. Note that this specializes to $\underline{\mathbf{Spc}}_G$ when $T = \mathbf{O}_G$ because $\mathbf{O}_H \simeq (\mathbf{O}_G)^{/(G/H)}$; slice categories stand in for subgroups in our theory. With the theory of T-colimits advanced in this paper, we can then supply a universal property for $\underline{\mathbf{Spc}}_T$ *as a T-category.* Write $\underline{\mathrm{Fun}}_T$ for the internal hom in the ∞-category of T-categories, which is cartesian closed.

1.1. **Theorem.** *Suppose T is any ∞-category. Then $\underline{\mathbf{Spc}}_T$ is T-cocomplete, and for any T-category E which is T-cocomplete, the T-functor of evaluation at the T-final object*[3]

$$\underline{\mathrm{Fun}}^L_T(\underline{\mathbf{Spc}}_T, E) \longrightarrow \underline{\mathrm{Fun}}_T(*_T, E) \simeq E$$

induces an equivalence from the T-category of T-functors $\underline{\mathbf{Spc}}_T \longrightarrow E$ which strongly preserve T-colimits to E. In other words, $\underline{\mathbf{Spc}}_T$ is freely generated under T-colimits by the final T-category.

1.2. **Remark.** The notion of T-cocompleteness needed for the theorem is slightly more elaborate than one might naively expect. Namely, we say that a T-category C is *T-cocomplete* if for all $t \in T$, the pullback of C to a $T^{/t}$-category $C_{\underline{t}}$ (Notation 2.29) admits all (small) $T^{/t}$-colimits (Definition 5.13). Correspondingly, we say that a T-functor $F : C \longrightarrow D$ strongly preserves T-colimits if for all $t \in T$, the pulled-back $T^{/t}$-functor $F_{\underline{t}} : C_{\underline{t}} \longrightarrow D_{\underline{t}}$ preserves all $T^{/t}$-colimits (Definition 11.2).

When $T = \mathbf{O}_G$, this result was originally conjectured by Mike Hill.

To go further and define T-spectra, we need a condition on T so that it supports a theory of spectral Mackey functors. We say that T is *orbital* if T admits multipullbacks, by which we mean that its finite coproduct completion $\mathbf{F}_T$ admits pullbacks. The purpose of the orbitality assumption is to ensure that the effective Burnside category $A^{\mathit{eff}}(\mathbf{F}_T)$ is well-defined. Note that the slice categories $T_{/t}$ are orbital if T is. We define the *T-category of T-spectra* $\underline{\mathbf{Sp}}^T$ to be the functor $T^{op} \longrightarrow \mathbf{Cat}_\infty$ given by $t \mapsto \mathrm{Fun}^\oplus(A^{\mathit{eff}}(\mathbf{F}_{T_{/t}}), \mathbf{Sp})$. We then have the following theorem of Denis Nardin concerning $\underline{\mathbf{Sp}}^T$ from [16], which resolves our question:

1.3. **Theorem** ([16, Theorem 7.4])**.** *Suppose T is an atomic*[4] *orbital ∞-category. Then $\underline{\mathbf{Sp}}^T$ is T-stable, and for any pointed T-category C which has all finite T-colimits, the functor of postcomposition by Ω^∞*

$$(\Omega^\infty)_* : \mathrm{Fun}^{T-rex}_T(C, \underline{\mathbf{Sp}}^T) \longrightarrow \mathrm{Lin}^T(C, \underline{\mathbf{Spc}}_T)$$

induces an equivalence from the ∞-category of T-functors $C \longrightarrow \underline{\mathbf{Sp}}^T$ which preserve finite T-colimits to the ∞-category of T-linear functors $C \longrightarrow \underline{\mathbf{Spc}}_T$, i.e. those T-functors which are fiberwise linear and send finite T-coproducts to T-products.

We hope that the two aforementioned theorems will serve to impress upon the reader the utility of the purely ∞-categorical work that we undertake in this paper.

1.4. **Warning.** In contrast to this introduction thus far and the conventions adopted elsewhere (e.g. in [16]), we will henceforth speak of S-categories, S-colimits, etc. for $S = T^{op}$.

[3] We define $*_T$ to be the constant T-presheaf valued at $*$, which is the final object in the ∞-category of T-categories.

[4] This is an additional technical hypothesis which we do not explain here. It will not concern us in the body of the paper.

What is parametrized ∞-category theory? Roughly speaking, parametrized ∞-category theory is an interpretation of the familiar notions of ordinary or 'absolute' ∞-category theory within the $(\infty, 2)$-category of functors $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$, done relative to a fixed 'base' ∞-category S. By 'interpretation', we mean something along the lines of the program of Emily Riehl and Dominic Verity [17], which axiomatizes the essential properties of an $(\infty, 2)$-category that one needs to do formal category theory into the notion of an ∞-*cosmos*, of which $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ is an example. In an ∞-cosmos, one can write down in a formal way notions of limits and colimits, adjunctions, Kan extensions, and so forth. Working out what this means in the example of $\mathbf{Cat}_\infty$-valued functors is the goal of this paper. In the classical 2-categorical setting, such limits and colimits are referred to as "indexed" limits and colimits, so another perspective on this paper is that it extends indexed category theory to the ∞-categorical setting.

In contrast to Riehl–Verity, we will work within the model of *quasi-categories* and not hesitate to use special aspects of our model (e.g., combinatorial arguments involving simplicial sets). We are motivated in this respect by the existence of a highly developed theory of *cocartesian fibrations* due to Jacob Lurie, which we review in §2. Cocartesian fibrations are our preferred way to model $\mathbf{Cat}_\infty$-valued functors, for two reasons:

(1) The data of a functor $F : S \longrightarrow \mathbf{Cat}_\infty$ is overdetermined vs. that of a cocartesian fibration over S, in the sense that to define F one must prescribe an infinite hierarchy of coherence data, which under the functor-fibration correspondence amounts to prescribing an infinite sequence of compatible horn fillings.[5] Because of this, specifying a cocartesian fibration (which one ultimately needs to do in order to connect our theory to applications) is typically an easier task than specifying the corresponding functor to $\mathbf{Cat}_\infty$.

(2) The Grothendieck construction on a functor $S \longrightarrow \mathbf{Cat}_\infty$ is made visible in the cocartesian fibration setup, as the total category of the cocartesian fibration. Many of our arguments involve direct manipulation of the Grothendieck construction, in order to relate or reduce notions of parametrized ∞-category theory to absolute ∞-category theory.

We have therefore tailored our exposition to the reader familiar with the first five chapters of [10]; the only additional major prerequisite is the part of [12, App. B] dealing with variants of the cocartesian model structure of [10, §3] and functoriality in the base.

Linear overview. Let us now give a section-by-section summary of the contents of this paper.

(§2) We define an S-*category* as a cocartesian fibration over S, and then collect some necessary preliminaries on cocartesian fibrations and model structures on categories of marked simplicial sets. In particular, we recapitulate Lurie's theorem that establishes conditions under which change-of-base adjunctions are Quillen (Theorem 2.24); this theorem will allow us to efficiently verify the fibrancy of many of the simplicial set constructions introduced in this paper.

(§3) We first define and study the internal hom $\underline{\mathrm{Fun}}_S(-,-)$ of S-categories (Definition 3.2). We then recall the S-*category of* S-*objects* $\underline{E}_S$ in an ∞-category E from [2] (Construction 3.9), which computes the right adjoint to the forgetful functor $[C \longrightarrow S] \mapsto C$. When $S = \mathbf{O}_G^{op}$ and $E = \mathbf{Spc}$, this recovers the G-category of G-spaces $\underline{\mathbf{Spc}}_G$.

(§4) We first introduce the S-join $(- \star_S -)$ (Definition 4.1), which in terms of presheaves computes the fiberwise join. We then define and study two (canonically equivalent) S-slice constructions: for a S-functor $p : K \longrightarrow C$, we have S-undercategories $C_{(p,S)/}$ and $C^{(p,S)/}$ and S-overcategories $C_{/(p,S)}$ and $C^{/(p,S)}$. The 'lower' construction (Definition 4.17) is a direct generalization of Joyal's slice construction (cf. [10, Proposition 1.2.9.2]) and participates in a Quillen adjunction with the S-join. The 'upper' construction (Definition 4.26) proceeds by taking an S-fiber of the relevant map of S-functor categories. In practice, the upper S-slice is far easier to work with as its definition is less bound up with the intricate combinatorics of the S-join (which need to be thoroughly understood to even establish the fibrancy of the lower S-slice; cf. Proposition 4.11). However, it is easier to establish the universal mapping property of the S-slice using its lower incarnation (Proposition 4.25).

[5] It is for this reason that one speaks of *straightening* a cocartesian fibration to a functor.

(§5) We initiate our study of S-colimits and S-limits by giving the basic definition 5.2, and then discuss a few special cases: S-(co)limits in an S-category of S-objects, S-colimits indexed by constant S-diagrams, and S-colimits indexed by S-points (i.e., S-coproducts). We then explain how to deduce results about S-limits from S-colimits (or vice-versa) by means of the vertical opposite construction (Corollary 5.25).

(§6) Our main goal in this section is to establish an S-analogue of Joyal's cofinality theorem [10, Theorem 4.1.3.1]: an S-functor $C \longrightarrow D$ is *S-final* if and only if it is fiberwise final[6] (Theorem 6.7). Our strategy is to control the functoriality encoded by the S-slice category in terms of a construction, the *twisted slice* (Definition 6.5), fibered over the twisted arrow category $\widetilde{\mathcal{O}}(S)$; the right Kan extension of the latter will then obtain the former (Theorem 6.6). In fact, we first do the same for the internal hom $\underline{\mathrm{Fun}}_S$ itself (Equation 6.3.1). This may be thought of as a refinement of the end formula for an ∞-category of natural transformations (cf. Remark 6.4).

(§7) In this brief section, we introduce the notions of S-fibration, S-(co)cartesian fibration, and S-bifibration (Definition 7.1 and Definition 7.9). We also introduce the free S-(co)cartesian fibration as an example (Definition 7.6).

(§8) We recall Lurie's definition of a relative adjunction and specialize it to the notion of an S-adjunction (Definition 8.3). We then prove a number of fundamental results about S-adjunctions —most notably, the fact that a left S-adjoint preserve S-colimits (Corollary 8.9).

(§9) Given an S-cocartesian fibration $\phi : C \longrightarrow D$ and an S-functor $F : C \longrightarrow E$, we construct the left S-Kan extension $\phi_! F : D \longrightarrow E$, which will also call the *D-parametrized S-colimit* of F. With our assumption on ϕ, we have that for every object $x \in D_s$, $(\phi_! F)(x)$ is computed as the $S^{s/}$-colimit of the restriction of F to the $S^{s/}$-fiber $C_{\underline{x}}$; this is precisely analogous to the situation where the left Kan extension along a cocartesian fibration is computed by taking colimits fiberwise. In order to construct $\phi_! F$, we need to solve the coherence problem of assembling the individual $S^{s/}$-colimits of $F_{\underline{s}} : C_{\underline{x}} \longrightarrow E_{\underline{s}}$ (ranging over all $x \in D_s$) into a single S-functor out of D. We introduce the S-pairing construction 9.1, and subsequently the D-parametrized slice (Construction 9.8), to facilitate this. The problem of constructing $\phi_! F$ then ultimately reduces to choosing a section of a certain trivial Kan fibration defined in terms of the D-parametrized slice (Theorem 9.15).

(§10) We define left S-Kan extensions in general (Definition 10.1) and prove the basic existence and uniqueness result about them (Theorem 10.3). In contrast to the brutal simplex-by-simplex approach taken in [10, §4.3.2] to the construction of Kan extensions (cf. [10, Lemma 4.3.2.13]), we instead reduce to the solved coherence problem for D-parametrized S-colimits via factoring the S-functor $\phi : C \longrightarrow D$ to be extended along through the free S-cocartesian fibration on it. We remark that, to our knowledge, the approach of §9 and §10 gave a novel[7] and more conceptual construction of Kan extensions even in the context of ordinary ∞-category theory. Lurie has since independently written up a treatment of (relative) Kan extensions along these lines in Kerodon [13, Tag 02Y1].

(§11) We recall the S-category of presheaves $\mathbf{P}_S(-)$, prove the S-Yoneda lemma 11.1, discuss S-mapping spaces, and establish the universal property of $\mathbf{P}_S(-)$ as free S-cocompletion (Theorem 11.5), thereby proving Theorem 1.1.

(§12) We prove two Bousfield-Kan style[8] decomposition results that express an arbitrary S-colimit as a geometric realization of either S-coproducts or S-space-indexed S-colimits (Theorem 12.13 and Theorem 12.6). The essential content behind such formulas lies in replacing a given diagram C with one fibered over $\Delta^{op} \times S$ that possesses an S-final map to C. As a warmup, we first explain how this goes when S is a point (Corollary 12.3 and Corollary 12.5); the resulting formula appears to be new in the case of coproducts, whereas the case of spaces was first obtained by Aaron Mazel-Gee in [15]. We then apply the S-Bousfield-Kan formula to show

[6] We write final and initial for what Lurie calls (left) cofinal and right cofinal, respectively.

[7] All these results date to 2017.

[8] By this, we mean to refer to generalizations of the classical formula for writing a colimit as a coequalizer of coproducts, which were studied by Bousfield and Kan in the context of homotopy colimits with coequalizers replaced by geometric realization.

that, supposing S^{op} admits multipullbacks, an S-category is S-cocomplete if and only if it admits all S-(co)products and geometric realizations (Corollary 12.15).

Notation and conventions. Let C be an ∞-category. We write

$$\mathcal{O}(C) := \mathrm{Fun}(\Delta^1, C)$$

for the ∞-category of arrows in C. In this paper, we will frequently encounter fiber products of the form

$$A \times_{F,C,\mathrm{ev}_0} \mathcal{O}(C) \times_{\mathrm{ev}_1,C,G} B$$

where $F : A \longrightarrow C$ and $G : B \longrightarrow C$ are functors. To avoid notational clutter, we adopt the global convention that, unless otherwise decorated, fiber products with the source functor ev_0 are to be written on the left, and fiber products with the target functor ev_1 are to written on the right. Moreover, we will drop F and G from the notation if they are understood from context. For instance, we would write the preceding expression as $A \times_C \mathcal{O}(C) \times_C B$.

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2. Cocartesian fibrations and model categories of marked simplicial sets

Let S be an ∞-category. In this section, we give a rapid review of the theory of cocartesian fibrations and the surrounding apparatus of marked simplicial sets. This primarily serves to fix some of our notation and conventions for the remainder of the paper; for a more detailed exposition of these concepts, we refer the reader to [5]. In particular, the reader should be aware of our special notation (Notation 2.29) for the S-fibers of a S-functor.

Cocartesian fibrations. We begin with the basic definitions:

2.1. **Definition.** Let $\pi : X \longrightarrow S$ be a map of simplicial sets. Then π is a *cocartesian fibration* if

(1) π is an *inner fibration*: for every $n > 1$, $0 < k < n$ and commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \longrightarrow & S, \end{array}$$

the dotted lift exists.

(2) For every edge $\alpha : s_0 \to s_1$ in S and $x_0 \in X$ with $\pi(x_0) = s_0$, there exists an edge $e : x_0 \to x_1$ in X with $\pi(e) = \alpha$, such that e is π-*cocartesian*: for every $n > 1$ and commutative square

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{f} & X \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \longrightarrow & S \end{array}$$

with $f|_{\Delta^{\{0,1\}}} = e$, the dotted lift exists.

Dually, π is a *cartesian fibration* if π^{op} is a cocartesian fibration.

A cocartesian resp. cartesian fibration $\pi : X \longrightarrow S$ is said to be a *left* resp. *right* fibration if for every object $s \in S$ the fiber X_s is a Kan complex.

Now suppose $\pi : X \longrightarrow S$ and $\rho : Y \longrightarrow S$ are (co)cartesian fibrations. Then a *map of (co)cartesian fibrations* $f : X \longrightarrow Y$ is a map of simplicial sets such that $\rho \circ f = \pi$ and f carries π-(co)cartesian edges to ρ-(co)cartesian edges. The collection of cocartesian fibrations over S and maps thereof organize into a subcategory $\mathbf{Cat}^{cocart}_{\infty/S}$ of the overcategory $\mathbf{Cat}_{\infty/S}$.

In this paper, owing to the importance of these notions we see fit to introduce more concise and suggestive terminology for cocartesian fibrations and left fibrations over S.

2.2. **Definition.** An *S-category* resp. *S-space* C is a cocartesian resp. left fibration $\pi : C \longrightarrow S$. An *$S$-functor* $F : C \longrightarrow D$ between S-categories C and D is a map of cocartesian fibrations over S.

Given an S-category $\pi : C \longrightarrow S$, an *$S$-subcategory* $D \subset C$ is a subcategory such that the restriction $\pi|_D$ is a cocartesian fibration and an edge in D is $\pi|_D$-cocartesian if and only if it is π-cocartesian. The inclusion functor then necessarily preserves cocartesian edges, so is an S-functor. We further say that D is a *full S-subcategory* if $D \subset C$ is in addition a full subcategory, or equivalently, for every $s \in S$, $D_s \subset C_s$ is a full subcategory.

2.3. **Example** (Arrow ∞-categories)**.** The arrow ∞-category $\mathcal{O}(S)$ of S is cocartesian over S via the target morphism ev_1, and cartesian over S via the source morphism ev_0. An edge

$$e : [s_0 \to t_0] \longrightarrow [s_1 \to t_1]$$

in $\mathcal{O}(S)$ is ev_1-cocartesian resp. ev_0-cartesian if and only if $\mathrm{ev}_0(e)$ resp. $\mathrm{ev}_1(e)$ is an equivalence in S.

The fiber of $\mathrm{ev}_0 : \mathcal{O}(S) \longrightarrow S$ over s is isomorphic to Lurie's 'alternative' slice ∞-category $S^{s/}$. Using our knowledge of the ev_1-cocartesian edges, we see that ev_1 restricts to a left fibration $S^{s/} \longrightarrow S$. In the terminology of [10, Proposition 4.4.4.5], this is a *corepresentable* left fibration. We will refer to the corepresentable left fibrations as *S-points*. Further emphasizing this viewpoint, we will often let $\underline{s}$ denote $S^{s/}$.

To a beginner, the lifting conditions of Definition 2.1 can seem opaque. Under our standing assumption that S is an ∞-category, we have a reformulation of the definition of cocartesian edge, and hence that of cocartesian fibration, which serves to illuminate its homotopical meaning.

2.4. **Proposition.** *Let $\pi : X \longrightarrow S$ be an inner fibration (so X is an ∞-category). Then an edge $e : x_0 \to x_1$ in X is π-cocartesian if and only if for every $x_2 \in X$, the commutative square of mapping spaces*

$$\begin{array}{ccc} \mathrm{Map}_X(x_1, x_2) & \xrightarrow{e^*} & \mathrm{Map}_X(x_0, x_2) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Map}_S(\pi(x_1), \pi(x_2)) & \xrightarrow{\pi(e)^*} & \mathrm{Map}_S(\pi(x_0), \pi(x_2)) \end{array}$$

is homotopy cartesian.

With some work, Proposition 2.4 can be used to supply an alternative, model-independent definition of a cocartesian fibration: we refer to Mazel-Gee's paper [14] for an exposition along these lines.

2.5. **Example** ([10, §3.2.2])**.** Let $\mathbf{Cat}_\infty$ denote the (large) ∞-category of (small) ∞-categories. Then there exists a *universal cocartesian fibration* $\mathcal{U} \longrightarrow \mathbf{Cat}_\infty$, which is characterized up to contractible choice by the requirement that any cocartesian fibration $\pi : X \longrightarrow S$ (with essentially small fibers) fits into a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{U} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ S & \xrightarrow{F_\pi} & \mathbf{Cat}_\infty. \end{array}$$

Concretely, one can take $\mathcal{U}$ to be the subcategory of the arrow category $\mathcal{O}(\mathbf{Cat}_\infty)$ spanned by the representable right fibrations and morphisms thereof.

As suggested by Example 2.5, the functor

$$\mathrm{Fun}(S, \mathbf{Cat}_\infty) \longrightarrow \mathbf{Cat}_{\infty/S}^{cocart}$$

given by pulling back $\mathcal{U} \longrightarrow \mathbf{Cat}_\infty$ is an equivalence. The composition

$$\mathrm{Gr} : \mathrm{Fun}(S, \mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Cat}_{\infty/S}^{cocart} \subset \mathbf{Cat}_{\infty/S}$$

is the *Grothendieck construction* functor. Since equivalences in $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ are detected objectwise, Gr is conservative. Moreover, one can check that Gr preserves limit and colimits, so by the adjoint functor theorem Gr admits both a left and a right adjoint.

2.6. **Notation.** Let

$$\mathrm{Fr} \dashv \mathrm{Gr} \dashv H$$

denote the left and right adjoints of Gr.

We call Fr the *free cocartesian fibration* functor (see also [6]): concretely, one has

$$\mathrm{Fr}(X \longrightarrow S) = X \times_S \mathcal{O}(S) \xrightarrow{ev_1} S,$$

or as a functor $s \mapsto X \times_S S_{/s}$ with functoriality obtained from $S_{/(-)}$. The functor H can also be concretely described using its universal mapping property: since $\mathrm{Fr}(\{s\} \subset S) = S_{s/}$, the fiber $H(X)_s$ is given by $\mathrm{Fun}_{/S}(S_{s/}, X)$, and the functoriality in S is obtained from that of $S_{(-)/}$.

A model structure for cocartesian fibrations. We want a model structure which has as its fibrant objects the cocartesian fibrations over a fixed simplicial set. However, it is clear that to define it we need some way to remember the data of the cocartesian edges. This leads us to introduce *marked simplicial sets*.

2.7. **Definition.** A marked simplicial set $(X, \mathcal{E})$ is the data of a simplicial set X and a subset $\mathcal{E} \subset X_1$ of the edges of X, such that $\mathcal{E}$ contains all of the degenerate edges. We call $\mathcal{E}$ the set of *marked edges* of X. A map of marked simplicial sets $f : (X, \mathcal{E}) \longrightarrow (Y, \mathcal{F})$ is a map of simplicial sets $f : X \longrightarrow Y$ such that $f(\mathcal{E}) \subset \mathcal{F}$.

2.8. **Notation.** We introduce notation for certain classes of marked simplicial sets. Let X be a simplicial set.

- $X^\flat$ is X with only the degenerate edges marked. To avoid notational clutter, we will sometimes suppress this notation and simply write X for $X^\flat$.
- $X^\sharp$ is X with all of its edges marked.
- Suppose that X is an ∞-category. Then $X^\sim$ is X with its equivalences marked.
- Suppose that $\pi : X \longrightarrow S$ is an inner fibration. Then ${}_\natural X$ is X with its π-cocartesian edges marked, and $X^\natural$ is X with its π-cartesian edges marked.
- Let $n > 0$. Let ${}_\natural\Delta^n$ resp. ${}_\natural\Lambda^n_0$ denote Δ^n resp. Λ^n_0 with the edge $\{0, 1\}$ marked (if it exists, i.e. excluding Δ^0 and $\Lambda^1_0 = \{0\}$) along with the degenerate edges. Dually, let $\Delta^{n\natural}$ resp. $\Lambda^{n\natural}_n$ denote Δ^n resp. Λ^n_n with the edge $\{n-1, n\}$ marked.

Note that our choice of notation ${}_\natural\Delta^n$ and ${}_\natural\Lambda^n_0$ is not meant to be interpreted as a special instance of marking cocartesian edges (though the map $\Delta^n \longrightarrow \Delta^1$ given by $0 \mapsto 0$ and $1, ..., n \mapsto 1$ renders it as such for the former); rather, we mean to indicate that the relevant lifting problem for a cocartesian fibration as a marked simplicial set is to lift along the marked horn inclusion ${}_\natural\Lambda^n_0 \longrightarrow {}_\natural\Delta^n$ (cf. Definition 2.9 below), and vice-versa for cartesian fibrations and $\Lambda^{n\natural}_n \longrightarrow \Delta^{n\natural}$.

For the rest of this section, fix a marked simplicial set $(Z, \mathcal{E})$ where Z is an ∞-category and $\mathcal{E}$ contains all of the equivalences in Z; in our applications, Z will generally be some type of fibration over S. Let $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ denote the category of marked simplicial sets over $(Z, \mathcal{E})$; following Lurie [10, Notation 3.1.0.2], we will also denote $s\mathbf{Set}^+_{/Z^\sharp}$ more simply as $s\mathbf{Set}^+_{/Z}$. We will frequently abuse notation by referring an object $\pi : (X, \mathcal{F}) \longrightarrow (Z, \mathcal{E})$ of $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ by its domain $(X, \mathcal{F})$, or even just by X.

2.9. **Definition.** An object $(X, \mathcal{F})$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ is $(Z, \mathcal{E})$-*fibered*[9] if

(1) $\pi : X \longrightarrow Z$ is an inner fibration.
(2) For every $n > 0$ and commutative square

$$\begin{array}{ccc} {}_\natural\Lambda^n_0 & \longrightarrow & (X, \mathcal{F}) \\ \downarrow & \nearrow & \downarrow \\ {}_\natural\Delta^n & \longrightarrow & (Z, \mathcal{E}), \end{array}$$

[9]This differs from [12, Definition B.0.19], but nonetheless defines the correct class of anodyne morphisms [12, Definition B.1.1].

a dotted lift exists. In other words, letting $n = 1$, π-cocartesian lifts exist over marked edges in Z, and letting $n > 1$, marked edges in X are π-cocartesian.[10]

(3) For every commutative square

$$\begin{array}{ccc} (\Lambda^2_1)^\sharp \cup_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat & \longrightarrow & (X, \mathcal{F}) \\ \downarrow & \nearrow & \downarrow \\ (\Delta^2)^\sharp & \longrightarrow & (Z, \mathcal{E}), \end{array}$$

a dotted lift exists. In other words, marked edges are closed under composition.[11]

(4) Let $Q = \Delta^0 \coprod_{\Delta^{\{0,2\}}} \Delta^3 \coprod_{\Delta^{\{1,3\}}} \Delta^0$. For every commutative square

$$\begin{array}{ccc} Q^\flat & \longrightarrow & (X, \mathcal{F}) \\ \downarrow & \nearrow & \downarrow \\ Q^\sharp & \longrightarrow & (Z, \mathcal{E}), \end{array}$$

a dotted lift exists. Since we assumed that $\mathcal{E}$ contains all equivalences in Z, this implies that all equivalences in X are marked.

2.10. **Example.** Let $\pi : X \longrightarrow Z$ be an inner fibration. Comparing with Definition 2.1, it is clear that $(X, \mathcal{F})$ is $Z^\sharp$-fibered if and only if π is a cocartesian fibration and $(X, \mathcal{F}) = {}_\natural X$. At the other extreme, $(X, \mathcal{F})$ is $Z^\sim$-fibered if and only if π is a categorical fibration and $(X, \mathcal{F}) = X^\sim$.

Recall that a model structure, if it exists, is determined by its cofibrations and fibrant objects. Collecting results of Lurie from [12, App. B], we now define a model structure on $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ with cofibrations the monomorphisms and fibrant objects given by the $(Z, \mathcal{E})$-fibered objects.

2.11. **Definition.** Define functors[12]

$$\mathrm{Map}_Z(-,-) : s\mathbf{Set}^{+\,op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}$$
$$\mathrm{Fun}_Z(-,-) : s\mathbf{Set}^{+\,op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}$$

by

$$\mathrm{Hom}(A, \mathrm{Map}_Z(X, Y)) = \mathrm{Hom}_{/(Z,\mathcal{E})}(A^\sharp \times X, Y),$$
$$\mathrm{Hom}(A, \mathrm{Fun}_Z(X, Y)) = \mathrm{Hom}_{/(Z,\mathcal{E})}(A^\flat \times X, Y).$$

2.12. **Definition.** A map $f : A \longrightarrow B$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ is a *cocartesian equivalence* (with respect to $(Z, \mathcal{E})$) if the following equivalent conditions obtain:

(1) For all $(Z, \mathcal{E})$-fibered X, $f^* : \mathrm{Map}_Z(B, X) \longrightarrow \mathrm{Map}_Z(A, X)$ is an equivalence of Kan complexes.
(2) For all $(Z, \mathcal{E})$-fibered X, $f^* : \mathrm{Fun}_Z(B, X) \longrightarrow \mathrm{Fun}_Z(A, X)$ is an equivalence of ∞-categories.

2.13. **Theorem** ([12, Theorem B.0.20])**.** *There exists a left proper combinatorial model structure on the category $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$, which we call the* ***cocartesian model structure****, such that:*

(1) The cofibrations are the monomorphisms.
(2) The weak equivalences are the cocartesian equivalences.
(3) The fibrant objects are the $(Z, \mathcal{E})$-fibered objects.

Dually, we define the ***cartesian model structure*** *on $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ to be the cocartesian model structure on $s\mathbf{Set}^+_{/(Z,\mathcal{E})^{op}}$ under the isomorphism given by taking opposites.*

[10]Note that condition (2) already guarantees that $X \longrightarrow Z$ is a cocartesian fibration if $\mathcal{E} = Z_1$; however, one additionally needs condition (4) to ensure that *all* of the π-cocartesian edges are marked in X.

[11]Strictly speaking, condition (3) by itself only guarantees that for any pair of composable marked edges, there exists a composite that is again marked. One additionally needs condition (4) to ensure that *all* compositions of marked edges are again marked.

[12]In [12, App. B], these functors are denoted as $\mathrm{Map}^\sharp_Z$ and $\mathrm{Map}^\flat_Z$ respectively.

2.14. **Remark.** The underlying ∞-category of $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ identifies as the subcategory of $\mathbf{Cat}_{\infty/Z}$ on those isofibrations[13] $X \longrightarrow Z$ that admit cocartesian lifts over $\mathcal{E}$, and with morphisms preserving cocartesian edges. In particular, passing to the closure of $\mathcal{E}$ under composition does not change the underlying ∞-category.

We have the following characterization of the cocartesian equivalences between fibrant objects (which is unsurprising, in light of the equivalence $\mathbf{Cat}^{cocart}_{\infty/Z} \simeq \mathrm{Fun}(Z, \mathbf{Cat}_\infty)$).

2.15. **Proposition** ([12, Lemma B.2.4])**.** *Let X and Y be fibrant objects in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ equipped with the cocartesian model structure, and let $f : X \longrightarrow Y$ be a map in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$. Then the following are equivalent:*

(1) f is a cocartesian equivalence.
(2) f is a homotopy equivalence, i.e. f admits a homotopy inverse: there exists a map $g : Y \longrightarrow X$ and homotopies $h : (\Delta^1)^\sharp \times X \longrightarrow X$, $h' : (\Delta^1)^\sharp \times Y \longrightarrow Y$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ connecting $g \circ f$ to id_X and $f \circ g$ to id_Y, respectively.
(3) f is a categorical equivalence.
(4) For every (not necessarily marked) edge $\alpha : \Delta^1 \longrightarrow Z$, $f_\alpha : \Delta^1 \times_Z X \longrightarrow \Delta^1 \times_Z Y$ is a categorical equivalence.

If every edge of Z is marked, then (4) can be replaced by the following apparently weaker condition:

(4′) For every object $z \in Z$, $f_z : X_z \longrightarrow Y_z$ is a categorical equivalence.

We also have the following characterization of the fibrations between fibrant objects.

2.16. **Proposition** ([12, Proposition B.2.7])**.** *Let $Y = (Y, \mathcal{F})$ be a fibrant object in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ equipped with the cocartesian model structure, and let $f : X \longrightarrow Y$ be a map in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$. Then the following are equivalent:*

(1) f is a fibration.
(2) X is fibrant, and f is a categorical fibration.
(3) f is fibrant in $s\mathbf{Set}^+_{/(Y,\mathcal{F})}$.

2.17. **Corollary.** *Suppose $Z \longrightarrow S$ is a cocartesian fibration. Then the cocartesian model structure $s\mathbf{Set}^+_{/_\natural Z}$ coincides with the 'slice' model structure on $(s\mathbf{Set}^+_{/S})_{/_\natural Z}$ created by the forgetful functor to $s\mathbf{Set}^+_{/S}$ equipped with its cocartesian model structure.*

Proof. This immediately follows from Proposition 2.16. □

2.18. **Example.** Suppose that Z is a Kan complex. Then the cocartesian and cartesian model structures on $s\mathbf{Set}^+_{/Z}$ coincide. In particular, taking $Z = \Delta^0$, we will also refer to the cocartesian model structure on $s\mathbf{Set}^+$ as the *marked model structure*. Since this model structure on $s\mathbf{Set}^+$ is unambiguous, we will always regard $s\mathbf{Set}^+$ as equipped with it. Then the fibrant objects of $s\mathbf{Set}^+$ are precisely the ∞-categories with their equivalences marked.

2.19. **Example.** Suppose that $(Z, \mathcal{E}) = Z^\sim$. Then the cocartesian and cartesian model structures on $s\mathbf{Set}^+_{/Z^\sim}$ coincide. Moreover, we have a Quillen equivalence

$$(-)^\flat : (s\mathbf{Set}_{\mathrm{Joyal}})_{/Z} \rightleftarrows s\mathbf{Set}^+_{/Z^\sim} : U$$

where the functor U forgets the marking. In particular, $(-)^\flat$ sends categorical equivalences to marked equivalences.

2.20. **Example.** The inclusion functor $\mathbf{Spc} \subset \mathbf{Cat}_\infty$ admits left and right adjoints B and ι, where B is the classifying space functor that inverts all edges and ι is the 'core' functor that takes the maximal sub-∞-groupoid. These two adjunctions are modeled by the two Quillen adjunctions

$$U : s\mathbf{Set}^+ \rightleftarrows s\mathbf{Set}_{\mathrm{Quillen}} : (-)^\sharp,$$

[13]Note that with this choice, the resulting subcategory is not stable under equivalence. One could alternatively appeal to a homotopy-invariant notion of cocartesian fibration and instead replace isofibrations with functors – cf. [14], which admits an obvious generalization to this setting.

$$(-)^\sharp\colon s\mathbf{Set}_{\mathrm{Quillen}} \rightleftarrows s\mathbf{Set}^+ :M.$$

Here $M(X,\mathcal{E})$ is the maximal sub-simplicial set of X such that all of its edges are marked. In particular, $(-)^\sharp$ sends weak homotopy equivalences to marked equivalences.

2.21. **Proposition** ([12, Remark B.2.5]). *The bifunctor*

$$-\times - : s\mathbf{Set}^+_{/(Z_1,\mathcal{E}_1)} \times s\mathbf{Set}^+_{/(Z_2,\mathcal{E}_2)} \longrightarrow s\mathbf{Set}^+_{/(Z_1\times Z_2,\mathcal{E}_1\times\mathcal{E}_2)}$$

is left Quillen. Consequently, the bifunctors

$$\begin{aligned} \mathrm{Map}_Z(-,-) &: s\mathbf{Set}^{+\,op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}_{Quillen} \\ \mathrm{Fun}_Z(-,-) &: s\mathbf{Set}^{+\,op}_{/(Z,\mathcal{E})} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \longrightarrow s\mathbf{Set}_{Joyal} \end{aligned}$$

are right Quillen, so $s\mathbf{Set}^+_{/(Z,E)}$ is both a $s\mathbf{Set}_{Quillen}$-enriched model category (with respect to Map_Z) and $s\mathbf{Set}_{Joyal}$-enriched model category (with respect to Fun_Z).

2.22. **Remark.** As explained in [17, Digression 1.2.13], by Proposition 2.21 the full subcategory of $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ spanned by the fibrant objects is an example of an ∞-cosmos [17, Definition 1.2.1].

Finally, we explain how the formalism of marked simplicial sets can be used to extract the pushforward functors implicitly defined by a cocartesian fibration. First, we need a lemma.

2.23. **Lemma.** *For $n>0$, the inclusion $i_n : \Delta^{n-1} \cong \Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}} \longrightarrow {}_\natural\Delta^n$ is left marked anodyne. Consequently, for a cocartesian fibration $C \longrightarrow S$, the map*

$$\mathrm{Fun}({}_\natural\Delta^n, {}_\natural C) \longrightarrow \mathrm{Fun}(\Delta^{n-1}, C) \times_{\mathrm{Fun}(\Delta^{n-1},C)} \mathrm{Fun}(\Delta^n, S)$$

induced by i_n is a trivial Kan fibration.

Proof. We proceed by induction on n, the base case $n=1$ being the left marked anodyne map $\Delta^{\{0\}} \longrightarrow {}_\natural\Delta^1 = (\Delta^1)^\sharp$. Consider the commutative diagram

$$\begin{array}{ccc} \Delta^{\{0\}} \star \partial\Delta^{n-2} & \longrightarrow & \Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}} \\ \big\downarrow \cup\, i_{n-1} & & \big\downarrow \quad \searrow i_n \\ (\Delta^{\{0\}} \star \Lambda^{n-1}_0, \mathcal{E}) & \longrightarrow & {}_\natural\Lambda^n_0 \\ & & \big\downarrow \\ & & {}_\natural\Delta^n \end{array}$$

where $\mathcal{E}$ is the collection of edges $\{0,i\}$, $0<i\le n$ (and the degenerate edges). The square is a pushout, and by the inductive hypothesis, the lefthand vertical map is left marked anodyne. We deduce that i_n is left marked anodyne. The second statement now follows because the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Fun}({}_\natural\Delta^n, {}_\natural C) \\ \big\downarrow & \nearrow & \big\downarrow \\ B & \longrightarrow & \mathrm{Fun}(\Delta^{n-1}, C) \times_{\mathrm{Fun}(\Delta^{n-1},C)} \mathrm{Fun}(\Delta^n, S) \end{array}$$

transposes to

$$\begin{array}{ccc} A \times {}_\natural\Delta^n \displaystyle\bigcup_{A\times\Delta^{n-1}} B\times\Delta^{n-1} & \longrightarrow & {}_\natural C \\ \big\downarrow & \nearrow & \big\downarrow \\ B \times {}_\natural\Delta^n & \longrightarrow & S \end{array}$$

and the lefthand vertical map is left marked anodyne for any cofibration $A \longrightarrow B$ by [10, Proposition 3.1.2.3]. □

The main case of interest in Lemma 2.23 is when $n = 1$, which shows that

$$\mathcal{O}^{cocart}(C) \longrightarrow C \times_S \mathcal{O}(S)$$

is a trivial Kan fibration. Let

$$P : C \times_S \mathcal{O}(S) \longrightarrow \mathcal{O}^{cocart}(C)$$

be a section that fixes the inclusion $C \subset \mathcal{O}^{cocart}(C)$ (for this, note that $C \subset C \times_S \mathcal{O}(S)$ is a cofibration as it is a monomorphism of simplicial sets). Then we say that P or the further composite $P' = \mathrm{ev}_1 \circ P$ is a *cocartesian pushforward* for $C \longrightarrow S$. Given an edge α of S, $P'_\alpha : C_s \longrightarrow C_t$ is the pushforward functor $\alpha_!$ determined under the equivalence $\mathbf{Cat}^{cocart}_{\infty/S} \simeq \mathrm{Fun}(S, \mathbf{Cat}_\infty)$.

Functoriality in the base. Let $\pi : X \longrightarrow Z$ be a map of simplicial sets. Then the pullback functor $\pi^* : s\mathbf{Set}_{/Z} \longrightarrow s\mathbf{Set}_{/X}$ admits a left adjoint $\pi_!$, given by postcomposing with π. In addition, since $s\mathbf{Set}$ is a topos, π^* also admits a right adjoint π_*, which may be thought of as the functor of relative sections because $\mathrm{Hom}_{/X}(A, \pi_*(B)) \cong \mathrm{Hom}_{/Z}(A \times_X Z, B)$.

Now supposing that π is a map of marked simplicial sets, π^*, $\pi_!$, and π_* extend to functors of marked simplicial sets over X or Y in an evident manner. We then seek conditions under which the adjunctions $\pi_! \dashv \pi^*$ and $\pi^* \dashv \pi_*$ are Quillen with respect to the cocartesian model structures. To this end, we have the following theorem of Lurie:

2.24. **Theorem** ([12, Theorem B.4.2])**.** *Let*

$$(Z, \mathcal{E}) \xleftarrow{\pi} (X, \mathcal{F}) \xrightarrow{\rho} (X', \mathcal{F}')$$

be a span of marked simplicial sets such that Z, X, X' are ∞-categories and the collections of markings contain all the equivalences.

(i) The adjunction

$$\rho_! \colon s\mathbf{Set}^+_{/(X,\mathcal{F})} \rightleftarrows s\mathbf{Set}^+_{/(X',\mathcal{F}')} : \rho^*$$

is Quillen with respect to the cocartesian model structures.

(ii) Further suppose that

(1) For every object $x \in X$ and marked edge $f : z \to \pi(x)$ in Z, there exists a locally π-cartesian edge $x_0 \to x$ in X lifting f.

(2) π is a flat categorical fibration.

(3) $\mathcal{E}$ and $\mathcal{F}$ are closed under composition.

(4) Suppose given a commutative diagram

$$\begin{array}{ccccc} & & x_1 & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ x_0 & & \xrightarrow{\quad h \quad} & & x_2 \end{array}$$

in X where g is locally π-cartesian, $\pi(g)$ is marked, and $\pi(f)$ is an equivalence. Then f is marked if and only if h is marked. (Note in particular that, taking f to be an identity morphism, every locally π-cartesian edge lying over a marked edge is itself marked.)

Then the adjunction

$$\pi^* \colon s\mathbf{Set}^+_{/(X,\mathcal{F})} \rightleftarrows s\mathbf{Set}^+_{/(Z,\mathcal{E})} : \pi_*$$

is Quillen with respect to the cocartesian model structures.

We formulated Theorem 2.24 as a theorem concerning a span $Z \xleftarrow{\pi} X \xrightarrow{\rho} X'$ because in applications we will typically be interested in the composite Quillen adjunction

$$\rho_! \pi^* \colon s\mathbf{Set}^+_{/(Z,\mathcal{E})} \rightleftarrows s\mathbf{Set}^+_{/(X',\mathcal{F}')} : \pi_* \rho^*.$$

Here are two examples.

2.25. **Example** (Pairing cartesian and cocartesian fibrations)**.** Let $\pi : X \longrightarrow Z$ be a cartesian fibration. Then the span

$$Z^\sharp \xleftarrow{\pi} X^\natural \xrightarrow{\pi} Z^\sharp$$

satisfies the hypotheses of Theorem 2.24. Now given a cocartesian fibration $Y \longrightarrow Z$, define

$$\widetilde{\mathrm{Fun}}_Z(X,Y) := (\pi_* \pi^*)({}_\natural Y \to Z^\sharp).$$

Then the fiber of $\widetilde{\mathrm{Fun}}_Z(X,Y)$ over an object $z \in Z$ is $\mathrm{Fun}(X_z, Y_z)$, and given a morphism $\alpha : z_0 \longrightarrow z_1$, the pushforward functor

$$\alpha_! : \mathrm{Fun}(X_{z_0}, Y_{z_0}) \longrightarrow \mathrm{Fun}(X_{z_1}, Y_{z_1})$$

is given by precomposition in the source and postcomposition in the target. Note how this example highlights the relevance of condition (1) in Theorem 2.24(ii).

2.26. **Example** (Right Kan extension)**.** Let $f : Y \longrightarrow Z$ be a functor. We can apply Theorem 2.24 to perform the operation of right Kan extension at the level of cocartesian fibrations. Consider the span

$$Z^\sharp \xleftarrow{\mathrm{ev}_0} (\mathcal{O}(Z) \times_{Z,f} Y)^\sharp \xrightarrow{\mathrm{pr}_Y} Y^\sharp.$$

Then the conditions of Theorem 2.24 are satisfied, so we obtain a Quillen adjunction

$$(\mathrm{pr}_Y)_! (\mathrm{ev}_0)^* \colon s\mathbf{Set}^+_{/Z} \rightleftarrows s\mathbf{Set}^+_{/Y} : (\mathrm{ev}_0)_* (\mathrm{pr}_Y)^*.$$

In addition, the map $C \times_Z Y^\sharp \longrightarrow C \times_Z \mathcal{O}(Z)^\sharp \times_Z Y^\sharp$ induced by the identity section $\iota : Z \longrightarrow \mathcal{O}(Z)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/Y}$ for $C \longrightarrow Z$ *fibrant* in $s\mathbf{Set}^+_{/Z}$, by [3, Lemma 9.8]. Consequently, the induced adjunction of ∞-categories

$$(\mathrm{pr}_Y)_! (\mathrm{ev}_0)^* \colon \mathbf{Cat}^{cocart}_{\infty/Z} \rightleftarrows \mathbf{Cat}^{cocart}_{\infty/Y} : (\mathrm{ev}_0)_* (\mathrm{pr}_Y)^*$$

is equivalent to

$$f^* \colon \mathrm{Fun}(Z, \mathbf{Cat}_\infty) \rightleftarrows \mathrm{Fun}(Y, \mathbf{Cat}_\infty) : f_*$$

under the straightening/unstraightening equivalence (which is natural with respect to pullback).

Note that as a special case, if $Z = \Delta^0$ we recover the formula $\mathrm{Fun}_Y(Y^\sharp, {}_\natural C) \simeq \varprojlim F_C$ of [10, Corollary 3.3.3.2] (where $C \longrightarrow Y$ is a cocartesian fibration and $F_C : Y \longrightarrow \mathbf{Cat}_\infty$ the corresponding functor). Indeed, this construction of the right Kan extension of a cocartesian fibration is suggested by that result and the pointwise formula for a right Kan extension.

Finally, we will use the following two observations concerning the interaction of Theorem 2.24 with compositions and homotopy equivalences of spans (which we also recorded in [5]).

2.27. **Lemma.** *Suppose we have spans of marked simplicial sets*

$$X_0 \xleftarrow{\pi_0} Z_0 \xrightarrow{\rho_0} X_1$$

and

$$X_1 \xleftarrow{\pi_1} Z_1 \xrightarrow{\rho_1} X_2$$

which each satisfy the hypotheses of Theorem 2.24. Then the span

$$Z_0 \xleftarrow{\mathrm{pr}_0} Z_0 \times_{X_1} Z_1 \xrightarrow{\mathrm{pr}_1} Z_1$$

also satisfies the hypothesis of Theorem 2.24.[14] *Consequently, we obtain a Quillen adjunction*

$$(\rho_1 \circ \mathrm{pr}_1)_! (\pi_0 \circ \mathrm{pr}_0)^* \colon s\mathbf{Set}^+_{/X_0} \rightleftarrows s\mathbf{Set}^+_{X_2} : (\pi_0 \circ \mathrm{pr}_0)_* (\rho_1 \circ \mathrm{pr}_1)^*,$$

which is the composite of the Quillen adjunction from $s\mathbf{Set}^+_{/X_0}$ to $s\mathbf{Set}^+_{/X_1}$ with the one from $s\mathbf{Set}^+_{/X_1}$ to $s\mathbf{Set}^+_{/X_2}$.

Proof. The assertion that the span satisfies the hypotheses of Theorem 2.24 is by inspection. The other assertion that the Quillen adjunction factors as a composite follows from the base-change isomorphism $\rho_0^* \pi_{1,*} \cong \mathrm{pr}_{0,*} \circ \mathrm{pr}_1^*$. □

[14] However, one should beware that the "long" span

$$X_0 \longleftarrow Z_0 \times_{X_1} Z_1 \longrightarrow X_2$$

may fail to satisfy the hypotheses of Theorem 2.24, because the composition of locally cartesian fibrations may fail to again be locally cartesian; this explains the roundabout formulation of the statement.

2.28. **Lemma.** *Suppose a morphism of spans of marked simplicial sets*

$$\begin{array}{ccccc} & & Z & & \\ & {}^{\pi}\swarrow & \downarrow{\scriptstyle f} & \searrow^{\rho} & \\ X & \xleftarrow[\pi']{} & Z' & \xrightarrow[\rho']{} & X' \end{array}$$

where $\rho_! \pi^*$ and $(\rho')_!(\pi')^*$ are left Quillen with respect to the cocartesian model structures on X and X'. Suppose moreover that f is a homotopy equivalence in $s\mathbf{Set}^+_{/X'}$, so that there exists a homotopy inverse g and homotopies

$$h\colon \mathrm{id} \simeq g \circ f \quad \textit{and} \quad k\colon \mathrm{id} \simeq f \circ g.$$

Then the natural transformation $\rho_! \pi^* \longrightarrow (\rho')_!(\pi')^*$ induced by f is a cocartesian equivalence on all objects, and, consequently, the adjoint natural transformation $(\pi')_*(\rho')^* \longrightarrow \pi_* \rho^*$ is a cocartesian equivalence on all fibrant objects.

Proof. The homotopies h and k pull back to show that for all $X \longrightarrow C$, the map

$$\mathrm{id}_X \times_C f\colon X \times_C K \longrightarrow X \times_C L$$

is a homotopy equivalence with inverse $\mathrm{id}_X \times_C g$. The last statement now follows from [8, Corollary 1.4.4(b)]. $\square$

Parametrized fibers. In this brief subsection, we record notation for the S-fibers of an S-functor.

2.29. **Notation.** Given an S-category $\pi : D \longrightarrow S$ and an object $x \in D$, define

$$\mathcal{O}_{x\to}(D) := \{x\} \times_D \mathcal{O}(D).$$

For the full subcategory of cocartesian edges $\mathcal{O}^{cocart}(D) \subset \mathcal{O}(D)$, also define

$$\underline{x} := \{x\} \times_D \mathcal{O}^{cocart}(D).$$

Given an S-functor $\phi : C \longrightarrow D$, define

$$C_{\underline{x}} := \underline{x} \times_{D,\phi} C.$$

Note that by definition, the objects of $\underline{x}$ are π-cocartesian edges in D with source x. Then by the right cancellative property of π-cocartesian edges [10, Lemma 2.4.2.7], the morphisms in $\underline{x}$ are 2-simplices of cocartesian edges with source x, hence $\underline{x}$ is an S-space (via the map $\mathrm{ev}_1 : \underline{x} \longrightarrow S$). In fact, by Lemma 12.10, $\mathrm{ev}_1 : \underline{x} \longrightarrow S^{\pi x/}$ is a trivial fibration, so we may think of $\underline{x}$ as an 'S-point' of D.

In view of this, we will also regard $C_{\underline{x}}$ as a $S^{\pi x/}$-category (and we will sometimes be cavalier about the distinction between $\underline{x}$ and $S^{\pi x/}$). Note however, that the functor $\underline{x} \longrightarrow D$ is canonical in our setup, whereas we need to make a choice of cocartesian pushforward to choose a S-functor $S^{\pi x/} \longrightarrow D$ that selects $x \in D$.

3. Functor categories

Let S be an ∞-category. Then $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ is cartesian closed, so it possesses an internal hom. As a basic application of Theorem 2.24, we will define this internal hom at the level of cocartesian fibrations over S.

3.1. **Proposition.** *Let $C \longrightarrow S$ be a cocartesian fibration. Let $\mathrm{ev}_0, \mathrm{ev}_1 : \mathcal{O}(S) \times_S C \longrightarrow S$ denote the source and target maps. Then the functor*

$$(\mathrm{ev}_1)_!(\mathrm{ev}_0)^* : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{/\mathcal{O}(S)^\sharp \times_S {}^\natural C} \longrightarrow s\mathbf{Set}^+_{/S}$$

is left Quillen with respect to the cocartesian model structures.

Proof. We verify the hypotheses of Theorem 2.24 as applied to the span $S^\sharp \xleftarrow{\mathrm{ev}_0} \mathcal{O}(S)^\sharp \times_S {}^\natural C \xrightarrow{\mathrm{ev}_1} S^\sharp$. By [10, Corollary 2.4.7.12], ev_0 is a cartesian fibration and an edge e in $\mathcal{O}(S) \times_S C$ is ev_0-cartesian if and only if its projection to C is an equivalence. (1) thus holds. (2) holds since cartesian fibrations are flat categorical fibrations. (3) is obvious. (4) follows from the stability of cocartesian edges under equivalence. $\square$

3.2. **Definition.** In the statement of Proposition 3.1, let

$$\underline{\mathrm{Fun}}_S(C,-) := (\mathrm{ev}_0)_*(\mathrm{ev}_1)^* : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{/S}.$$

We will also write this as $\underline{\mathrm{Fun}}_S({}_\natural C,-)$ if we wish to emphasize the marking.

Proposition 3.1 implies that if $D \longrightarrow S$ is a cocartesian fibration, $\underline{\mathrm{Fun}}_S(C,D) \longrightarrow S$ is a cocartesian fibration. Unwinding the definitions, we see that an object of $\underline{\mathrm{Fun}}_S(C,D)$ over $s \in S$ is a $S^{s/}$-functor

$$S^{s/} \times_S C \longrightarrow S^{s/} \times_S D,$$

and a cocartesian edge of $\underline{\mathrm{Fun}}_S(C,D)$ over an edge $e : \Delta^1 \longrightarrow S$ is a $\Delta^1 \times_S \mathcal{O}(S)$-functor

$$\Delta^1 \times_S \mathcal{O}(S) \times_S C \longrightarrow \Delta^1 \times_S \mathcal{O}(S) \times_S D.$$

Our first goal is to prove that the construction $\underline{\mathrm{Fun}}_S(C,-)$ implements the internal hom at the level of underlying ∞-categories. To this end, we have the following lemma and proposition.

3.3. **Lemma.** *Let $\iota : S \longrightarrow \mathcal{O}(S)$ be the identity section and regard $\mathcal{O}(S)^\sharp$ as a marked simplicial set over S via the target map. Then*

(1) For every marked simplicial set $X \longrightarrow S$ and cartesian fibration $C \longrightarrow S$,

$$\mathrm{id}_X \times \iota \times \mathrm{id}_C : X \times_S C^\natural \longrightarrow X \times_S \mathcal{O}(S)^\sharp \times_S C^\natural$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.

(1′) For every marked simplicial set $X \longrightarrow S$ and cartesian fibration $C \longrightarrow S$,

$$\iota \times \mathrm{id}_C : X \times_S C^\natural \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, X) \times_S C^\natural$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, where the marked edges in $\mathrm{Fun}((\Delta^1)^\sharp, X)$ are the marked squares in X.

(2) For every marked simplicial set $X \longrightarrow S$ and cocartesian fibration $C \longrightarrow S$,

$$\mathrm{id}_C \times \iota \times \mathrm{id}_X : {}_\natural C \times_S X \longrightarrow {}_\natural C \times_S \mathcal{O}(S)^\sharp \times_S X$$

is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$.

Proof. (1) Because $- \times_S C^\natural$ preserves cocartesian equivalences, we reduce to the case where $C = S$. By definition, $X \longrightarrow X \times_S \mathcal{O}(S)^\sharp$ is a cocartesian equivalence if and only if for every cocartesian fibration $Z \longrightarrow S$, $\mathrm{Map}^\sharp_S(X \times_S \mathcal{O}(S)^\sharp, {}_\natural Z) \longrightarrow \mathrm{Map}^\sharp_S(X, {}_\natural Z)$ is a trivial Kan fibration. In other words, for every monomorphism of simplicial sets $A \longrightarrow B$ and cocartesian fibration $Z \longrightarrow S$, we need to provide a lift in the following commutative square

$$\begin{array}{ccc} B^\sharp \times X \bigsqcup_{A^\sharp \times X} (A^\sharp \times X) \times_S \mathcal{O}(S)^\sharp & \xrightarrow{\phi} & {}_\natural Z \\ \downarrow & \nearrow & \downarrow \\ (B^\sharp \times X) \times_S \mathcal{O}(S)^\sharp & \longrightarrow & S^\sharp \end{array}$$

Define $h_0 : \mathcal{O}(S)^\sharp \times (\Delta^1)^\sharp \longrightarrow \mathcal{O}(S)^\sharp$ to be the adjoint to the map $\mathcal{O}(S)^\sharp \longrightarrow \mathcal{O}(\mathcal{O}(S))^\sharp$ obtained by precomposing by the map of posets $\Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ which sends $(1,1)$ to 1 and the other vertices to 0. Precomposing ϕ by $\mathrm{id}_{A^\sharp \times X} \times h_0$, define a homotopy

$$h : (A^\sharp \times X) \times_S \mathcal{O}(S)^\sharp \times (\Delta^1)^\sharp \longrightarrow {}_\natural Z$$

from $\phi|_{A^\sharp \times X} \circ \mathrm{pr}_{A^\sharp \times X}$ to $\phi|_{(A^\sharp \times X) \times_S \mathcal{O}(S)^\sharp}$. Using h and $\phi|_{B^\sharp \times X}$, define a map

$$\psi : B^\sharp \times X \bigsqcup_{A^\sharp \times X} (A^\sharp \times X) \times_S \mathcal{O}(S)^\sharp \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z)$$

such that $\psi|_{B^\sharp \times X}$ is adjoint to $\phi|_{B^\sharp \times X} \circ \mathrm{pr}_{B^\sharp \times X}$ and $\psi|_{(A^\sharp \times X)\times_S \mathcal{O}(S)^\sharp}$ is adjoint to h. Then we may factor the above square through the trivial fibration $\mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z) \longrightarrow {}_\natural Z \times_S \mathcal{O}(S)^\sharp$ to obtain the commutative rectangle

$$\begin{array}{ccccc} B^\sharp \times X \coprod_{A^\sharp \times X} (A^\sharp \times X) \times_S \mathcal{O}(S)^\sharp & \xrightarrow{\psi} & \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z) & \xrightarrow{e_1} & {}_\natural Z \\ \downarrow & \overset{\widetilde{\psi}}{\nearrow} & \downarrow \simeq & & \downarrow \\ (B^\sharp \times X) \times_S \mathcal{O}(S)^\sharp & \xrightarrow[\phi|_{B^\sharp \times X} \times \mathrm{id}]{} & {}_\natural Z \times_S \mathcal{O}(S)^\sharp & \xrightarrow{e_1} & S^\sharp. \end{array}$$

The dotted lift $\widetilde{\psi}$ exists, and $e_1 \circ \widetilde{\psi}$ is our desired lift.

(1') Repeat the argument of (1) with $\mathrm{Fun}((\Delta^1)^\sharp, X)$ in place of $\mathcal{O}(S)^\sharp$.

(2) Let $p : C \longrightarrow S$ denote the structure map and let P be a lift in the commutative square

$$\begin{array}{ccc} {}_\natural C & \xrightarrow{\iota_C} & \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \\ \downarrow & \overset{P}{\nearrow} & \simeq \downarrow (e_0, \mathcal{O}(p)) \\ {}_\natural C \times_S \mathcal{O}(S)^\sharp & \xrightarrow{=} & {}_\natural C \times_S \mathcal{O}(S)^\sharp. \end{array}$$

Let

$$g = (e_1 \times \mathrm{id}_X) \circ (P \times \mathrm{id}_X) : {}_\natural C \times_S \mathcal{O}(S)^\sharp \times_S X \longrightarrow {}_\natural C \times_S X$$

and note that g is a map over S. We claim that g is a marked homotopy inverse of $f = \mathrm{id}_C \times \iota \times \mathrm{id}_X$. By construction, $g \circ f = \mathrm{id}$. For the other direction, define

$$h_0 : \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \times (\Delta^1)^\sharp \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C)$$

as the adjoint of the map $\mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \longrightarrow \mathrm{Fun}((\Delta^1 \times \Delta^1)^\sharp, {}_\natural C)$ obtained by precomposing by the map of posets $\Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ which sends $(0,0)$ to 0 and the other vertices to 1. Define

$$h : {}_\natural C \times_S \mathcal{O}(S)^\sharp \times_S X \times (\Delta^1)^\sharp \longrightarrow {}_\natural C \times_S \mathcal{O}(S)^\sharp \times_S X$$

as the composite $((e_0, \mathcal{O}(p)) \times X) \circ (h_0 \times X) \circ (P \times \mathrm{id}_{X \times (\Delta^1)^\sharp})$. Then h is a homotopy over S from id to $f \circ g$.

□

3.4. **Proposition.** *Let $C, C', D \longrightarrow S$ be cocartesian fibrations and let $F : C \longrightarrow C'$ be a monomorphism of cocartesian fibrations over S (so preserving cocartesian edges). For all marked simplicial sets Y over S, the map*

$$\mathrm{Fun}_S({}_\natural D, \underline{\mathrm{Fun}}_S({}_\natural C', Y)) \longrightarrow \mathrm{Fun}_S({}_\natural D \times_S {}_\natural C', Y) \times_{\mathrm{Fun}_S({}_\natural D \times_S {}_\natural C, Y)} \mathrm{Fun}_S({}_\natural D, \underline{\mathrm{Fun}}_S({}_\natural C, Y))$$

which precomposes by F is a trivial Kan fibration.

Proof. From the defining adjunction, for all $X, Y \in s\mathbf{Set}^+_{/S}$ we have a natural isomorphism

$$\mathrm{Fun}_S(X, \underline{\mathrm{Fun}}_S({}_\natural C, Y)) \cong \mathrm{Fun}_S(X \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural C, Y)$$

of simplicial sets. Since $\mathrm{Fun}_S(-,-)$ is a right Quillen bifunctor, the assertion reduces to showing that

$${}_\natural D \times_S {}_\natural C' \coprod_{{}_\natural D \times_S {}_\natural C} {}_\natural D \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural C \longrightarrow {}_\natural D \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural C'$$

is a trivial cofibration in $s\mathbf{Set}^+_{/S}$, which follows from Lemma 3.3(2). □

In Proposition 3.4, letting $C = \emptyset$ and $Y = {}_\natural E$ for another cocartesian fibration $E \longrightarrow S$, we deduce that $\underline{\mathrm{Fun}}_S(C', -)$ is right adjoint to $C' \times_S -$ as an endofunctor of $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$. Further setting $D = S$, we deduce that the category of cocartesian sections of $\underline{\mathrm{Fun}}_S({}_\natural C, {}_\natural E)$ is equivalent to $\mathrm{Fun}_S({}_\natural C, {}_\natural E)$. We will employ the following notation to explicitly track objects under this correspondence.

3.5. **Notation.** Given a map $f : {}_\natural C \longrightarrow {}_\natural E$, let σ_f denote the cocartesian section $S^\sharp \longrightarrow \underline{\mathrm{Fun}}_S({}_\natural C, {}_\natural E)$ given by adjointing the map $\mathcal{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{\mathrm{pr}_C} {}_\natural C \xrightarrow{f} {}_\natural E$.

We next study varying the second variable in the construction $\underline{\mathrm{Fun}}_S(-,-)$.

3.6. **Lemma.** *Let $C \longrightarrow D$ be a fibration of marked simplicial sets over S.*

(1) Let $K \longrightarrow S$ be a cocartesian fibration. Then
$$\underline{\mathrm{Fun}}_S({}_\natural K, C) \longrightarrow \underline{\mathrm{Fun}}_S({}_\natural K, D) \times_D C$$
is a fibration in $s\mathbf{Set}^+_{/S}$.

(2) The map
$$\underline{\mathrm{Fun}}_S(S^\sharp, C) \longrightarrow \underline{\mathrm{Fun}}_S(S^\sharp, D) \times_D C$$
is a trivial fibration in $s\mathbf{Set}^+_{/S}$.

Proof. Let $i : A \longrightarrow B$ be a map of marked simplicial sets. For (1), we use that if i is a trivial cofibration, then
$$B \coprod_A A \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural K \longrightarrow B \times_S \mathcal{O}(S) \times_S {}_\natural K$$
is a trivial cofibration, which follows from Proposition 3.1. For (2), we use that if i is a cofibration, then
$$B \coprod_A A \times_S \mathcal{O}(S)^\sharp \longrightarrow B \times_S \mathcal{O}(S)$$
is a trivial cofibration, which follows from Lemma 3.3(1). □

The following proposition indicates that we can promote the conclusion $\underline{\mathrm{Fun}}_S(S, -) \simeq \mathrm{id}$ (as an endofunctor of $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$) of Proposition 3.4 to the level of cocartesian model structures. It will not be used in the sequel and is included only for illustrative purposes.

3.7. **Proposition.** *The Quillen adjunction*
$$- \times_S \mathcal{O}(S)^\sharp : s\mathbf{Set}^+_{/S} \rightleftarrows s\mathbf{Set}^+_{/S} : \underline{\mathrm{Fun}}_S(S^\sharp, -)$$
is a Quillen equivalence.

Proof. We first check that for every cocartesian fibration $C \longrightarrow S$, the counit map
$$\underline{\mathrm{Fun}}_S(S^\sharp, {}_\natural C) \times_S \mathcal{O}(S)^\sharp \longrightarrow {}_\natural C$$
is a cocartesian equivalence. By Lemma 3.3(1), it suffices to show that
$$\underline{\mathrm{Fun}}_S(S^\sharp, {}_\natural C) \longrightarrow {}_\natural C$$
is a trivial marked fibration, which follows from Lemma 3.6(2) (taking $D = S$). We now complete the proof by checking that $- \times_S \mathcal{O}(S)^\sharp$ reflects cocartesian equivalences: i.e., given the commutative square
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \times_S \mathcal{O}(S)^\sharp & \longrightarrow & B \times_S \mathcal{O}(S)^\sharp. \end{array}$$
if the lower horizontal map is a cocartesian equivalence over S (with respect to the target map) then the upper horizontal map is a cocartesian equivalence over S. But the vertical maps are cocartesian equivalences by Lemma 3.3(1). □

The construction $\underline{\mathrm{Fun}}_S(-, -)$ does not make homotopical sense when the first variable is not fibrant, so it does not yield a Quillen bifunctor. Nevertheless, we can say the following about varying the first variable.

3.8. **Proposition.** *Let K, L, and C be fibrant marked simplicial sets over S, let $f : K \longrightarrow L$ be a map and let*
$$f^* : \underline{\mathrm{Fun}}_S(L, C) \longrightarrow \underline{\mathrm{Fun}}_S(K, C)$$
denote the induced map.

(1) Suppose that f is a cocartesian equivalence over S. Then f^* is a cocartesian equivalence over S.

(2) Suppose that f is a cofibration. Then f^* is a fibration in $s\mathbf{Set}^+_{/S}$.

Proof. (1): It suffices to check that for all $s \in S$, f^* induces a categorical equivalence between the fibers over s, i.e. that

$$\mathrm{Fun}_S((S^{s/})^\sharp \times_S L, C) \longrightarrow \mathrm{Fun}_S((S^{s/})^\sharp \times_S K, C)$$

is a categorical equivalence. Our assumption implies that $(S^{s/})^\sharp \times_S K \longrightarrow (S^{s/})^\sharp \times_S L$ is a cocartesian equivalence over S, so this holds.

(2): For any trivial cofibration $A \longrightarrow B$ in $s\mathbf{Set}^+_S$, we need to check that

$$A \times_S \mathcal{O}(S) \times_S L \coprod_{A \times_S \mathcal{O}(S) \times_S K} B \times_S \mathcal{O}(S) \times_S K \longrightarrow B \times_S \mathcal{O}(S) \times_S L$$

is a trivial cofibration in $s\mathbf{Set}^+_{/S}$. By Proposition 3.1, $- \times_S \mathcal{O}(S) \times_S K$ preserves trivial cofibrations and ditto for L. The result then follows. □

A final word on notation: since $\underline{\mathrm{Fun}}_S(-,-)$ is only homotopically meaningful (and fibrant) when both variables are fibrant, we will henceforth cease to denote the markings on the variables.

S-categories of S-objects. For the convenience of the reader, we briefly review the construction and basic properties of the S-category of S-objects in an ∞-category C. This is a construction, at the level of marked simplicial sets, of the right adjoint to the Grothendieck construction functor[15]

$$\mathrm{Gr}_U : \mathbf{Cat}^{cocart}_{\infty/S} \longrightarrow \mathbf{Cat}_\infty, \quad (C \longrightarrow S) \mapsto C.$$

This material is originally due to Denis Nardin in [3, §7].

3.9. **Construction** ([3, Definition 7.4])**.** The span

$$S^\sharp \xleftarrow{\mathrm{ev}_0} \mathcal{O}(S)^\natural \xrightarrow{\rho} \Delta^0$$

defines a right Quillen functor $(\mathrm{ev}_0)_* \rho^* : s\mathbf{Set}^+ \longrightarrow s\mathbf{Set}^+_{/S}$, which sends an ∞-category E to $\widetilde{\mathrm{Fun}}_S(\mathcal{O}(S), E \times S)$ (cf. Example 2.25). This is the *S-category of objects in E*, which we will denote by $\underline{E}_S$.

The next proposition shows that the functor $E \mapsto \underline{E}_S$ indeed implements the right adjoint to Gr_U.

3.10. **Proposition.** *Suppose C a S-category and E an ∞-category. Then we have an equivalence*

$$\psi : \mathrm{Fun}_S(C, \underline{E}_S) \xrightarrow{\simeq} \mathrm{Fun}(C, E)$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} C^\sim & \longrightarrow & \mathcal{O}(S)^\natural & \longrightarrow & \Delta^0 . \\ \downarrow & & \downarrow & & \\ {}_\natural C & \longrightarrow & S^\sharp & & \\ \downarrow & & & & \\ \Delta^0 & & & & \end{array}$$

Given an ∞-category E, travelling along the outer span (i.e., pulling back and then pushing forward) yields $\mathrm{Fun}(C, E)$, travelling along the two inner spans yields $\mathrm{Fun}_S(C, \underline{E}_S)$, and the comparison functor ψ is induced by the map $\iota : C^\sim \longrightarrow {}_\natural C \times_S \mathcal{O}(S)^\natural$. By [3, Proposition 6.2], ι is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$. Therefore, combining Lemma 2.27 and Lemma 2.28, we deduce the claim. □

3.11. **Notation.** Given a S-functor $p : C \longrightarrow \underline{E}_S$, let $p^\dagger : C \longrightarrow E$ denote the corresponding functor under the equivalence of Proposition 3.10.

[15] We write Gr_U to distinguish from Notation 2.6.

3.12. **Example.** Let $E = \mathbf{Spc}$ or $\mathbf{Cat}_\infty$. Then $\underline{\mathbf{Spc}}_S$ resp. $\underline{\mathbf{Cat}}_{\infty,S}$ is the S-category of S-spaces resp. S-categories. In particular, suppose $E = \mathbf{Spc}$ and $S = \mathbf{O}_G^{op}$. Then we also call $\underline{\mathbf{Spc}}_{\mathbf{O}_G^{op}}$ the G-∞-category of G-spaces. Note that the fiber of this cocartesian fibration over a transitive G-set G/H is equivalent to the ∞-category of H-spaces $\mathrm{Fun}(\mathbf{O}_H^{op}, \mathbf{Spc})$, and the pushforward functors are given by restriction along a subgroup and conjugation.

3.13. **Remark.** Let C be an S-category and $\pi : X \longrightarrow C$ a left fibration. Then π straightens to a functor $F : C \longrightarrow \mathbf{Spc}$, which under the equivalence of Proposition 3.10 corresponds to a S-functor $F' : C \longrightarrow \underline{\mathbf{Spc}}_S$. We will say that π *S-straightens* to F'. Similarly, if π is a cocartesian fibration, then π S-straightens to a S-functor valued in $\underline{\mathbf{Cat}}_{\infty,S}$.

4. Join and slice

The join and slice constructions are at the heart of the ∞-categorical approach to limits and colimits. In this section, we introduce relative join and slice constructions and explore their homotopical properties.

The S-join.

4.1. **Definition.** Let $\iota : S \times \partial\Delta^1 \hookrightarrow S \times \Delta^1$ be the inclusion. Define the *S-join* to be the functor

$$(- \star_S -) := \iota_* : s\mathbf{Set}_{/S\times\partial\Delta^1} \longrightarrow s\mathbf{Set}_{/S\times\Delta^1}.$$

Define the *marked S-join* to be the functor

$$(- \star_S -) := \iota_* : s\mathbf{Set}^+_{/S^\sharp\times(\partial\Delta^1)^\flat} \longrightarrow s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}.$$

4.2. **Notation.** Given X, Y marked simplicial sets over S, we will usually refer to the structure maps to S by $\pi_1 : X \longrightarrow S$, $\pi_2 : Y \longrightarrow S$, and $\pi : X \star_S Y \longrightarrow S$. Explicitly, a $(i+j+1)$-simplex λ of $X \star_S Y$ is the data of simplices $\sigma : \Delta^i \longrightarrow X$, $\tau : \Delta^j \longrightarrow Y$, and $\lambda' : \Delta^i \star \Delta^j \longrightarrow S$ such that the diagram

$$\begin{array}{ccccc} \Delta^i & \longrightarrow & \Delta^i \star \Delta^j & \longleftarrow & \Delta^j \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \tau} \\ X & \xrightarrow{\pi_1} & S & \xleftarrow{\pi_2} & Y \end{array}$$

commutes; we then have that $\lambda' = \pi \circ \lambda$. We will sometimes write $\lambda = (\sigma, \tau)$ so as to remember the data of the i-simplex of X and the j-simplex of Y in the notation. If given an n-simplex of $X \star_S Y$, we will indicate the decomposition of Δ^n given by the structure map to Δ^1 as $\Delta^{n_0} \star \Delta^{n_1}$ (with either side possibly empty).

4.3. **Proposition.** *Let $\iota : S \times \partial\Delta^1 \hookrightarrow S \times \Delta^1$ be the inclusion. Then*

(a) $\iota_* : s\mathbf{Set}_{/S\times\partial\Delta^1} \longrightarrow s\mathbf{Set}_{/S\times\Delta^1}$ is a right Quillen functor.
(b) $\iota_* : s\mathbf{Set}^+_{/S^\sharp\times(\partial\Delta^1)^\flat} \longrightarrow s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}$ is a right Quillen functor.

Consequently, if X and Y are categorical resp. cocartesian fibrations over S, then $X \star_S Y$ is a categorical resp. cocartesian fibration over S, with the cocartesian edges given by those in X and Y.

Proof. For (b), we verify the hypotheses of Theorem 2.24(ii). All of the requirements are immediate except for (1) and (2).

(1): Let (s, i) be a vertex of $S^\sharp \times (\partial\Delta^1)^\flat$, $i = 0$ or 1. Let $f : (s', i') \longrightarrow (s, i)$ be a marked edge in $S^\sharp \times (\Delta^1)^\flat$. Then $i' = i$ and f viewed as an edge in $S^\sharp \times (\partial\Delta^1)^\flat$ is locally ι-cartesian.

(2): It is obvious that $\partial\Delta^1 \hookrightarrow \Delta^1$ is a flat categorical fibration, so by stability of flat categorical fibrations under base change, $S \times \partial\Delta^1 \hookrightarrow S \times \Delta^1$ is a flat categorical fibration.

(a) also follows from (2) by [12, Proposition B.4.5]. By (a), if X and Y are categorical fibrations over S, $X \star_S Y$ is a categorical fibration over $S \times \Delta^1$. The projection map $S \times \Delta^1 \longrightarrow S$ is a categorical fibration, so $X \star_S Y$ is also a categorical fibration over S. By (b), if X and Y are cocartesian fibrations over S, ${}_\natural X \star_S {}_\natural Y$ is fibrant in $s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}$. Since $S^\sharp \times (\Delta^1)^\flat$ is marked as a cocartesian fibration over S, ${}_\natural X \star_S {}_\natural Y$ is marked as a cocartesian fibration over S. $\square$

We have the compatibility of the relative join with base change.

4.4. **Lemma.** *Let $f : T \longrightarrow S$ be a functor and let X and Y be (marked) simplicial sets over S. Then we have a canonical isomorphism*

$$(X \star_S Y) \times_S T \cong (X \times_S T) \star_T (Y \times_S T).$$

Proof. From the pullback square

$$\begin{array}{ccc} T \times \partial\Delta^1 & \xrightarrow{\iota_T} & T \times \Delta^1 \\ \downarrow{\scriptstyle f\times \mathrm{id}} & & \downarrow{\scriptstyle f\times \mathrm{id}} \\ S \times \partial\Delta^1 & \xrightarrow{\iota_S} & S \times \Delta^1 \end{array}$$

we obtain the base-change isomorphism $f^*(\iota_S)_* \cong (\iota_T)_* f^*$. □

In [10, §4.2.2], Lurie introduces the relative 'diamond' join operation $\diamond_S$, which we now recall. Given X and Y marked simplicial sets over S, define

$$X \diamond_S Y = X \sqcup_{X \times_S Y \times \{0\}} X \times_S Y \times (\Delta^1)^\flat \sqcup_{X \times_S Y \times \{1\}} Y.$$

There is a comparison map

$$\psi_{(X,Y)} : X \diamond_S Y \longrightarrow X \star_S Y = \iota_*(X, Y),$$

adjoint to the isomorphism $\iota^*(X \star_S Y) \cong (X, Y)$.

4.5. **Lemma.** *Let X be a marked simplicial set. Then $\psi_{(X,S)} : X \diamond_S S^\sharp \longrightarrow X \star_S S^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Dually, if X is in addition fibrant, then $\psi_{(S,X)} : S^\sharp \diamond_S X \longrightarrow S^\sharp \star_S X$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof. We first address the map $\psi_{(X,S)}$. By left properness of the cocartesian model structure, the defining pushout for $X \diamond_S S^\sharp$ is a homotopy pushout. By Theorem 4.16,[16] $- \star_S S^\sharp$ preserves cocartesian equivalences. Therefore, choosing a fibrant replacement for X and using naturality of the comparison map $\psi_{(X,S)}$, we may reduce to the case that X is fibrant. Then we have to check that

$$\begin{array}{ccc} X \times \{1\} & \longrightarrow & X \times (\Delta^1)^\flat \\ \downarrow & & \downarrow \\ S^\sharp & \longrightarrow & X \star_S S^\sharp \end{array}$$

is a homotopy pushout square. Since this is a square of fibrant objects, this assertion can be checked fiberwise, in which case it reduces to the equivalence $X_s \diamond \Delta^0 \xrightarrow{\simeq} X^\triangleright$ of [10, Proposition 4.2.1.2].

The second statement concerning $\psi_{(S,X)}$ follows by the same type of argument, but without the reduction step. □

4.6. **Warning.** In general, $\psi_{(X,Y)}$ is not a cocartesian equivalence. As a counterexample, consider $S = \Delta^1$, $X = \{0\}$, and $Y = \{1\}$. Then $\psi_{(X,Y)}$ is the inclusion of $X \diamond_S Y \cong \Delta^{\{0\}} \sqcup \Delta^{\{1\}}$ into $X \star_S Y \cong \Delta^1$, which is not a cocartesian equivalence over Δ^1.

We will later need the following strengthening of the conclusion of Proposition 4.3.

4.7. **Proposition.** *(1) Let $C, C', D \longrightarrow S$ be inner fibrations and let $C, C' \longrightarrow D$ be functors over S. Then $C \star_D C' \longrightarrow S$ is an inner fibration.*

(2) Let $C, C', D \longrightarrow S$ be S-categories and let $C, C' \longrightarrow D$ be S-functors. Then $C \star_D C' \longrightarrow S$ is a S-category with cocartesian edges given by those in C or C', and $C \star_D C' \longrightarrow D$ is a S-functor.

[16] Note there is no circularity since Lemma 4.5 is only later referenced in this paper at the beginning of §9.

Proof. (1) Let $0 < k < n$. We need to solve the lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\lambda_0} & C \star_D C' \\ \downarrow & \overset{\lambda}{\nearrow} & \downarrow \\ \Delta^n & \longrightarrow & S. \end{array}$$

Let $\overline{\lambda} : \Delta^n \longrightarrow D$ be a lift in the commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & D \\ \downarrow & \overset{\overline{\lambda}}{\nearrow} & \downarrow \\ \Delta^n & \longrightarrow & S. \end{array}$$

Define λ using the data $(\lambda_0|_{\Delta^{n_0}}, \lambda_0|_{\Delta^{n_1}}, \overline{\lambda})$. Then λ is a valid lift.

(2) Consider $C \star_D C'$ as a marked simplicial set with marked edges those in ${}_\natural C$ or in ${}_\natural C'$. We need to solve the lifting problem

$$\begin{array}{ccc} {}_\natural\Lambda^n_0 & \xrightarrow{\lambda_0} & C \star_D C' \\ \downarrow & \overset{\lambda}{\nearrow} & \downarrow \\ {}_\natural\Delta^n & \longrightarrow & S. \end{array}$$

Let $\overline{\lambda} : \Delta^n \longrightarrow D$ be a lift in the commutative square

$$\begin{array}{ccc} {}_\natural\Lambda^n_0 & \longrightarrow & {}_\natural D \\ \downarrow & \overset{\overline{\lambda}}{\nearrow} & \downarrow \\ {}_\natural\Delta^n & \longrightarrow & S. \end{array}$$

Define λ using the data $(\lambda_0|_{\Delta^{n_0}}, \lambda_0|_{\Delta^{n_1}}, \overline{\lambda})$. Then λ is a valid lift. Finally, note that we may obviously lift against classes (3) and (4) of [10, Definition 3.1.1.1]. We conclude that $C \star_D C' \longrightarrow S$ is fibrant in $s\mathbf{Set}^+_{/S}$, hence an S-category with cocartesian edges as marked.

□

Since the S-join is defined as a right Kan extension, it is simple to map into. In the other direction, we can offer the following lemma.

4.8. **Lemma.** *Let C, C', D, and E be S-categories and let $C, C' \longrightarrow D$ be S-functors. Then*

$$\mathrm{Fun}_S(C \star_D C', E) \longrightarrow \mathrm{Fun}_S(C, E) \times \mathrm{Fun}_S(C', E)$$

is a bifibration [10, Definition 2.4.7.2]. *Consequently,*

$$\mathrm{Fun}_S(C \star_D C', E) \longrightarrow \mathrm{Fun}_S(C, E)$$

is a cartesian fibration with cartesian edges those sent to equivalences in $\mathrm{Fun}_S(C', E)$*, and*

$$\mathrm{Fun}_S(C \star_D C', E) \longrightarrow \mathrm{Fun}_S(C', E)$$

is a cocartesian fibration with cocartesian edges those sent to equivalences in $\mathrm{Fun}_S(C', E)$*.*

Proof. By inspection, the span

$$(\Delta^1)^\flat \xleftarrow{\pi} {}_\natural(C \star_D C') \xrightarrow{\pi'} S^\sharp$$

satisfies the hypotheses of Theorem 2.24. Therefore, $\pi_* \pi'^* ({}_\natural E \longrightarrow S)$ is a categorical fibration over Δ^1. The claim now follows from [10, Proposition 2.4.7.10], and the consequence from [10, Lemma 2.4.7.5] and its opposite. □

The Quillen adjunction between S-join and S-slice. Our next goal is to obtain a relative join and slice Quillen adjunction. To this end, we need a good understanding of the combinatorics of the relative join (Proposition 4.11). We prepare for the proof of that proposition with a few lemmas.

4.9. **Lemma.** *Let $i, l \geq -1$ and $j, k \geq 0$. Then the map*

$$\Delta^i \star \Delta^j \star \partial\Delta^k \star \Delta^l \coprod_{\Delta^j \star \partial\Delta^k \star \Delta^l} \Delta^{j+k+l+2} \hookrightarrow \Delta^{i+j+k+l+3}$$

is inner anodyne.

Proof. Let $f : \Delta^{j-1} \hookrightarrow \Delta^i \star \Delta^{j-1}$ and $g : \Lambda^{k+1}_0 \hookrightarrow \Delta^{k+1}$. The map in question is $f \star g \star \Delta^l$, so is inner anodyne by [10, Lemma 2.1.2.3]. □

By [10, Lemma 2.1.2.4], the join of a left anodyne map and an inclusion is left anodyne. We need a slight refinement of this result:

4.10. **Lemma.** *Let $f : A_0 \hookrightarrow A$ be a cofibration of simplicial sets.*

(1) Let $g : B_0 \hookrightarrow B$ be a right marked anodyne map between marked simplicial sets. Then

$$f^\flat \star g : A_0^\flat \star B \coprod_{A_0^\flat \star B_0} A^\flat \star B_0 \hookrightarrow A^\flat \star B$$

is a right marked anodyne map.

(2) Let $g : B_0 \hookrightarrow B$ be a left marked anodyne map between marked simplicial sets. Then

$$g \star f^\flat : B \star A_0^\flat \coprod_{B_0 \star A_0^\flat} B_0 \star A^\flat \hookrightarrow B \star A^\flat$$

is a left marked anodyne map.

Proof. We prove (1); the dual assertion (2) is proven by a similar argument. f lies in the weakly saturated closure of the inclusions $i_m : \partial\Delta^m \hookrightarrow \Delta^m$, so it suffices to check that $i_m^\flat \star g$ is right marked anodyne for the four classes of morphisms enumerated in [10, Definition 3.1.1.1]. For $g : (\Lambda^n_i)^\flat \hookrightarrow (\Delta^n)^\flat$, $0 < i < n$, $i_m^\flat \star g$ obtained from an inner anodyne map by marking common edges, so is marked right anodyne. For $g : \Lambda^{n\natural}_n \hookrightarrow \Delta^{n\natural}$, $i_m^\flat \star g$ is $\Lambda^{n+m+1\natural}_{n+m+1} \hookrightarrow \Delta^{n+m+1\natural}$, so $i_m^\flat \star g$ is marked right anodyne. For the remaining two classes, $i_m^\flat \star g$ is the identity because no markings are introduced when joining two marked simplicial sets. □

The following proposition reveals a basic asymmetry of the relative join, which is related to our choice of *cocartesian* fibrations to model functors.

4.11. **Proposition.** *Let K be a marked simplicial set over S.*

(1) For every marked left horn inclusion ${}_\natural\Lambda^n_0 \hookrightarrow {}_\natural\Delta^n$ over S, the induced map

$$K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural) \hookrightarrow K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural)$$

is left marked anodyne, where the pullbacks ${}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural$ and ${}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural$ are formed with respect to the source map e_0 and are regarded as marked simplicial sets over S via the target map e_1.

(1') For every left horn inclusion $\Lambda^n_0 \hookrightarrow \Delta^n$ over S, the induced map

$$\Delta^n \times_S \mathcal{O}(S) \coprod_{\Lambda^n_0 \times_S \mathcal{O}(S)} K \star_S (\Lambda^n_0 \times_S \mathcal{O}(S)) \hookrightarrow K \star_S (\Delta^n \times_S \mathcal{O}(S))$$

is an inner anodyne map.

(2) Let $e_0 : C \longrightarrow S$ be a cartesian fibration over S and let $e_1 : C \longrightarrow S$ be any map of simplicial sets. For every inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, $0 < k < n$ over S, the induced map

$$K \star_S (\Lambda^n_k \times_S C) \hookrightarrow K \star_S (\Delta^n \times_S C)$$

is inner anodyne, where the pullbacks $\Lambda^n_k \times_S C$ and $\Delta^n \times_S C$ are formed with respect to e_0 and are regarded as simplicial sets over S via e_1.

(3) *For every marked right horn inclusion $\Lambda^{n\natural}_n \hookrightarrow \Delta^{n\natural}$ over S, the induced map*
$$K \star_S \Lambda^{n\natural}_n \hookrightarrow K \star_S \Delta^{n\natural}$$
is right marked anodyne.

Proof. Let I be the set of simplices of K endowed with a total order such that $\sigma < \sigma'$ if the dimension of σ is less than that of σ', where we view the empty set as a simplex of dimension -1. Let J be the set of epimorphisms $\chi : \Delta^j \twoheadrightarrow \Delta^{n-1}$ endowed with a total order such that $\chi < \chi'$ if the dimension of χ is less than that of χ'. Order $I \times J$ by $(\sigma, \chi) < (\sigma', \chi')$ if $\sigma < \sigma'$ or $\sigma = \sigma'$ and $\chi < \chi'$. For any simplex $\tau : \Delta^j \longrightarrow \Delta^n$, we let $r_k(\tau)$ be the pullback

$$\begin{array}{ccc} \Delta^{r_k(\tau)_0} & \xrightarrow{r_k(\tau)} & \Delta^{n-1} \\ \downarrow & & \downarrow d_k \\ \Delta^j & \xrightarrow{\tau} & \Delta^n \end{array}$$

We will let ι denote the map under consideration. We first prove (1). Given $\sigma \in I$ and $\chi \in J$, let $X_{\sigma,\chi}$ be the sub-marked simplicial set of $K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural)$ on $K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural)$ and simplices $(\sigma', \tau') : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S \mathcal{O}(S))$ not in $K \star_S (\Lambda^n_0 \times_S \mathcal{O}(S))$ with $(\sigma', r_0(e_0 \circ \tau')) \leq (\sigma, \chi)$. If $(\sigma, \chi) < (\sigma', \chi')$, then we have an obvious inclusion $X_{\sigma,\chi} \hookrightarrow X_{\sigma',\chi'}$, and we let
$$X_{<(\sigma,\chi)} = ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural) \bigcup (\cup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma',\chi'}).$$

Since $K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural) = \operatorname{colim}_{(\sigma,\chi)\in I\times J} X_{\sigma,\chi}$, in order to show that ι is left marked anodyne it suffices to show that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is left marked anodyne for all $(\sigma, \chi) \in I \times J$. We will say that a simplex of $X_{\sigma,\chi}$ is *new* if it does not belong to $X_{<(\sigma,\chi)}$.

Let $\sigma : \Delta^i \longrightarrow K$ be an element of I and $\chi : \Delta^j \twoheadrightarrow \Delta^{n-1}$ an element of J. Let $\lambda = (\sigma, \tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S \mathcal{O}(S))$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_0(e_0 \circ \tau) = \chi$. Let $\bar{\chi} : \Delta^{j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_0(\bar{\chi}) = \chi$ and let $e : \Delta^1 \longrightarrow \Delta^n \times_S \mathcal{O}(S)$ be a cartesian edge over $\{0, 1\}$ with $e(1) = \tau(0)$. The inclusion $(\Delta^1)^\sharp \coprod_{\Delta^0} \Delta^j \hookrightarrow {}_\natural\Delta^{j+1}$ is right marked anodyne, so we have a lift $\bar{\tau}$ in the following diagram

$$\begin{array}{ccc} \Delta^1 \coprod_{\Delta^0} \Delta^j & \xrightarrow{e \cup \tau} & \Delta^n \times_S \mathcal{O}(S) \\ \downarrow & \overset{\bar{\tau}}{\nearrow} & \downarrow \\ \Delta^{j+1} & \xrightarrow{\bar{\chi}} & \Delta^n. \end{array}$$

By Lemma 4.10,
$$\Delta^i \star \Delta^j \coprod_{\Delta^j} {}_\natural\Delta^{j+1} \hookrightarrow \Delta^i \star {}_\natural\Delta^{j+1}$$
is right marked anodyne. Using that $(e_1 \circ \bar{\tau})(e)$ is an equivalence, we obtain a lift

$$\begin{array}{ccc} \Delta^i \star \Delta^j \coprod_{\Delta^j} {}_\natural\Delta^{j+1} & \xrightarrow{\pi\lambda \cup e_1\bar{\tau}} & S^\sim \\ \downarrow & \nearrow & \\ \Delta^i \star {}_\natural\Delta^{j+1} & & \end{array}$$

which allows us to define $\bar{\lambda} : \Delta^i \star \Delta^{j+1} \longrightarrow K \star_S (\Delta^n \times_S \mathcal{O}(S))$ extending λ and $\bar{\tau}$. Then $\bar{\lambda}$ is a nondegenerate new simplex of $X_{\sigma,\chi}$ and every face of $\bar{\lambda}$ except for $\lambda = d_{i+1}(\bar{\lambda})$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{array}{ccc} \coprod_\lambda (\Lambda^{i+j+2}_{i+1}, \{i+1, i+2\}) & \longrightarrow & X_{<(\sigma,\chi)} \\ \downarrow & & \downarrow \\ \coprod_\lambda (\Delta^{i+j+2}, \{i+1, i+2\}) & \longrightarrow & X_{<(\sigma,\chi),1} \end{array}$$

which factors the inclusion $X_{<(\sigma,\chi)} \hookrightarrow X_{(\sigma,\chi)}$ as the composition of a left marked anodyne map and an inclusion (there is one further complication involving markings: in the special case $n = 1$, $\sigma = \emptyset$,

$j=1$, we may have that $\lambda=\tau$ is a marked edge, i.e. an equivalence over 1. Then the edges of $\bar{\tau}$ are all marked, so we should form the pushout via maps $(\Lambda^2_0)^\sharp \hookrightarrow (\Delta^2)^\sharp$, which are left marked anodyne by [10, Corollary 3.1.1.7]).

Now for the inductive step suppose that we have defined a sequence of left marked anodyne maps

$$X_{<(\sigma,\chi)} \hookrightarrow \dots \hookrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0<l\leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i+l+j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to a $i+l+j+1$-simplex with the edge $\{i+l,i+l+1\}$ marked in $X_{<(\sigma,\chi),l}$, and no new nondegenerate simplices of dimension $>i+l+j+1$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda=(\sigma,\tau)$ be any new nondegenerate $i+m+j+1$-simplex not in $X_{<(\sigma,\chi),m}$. For $0\leq l<m$ let $\lambda_l=(\sigma,\tau_l)$ be a nondegenerate $i+m+j+1$-simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m}(\lambda_l)=d_{i+l+1}(\lambda)$ and edge $\{i+m,i+m+1\}$ marked. τ and $\tau_0,...,\tau_{m-1}$ together define a map

$$\tau' : \Lambda^{m+1}_{m+1} \star \Delta^{j-1} \longrightarrow \Delta^n \times_S \mathcal{O}(S)$$

where the domain of τ is the subset $\{0,...,m-1,m+1,...,m+j+1\}$ and the domain of τ_l is the subset $\{0,...,\widehat{l},...,m+j+1\}$. Observe that the map $\Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \hookrightarrow \Delta^{m+1\natural} \star \Delta^{j-1}$ is right marked anodyne, since it factors as

$$\Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \hookrightarrow \Delta^{m+1\natural} \coprod_{\Lambda^{m+1\natural}_{m+1}} \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \hookrightarrow \Delta^{m+1\natural} \star \Delta^{j-1}$$

where the first map is obtained as the pushout of the right marked anodyne map $\Lambda^{m+1\natural}_{m+1} \hookrightarrow \Delta^{m+1\natural}$ along the inclusion $\Lambda^{m+1\natural}_{m+1} \hookrightarrow \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1}$ and the second map is obtained by marking a common edge of an inner anodyne map. Let $\bar{\chi}:\Delta^{m+j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_0(\bar{\chi})=\chi$. Then we have a lift $\bar{\tau}$ in the following commutative diagram

$$\begin{array}{ccc} \Lambda^{m+1}_{m+1} \star \Delta^{j-1} & \xrightarrow{\tau'} & \Delta^n \times_S \mathcal{O}(S) \\ \downarrow & \overset{\bar{\tau}}{\nearrow} & \downarrow \\ \Delta^{m+1} \star \Delta^{j-1} & \xrightarrow{\bar{\chi}} & \Delta^n. \end{array}$$

By Lemma 4.10, the map

$$\Delta^i \star \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1} \coprod_{\Lambda^{m+1\natural}_{m+1} \star \Delta^{j-1}} \Delta^{m+1\natural} \star \Delta^{j-1} \hookrightarrow \Delta^i \star \Delta^{m+1\natural} \star \Delta^{j-1}$$

is right marked anodyne. Since $(e_1 \circ \bar{\tau})(\{m,m+1\})$ is an equivalence, we may extend $(\cup_l \pi\lambda_l) \cup \pi\lambda \cup e_1\bar{\tau}$ to a map $\Delta^{i+m+j+2} \longrightarrow S$, which defines a nondegenerate $(i+m+j+2)$-simplex $\bar{\lambda}$ with λ as its $(i+m+1)$th face and which extends $\bar{\tau}$. By construction every other face of $\bar{\lambda}$ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{array}{ccc} \coprod_\lambda (\Lambda^{i+m+j+2}_{i+m+1}, \{i+m+1,i+m+2\}) & \longrightarrow & X_{<(\sigma,\chi),m} \\ \downarrow & & \downarrow \\ \coprod_\lambda (\Delta^{i+m+j+2}, \{i+m+1,i+m+2\}) & \longrightarrow & X_{<(\sigma,\chi),m+1} \end{array}$$

and complete the inductive step (again, there is one further complication involving markings: in the special case $i=-1$, $n=1$, $j=0$, $m=1$, we may have that λ is marked. Then every edge of $\bar{\lambda}$ is marked since $(\Lambda^2_2)^\sharp \hookrightarrow (\Delta^2)^\sharp$ is right marked anodyne, and we form the pushout along maps $(\Lambda^2_1)^\sharp \hookrightarrow (\Delta^2)^\sharp$). Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is marked left anodyne, which completes the proof.

For (1'), simply observe that if $i>-1$ we are attaching along inner horns.

We now modify the above proof to prove (2). Let $X_{\sigma,\chi}$ be the sub-simplicial set of $K \star_S (\Delta^n \times_S C)$ on $K \star_S (\Lambda^n_k \times_S C)$ and simplices $(\sigma',\tau') : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S C)$ not in $K \star_S (\Lambda^n_k \times_S C)$ with $(\sigma', r_k(e_0 \circ \tau')) \leq (\sigma,\chi)$. Let $X_{<(\sigma,\chi)} = (K \star (\Lambda^n_k \times_S C)) \bigcup (\cup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma',\chi'})$. We will show that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is inner anodyne for all $(\sigma,\chi) \in I \times J$.

Let $\sigma : \Delta^i \longrightarrow K$ be an element of I, $\chi : \Delta^j \twoheadrightarrow \Delta^{n-1}$ an element of J, and let k' be the first vertex of χ with $\chi(k') = k$. Let $\lambda = (\sigma, \tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S (\Delta^n \times_S C)$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_k(e_0 \circ \tau) = \chi$. Let $\bar{\chi} : \Delta^{j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_k(\bar{\chi}) = \chi$. Combining [10, Lemma 2.1.2.3] and Lemma 4.10, we see that the inclusion

$$d_{k'} : \Delta^j = \Delta^{k'-1} \star \Delta^{j-k'} \hookrightarrow \Delta^{k'-1} \star_{\natural} \Delta^{j-k'+1}$$

is right marked anodyne, so we have a lift $\bar{\tau}$ in the following diagram

$$\begin{array}{ccc} \Delta^j & \xrightarrow{\tau} & \Delta^n \times_S C \\ \downarrow & \overset{\bar{\tau}}{\nearrow} & \downarrow \\ \Delta^{j+1} & \xrightarrow{\bar{\chi}} & \Delta^n \end{array}$$

where $\bar{\tau}(\{k', k'+1\})$ is a cartesian edge. By Lemma 4.9, $\Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} \hookrightarrow \Delta^i \star \Delta^{j+1}$ is inner anodyne. We thus obtain an extension

$$\begin{array}{ccc} \Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} & \xrightarrow{\pi\lambda \cup e_1 \bar{\tau}} & S \\ \downarrow & \nearrow & \\ \Delta^i \star \Delta^{j+1} & & \end{array}$$

which allows us to define $\bar{\lambda} : \Delta^i \star \Delta^{j+1} \longrightarrow K \star_S (\Delta^n \times_S C)$ extending λ and $\bar{\tau}$. Then $\bar{\lambda}$ is nondegenerate and every face of $\bar{\lambda}$ except for $\lambda = d_{i+k'+1}(\bar{\lambda})$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{array}{ccc} \coprod_\lambda \Lambda^{i+j+2}_{i+k'+1} & \longrightarrow & X_{<(\sigma,\chi)} \\ \downarrow & & \downarrow \\ \coprod_\lambda \Delta^{i+j+2} & \longrightarrow & X_{<(\sigma,\chi),1} \end{array}$$

which factors the inclusion $X_{<(\sigma,\chi)} \hookrightarrow X_{(\sigma,\chi)}$ as the composition of an inner anodyne map and an inclusion.

Now for the inductive step suppose that we have defined a sequence of inner anodyne maps

$$X_{<(\sigma,\chi)} \hookrightarrow ... \hookrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i + l + j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to a $i+l+j+1$-simplex such that the edge $\{i+k'+l, i+k'+l+1\}$ is sent to a cartesian edge of $\Delta^n \times_S C$, and no new nondegenerate simplices of dimension $> i+l+j+1$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma, \tau)$ be any new nondegenerate $i+m+j+1$-simplex not in $X_{<(\sigma,\chi),m}$. For $0 \leq l < m$ let $\lambda_l = (\sigma, \tau_l)$ be a nondegenerate $i+m+j+1$-simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m+k'}(\lambda_l) = d_{i+l+k'+1}(\lambda)$. τ and $\tau_0, ..., \tau_{m-1}$ together define a map

$$\tau' : \Delta^{k'-1} \star \Lambda^{m+1}_{m+1} \star \Delta^{j-k'-1} \longrightarrow \Delta^n \times_S C$$

where the domain of τ is the subset $\{0, ..., k'+m-1, k'+m+1, ..., m+j+1\}$ and the domain of τ_l is the subset $\{0, ..., \widehat{k'+l}, ..., m+j+1\}$. The map

$$\Delta^{k'-1} \star \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-k'-1} \hookrightarrow \Delta^{k'-1} \star \Delta^{m+1\natural} \star \Delta^{j-k'-1}$$

is $\Delta^{k'-1}$ joined with a right marked anodyne map, so is right marked anodyne by Lemma 4.10. Let $\bar{\chi} : \Delta^{m+j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_k(\bar{\chi}) = \chi$. Then we have a lift $\bar{\tau}$ in the following commutative diagram

$$\begin{array}{ccc} \Delta^{k'-1} \star \Lambda^{m+1}_{m+1} \star \Delta^{j-k'-1} & \xrightarrow{\tau'} & \Delta^n \times_S C \\ \downarrow & \overset{\bar{\tau}}{\nearrow} & \downarrow \\ \Delta^{m+j+1} & \xrightarrow{\bar{\chi}} & \Delta^n \end{array}$$

such that $\bar{\tau}(\{k'+m, k'+m+1\})$ is a cartesian edge. By Lemma 4.9, the map

$$\Delta^i \star \Delta^{k'-1} \star \partial\Delta^m \star \Delta^{j-k'} \coprod_{\Delta^{k'-1} \star \partial\Delta^m \star \Delta^{j-k'}} \Delta^{m+j+1} \hookrightarrow \Delta^{i+m+j+2}$$

is inner anodyne. Therefore, we may extend $(\cup_l \pi\lambda_l) \cup \pi\lambda \cup e_1\bar{\tau}$ to a map $\Delta^{i+m+j+2} \longrightarrow S$, which defines a nondegenerate $(i+m+j+2)$-simplex $\bar{\lambda}$ with λ as its $(i+k'+m+1)$th face and which extends $\bar{\tau}$. By construction every other face of $\bar{\lambda}$ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{array}{ccc} \coprod_\lambda \Lambda^{i+m+j+2}_{i+k'+m+1} & \longrightarrow & X_{<(\sigma,\chi),m} \\ \downarrow & & \downarrow \\ \coprod_\lambda \Delta^{i+m+j+2} & \longrightarrow & X_{<(\sigma,\chi),m+1} \end{array}$$

and complete the inductive step. Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is inner anodyne, which completes the proof.

We finally modify the above proof to prove (3). Given $\sigma \in I$ and $\chi \in J$, let $X_{\sigma,\chi}$ be the sub-marked simplicial set of $K \star_S \Delta^{n\natural}$ on $K \star_S \Lambda^{n\natural}_n$ and simplices $(\sigma', \tau') : \Delta^i \star \Delta^j \longrightarrow K \star_S \Delta^{n\natural}$ not in $K \star_S \Lambda^{n\natural}_n$ with $(\sigma', r_n(\tau')) \leq (\sigma, \chi)$. Let $X_{<(\sigma,\chi)} = (K \star_S \Lambda^{n\natural}_n) \bigcup (\cup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma',\chi'})$. We will show that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is right marked anodyne for all $(\sigma,\chi) \in I \times J$.

Let $\sigma : \Delta^i \longrightarrow K$ be an element of I and $\chi : \Delta^j \twoheadrightarrow \Delta^{n-1}$ an element of J. Let $\lambda = (\sigma, \tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S \Delta^{n\natural}$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_n(\tau) = \chi$. Let $\bar{\chi} : \Delta^{j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_n(\bar{\chi}) = \chi$. By Lemma 4.9, the inclusion

$$\Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} \hookrightarrow \Delta^i \star \Delta^{j+1}$$

is inner anodyne, so we have an extension in the following diagram

$$\begin{array}{ccc} \Delta^i \star \Delta^j \coprod_{\Delta^j} \Delta^{j+1} & \xrightarrow{\pi\lambda \cup \pi_2\bar{\chi}} & S \\ \downarrow & \nearrow & \\ \Delta^i \star \Delta^{j+1} & & \end{array}$$

which allows us to define $\bar{\lambda} : \Delta^i \star \Delta^{j+1} \longrightarrow K \star_S \Delta^{n\natural}$ extending λ and $\bar{\chi}$. Then $\bar{\lambda}$ is nondegenerate and every face of $\bar{\lambda}$ except for $\lambda = d_{i+j+2}(\bar{\lambda})$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{array}{ccc} \coprod_\lambda \Lambda^{i+j+2\natural}_{i+j+2} & \longrightarrow & X_{<(\sigma,\chi)} \\ \downarrow & & \downarrow \\ \coprod_\lambda \Delta^{i+j+2\natural} & \longrightarrow & X_{<(\sigma,\chi),1} \end{array}$$

which factors the inclusion $X_{<(\sigma,\chi)} \longrightarrow X_{(\sigma,\chi)}$ as the composition of a right marked anodyne map and an inclusion.

Now for the inductive step suppose that we have defined a sequence of right marked anodyne maps

$$X_{<(\sigma,\chi)} \hookrightarrow \ldots \hookrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i+l+j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to a $i+l+j+1$-simplex, and no new nondegenerate simplices of dimension $> i+l+j+1$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma,\tau)$ be any new nondegenerate $i+m+j+1$-simplex not in $X_{<(\sigma,\chi),m}$. For $0 < l \leq m$ let $\lambda_l = (\sigma, \tau_l)$ be a nondegenerate $i+m+j+1$-simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m+j+1}(\lambda_l) = d_{i+j+l+1}(\lambda)$ (note that $\tau_l = \tau$). By Lemma 4.9, the map

$$\Delta^i \star \Delta^j \star \partial\Delta^m \coprod_{\Delta^j \star \partial\Delta^m} \Delta^j \star \Delta^m \hookrightarrow \Delta^i \star \Delta^j \star \Delta^m$$

is inner anodyne. Therefore, we may extend $\pi\lambda \cup (\cup_l \pi\lambda_l)$ to a map $\Delta^{i+j+m+2} \longrightarrow S$ and define a $(i+j+m+2)$-simplex $\bar{\lambda}$ of $K \star \Delta^{n\natural}$ with $d_{i+j+m+2}\bar{\lambda} = \lambda$ and $d_{i+j+l+1}\bar{\lambda} = \lambda + l$. By construction every face of $\bar{\lambda}$ except for λ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{array}{ccc} \coprod_\lambda \Lambda^{i+j+m+2\natural}_{i+j+m+2} & \longrightarrow & X_{<(\sigma,\chi),m} \\ \downarrow & & \downarrow \\ \coprod_\lambda \Delta^{i+j+m+2\natural} & \longrightarrow & X_{<(\sigma,\chi),m+1} \end{array}$$

and complete the inductive step. Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is right marked anodyne, which completes the proof. $\square$

4.12. **Remark.** The proof of Proposition 4.11 can be adapted to show that for any cartesian fibration $C \longrightarrow S$, ${}_\natural\Lambda^n_0 \times_S C^\natural \hookrightarrow {}_\natural\Delta^n \times_S C^\natural$ is marked left anodyne (in the $\sigma = \emptyset$ case, we only use that $e_0 : \mathcal{O}(S) \longrightarrow S$ is a cartesian fibration). As well, letting $K = \emptyset$, part (2) of Proposition 4.11 shows that $\Lambda^n_k \times_S C \hookrightarrow \Delta^n \times_S C$ is inner anodyne. This refines the theorem that marked left anodyne maps resp. inner anodyne maps pullback to cocartesian equivalences resp. categorical equivalences along cartesian fibrations.

For later use, we state a criterion for showing that a functor is left Quillen.

4.13. **Lemma.** *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $F : \mathcal{M} \longrightarrow \mathcal{N}$ be a functor which preserves cofibrations. Let I be a weakly saturated* [10, Definition A.1.2.2] *subset of the trivial cofibrations in $\mathcal{M}$ such that for every object $A \in \mathcal{M}$, we have a map $f : A \longrightarrow A'$ where $f \in I$ and A' is fibrant. Then F preserves trivial cofibrations if and only if*

(1) For every $f \in I$, $F(f)$ is a trivial cofibration.
(2) F preserves trivial cofibrations between fibrant objects.

Proof. The 'only if' direction is obvious. For the other direction, let $A \longrightarrow B$ be a trivial cofibration in $\mathcal{M}$. We may form the diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & & \\ \downarrow & & \downarrow & \searrow & \\ A' & \longrightarrow & A' \coprod_A B & \longrightarrow & (A' \coprod_A B)' \end{array}$$

where the vertical and lower right horizontal arrows are in I. Then our two assumptions along with the two-out-of-three property of the weak equivalences shows that $F(A) \longrightarrow F(B)$ is a trivial cofibration. $\square$

4.14. **Lemma.** *Let K be a simplicial set over S. Then*

$$K \star_S -, - \star_S K : s\mathbf{Set}_{/S} \longrightarrow s\mathbf{Set}_{K//S}$$

are left adjoints. Similarly, for K a marked simplicial set over S,

$$K \star_S -, - \star_S K : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{K//S}$$

are left adjoints.

Proof. We will prove that $K \star_S -$ is a left adjoint in the unmarked case and leave the other cases to the reader. Let F denote $K \star_S -$ and define a functor $G : s\mathbf{Set}_{K//S} \longrightarrow s\mathbf{Set}_{/S}$ by letting $G(K \longrightarrow C)$ be the simplicial set over S which satisfies

$$\mathrm{Hom}_{/S}(\Delta^n, G(K \longrightarrow C)) = \mathrm{Hom}_{K//S}(K \star_S \Delta^n, C);$$

this is evidently natural in $K \longrightarrow C$. Define a unit map $\eta : \mathrm{id} \longrightarrow GF$ on objects X by sending $\sigma : \Delta^n \longrightarrow X$ to $K \star_S \sigma : K \star_S \Delta^n \longrightarrow K \star_S X$, which corresponds to $\Delta^n \longrightarrow G(K \star_S X)$. Define a counit map $\eta : FG \longrightarrow \mathrm{id}$ on objects $K \longrightarrow C$ by sending $\lambda = (\sigma, \tau) : \Delta^i \star \Delta^j \longrightarrow K \star_S G(K \longrightarrow C)$ to $\Delta^i \star \Delta^j \xrightarrow{(\sigma,\mathrm{id})} K \star_S \Delta^j \xrightarrow{\tau'} C$, where τ' corresponds to $\tau : \Delta^j \longrightarrow G(K \longrightarrow C)$. Then it is straightforward to verify the triangle identities, so F is adjoint to G. $\square$

For the following pair of results, endow $s\mathbf{Set}^+_{/S}$ with the cocartesian model structure and $s\mathbf{Set}^+_{K//S} = (s\mathbf{Set}^+_{/S})_{K/}$ with the model structure created by the forgetful functor to $s\mathbf{Set}^+_{/S}$.

4.15. **Theorem.** *Let K be a marked simplicial set over S. The functor*

$$K \star_S (- \times_S \mathcal{O}(S)^\sharp) : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{K//S}$$

is left Quillen.

Proof. We will denote the functor in question by F. First observe that F is the composite of the three left adjoints e_0^*, $e_{1!}$, and $K \star_S -$, so F is a left adjoint. F evidently preserve cofibrations, so it only remains to check that F preserves the trivial cofibrations. We first verify that F preserves the left marked anodyne maps. Since F preserves colimits it suffices to check that F preserves a collection of morphisms which generate the left marked anodyne maps as a weakly saturated class. We verify that F preserves the four classes of maps enumerated in [10, Definition 3.1.1.1].

(1): For $\iota : (\Lambda^n_k)^\flat \longrightarrow (\Delta^n)^\flat$, $0 < k < n$, the underlying map of simplicial sets of $F(\iota)$ is inner anodyne by Proposition 4.11. $F(\iota)$ is obtained by marking common edges of an inner anodyne map, so is left marked anodyne.

(2): For $\iota : {}_\natural\Lambda^n_0 \longrightarrow {}_\natural\Delta^n$, we observe that the map

$$K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\sharp) \coprod_{K\star_S({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural)} K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural) \longrightarrow K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\sharp)$$

in the case $n = 1$ is marked left anodyne, since every marked edge in the codomain factors as a composite of two marked edges in the domain, and is the identity if $n > 1$. It thus suffices to show that $K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural) \longrightarrow K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural)$ is left marked anodyne, which is the content of part 1 of 4.11.

(3) and (4): In both of these cases one has a map of marked simplicial sets $A \longrightarrow B$ whose underlying map is an isomorphism of simplicial sets. Then

$$\begin{array}{ccc} A & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ B & \longrightarrow & F(B) \end{array}$$

is a pushout square, so $F(A) \longrightarrow F(B)$ is left marked anodyne if $A \longrightarrow B$ is.

Next, let $f : {}_\natural C \longrightarrow {}_\natural D$ be a cocartesian equivalence between cocartesian fibrations over S. Let $g : {}_\natural D \longrightarrow {}_\natural C$ be a homotopy inverse of f, so that there exists a homotopy $h : {}_\natural C \times (\Delta^1)^\sharp \longrightarrow {}_\natural C$ over S from id_C to $g \circ f$. Define a map

$$\phi : (K \star_S ({}_\natural C \times_S \mathcal{O}(S)^\sharp)) \times (\Delta^1)^\sharp \longrightarrow K \star_S (({}_\natural C \times_S \mathcal{O}(S)^\sharp) \times (\Delta^1)^\sharp)$$

by sending a $(i + j + 1)$-simplex (λ, α) given by the data $\sigma : \Delta^i \longrightarrow K$, $\tau : \Delta^j \longrightarrow {}_\natural C \times_S \mathcal{O}(S)^\sharp$, $\pi \circ \lambda : \Delta^{i+j+1} \longrightarrow \Delta^1$, $\alpha : \Delta^{i+j+1} \longrightarrow \Delta^1$ to a $i + j + 1$-simplex λ' given by the data σ, $(\tau, \alpha \circ \iota)$, $\pi \circ \lambda$ where $\iota : \Delta^j \longrightarrow \Delta^i \star \Delta^j$ is the inclusion. It is easy to see that ϕ restricts to an isomorphism on $(K \star_S ({}_\natural C \times_S \mathcal{O}(S)^\sharp)) \times \partial\Delta^1$. We deduce that $F(h) \circ \phi$ is a homotopy from $F(g \circ f)$ to the identity. A similar argument concerning a chosen homotopy from $f \circ g$ to id_D shows that $F(f)$ is a cocartesian equivalence.

Finally, invoking Lemma 4.13 completes the proof. □

4.16. **Theorem.** *Let K be a marked simplicial set over S. The functor*

$$- \star_S K : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{K//S}$$

is left Quillen.

Proof. As with the proof of Theorem 4.15, the proof will be an application of Lemma 4.13. We first verify that $- \star_S K$ preserves the four classes of left marked anodyne maps enumerated in [10, Definition 3.1.1.1]. (1) is handled by the dual of part (2) of Proposition 4.11. (2) is handled by the dual of part (3) of Proposition 4.11. (3) and (4) are handled as in the proof of Theorem 4.15. Finally, the case of $A \longrightarrow B$ a cocartesian equivalence between fibrant objects is also handled as in the proof of Theorem 4.15. □

4.17. **Definition.** Let $K, C \longrightarrow S$ be marked simplicial sets over S and let $p : K \longrightarrow C$ be a map over S. Define the marked simplicial set $C_{(p,S)/} \longrightarrow S$ as the value of the right adjoint to $K \star_S (- \times_S \mathcal{O}(S)^\sharp)$ on $K \longrightarrow C \longrightarrow S$ in $s\mathbf{Set}^+_{K//S}$. By Theorem 4.15, if $C \longrightarrow S$ is a S-category, then $C_{(p,S)/} \longrightarrow S$ is a S-category. We will refer to $C_{(p,S)/}$ as a *S-undercategory* of C.

Dually, define the marked simplicial set $C_{/(p,S)} \longrightarrow S$ as the value of the right adjoint to $- \star_S (K \times_S \mathcal{O}(S)^\sharp)$ on $K \longrightarrow C \longrightarrow S$ in $s\mathbf{Set}^+_{K//S}$. By Theorem 4.16 applied to $K \times_S \mathcal{O}(S)^\sharp$, if $C \longrightarrow S$ is a S-category, then $C_{/(p,S)} \longrightarrow S$ is a S-category. We will refer to $C_{/(p,S)}$ as a *S-overcategory* of C.

In the sequel, we will focus our attention on the S-undercategory and leave proofs of the evident dual assertions to the reader.

Functoriality in the diagram. We now study the functoriality of the S-undercategory with respect to the diagram category. Given maps $f : K \longrightarrow L$ and $p : L \longrightarrow X$ of marked simplicial sets over S, we have an induced map $X_{(p,S)/} \longrightarrow X_{(pf,S)/}$, which in terms of the functors that $X_{(p,S)/}$ and $X_{(pf,S)/}$ represent is given by precomposing $L \star_S (A \times_S \mathcal{O}(S)^\sharp) \longrightarrow X$ by $f \star_S \mathrm{id}$.

Recall that for a category $\mathcal{M}$ admitting pushouts and a map $f : K \longrightarrow L$, we have an adjunction

$$f_! : \mathcal{M}_{K/} \rightleftarrows \mathcal{M}_{L/} : f^*$$

where $f_!(K \longrightarrow X) = X \coprod_K L$ and $f^*(L \xrightarrow{p} X) = p \circ f$. If $\mathcal{M}$ is a model category and $\mathcal{M}_{K/}$, $\mathcal{M}_{L/}$ are provided with the model structures induced from $\mathcal{M}$, then $(f_!, f^*)$ is a Quillen adjunction. Moreover, if $\mathcal{M}$ is a left proper model category and f is a weak equivalence, then $(f_!, f^*)$ is a Quillen equivalence.

4.18. **Proposition.** *Let $f : K \longrightarrow L$ be a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Let C be a S-category and let $p : L \longrightarrow {}_\natural C$ be a map. Then ${}_\natural C_{(p,S)/} \longrightarrow {}_\natural C_{(pf,S)/}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof. Let $F = f_! \circ (K \star_S (- \times_S \mathcal{O}(S)^\sharp))$ and let $F' = L \star_S (- \times_S \mathcal{O}(S)^\sharp)$. Let G and G' be the right adjoints to F and F', respectively. Let $\alpha : F \longrightarrow F'$ be the evident natural transformation and let $\beta : G' \longrightarrow G$ be the dual natural transformation, defined by $G' \xrightarrow{\eta_{G'}} GFG' \xrightarrow{G\alpha G'} GF'G' \xrightarrow{G\epsilon'} G$. Then $\beta_C : {}_\natural C_{(p,S)/} \longrightarrow {}_\natural C_{(pf,S)/}$ is the map under consideration. By Theorem 4.16, α_X is a cocartesian equivalence for all $X \in s\mathbf{Set}^+_{/S}$. Therefore, by [8, Corollary 1.4.4(b)], β_C is a cocartesian equivalence. □

4.19. **Proposition.** *Consider a commutative diagram of marked simplicial sets*

$$\begin{array}{ccc} K & \longrightarrow & C \\ {\scriptstyle i}\downarrow & \overset{p}{\nearrow} & \downarrow{\scriptstyle q} \\ L & \longrightarrow & D \end{array}$$

where i is a cofibration and q is a fibration.

1. *The map*
$$C_{(p,S)/} \longrightarrow C_{(pi,S)/} \times_{D_{(qpi,S)/}} D_{(qp,S)/}$$
is a fibration.
2. *Let $K = \emptyset$ and $D = S^\sharp$. Then the map*
$$C_{(p,S)/} \longrightarrow C_{(pi,S)/} \cong \underline{\mathrm{Fun}}_S(S^\sharp, C)$$
is a left fibration (of the underlying simplicial sets).

Proof. (1) Given a trivial cofibration $A \longrightarrow B$, we need to solve lifting problems of the form

$$\begin{array}{ccc} L \star_S (A \times_S \mathcal{O}(S)^\sharp) \coprod_{K \star_S (A \times_S \mathcal{O}(S)^\sharp)} K \star_S (B \times_S \mathcal{O}(S)^\sharp) & \longrightarrow & C \\ \downarrow & & \downarrow \\ L \star_S (B \times_S \mathcal{O}(S)^\sharp) & \longrightarrow & D. \end{array}$$

But the lefthand map is a trivial cofibration by Theorem 4.15.

(2) We need to solve lifting problems of the form

$$\begin{array}{ccc} (\Delta^n)^\flat \times_S \mathcal{O}(S)^\sharp \coprod_{(\Lambda^n_i)^\flat} K \star_S ((\Lambda^n_i)^\flat \times_S \mathcal{O}(S)^\sharp) & \longrightarrow & C \\ \downarrow & & \downarrow \\ K \star_S ((\Delta^n)^\flat \times_S \mathcal{O}(S)^\sharp) & \longrightarrow & S \end{array}$$

where $0 \leq i < n$. But the lefthand map is a trivial cofibration by Proposition 4.11 (1') and (2). □

Combining (2) of the above proposition with Lemma 3.6 (2) (which supplies a trivial marked fibration $\underline{\mathrm{Fun}}_S(S^\sharp, C) \longrightarrow C$), we obtain a map $C_{(p,S)/} \longrightarrow C$ which is a marked fibration and a left fibration, and such that for any $f : K \longrightarrow L$, the triangle

$$\begin{array}{ccc} C_{(p,S)/} & \longrightarrow & C_{(pf,S)/} \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

commutes.

The universal mapping property of the S-slice. Because the S-join and slice Quillen adjunction is not simplicial, we do not immediately obtain a universal mapping property characterizing the S-slice. Our goal in this subsection is to supply such a universal mapping property (Proposition 4.25). We first recall how to slice Quillen bifunctors. Suppose $\mathcal{V}$ is a closed symmetric monoidal category and $\mathcal{M}$ is enriched, tensored, and cotensored over $\mathcal{V}$. Denote the internal hom by

$$\underline{\mathrm{Hom}}(-,-) : \mathcal{M}^{op} \times \mathcal{M} \longrightarrow \mathcal{V}.$$

Define bifunctors

$$\underline{\mathrm{Hom}}_{x/}(-,-) : \mathcal{M}^{op}_{x/} \times \mathcal{M}_{x/} \longrightarrow \mathcal{V}$$
$$\underline{\mathrm{Hom}}_{/x}(-,-) : \mathcal{M}^{op}_{/x} \times \mathcal{M}_{/x} \longrightarrow \mathcal{V}$$

on objects $f : x \longrightarrow a, g : x \longrightarrow b$ and $f' : a \longrightarrow x, g' : b \longrightarrow x$ to be pullbacks

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{x/}(f,g) & \longrightarrow & \underline{\mathrm{Hom}}(a,b) \\ \downarrow & & \downarrow f^* \\ 1 & \xrightarrow{g} & \underline{\mathrm{Hom}}(x,b) \end{array} \qquad \begin{array}{ccc} \underline{\mathrm{Hom}}_{/x}(f',g') & \longrightarrow & \underline{\mathrm{Hom}}(a,b) \\ \downarrow & & \downarrow g'_* \\ 1 & \xrightarrow{f'} & \underline{\mathrm{Hom}}(a,x) \end{array}$$

and on morphisms in the obvious way (we abusively denote by $g : 1 \longrightarrow \underline{\mathrm{Hom}}(x,b)$ the map corresponding to g under the natural isomorphisms $\underline{\mathrm{Hom}}(1, \underline{\mathrm{Hom}}(x,b)) \cong \underline{\mathrm{Hom}}(1 \otimes x, b) \cong \underline{\mathrm{Hom}}(x,b)$, and likewise for f'). It is easy to see that $\underline{\mathrm{Hom}}_{x/}$ and $\underline{\mathrm{Hom}}_{/x}$ preserve limits separately in each variable.

4.20. **Lemma.** *In the above situation let $\mathcal{M}$ be a model category and $\mathcal{P}$ be a monoidal model category. If $\underline{\mathrm{Hom}}(-,-)$ is a right Quillen bifunctor, then $\underline{\mathrm{Hom}}_{x/}(-,-)$ and $\underline{\mathrm{Hom}}_{/x}(-,-)$ are right Quillen bifunctors, where we endow $\mathcal{M}_{x/}$ and $\mathcal{M}_{/x}$ with the model structures created by the forgetful functor to $\mathcal{M}$.*

Proof. We prove the assertion for $\underline{\mathrm{Hom}}_{x/}(-,-)$, the proof for $\underline{\mathrm{Hom}}_{/x}(-,-)$ being identical. Let $i : a \longrightarrow b$ and $f : c \longrightarrow d$ be morphisms in $\mathcal{M}_{x/}$ (so they are compatible with the structure maps $\pi_a, ..., \pi_d$). In the commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{x/}(\pi_b, \pi_c) & \longrightarrow & \underline{\mathrm{Hom}}(b,c) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{x/}(\pi_a, \pi_c) \times_{\underline{\mathrm{Hom}}_{x/}(\pi_a,\pi_d)} \underline{\mathrm{Hom}}_{x/}(\pi_b, \pi_d) & \longrightarrow & \underline{\mathrm{Hom}}(a,c) \times_{\underline{\mathrm{Hom}}(a,d)} \underline{\mathrm{Hom}}(b,d) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \underline{\mathrm{Hom}}(x,c) \end{array}$$

it is easy to see that the lower square and the rectangle are pullback squares, so the upper square is a pullback square. It is now clear that if $\underline{\mathrm{Hom}}(-,-)$ is a right Quillen bifunctor, then $\underline{\mathrm{Hom}}_{x/}(-,-)$ is as well. □

We apply Lemma 4.20 to the bifunctors

$$\begin{aligned}\mathrm{Map}_{K//S}(-,-) :& s\mathbf{Set}^{+\,op}_{K//S} \times s\mathbf{Set}^{+}_{K//S} \longrightarrow s\mathbf{Set}_{\mathrm{Quillen}}\\ \mathrm{Fun}_{K//S}(-,-) :& s\mathbf{Set}^{+\,op}_{K//S} \times s\mathbf{Set}^{+}_{K//S} \longrightarrow s\mathbf{Set}_{\mathrm{Joyal}}\end{aligned}$$

induced by $\mathrm{Map}_S(-,-)$ and $\mathrm{Fun}_S(-,-)$.

4.21. **Lemma.** *Let K, A, and B be simplicial sets and define a map*

$$A \times (K \star B) \longrightarrow K \star (A \times B)$$

by sending the data $(\Delta^n \longrightarrow A, \Delta^k \longrightarrow K, \Delta^{n-k-1} \longrightarrow B)$ of a n-simplex of $A \times (K \star B)$ to the data $(\Delta^k \longrightarrow K, \Delta^{n-k-1} \longrightarrow A \times B)$ of a n-simplex of $K \star (A \times B)$. Then

$$\phi : A \times (K \star B) \coprod_{A \times K} K \longrightarrow K \star (A \times B)$$

is a categorical equivalence.

Proof. Recall [10, Proposition 4.2.1.2] that there is a map

$$\eta_{X,Y} : X \diamond Y = X \coprod_{X \times Y \times \{0\}} X \times Y \times \Delta^1 \coprod_{X \times Y \times \{1\}} Y \longrightarrow X \star Y$$

natural in X and Y which is always a categorical equivalence. Thus

$$f = (A \times \eta_{K,B}) \sqcup \mathrm{id}_K : A \times (K \diamond B) \coprod_{A \times K} K \longrightarrow A \times (K \star B) \coprod_{A \times K} K$$

is a categorical equivalence. The domain is isomorphic to $K \diamond (A \times B)$, and it is easy to check that the map $\eta_{K, A\times B}$ is the composite

$$K \diamond (A \times B) \xrightarrow{f} A \times (K \star B) \coprod_{A \times K} K \xrightarrow{\phi} K \star (A \times B).$$

Using the 2 out of 3 property of the categorical equivalences, we deduce that ϕ is a categorical equivalence. □

4.22. **Lemma.** *For all $L \in s\mathbf{Set}^+_{/S}$, we have a natural equivalence*

$$\phi : \mathrm{Fun}_S(L, {}_\natural C_{(p,S)/}) \xrightarrow{\simeq} \mathrm{Fun}_{K//S}(K \star_S (L \times_S \mathcal{O}(S)^\sharp), {}_\natural C).$$

Proof. Define bisimplicial sets $X, Y : \Delta^{op} \longrightarrow s\mathbf{Set}$ by

$$\begin{aligned}X_n &= \mathrm{Map}_{K//S}(K \star_S ((\Delta^n)^\flat \times L \times_S \mathcal{O}(S)^\sharp), {}_\natural C)\\ Y_n &= \mathrm{Map}(\Delta^n, \mathrm{Fun}_{K//S}(K \star_S (L \times_S \mathcal{O}(S)^\sharp), {}_\natural C))\\ &\cong \mathrm{Map}_{K//S}((\Delta^n)^\flat \times (K \star_S (L \times_S \mathcal{O}(S)^\sharp) \coprod_{(\Delta^n)^\flat \times K} K, {}_\natural C).\end{aligned}$$

and define a map of bisimplicial sets $\Phi : X \longrightarrow Y$ by precomposing levelwise by the map

$$g_{L,n} : (\Delta^n)^\flat \times (K \star_S (L \times_S \mathcal{O}(S)^\sharp)) \coprod_{(\Delta^n)^\flat \times K} K \longrightarrow K \star_S ((\Delta^n)^\flat \times L \times_S \mathcal{O}(S)^\sharp)$$

adjoint as a map over $S \times \Delta^1$ to the identity over $S \times \partial\Delta^1$. Taking levelwise zero simplices then defines the map ϕ, which is clearly natural in L, K, and C. By Theorem 4.16, taking a fibrant replacement of K we may suppose that K is fibrant. We first check that X and Y are complete Segal spaces. By [9, Theorem 4.12], Y is a complete Segal space as it arises from a ∞-category. For X, since $\mathrm{Map}_{K//S}(-,-)$ is a right Quillen bifunctor, we only have to observe that:

- Every monomorphism $A \longrightarrow B$ of simplicial sets induces a cofibration

$$K \star_S (A^\flat \times L \times_S \mathcal{O}(S)^\sharp) \longrightarrow K \star_S (B^\flat \times L \times_S \mathcal{O}(S)^\sharp)$$

so X is Reedy fibrant.

- The spine inclusion $\iota_n : \mathrm{Sp}(n) \longrightarrow \Delta^n$ induces a trivial cofibration
$$K \star_S (\mathrm{Sp}(n)^\flat \times L \times_S \mathcal{O}(S)^\sharp) \longrightarrow K \star_S ((\Delta^n)^\flat \times L \times_S \mathcal{O}(S)^\sharp);$$
ι_n is inner anodyne, so this follows from Theorem 4.15 and [10, Proposition 3.1.4.2].
- The map $\pi : E \longrightarrow \Delta^0$ where E is the nerve of the contractible groupoid with two elements induces a cocartesian equivalence
$$K \star_S (E^\flat \times L \times_S \mathcal{O}(S)^\sharp) \longrightarrow K \star_S (L \times_S \mathcal{O}(S)^\sharp);$$
$\pi^\flat$ is a cocartesian equivalence (as the composite of $E^\flat \longrightarrow E^\sharp$ and $E^\sharp \longrightarrow \Delta^0$), so this also follows from Theorem 4.15 and [10, Proposition 3.1.4.2].

We next prove that Φ is an equivalence in the complete Segal model structure. For this, we will prove that each map $g_{L,n}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Both sides preserves colimits as a functor of L (valued in $s\mathbf{Set}^+_{K//S}$), so by left properness and the stability of cocartesian equivalences under filtered colimits we reduce to the case L is an m-simplex with some marking. In particular, $(\Delta^m)^\flat \times_S \mathcal{O}(S)^\sharp \longrightarrow S$ is fibrant in $s\mathbf{Set}^+_{/S}$. By [10, Theorem 4.2.4.1] we may check that the square of fibrant objects

$$\begin{array}{ccc}
(\Delta^n)^\flat \times K & \longrightarrow & K \\
\downarrow & & \downarrow \\
(\Delta^n)^\flat \times (K \star_S ((\Delta^m)^\flat \star_S \mathcal{O}(S)^\sharp)) & \longrightarrow & K \star_S ((\Delta^n)^\flat \times (\Delta^m)^\flat \times_S \mathcal{O}(S)^\sharp)
\end{array}$$

is a homotopy pushout square in the underlying ∞-category $\mathbf{Cat}^{cocart}_{\infty,S} \simeq \mathrm{Fun}(S, \mathbf{Cat}_\infty)$, where colimits are computed objectwise. In other words, we may check that for every $s \in S$, the fiber of the square over s is a homotopy pushout square in $s\mathbf{Set}$, which holds by Lemma 4.21. Pushing out along the cofibration $(\Delta^m)^\flat \times_S \mathcal{O}(S)^\sharp \longrightarrow L \times_S \mathcal{O}(S)^\sharp$ and using left properness, we deduce that $g_{L,m}$ is a cocartesian equivalence. Finally, we invoke [9, Theorem 4.11] to deduce that ϕ is a categorical equivalence. □

4.23. **Lemma.** *Let $L \longrightarrow S$ be a cocartesian fibration. Then* $\mathrm{id}_K \star \iota_L : K \star_S {}_\natural L \longrightarrow K \star_S ({}_\natural L \times_S \mathcal{O}(S)^\sharp)$ *is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof. By Theorem 4.16, taking a fibrant replacement of K we may suppose that K is fibrant. By Proposition 13.4, it suffices to show that for every $s \in S$, $K^\sim_s \star L^\sim_s \longrightarrow K^\sim_s \star ({}_\natural L \times_S (S^{/s})^\sharp)$ is a marked equivalence in $s\mathbf{Set}^+$. Observe that the *cartesian* equivalence $\{s\} \longrightarrow (S^{/s})^\sharp$ pulls back by the cocartesian fibration ${}_\natural L \longrightarrow S^\sharp$ to a marked equivalence $L^\sim_s \longrightarrow {}_\natural L \times_S (S^{/s})^\sharp$. Then by Theorem 4.15 for $S = \Delta^0$, $K^\sim_s \star -$ preserves marked equivalences, which concludes the proof. □

4.24. **Notation.** Suppose we have a commutative square of S-categories and S-functors

$$\begin{array}{ccc}
K & \xrightarrow{G} & D \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \pi} \\
C & \xrightarrow{\rho} & M.
\end{array}$$

Define $\underline{\mathrm{Fun}}_{K//M,S}(C,D)$ to be the pullback

$$\begin{array}{ccc}
\underline{\mathrm{Fun}}_{K//M,S}(C,D) & \longrightarrow & \underline{\mathrm{Fun}}_S(C,D) \\
\downarrow & & \downarrow{\scriptstyle (F^*,\pi_*)} \\
S & \xrightarrow{\sigma_{\pi G}} & \underline{\mathrm{Fun}}_S(K,M).
\end{array}$$

If $K = \emptyset$, we will also denote $\underline{\mathrm{Fun}}_{K//M,S}(C,D)$ as $\underline{\mathrm{Fun}}_{/M,S}(C,D)$. If $M = S$, we will write $\underline{\mathrm{Fun}}_{K//S}(C,D)$ in place of $\underline{\mathrm{Fun}}_{K//S,S}(C,D)$.

Note that by Proposition 3.8 and Proposition 2.16, the defining pullback square is a homotopy pullback square if F is a monomorphism and π is a categorical fibration.

4.25. **Proposition.** *Let K, L, C be S-categories and let $p : K \longrightarrow C$, $q : L \longrightarrow C$ be S-functors.*

(1) We have an equivalence

$$\psi : \underline{\mathrm{Fun}}_S(L, C_{(p,S)/}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{K//S}(K \star_S L, C).$$

(2) We have an equivalence

$$\psi' : \underline{\mathrm{Fun}}_S(L, C_{/(q,S)}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{L//S}(K \star_S L, C)$$

(3) We have equivalences

$$\underline{\mathrm{Fun}}_{/C,S}(L, C_{(p,S)/}) \xrightarrow[\simeq]{\psi_q} \underline{\mathrm{Fun}}_{K \sqcup L//S}(K \star_S L, C) \xleftarrow[\simeq]{\psi'_p} \underline{\mathrm{Fun}}_{/C,S}(K, C_{/(q,S)}).$$

Proof. (1) Define the S-functor ψ as follows: suppose given a marked simplicial set A and a map $A \longrightarrow \underline{\mathrm{Fun}}_S(L, C_{(p,S)/})$ over S. This is equivalently given by the datum of a map

$$f_A : {}_\natural K \star_S ((A \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural L) \times_S \mathcal{O}(S)^\sharp) \longrightarrow {}_\natural C$$

under K and over S. Let

$${}_\natural K \coprod_{A \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural K} (A \times_S \mathcal{O}(S)^\sharp) \times_S ({}_\natural K \star_S ({}_\natural L \times_S \mathcal{O}(S)^\sharp)) \longrightarrow K \star_S (A \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural L \times_S \mathcal{O}(S)^\sharp)$$

be the map over $S \times \Delta^1$ adjoint to the identity over $S \times \partial\Delta^1$. Precomposing f_A by this and $\iota_L : {}_\natural L \longrightarrow {}_\natural L \times_S \mathcal{O}(S)^\sharp$ on that factor defines the desired map $A \longrightarrow \underline{\mathrm{Fun}}_{K//S}(K \star_S L, C)$.

Now to check that ψ is an equivalence, we may work fiberwise and combine Lemma 4.22 and Lemma 4.23.

(2) This follows by a parallel argument to the proof of (1).

(3) We prove that ψ_q is an equivalence; a parallel argument will work for ψ'_p. $\underline{\mathrm{Fun}}_{K \sqcup L//S}(K \star_S L, C)$ fits into a diagram

$$\begin{array}{ccccc}
\underline{\mathrm{Fun}}_{K \sqcup L//S}(K \star_S L, C) & \longrightarrow & \underline{\mathrm{Fun}}_{K//S}(K \star_S L, C) & \longrightarrow & \underline{\mathrm{Fun}}_S(K \star_S L, C) \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & \underline{\mathrm{Fun}}_{K//S}(K \coprod L, C) & \longrightarrow & \underline{\mathrm{Fun}}_S(K \coprod L, C) \\
 & & \downarrow & & \downarrow \\
 & & S & \xrightarrow{\sigma_p} & \underline{\mathrm{Fun}}_S(K, C)
\end{array}$$

in which every square is a pullback square. The map ψ_q is then defined to be the pullback of the map of spans

$$\begin{array}{ccccc}
\underline{\mathrm{Fun}}_S(L, C_{(p,S)/}) & \longrightarrow & \underline{\mathrm{Fun}}_S(L, C) & \xleftarrow{\sigma_q} & S \\
\downarrow \psi & & \downarrow p \sqcup - & & \downarrow = \\
\underline{\mathrm{Fun}}_{K//S}(K \star_S L, C) & \longrightarrow & \underline{\mathrm{Fun}}_{K//S}(K \coprod L, C) & \longleftarrow & S
\end{array}$$

in which the vertical arrows are equivalences. By Proposition 4.19 and $\underline{\mathrm{Fun}}_S(L, -)$ being right Quillen, the top left horizontal arrow is a S-fibration, and by Proposition 3.8, the bottom left horizontal arrow is a S-fibration. It follows that ψ_q is an equivalence.

□

In light of Proposition 4.25, we have evident 'alternative' S-slice S-categories, whose definition more closely adheres to the intuition that a slice category is a category of extensions.

4.26. **Definition.** Let $p : K \longrightarrow C$ be a S-functor. We define the *alternative S-undercategory*

$$C^{(p,S)/} := \underline{\mathrm{Fun}}_{K//S}(K \star_S S, C).$$

Similarly, we define the *alternative S-overcategory*

$$C^{/(p,S)} := \underline{\mathrm{Fun}}_{K//S}(S \star_S K, C).$$

4.27. **Corollary.** *Let $p : K \longrightarrow C$ and $q : L \longrightarrow C$ be S-functors.*

(1) We have equivalences $C_{(p,S)/} \xrightarrow{\simeq} C^{(p,S)/}$ *and* $C_{/(q,S)} \xrightarrow{\simeq} C^{/(q,S)}$.
(2) We have an equivalence $\underline{\mathrm{Fun}}_{/C,S}(L, C^{(p,S)/}) \simeq \underline{\mathrm{Fun}}_{/C,S}(K, C^{/(q,S)})$ *through a natural zig-zag.*

Proof. For (1), let $L = S$ and $K = S$ in Proposition 4.25(1) and (2), respectively. For (2), combine the preceding (1) and Proposition 4.25(3). □

4.28. **Warning.** When $S = \Delta^0$, the alternative S-undercategory $C^{(p,S)/} \cong \{p\} \times_{\mathrm{Fun}(K,C)} \mathrm{Fun}(K^{\rhd}, C)$ differs from Lurie's alternative undercategory $C^{p/}$. However, we have a comparison functor

$$\{p\} \times_{\mathrm{Fun}(K,C)} \mathrm{Fun}(K^{\rhd}, C) \longrightarrow C^{p/}$$

which is a categorical equivalence and which factors through the categorical equivalence $C_{p/} \longrightarrow C^{p/}$ of [10, Proposition 4.2.1.5].

Slicing over and under S-points. We give a smaller model for slicing over and under S-points in an S-category C.

4.29. **Notation.** Suppose C an S-category. Let

$$\mathcal{O}_S(C) := \widetilde{\mathrm{Fun}}_S(S \times \Delta^1, C) \cong S \times_{\mathcal{O}(S)} \mathcal{O}(C)$$

denote the fiberwise arrow S-category of C. Given an object $x \in C$, let

$$C^{/\underline{x}} := \mathcal{O}_S(C) \times_C \underline{x}\,, \quad C^{\underline{x}/} := \underline{x} \times_C \mathcal{O}_S(C).$$

4.30. **Proposition.** *Let $x \in C$ be an object and denote by $i_x : \underline{x} \longrightarrow C_{\underline{x}}$ the $\underline{x}$-functor defined by x. We have natural equivalences of $\underline{x}$-categories*

$$C_{\underline{x}}{}^{/(\underline{x}, i_x)} \simeq C^{/\underline{x}}$$
$$C_{\underline{x}}{}^{/(i_x, \underline{x})} \simeq C^{\underline{x}/}.$$

Proof. For any functor $S' \longrightarrow S$ and S-category C, $\mathcal{O}_S(C) \times_S S' \cong \mathcal{O}_{S'}(C \times_S S')$. Therefore, $\mathcal{O}_S(C) \times_C \underline{x} \cong \mathcal{O}_{\underline{x}}(C_{\underline{x}}) \times_{C_{\underline{x}}} \underline{x}$ and likewise for $\underline{x} \times_C \mathcal{O}_S(C)$. Changing base to $\underline{x}$, we may suppose $S = \underline{x}$ and $i_x = i : S \longrightarrow C$ is any S-functor. The identity section $S \longrightarrow \mathcal{O}(S)$ induces a morphism of spans

$$\begin{array}{ccccc} S & \xrightarrow{\sigma_i} & \underline{\mathrm{Fun}}_S(S, C) & \longleftarrow & \underline{\mathrm{Fun}}_S(S \times \Delta^1, C) \\ \downarrow {=} & & \downarrow & & \downarrow \\ S & \xrightarrow{i} & C & \longleftarrow & \widetilde{\mathrm{Fun}}_S(S \times \Delta^1, C) \end{array}$$

with the vertical maps equivalences. Taking pullbacks now yields the claim (where we use the isomorphism $S \star_S S \cong S \times \Delta^1$ to identify the upper pullback with the S-slice category in question). □

4.31. **Proposition.** *We have a natural equivalence $C^{\underline{x}/} \simeq C^{x/}$ of left fibrations over C.*

Proof. Using the marked left anodyne map ${}_\natural\Lambda^2_1 \longrightarrow {}_\natural\Delta^2$ and the map of Lemma 2.23 for $n = 2$, we obtain a span

$$\mathrm{Fun}((\Delta^{\{0,1\}})^\sharp, {}_\natural C) \times_{C^{\{1\}}} \mathrm{Fun}(\Delta^{\{1,2\}}, C) \overset{\simeq}{\twoheadleftarrow} \mathrm{Fun}({}_\natural\Delta^2, {}_\natural C) \overset{\simeq}{\twoheadrightarrow} \mathrm{Fun}(\Delta^{\{0,2\}}, C) \times_{S^{\{0,2\}}} \mathrm{Fun}(\Delta^2, S).$$

Pulling back via $\{x\} \times_{C^{\{0\}}} -$ on the left and $- \times_{S^{\{1,2\}}} S$ on the right, and using that the inclusion $\Delta^{\{0,2\}} \longrightarrow \Delta^2 \cup_{\Delta^{\{1,2\}}} \Delta^0$ is a categorical equivalence, we get

$$C^{\underline{x}/} \overset{\simeq}{\twoheadleftarrow} \{x\} \times_{C^{\{0\}}} \mathrm{Fun}({}_\natural\Delta^2, {}_\natural C) \times_{S^{\{1,2\}}} S \overset{\simeq}{\twoheadrightarrow} C^{x/}.$$

□

5. Limits and colimits

In this section, we introduce S-colimits and study their basic properties. We then study the correspondence between S-colimits and S-limits through the vertical opposite construction of [4].

5.1. **Definition.** Let C be a S-category and $\sigma : S \longrightarrow C$ be a cocartesian section. We say that σ is a S-*initial object* if $\sigma(s)$ is an initial object for all objects $s \in S$. Dually, σ is a S-*final object* if $\sigma(s)$ is a final object for all $s \in S$.

5.2. **Definition.** Let K and C be S-categories. Let $\overline{p} : K \star_S S \longrightarrow C$ be an extension of a S-functor $p : K \longrightarrow C$. From the commutativity of the diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma_{\overline{p}}} & \underline{\mathrm{Fun}}_S(K \star_S S, C) \\ \downarrow{=} & & \downarrow \\ S & \xrightarrow{\sigma_p} & \underline{\mathrm{Fun}}_S(K, C) \end{array}$$

(recall Notation 3.5 for $\sigma_{(-)}$) we see that $\sigma_{\overline{p}}$ defines a cocartesian section of $C^{(p,S)/}$ (Definition 4.26), which we also denote by $\sigma_{\overline{p}}$. We say that $\overline{p}$ is a S-*colimit diagram* if $\sigma_{\overline{p}}$ is a S-initial object. If $\overline{p}$ is a S-colimit diagram, then $\overline{p}|_S : S \longrightarrow C$ is said to be a S-*colimit* of p. If S admits an initial object s, we will also identify the S-colimit with its value on s.

Dually, substituting $S \star_S K$ for $K \star_S S$ leads in a parallel way to the definition of an S-*limit diagram* and an S-*limit*.

5.3. **Remark.** In view of the comparison result Corollary 4.27, we could also use the S-slice category $C_{(p,S)/}$ to make the definition of a S-colimit diagram. This would yield some additional generality, in that $C_{(p,S)/}$ is defined for an arbitrary marked simplicial set K. However, the construction $C^{(p,S)/}$ is easier to relate to functor categories, which we need to do to show that the left adjoint to the restriction along $K \subset K \star_S S$ computes colimits (a special case of Corollary 9.16).

5.4. **Remark.** Suppose K and C are ∞-categories, and write $\pi : K \longrightarrow *$ for the map to a point. One may define the K-indexed colimit 'globally' as the (partially defined) left adjoint $\pi_!$ to the restriction functor $\pi^* : C \longrightarrow \mathrm{Fun}(K, C)$. Given a diagram $p : K \longrightarrow C$ that admits an extension to a colimit diagram $\overline{p} : K^{\rhd} \longrightarrow C$ with cone point $\{v\}$, one then has $\overline{p}|_{\{v\}} \simeq \pi_!(p)$.

To establish a parallel picture for S-colimits, we will first need to introduce the concept of S-adjunctions (Definition 8.3). If we now let K and C be S-categories and $\pi : K \longrightarrow S$ denote the structure map, we will show that if for all $s \in S$, $C_{\underline{s}}$ admits $K_{\underline{s}}$-indexed $S^{s/}$-colimits, then the restriction S-functor $\pi^* : C \longrightarrow \underline{\mathrm{Fun}}_S(K, C)$ admits a left S-adjoint $\pi_!$ such that

$$(\pi_!)_s : \mathrm{Fun}_{S^{s/}}(K_{\underline{s}}, C_{\underline{s}}) \longrightarrow C_s$$

computes the $S^{s/}$-colimit (Theorem 10.5 in the special case $\phi = \pi$). Furthermore, taking cocartesian sections of this S-adjunction then yields an adjunction, which we may abusively denote as

$$\pi_! : \mathrm{Fun}_S(K, C) \rightleftarrows \mathrm{Fun}_S(S, C) : \pi_*,$$

in which $\pi_!$ computes the S-colimit.

In proving some of the assertions in this subsection (Corollary 5.9, Proposition 5.11, and Proposition 5.12), it will be convenient to have this relationship between S-colimits and S-adjunctions established. We note that there is no danger of circularity here since the proof of Theorem 10.5 (or its simpler predecessor Theorem 9.15) doesn't use any of the remainder of this subsection (which, apart from S-(co)limits in an S-category of S-objects, is only devoted to working out special classes of diagrams in the theory).

There are a couple instances where the notion of S-colimit specializes to a notion of ordinary category theory. For example, we have the following pair of propositions computing S-colimits and S-limits in an S-category of objects $\underline{C}_S$ as left or right Kan extensions in C; the asymmetry in their formulations arises due to working with *cocartesian* fibrations instead of cartesian fibrations to model S-categories. In the statements, recall Notation 3.11 for the meaning of $(-)^{\dagger}$.

5.5. **Proposition.** *Let $\overline{p} : K \star_S S \longrightarrow \underline{C}_S$ be a S-functor extending $p : K \longrightarrow \underline{C}_S$. Suppose further that a left Kan extension of $p^\dagger : K \longrightarrow C$ to a functor $K \star_S S \longrightarrow C$ exists. Then the following are equivalent:*

(1) $\overline{p}$ is a S-colimit diagram.
(2) $\overline{p}^\dagger$ is a left Kan extension of $p^\dagger$.
(3) $\overline{p}^\dagger|_{K_s^\rhd}$ is a colimit diagram for all $s \in S$.

Proof. (2) and (3) are equivalent because left Kan extensions along cocartesian fibrations are computed fiberwise. Suppose (3). To prove (1), we want to show that for every $s \in S$, $\overline{p}_{\underline{s}}$ is an initial object in $((\underline{C}_S)^{(p,S)/})_s$. But $((\underline{C}_S)^{(p,S)/})_s$ is equivalent to the fiber of $\mathrm{Fun}(K_{\underline{s}} \star_{\underline{s}} \underline{s}, C) \longrightarrow \mathrm{Fun}(K_{\underline{s}}, C)$ over $p^\dagger|_{K_{\underline{s}}}$, so to prove the claim it suffices to show that the functor $\overline{p}^\dagger|_{K_{\underline{s}}}$ is a left Kan extension of $p|_{K_{\underline{s}}}$. This holds by the equivalence of (2) and (3) for $S^{s/}$.

Conversely, suppose (1). Since we supposed that a left Kan extension of $p^\dagger$ exists, left Kan extensions of $p^\dagger|_{K_s}$ all exist and *any* initial object in the fiber of $\mathrm{Fun}(K_{\underline{s}} \star_{\underline{s}} \underline{s}, C) \longrightarrow \mathrm{Fun}(K_{\underline{s}}, C)$ over $p^\dagger|_{K_{\underline{s}}}$ is a left Kan extension of $p^\dagger|_{K_{\underline{s}}}$, necessarily a fiberwise colimit diagram (we need this hypothesis because Kan extensions as defined in [10, §4.3.2] are always *pointwise* Kan extensions). This implies (3). □

5.6. **Proposition.** *Let $\overline{p} : S \star_S K \longrightarrow \underline{C}_S$ be a S-functor extending $p : K \longrightarrow \underline{C}_S$. Suppose further that a right Kan extension of $p^\dagger : K \longrightarrow C$ to a functor $S \star_S K \longrightarrow C$ exists. Then the following are equivalent:*

(1) $\overline{p}$ is a S-limit diagram.
(2) $\overline{p}^\dagger$ is a right Kan extension of $p^\dagger$.
(2') $\overline{p}^\dagger|_{\underline{s} \star_{\underline{s}} K_{\underline{s}}}$ is a right Kan extension of $p^\dagger|_{K_{\underline{s}}}$ for all $s \in S$.
(3) $\overline{p}^\dagger|_{K_{\underline{s}}^\lhd}$ is a limit diagram for all $s \in S$.

Proof. We first observe that because the inclusion $S \longrightarrow S \star_S K$ is left adjoint to the structure map $S \star_S K \longrightarrow S$ of the cocartesian fibration,

$$(S \star_S K)^{s/} \simeq S^{s/} \times_S (S \star_S K) \cong \underline{s} \star_{\underline{s}} K_{\underline{s}}.$$

The equivalence of (2) and (2') now follows from the formula for a right Kan extension. Also, if we view $K_{\underline{s}}^\lhd$ as mapping to $S \star_S K$ via $\{s\} \star K_{\underline{s}} \longrightarrow \underline{s} \star_{\underline{s}} K_{\underline{s}} \longrightarrow S \star_S K$ where the first map is adjoint to $(\{s\} \longrightarrow \underline{s}, \mathrm{id})$, then (2) and (3) are also equivalent by the same argument. Finally, (2') implies (1) by definition, and (1) implies (2') under our additional assumption that a right Kan extension of $p^\dagger$ exists (for the same reason as given in the proof of Proposition 5.5). □

If S is a Kan complex, then the notion of S-colimit reduces to the usual notion of colimit.

5.7. **Proposition.** *Let S be a Kan complex. Then a S-functor $\overline{p} : K \star_S S \longrightarrow C$ is a S-colimit diagram if and only if for every object $s \in S$, $\overline{p}|_s : (K_s)^\rhd \longrightarrow C_s$ is a colimit diagram.*

Proof. If S is a Kan complex, then for every $s \in S$, $S^{s/}$ is a contractible Kan complex. Therefore, for all $s \in S$ we have $(C^{(p,S)/})_s \simeq \{p_s\} \times_{\mathrm{Fun}(K_s, C_s)} \mathrm{Fun}(K_s^\rhd, C_s)$, which proves the claim. □

We say that K is a *constant* S-category if it is equivalent to $S \times L$ for L an ∞-category. We have an isomorphism $L^\rhd \times S \longrightarrow (L \times S) \star_S S$ (defined as a map over $S \times \Delta^1$ to be the adjoint to the identity on $(L \times S, S)$).

5.8. **Proposition.** *A S-functor $\overline{p} : L^\rhd \times S \longrightarrow C$ is a S-colimit diagram if and only if for every object $s \in S$, $\overline{p}_s : L^\rhd \longrightarrow C_s$ is a colimit diagram.*

Proof. Observe that

$$(C^{(p,S)/})_s = \{p_{\underline{s}}\} \times_{\mathrm{Fun}_{S^{s/}}(L \times S^{s/}, C_{\underline{s}})} \mathrm{Fun}_{S^{s/}}(L^\rhd \times S^{s/}, C_{\underline{s}}) \simeq \{p_s\} \times_{\mathrm{Fun}(L, C_s)} \mathrm{Fun}(L^\rhd, C_s).$$

Therefore, $\sigma_{\overline{p}} : S \longrightarrow C^{(p,S)/}$ is S-initial if and only if for all $s \in S$, $\{\overline{p}_s\} \in \{p_s\} \times_{\mathrm{Fun}(L, C_s)} \mathrm{Fun}(L^\rhd, C_s)$ is an initial object, which is the claim. □

5.9. **Corollary.** *Suppose C is a S-category such that C_s admits all colimits for every object $s \in S$ and the pushforward functors $\alpha_! : C_s \longrightarrow C_t$ preserve all colimits for every morphism $\alpha : s \to t$ in S. Then C admits all S-colimits indexed by constant diagrams.*

Proof. First suppose that S has an initial object s. Suppose that $p : L \times S \longrightarrow C$ is a S-functor. Let $\overline{p_s} : L^{\rhd} \longrightarrow C_s$ be a colimit diagram extending p_s. Let $\overline{p} : L^{\rhd} \times S \longrightarrow C$ be a S-functor corresponding to $\overline{p_s}$ under the equivalence $\mathrm{Fun}_S(L^{\rhd} \times S, C) \simeq \mathrm{Fun}(L^{\rhd}, C_s)$, which we may suppose extends p. By Proposition 5.8, $\overline{p}$ is a S-colimit diagram.

The general case now follows from Theorem 9.15, taking $\phi : C \longrightarrow D$ to be $L \times S \longrightarrow S$. □

We now turn to the example of corepresentable fibrations.

5.10. **Definition.** Let $s \in S$ be an object and let K be an $S^{s/}$-category which is equivalent to a coproduct of corepresentable fibrations

$$\coprod_{i \in I} S^{\alpha_i/} \simeq \coprod_{i \in I} S^{t_i/} \xrightarrow{\coprod \alpha_i^*} S^{s/}$$

for $\{\alpha_i : s \to t_i\}_{i \in I}$ a collection of morphisms in S. Let $p : K \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-functor, so p is precisely the data of objects $\{x_i \in C_{t_i}\}_{i \in I}$. Let $\overline{p} : K \star_{S^{s/}} S^{s/} \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-colimit diagram extending p, and let $y = \overline{p}(v) \in C_s$ for $v = \mathrm{id}_s$ the cone point. Then we say that y is the *S-coproduct* of $\{x_i\}_{i \in I}$ along $\{\alpha_i\}_{i \in I}$, and we adopt the notation $y = \coprod_{\alpha_i} x_i$.

Our choice of terminology is guided by the following result, which shows that a $S^{s/}$-colimit of a $S^{s/}$-functor $p : S^{\alpha/} \simeq S^{t/} \longrightarrow C$ obtains the value of a left adjoint to the pushforward functor $\alpha_!$ on $p(t)$. In the case of $S = \mathbf{O}_G^{op}$, $C = \underline{\mathbf{Spc}}_G$ or $\underline{\mathbf{Sp}}^G$, and $K = \mathbf{O}_H^{op}$, this is the induction or indexed coproduct functor from H to G.

5.11. **Proposition.** *Let C be a S-category, let $\alpha : s \to t$ be a morphism in C, and let $\pi : M \longrightarrow \Delta^1$ be a* ***cartesian*** *fibration classified by the pushforward functor $\alpha_! : C_s \longrightarrow C_t$. Let $p : S^{t/} \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-functor and let $x = p(\mathrm{id}_t) \in C_t$. Then the data of a $S^{s/}$-colimit diagram extending p yields a π-cocartesian edge e in M with $d_0(e) = x$ and lifting $0 \to 1$.*

Proof. Let $\overline{p} : S^{t/} \star_{S^{s/}} S^{s/} \longrightarrow C \times_S S^{s/}$ be a $S^{s/}$-colimit diagram extending p. Let $y = \overline{p}(\mathrm{id}_s)$ and let $f' : \Delta^1 \longrightarrow S^{t/} \star_{S^{s/}} S^{s/}$ be the edge connecting id_t to α. We may suppose that M is given by the relative nerve of $\alpha_!$, so that edges in M over Δ^1 are given by commutative squares

$$\begin{array}{ccc} \{1\} & \longrightarrow & C_s \\ \downarrow & & \downarrow{\scriptstyle \alpha_!} \\ \Delta^1 & \longrightarrow & C_t. \end{array}$$

Then let e be the edge in M determined by y and $f = \overline{p} \circ f' : x \to \alpha_! y$. By definition, $d_0(e) = x$.

We claim that e is π-cocartesian. This holds if and only if for every $y' \in C_s$ the map

$$\mathrm{Map}_{C_s}(y, y') \longrightarrow \mathrm{Map}_{C_t}(x, \alpha_! y')$$

induced by f is an equivalence. But the local variant of the adjunction of Theorem 10.5 implies this (passing to global sections). □

S-coproducts also satisfy a base-change condition. This is awkward to articulate in general, because the pullback of a corepresentable fibration along another need not be corepresentable. However, if we impose the additional hypothesis that $T = S^{op}$ admits multipullbacks, then a pullback of a corepresentable fibration decomposes as a finite coproduct of corepresentable fibrations. In this case, we have the following useful reformulation of the base-change condition. Recall from the introduction that we let $\mathbf{F}_T$ denote the finite coproduct completion of T. Let $X \subset \mathcal{O}(\mathbf{F}_T)$ be the full subcategory on those arrows whose source lies in T and consider the span

$$(\mathbf{F}_T)^{\sharp} \xleftarrow{\mathrm{ev}_1} {}_{\natural}X \xrightarrow{\mathrm{ev}_0} T^{\sharp}.$$

This satisfies the dual of the hypotheses of Theorem 2.24, so

$$C^{\times} := (\mathrm{ev}_0)_* (\mathrm{ev}_1)^* ((C^{\vee})^{\natural})$$

is a cartesian fibration over $\mathbf{F}_T$ (with the cartesian edges marked), where $C^\vee \longrightarrow T$ is the dual cartesian fibration of [4]. Unwinding the definitions, given a finite T-set $U = \coprod_i s_i$, we have that the fiber

$$(C^\times)_U \simeq \mathrm{Fun}_T(\coprod_i T^{/s_i}, C^\vee) \simeq \prod_i C_{s_i}$$

(where $\mathrm{Fun}_T(-,-)$ denotes those functors over T that preserve cartesian edges), and given a morphism of T-sets $\alpha : U \longrightarrow V$, the pullback functor $\alpha^* : (C^\times)_U \longrightarrow (C^\times)_V$ is induced by restriction.

5.12. **Proposition.** *C admits finite S-coproducts if and only if $\pi : C^\times \longrightarrow \mathbf{F}_T$ is a **Beck-Chevalley fibration**, i.e. π is both cocartesian and cartesian, and for every pullback square*

$$\begin{array}{ccc} W & \xrightarrow{\alpha'} & V' \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\ U & \xrightarrow{\alpha} & V \end{array}$$

in $\mathbf{F}_T$, the natural transformation

$$(*) \qquad (\alpha')_!(\beta')^* \longrightarrow \beta^* \alpha_!$$

adjoint to the equivalence $(\beta')^*\alpha^* \simeq (\alpha')^*\beta^*$ is itself an equivalence.

Proof. By Theorem 10.5, C admits finite S-coproducts if and only if for every finite collection of morphisms $\{\alpha_i : s \to t_i\}$, the restriction functor

$$(\coprod \alpha_i)^* : \underline{\mathrm{Fun}}_S(S^{s/}, C) \longrightarrow \underline{\mathrm{Fun}}_S(\coprod_i S^{t_i/}, C)$$

admits a left S-adjoint, in which case that left S-adjoint is computed by the S-coproduct along the α_i. This in turn is immediately equivalent to π being additionally cocartesian and $(*)$ being an equivalence for $\alpha = \coprod \alpha_i : \coprod t_i \to s$ and all morphisms $\beta : s' \to s$ in T. Finally, note that the apparently more general case of $(*)$ being an equivalence for any pullback square is actually determined by this, because any map $\alpha : U = \coprod t_i \to V = \coprod s_j$ is the data of $f : I \to J$ and $\{\alpha_{ij} : s_j \to t_i\}_{i \in f^{-1}(j)}$, whence $\alpha^* = (\alpha_{ij})^* : \prod_j C_{s_j} \longrightarrow \prod_i C_{t_i}$, etc. yields a decomposition of the map $(*)$ in terms of the 'basic' squares that we already handled. □

We conclude this subsection by introducing a bit of useful terminology.

5.13. **Definition.** Let C be a S-category. We say that C is *S-cocomplete* if, for every object $s \in S$ and $S^{s/}$-diagram $p : K \longrightarrow C_{\underline{s}}$ (with K fiberwise small), p admits a $S^{s/}$-colimit.

5.14. **Remark.** Suppose that E is S-cocomplete. Then taking $D = S$ in Theorem 9.15, E admits all (small) S-colimits. However, the converse may fail: if we suppose that E admits all S-colimits, then any $S^{s/}$-diagram $K_{\underline{s}} \longrightarrow E_{\underline{s}}$ pulled back from a S-diagram $K \longrightarrow E$ admits a $S^{s/}$-colimit; however, not every $S^{s/}$-diagram need be of this form.

Vertical opposites. In this subsection we study the vertical opposite construction of [4], with the goal of justifying our intuition that the theory of S-limits can be recovered from that of S-colimits, and vice-versa (Corollary 5.25). We first recall the definition of the twisted arrow ∞-category from [1, §2].

5.15. **Definition.** Given a simplicial set X, we define $\widetilde{\mathcal{O}}(X)$ to be the simplicial set whose n-simplices are given by the formula

$$\widetilde{\mathcal{O}}(X)_n := \mathrm{Hom}((\Delta^n)^{op} \star \Delta^n, X).$$

If X is an ∞-category, then $\widetilde{\mathcal{O}}(X)$ is the twisted arrow ∞-category of X.

5.16. **Warning.** By definition, $\widetilde{\mathcal{O}}(X)$ comes equipped with a source functor $\mathrm{ev}_0 : \widetilde{\mathcal{O}}(X) \longrightarrow X^{op}$ and a target functor $\mathrm{ev}_1 : \widetilde{\mathcal{O}}(X) \longrightarrow X$. In other words, twisted arrows are *contravariant* in the source and *covariant* in the target. This convention is opposite to that in [12], but agrees with [4].

5.17. **Recollection.** Suppose $X \longrightarrow T$ a cocartesian fibration. Then the simplicial set X^{vop} is defined to have n-simplices

$$\begin{array}{ccc} {}_\natural\widetilde{\mathcal{O}}(\Delta^n) & \longrightarrow & {}_\natural X \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow \\ (\Delta^n)^\sharp & \longrightarrow & T^\sharp. \end{array}$$

The forgetful map $X^{vop} \longrightarrow T$ is a cocartesian fibration with cocartesian edges given by $\widetilde{\mathcal{O}}(\Delta^1)^\sharp \longrightarrow {}_\natural X$. For every $t \in T$, we have an equivalence $(X_t)^{op} \xrightarrow{\simeq} (X^{vop})_t$ implemented by the map which precomposes by $\mathrm{ev}_0 : {}_\natural\widetilde{\mathcal{O}}(\Delta^n) \longrightarrow ((\Delta^n)^{op})^\flat$, which is an equivalence in $s\mathbf{Set}^+$.

Dually, suppose $Y \longrightarrow T$ a cartesian fibration. Then the simplicial set Y^{vop} is defined to have n-simplices

$$\begin{array}{ccc} (\widetilde{\mathcal{O}}(\Delta^n)^{op})^\natural & \longrightarrow & Y^\natural \\ {\scriptstyle \mathrm{ev}_0^{op}}\downarrow & & \downarrow \\ (\Delta^n)^\sharp & \longrightarrow & T^\sharp. \end{array}$$

and similarly the forgetful map $Y^{vop} \longrightarrow T$ is a cartesian fibration with fibers $(Y^{vop})_t \xleftarrow{\simeq} (Y_t)^{op}$. As a warning, note that the definition of the underlying simplicial set of $(-)^{vop}$ changes depending on whether the input is a cocartesian or cartesian fibration; in particular, the notation is potentially ambiguous for a bicartesian fibration. We will not apply $(-)^{vop}$ to bicartesian fibrations in this paper.

Define a functor $\widetilde{\mathcal{O}}'(-) : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+_{/S}$ by

$$\widetilde{\mathcal{O}}'(A \xrightarrow{\pi} S) = (\widetilde{\mathcal{O}}(A), \mathcal{E}_A) \xrightarrow{\pi \circ \mathrm{ev}_1} S$$

where an edge e is in $\mathcal{E}_A$ just in case $\mathrm{ev}_0(e)$ is marked in A^{op}. Note that $\widetilde{\mathcal{O}}(-)$ preserves colimits since it is defined as precomposition by $\Delta^{op} \xrightarrow{(rev \star \mathrm{id})^{op}} \Delta^{op}$, and from this it easily follows that $\widetilde{\mathcal{O}}'(-)$ also preserves colimits. By the adjoint functor theorem, $\widetilde{\mathcal{O}}'(-)$ admits a right adjoint, which we label $(-)^{vop}$; this agrees with the previously defined $(-)^{vop}$ for cocartesian fibrations ${}_\natural X \longrightarrow S^\sharp$.

5.18. **Proposition.** *The adjunction*

$$\widetilde{\mathcal{O}}'(-) \colon s\mathbf{Set}^+_{/S} \rightleftarrows s\mathbf{Set}^+_{/S} : (-)^{vop}$$

is a Quillen equivalence with respect to the cocartesian model structure on $s\mathbf{Set}^+_{/S}$.

Proof. We first prove the adjunction is Quillen by employing the criterion of Lemma 4.13. Consider the four classes of maps which generate the left marked anodyne maps:

(1) $i : \Lambda^n_k \hookrightarrow \Delta^n$, $0 < k < n$: By [1, Lemma 12.15], $\widetilde{\mathcal{O}}(\Lambda^n_k) \hookrightarrow \widetilde{\mathcal{O}}(\Delta^n)$ is inner anodyne, so $\widetilde{\mathcal{O}}'(i)$ is left marked anodyne.
(2) $i : {}_\natural\Lambda^n_0 \hookrightarrow {}_\natural\Delta^n$: We can adapt the proof of [1, Lemma 12.16] to show that $\widetilde{\mathcal{O}}'(i)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ (even though it fails to be left marked anodyne). The basic fact underlying this is that a *right* marked anodyne map is an equivalence in $s\mathbf{Set}^+$, so in $s\mathbf{Set}^+_{/S}$ if it lies entirely over an object; details are left to the reader.
(3) $i : K^\flat \hookrightarrow K^\sharp$ for K a Kan complex: Because $\widetilde{\mathcal{O}}(K) \longrightarrow K^{op} \times K$ is a left fibration, $\widetilde{\mathcal{O}}(K)$ is then again a Kan complex. It follows that $\widetilde{\mathcal{O}}'(i)$ is left marked anodyne.
(4) $(\Lambda^2_1)^\sharp \cup_{\Lambda^2_1} (\Delta^2)^\flat \hookrightarrow (\Delta^2)^\sharp$: Obvious from the definitions.

It remains to show that for a trivial cofibration $f : {}_\natural X \hookrightarrow {}_\natural Y$ between fibrant objects, $\widetilde{\mathcal{O}}'(f)$ is again a trivial cofibration. Since $\widetilde{\mathcal{O}}(X) \longrightarrow \widetilde{\mathcal{O}}(Y)$ is a map of cocartesian fibrations over S and the marking on $\widetilde{\mathcal{O}}'(-)$ contains these cocartesian edges, by Proposition 13.4 it suffices to show that for every object

$s \in S$, $\widetilde{\mathcal{O}}'(X)_s \longrightarrow \widetilde{\mathcal{O}}'(Y)_s$ is an equivalence in $s\mathbf{Set}^+$. We have a commutative square

$$\begin{array}{ccc} \widetilde{\mathcal{O}}'(X)_s & \longrightarrow & \widetilde{\mathcal{O}}'(Y)_s \\ \downarrow & & \downarrow \\ X_s^\sharp & \xrightarrow{f_s} & Y_s^\sharp \end{array}$$

where the vertical maps are left fibrations and the bottom map is an equivalence in $s\mathbf{Set}^+$. Therefore, the map $X_s^\sharp \times_{Y_s^\sharp} \widetilde{\mathcal{O}}'(Y)_s \longrightarrow \widetilde{\mathcal{O}}'(Y)_s$ is an equivalence in $s\mathbf{Set}^+$. Applying Proposition 13.4 once more, we reduce to showing that for every object $x_1 \in X$, $\widetilde{\mathcal{O}}'(X)_{x_1} \longrightarrow \widetilde{\mathcal{O}}'(Y)_{f(x_1)}$ is an equivalence in $s\mathbf{Set}^+$.

Now employing the source maps, we have a commutative square

$$\begin{array}{ccc} \widetilde{\mathcal{O}}'(X)_{x_1} & \longrightarrow & \widetilde{\mathcal{O}}'(Y)_{f(x_1)} \\ \downarrow & & \downarrow \\ X^{op\natural} & \xrightarrow{f^{op}} & Y^{op\natural} \end{array}$$

where the vertical maps are left fibrations and the bottom horizontal map is a *cartesian* equivalence in $s\mathbf{Set}^+_{/S^{op}}$. Therefore, the map $X^{op} \times_{Y^{op}} \widetilde{\mathcal{O}}'(Y)_s \longrightarrow \widetilde{\mathcal{O}}'(Y)_s$ is a cartesian equivalence. By a third application of Proposition 13.4, we reduce to showing that for every object $x_0 \in X$, $\widetilde{\mathcal{O}}'(X)_{(x_0,x_1)} \longrightarrow \widetilde{\mathcal{O}}'(Y)_{(f(x_0),f(x_1))}$ is an equivalence. But now both sides are endowed with the maximal marking and the map is equivalent to $\mathrm{Map}_X(x_0,x_1) \xrightarrow{f_*} \mathrm{Map}_Y(f(x_0),f(x_1))$, which is an equivalence by assumption.

The fact that this Quillen adjunction is an equivalence follows immediately from [4, Theorem 1.4]. □

5.19. **Lemma.** *Let $C \longrightarrow S$ be a cocartesian fibration.*

(1) Let $f : S' \longrightarrow S$ be a functor. Then we have an isomorphism $f^*(C^{vop}) \cong f^*(C)^{vop}$.
(2) Let $g : S \longrightarrow T$ be a cartesian fibration and let C be a S-category. Then there is a T-functor $\chi : g_*(C)^{vop} \longrightarrow g_*(C^{vop})$ natural in C which is an equivalence.

Proof. (1) is obvious from the definitions. For (2), the map χ is defined as follows: an n-simplex of $g_*(C)^{vop}$ over $\sigma \in T_n$ is given by the data of a commutative diagram

$$\begin{array}{ccc} {}_\natural\widetilde{\mathcal{O}}(\Delta^n) \times_{T^\sharp} S^\sharp & \longrightarrow & {}_\natural C \\ \downarrow & & \downarrow \\ (\Delta^n \times_T S)^\sharp & \xrightarrow{g^*\sigma} & S^\sharp \end{array}$$

and precomposition by the obvious map $\widetilde{\mathcal{O}}(\Delta^n \times_T S) \longrightarrow \widetilde{\mathcal{O}}(\Delta^n) \times_T S$ yields an n-simplex of $g_*(C^{vop})$.

We now show that for all $t \in T$, χ_t is a categorical equivalence. Because χ_t is obtained by taking levelwise 0-simplices of the map of complete Segal spaces

$$\mathrm{Map}_S({}_\natural\widetilde{\mathcal{O}}(\Delta^\bullet) \times S_t^\sharp, {}_\natural C) \longrightarrow \mathrm{Map}_S({}_\natural\widetilde{\mathcal{O}}(\Delta^\bullet) \times \widetilde{\mathcal{O}}(S_t)^\sharp, {}_\natural C),$$

it suffices to show that for all n, ${}_\natural\widetilde{\mathcal{O}}(\Delta^n) \times \widetilde{\mathcal{O}}(S_t)^\sharp \longrightarrow {}_\natural\widetilde{\mathcal{O}}(\Delta^n) \times S_t^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. As a special case of Proposition 6.3, $\widetilde{\mathcal{O}}(S_t)^\sharp \longrightarrow S_t^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S_t}$, so the claim follows. □

5.20. **Lemma.** *The map* $\mathrm{ev}^{op} : (\widetilde{\mathcal{O}}(\Delta^n)^{op})^\natural \longrightarrow (\Delta^n)^\sharp \times ((\Delta^n)^{op})^\flat$ *is left marked anodyne.*

Proof. For convenience, we will relabel $\widetilde{\mathcal{O}}(\Delta^n)^{op}$ as the nerve of the poset I_n with objects ij, $0 \le i \le j \le n$ and maps $ij \longrightarrow kl$ for $i \le k$ and $j \le l$. Then an edge $ij \to kl$ is marked in I_n just in case $j = l$, and the map ev^{op} becomes the projection $\rho_n : I_n \longrightarrow (\Delta^n)^\sharp \times (\Delta^n)^\flat$, $ij \mapsto (i,j)$. Let $f_n : (\Delta^n)^\flat \longrightarrow I_n$ be the map which sends i to $0i$. Then $\rho_n \circ f_n : \{0\} \times (\Delta^n)^\flat \longrightarrow (\Delta^n)^\sharp \times (\Delta^n)^\flat$ is left marked anodyne, so by the right cancellativity of left marked anodyne maps it suffices to show that i_n is left marked anodyne. For this, we factor f_n as the composition

$$(\Delta^n)^\flat = I_{n,-1} \longrightarrow I_{n,0} \longrightarrow \dots \longrightarrow I_{n,n} = I_n$$

where $I_{n,k} \subset I_n$ is the subcategory on objects ij, $i = 0$ or $j \leq k$ (and inherits the marking from I_n), and argue that each inclusion $g_k : I_{n,k} \subset I_{n,k+1}$ is left marked anodyne. For this, note that g_k fits into a pushout square

$$\begin{array}{ccc} \{0\} \times (\Delta^{k+1})^\flat \bigcup_{\{0\}\times(\Delta^k)^\flat} (\Delta^{n-k-1})^\sharp \times (\Delta^k)^\flat & \longrightarrow & (\Delta^{n-k-1})^\sharp \times (\Delta^{k+1})^\flat \\ \downarrow & & \downarrow \\ I_{n,k} & \xrightarrow{g_k} & I_{n,k+1} \end{array}$$

with the upper horizontal map marked left anodyne. □

5.21. **Construction.** Suppose T an ∞-category, $X, Z \longrightarrow T$ cocartesian fibrations, $Y \longrightarrow T$ a cartesian fibration, and a map $\mu : {}_\natural X \times_T Y^\natural \longrightarrow {}_\natural Z$ of marked simplicial sets over T. We define a map

$$\mu^{vop} : {}_\natural X^{vop} \times_T Y^{vop\natural} \longrightarrow {}_\natural Z^{vop}$$

by the following process:

Let J_n be the nerve of the poset with objects ij, $0 \leq i \leq n$, $-n \leq j \leq n$ and $-j \leq i$ and maps $ij \to kl$ if $i \leq k$, $j \leq l$. Mark edges $ij \longrightarrow kl$ if $j = l$. Let $I_n \subset J_n$ be the subcategory on ij with $j \geq 0$ and $I'_n \subset J_n$ be the subcategory on ij with $j \leq 0$; also give I_n, I'_n the induced markings. We have an inclusion $(\Delta^n)^\sharp \longrightarrow J_n$ given by $i \mapsto i0$ which restricts to inclusions $(\Delta^n)^\sharp \longrightarrow I_n$, $(\Delta^n)^\sharp \longrightarrow I'_n$ and induces a map $\gamma_n : I_n \cup_{(\Delta^n)^\sharp} I'_n \subset J_n$.

Define auxiliary (unmarked) simplicial sets $Z' \longrightarrow T$ by $\mathrm{Hom}_{/T}(\Delta^n, Z') = \mathrm{Hom}_{/T}(J_n, {}_\natural Z)$ and $Z'' \longrightarrow T$ by $\mathrm{Hom}_{/T}(\Delta^n, Z'') = \mathrm{Hom}_{/T}(I_n \cup_{(\Delta^n)^\sharp} I'_n, {}_\natural Z)$, where $J_n \longrightarrow \Delta^n$ via $ij \mapsto i$. We have a map $r : Z' \longrightarrow Z''$ given by restriction along the γ_n, which we claim is a trivial fibration. By a standard reduction, for this it suffices to show that γ_n is left marked anodyne. Indeed, this follows from Lemma 5.20 applied to $I_n \longrightarrow (\Delta^n)^\sharp \times \Delta^n$ and the observation that the map $\Delta^n \times \Delta^n \cup_{\Delta^n} I'_n \longrightarrow J_n$ is inner anodyne, whose proof we leave to the reader.

Define also a map $Z' \longrightarrow Z^{vop}$ over T by restriction along the map ${}_\natural\widetilde{\mathcal{O}}(\Delta^n) \longrightarrow J_n$ which sends ij to jn if $i = 0$ and $j(-i)$ otherwise. Finally, define a map $X^{vop} \times_T Y^{vop} \longrightarrow Z''$ over T as follows: a map $\Delta^n \longrightarrow X^{vop} \times_T Y^{vop}$ is given by the data

$$\begin{array}{ccc} {}_\natural\widetilde{\mathcal{O}}(\Delta^n) & \longrightarrow & {}_\natural X \\ \downarrow & & \downarrow \\ (\Delta^n)^\sharp & \longrightarrow & T^\sharp \end{array} , \quad \begin{array}{ccc} \left(\widetilde{\mathcal{O}}(\Delta^n)^{op}\right)^\natural & \longrightarrow & Y^\natural \\ \downarrow & & \downarrow \\ (\Delta^n)^\sharp & \longrightarrow & T^\sharp. \end{array}$$

We have isomorphisms ${}_\natural\widetilde{\mathcal{O}}(\Delta^n) \cong I'_n$ and $\left(\widetilde{\mathcal{O}}(\Delta^n)^{op}\right)^\natural \cong I_n$, and obvious retractions $I_n \cup_{(\Delta^n)^\sharp} I'_n \longrightarrow I_n, I'_n$ given by collapsing the complementary part onto Δ^n. Using this, we may define

$$I_n \cup_{(\Delta^n)^\sharp} I'_n \longrightarrow {}_\natural X \times_T Y^\natural \longrightarrow {}_\natural Z$$

which is an n-simplex of Z''.

Choosing a section of r, we may compose these maps to define μ^{vop}, which is then easily checked to also preserve the indicated markings. For example, μ^{vop} on edges is given by

$$\left(\begin{array}{ccc} & & x_{11} \\ & & \downarrow \\ x_{00} & \longrightarrow & x_{01}, \\ & & \\ y_{01} & \longrightarrow & y_{11} \\ \downarrow & & \\ y_{00} & & \end{array}\right) \mapsto \left(\begin{array}{ccc} & & \mu(x_{11}, y_{11}) \\ & & \downarrow \\ \mu(x_{00}, y_{01}) & \longrightarrow & \mu(x_{01}, y_{11}) \\ \downarrow & & \downarrow \\ \mu(x_{00}, y_{00}) & \longrightarrow & \alpha_! \mu(x_{00}, y_{00}) \end{array}\right) \mapsto \left(\begin{array}{ccc} & & \mu(x_{11}, y_{11}) \\ & & \downarrow \\ \mu(x_{00}, y_{00}) & \longrightarrow & \alpha_! \mu(x_{00}, y_{00}) \end{array}\right)$$

where $\alpha_! \mu(x_{00}, y_{00})$ is a choice of pushforward for the edge α in T that the diagrams are vertically over.

5.22. **Lemma.** *Let $C \longrightarrow T$ be a cartesian fibration and let $D \longrightarrow T$ be a cocartesian fibration. There exists a T-equivalence $\psi : \widetilde{\mathrm{Fun}}_T(C, D)^{vop} \longrightarrow \widetilde{\mathrm{Fun}}_T(C^{vop}, D^{vop})$.*

Proof. We have a map $\mu : \widetilde{\mathrm{Fun}}_T(C, D) \times_T C \longrightarrow D$ adjoint to the identity. Employing Construction 5.21 on μ and then adjointing, we obtain our desired T-functor ψ. A chase of the definitions then shows that for all objects $t \in T$, ψ_t is homotopic to the known equivalence $\mathrm{Fun}(C_t, D_t)^{op} \simeq \mathrm{Fun}(C_t^{op}, D_t^{op})$. □

5.23. **Lemma.** *Let K and L be S-categories. Then there exists a S-equivalence*

$$\psi : (K \star_S L)^{vop} \xrightarrow{\simeq} L^{vop} \star_S K^{vop}$$

over $S \times \Delta^1$.

Proof. Note that $(S \times \Delta^1)^{vop} \cong S \times (\Delta^1)^{op}$. View $(K \star_S L)^{vop}$ as lying over $S \times \Delta^1$ via the isomorphism $(\Delta^1)^{op} \cong \Delta^1$. Since $(K \star_S L)_0^{vop} \cong L^{vop}$ and $(K \star_S L)_1^{vop} \cong K^{vop}$, we have our S-functor ψ as adjoint to the identity over $S \times \partial \Delta^1$. Fiberwise, ψ_s is homotopic to the known isomorphism $(K_s \star L_s)^{op} \cong L_s^{op} \star K_s^{op}$, so ψ is an equivalence. □

5.24. **Proposition.** *Suppose S-categories K and C.*

(1) The adjoint of the vertical opposite of the evaluation map induces a equivalence

$$\underline{\mathrm{Fun}}_S(K, C)^{vop} \xrightarrow{\simeq} \underline{\mathrm{Fun}}_S(K^{vop}, C^{vop}).$$

(2) Suppose a S-functor $p : K \longrightarrow C$. We have equivalences

$$(C^{(p,S)/})^{vop} \simeq (C^{vop})^{/(p^{vop}, S)}, \quad (C^{/(p,S)})^{vop} \simeq (C^{vop})^{(p^{vop}, S)/}.$$

Proof. (1) Recall from 6.3.1 the equivalence $\underline{\mathrm{Fun}}_S(K, C) \simeq \pi_* \pi'^* \{K, C\}_S$. By Lemma 5.22 and Lemma 5.19(1),

$$\{K, C\}_S^{vop} \simeq \{K^{vop}, C^{vop}\}_S.$$

By Lemma 5.19(1) and (2),

$$\pi_* \pi'^* \{K, C\}_S^{vop} \simeq (\pi_* \pi'^* \{K, C\}_S)^{vop}.$$

Combining these equivalences supplies an equivalence $\underline{\mathrm{Fun}}_S(K, C)^{vop} \simeq \underline{\mathrm{Fun}}_S(K^{vop}, C^{vop})$. It is straightforward but tedious to verify that the adjoint of the vertical opposite of the evaluation map $\underline{\mathrm{Fun}}_S(K, C)^{vop} \times_S K^{vop} \longrightarrow C^{vop}$ is homotopic to this equivalence.

(2) Combine (1), Lemma 5.23, Proposition 5.18 (which shows in particular that $(-)^{vop}$ is right Quillen), and the definition of the S-slice category.

□

5.25. **Corollary.** *Let $\overline{p} : S \star_S K \longrightarrow C$ be a S-functor. Then $\overline{p}$ is a S-limit diagram if and only if $\overline{p}^{vop} : K^{vop} \star_S S \longrightarrow C^{vop}$ is a S-colimit diagram.*

This allows us to deduce statements about S-limits from statements about S-colimits, and vice-versa. For this reason, we will primarily concentrate our attention on proving statements concerning S-colimits (and eventually, S-left Kan extensions), leaving the formulation of the dual results to the reader.

5.26. **Warning.** Even with Corollary 5.25, it seems difficult to deduce Proposition 5.6 concerning S-limits in an S-category of objects $\underline{C}_S$ directly from Proposition 5.5 on S-colimits in $\underline{C}_S$. This is because the formation of vertical opposites $\underline{C}_S \mapsto (\underline{C}_S)^{vop}$ doesn't intertwine with any operation at the level of the ∞-category C.

6. Assembling S-slice categories from ordinary slice categories

Suppose a S-functor $p : K \longrightarrow C$. For every morphism $\alpha : s \to t$ in S, we have a functor $p_\alpha : K_s \longrightarrow C_t$, and we may consider the collection of 'absolute' slice categories $C_{p_\alpha/}$ and examine the functoriality that they satisfy. For this, we have the following basic observation: given a morphism $f : t \to t'$, *covariant* functoriality of slice categories in the target yields a functor $C_{p_\alpha/} \longrightarrow C_{p_{f\alpha}/}$, and given a morphism $g : s' \to s$, *contravariant* functoriality in the source yields a functor $C_{p_\alpha/} \longrightarrow C_{p_{\alpha g}/}$. Elaborating, we will show in this section that there exists a functor

$$F := F(p : K \longrightarrow C) : \widetilde{\mathcal{O}}(S) \longrightarrow \mathbf{Cat}_\infty$$

out of the twisted arrow category $\widetilde{\mathcal{O}}(S)$ such that $F(\alpha) \simeq C_{p_\alpha/}$, which encodes all of this functoriality (Definition 6.5). Moreover, the right Kan extension of F along the target functor $\widetilde{\mathcal{O}}(S) \longrightarrow S$ is $C_{(p,S)/}$ (Theorem 6.6). We will end with some applications of this result to the theory of cofinality and presentability (Theorem 6.7 and Remark 6.11).

We first record a cofinality result which implies that the values of a right Kan extension along $\mathrm{ev}_1 : \widetilde{\mathcal{O}}(S) \longrightarrow S$ are computed as ends.

6.1. **Lemma.** *The functor $\widetilde{\mathcal{O}}(S^{s/}) \longrightarrow \widetilde{\mathcal{O}}(S) \times_S S^{s/}$ is initial.*

Proof. Let $(\alpha : u \longrightarrow t, \beta : s \longrightarrow t)$ be an object of $\widetilde{\mathcal{O}}(S) \times_S S^{s/}$. We will prove that

$$C = \widetilde{\mathcal{O}}(S^{s/}) \times_{\widetilde{\mathcal{O}}(S) \times_S S^{s/}} (\widetilde{\mathcal{O}}(S) \times_S S^{s/})_{/(\alpha,\beta)}$$

is weakly contractible. An object of C is the data of an edge

$$\begin{array}{ccc} & s & \\ {}^{f}\swarrow & & \searrow^{g} \\ x & \xrightarrow{h} & y \end{array}$$

in $S^{s/}$, which we will abbreviate as $f \xrightarrow{h} g$, and an edge

$$\left(\begin{array}{ccc} x & \xrightarrow{h} & y \\ {}^{\delta}\uparrow & & \downarrow^{\gamma} \\ u & \xrightarrow{\alpha} & t \end{array} , \quad \begin{array}{ccc} s & \xrightarrow{g} & y \\ & {}_{\beta}\searrow & \downarrow^{\gamma} \\ & & t \end{array} \right)$$

in $\widetilde{\mathcal{O}}(S) \times_S S^{s/}$, which we will abbreviate as $(h, g) \xrightarrow{(\delta,\gamma)} (\alpha, \beta)$.

Let $C_0 \subset C$ be the full subcategory on objects $c = ((f \xrightarrow{h} g), (h, g) \xrightarrow{(\delta,\gamma)} (\alpha, \beta))$ such that γ is a degenerate edge in $S^{s/}$. We will first show that C_0 is a reflective subcategory of C by verifying the first condition of [10, Proposition 5.2.7.8]. Given an object c of C, define c' to be $((f \xrightarrow{\gamma h} \beta), (\gamma h, \beta) \xrightarrow{(\delta,\mathrm{id}_t)} (\alpha, \beta))$ and let $e : c \longrightarrow c'$ be the edge given by

$$\left(\begin{array}{ccc} f & \xrightarrow{h} & g \\ {}^{\mathrm{id}_f}\uparrow & & \downarrow^{\gamma} \\ f & \xrightarrow{\gamma h} & \beta \end{array} , \quad \begin{array}{ccc} (h,g) & \xrightarrow{(\mathrm{id}_x,\gamma)} & (\gamma h, \beta) \\ & {}_{(\delta,\gamma))}\searrow \quad \swarrow_{(\delta,\mathrm{id}_t))} & \\ & (\alpha,\beta) & \end{array} \right).$$

We need to show that for all $d = ((f' \xrightarrow{h'} \beta), (h', \beta) \xrightarrow{(\delta',\mathrm{id})} (\alpha, \beta)) \in C_0$, $\mathrm{Map}_C(c', d) \xrightarrow{e^*} \mathrm{Map}_C(c, d)$ is a homotopy equivalence. The space $\mathrm{Map}_C(c, d)$ lies in a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_C(c,d) & \longrightarrow & \mathrm{Map}_{\widetilde{\mathcal{O}}(S^{s/})}(f \xrightarrow{h} g, f' \xrightarrow{h'} \beta) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{(\widetilde{\mathcal{O}}(S) \times_S S^{s/})_{/(\alpha,\beta)}}((h,g),(h',\beta)) & \longrightarrow & \mathrm{Map}_{\widetilde{\mathcal{O}}(S) \times_S S^{s/}}((h,g),(h',\beta)) \\ \downarrow & & \downarrow{\scriptstyle (\delta',\mathrm{id})_*} \\ \Delta^0 & \xrightarrow{(\delta,\gamma)} & \mathrm{Map}_{\widetilde{\mathcal{O}}(S) \times_S S^{s/}}((h,g),(\alpha,\beta)) \end{array}$$

where the two squares are homotopy pullback squares. We also have the analogous diagram for $\mathrm{Map}_C(c', d)$, and the map e^* is induced by a natural transformation of these diagrams. The assertion then reduces to checking that the upper square in the diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\widetilde{\mathcal{O}}(S^{s/})}(f \xrightarrow{\gamma h} \beta, f' \xrightarrow{h'} \beta) & \xrightarrow{(\mathrm{id}_f, \gamma)^*} & \mathrm{Map}_{\widetilde{\mathcal{O}}(S^{s/})}(f \xrightarrow{h} g, f' \xrightarrow{h'} \beta) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\widetilde{\mathcal{O}}(S) \times_S S^{s/}}((\gamma h, \beta), (\alpha, \beta)) & \xrightarrow{(\mathrm{id}_x, \gamma))^*} & \mathrm{Map}_{\widetilde{\mathcal{O}}(S) \times_S S^{s/}}((h, g), (\alpha, \beta)) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{S^{s/}}(\beta, \beta) & \xrightarrow{\gamma^*} & \mathrm{Map}_{S^{s/}}(g, \beta)
\end{array}$$

is a homotopy pullback square. Since (id_x, γ) and (id_f, γ) are ev_1-cocartesian edges in $\widetilde{\mathcal{O}}(S)$ and $\widetilde{\mathcal{O}}(S^{s/})$ respectively, the lower and outer squares are homotopy pullback squares (where we implicitly use that the map (δ', id) covers the identity in $S^{s/}$ to identify the long vertical maps with those induced by ev_1), and the claim is proven.

To complete the proof, we will show that $c = (\beta = \beta, (\mathrm{id}_t, \beta) \xrightarrow{(\alpha, \mathrm{id}_t)} (\alpha, \beta))$ is an initial object in C_0. Let $d \in C_0$ be as above. In the diagram

$$\begin{array}{ccccc}
\Delta^0 & \xrightarrow{(h', \mathrm{id}_\beta)} & \mathrm{Map}_{\widetilde{\mathcal{O}}(S^{s/})}(\beta = \beta, f' \xrightarrow{h'} \beta) & & \\
\downarrow & & \downarrow & & \\
\Delta^0 & \xrightarrow{(\alpha, \mathrm{id}_t)} & \mathrm{Map}_{\widetilde{\mathcal{O}}(S) \times_S S^{s/}}((\mathrm{id}_t, \beta), (\alpha, \beta)) & \longrightarrow & \mathrm{Map}_{\widetilde{\mathcal{O}}(S)}(\mathrm{id}_t, \alpha) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^0 & \xrightarrow{\mathrm{id}_\beta} & \mathrm{Map}_{S^{s/}}(\beta, \beta) & \longrightarrow & \mathrm{Map}_S(t, t)
\end{array}$$

we need to show that the upper square is a homotopy pullback square in order to prove that $\mathrm{Map}_C(c, d) \simeq *$. The fiber of $\widetilde{\mathcal{O}}(S)$ over $t \in S$ is equivalent to $(S_{/t})^{op}$; in particular, id_t is an initial object in the fiber over t. Therefore, the two outer squares are both homotopy pullbacks. Since the lower right square is a homotopy pullback, this shows that all squares in the diagram are homotopy pullbacks, as desired. □

Let K be an S-category. Let J_n be the poset with objects ij for $0 \leq i \leq j \leq 2n+1$ which has a unique morphism $ij \longrightarrow kl$ if and only if $k \leq i \leq j \leq l$. Let $I_n \subset J_n$ be the full subcategory on objects ij such that $i \leq n$. In view of the isomorphisms

$$J_n \cong \widetilde{\mathcal{O}}(\Delta^{2n+1}) \cong \widetilde{\mathcal{O}}((\Delta^n)^{op} \star \Delta^n),$$

the I_n and J_n extend to functors

$$I_\bullet \subset J_\bullet \cong \widetilde{\mathcal{O}}((\Delta^\bullet)^{op} \star \Delta^\bullet) : \Delta \longrightarrow s\mathbf{Set}.$$

Viewing I_n and J_n as marked simplicial sets where $ij \longrightarrow kl$ is marked just in case $k = i$, we moreover have functors to $s\mathbf{Set}^+$. Define the simplicial set $X : \Delta^{op} \longrightarrow \mathbf{Set}$ to be the functor

$$\mathrm{Hom}_{s\mathbf{Set}^+}(I_\bullet, \natural K) \times_{\mathrm{Hom}(I_\bullet, S)} \mathrm{Hom}((\Delta^\bullet)^{op} \star \Delta^\bullet, S)$$

where $I_\bullet \subset J_\bullet \longrightarrow (\Delta^\bullet)^{op} \star \Delta^\bullet$ is given by the target map. An n-simplex of X is thus the data of a diagram

$$\begin{array}{ccccccccccccc}
 & & & & & & k_{nn} & \longrightarrow & k_{n(n+1)} & \longrightarrow & \dots & \longrightarrow & k_{n(2n+1)} \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \iddots & & & & \vdots & & \vdots & & \dots & & \vdots \\
 & & k_{11} & \longrightarrow & \dots & \longrightarrow & k_{1n} & \longrightarrow & k_{1(n+1)} & \longrightarrow & \dots & \longrightarrow & k_{1(2n+1)} \\
 & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
k_{00} & \longrightarrow & k_{01} & \longrightarrow & \dots & \longrightarrow & k_{0n} & \longrightarrow & k_{0(n+1)} & \longrightarrow & \dots & \longrightarrow & k_{0(2n+1)}
\end{array}$$

where the horizontal edges are cocartesian in K and the vertical edges lie over degeneracies in S.

Declare an edge e in X to be marked if the corresponding map $I_1 \longrightarrow {}_\natural K$ sends all edges to marked edges. We have a commutative square of marked simplicial sets

$$\begin{array}{ccc}
X & \longrightarrow & \widetilde{\mathcal{O}}(S)^\sharp \\
\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\
(K^\vee)^\natural & \longrightarrow & (S^{op})^\sharp
\end{array}$$

where $K^\vee = (K^{vop})^{op} \longrightarrow S^{op}$ is the dual cartesian fibration and the map $X \longrightarrow K^\vee$ is defined by restricting $I_n \longrightarrow K$ to $I'_n \longrightarrow K$ (where I'_n is the full subcategory of I_n on ij with $j \leq n$). Let ψ denote the resulting map from X to the pullback.

6.2. **Lemma.** $\psi : X \longrightarrow (K^\vee)^\natural \times_{(S^{op})^\sharp} \widetilde{\mathcal{O}}(S)^\sharp$ *is a trivial fibration of marked simplicial sets.*

Proof. Since any lift of a marked edge in $(K^\vee)^\natural \times_{(S^{op})^\sharp} \widetilde{\mathcal{O}}(S)^\sharp$ to an edge in X is marked, it suffices to prove that the underlying map of simplicial sets is a trivial fibration.

We first show that $I'_n \subset I_n$ is left marked anodyne. Let $I_{n,k} \subset I_n$ be the full subcategory on objects ij with $i \leq k$ and similarly for $I'_{n,k}$. For $0 \leq k < n$ we have a pushout decomposition

$$\begin{array}{ccc}
((\Delta^{n-k})^{op})^\flat \times (\Delta^k)^\sharp \displaystyle\bigcup_{((\Delta^{n-k-1})^{op})^\flat \times (\Delta^k)^\sharp} ((\Delta^{n-k-1})^{op})^\flat \times (\Delta^{n+k+1})^\sharp & \longrightarrow & I'_{n,n-k} \displaystyle\bigcup_{I'_{n,n-k-1}} I_{n,n-k-1} \\
\downarrow & & \downarrow \\
((\Delta^{n-k})^{op})^\flat \times (\Delta^{n+k+1})^\sharp & \longrightarrow & I_{n,n-k},
\end{array}$$

and the lefthand map is left marked anodyne by [10, Proposition 3.1.2.3]. It thus suffices to show that

$$I'_{n,0} \cong (\Delta^n)^\sharp \longrightarrow I_{n,0} \cong (\Delta^{2n+1})^\sharp$$

is left marked anodyne, and this is clear.

We now explain how to solve the lifting problem

$$\begin{array}{ccc}
\partial\Delta^n & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta^n & \longrightarrow & K^\vee \times_{S^{op}} \widetilde{\mathcal{O}}(S).
\end{array}$$

To supply the dotted arrow we must provide a lift in the commutative square

$$\begin{array}{ccc}
\partial I_n \cup_{\partial I'_n} I'_n & \longrightarrow & {}_\natural K \\
\downarrow{\scriptstyle f} & \nearrow & \downarrow \\
I_n & \longrightarrow & S^\sharp.
\end{array}$$

where $\partial I_n = \bigcup_{[n-1]\subset[n]} I_{n-1}$ as a simplicial subset of I_n and likewise for $\partial I'_n$. Then since $I'_n \longrightarrow \partial I_n \cup_{\partial I'_n} I'_n$ and $I'_n \longrightarrow I_n$ are left marked anodyne, f is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, and the lift exists. $\square$

For all $s \in S$, we have trivial cofibrations $i_s : K_s \xrightarrow{\simeq} (K^\vee)_s$, and thus commutative squares

$$\begin{array}{ccc} K_s & \xrightarrow{\mathrm{id}_s} & \widetilde{\mathcal{O}}(S) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_0} \\ K^\vee & \longrightarrow & S^{op}. \end{array}$$

From this we obtain a cofibration

$$\iota : \coprod_{s\in S} K_s \hookrightarrow K^\vee \times_{S^{op}} \widetilde{\mathcal{O}}(S).$$

We have an explicit lift ι' of ι to X, where $K_s \longrightarrow X$ is given by precomposition by $I_n \longrightarrow \Delta^n$, $ij \mapsto n-i$.

By Lemma 6.2, there exists a lift σ in the commutative square

$$\begin{array}{ccc} \coprod_{s\in S} K_s & \xrightarrow{\iota'} & X \\ \big\downarrow{\scriptstyle \iota} & \overset{\sigma}{\nearrow} & \big\downarrow{\scriptstyle \psi} \\ K^\vee \times_{S^{op}} \widetilde{\mathcal{O}}(S) & \xrightarrow{=} & K^\vee \times_{S^{op}} \widetilde{\mathcal{O}}(S). \end{array}$$

Let $\chi : X \longrightarrow K$ be the functor induced by $\Delta^n \longrightarrow I_n$, $i \mapsto (n-i)(n+i)$. Define the *twisted pushforward*

$$\widetilde{P} : K^\vee \times_{S^{op}} \widetilde{\mathcal{O}}(S) \longrightarrow K$$

to be the map over S given by the composite $\chi \circ \sigma$. Then for every object $\alpha : s \longrightarrow t$ in $\widetilde{\mathcal{O}}(S)$, $\widetilde{P}_\alpha \circ i_s : K_s \longrightarrow K_t$ is a choice of pushforward functor over α, which is chosen to be the identity if $\alpha = \mathrm{id}_s$.

6.3. **Proposition.** *For all $A \in s\mathbf{Set}_{/S}$,*

$$\widetilde{P} \times_S \mathrm{id}_A : (K^\vee)^\natural \times_{(S^{op})^\sharp} \widetilde{\mathcal{O}}(S)^\sharp \times_S A^\sharp \longrightarrow {}_\natural K \times_S A^\sharp$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/A}$.

Proof. Let (Z, E) denote the marked simplicial set $(K^\vee)^\natural \times_{(S^{op})^\sharp} \widetilde{\mathcal{O}}(S)^\sharp$. Viewing Z as $\widetilde{\mathcal{O}}(S) \times_{S^{op}\times S} (K^\vee \times S)$, we see that $Z \longrightarrow S$ is a cocartesian fibration with the cocartesian edges a subset of E. Moreover, every edge in E factors as a cocartesian edge followed by an edge in E in the fiber over S. By Proposition 13.4, it suffices to verify that for all $s \in S$, $\widetilde{P}_s$ is a cocartesian equivalence in $s\mathbf{Set}^+$. Since id_s is an initial object in $\widetilde{\mathcal{O}}(S) \times_S \{s\}$, the inclusion of the fiber $(K^\vee)^\sim_s \subset (Z_s, E_s)$ is a cocartesian equivalence in $s\mathbf{Set}^+$ by [10, Lemma 3.3.4.1]. We chose $\widetilde{P}$ so as to split the inclusion of K_s in Z, so this completes the proof. $\square$

Consider the commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{O}(S)^\sharp \times_S {}_\natural K & & & \xrightarrow{\qquad\qquad\qquad\qquad \mathrm{ev}_1 \qquad\qquad\qquad\qquad} & & & & & S^\sharp \\
& \nwarrow^{\mathrm{id}_{\mathcal{O}(S)}\times_S \widetilde{P}} & & & & & & & \\
& & \mathcal{O}(S)^\sharp \times_{\mathrm{ev}_1,S,\mathrm{ev}_1} ((K^\vee)^\natural \times_{S^{op}} \widetilde{\mathcal{O}}(S)^\sharp) & \xrightarrow{\mathrm{pr}} & (K^\vee)^\natural \times_{S^{op}} \widetilde{\mathcal{O}}(S)^\sharp & \xrightarrow{\mathrm{id}\times\mathrm{ev}_1} & (K^\vee)^\natural \times S^\sharp & \xrightarrow{\mathrm{pr}_S} & S^\sharp \\
& & \downarrow{\scriptstyle\mathrm{pr}} & & \downarrow{\scriptstyle\mathrm{pr}} & & \downarrow{\scriptstyle q^\vee\times\mathrm{id}} & & \\
& & \mathcal{O}(S)^\sharp \times_{\mathrm{ev}_1,S,\mathrm{ev}_1} \widetilde{\mathcal{O}}(S)^\sharp & \xrightarrow{\pi'} & \widetilde{\mathcal{O}}(S)^\sharp & \xrightarrow{\mathrm{ev}} & (S^{op})^\sharp \times S^\sharp & & \\
& \searrow_{\mathrm{ev}_0} & \downarrow{\scriptstyle\pi} & & & & & & \\
& & S^\sharp. & & & & & &
\end{array}$$

Here, $\pi = \mathrm{ev}_0 \circ \mathrm{pr}_{\mathcal{O}(S)}$ and $\pi' = \mathrm{pr}_{\widetilde{\mathcal{O}}(S)}$. Since $K^\vee \longrightarrow S^{op}$ is a cartesian fibration, by Theorem 2.24 $(q^\vee \times \mathrm{id})_*$ is right Quillen. Therefore, given a S-category C, we obtain a $\widetilde{\mathcal{O}}(S)$-category

$$\{K, C\}_S := (\mathrm{ev}^* \circ (q^\vee \times \mathrm{id})_* \circ \mathrm{pr}_S^*)({}_\natural C).$$

Moreover, we saw in Example 2.26 that $\pi_* \pi'^*$ is right Quillen and computes right Kan extension along $\mathrm{ev}_1 : \widetilde{\mathcal{O}}(S) \longrightarrow S$. Finally, the map $\mathrm{id}_{\mathcal{O}(S)} \times_S \widetilde{P}$ induces a S-functor

$$\theta : \underline{\mathrm{Fun}}_S(K, C) \longrightarrow \pi_* \pi'^* \{K, C\}_S, \tag{6.3.1}$$

natural in K and C. By Proposition 6.3 applied to $A = S^{s/}$ for all $s \in S$, θ is an equivalence.

6.4. **Remark.** As a corollary, the global sections of $\{K, C\}_S$ are equivalent to $\mathrm{Fun}_S(K, C)$. If we knew that under the straightening functor St, $\{K, C\}_S$ was equivalent to the composite

$$\widetilde{\mathcal{O}}(S) \longrightarrow S^{op} \times S \xrightarrow{\mathrm{St}_S(K)^{op} \times \mathrm{St}_S(C)} \mathbf{Cat}_\infty^{op} \times \mathbf{Cat}_\infty \xrightarrow{\mathrm{Fun}} \mathbf{Cat}_\infty,$$

then this would yield another proof of the end formula for the ∞-category of natural transformations, as proven in [6, §6]. As we manage to always stay within the environment of cocartesian fibrations, this identification is not necessary for our purposes.

6.5. **Definition.** Given a S-functor $p : K \longrightarrow C$ and a choice of twisted pushforward $\widetilde{P}$ for K, define the cocartesian section $\omega_p : \widetilde{\mathcal{O}}(S) \longrightarrow \{K, C\}_S$ to be the adjoint to

$$p \circ \widetilde{P} : K^{\vee\natural} \times_{S^{op}} \widetilde{\mathcal{O}}(S)^\sharp \longrightarrow {}_\natural K \longrightarrow {}_\natural C.$$

For objects $[\alpha : s \to t]$ in $\widetilde{\mathcal{O}}(S)$, $\omega_p(\alpha) \in \mathrm{Fun}((K^\vee)_s, C_t)$ is the functor

$$p_t \circ \widetilde{P}_\alpha : (K^\vee)_s \longrightarrow K_t \longrightarrow C_t.$$

Define the *twisted slice* $\widetilde{\mathcal{O}}(S)$*-category* to be

$$C^{\widetilde{(p,S)}/} := \widetilde{\mathcal{O}}(S) \times_{\{K,C\}_S} \{K \star_S S, C\}_S.^{17}$

Note that the fiber of $C^{\widetilde{(p,S)}/}$ over an object $[\alpha : s \to t]$ is $C^{p_t \circ \widetilde{P}_\alpha /}$.

We now connect the constructions $C^{\widetilde{(p,S)}/}$ and $C^{(p,S)/}$. A check of the definitions reveals that $\theta \circ \sigma_p = \pi_* \pi'^*(\omega_p)$ for the canonical cocartesian section $\sigma_p : S \longrightarrow \underline{\mathrm{Fun}}_S(K, C)$. We thus have a morphism of spans

$$\begin{array}{ccccc}
S & \xrightarrow{\sigma_p} & \underline{\mathrm{Fun}}_S(K, C) & \longleftarrow & \underline{\mathrm{Fun}}_S(K \star_S S, C) \\
\downarrow{=} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
S & \xrightarrow{\pi_* \pi'^*(\omega_p)} & \pi_* \pi'^* \{K, C\}_S & \longleftarrow & \pi_* \pi'^* \{K \star_S S, C\}_S
\end{array}$$

with all objects fibrant and the right horizontal maps fibrations by a standard argument. Taking pullbacks, we deduce:

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[17] We omit the dependence on $\widetilde{P}$ from the notation.

6.6. **Theorem.** *We have an equivalence*

$$\pi_* \pi'^* (\widetilde{C^{(p,S)/}}) \xrightarrow{\simeq} C^{(p,S)/}.$$

In other words, the right Kan extension of $\widetilde{C^{(p,S)/}}$ *along the target functor* $\mathrm{ev}_1 : \widetilde{\mathcal{O}}(S) \longrightarrow S$ *is equivalent to* $C^{(p,S)/}$.

Proof. Our interpretation of this equivalence is by Example 2.26. □

Relative cofinality. Let us now apply Theorem 6.6. We have the S-analogue of the basic cofinality result [10, Proposition 4.1.1.8].

6.7. **Theorem.** *Let* $f : K \longrightarrow L$ *be a* S*-functor. The following conditions are equivalent:*

(1) *For every object* $s \in S$, $f_s : K_s \longrightarrow L_s$ *is final.*
(2) *For every* S*-functor* $p : L \longrightarrow C$*, the functor* $f^* : C^{(p,S)/} \longrightarrow C^{(pf,S)/}$ *is an equivalence.*
(3) *For every* S*-colimit diagram* $\overline{p} : L \star_S S \longrightarrow C$, $\overline{p} \circ f^{\rhd} : K \star_S S \longrightarrow C$ *is a* S*-colimit diagram.*

Proof. (1) ⇒ (2): Factoring f as the composition of a cofibration and a trivial fibration, we may suppose that f is a cofibration, in which case we may choose compatible twisted pushforward functors $\widetilde{P}_K$ and $\widetilde{P}_L$. Let $p : L \longrightarrow C$ be a S-functor. Precomposition by f yields a $\widetilde{\mathcal{O}}(S)$-functor $\widetilde{f^*}$: $\widetilde{C^{(p,S)/}} \longrightarrow \widetilde{C^{(pf,S)/}}$. Passing to the fiber over an object $\alpha : s \longrightarrow t$, the compatibility of $\widetilde{P}_K$ and $\widetilde{P}_L$ implies that the diagram

$$\begin{array}{ccccc} (K^\vee)_s & \xrightarrow{(\widetilde{P}_K)_\alpha} & K_t & & \\ \downarrow{\scriptstyle (f^\vee)_s} & & \downarrow{\scriptstyle f_t} & \searrow^{(pf)_t} & \\ (L^\vee)_s & \xrightarrow{(\widetilde{P}_L)_\alpha} & L_t & \xrightarrow{p_t} & C_t \end{array}$$

commutes and that

$$(\widetilde{f^*})_\alpha = (f^\vee)^*_s : C^{p_t \circ (\widetilde{P}_L)_\alpha /} \longrightarrow C^{(pf)_t \circ (\widetilde{P}_K)_\alpha /}.$$

By [10, Corollary 4.1.1.10], $(f^\vee)_s$ is final, so by [10, Proposition 4.1.1.8], $(f^\vee)^*_s$ is an equivalence. Consequently, $\widetilde{f^*}$ is an equivalence. Now by Theorem 6.6, f^* is an equivalence.

(2) ⇒ (3): Immediate from the definition.

(3) ⇒ (1): Let $s \in S$ be any object and $\overline{p_s} : L_s^{\rhd} \longrightarrow \mathbf{Spc}$ a colimit diagram. Let $\overline{p} : (L \star_S S)_{\underline{s}} \longrightarrow \mathbf{Spc}$ be a left Kan extension of $\overline{p_s}$ along the full and faithful inclusion $L_s^{\rhd} \subset (L \star_S S)_{\underline{s}}$. By transitivity of left Kan extensions, $\overline{p}$ is a left Kan extension of its restriction to $L_{\underline{s}}$. By Proposition 5.5, under the equivalence $\mathrm{Fun}(L, \mathbf{Spc}) \simeq \mathrm{Fun}_S(L, \underline{\mathbf{Spc}}_S)$, $\overline{p}$ is a $S^{s/}$-colimit diagram. By assumption, $\overline{p} \circ (f^{\rhd})_{\underline{s}}$ is a $S^{s/}$-colimit diagram. By Proposition 5.5 again, $\overline{p_s} \circ f_s$ is a colimit diagram, as desired. □

6.8. **Definition.** Let $f : K \longrightarrow L$ be a S-functor. We say that f is *S-final* if it satisfies the equivalent conditions of Theorem 6.7. We say that f is *S-initial* if f^{vop} is S-final.

6.9. **Example.** Let $F : C \rightleftarrows D : G$ be a S-adjunction (Definition 8.3). Then F is S-initial and G is S-final.

6.10. **Remark.** Let C, D be S-categories and $F : C \longrightarrow D$ an S-functor.

(1) Suppose F is fiberwise a weak homotopy equivalence. Then F is a weak homotopy equivalence by [10, Proposition 4.1.2.15], [10, Proposition 4.1.2.18], and [10, Proposition 3.1.5.7].
(2) Suppose F is S-final. Then F is final. Indeed, for any diagram $p : D \longrightarrow \mathbf{Spc}$, we have that
$$\operatorname*{colim}_{d \in D} p(d) \simeq \operatorname*{colim}_{s \in S} \operatorname*{colim}_{d \in D_s} p(d) \simeq \operatorname*{colim}_{s \in S} \operatorname*{colim}_{c \in C_s} pF(c) \simeq \operatorname*{colim}_{c \in C} pF(c).$$
(3) Suppose F is S-initial. Then F is initial. To show this, by (the dual of) [10, Theorem 4.1.3.1] it suffices to show that for every $d \in D$, $C \times_D D^{/d}$ is weakly contractible. Let s be the image of d in S. By Lemma 10.9, the inclusion $C_s \times_{D_s} (D_s)^{/d} \longrightarrow C \times_D D^{/d}$ is final, so in particular is a weak homotopy equivalence. Hence the desired conclusion follows by our assumption that F is S-initial and [10, Theorem 4.1.3.1] again.

We conclude by using the twisted slice $\widetilde{\mathcal{O}}(S)$-category to give a criterion for the presentability of the S-slice.

6.11. **Remark** (Presentability of the parametrized slice)**.** Suppose the functor $S \longrightarrow \mathbf{Cat}_\infty$ classifying the cocartesian fibration $C \longrightarrow S$ factors through $\mathbf{Pr}^R$, i.e. $C \longrightarrow S$ is a *right presentable fibration.* For any X a presentable ∞-category and diagram $f : A \longrightarrow X$, $X^{f/}$ is again presentable and the forgetful functor $X^{f/} \longrightarrow X$ creates limits and filtered colimits. Therefore, the twisted slice $\widetilde{\mathcal{O}}(S)$-category $C^{\widetilde{(p,S)}/}$ is a right presentable fibration. Since the forgetful functor $\mathbf{Pr}^R \longrightarrow \mathbf{Cat}_\infty$ creates limits, by Theorem 6.6 we deduce that $C^{(p,S)/}$ is a right presentable fibration. In particular, in every fiber there exists an initial object. However, these initial objects may fail to be preserved by the pushforward functors. In fact, even if we assume that $C \longrightarrow S$ is both left and right presentable, C may fail to be S-cocomplete.

7. Types of S-fibrations

In this section we introduce some additional classes of fibrations which are all defined relative to S.

7.1. **Definition.** Let $\phi : C \longrightarrow D$ be an S-functor. We say that ϕ is an *S-fibration* if it is a categorical fibration. We then say that ϕ is an *S-cocartesian fibration* if it is an S-fibration such that for every object $s \in S$, $\phi_s : C_s \longrightarrow D_s$ is a cocartesian fibration, and for every square in C

$$\begin{array}{ccc} x_s & \xrightarrow{h} & x_t \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ y_s & \xrightarrow{k} & y_t \end{array}$$

with h and k ϕ-cocartesian edges over $\phi(h) = \phi(k) : s \longrightarrow t$, if f is a ϕ_s-cocartesian edge then g is a ϕ_t-cocartesian edge.

Dually, we say that ϕ is an *S-cartesian fibration* if it is an S-fibration such that for every object $s \in S$, $\phi_s : C_s \longrightarrow D_s$ is a cartesian fibration, and for every square in C labeled as above, but now with h and k ϕ-cartesian edges over $\phi(h) = \phi(k) : s \longrightarrow t$, if f is a ϕ_s-cartesian edge then g is a ϕ_t-cartesian edge.

Equivalently, $\phi : C \longrightarrow D$ is S-(co)cartesian if it is a categorical fibration, fiberwise a (co)cartesian fibration, and for every edge in S, the cocartesian pushforward along that edge preserves (co)cartesian edges in the fibers. We formulate our definition as above so as to avoid having to make any 'straightening' constructions such as choosing pushforward functors.

7.2. **Remark.** Declare a morphism of S-cocartesian fibrations $[C \xrightarrow{\phi} D] \longrightarrow [C' \xrightarrow{\phi'} D']$ to be a commutative square of S-functors

$$\begin{array}{ccc} C & \xrightarrow{F} & C' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ D & \xrightarrow{G} & D' \end{array}$$

in which for all $s \in S$, F_s sends ϕ_s-cocartesian edges to ϕ'_s cocartesian edges. Let $\mathcal{O}^{\text{cocart.fib}}(\mathbf{Cat}^{cocart}_{\infty/S})$ be the ∞-category of S-cocartesian fibrations and morphisms thereof. Then one has the straightening equivalence

$$\mathcal{O}^{\text{cocart.fib}}(\mathbf{Cat}^{cocart}_{\infty/S}) \simeq \text{Fun}(S, \mathcal{O}^{\text{cocart.fib}}(\mathbf{Cat}_\infty)).$$

7.3. **Remark.** $\phi : C \longrightarrow D$ is a S-fibration if and only if $\phi : {}_\natural C \longrightarrow {}_\natural D$ is a marked fibration.

7.4. **Remark.** In view of [10, Proposition 2.4.2.11], [10, Lemma 2.4.2.7], and [10, Proposition 2.4.2.8], $\phi : C \longrightarrow D$ is an S-cocartesian fibration if and only if ϕ is a cocartesian fibration. However, there is no corresponding simplification of the definition of an S-cartesian fibration.

7.5. **Lemma.** *Let $\phi : C \longrightarrow D$ be a S-cartesian fibration and let $f : x \longrightarrow y$ be a ϕ_s-cartesian edge in C_s. Then f is a ϕ-cartesian edge.*

Proof. The property of being ϕ-cartesian may be checked after base-change to the 2-simplices of D. Consequently, we may suppose that $S = \Delta^1$ and $s = \{1\}$. We have to verify that for every object $w \in C$ we have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{Map}_C(w,x) & \xrightarrow{f_*} & \mathrm{Map}_C(w,y) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{Map}_D(\phi w, \phi x) & \xrightarrow{\phi(f)_*} & \mathrm{Map}_D(\phi w, \phi y). \end{array}$$

If $w \in C_0$, for any choice of cocartesian edge $w \longrightarrow w'$ over $0 \longrightarrow 1$, the square is equivalent to

$$\begin{array}{ccc} \mathrm{Map}_{C_1}(w',x) & \xrightarrow{f_*} & \mathrm{Map}_{C_1}(w',y) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{Map}_{D_1}(\phi w', \phi x) & \xrightarrow{\phi(f)_*} & \mathrm{Map}_{D_1}(\phi w', \phi y). \end{array}$$

Hence we may suppose that $w \in C_1$, in which case the square is a homotopy pullback square since f is a ϕ_1-cartesian edge. □

We next discuss an important example of S-(co)cartesian fibrations. Recall (Notation 4.29) the fiberwise arrow S-category $\mathcal{O}_S(D)$. Fix $\phi : C \longrightarrow D$ a S-functor.

7.6. **Definition.** The *free S-cocartesian* and *free S-cartesian* fibrations on ϕ are the S-functors

$$\mathrm{Fr}^{cocart}(\phi) := \mathrm{ev}_1 \circ \mathrm{pr}_2 : C \times_D \mathcal{O}_S(D) \longrightarrow D,$$

$$\mathrm{Fr}^{cart}(\phi) := \mathrm{ev}_0 \circ \mathrm{pr}_1 : \mathcal{O}_S(D) \times_D C \longrightarrow D.$$

7.7. **Proposition.** *$Fr^{cocart}(\phi)$ is a S-cocartesian fibration. Dually, $Fr^{cart}(\phi)$ is a S-cartesian fibration.*

Proof. We prove the second assertion, the proof of the first being similar but easier. First note that $\mathcal{O}_S(D) \times_D C$ is a subcategory of $\mathcal{O}(D) \times_D C$ stable under equivalences. Therefore, since $\mathrm{ev}_0 : \mathcal{O}(D) \times_D C \longrightarrow D$ is a cartesian fibration, $\mathrm{Fr}^{cart}(\phi)$ is a categorical fibration. Moreover, for every object $s \in S$, $\mathrm{Fr}^{cart}(\phi)_s : \mathcal{O}(D_s) \times_{D_s} C_s$ is the free cartesian fibration on $\phi_s : C_s \longrightarrow D_s$. It remains to show that for every square

$$\begin{array}{ccc} (a \to \phi x, x) & \xrightarrow{h} & (b \to \phi y, y) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ (a' \to \phi x', x') & \xrightarrow{k} & (b' \to \phi y', y') \end{array}$$

in $\mathcal{O}_S(D) \times_D C$ with the horizontal edges cocartesian over S and the left vertical edge $\mathrm{Fr}^{cart}(\phi)_s$-cartesian, the right vertical edge is $\mathrm{Fr}^{cart}(\phi)_t$-cartesian. This amounts to verifying that $y \to y'$ is an equivalence in C_t. The above square yields a square

$$\begin{array}{ccc} x & \xrightarrow{h} & y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ x' & \xrightarrow{k} & y' \end{array}$$

in C with $x \to x'$ an equivalence and the horizontal edges cocartesian over S, from which the claim follows. □

We conclude this section with an observation about the interaction between S-joins and S-cocartesian fibrations which will be used in the sequel.

7.8. **Lemma.** *Let C, C', and D be S-categories and let $\phi, \phi' : C, C' \longrightarrow D$ be S-functors. If ϕ and ϕ' are S-(co)cartesian, then $\phi \star \phi' : C \star_D C' \longrightarrow D$ is S-(co)cartesian.*

Proof. This is an easy corollary of Proposition 4.7. □

7.9. **Definition.** We say that a S-functor $F : C \longrightarrow D \times_S E$ is a *S-bifibration* if for all objects $s \in S$, F_s is a bifibration. Observe it is then automatic that $\mathrm{pr}_D F$ is S-cartesian and $\mathrm{pr}_E F : C \longrightarrow E$ is S-cocartesian.

7.10. **Example.** The S-functor

$$\underline{\mathrm{Fun}}_S(K \star_S L, C) \longrightarrow \underline{\mathrm{Fun}}_S(K, C) \times_S \underline{\mathrm{Fun}}_S(L, C)$$

is a S-bifibration by Lemma 4.8. In particular, for a S-functor $p : K \longrightarrow C$, the S-functors $C^{(p,S)/} \longrightarrow C$ and $C^{/(p,S)} \longrightarrow C$ are S-cocartesian resp. S-cartesian.

8. Relative adjunctions

In [12, §7.3.2], Lurie introduces the notion of a *relative adjunction*.

8.1. **Definition** ([12, Definition 7.3.2.2])**.** Suppose given categorical fibrations $q : C \longrightarrow S$, $p : D \longrightarrow S$ and functors $F : C \longrightarrow D$, $G : D \longrightarrow C$ over S. Suppose there exists a natural transformation $u : \mathrm{id}_C \longrightarrow GF$ such that

1. u exhibits F as a left adjoint to G, and
2. $q(u)$ is the identity transformation from q to itself.

Then we say that the adjunction $F \dashv G$ is a *relative adjunction* with respect to S.

8.2. **Recollection.** By [12, Proposition 7.3.2.5], relative adjunctions are stable under base-change; in particular, they restrict to adjunctions over every fiber.

8.3. **Definition.** Let C and D be S-categories. We call a relative adjunction (with respect to S)

$$F\colon C \rightleftarrows D : G$$

an *S-adjunction* if F and G are S-functors.

We prove some basic results about S-adjunctions in this section. Let us first reformulate the definition of a relative adjunction in terms of a correspondence. Let $F : C \longrightarrow D$ be a S-functor. By the relative nerve construction, F defines a cocartesian fibration $M \longrightarrow \Delta^1$ by prescribing, for every $\Delta^n \cong \Delta^{n_0} \star \Delta^{n_1} \longrightarrow \Delta^1$, the set $\mathrm{Hom}_{\Delta^1}(\Delta^n, M)$ to be the collection of commutative squares

$$\begin{array}{ccc} \Delta^{n_0} & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle F} \\ \Delta^n & \longrightarrow & D \end{array}$$

for $n_1 \geq 0$, and setting $\mathrm{Hom}_{\Delta^1}(\Delta^n, M) = \mathrm{Hom}(\Delta^n, C)$ for $n_1 = -1$. Moreover, the structure maps for C and D to S define a functor $M \longrightarrow S$ by sending $\Delta^n \longrightarrow M$ to $\Delta^n \longrightarrow D \longrightarrow S$ if $n_1 \geq 0$, and $\Delta^n \longrightarrow C \longrightarrow S$ if $n_1 < 0$. Then M is a S-category, $M \longrightarrow S \times \Delta^1$ is a S-cocartesian fibration, and F admits a right S-adjoint if and only if $M \longrightarrow S \times \Delta^1$ is a S-cartesian fibration.

8.4. **Proposition.** *Let $F\colon C \rightleftarrows D : G$ be a S-adjunction and let I be a S-category. Then we have adjunctions*

$$F_*\colon \mathrm{Fun}_S(I, C) \rightleftarrows \mathrm{Fun}_S(I, D) : G_*$$
$$G^*\colon \mathrm{Fun}_S(C, I) \rightleftarrows \mathrm{Fun}_S(D, I) : F^*$$

Proof. Let $M \longrightarrow S \times \Delta^1$ be the S-functor obtained from F. We first produce the adjunction $F_* \dashv G_*$. Invoking Theorem 2.24 on the span

$$(\Delta^1) \xleftarrow{\pi} {}_\natural I \times (\Delta^1)^\sharp \xrightarrow{\pi'} S^\sharp \times (\Delta^1)^\sharp$$

we deduce that $\pi_* \pi'^* : s\mathbf{Set}^+_{/(S^\sharp \times (\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $N = \pi_* \pi'^*(M)$. Then $N \longrightarrow \Delta^1$ is a cocartesian fibration classified by the functor

$$F_* : \mathrm{Fun}_S(I, C) \longrightarrow \mathrm{Fun}_S(I, D).$$

Now invoking Theorem 2.24 on the span

$$((\Delta^1)^\sharp)^{op} \xleftarrow{\rho} (I^\sim \times (\Delta^1)^\sharp)^{op} \xrightarrow{\rho'} (S^\sim \times (\Delta^1)^\sharp)^{op}$$

we deduce that with respect to the cartesian model structures $\rho_*\rho'^* : s\mathbf{Set}^+_{/(S^\sim\times(\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $N' = \rho_*\rho'^*M$. Since G is right S-adjoint to F, $N' \longrightarrow \Delta^1$ is a cartesian fibration classified by the functor

$$G_* : \mathrm{Fun}_{/S}(I, D) \longrightarrow \mathrm{Fun}_{/S}(I, C)$$

where we view I, C, D as categorical fibrations over S. N is a subcategory of N', and the cartesian edges e in N' with $d_0(e) \in N$ are in N. Hence $N \longrightarrow \Delta^1$ is also a cartesian fibration classified by the functor

$$G_* : \mathrm{Fun}_S(I, D) \longrightarrow \mathrm{Fun}_S(I, C).$$

We now produce the adjunction $G^* \dashv F^*$ by similar methods. Let $\mathcal{E}_0$ be the collection of edges $e : x \longrightarrow y$ in M such that e admits a factorization as a cocartesian edge over S followed by a cartesian edge in the fiber. Note that since $M \longrightarrow S \times \Delta^1$ is a S-cartesian fibration, $\mathcal{E}_0$ is closed under composition of edges. Invoking Theorem 2.24 on the span

$$(\Delta^1)^\sharp \xleftarrow{\mu} (M, \mathcal{E}_0) \xrightarrow{\mu'} S^\sharp \times (\Delta^1)^\sharp$$

we deduce that $\mu_*\mu'^* : s\mathbf{Set}^+_{/(S^\sharp\times(\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $P = \mu_*\mu'^*({}_\natural I \times (\Delta^1)^\sharp)$. Then $P \longrightarrow \Delta^1$ is a cocartesian fibration classified by the functor

$$G^* : \mathrm{Fun}_S(C, I) \longrightarrow \mathrm{Fun}_S(D, I).$$

Let $\mathcal{E}_1$ be the collection of edges $e : x \longrightarrow y$ in M such that e is a cocartesian edge over an equivalence in S. Now invoking Theorem 2.24 on the span

$$((\Delta^1)^\sharp)^{op} \xleftarrow{\nu} (M, \mathcal{E}_1)^{op} \xrightarrow{\nu'} (S^\sim \times (\Delta^1)^\sharp)^{op}$$

we deduce that with respect to the cartesian model structures $\nu_*\nu'^* : s\mathbf{Set}^+_{/(S^\sim\times(\Delta^1)^\sharp)} \longrightarrow s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $P' = \nu_*\nu'^*(I^\sim \times (\Delta^1)^\sharp)$. $P' \longrightarrow \Delta^1$ is a cartesian fibration with P as a subcategory. One may check that $P \longrightarrow \Delta^1$ inherits the property of being a cartesian fibration, which is classified by the functor $F^* : \mathrm{Fun}_S(D, I) \longrightarrow \mathrm{Fun}_S(C, I)$. □

8.5. **Corollary.** *Let $F : C \rightleftarrows D : G$ be a S-adjunction and let I be a S-category. Then we have S-adjunctions*

$$F_* : \underline{\mathrm{Fun}}_S(I, C) \rightleftarrows \underline{\mathrm{Fun}}_S(I, D) : G_*$$
$$G^* : \underline{\mathrm{Fun}}_S(C, I) \rightleftarrows \underline{\mathrm{Fun}}_S(D, I) : F^*$$

Proof. By Proposition 8.4, for every $s \in S$

$$F_* : \mathrm{Fun}_{S^{s/}}(I \times_S S^{s/}, C \times_S S^{s/}) \rightleftarrows \mathrm{Fun}_{S^{s/}}(I \times_S S^{s/}, D \times_S S^{s/}) : G_*$$

is an adjunction, and similarly for the contravariant case. □

To state the next corollary, it is convenient to introduce a definition.

8.6. **Definition.** Suppose $\pi : C \longrightarrow D$ a S-fibration. Define the ∞-category $\mathbf{Sect}_{D/S}(\pi)$ of S-sections of π to be the pullback

$$\begin{array}{ccc} \mathbf{Sect}_{D/S}(\pi) & \longrightarrow & \mathrm{Fun}_S(D, C) \\ \downarrow & & \downarrow \pi_* \\ \Delta^0 & \xrightarrow{\mathrm{id}_D} & \mathrm{Fun}_S(D, D). \end{array}$$

Define the S-category $\underline{\mathbf{Sect}}_{D/S}(\pi)$ to be the pullback

$$\begin{array}{ccc} \underline{\mathbf{Sect}}_{D/S}(\pi) & \longrightarrow & \underline{\mathrm{Fun}}_S(D, C) \\ \downarrow & & \downarrow \pi_* \\ S & \xrightarrow{\sigma_{\mathrm{id}_D}} & \underline{\mathrm{Fun}}_S(D, D). \end{array}$$

We will often denote $\mathbf{Sect}_{D/S}(\pi)$ by $\mathbf{Sect}_{D/S}(C)$, the S-functor π being left implicit.

Note that for any object $s \in S$, the fiber $\underline{\mathbf{Sect}}_{D/S}(\pi)_s$ is isomorphic to $\mathbf{Sect}_{D_{\underline{s}}/\underline{s}}(\pi_{\underline{s}})$.

8.7. **Corollary.** *Let $p : C \longrightarrow E$ and $q : D \longrightarrow E$ be S-fibrations. Let $F\colon C \rightleftarrows D : G$ be an adjunction relative to E where F and G are S-functors. Then for any S-category I,*

$$F_*\colon \mathrm{Fun}_S(I,C) \rightleftarrows \mathrm{Fun}_S(I,D) : G_*$$

is an adjunction relative to $\mathrm{Fun}_S(I,E)$. In particular, taking $I = E$ and the fiber over the identity, we deduce that

$$F_*\colon \mathbf{Sect}_{E/S}(p) \rightleftarrows \mathbf{Sect}_{E/S}(q) : G_*$$

is an adjunction, and also that

$$F_*\colon \underline{\mathbf{Sect}}_{E/S}(p) \rightleftarrows \underline{\mathbf{Sect}}_{E/S}(q) : G_*$$

is a S-adjunction.

Proof. The proof of Proposition 8.4 shows that the unit for the adjunction $F_* \dashv G_*$ is sent by p_* to a natural transformation through equivalences. □

8.8. **Lemma.** *Let $F\colon C \rightleftarrows D : G$ be a S-adjunction. For every S-functor $p : K \longrightarrow D$, we have a homotopy pullback square in $s\mathbf{Set}^+_{/S}$*

$$\begin{array}{ccc} C^{/(Gp,S)} & \longrightarrow & D^{/(p,S)} \\ \downarrow{\scriptstyle \mathrm{ev}_0^C} & & \downarrow{\scriptstyle \mathrm{ev}_0^D} \\ C & \xrightarrow{F} & D \end{array}$$

where the upper horizontal map is defined to be the composite $C^{/(Gp,S)} \xrightarrow{F} C^{/(FGp,S)} \xrightarrow{\epsilon(p)_!} D^{/(p,S)}$. Dually, for every S-functor $p : K \longrightarrow D$, we have a homotopy pullback square in $s\mathbf{Set}^+_{/S}$

$$\begin{array}{ccc} D^{(Fp,S))/} & \longrightarrow & C^{(p,S)/} \\ \downarrow{\scriptstyle \mathrm{ev}_1^D} & & \downarrow{\scriptstyle \mathrm{ev}_1^C} \\ D & \xrightarrow{G} & C. \end{array}$$

where the upper horizontal map is defined to be the composite $D^{(Fp,S))/} \xrightarrow{G} C^{(GFp,S)/} \xrightarrow{\eta(p)^*} C^{(p,S)/}$.

Proof. We prove the first assertion; the second then follows by taking vertical opposites. We first explain how to define the map $\epsilon(p)_!$. Choose a counit transformation $\epsilon : D \times \Delta^1 \longrightarrow D$ for $F \dashv G$ such that $\pi_D \circ \epsilon$ is the identity natural transformation from π_D to itself. Then $\epsilon \circ (p \times \mathrm{id})$ is adjoint to a S-functor $\epsilon(p) : S \times \Delta^1 \longrightarrow \underline{\mathrm{Fun}}_S(K,D)$ with $\epsilon(p)_0 = \sigma_{FGp}$ and $\epsilon(p)_0 = \sigma_p$. Because $\underline{\mathrm{Fun}}_S(S \star_S K, D) \longrightarrow D \times_S \underline{\mathrm{Fun}}_S(K,D)$ is an S-bifibration, from $\epsilon(p)$ we obtain a pushforward S-functor $\epsilon(p)_! : D^{/(FGp,S)} \longrightarrow D^{/(p,S)}$ compatible with the source maps to D.

We need to check that for every object $s \in S$, passage to the fiber over s yields a homotopy pullback square of ∞-categories. Because $(D^{/(p,S)})_s \cong (D_{\underline{s}}^{/(p_{\underline{s}},\underline{s})})_s$, we may replace S by $S^{s/}$ and thereby suppose that s is an initial object in S.

Let $r : \{s\} \star S \longrightarrow S$ be a left Kan extension of the identity $S \longrightarrow S$. By the formula for a left Kan extension, $r(s)$ is an initial object in S, which without loss of generality we may suppose to be s. Using $r \circ (\mathrm{id} \star \pi_K)$ as the structure map for $\{s\} \star K$ over S, define $\phi' : \{s\} \star {}_\natural K \longrightarrow \{s\} \star_S {}_\natural K$ as adjoint to the identity over $S \times \partial\Delta^1$. It is easy to show that ϕ' is a trivial cofibration in $s\mathbf{Set}^+_{/S}$. Moreover, since the inclusion $\{s\} \longrightarrow S^\sharp$ is a trivial cofibration, $\{s\} \star_S {}_\natural K \longrightarrow S^\sharp \star_S {}_\natural K$ is a trivial cofibration in $s\mathbf{Set}^+_{/S}$ by Theorem 4.16. Let ϕ be the composition of these two maps. Then because $\mathrm{Fun}_S(-,-)$ is a right Quillen bifunctor, $\phi^* : \mathrm{Fun}_S(S^\sharp \star_S {}_\natural K, {}_\natural D) \longrightarrow \mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D)$ is a trivial Kan fibration.

We further claim that the inclusion $j : \mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D) \longrightarrow D_s \times_D \mathrm{Fun}(\{s\} \star K, D) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}_S({}_\natural K, {}_\natural D)$ is an equivalence. Indeed, we have the pullback square

$$\begin{array}{ccc} \mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D) & \longrightarrow & D_s \times_D \mathrm{Fun}(\{s\} \star K, D) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}_S({}_\natural K, {}_\natural D) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{r \circ (\mathrm{id} \star \pi_K)} & \{s\} \times_S \mathrm{Fun}(\{s\} \star K, S) \times_{\mathrm{Fun}(K,S)} \{\pi_K\} \end{array}$$

and the term in the lower right is contractible since it is equivalent to the full subcategory $\mathrm{Fun}'(\{s\}\star K, S) \subset \mathrm{Fun}(\{s\}\star K, S)$ of functors which are left Kan extensions of π_K.

Now taking the pullback of the composition $j \circ \phi^*$ over $\{p\}$, we obtain an equivalence

$$(D^{/(p,S)})_s \longrightarrow D_s \times_D D^{/p}.$$

Similarly, we have an equivalence

$$(C^{/(Gp,S)})_s \longrightarrow C_s \times_C C^{/Gp}.$$

Since $F \dashv G$ is in particular an adjunction, by [10, Lemma 5.2.5.5] $C^{/Gp} \longrightarrow C \times_D D^{/p}$ is an equivalence. Taking the fiber over s, we deduce the claim. □

8.9. **Corollary.** *Let $F: C \rightleftarrows D: G$ be a S-adjunction. Then F preserves S-colimits and G preserves S-limits.*

Proof. Let $\overline{p} : K \star_S S \longrightarrow C$ be a S-colimit diagram. To show that $F\overline{p}$ is a S-colimit diagram, it suffices to prove that the restriction map $D^{(F\overline{p},S)/} \longrightarrow D^{(Fp,S)/}$ is an equivalence. We have the commutative square

$$\begin{array}{ccc} D^{(F\overline{p},S)/} & \longrightarrow & C^{(\overline{p},S)/} \times_C D \\ \downarrow & & \downarrow \\ D^{(Fp,S)/} & \longrightarrow & C^{(p,S)/} \times_C D \end{array}$$

(here we suppress some details about the naturality of $\epsilon(-)_!$). The righthand vertical map is an equivalence by assumption, and the horizontal maps are equivalences by Lemma 8.8. Thus the lefthand vertical map is an equivalence. □

Free S-(co)cartesian fibrations revisited. With the theory of S-adjunctions, we can now establish a key property of the free S-(co)cartesian fibration (Definition 7.6). Let $\phi : C \longrightarrow D$ be an S-functor and define S-functors

$$\iota_0 : C \longrightarrow C \times_D \mathcal{O}_S(D), \quad \iota_1 : C \longrightarrow \mathcal{O}_S(D) \times_D C$$

via the commutative square

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{O}_S(D) \\ \downarrow{=} & & \downarrow{\mathrm{ev}_i} \\ C & \xrightarrow{\phi} & D \end{array}$$

where the upper horizontal map is the composite $C \xrightarrow{\iota} \mathcal{O}_S(C) \longrightarrow \mathcal{O}_S(D)$.

8.10. **Proposition.** *ι_0 is left S-adjoint to pr_C. Dually, ι_1 is right S-adjoint to pr_C.*

Proof. We prove the first assertion, the proof of the second being similar. To prove that we have a relative S-adjunction $\iota_0 \dashv \mathrm{pr}_C$, we must prove that for each $s \in S$ we have an adjunction $(\iota_0)_s \dashv (\mathrm{pr}_C)_s$. So suppose that $S = \Delta^0$. Since $\mathrm{pr}_C \circ \iota_0 = \mathrm{id}$, it suffices by [10, Proposition 5.2.2.8] to check that the identity is a unit transformation: that is, for every $x \in C$ and $(y, \phi y \to a) \in C \times_D \mathcal{O}(D)$,

$$\mathrm{pr}_C : \mathrm{Map}_{C\times_D \mathcal{O}(D)}((x, \mathrm{id}_{\phi x}), (y, \phi y \to a)) \longrightarrow \mathrm{Map}_C(x, y)$$

is an equivalence. Under the fiber product decomposition

$$\mathrm{Map}_{C\times_D \mathcal{O}(D)}((x, \mathrm{id}_{\phi x}), (y, \phi y \to a)) \simeq \mathrm{Map}_C(x, y) \times_{\mathrm{Map}_D(\phi x, \phi y)} \mathrm{Map}_{\mathcal{O}(D)}((\mathrm{id}_{\phi x}), (\phi y \to b))$$

the map pr_C is projection onto the first factor. The adjunction $\iota: D \rightleftarrows \mathcal{O}(D) : \mathrm{ev}_0$ obtained by exponentiating the adjunction $i_0: \{0\} \rightleftarrows \Delta^1 : p$ implies that

$$\mathrm{Map}_{\mathcal{O}(D)}((\mathrm{id}_{\phi x}), (\phi y \to b)) \longrightarrow \mathrm{Map}_D(\phi x, \phi y)$$

is an equivalence, so the claim follows. □

8.11. **Remark** (Universal property of the free S-cocartesian fibration)**.** Let $\phi : C \longrightarrow D$ be an S-functor and $\psi : E \longrightarrow D$ be an S-cocartesian fibration. Then we would like to show that the restriction functor

$$\mathrm{Fun}^{cocart}_{/D}(C \times_D \mathcal{O}_S(D), E) \longrightarrow \mathrm{Fun}_{/D,S}(C, E) = S \times_{\sigma_\phi, \mathrm{Fun}_S(C,D), \psi_*} \mathrm{Fun}_S(C, E)$$

is an equivalence of ∞-categories.[18] We prove this as [18, Example 3.8] as an application of the theory of parametrized factorization systems.

9. Parametrized colimits

In this section, we first introduce a parametrized generalization of Lurie's pairing construction [10, Corollary 3.2.2.13]. We then employ it to study *D-parametrized S-(co)limits.* This material recovers and extends [10, §4.2.2] (in view of Lemma 4.5). It is a precursor to our study of Kan extensions.

An S-pairing construction.

9.1. **Construction.** Let $p : C \longrightarrow S$, $q : D \longrightarrow S$ be S-categories and let $\phi : C \longrightarrow D$ be a S-functor. Let $\pi, \pi' : \mathcal{O}^{cocart}(D) \times_D C \longrightarrow D$ be given by $\pi = \mathrm{ev}_0 \circ \mathrm{pr}_1$, $\pi' = \mathrm{ev}_1 \circ \mathrm{pr}_1$. Let $\mathcal{E}$ denote the collection of edges e in $\mathcal{O}^{cocart}(D) \times_D C$ such that $\pi(e)$ is q-cocartesian and $\mathrm{pr}_2(e)$ is p-cocartesian (so $\pi'(e)$ is q-cocartesian). Then the span

$$\natural D \xleftarrow{\pi} (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}) \xrightarrow{\pi'} \natural D$$

defines a functor

$$\pi_* \pi'^* : s\mathbf{Set}^+_{/\natural D} \longrightarrow s\mathbf{Set}^+_{/\natural D}.$$

For a S-category E and a S-functor $\psi : E \longrightarrow D$, define

$$(\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow \natural D) := \pi_* \pi'^*(\natural E \xrightarrow{\psi} \natural D).$$

9.2. **Lemma.** *Let $q : D \longrightarrow S$ be a S-category.*

(1) $\mathrm{ev}_0 : \mathcal{O}^{cocart}(D) \longrightarrow D$ is a cartesian fibration, and an edge e in $\mathcal{O}^{cocart}(D)$ is ev_0-cartesian if and only $(\mathrm{ev}_{S,1} \circ q)(e)$ is an equivalence in S. In particular, if $\mathrm{ev}_0(e)$ is q-cocartesian, then e is ev_0-cartesian if and only if $\mathrm{ev}_1(e)$ is an equivalence in D.

(2) If $f : x \longrightarrow y$ is an edge in D such that $q(f)$ is an equivalence, then there exists a ev_0-cocartesian edge e over f. Moreover, an edge e over f is ev_0-cocartesian if and only if it is ev_0-cartesian.

Proof. $\mathrm{ev}_0 : \mathcal{O}^{cocart}(D) \longrightarrow D$ factors as

$$\mathcal{O}^{cocart}(D) \longrightarrow D \times_S \mathcal{O}(S) \longrightarrow D$$

where the first functor is a trivial fibration and the second is a cartesian fibration, as the pullback of $\mathrm{ev}_{S,0} : \mathcal{O}(S) \longrightarrow S$. Thus ev_0 is a cartesian fibration with cartesian edges as indicated. Moreover, since $\mathrm{ev}_{S,0} : \mathcal{O}(S) \longrightarrow S$ is a categorical fibration, the second claim follows from [12, Proposition B.2.9]. $\square$

We have designed our construction so that for any object $x \in D$ and cocartesian section $S^{qx/} \longrightarrow D$, the fiber of $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ over x is equivalent to $\mathrm{Fun}_{S^{qx/}}(C \times_D S^{qx/}, E \times_D S^{qx/})$. For this reason, we think of $\widetilde{\mathrm{Fun}}_{D/S}(-,-)$ as the parametrized generalization of the pairing construction $\widetilde{\mathrm{Fun}}_D(-,-)$, to which it reduces when $S = \Delta^0$.

9.3. **Theorem.** *Notation as in Construction 9.1, $\widetilde{\mathrm{Fun}}_{D/S}(C, E)$ enjoys the following functoriality:*

(1) If ϕ is either a S-cartesian fibration or a S-cocartesian fibration and ψ is a categorical fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow S$ is a S-category with cocartesian edges marked as indicated in Construction 9.1, and $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ is a categorical fibration.

(2) If ϕ is a S-cartesian fibration and ψ is a S-cocartesian fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \longrightarrow D$ is a S-cocartesian fibration.

[18] We use Remark 7.4 to simplify the appearance of the lefthand side, which would otherwise be denoted as $\mathrm{Fun}^{cocart}_{/D,S}(C \times_D \mathcal{O}_S(D), E)$.

(3) If ϕ is a S-cocartesian fibration and ψ is a S-cartesian fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C,E) \longrightarrow D$ is a S-cartesian fibration.

Proof. (1) It suffices to check that Theorem 2.24 applies to the span

$$\natural D \xleftarrow{\pi} (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}) \xrightarrow{\pi'} \natural D.$$

In the remainder of this proof we will verify that $\mathcal{O}^{cocart}(D) \times_D C \longrightarrow D$ is a flat categorical fibration. For condition (4) we appeal to Lemma 9.2. The rest of the conditions are easy verifications.

(2) By Lemma 9.2 and 7.5, $\pi : \mathcal{O}^{cocart}(D) \times_D C \longrightarrow D$ is a cartesian fibration (hence flat) with an edge e π-cartesian if and only if $\mathrm{pr}_1(e)$ is ev_0-cartesian and $\mathrm{pr}_2(e)$ is ϕ-cartesian. Let $\mathcal{E}'$ be the collection of edges e in $\mathcal{O}^{cocart}(D) \times_{\mathrm{ev}_1, D} C$ such that for any π-cartesian lift e' of $\pi(e)$, the induced edge $d_1(e) \longrightarrow d_1(e')$ is in $\mathcal{E}$. Note that since ϕ is S-cartesian (and not just fiberwise cartesian), $\mathcal{E}'$ is closed under composition. Invoking Theorem 2.24 on the span

$$D^\sharp \xleftarrow{\pi} (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}') \xrightarrow{\pi'} D^\sharp$$

we deduce that

$$\pi_* \pi'^* : s\mathbf{Set}^+_{/D} \longrightarrow s\mathbf{Set}^+_{/D}$$

is right Quillen. Note that there is no conflict of notation with the functor $\pi_* \pi'^*$ defined before on $s\mathbf{Set}^+_{/\natural D}$ since $\mathcal{E} \subset \mathcal{E}'$ and the two restrict to the same collections of marked edges in the fibers of π. Since S-cocartesian fibrations are cocartesian fibrations over D (Remark 7.4), we conclude.

(3) First note that π factors as a cocartesian fibration followed by a cartesian fibration, so is flat. Let $\mathcal{F}$ be the collection of edges f in D such that $q(f)$ is an equivalence. By Lemma 9.2, we have that $\pi : \mathcal{O}^{cocart}(D) \times_{\mathrm{ev}_1, D} C \longrightarrow D$ admits cocartesian lifts of edges in $\mathcal{F}$. Let $\mathcal{E}''$ be the collection of those π-cocartesian edges. Invoking Theorem 2.24 on the span

$$(D, \mathcal{F})^{op} \xleftarrow{\rho} (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}'')^{op} \xrightarrow{\rho'} (D, \mathcal{F})^{op},$$

where $\rho = \pi^{op}$ and $\rho' = \pi'^{op}$, we deduce that with respect to the cartesian model structures

$$\rho_* \rho'^* : s\mathbf{Set}^+_{/(D,\mathcal{F})} \longrightarrow s\mathbf{Set}^+_{/(D,\mathcal{F})}$$

is right Quillen. We have that $\widetilde{\mathrm{Fun}}_{D/S}(C,E)$ is a full subcategory of $\rho_* \rho'^*(\psi)$. Moreover, the compatibility condition in the definition of a S-cartesian fibration ensures that $\widetilde{\mathrm{Fun}}_{D/S}(C,E) \longrightarrow D$ inherits the property of being fibrant in $s\mathbf{Set}^+_{/(D,\mathcal{F})}$. Another routine verification shows that $\widetilde{\mathrm{Fun}}_{D/S}(C,E) \longrightarrow D$ is indeed S-cartesian.

□

9.4. **Lemma.** *Let $C \longrightarrow C'$ be a monomorphism between S-cartesian or S-cocartesian fibrations over D and let $E \longrightarrow D$ be a S-fibration. Then the induced functor*

$$\widetilde{\mathrm{Fun}}_{D/S}(C',E) \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C,E)$$

is a categorical fibration.

Proof. Given a trivial cofibration $A \longrightarrow B$ in $s\mathbf{Set}_{\mathrm{Joyal}}$, we need to solve the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{\mathrm{Fun}}_{D/S}(C',E) \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \widetilde{\mathrm{Fun}}_{D/S}(C,E). \end{array}$$

This diagram transposes to

$$\begin{array}{ccc} A \times_D \mathcal{O}^{cocart}(D) \times_D C' \bigcup_{A \times_D \mathcal{O}^{cocart}(D) \times_D C} B \times_D \mathcal{O}^{cocart}(D) \times_D C & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ B \times_D \mathcal{O}^{cocart}(D) \times_D C' & \longrightarrow & D. \end{array}$$

By the proof of Theorem 9.3, $\mathcal{O}^{cocart}(D) \times_D C \longrightarrow D$ is a flat categorical fibration. Therefore, by [12, Proposition B.4.5] the left vertical arrow is a trivial cofibration in $s\mathbf{Set}_{\mathrm{Joyal}}$. □

For later use, we analyze some degenerate instances of the S-pairing construction.

9.5. **Lemma.** *There is a natural equivalence $\widetilde{\mathrm{Fun}}_{D/S}(D, E) \xrightarrow{\simeq} E$ of S-categories over D.*

Proof. The map is induced by the identity section $\iota_D : D \longrightarrow \mathcal{O}^{cocart}(D)$ fitting into a morphism of spans

$$\begin{array}{ccccc} & & {}_\natural D & & \\ & \swarrow^{=} & \downarrow{\scriptstyle \iota_D} & \searrow^{=} & \\ {}_\natural D & \longleftarrow & (\mathcal{O}^{cocart}(D), \mathcal{E}) & \longrightarrow & {}_\natural D. \end{array}$$

By Lemma 3.3(1'), ι_D is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ via the target map. Since the cocartesian model structure on $s\mathbf{Set}^+_{/{}_\natural D}$ is created by the forgetful functor to $s\mathbf{Set}^+_{/S}$, the assertion follows. □

9.6. **Lemma.** *Let $C' \longrightarrow D'$ be a cartesian fibration of ∞-categories and let E' be a S-category. For all $s \in S$, there is a natural equivalence*

$$\widetilde{\mathrm{Fun}}_{D' \times S/S}(C' \times S, D' \times E')_s \xrightarrow{\simeq} \widetilde{\mathrm{Fun}}_{D'}(C', D' \times E'_s)$$

of cartesian fibrations over D'.

Proof. The lefthand side is defined using the span

$$(D')^\sharp \times \{s\} \longleftarrow ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathcal{O}^{cocart}(D' \times S) \times_{D'} C', \mathcal{E}') \longrightarrow S^\sharp$$

with $\mathcal{E}'$ as in the proof of Theorem 9.3. Cocartesian edges (over S) in $D' \times S$ are precisely those edges which become equivalences when projected to D', so $\mathcal{O}^{cocart}(D' \times S) \cong \mathrm{Fun}((\Delta^1)^\sharp, (D')^\sim) \times \mathcal{O}(S)$, and the identity section $\iota_{D'} : D' \longrightarrow \mathrm{Fun}((\Delta^1)^\sharp, (D')^\sim)$ is a categorical equivalence. Therefore, the map

$$(D' \times S^{s/})^\sharp \longrightarrow ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathcal{O}^{cocart}(D' \times S), \mathcal{E})$$

induced by $\iota_{D'}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Since $C' \times S \longrightarrow D' \times S$ is a cartesian fibration, it follows that

$$(C')^\natural \times (S^{s/})^\sharp \longrightarrow ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathcal{O}^{cocart}(D' \times S) \times_{D'} C', \mathcal{E}')$$

is also a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Finally, using the inclusion $C' \times \{s\} \longrightarrow C' \times S^{s/}$, we obtain a morphism from the span

$$(D')^\sharp \longleftarrow (C')^\natural \longrightarrow \{s\} \subset S^\sharp$$

through a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. This yields the equivalence of the lemma. □

Directly from the definition, we have that for an object $x \in D$, the fiber $\widetilde{\mathrm{Fun}}_{D/S}(C, E)_x$ is isomorphic to $\mathrm{Fun}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$. We now proceed to identify the S-fiber $\widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}$.

9.7. **Proposition.** *There is a $\underline{x}$-functor*

$$\epsilon^* : \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}} \longrightarrow \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$$

which is a cocartesian equivalence in $s\mathbf{Set}^+_{/\underline{x}}$.

Proof. We first define the $\underline{x}$-functor ϵ^*. The data of maps of marked simplicial sets

$$A \longrightarrow {}_\natural\widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}$$
$$A \longrightarrow {}_\natural\underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, (E \times_S D)_{\underline{x}})$$

over $\underline{x}$ is identical to the data of maps

$$A \times_{\underline{x}} \underline{x}^\sharp \times_D (\mathcal{O}^{cocart}(D), \mathcal{E}) \times_D {}_\natural C \longrightarrow {}_\natural E$$
$$A \times_{\underline{x}} \mathcal{O}(\underline{x})^\sharp \times_{\mathrm{ev}_1 \circ \mathrm{ev}_1, D} {}_\natural C \longrightarrow {}_\natural E$$

over $\natural D$ (where $\mathcal{E}$ is the collection of edges e in $\mathcal{O}^{cocart}(D)$ such that $\mathrm{ev}_0(e)$ and $\mathrm{ev}_1(e)$ are cocartesian). We have a commutative square

$$\begin{array}{ccc} \mathcal{O}(\underline{x})^\sharp & \xrightarrow{\mathrm{ev}_0} & \underline{x}^\sharp \\ \downarrow{\scriptstyle \mathcal{O}(\mathrm{ev}_1)} & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ (\mathcal{O}^{cocart}(D), \mathcal{E}) & \xrightarrow{\mathrm{ev}_0} & \natural D \end{array}$$

which defines the functor $\epsilon : \mathcal{O}(\underline{x}) \longrightarrow \underline{x} \times_D \mathcal{O}^{cocart}(D)$, and this in turn induces the functor ϵ^*. To show that ϵ^* is a cocartesian equivalence, it will suffice to show that ϵ is a trivial fibration, for then a choice of section σ and homotopy $\sigma \circ \epsilon \simeq \mathrm{id}$ will furnish a strong homotopy inverse to ϵ^* in the sense of [10, Proposition 3.1.3.5]. Since we have a pullback diagram

$$\begin{array}{ccc} \mathcal{O}(\underline{x}) & \longrightarrow & D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon'} \\ \underline{x} \times_D \mathcal{O}^{cocart}(D) & \longrightarrow & \mathrm{Fun}(\Lambda^2_1, D) \end{array}$$

it will further suffice to show that ϵ' is a trivial Kan fibration. ϵ' factors as the composition

$$D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \xrightarrow{\epsilon''} \mathrm{Fun}(\Delta^2, D) \xrightarrow{\epsilon'''} \mathrm{Fun}(\Lambda^2_1, D)$$

where ϵ'' is defined by precomposing by the inclusion $i : \Delta^2 \longrightarrow \Delta^1 \times \Delta^1$ which avoids the degenerate edge for objects in $D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D)$, and ϵ''' is precomposition by $\Lambda^2_1 \longrightarrow \Delta^2$. ϵ''' is a trivial fibration since $\Lambda^2_1 \longrightarrow \Delta^2$ is inner anodyne. To argue that ϵ'' is a trivial fibration, first note that ϵ'' inherits the property of being a categorical fibration from $i^* : \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \longrightarrow \mathrm{Fun}(\Delta^2, D)$. Define an inverse σ'' by precomposing by the unique retraction $r : \Delta^1 \times \Delta^1 \longrightarrow \Delta^2$ chosen so that $r \circ i = \mathrm{id}$. Then σ'' is a section of ϵ'' and one can write down an explicit homotopy through equivalences of the identity functor on $D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D)$ to $\sigma'' \circ \epsilon''$, so ϵ'' is a trivial fibration. □

D-parametrized slice. We now study another slice construction defined using the S-pairing construction.

9.8. **Construction.** Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration, let $E \longrightarrow D$ be a S-fibration, and let $F : C \longrightarrow E$ be a S-functor over D. Then F defines a section S-functor

$$\tau_F : D \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)$$

as adjoint to the functor $\mathcal{O}^{cocart}(D) \times_{ev_1, D} C \longrightarrow C \xrightarrow{F} E$. Define

$$E^{(\phi,F)/S} := D \times_{\tau_F, \widetilde{\mathrm{Fun}}_{D/S}(C,E)} \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)$$

and let $\pi_{(\phi,F)}$ denote the projection $E^{(\phi,F)/S} \longrightarrow D$.

Given an object $x \in D$, the functor $\tau_F : D \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)$ induces via pullback a $\underline{x}$-functor

$$\tau_{F_{\underline{x}}} : \underline{x} \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}.$$

We also have the $\underline{x}$-functor

$$\sigma_{F_{\underline{x}}} : \underline{x} \longrightarrow \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$$

adjoint to

$$\mathcal{O}(\underline{x}) \times_{\underline{x}} C_{\underline{x}} \xrightarrow{\mathrm{pr}_2} C_{\underline{x}} \xrightarrow{F_{\underline{x}}} E_{\underline{x}}.$$

An inspection of the definition of the comparison functor ϵ^* of 9.7 shows that the triangle

$$\begin{array}{ccc} \underline{x} & \xrightarrow{\tau_{F_{\underline{x}}}} & \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}} \\ & \searrow{\scriptstyle \sigma_{F_{\underline{x}}}} & \downarrow{\scriptstyle \epsilon^*} \\ & & \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}}) \end{array}$$

commutes. Recalling the definitions

$$(E^{(\phi,F)/S})_{\underline{x}} = \underline{x} \times_{\widetilde{\mathrm{Fun}}_{D/S}(C,E)_{\underline{x}}} \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_{\underline{x}},$$

$$(E_{\underline{x}})^{F_{\underline{x}}/\underline{x}} = \underline{x} \times_{\underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}},E_{\underline{x}})} \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}} \star_{\underline{x}} \underline{x}, E_{\underline{x}}),$$

we therefore obtain a comparison $\underline{x}$-functor

$$\psi : (E^{(\phi,F)/S})_{\underline{x}} \longrightarrow (E_{\underline{x}})^{F_{\underline{x}}/\underline{x}}.$$

9.9. **Corollary.** *The functor ψ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\underline{x}}$.*

Proof. By [10, Proposition 3.3.1.5], we have to verify that ψ induces a categorical equivalence on the fibers. But after passage to the fiber over an object $e = [x \longrightarrow y]$ in $\underline{x}$, by Lemma 4.8 ψ_e is a functor between two pullback squares in which one leg is a cartesian fibration. Therefore, by Proposition 9.7 and [10, Corollary 3.3.1.4], ψ_e is a categorical equivalence. □

9.10. **Proposition.** *Setup as in Construction 9.8, suppose in addition that $E \longrightarrow D$ is a S-cartesian fibration. Then $\pi_{(\phi,F)} : E^{(\phi,F)/S} \longrightarrow D$ is a S-cartesian fibration.*

Proof. By Lemma 9.4, $\pi_{(\phi,F)}$ is a categorical fibration. By Theorem 9.3, Lemma 9.4, and Lemma 4.8, the functor

$$(\iota_C^*)_s : \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_s \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C, E)_s$$

over D_s satisfies the hypotheses of [10, Proposition 2.4.2.11], hence is a locally cartesian fibration. To then show that $(\iota_C^*)_s$ is a cartesian fibration, it suffices to check that for every square

$$\begin{array}{ccc} \left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] & \longrightarrow & \left[G' : C_{\underline{y}} \star_{\underline{y}} \underline{y} \longrightarrow E_{\underline{y}}\right] \\ \downarrow & & \downarrow \\ \left[H : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] & \longrightarrow & \left[H' : C_{\underline{y}} \star_{\underline{y}} \underline{y} \longrightarrow E_{\underline{y}}\right] \end{array}$$

in $\widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_s$ lying over an edge $e : x \longrightarrow y$ in D_s, if the horizontal edges are cartesian lifts over e and the right vertical edge is $(\iota_C^*)_{s,y}$-cartesian, then the left vertical edge is $(\iota_C^*)_{s,x}$-cartesian. In other words, if we let $e_! : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow C_{\underline{y}} \star_{\underline{y}} \underline{y}$ and $e^* : E_{\underline{y}} \longrightarrow E_{\underline{x}}$ denote choices of pushforward and pullback functors, then we want to show that given $G \simeq e^* \circ G' \circ e_!$, $H \simeq e^* \circ H' \circ e_!$, and $G'|_{\underline{y}} \simeq H'|_{\underline{y}}$, we have that $G|_{\underline{x}} \simeq H|_{\underline{x}}$. But this is clear. We deduce that $(\pi_{(\phi,F)})_s$, being pulled back from $(\iota_C^*)_s$, is a cartesian fibration.

For the final verification, let us abbreviate objects

$$(x \in D, \left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] : G|_{C_{\underline{x}}} = F_{\underline{x}}) \in E^{(\phi,F)/S}$$

as $\left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right]$, the restriction to $C_{\underline{x}}$ equaling $F_{\underline{x}}$ being left implicit. We must check that given a square

$$\begin{array}{ccc} x & \xrightarrow{\widetilde{\alpha}_x} & x' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ y & \xrightarrow{\widetilde{\alpha}_y} & y' \end{array}$$

in D lying over $\alpha : s \longrightarrow t$ with the vertical edges in the fiber and the horizontal edges cocartesian lifts of α, and given a lift of that square to a square

$$\begin{array}{ccc} \left[G : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{x}}\right] & \longrightarrow & \left[G' : C_{\underline{x'}} \star_{\underline{x'}} \underline{x'} \longrightarrow E_{\underline{x'}}\right] \\ \downarrow & & \downarrow \\ \left[H : C_{\underline{y}} \star_{\underline{y}} \underline{y} \longrightarrow E_{\underline{y}}\right] & \longrightarrow & \left[H' : C_{\underline{y'}} \star_{\underline{y'}} \underline{y'} \longrightarrow E_{\underline{y'}}\right] \end{array}$$

in $E^{(\phi,F)/S}$ with the horizontal edges cocartesian lifts of α and the left vertical edge $(\pi_{(\phi,F)})_s$-cartesian, then the right vertical edge is $(\pi_{(\phi,F)})_t$-cartesian. We will once more translate this compatibility statement into a more obvious looking one so as to conclude. Let $e_!, e^*, e'_!, e'^*$ be defined as above. Let $\alpha^* : \underline{x'} \longrightarrow \underline{x}$, $\alpha^* : \underline{y'} \longrightarrow \underline{y}$ be choices of pullback functors (e.g. the first sends a cocartesian edge $f : x' \longrightarrow z$ to $f \circ \widetilde{\alpha}_x : x \longrightarrow z$), and also label related functors by α^*. Then the cocartesianness of the horizontal edges amounts to the equivalences $G' \simeq G \circ \alpha^*$ and $H' \simeq H \circ \alpha^*$, and the cartesianness of the left vertical edge amounts to the equivalence $G|_{\underline{x}} \simeq (e^* \circ H \circ e_!)|_{\underline{x}}$. Our desired assertion now is implied by the homotopy commutativity of the diagram

$$\begin{array}{ccccc} \underline{x'} & \xrightarrow{\alpha^*} & \underline{x} & \xrightarrow{G|_{\underline{x}}} & E_{\underline{x}} \\ \downarrow{\scriptstyle e'_!} & & \downarrow{\scriptstyle e_!} & & \uparrow{\scriptstyle e^*} \\ \underline{y'} & \xrightarrow{\alpha^*} & \underline{y} & \xrightarrow{H|_{\underline{y}}} & E_{\underline{y}} \end{array}$$

(the content being in the commutativity of the first square), for this demonstrates that $G'|_{\underline{x'}} \simeq (e'^* \circ H' \circ e'_!)|_{\underline{x'}}$. ∎

9.11. **Lemma.** *Let $p : W \longrightarrow S$, $q : D \longrightarrow S$ be S-categories and let $\pi : W \longrightarrow D$ be a S-fibration such that for every object $s \in S$, π_s is a cartesian fibration.*

(1) Suppose that:
   (a) For every object $x \in D$, there exists an initial object in W_x.
   (b) For every p-cocartesian edge $w \to w'$ in W, if w is an initial object in $W_{\pi(w)}$, then w' is an initial object in $W_{\pi(w')}$.

   Let $W' \subset W$ be the full simplicial subset of W spanned by those objects $w \in W$ which are initial in $W_{\pi(w)}$ and let $\pi' = \pi|_{W'}$. Then W' is a full S-subcategory of W and π' is a trivial fibration.

(2) Let $\sigma : D \longrightarrow W$ be a S-functor which is a section of π. Then σ is a left adjoint of π relative to D if and only if, for every object $x \in D$, $\sigma(x)$ is an initial object of W_x.

Proof. (1) Condition (b) ensures that W' is a S-subcategory of W. By [10, Proposition 2.4.4.9], for every object $s \in S$, π'_s is a trivial fibration. In particular, π' is S-cocartesian fibration (the compatibility condition being vacuous since all edges in W'_s are π'_s-cocartesian). By Remark 7.4, π' is a cocartesian fibration. As a cocartesian fibration with contractible fibers, π' is a trivial fibration.

(2) Since relative adjunctions are stable under base change, if σ is a left adjoint of π relative to D, passage to the fiber over $x \in D$ shows that $\sigma(x)$ is an initial object of W_x. Conversely, if for all $x \in D$, $\sigma(x)$ is an initial object of W_x, then by [10, Proposition 5.2.4.3], σ_s is left adjoint to π_s for all $s \in S$. Since σ is already given as a S-functor, this implies that σ is S-left adjoint to π; in particular, σ is left adjoint to π. The existence of σ implies the hypotheses of (1), so σ is fully faithful. Now by definition, σ is left adjoint to π relative to D.

∎

We now connect the construction $\widetilde{\mathrm{Fun}}_{D/S}(-,-)$ with $\underline{\mathrm{Fun}}_S(-,-)$. To this end, consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(S)^\sharp \times_S {}^\natural C & & \longrightarrow & & S^\sharp \\ & \searrow^{i} & & & \\ & \mathcal{O}(S)^\sharp \times_S (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}) & \longrightarrow & (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}) & \longrightarrow S^\sharp \\ & \downarrow & & \downarrow & \\ & \mathcal{O}(S)^\sharp \times_S {}^\natural D & \xrightarrow{\mathrm{pr}_D} & {}^\natural D & \\ & \downarrow{\scriptstyle \mathrm{ev}_0} & & & \\ & S^\sharp, & & & \end{array}$$

where the map i is induced by the identity section $D \longrightarrow \mathcal{O}^{cocart}(D)$.

9.12. **Lemma.** *i is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$ (considered over S via $p : C \longrightarrow S$).*

Proof. Define a map $h' : \mathcal{O}(S) \times_S \mathcal{O}^{cocart}(D) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{O}(S) \times_S \mathcal{O}^{cocart}(D))$ to be the product of the following three maps:

(1) Choose a lift σ

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^{\{0,1\}}, S) & \xrightarrow{s_1} & \mathrm{Fun}(\Delta^2, S) \\ \downarrow & \overset{\sigma}{\nearrow} & \downarrow \sim \\ \mathrm{Fun}(\Lambda^2_1, S) & \xrightarrow{=} & \mathrm{Fun}(\Lambda^2_1, S) \end{array}$$

and let $\Delta^1 \times \Delta^1 \longrightarrow \Delta^2$ be the unique map so that the induced map $\mathrm{Fun}(\Delta^2, S) \longrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, S) \cong \mathrm{Fun}(\Delta^1, \mathcal{O}(S))$ sends $(s \longrightarrow t \longrightarrow u)$ to $[s \longrightarrow t] \longrightarrow [s \longrightarrow u]$. Use these two maps to define

$$\mathcal{O}(S) \times_S \mathcal{O}^{cocart}(D) \times_D C \longrightarrow \mathcal{O}(S) \times_S \mathcal{O}(S) \cong \mathrm{Fun}(\Lambda^2_1, S) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{O}(S)).$$

(2) Use the unique map $\Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ which sends $(0,0)$ to 0 and all other vertices to 1 to define

$$\mathcal{O}(S) \times_S \mathcal{O}^{cocart}(D) \times_D C \longrightarrow \mathcal{O}^{cocart}(D) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{O}^{cocart}(D)).$$

(3) The degeneracy map $s_0 : C \longrightarrow \mathrm{Fun}(\Delta^1, C)$ defines

$$\mathcal{O}(S) \times_S \mathcal{O}^{cocart}(D) \times_D C \longrightarrow C \longrightarrow \mathrm{Fun}(\Delta^1, C).$$

Then h' is adjoint to a map of marked simplicial sets over S

$$h : (\Delta^1)^\sharp \times \mathcal{O}(S)^\sharp \times_S (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}) \longrightarrow \mathcal{O}(S)^\sharp \times_S (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E})$$

such that $h_0 = \mathrm{id}$ and h_1 factors as a composition

$$\mathcal{O}(S)^\sharp \times_S (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}) \xrightarrow{r} \mathcal{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{i} \mathcal{O}(S)^\sharp \times_S (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E})$$

where r is defined by

$$\mathcal{O}(S)^\sharp \times_S (\mathcal{O}^{cocart}(D) \times_D C, \mathcal{E}) \longrightarrow \mathrm{Fun}(\Lambda^2_1, S)^\sharp \times_S {}_\natural C \xrightarrow{d_1 \circ \sigma} \mathcal{O}(S)^\sharp \times_S {}_\natural C.$$

Our choice of σ ensures that $r \circ i = \mathrm{id}$, completing the proof. □

Note that for any S-fibration $\pi : X \longrightarrow D$, the S-category $\underline{\mathbf{Sect}}_{D/S}(\pi)$ defined in 8.6 may be identified with $(\mathrm{ev}_0)_*(\mathrm{pr}_D)^*({}_\natural X \xrightarrow{\pi} {}_\natural D)$. Combining Lemma 9.12, Lemma 2.27, and Lemma 2.28, we see that if E is a S-category and $C \longrightarrow D$ is S-cocartesian or S-cartesian, then the map induced by i

$$i^* : \underline{\mathbf{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C, E \times_S D)) \longrightarrow \underline{\mathrm{Fun}}_S(C, E)$$

is a equivalence of S-categories. Moreover, a chase of the definitions reveals that for every S-functor $F : C \longrightarrow E$, we have an identification

$$i^* \circ \underline{\mathbf{Sect}}_{D/S}(\tau_{F \times \phi}) = \sigma_F : S \longrightarrow \underline{\mathrm{Fun}}_S(C, E).$$

We thus have a morphism of spans

$$\begin{array}{ccccc} S & \xrightarrow{\underline{\mathbf{Sect}}_{D/S}(\tau_{F\times\phi})} & \underline{\mathbf{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C, E \times_S D)) & \longleftarrow & \underline{\mathbf{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E \times_S D)) \\ \downarrow = & & \downarrow \simeq & & \downarrow \simeq \\ S & \xrightarrow{\sigma_F} & \underline{\mathrm{Fun}}_S(C, E) & \longleftarrow & \underline{\mathrm{Fun}}_S(C \star_D D, E). \end{array}$$

The right horizontal maps are S-fibrations by Lemma 9.4 and [3, Proposition 9.11(2)], so taking pullbacks yields an equivalence

$$\underline{\mathbf{Sect}}_{D/S}((E \times_S D)^{(\phi, F\times\phi)/S}) \xrightarrow{\simeq} S \times_{\sigma_F, \underline{\mathrm{Fun}}_S(C,E)} \underline{\mathrm{Fun}}_S(C \star_D D, E). \tag{9.12.1}$$

We are now prepared to introduce the main definition of this section.

9.13. **Definition.** Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration. A S-functor $\overline{F} : C \star_D D \longrightarrow E$ is a *D-parametrized S-colimit diagram* if for every object $x \in D$, the $\underline{x}$-functor $\overline{F}|_{C_{\underline{x}} \star_{\underline{x}} \underline{x}} : C_{\underline{x}} \star_{\underline{x}} \underline{x} \longrightarrow E_{\underline{s}}$ is a $\underline{s}$-colimit diagram.

9.14. **Proposition.** *Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration, let $F : C \longrightarrow E$ be a S-functor, and let $\overline{F} : C \star_D D \longrightarrow E$ be a D-parametrized S-colimit diagram extending F. Then the section*

$$\mathrm{id}_S \times \sigma_{\overline{F}} : S \longrightarrow S \times_{\sigma_F, \underline{\mathrm{Fun}}_S(C,E)} \underline{\mathrm{Fun}}_S(C \star_D D, E)$$

is a S-initial object.

Proof. Combine Equation 9.12.1, Lemma 9.11(2), and Corollary 8.7. □

We have the following existence and uniqueness result for D-parametrized S-colimits.

9.15. **Theorem.** *Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration and let $F : C \longrightarrow E$ be a S-functor. Suppose that for every object $x \in D$, the $\underline{s}$-functor $F|_{C_{\underline{x}}} : C_{\underline{x}} \longrightarrow E_{\underline{s}}$ admits a $\underline{s}$-colimit. Then there exists a D-parametrized S-colimit diagram $\overline{F} : C \star_D D \longrightarrow E$ extending F. Moreover, the full subcategory of $\{F\} \times_{\mathrm{Fun}_S(C,E)} \mathrm{Fun}_S(C \star_D D, E)$ spanned by the D-parametrized S-colimit diagrams coincides with that spanned by the initial objects.*

Proof. By Proposition 9.10 and Corollary 9.9, the functor

$$\pi_{(\phi, F\times\phi)} : (E \times_S D)^{(\phi, F\times\phi)/S} \longrightarrow D$$

is a S-cartesian fibration with $\underline{x}$-fibers equivalent to $(E_{\underline{s}})^{(F|_{C_{\underline{x}}}, \underline{s})/}$. Our hypothesis ensures that the conditions of Lemma 9.11(1) are satisfied, so $\pi_{(\phi,F\times\phi)}$ admits a section σ which is a S-functor that selects an initial object in each fiber. The resulting S-functor $D \longrightarrow \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E \times_S D)$ covering $\tau_{F\times\phi}$ is adjoint to a S-functor $\overline{F} : C \star_D D \longrightarrow E$ extending F, which is a D-parametrized S-colimit diagram. Having proven existence, the second statement now follows from Proposition 9.14. □

Theorem 9.15 also admits the following 'global' consequence.

9.16. **Corollary.** *Let $\phi : C \longrightarrow D$ be a S-cocartesian fibration and E be an S-category. Suppose that for every $s \in S$ and $x \in D_s$, $E_{\underline{s}}$ admits all $S^{s/}$-colimits of shape $C_{\underline{x}}$. Then $U : \underline{\mathrm{Fun}}_S(C \star_D D, E) \longrightarrow \underline{\mathrm{Fun}}_S(C, E)$ admits a left S-adjoint L which is a section of U such that for every object $F : C_{\underline{s}} \longrightarrow E_{\underline{s}}$, $L(F)$ is a $D_{\underline{s}}$-parametrized $S^{s/}$-colimit diagram.*

Proof. By Example 7.10, Theorem 9.15 and the stability of parametrized colimit diagrams under base change, the conditions of Lemma 9.11(1) are satisfied for U. Thus U admits a section L which selects an initial object in each fiber, necessarily a parametrized colimit diagram. By Lemma 9.11(2), L is a left adjoint of U relative to $\underline{\mathrm{Fun}}_S(C, E)$; in particular, L is S-left adjoint to U. □

Application: Functor categories.

9.17. **Proposition.** *Let K, I, and C be S-categories.*

1. *Suppose that for all $s \in S$, $C_{\underline{s}}$ admits all $K_{\underline{s}}$-indexed colimits. $\overline{p} : K \star_S S \longrightarrow \underline{\mathrm{Fun}}_S(I, C)$ is a S-colimit diagram if and only if, for every object $x \in I$ over s,*

$$K_{\underline{s}} \star_{\underline{s}} \underline{s} \xrightarrow{\overline{p}_{\underline{s}}} \underline{\mathrm{Fun}}_{\underline{s}}(I_{\underline{s}}, C_{\underline{s}}) \xrightarrow{\mathrm{ev}_x} C_{\underline{s}}$$

   is a $S^{s/}$-colimit diagram.
2. *A S-functor $p : K \longrightarrow \underline{\mathrm{Fun}}_S(I, C)$ admits an extension to a S-colimit diagram $\overline{p}$ if for all $x \in I$, $\mathrm{ev}_x \circ p_{\underline{s}}$ admits an extension to a $S^{s/}$-colimit diagram.*

Proof. We prove (1), the proof for (2) being similar. Let

$$\overline{p'} : (K \times_S I) \star_I I \cong (K \star_S S) \times_S I \longrightarrow C$$

be a choice of adjoint of p under the equivalence

$$\mathrm{Fun}_S(K \star_S S, \underline{\mathrm{Fun}}_S(I, C)) \simeq \mathrm{Fun}_S((K \star_S S) \times_S I, C).$$

By Theorem 9.15 applied to the S-cocartesian fibration $K \times_S I \longrightarrow I$ and the hypothesis on C, there exists an I-parametrized S-colimit diagram p'' extending $p' = \overline{p'}|_{K \times_S I}$. By Proposition 9.14, p'' defines an S-initial object in

$$S \times_{\underline{\mathrm{Fun}}_S(K \times_S I, C)} \underline{\mathrm{Fun}}_S((K \times_S I) \star_I I, C) \simeq \underline{\mathrm{Fun}}_S(I, C)^{(p,S)/}$$

so *its* adjoint is a S-colimit diagram. For the 'if' direction, supposing that $\overline{p}$ is a S-colimit diagram, then by the uniqueness of S-initial objects, p'' is equivalent to $\overline{p'}$. Then $\mathrm{ev}_x \circ \overline{p}_{\underline{s}}$ is equivalent to $p''_{\underline{x}}$, which is a $S^{s/}$-colimit diagram by definition of I-parametrized S-colimit diagram. For the 'only if' direction, supposing that all the $\mathrm{ev}_x \, \overline{p}_{\underline{s}}$ are $S^{s/}$-colimit diagrams, we get that $\overline{p'}$ is a I-parametrized S-colimit diagram, so is equivalent to p''. □

9.18. **Corollary.** *Suppose C is S-cocomplete and I is a S-category. Then $\underline{\mathrm{Fun}}_S(I, C)$ is S-cocomplete.*

10. Kan extensions

We now combine the theory of S-colimits parametrized by a base S-category D and that of free S-cocartesian fibrations to establish the theory of left S-Kan extensions.

10.1. **Definition.** Suppose a diagram of S-categories

$$\begin{array}{ccc} C & \xrightarrow{\quad F \quad} & E \\ {\scriptstyle \phi}\big\downarrow & {\scriptstyle \eta}\Downarrow \nearrow_{G} & \\ D & & \end{array}$$

where by the '2-cell' η we mean exactly the datum of a S-functor $\eta : C \times \Delta^1 \longrightarrow E$ restricting to F on 0 and $G \circ \phi$ on 1. Let

$$G' : (C \times_D \mathcal{O}_S(D)) \star_D D \xrightarrow{\pi_D} D \xrightarrow{G} E,$$

let

$$\theta : (C \times_D \mathcal{O}_S(D)) \times \Delta^1 \longrightarrow E$$

be the natural transformation adjoint to $G_* : C \times_D \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(E)$, let

$$\eta' : (C \times_D \mathcal{O}_S(D)) \times \Delta^1 \longrightarrow C \times \Delta^1 \xrightarrow{\eta} E$$

be the natural transformation obtained from η, and let $\theta' = \theta \circ \eta'$ be a choice of composition in $\mathrm{Fun}_S(C \times_D \mathcal{O}_S(D), E)$. Let

$$r : \mathrm{Fun}_S((C \times_D \mathcal{O}_S(D)) \star_D D, E) \longrightarrow \mathrm{Fun}_S(C \times_D \mathcal{O}_S(D), E)$$

denote the restriction functor. By Lemma 4.8, we may select a r-cartesian edge e in $\mathrm{Fun}_S((C \times_D \mathcal{O}_S(D)) \star_D D, E)$ with $d_0(e) = G'$ covering θ', chosen so that $e|_D$ is degenerate. Let $G'' = d_1(e)$.

We say that G is a *left S-Kan extension* of F along ϕ if G'' is a D-parametrized S-colimit diagram.

10.2. **Remark.** The following are equivalent:

(1) G is a left S-Kan extension of F along ϕ.
(2) For all $s \in S$, $G_{\underline{s}}$ is a left $S^{s/}$-Kan extension of $F_{\underline{s}}$ along $\phi_{\underline{s}}$.
(3) For all $s \in S$ and $x \in D_s$, $G|_{\underline{x}} : \underline{x} \longrightarrow E_{\underline{s}}$ is a left $S^{s/}$-Kan extension of $F|_{C_{\underline{x}}} : C_{\underline{x}} \longrightarrow E_{\underline{s}}$ along $\phi_{\underline{x}} : C_{\underline{x}} \longrightarrow \underline{x}$.

In other words, our notion of S-Kan extension generalizes the concept of *pointwise* Kan extensions.

We can bootstrap Theorem 9.15 to prove existence and uniqueness of left S-Kan extensions.

10.3. **Theorem.** *Let $\phi : C \longrightarrow D$ and $F : C \longrightarrow E$ be S-functors. Suppose that for every object $x \in D$, the $S^{s/}$-functor*

$$C \times_D D^{/\underline{x}} \longrightarrow C_{\underline{s}} \xrightarrow{F_{\underline{s}}} E_{\underline{s}}$$

admits a $S^{s/}$-colimit. Then there exists a left S-Kan extension $G : D \longrightarrow E$ of F along ϕ, uniquely specified up to contractible choice.

Proof. We spell out the details of existence and leave the proof of uniqueness to the reader. By Theorem 9.15, there exists a D-parametrized S-colimit diagram

$$\overline{F} : (C \times_D \mathcal{O}_S(D)) \star_D D \longrightarrow E$$

extending $C \times_D \mathcal{O}_S(D) \longrightarrow C \xrightarrow{F} E$. Let $G = \overline{F}|_D$. Define a map

$$h : C \times \Delta^1 \longrightarrow (C \times_D \mathcal{O}_S(D)) \star_D D$$

over $D \times \Delta^1$ as adjoint to $(C \xrightarrow{(\mathrm{id}, \iota\phi)} C \times_D \mathcal{O}_S(D), C \xrightarrow{\phi} D)$ and let $\eta = \overline{F} \circ h$, so that η is a natural transformation from F to $G \circ \phi$.

We claim that η exhibits G as a left Kan extension of F along ϕ. To show this, we will exhibit a r-cartesian edge e from $\overline{F}$ to G' such that the restriction $r(e)$ of e to $C \times_D \mathcal{O}_S(D)$ is a choice of composition $\theta \circ \eta'$. Define

$$e' : (C \times_D \mathcal{O}_S(D)) \star_D D \times \Delta^1 \longrightarrow (C \times_D \mathcal{O}_S(D)) \star_D D$$

over $D \times \Delta^1$ as adjoint to (id, π_D), and let $e = \overline{F} \circ e'$, so that e is an edge from $\overline{F}$ to G'. Since $(\pi_D)|_D = \mathrm{id}_D$, $e|_D$ is a degenerate edge in $\mathrm{Fun}_S(D, E)$, so e is r-cartesian.

To finish the proof, we need to introduce a few more maps. Define

$$\alpha = (\mathrm{pr}_C, \alpha') : C \times_D \mathcal{O}_S(D) \times \Delta^1 \longrightarrow C \times_D \mathcal{O}_S(D)$$

where α' is adjoint to

$$C \times_D \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D) = \widetilde{\mathrm{Fun}}_S(S \times \Delta^1, D) \xrightarrow{\min^*} \widetilde{\mathrm{Fun}}_S(S \times \Delta^1 \times \Delta^1, D).$$

Here $\min : \Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ is the functor which takes the minimum. Define

$$\beta : C \times_D \mathcal{O}_S(D) \times \Delta^1 \longrightarrow \mathcal{O}_S(D) \times \Delta^1 \xrightarrow{\mathrm{ev}} D.$$

Use α and β to define

$$\gamma : C \times_D \mathcal{O}_S(D) \times \Delta^1 \times \Delta^1 \longrightarrow (C \times_D \mathcal{O}_S(D)) \star_D D$$

so that on objects $(c, \phi c \xrightarrow{f} d)$, γ sends $\Delta^1 \times \Delta^1$ to the square

$$\begin{array}{ccc} (c, \phi c = \phi c) & \longrightarrow & \phi c \\ \downarrow{\scriptstyle (\mathrm{id}, f)} & & \downarrow{\scriptstyle f} \\ (c, \phi c \xrightarrow{f} d) & \longrightarrow & d. \end{array}$$

Then $\overline{F} \circ \gamma$ defines a square

$$\begin{array}{ccc} F \circ \mathrm{pr}_C & \xrightarrow{\eta'} & G \circ \phi \circ \mathrm{pr}_C \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \theta} \\ F \circ \mathrm{pr}_C & \xrightarrow{r(e)} & G'. \end{array}$$

in $\mathrm{Fun}_S(C \times_D \mathcal{O}_S(D), E)$, which proves that $r(e) \simeq \theta \circ \eta'$. □

We also have the Kan extension counterpart to Corollary 9.16.

10.4. **Definition.** Let $\phi : C \longrightarrow D$ be a S-functor and E a S-category. We say that E *admits the relevant S-colimits for ϕ* if for every $s \in S$ and $x \in D_s$, $E_{\underline{s}}$ admits all $S^{s/}$-colimits of shape $C \times_D D^{/\underline{x}}$.

10.5. **Theorem.** *Let $\phi : C \longrightarrow D$ be a S-functor and E a S-category. Suppose that E admits the relevant S-colimits for ϕ. Then the S-functor*

$$\phi^* : \underline{\mathrm{Fun}}_S(D, E) \longrightarrow \underline{\mathrm{Fun}}_S(C, E)$$

given by restriction along ϕ admits a left S-adjoint $\phi_!$ such that for every S-functor $F : C \longrightarrow E$, the unit map $F \longrightarrow \phi^* \phi_! F$ exhibits $\phi_! F$ as a left S-Kan extension of F along ϕ.

Proof. Factor ϕ as the composition

$$C \xrightarrow{\iota_C} C \times_D \mathcal{O}_S(D) \xrightarrow{i} (C \times_D \mathcal{O}_S(D)) \star_D D \xrightarrow{\pi_D} D.$$

Then ϕ^* factors as the composition

$$\underline{\mathrm{Fun}}_S(D, E) \xrightarrow{\pi_D^*} \underline{\mathrm{Fun}}_S((C \times_D \mathcal{O}_S(D)) \star_D D, E) \xrightarrow{i^*} \underline{\mathrm{Fun}}_S(C \times_D \mathcal{O}_S(D), E) \xrightarrow{\iota_C^*} \underline{\mathrm{Fun}}_S(C, E).$$

By Proposition 8.10 and Corollary 8.5, pr_C^* is left S-adjoint to ι_C^*. Since i_D is right S-adjoint to π_D, by Corollary 8.5 again i_D^* is left S-adjoint to π_D^*. By Corollary 9.16, i^* admits a left S-adjoint L which extends functors to D-parametrized S-colimit diagrams. Let $\phi_!$ be the composite of these three functors. The proof of Theorem 10.3 shows that $\phi_!(F)$ is as asserted. □

The next proposition permits us to eliminate the datum of the natural transformation η from the definition of a left S-Kan extension when ϕ is fully faithful.

10.6. **Proposition.** *Suppose $\phi : C \longrightarrow D$ is the inclusion of a full S-subcategory. Then for any left S-Kan extension G of $F : C \longrightarrow E$ along ϕ, η is a natural transformation through equivalences. Consequently, G is homotopic to a functor $\overline{F} : D \longrightarrow E$ which is both an extension of F and a left S-Kan extension (with the natural transformation $F \longrightarrow \overline{F} \circ \phi = F$ chosen to be the identity).*

Proof. Let $G'' : (C \times_D \mathcal{O}_S(D)) \star_D D \longrightarrow E$ be as in the definition of a left S-Kan extension. Because D-parametrized S-colimit diagrams are stable under restriction to S-subcategories,

$$(G'')_C : (C \times_D \mathcal{O}_S(D) \times_D C) \star_C C \longrightarrow E$$

is a C-parametrized S-colimit diagram. The additional assumption that C is a full S-subcategory has the consequence that $(C \times_D \mathcal{O}_S(D) \times_D C) \cong \mathcal{O}_S(C)$. Also, for any object $x \in C$, the inclusion $\underline{x}$-functor $i_x : \underline{x} \longrightarrow C^{/\underline{x}}$ is $\underline{x}$-final, using the first criterion of Theorem 6.7. Therefore, $\mathcal{O}_S(C) \star_C C \xrightarrow{\pi_C} C \xrightarrow{F} E$ is a C-parametrized S-colimit diagram extending $\mathcal{O}_S(C) \xrightarrow{\mathrm{ev}_0} C \xrightarrow{F} E$, so $(G'')_C \simeq F \circ \pi_C$.

The map h in the proof of Theorem 10.3 factors as

$$C \times \Delta^1 \xrightarrow{h'} \mathcal{O}_S(C) \star_C C \longrightarrow (C \times_D \mathcal{O}_S(D)) \star_D D.$$

We have the chain of equivalences

$$\eta \simeq G'' \circ h \simeq F \circ \pi_C \circ h' = F \circ \mathrm{pr}_C,$$

proving the first assertion. For the second assertion, use that

$$(\natural D \times \{1\}) \cup_{\natural C \times \{1\}} (\natural C \times (\Delta^1)^\sharp) \longrightarrow \natural D \times (\Delta^1)^\sharp$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ to extend (G, η) to a homotopy between G and an extension $\overline{F}$, which is then necessarily a left S-Kan extension. □

10.7. **Corollary.** *Suppose $\phi : C \longrightarrow D$ a fully faithful S-functor and E a S-cocomplete S-category. Then the left S-adjoint $\phi_!$ to the restriction S-functor ϕ^* exists and is fully faithful.*

Proof. Combine Theorem 10.5 and Proposition 10.6. □

As expected, S-colimit diagrams are examples of S-left Kan extensions.

10.8. **Proposition.** *Suppose $\phi : C \longrightarrow D$ a S-cocartesian fibration and $\overline{F} : C \star_D D \longrightarrow E$ a S-functor extending $F : C \longrightarrow E$. Then $\overline{F}$ is a D-parametrized S-colimit diagram if and only if $\overline{F}$ is a S-left Kan extension of F.*

Proof. We may check the assertion objectwise on D, so let $x \in D_s$. Consider the commutative diagram

$$\begin{array}{ccc} C_{\underline{x}} & \hookrightarrow & C_{\underline{s}} \\ {\scriptstyle\theta}\downarrow & \nearrow{\scriptstyle \mathrm{pr}_C} & \downarrow{\scriptstyle F_{\underline{s}}} \\ C \times_{C \star_D D} (C \star_D D)^{/\underline{x}} & \longrightarrow & E_{\underline{s}}. \end{array}$$

The value of a D-parametrized colimit of F on x is computed as the $S^{s/}$-colimit of $(F_{\underline{s}})|_{C_{\underline{x}}}$, and that of a S-left Kan extension of F as the $S^{s/}$-colimit of $F_{\underline{s}} \circ \mathrm{pr}_C$. Therefore, it suffices to prove that θ is

$\underline{x}$-final. Let $f : x \to y$ be an object in $\underline{x}$, i.e. a cocartesian edge in D, which lies over $s \longrightarrow t$. Then θ_f is equivalent to the inclusion

$$C_y \cong C_y \times_{(C_y)^{\rhd}} ((C_y)^{\rhd})^{/\{\infty\}} \longrightarrow C_t \times_{C_t \star_{D_t} D_t} (C_t \star_{D_t} D_t)^{/y}.$$

Applying Lemma 10.9 to the map $C_t \longrightarrow C_t \star_{D_t} D_t$ of cocartesian fibrations over D_t, we deduce that θ_f is final. □

10.9. **Lemma.** *Let $X \longrightarrow Y$ be a map of cocartesian fibrations over Z and let $y \in Y$ be an object over $z \in S$. Then the inclusion $X_z \times_{Y_z} Y_z^{/y} \longrightarrow X \times_Y Y^{/y}$ is final.*

Proof. By the dual of [12, Lemma 3.4.1.10], $X \times_Y Y^{/y} \longrightarrow Z^{/z}$ is a cocartesian fibration. We have a pullback square

$$\begin{array}{ccc} X_z \times_{Y_z} Y_z^{/y} & \longrightarrow & X \times_Y Y^{/y} \\ \big\downarrow & & \big\downarrow \\ \{z\} & \xrightarrow{\mathrm{id}_z} & Z^{/z}. \end{array}$$

Since the bottom horizontal map is final and cocartesian fibrations are smooth, the top horizontal map is final. □

As with S-colimits, S-left Kan extensions reduce to the usual notion of left Kan extension when taken in a S-category of objects.

10.10. **Proposition.** *Suppose a diagram of S-categories*

$$\begin{array}{ccc} C & \xrightarrow{F} & \underline{E}_S \\ {\scriptstyle\phi}\big\downarrow & \overset{\eta}{\Downarrow} \nearrow_{G} & \\ D & & \end{array}.$$

The following are equivalent:

1. *(1) G is a left S-Kan extension of F along ϕ.*
2. *(2) $G^\dagger$ is a left Kan extension of $F^\dagger$ along ϕ.*
3. *(3) For all objects $s \in S$, $G^\dagger|_{D_s}$ is a left Kan extension of $F^\dagger|_{C_s}$ along ϕ_s.*

Proof. We first prove that (1) and (2) are equivalent. Factor $\phi : C \longrightarrow D$ through the free S-cocartesian fibration on ϕ:

$$\phi : C \xrightarrow{\iota_C} C \times_D \mathcal{O}_S D \xrightarrow{\mathrm{Fr}^{cocart}(\phi)} D.$$

Since ι_C is S-left adjoint to pr_C, it is also left adjoint. Therefore, the S-left Kan extension resp. the left Kan extension of F resp. $F^\dagger$ along ι_C is computed by $F \circ \mathrm{pr}_C$ resp. $F^\dagger \circ \mathrm{pr}_C$. By transitivity of Kan extensions, we thereby reduce to the case that ϕ is S-cocartesian. The claim now follows easily by combining Proposition 5.5 and Proposition 10.8.

We next prove that (2) and (3) are equivalent. For this, it suffices to observe that for all objects $d \in D$ over some $s \in S$, $C_s \times_{D_s} D_s^{/d} \longrightarrow C \times_D D^{/d}$ is final by Lemma 10.9 applied to $C \longrightarrow D$. □

11. Yoneda lemma

By Proposition 5.5, $\underline{\mathbf{Spc}}_S$ is S-cocomplete, so by Corollary 9.18, the S-category of presheaves

$$\mathbf{P}_S(C) := \underline{\mathrm{Fun}}_S(C^{vop}, \underline{\mathbf{Spc}}_S)$$

is S-cocomplete. The S-Yoneda embedding $j : C \longrightarrow \mathbf{P}_S(C)$ was constructed in [3, §10] via S-straightening the left fibration $\widetilde{\mathcal{O}}_S(C) \longrightarrow C^{vop} \times_S C$ given fiberwise by twisted arrows. It was shown there that j is fully faithful [2, Theorem 10.4]. In this section, we first prove the S-Yoneda lemma and then establish the universal property of $\mathbf{P}_S(C)$ as the free S-cocompletion of C.

11.1. **Lemma** (S-Yoneda lemma)**.** *Let $j : C \longrightarrow \mathbf{P}_S(C)$ denote the S-Yoneda embedding. Then the identity on $\mathbf{P}_S(C)$ is a S-left Kan extension of j along itself.*

Proof. By Proposition 9.17, it suffices to show that for every $s \in S$ and object $x \in C_s$, $\mathrm{ev}_x : \mathbf{P}_{\underline{s}}(C_{\underline{s}}) \longrightarrow \underline{\mathbf{Spc}}_{\underline{s}}$ is a $S^{s/}$-left Kan extension of $\mathrm{ev}_x\, j_{\underline{s}}$. To ease notation, let us replace $S^{s/}$ by S and suppose that $s \in S$ is an initial object.

We claim that $(\mathrm{ev}_x\, j)^\dagger : C \longrightarrow \mathbf{Spc}$ is homotopic to $\mathrm{Map}_C(x, -)$. By definition of the S-Yoneda embedding, $(\mathrm{ev}_x\, j)^\dagger$ classifies the left fibration

$$\mathrm{ev}_1 : \widetilde{\mathcal{O}}_S(C)_{x\to} \longrightarrow C$$

pulled back from $\widetilde{\mathcal{O}}_S(C) \longrightarrow C^{vop} \times_S C$ via the cocartesian section $\sigma : S \longrightarrow C^{vop}$ defined by $\sigma(s) = x$. By [10, Proposition 4.4.4.5], it suffices to show that id_x is an initial object in $\widetilde{\mathcal{O}}_S(C)_{x\to}$. For this, because $s \in S$ is an initial object we reduce to checking that for all edges $\alpha : s \to t$, the pushforward of id_x by α is an initial object in the fiber $(\widetilde{\mathcal{O}}_S(C)_{x\to})_t$. But this fiber is equivalent to $\widetilde{\mathcal{O}}(C_t)_{\alpha_! x\to} \simeq (C_t)^{\alpha_! x/}$.

Applying Proposition 10.10, we reduce to showing that for all $t \in S$, $(\mathrm{ev}_x)^\dagger|_{\mathbf{P}_S(C)_t}$ is a left Kan extension of $(\mathrm{ev}_x\, j)^\dagger|_{C_t}$. Note that for y any cocartesian pushforward of x over the essentially unique edge $s \to t$, we have both that $(\mathrm{ev}_x\, j)^\dagger|_{C_t}$ is homotopic to $\mathrm{Map}_{C_t}(y, -)$ and $(\mathrm{ev}_x)^\dagger|_{\mathbf{P}_S(C)_t}$ is homotopic to ev_y (regarding y as an object in $C_{\underline{t}}^{vop}$). The inclusion

$$C_t \longrightarrow \mathbf{P}_S(C)_t \simeq \mathrm{Fun}(C_{\underline{t}}^{vop}, \mathbf{Spc})$$

factors through $\mathbf{P}(C_t)$ with $\mathbf{P}(C_t) \longrightarrow \mathrm{Fun}(C_{\underline{t}}^{vop}, \mathbf{Spc})$ left adjoint to precomposition by the inclusion $i : C_t^{op} \longrightarrow C_{\underline{t}}^{vop}$. By the usual Yoneda lemma for ∞-categories, $\mathrm{ev}_y : \mathbf{P}(C_t) \longrightarrow \mathbf{Spc}$ is the left Kan extension of $\mathrm{Map}_{C_t}(y, -)$. The left Kan extension of ev_y to $\mathbf{P}_S(C)_t$ is then given by precomposition by i, so is again ev_y. □

To state the universal property of $\mathbf{P}_S(C)$, we need to introduce a bit of terminology.

11.2. **Definition.** Let $F : C \longrightarrow D$ be a S-functor. We say that F *strongly preserves S-(co)limits* if for all $s \in S$, $F_{\underline{s}}$ preserves $S^{s/}$-(co)limits.

11.3. **Remark.** If F strongly preserves S-colimits then F preserves S-colimits. However, the converse is not necessarily true.

11.4. **Notation.** Suppose that C and D are S-cocomplete S-categories. Let $\mathrm{Fun}^L_S(C, D)$ denote the full subcategory of $\mathrm{Fun}_S(C, D)$ on the S-functors F which strongly preserve S-colimits. Let $\underline{\mathrm{Fun}}^L_S(C, D)$ denote the full S-subcategory of $\underline{\mathrm{Fun}}_S(C, D)$ with fibers $\mathrm{Fun}^L_{S^{s/}}(C, D)$ over $s \in S$.

11.5. **Theorem.** *Let E be a S-cocomplete S-category. Then restriction along the S-Yoneda embedding defines equivalences*

$$\mathrm{Fun}^L_S(\mathbf{P}_S(C), E) \xrightarrow{\simeq} \mathrm{Fun}_S(C, E)$$
$$\underline{\mathrm{Fun}}^L_S(\mathbf{P}_S(C), E) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_S(C, E)$$

with the inverse given by S-left Kan extension.

We prepare for the proof of Theorem 11.5 with some necessary results concerning S-mapping spaces. Recall that given an ∞-category C, we have a number of equivalent options for describing mapping spaces in C. The relevant ones to consider for us are:

(1) Straightening the left fibration $\widetilde{\mathcal{O}}(C) \longrightarrow C^{op} \times C$, we obtain the mapping space functor
$$\mathrm{Map}_C(-, -) : C^{op} \times C \longrightarrow \mathbf{Spc};$$
(2) Fixing an object $x \in C$, straightening the left fibration $C^{x/} \longrightarrow C$ also yields the functor
$$\mathrm{Map}_C(x, -) : C \longrightarrow \mathbf{Spc};$$
(3) Fixing objects $x, y \in C$, we have that the space $\mathrm{Map}_C(x, y)$ is given by
$$\{x\} \times_C \mathcal{O}(C) \times_C \{y\}.$$

Likewise, given a S-category C, we have these possibilities:

(1) The S-functor
$$\underline{\mathrm{Map}}_C(-,-) : C^{vop} \times_S C \longrightarrow \underline{\mathbf{Spc}}_S$$
given by the S-straightening of $\widetilde{\mathcal{O}}_S(C) \longrightarrow C^{vop} \times_S C$;

(2) Fixing an object $x \in C$, we have the left fibration $C^{\underline{x}/} = \underline{x} \times_C \mathcal{O}_S(C) \longrightarrow C$, which S-straightens to
$$\underline{\mathrm{Map}}_C(x,-) : C \longrightarrow \underline{\mathbf{Spc}}_S;$$

(3) Fixing an object $x \in C$, we have the left fibration $C^{x/} \longrightarrow C$, which S-straightens to
$$\underline{\mathrm{Map}}_C(x,-) : C \longrightarrow \underline{\mathbf{Spc}}_S;$$

(4) Fixing objects $x \in C$ and $y \in C_s$, we have the $S^{s/}$-space
$$\underline{\mathrm{Map}}_C(x,y) = \underline{x} \times_C \mathcal{O}_S(C) \times_C \underline{y} \longrightarrow \underline{y} \xrightarrow{\simeq} S^{s/}.$$

In the proof of Lemma 11.1, we showed that (1) and (3) were equivalent, and by Proposition 4.31, (2) and (3) are equivalent. Finally, (2) specializes to (4) by definition. We are thus justified in our abuse of notation when we interchangeably refer to any of these options by $\underline{\mathrm{Map}}_C(-,-)$.

Our next goal is to prove that $\underline{\mathrm{Map}}_C(-,-)$ preserves S-limits in the second variable, and dually, takes S-colimits in the first variable to S-limits. For this, we need a few lemmas.

11.6. **Lemma.** *Let $F : X \longrightarrow Y$ be a map of S-cocartesian or S-cartesian fibrations over an S-category C. The following are equivalent:*

(1) F is an equivalence.

(2) For all $s \in S$ and $S^{s/}$-functors $Z \longrightarrow C_{\underline{s}}$,
$$\underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(Z, X_{\underline{s}}) \longrightarrow \underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(Z, Y_{\underline{s}})$$
is an equivalence.

(3) For all $s \in S$ and $c \in C_s$,
$$\underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(\underline{c}, X_{\underline{s}}) \longrightarrow \underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(\underline{c}, Y_{\underline{s}})$$
is an equivalence.

(4) For all $c \in C$, $F_c : X_c \longrightarrow Y_c$ is an equivalence.

If X and Y are S-left or S-right fibrations over C, then all instances of $\underline{\mathrm{Fun}}$ can be replaced by $\underline{\mathrm{Map}}$.[19]

Proof. (1) ⇒ (2): If F is an equivalence, so is $F_{\underline{s}}$ for all $s \in S$. The map in question is then induced by a map of pullbacks through equivalences in which two matching legs are S-fibrations, so is an equivalence.

(2) ⇒ (3) is obvious.

(3) ⇒ (4): Given $c \in C_s$, take fibers over $\{s\} \in \underline{s}$ and note that
$$\underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(\underline{c}, X_{\underline{s}})_s \simeq \mathrm{Fun}_{/C_c}(\{c\}, X_s) \simeq X_c.$$

(4) ⇒ (1): We must check that F_s is an equivalence for all $s \in S$, for which it suffices to check fiberwise over C_s by the hypothesis. □

11.7. **Lemma.** *Let $\overline{q} : S \star_S K \longrightarrow \underline{\mathbf{Spc}}_S$ be a S-functor which extends $q : K \longrightarrow \underline{\mathbf{Spc}}_S$. Let $\overline{X} \longrightarrow S \star_S K$ be a left fibration which is an unstraightening of $\overline{q}^\dagger$, and let $X = \overline{X} \times_{S \star_S K} K$. Then $\overline{q}$ is a S-limit diagram if and only if the restriction S-functor*
$$R : \underline{\mathrm{Map}}_{/S \star_S K,S}(S \star_S K, \overline{X}) \longrightarrow \underline{\mathrm{Map}}_{/S \star_S K,S}(K, \overline{X}) \cong \underline{\mathrm{Map}}_{/K,S}(K, X)$$
is an equivalence.

Proof. In view of [10, Corollary 3.3.3.4], R_s is a map from the limit of $\overline{q}^\dagger|_{\underline{s} \star_{\underline{s}} K_{\underline{s}}}$ to the limit of $q^\dagger|_{K_{\underline{s}}}$ induced by precomposition on the diagram. But by Proposition 5.6, $\overline{q}$ is a S-limit diagram if and only if $\overline{q}^\dagger$ is a right Kan extension of $q^\dagger$, in which case both of the limits in question are equivalent to $\overline{q}^\dagger(s)$. The assertion now follows. □

[19] $\underline{\mathrm{Map}}$ refers here to the maximal sub-left fibration of $\underline{\mathrm{Fun}}$ and not the S-mapping space functor.

11.8. **Proposition.** *Let $\overline{p} : S \star_S K \longrightarrow C$ be a S-functor. The following are equivalent:*

(1) *$\overline{p}$ is a S-limit diagram.*
(2) *For all $s \in S$ and $c \in C_s$,*

$$\underline{\mathrm{Map}}_{C_{\underline{s}}}(c, \overline{p}_{\underline{s}}(-)) : \underline{s} \star_{\underline{s}} K_{\underline{s}} \longrightarrow \underline{\mathbf{Spc}}_{S^{s/}}$$

is a $S^{s/}$-limit diagram.
(3) *For all $s \in S$ and $c \in C_s$,*

$$\underline{\mathrm{Map}}_{/C_{\underline{s}}, S^{s/}}(\underline{c}, C_{\underline{s}}^{/(\overline{p}_{\underline{s}}, S^{s/})}) \longrightarrow \underline{\mathrm{Map}}_{/C_{\underline{s}}, S^{s/}}(\underline{c}, C_{\underline{s}}^{/(p_{\underline{s}}, S^{s/})})$$

is an equivalence.

Moreover, if the above conditions obtain, then

$$\underline{\mathrm{Map}}_{/C_{\underline{s}}, S^{s/}}(\underline{c}, C_{\underline{s}}^{/(p_{\underline{s}}, S^{s/})}) \simeq \underline{\mathrm{Map}}_{C_{\underline{s}}}(c, \overline{p}_{\underline{s}}(v))$$

where v is the cone point $\{s\} \in \underline{s} \star_{\underline{s}} K_{\underline{s}}$.

Proof. (2) $\Leftrightarrow$ (3): We will show that the statements match after fixing $c \in C_s$. To ease notation, let us replace $S^{s/}$ by S and suppose that $s \in S$ is an initial object. By Lemma 11.7 and using that $C^{\underline{c}/}$ is the S-unstraightening of $\underline{\mathrm{Map}}_C(c, -)$, $\underline{\mathrm{Map}}_C(c, \overline{p}(-))$ is a S-limit diagram if and only if

$$\underline{\mathrm{Map}}_{/C,S}(S \star_S K, C^{\underline{c}/}) \longrightarrow \underline{\mathrm{Map}}_{/C,S}(K, C^{\underline{c}/})$$

is an equivalence. By Corollary 4.27, this map is equivalent by a zig-zag to the map

$$\underline{\mathrm{Map}}_{/C,S}(\underline{c}, C^{/(\overline{p},S)}) \longrightarrow \underline{\mathrm{Map}}_{/C,S}(\underline{c}, C^{/(p,S)}).$$

The assertion now follows. The last assertion also follows in view of the equivalence $C^{/(\overline{p},S)} \simeq C^{/\underline{\overline{p}(v)}}$ and $\underline{\mathrm{Map}}_{/C,S}(\underline{c}, C^{/\underline{\overline{p}(v)}}) \simeq \underline{c} \times_C C^{/\underline{\overline{p}(v)}} \simeq \underline{\mathrm{Map}}_C(c, \overline{p}(v))$.

(1) $\Leftrightarrow$ (3): This follows from Lemma 11.6 applied to $C^{/(\overline{p},S)} \longrightarrow C^{/(p,S)}$, which is a map of S-right fibrations over C. □

11.9. **Corollary.** *Let $F : C \longrightarrow D$ be a S-functor. Then*

(1) *F strongly preserves S-limits if and only if for all $s \in S$ and $d \in D_s$,*

$$\underline{\mathrm{Map}}_{D_{\underline{s}}}(d, F_{\underline{s}}(-)) : C_{\underline{s}} \longrightarrow \underline{\mathbf{Spc}}_{S^{s/}}$$

preserves $S^{s/}$-limits.
(2) *F strongly preserves S-colimits if and only if for all $s \in S$ and $d \in D_s$,*

$$\underline{\mathrm{Map}}_{D_{\underline{s}}}(F_{\underline{s}}(-), d) = \underline{\mathrm{Map}}_{D_{\underline{s}}^{vop}}(d, F_{\underline{s}}^{vop}(-)) : C_{\underline{s}}^{vop} \longrightarrow \underline{\mathbf{Spc}}_{S^{s/}}$$

preserves $S^{s/}$-limits.

11.10. **Corollary.** *Let C be a S-category. The Yoneda embedding $j : C \longrightarrow \mathbf{P}_S(C)$ strongly preserves and detects S-limits.*

Proof. Combine Proposition 11.8 and Proposition 9.17. □

Proof of Theorem 11.5. By Theorem 10.5, we have a S-adjunction

$$j_! : \underline{\mathrm{Fun}}_S(C, E) \rightleftarrows \underline{\mathrm{Fun}}_S(\mathbf{P}_S(C), E) : j^*$$

with $j^* j_! \simeq \mathrm{id}$ and the essential image of $j_!$ spanned by the left $S^{s/}$-Kan extensions ranging over all $s \in S$. By Proposition 8.4, taking cocartesian sections yields an adjunction

$$j_! : \mathrm{Fun}_S(C, E) \rightleftarrows \mathrm{Fun}_S(\mathbf{P}_S(C), E) : j^*$$

again with $j^* j_! \simeq \mathrm{id}$ and the essential image of $j_!$ spanned by the left S-Kan extensions. Both assertions will therefore follow if we prove that for a S-functor $F : \mathbf{P}_S(C) \longrightarrow E$, F strongly preserves S-colimits if and only if F is a left S-Kan extension of its restriction $f = F|_C$.

For the 'only if' direction, because $\mathrm{id}_{\mathbf{P}_S(C)}$ is a S-left Kan extension of j by the S-Yoneda lemma 11.1, $F = F \circ \mathrm{id}_{\mathbf{P}_S(C)}$ is a left S-Kan extension as it is the postcomposition of $\mathrm{id}_{\mathbf{P}_S(C)}$ with a strongly S-colimit preserving functor.

For the 'if' direction, we use the criterion of Corollary 11.9. Replacing $S^{s/}$ by S and supposing that $s \in S$ is an initial object, we reduce to showing that for all $x \in E_s$, $\underline{\mathrm{Map}}_E(F(-), x) : \mathbf{P}_S(C)^{vop} \longrightarrow \underline{\mathbf{Spc}}_S$ preserves S-limits. We first observe that F^{vop} is a S-right Kan extension (of f^{vop}), hence so is $\underline{\mathrm{Map}}_E(F(-), x) = \underline{\mathrm{Map}}_{E^{vop}}(x, -) \circ F^{vop}$ as the postcomposition of a S-right Kan extension with a strongly S-limit preserving functor. However, by the vertical opposite of the S-Yoneda lemma, for any S-functor $G : C^{vop} \longrightarrow \underline{\mathbf{Spc}}_S$, the strongly S-limit preserving S-functor $\underline{\mathrm{Map}}_{\mathbf{P}_S(C)}(-, G)$ is a S-right Kan extension of G. Applying this for $G = \underline{\mathrm{Map}}_E(f(-), x)$, we conclude. □

12. Bousfield–Kan formula

In this section, we prove two decomposition formulas for S-colimits which resemble the classical Bousfield–Kan formula for computing homotopy colimits. We first study the situation when $S = \Delta^0$.

12.1. **Notation.** Let K be a simplicial set and let $\Delta_{/K}$ be the nerve of the category of simplices of K. We denote the first vertex map by $\upsilon_K : \Delta^{op}_{/K} \longrightarrow K$ and the last vertex map by $\mu_K : \Delta_{/K} \longrightarrow K$.

By [10, Proposition 4.2.3.14], μ_K is final. Unfortunately, this is the wrong direction for the purposes of obtaining a Bousfield–Kan type formula, since $\Delta_{/K}$ is a *cartesian* fibration over Δ. To rectify this state of affairs, we prove that υ_K is in fact final.

12.2. **Proposition.** *Let K be a simplicial set. Then the first vertex map $\upsilon_K : \Delta^{op}_{/K} \longrightarrow K$ is final. Equivalently, the last vertex map $\mu_{K^{op}}$ is initial.*

Proof. Note that υ_K is natural in K and that $\Delta^{op}_{/(-)} : s\mathbf{Set} \longrightarrow s\mathbf{Set}$ preserves colimits. Recall from [10, Proposition 4.1.2.5] that a map $f : X \longrightarrow Y$ is final if and only if it is a contravariant equivalence in $s\mathbf{Set}_{/Y}$. It follows that the class of final maps is stable under filtered colimits, so we may suppose that K has finitely many nondegenerate simplices. Using left properness of the contravariant model structure, by induction we reduce to the assertion for $K = \Delta^n$. But in this case υ_K is final by the proof of [10, Variant 4.2.3.15] (which proves the result when K is the nerve of a category).

For the second assertion, we note that the reversal isomorphism $\Delta_{/K^{op}} \cong \Delta_{/K}$ interchanges $\mu_{K^{op}}$ and $(\upsilon_K)^{op}$. □

12.3. **Corollary** (Bousfield–Kan formula)**.** *Suppose that C admits (finite) coproducts. Then for a (finite) simplicial set K and a map $p : K \longrightarrow C$, the colimit of p exists if and only if the geometric realization*

$$\left| \coprod_{x \in K_0} p(x) \Longleftarrow \coprod_{\alpha \in K_1} p(\alpha(0)) \Lleftarrow \coprod_{\sigma \in K_2} p(\sigma(0)) \ \dots \right|$$

exists, in which case the colimit of p is computed by the geometric realization.

Proof. The fibers of the cocartesian fibration $\pi_K : \Delta^{op}_{/K} \longrightarrow \Delta^{op}$ are the discrete sets K_n. Therefore, the left Kan extension of $p \circ \upsilon_K$ along π_K exists. By Proposition 12.2, $\operatorname{colim} p \simeq \operatorname{colim} p \circ \upsilon_K$, and the latter is computed as the colimit of $(\pi_K)_!(p \circ \upsilon_K)$ by the transitivity of left Kan extensions. □

We also have a variant of Cor 12.3 where the coproducts over K_n are replaced by colimits indexed by the spaces $\mathrm{Map}(\Delta^n, K)$. To formulate this, we need to introduce some auxiliary constructions. Let $\xi : W \longrightarrow \Delta^{op}$ be the opposite of the relative nerve of the inclusion $\Delta \longrightarrow s\mathbf{Set}$; this is a cartesian fibration which is an explicit model for the tautological cartesian fibration over Δ^{op} pulled back from the universal cartesian fibration over $\mathbf{Cat}^{op}_\infty$. Let $\lambda : \Delta^{op} \longrightarrow W$ be the 'first vertex' section of ξ which sends an n-simplex $\Delta^{a_0} \leftarrow ... \leftarrow \Delta^{a_n}$ to the n-simplex

$$\begin{array}{ccccccc}
\Delta^n & \longleftarrow & \dots & \longleftarrow & \Delta^{\{n-1,n\}} & \longleftarrow & \Delta^{\{n\}} \\
\downarrow{\scriptstyle (\lambda a)_0} & & & & \downarrow{\scriptstyle (\lambda a)_{n-1}} & & \downarrow{\scriptstyle (\lambda a)_n} \\
\Delta^{a_0} & \longleftarrow & \dots & \longleftarrow & \Delta^{a_{n-1}} & \longleftarrow & \Delta^{a_n}
\end{array}$$

of W specified by $(\lambda a)_i(0) = 0$ for all $0 \leq i \leq n$.

For an ∞-category C, let $Z_C = \widetilde{\mathrm{Fun}}_{\Delta^{op}}(W, C \times \Delta^{op})$ and let $Z'_C \subset Z_C$ be the sub-simplicial set on the simplices σ such that every edge of σ is cocartesian (with respect to the structure map to Δ^{op}), so that $Z'_C \longrightarrow \Delta^{op}$ is the maximal sub-left fibration in $Z_C \longrightarrow \Delta^{op}$. Define a Δ^{op}-functor $\Delta^{op}_{/C} \longrightarrow Z_C$ as adjoint to the map $\Delta^{op}_C \times_{\Delta^{op}} W \longrightarrow C$ which sends an n-simplex

$$\begin{array}{ccccccc}
\Delta^n & \longleftarrow & \dots & \longleftarrow & \Delta^{\{n-1,n\}} & \longleftarrow & \Delta^{\{n\}} \\
\downarrow{\scriptstyle (\lambda a)_0} & & & & \downarrow{\scriptstyle (\lambda a)_{n-1}} & & \downarrow{\scriptstyle (\lambda a)_n} \\
\Delta^{a_0} & \longleftarrow & \dots & \longleftarrow & \Delta^{a_{n-1}} & \longleftarrow & \Delta^{a_n} \\
\downarrow{\scriptstyle \tau} & \swarrow & & \swarrow & & & \\
C & & & & & &
\end{array}$$

to $\tau \circ (\lambda a)_0 \in C_n$. Note that since $\Delta^{op}_{/C} \longrightarrow \Delta^{op}$ is a left fibration, this functor factors through Z'_C.

Define a 'first vertex' functor $\Upsilon_C : Z_C \longrightarrow C$ by precomposition with ι (using the isomorphism $\widetilde{\mathrm{Fun}}_{\Delta^{op}}(\Delta^{op}, C \times \Delta^{op}) \cong C \times \Delta^{op}$). We then have a factorization of the first vertex map as

$$\Delta^{op}_{/C} \longrightarrow Z'_C \longrightarrow Z_C \xrightarrow{\Upsilon_C} C.$$

12.4. **Proposition.** *The functors Υ_C and $\Upsilon'_C = (\Upsilon_C)|_{Z'_C}$ are final.*

Proof. We first prove that Υ_C is final by verifying the hypotheses of [10, Theorem 4.1.3.1]. Let $c \in C$. The map $Z_C \longrightarrow C$ is functorial in C, so we have a map $Z_{C_{c/}} \longrightarrow Z_C \times_C C_{c/}$. We claim that this map is a trivial Kan fibration. Unwinding the definitions, this amounts to showing that for every cofibration $A \longrightarrow B$ of simplicial sets over Δ^{op}, we can solve the lifting problem

$$\begin{array}{ccc}
B \cup_A A \times_{\Delta^{op}} W & \longrightarrow & C_{c/} \\
\downarrow & \nearrow & \downarrow \\
B \times_{\Delta^{op}} W & \longrightarrow & C.
\end{array}$$

Since the class of left anodyne morphisms is right cancellative, we may suppose $A = \varnothing$. It thus suffices to prove that $\lambda_B = B \times_{\Delta^{op}} \lambda : B \longrightarrow B \times_{\Delta^{op}} W$ is left anodyne for any map of simplicial sets $B \longrightarrow \Delta^{op}$. Observe that even though λ is not a cartesian section, it is a left adjoint relative to Δ^{op} to ξ by [12, Proposition 7.3.2.6] and the uniqueness of adjoints, since on the fibers it restricts to the adjunction $\{0\} \rightleftarrows \Delta^n$. Consequently, for any ∞-category B and functor $B \longrightarrow \Delta^{op}$, by [12, Proposition 7.3.2.5] λ_B is a left adjoint, hence left anodyne. From this, we deduce the general case by using the characterization in [10, Proposition 4.1.2.1] of the left anodyne maps $X \longrightarrow Y$ as the trivial cofibrations in $s\mathbf{Set}_{/Y}$ equipped with the covariant model structure. Indeed, arguing as in the proof of Proposition 12.2, by induction on the nondegenerate simplices of B we reduce to the known case $B = \Delta^n$.

We next prove that Z_C is weakly contractible if C is, which will conclude the proof for Υ_C. For this, another application of (the opposite of) [12, Proposition 7.3.2.6] shows that the Δ^{op}-functor $C \times \Delta^{op} \longrightarrow Z_C$ defined by precomposition by ξ is a left adjoint relative to Δ^{op} to the functor $(\Upsilon_C, \mathrm{id}_{\Delta^{op}})$, because it restricts to the adjunction $\iota\colon C \rightleftarrows \mathrm{Fun}(\Delta^n, C) :\mathrm{ev}_0$ on the fibers. Hence, $|Z_C| \simeq |C \times \Delta^{op}| \simeq |C|$, and the latter is contractible by hypothesis.

We employ the same strategy to show that Υ'_C is final. Since $C_{c/} \longrightarrow C$ is conservative, the trivial Kan fibration above restricts to yield a trivial Kan fibration $Z'_{C_{c/}} \longrightarrow Z'_C \times_C C_{c/}$. Thus it suffices to show that Z'_C is weakly contractible if C is. By (the opposite of) [6, Proposition 7.3], the cocartesian fibration $Z'_C \longrightarrow \Delta^{op}$ is classified by the functor

$$\Delta^{op} \xrightarrow{i^{op}} \mathbf{Cat}_\infty \xrightarrow{\mathrm{Map}(-,C)} \mathbf{Spc}.$$

Let R denote the right adjoint to the colimit-preserving functor $L : \mathrm{Fun}(\Delta^{op}, \mathbf{Spc}) \longrightarrow \mathbf{Cat}_\infty$ left Kan extended from the inclusion $i : \Delta \subset \mathbf{Cat}_\infty$; R sends an ∞-category to its corresponding complete

Segal space. Then $R(C) \simeq \mathrm{Map}(-, C) \circ i^{op}$. For any $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathbf{Spc})$, we have $\operatorname{colim} X \simeq |L(X_\bullet)|$, hence

$$\operatorname{colim} R(C) \simeq |(L \circ R)(C)| \simeq |C|,$$

where $L \circ R \simeq \mathrm{id}$ by [11, Corollary 4.3.16]. By [10, Corollary 3.3.4.6], $|Z'_C| \simeq \operatorname{colim} R(C)$, so we conclude that $|Z'_C|$ is contractible. $\square$

The following corollary was previously proven by Mazel-Gee in [15].

12.5. **Corollary** (Bousfield–Kan formula, 'simplicial' variant)**.** *Suppose that C admits colimits indexed by spaces. Then for any ∞-category K and functor $p : K \longrightarrow C$, the colimit of p exists if and only if the geometric realization*

$$\left| \operatorname*{colim}_{x \in \mathrm{Map}(\Delta^0, K)} p(x) \leftleftarrows \operatorname*{colim}_{\alpha \in \mathrm{Map}(\Delta^1, K)} p(\alpha(0)) \,\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\, \operatorname*{colim}_{\sigma \in \mathrm{Map}(\Delta^2, K)} p(\sigma(0)) \;\cdots \right|$$

exists, in which case the colimit of p is computed by the geometric realization.

Proof. Using Proposition 12.4, we may repeat the proof of Corollary 12.3, now using the span

$$\Delta^{op} \leftarrow Z'_K \xrightarrow{\Upsilon'_K} K.$$

$\square$

We now proceed to relativize the above picture, starting with the map Υ_C. Let $C \longrightarrow S$ be a S-category. Define the map

$$\Upsilon_{C,S} : \widetilde{\mathrm{Fun}}_{\Delta^{op} \times S/S}(W \times S, \Delta^{op} \times C) \longrightarrow C$$

to be the composition of the map to $\widetilde{\mathrm{Fun}}_{\Delta^{op} \times S/S}(\Delta^{op} \times S, \Delta^{op} \times C)$ given by precomposition by $\lambda \times \mathrm{id}_S$, together with the equivalence of Lemma 9.5 of this to $\Delta^{op} \times C$ and the projection to C. Define $\Upsilon'_{C,S}$ to be the restriction of $\Upsilon_{C,S}$ to the maximal sub-left fibration (with respect to $\Delta^{op} \times S$).

12.6. **Theorem.** *The S-functors $\Upsilon_{C,S}$ and $\Upsilon'_{C,S}$ are S-final.*

Proof. For every object $s \in S$, we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathrm{Fun}}_{\Delta^{op} \times S/S}(W \times S, \Delta^{op} \times C)_s & \xrightarrow{(\lambda \times \mathrm{id}_S)^*_s} & \widetilde{\mathrm{Fun}}_{\Delta^{op} \times S/S}(\Delta^{op} \times S, \Delta^{op} \times C)_s & \longrightarrow & C_s \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow = \\
\widetilde{\mathrm{Fun}}_{\Delta^{op}}(W, \Delta^{op} \times C_s) & \xrightarrow{\lambda^*} & \widetilde{\mathrm{Fun}}_{\Delta^{op}}(\Delta^{op}, \Delta^{op} \times C_s) \cong \Delta^{op} \times C_s & \xrightarrow{\mathrm{pr}_{C_s}} & C_s
\end{array}$$

where the left two vertical maps are given by the natural categorical equivalences of Lemma 9.6; the only point to note is that the equivalences of Lemma 9.5 and Lemma 9.6 coincide when the first variable is trivial. By Proposition 12.4, Υ_{C_s} is final, so $(\Upsilon_{C,S})_s$ is final. By the S-cofinality Theorem 6.7, $\Upsilon_{C,S}$ is S-final. A similar argument shows that $\Upsilon'_{C,S}$ is S-final. $\square$

The process of relativizing υ_C is considerably more involved. We begin with some preliminaries on the relative nerve construction. Let J be a category.

12.7. **Lemma.** *The adjunctions*

$$\mathfrak{F}_J : s\mathbf{Set}_{/N(J)} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}) : N_J$$
$$\mathfrak{F}^+_J : s\mathbf{Set}^+_{/N(J)} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}^+) : N^+_J$$

of [10, §3.2.5] *are simplicial.*

Proof. Let $\underline{K} : J \longrightarrow s\mathbf{Set}$ denote the constant functor at a simplical set K. We have an obvious map $\chi_K : N(J) \times K \longrightarrow N_J(\underline{K})$ natural in K and hence a map

$$(\eta_X, \chi_K \circ \mathrm{pr}) : X \times K \longrightarrow N_J(\mathfrak{F}_J X \times \underline{K}) \cong N_J \mathfrak{F}_J X \times N_J(\underline{K})$$

natural in X and K. We want to show the adjoint

$$\theta_{X,K} : \mathfrak{F}_J(X \times K) \longrightarrow \mathfrak{F}_J(X) \times \underline{K}$$

is an isomorphism. Both sides preserve colimits separately in each variable, so we may suppose $X = \Delta^n \longrightarrow J$ and $K = \Delta^m$. By [10, Example 3.2.5.6], $\mathfrak{F}_I(I)(-) \cong N(I_{/-})$, and by [10, Remark 3.2.5.8], for any functor $f : I \longrightarrow J$, the square

$$\begin{array}{ccc} s\mathbf{Set}_{/N(I)} & \xrightarrow{f_!} & s\mathbf{Set}_{/N(J)} \\ \downarrow{\scriptstyle \mathfrak{F}_I} & & \downarrow{\scriptstyle \mathfrak{F}_J} \\ \mathrm{Fun}(I, s\mathbf{Set}) & \xrightarrow{f_!} & \mathrm{Fun}(J, s\mathbf{Set}) \end{array}$$

commutes. Letting $I = \Delta^n \times \Delta^m$ and $f : I \longrightarrow J$ be the structure map, we have

$$\mathfrak{F}_I(\Delta^n \times \Delta^m)(k,l) \cong (\Delta^n)_{/k} \times (\Delta^m)_{/l} \cong \Delta^k \times \Delta^l.$$

Factoring f as $\Delta^n \times \Delta^m \xrightarrow{g} \Delta^n \xrightarrow{h} J$, we then have

$$g_! \mathfrak{F}_I(\Delta^n \times \Delta^m)(k) \cong \Delta^i \times \Delta^m.$$

Let $G = g_! \mathfrak{F}_I(\Delta^n \times \Delta^m)$, so that $\mathfrak{F}_J(\Delta^n \times \Delta^m)(j) \cong (h_! G)(j)$. Then

$$(h_! G)(j) \cong \operatorname*{colim}_{\Delta^n \times_J J_{/j}} \left((k, h(k) \to j) \mapsto \Delta^k\right) \times \Delta^m \cong \mathfrak{F}_J(\Delta^n)(j) \times \Delta^m$$

and one can verify that $\theta_{X,K}$ implements this isomorphism. For the assertion about $\mathfrak{F}^+_J \dashv N^+_J$, recall that the simplicial tensor $s\mathbf{Set} \times s\mathbf{Set}^+ \longrightarrow s\mathbf{Set}^+$ is given by $(K, X) \mapsto K^\sharp \times X$. Consequently, in the above argument we may simply replace Δ^m by $(\Delta^m)^\sharp$ to conclude. □

Since $N^+_J(\underline{S^\sharp}) = N(J) \times S^\sharp$, the adjunction $\mathfrak{F}^+_J \dashv N^+_J$ lifts to an adjunction

$$\mathfrak{F}^+_{J,S} \colon s\mathbf{Set}^+_{/N(J)\times S} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) : N^+_{J,S}$$

between the overcategories. Moreover, for any functor $f : T \longrightarrow S$, the square

$$\begin{array}{ccc} \mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) & \xrightarrow{N^+_{J,S}} & s\mathbf{Set}^+_{/N(J)\times S} \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (\mathrm{id}\times f)^*} \\ \mathrm{Fun}(J, s\mathbf{Set}^+_{/T}) & \xrightarrow{N^+_{J,T}} & s\mathbf{Set}^+_{/N(J)\times T}, \end{array}$$

commutes.

12.8. **Proposition.** *Equip* $s\mathbf{Set}^+_{/N(J)\times S}$ *with the cocartesian model structure and* $\mathrm{Fun}(J, s\mathbf{Set}^+_{/S})$ *with the projective model structure, where* $s\mathbf{Set}^+_{/S}$ *has the cocartesian model structure. Then the adjunction*

$$\mathfrak{F}^+_{J,S} \colon s\mathbf{Set}^+_{/N(J)\times S} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) : N^+_{J,S}$$

is a Quillen equivalence.

Proof. We first prove that the adjunction is Quillen. Because this is a simplicial adjunction between left proper simplicial model categories, it suffices to show that $\mathfrak{F}^+_{J,S}$ preserves cofibrations and $N^+_{J,S}$ preserves fibrant objects. Observe that the slice model structure on

$$s\mathbf{Set}^+_{/N(J)\times S} \cong (s\mathbf{Set}^+_{/N(J)})_{/(N(J)\times S)^\sharp}$$

is a localization of the cocartesian model structure. Similarly, the slice model structure on

$$\mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) \cong \mathrm{Fun}(J, s\mathbf{Set}^+)_{/\underline{S^\sharp}}$$

is a localization of the projective model structure, since the trivial fibrations for the two model structures coincide and postcomposition by $\pi_! : s\mathbf{Set}^+_{/S} \longrightarrow s\mathbf{Set}^+$ gives a Quillen left adjoint between the projective model structures. Since the lift of a Quillen adjunction $L\colon M \rightleftarrows N : R$ to the adjunction $\widetilde{L}\colon M_{/R(x)} \rightleftarrows N_{/x} : \widetilde{R}$ is Quillen for the slice model structures, we deduce that $\mathfrak{F}^+_{J,S}$ preserves cofibrations.

Now suppose $F : J \longrightarrow s\mathbf{Set}^+_{/S}$ is fibrant. Since S is an ∞-category, $F \to \underline{S}$ is a fibration in $\mathrm{Fun}(J, s\mathbf{Set})$. Hence $N_{J,S}(F) \longrightarrow N(J) \times S$ is a categorical fibration. We verify that it is a cocartesian fibration (with every marked edge cocartesian) by solving the lifting problem ($n \geq 1$)

$$\begin{array}{ccc} {}_\natural\Lambda^n_0 & \longrightarrow & N^+_{J,S}(F) \\ \big\downarrow & \nearrow & \big\downarrow \\ {}_\natural\Delta^n & \xrightarrow[(j_\bullet, s_\bullet)]{} & (N(J) \times S)^\sharp. \end{array}$$

Unwinding the definitions, this amounts to solving the lifting problem

$$\begin{array}{ccc} {}_\natural\Lambda^n_0 & \longrightarrow & F(j_n) \\ \big\downarrow & \nearrow & \big\downarrow \\ {}_\natural\Delta^n & \xrightarrow[s_\bullet]{} & S^\sharp, \end{array}$$

and the dotted lift exists because $F(j_n)$ is cocartesian over S with the cocartesian edges marked. Finally, it is easy to see that marked edges compose and are stable under equivalence. We conclude that $N^+_{J,S}(F)$ is fibrant in $s\mathbf{Set}^+_{/N(J)\times S}$.

To prove that the Quillen adjunction is a Quillen equivalence, we will show that the induced adjunction of ∞-categories

$$\mathfrak{F}'^+_{J,S}\colon N((s\mathbf{Set}^+_{/N(J)\times S})^\circ) \rightleftarrows N(\mathrm{Fun}(J, s\mathbf{Set}^+_{/S})^\circ) : N'^+_{J,S}$$

is an adjoint equivalence, where $N'^+_{J,S}$ is the simplicial nerve of $N^+_{J,S}$ and $\mathfrak{F}'^+_{J,S}$ is any left adjoint to $N'^+_{J,S}$. We first check that $N'^+_{J,S}$ is conservative. Indeed, for this we may work in the model category: for a natural transformation $\alpha : F \to G$ in $\mathrm{Fun}(J, s\mathbf{Set}^+_{/S})$, $N^+_{J,S}(F) \longrightarrow N^+_{J,S}(G)$ on fibers is given by $F(j)_s \longrightarrow G(j)_s$, hence if F, G are fibrant and $N^+_{J,S}(\alpha)$ is an equivalence then α is as well. It now suffices to show that the unit transformation $\eta : \mathrm{id} \longrightarrow N'^+_{J,S}\mathfrak{F}'^+_{J,S}$ is an equivalence. We have the known equivalence $N((s\mathbf{Set}^+_{/N(J)\times S})^\circ) \simeq \mathrm{Fun}(N(J) \times S, \mathbf{Cat}_\infty)$ so it further suffices to check that the map

$$(\mathrm{id} \times i_s)^* \longrightarrow (\mathrm{id} \times i_s)^* N'^+_{J,S}\mathfrak{F}'^+_{J,S} \simeq N'^+_J i_s^* \mathfrak{F}'^+_{J,S}$$

is an equivalence for all $s \in S$, $i_s : \{s\} \longrightarrow S$ the inclusion. Equivalently, since $\mathfrak{F}^+_J \dashv N^+_J$ is a Quillen equivalence by [10, Proposition 3.2.5.18], we must show that the adjoint map

$$\mathfrak{F}'^+_J i_s^* \longrightarrow (\mathrm{id} \times i_s)^* \mathfrak{F}'^+_{J,S}$$

is an equivalence. This statement is in turn equivalent to the adjoint map

$$\theta : N'^+_{J,S}(i_s)_* \longrightarrow (\mathrm{id} \times i_s)_* N'^+_J$$

being an equivalence. Recall that for a functor $f : T \longrightarrow S$, $f_* : \mathrm{Fun}(T, \mathbf{Cat}_\infty) \longrightarrow \mathrm{Fun}(S, \mathbf{Cat}_\infty)$ is induced by $\pi_* \rho^* : s\mathbf{Set}^+_{/T} \longrightarrow s\mathbf{Set}^+_{/S}$ for the span

$$S^\sharp \xleftarrow{\pi} (\mathcal{O}(S) \times_S T)^\sharp \xrightarrow{\rho} T^\sharp$$

with π given by evaluation at 0 and ρ projection to T. Moreover, for a functor $\mathrm{id} \times f : U \times T \longrightarrow U \times S$, we may elect to use the span

$$(U \times S)^\sharp \xleftarrow{\mathrm{id}\times\pi} (U \times \mathcal{O}(S) \times_S T)^\sharp \xrightarrow{\mathrm{id}\times\rho} (U \times T)^\sharp$$

to model $(\mathrm{id} \times f)_*$. Letting $f = i_s$, we see that θ is induced by the map

$$N^+_{J,S}\pi_*\rho^* \longrightarrow (\mathrm{id} \times \pi)_* N^+_{J,S^{s/}}\rho^* \cong (\mathrm{id} \times \pi)_*(\mathrm{id} \times \rho)^* N^+_J.$$

where the first map is adjoint to the isomorphism $(\mathrm{id}\times\pi)^* N^+_{J,S} \cong N^+_{J,S^{s/}}\pi^*$. Direct computation reveals that this map is an equivalence on fibrant $F : J \longrightarrow s\mathbf{Set}^+$. □

We now return to the situation of interest. Let C be a S-category with structure map $\pi : C \longrightarrow S$. We first extend our existing notation $\underline{x}$ for objects $x \in C$.

12.9. **Notation.** For an n-simplex σ of C, define

$$\underline{\sigma} = \{\sigma\} \times_{\mathrm{Fun}(\Delta^n\times\{0\},C)} \mathrm{Fun}((\Delta^n)^\flat \times (\Delta^1)^\sharp, {}_\natural C) \times_{\mathrm{Fun}(\Delta^n\times\{1\},S)} S.$$

12.10. **Lemma.** *There exists a map $b_\sigma : \underline{\sigma} \longrightarrow \{\pi\sigma(n)\} \times_S \mathcal{O}(S) = S^{\pi\sigma(n)/}$ which is a trivial Kan fibration.*

Proof. First define a map $b'_\sigma : \underline{\sigma} \longrightarrow \underline{\pi\sigma}$ to be the pullback of the map

$$(e_0, \mathcal{O}(\pi))_* : \mathrm{Fun}(\Delta^n, \mathcal{O}^{cocart}(C)) \longrightarrow C^{\Delta^n} \times_{S^{\Delta^n}} \mathrm{Fun}(\Delta^n, \mathcal{O}(S))$$

over $\{\sigma\}$ and S. Since $(e_0, \mathcal{O}(\pi))$ is a trivial Kan fibration, so is b'_σ. Next, let K be the pushout $\Delta^n \times \{0\} \cup_{\{n\}\times\{0\}} \{n\} \times \Delta^1$. We claim that the map $\mathrm{Fun}(\Delta^n, \mathcal{O}(S)) \times_{S^{\Delta^n}} S \longrightarrow \mathrm{Fun}(K, S)$ induced by $K \subset \Delta^n \times \Delta^1$ is a trivial Kan fibration. For a monomorphism $A \longrightarrow B$, we need to solve the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Fun}(\Delta^n, \mathcal{O}(S)) \times_{S^{\Delta^n}} S \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \mathrm{Fun}(K, S). \end{array}$$

This transposes to

$$\begin{array}{ccc} A \times \Delta^n \bigcup_{A\times\{n\}} B \times \{n\} & \longrightarrow & \mathcal{O}(S) \\ \downarrow & \nearrow & \downarrow \mathrm{ev}_0 \\ B \times \Delta^n & \longrightarrow & S \end{array}$$

and the lefthand map is right anodyne by [10, Corollary 2.1.2.7], hence the dotted lift exists as ev_0 is a cartesian fibration. Now define b''_σ to be the pullback

$$\underline{\pi\sigma} = \{\pi\sigma\} \times_{S^{\Delta^n}} \mathrm{Fun}(\Delta^n, \mathcal{O}(S)) \times_{S^{\Delta^n}} S \longrightarrow \{\pi\sigma\} \times_{S^{\Delta^n}} \mathrm{Fun}(K, S) \cong S^{\pi\sigma(n)/};$$

this is also a trivial Kan fibration. Finally, let $b_\sigma = b''_\sigma \circ b'_\sigma$. □

We will regard $\underline{\sigma}$ as a $S^{\pi\sigma(n)/}$ or S-category via b_σ. We also have a target map $\underline{\sigma} \longrightarrow C^{\Delta^n}$ induced by $\Delta^n \times \{1\} \subset \Delta^n \times \Delta^1$. This covers the target map $S^{\pi\sigma(n)/} \longrightarrow S$ and is a S-functor.

Define a functor $F_C : \Delta^{op} \longrightarrow s\mathbf{Set}^+_{/S}$ on objects $[n]$ by

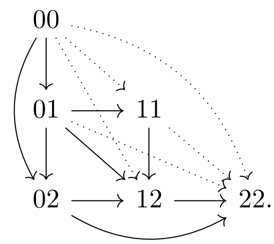
$$F_C([n]) = \bigsqcup_{\sigma\in C_n} \underline{\sigma}^\sharp$$

and on morphisms $\alpha : [m] \to [n]$ by the map $\underline{\sigma} \longrightarrow \underline{\sigma\alpha}$ induced by precomposition by $\alpha : \Delta^m \longrightarrow \Delta^n$.

12.11. **Remark.** The map $\underline{\sigma} \longrightarrow \underline{\sigma(n)}$ is compatible with the maps b_σ and $b_{\sigma(n)}$ of Lemma 12.10, hence is a categorical equivalence (in fact, a trivial Kan fibration). Consequently, given a morphism $f : x \to y$ in C, by choosing an inverse to $\underline{f} \xrightarrow{\simeq} \underline{y}$ we obtain a map $f^* : \underline{y} \longrightarrow \underline{x}$, unique up to contractible choice. Moreover, if f lies over an equivalence, then $\underline{f} \longrightarrow \underline{x}$ is a trivial Kan fibration, so we also obtain a map $f_! : \underline{x} \longrightarrow \underline{y}$.

In order to define the S-first vertex map $N^+_{\Delta^{op},S}(F_C) \longrightarrow C$, we need to introduce a few preliminary constructions. Let $A_n \subset \mathcal{O}(\Delta^n)$ be the sub-simplicial set where a k-simplex $x_0y_0 \to \ldots \to x_ky_k$ is in

A_n if and only if $x_k \leq y_0$. For the reader's aid we draw a picture of the inclusion $A_n \subset \mathcal{O}(\Delta^n)$ for $n = 2$, where dashed edges are not in A_2:



12.12. **Lemma.** *The inclusion* $A_n \longrightarrow \mathcal{O}(\Delta^n)$ *is inner anodyne.*

Proof. In this proof we adopt the notation $[x_0y_0, ..., x_ky_k]$ for a k-simplex of $\mathcal{O}(\Delta^n)$. Let E be the collection of edges $[ab, xy]$ in $\mathcal{O}(\Delta^n)$ where $x > b$, and choose a total ordering $\leq$ on E such that if we have a factorization

$$\begin{array}{ccc} ab & \longrightarrow & xy \\ \downarrow & & \downarrow \\ a'b' & \longrightarrow & x'y' \end{array}$$

then $[a'b', x'y'] \leq [ab, xy]$. Index edges in E by $I = \{0, ..., N\}$. Define simplicial subsets $A_{n,i}$ of $\mathcal{O}(\Delta^n)$ such that $A_{n,i}$ is obtained by expanding A_n to contain every k-simplex $[x_0y_0, ..., x_ky_k]$ with $[x_0y_0, x_ky_k]$ in $E_{<i}$. We will show that each inclusion $A_{n,i} \longrightarrow A_{n,i+1}$ is inner anodyne. We may divide the nondegenerate k-simplices $[x_0y_0, x_1y_1, ..., x_ky_k]$ in $A_{n,i+1}$ but not in $A_{n,i}$ into six classes:

- $A1$: $x_1y_1 \neq x_0(y_0+1)$ and $y_1 > y_0$.
- $A2$: $x_1y_1 = x_0(y_0+1)$.
- $B1$: $x_1y_1 = (x_0+1)y_0$, $y_2 > y_0$, and $x_2y_2 \neq (x_0+1)(y_0+1)$.
- $B2$: $x_1y_1 = (x_0+1)y_0$ and $x_2y_2 = (x_0+1)(y_0+1)$.
- $C1$: $x_1y_1 \neq (x_0+1)y_0$ and $y_1 = y_0$.
- $C2$: $x_1y_1 = (x_0+1)y_0$ and $y_2 = y_0$.

We have bijections between classes of form 1 and classes of form 2 given by

- A: $[x_0y_0, x_1y_1, ..., x_ky_k] \mapsto [x_0y_0, x_0(y_0+1), x_1y_1, ..., x_ky_k]$.
- B: $[x_0y_0, x_0+1y_1, x_2y_2, ..., x_ky_k] \mapsto [x_0y_0, (x_0+1)y_0, (x_0+1)(y_0+1), x_2y_2, ..., x_ky_k]$.
- C: $[x_0y_0, x_1y_1, ..., x_ky_k] \mapsto [x_0y_0, (x_0+1)y_0, x_1y_1, ..., x_ky_k]$.

Moreover, this identifies simplices in a class of form 1 as inner faces of simplices in the corresponding class of form 2. Let P be the collection of pairs $\tau \subset \tau'$ of nondegenerate $k-1$ and k-simplices matched by this bijection. Choose a total ordering on P where pairs are ordered first by the dimension of the smaller simplex, and then by $A < B < C$, and then randomly. Let $J = \{0, ..., M\}$ be the indexing set for P. We define a sequence of inner anodyne maps

$$A_{n,i} = A_{n,i,0} \longrightarrow A_{n,i,1} \longrightarrow ... \longrightarrow A_{n,i,M+1} = A_{n,i+1}$$

such that $A_{n,i,j+1}$ is obtained from $A_{n,i,j}$ by attaching the jth pair $\tau \subset \tau'$ along an inner horn. For this to be valid, we need the other faces of τ' to already be in $A_{n,i,j}$. The ordering on E was chosen so that the outer faces of τ' are in $A_{n,i}$. The argument for the inner faces proceeds by cases:

- τ' is in class $A2$: The other inner faces are also in class $A2$ since they contain $x_0(y_0+1)$, hence were added at some earlier stage.
- τ' is in class $B2$: The other inner faces of $[x_0y_0, (x_0+1)y_0, (x_0+1)(y_0+1), x_2y_2, ..., x_ky_k]$ are all in class $B2$, except for $[x_0y_0, (x_0+1)(y_0+1), x_2y_2, ..., x_ky_k]$, which is in class $A1$. Both of these were added at an earlier stage.
- τ' is in class $C2$: The other inner faces are in class $C2$ or $B1$ since they contain $(x_0+1)y_0$, hence were added at some earlier stage.

$\square$

Let $E_n \subset (A_n)_1 \subset \mathcal{O}(\Delta^n)_1$ be the subset of edges $x_0y_0 \to x_1y_1$ where $y_0 = y_1$. Define simplicial sets C' and C'' to be the pullbacks

$$\begin{array}{ccc} C'_\bullet & \longrightarrow & \mathrm{Hom}((\mathcal{O}(\Delta^\bullet), E_\bullet), {}_\natural C) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Delta^\bullet, S) & \xrightarrow{\mathrm{ev}_0^*} & \mathrm{Hom}(\mathcal{O}(\Delta^\bullet), S) \end{array}, \quad \begin{array}{ccc} C''_\bullet & \longrightarrow & \mathrm{Hom}((A_\bullet, E_\bullet), {}_\natural C) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Delta^\bullet, S) & \xrightarrow{\mathrm{ev}_0^*} & \mathrm{Hom}(A_\bullet, S). \end{array}$$

We now show that the map $C' \longrightarrow C''$ induced by precomposition by $A_\bullet \longrightarrow \mathcal{O}(\Delta^\bullet)$ is a trivial Kan fibration. Indeed, in order to solve the lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & C' \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & C'' \end{array}$$

we must supply a lift

$$\begin{array}{ccc} A_n \bigcup_{\cup A_{n-1}} (\bigcup \mathcal{O}(\Delta^{n-1})) & \longrightarrow & C \\ \downarrow & \nearrow & \downarrow \\ \mathcal{O}(\Delta^n) & \longrightarrow & S \end{array}$$

and the left vertical map is a trivial cofibration by Lemma 12.12. Let $\sigma : C'' \longrightarrow C'$ be any section. Also let $\delta : C' \longrightarrow C$ be the map induced by precomposition by the identity section $\Delta^\bullet \longrightarrow \mathcal{O}(\Delta^\bullet)$.

Define a map $\upsilon_{C,S} : N^+_{\Delta^{op},S}(F_C) \longrightarrow C$ over S as follows: the data of an n-simplex of $N^+_{\Delta^{op},S}(F_C)$ consists of

- an n-simplex $\Delta^{a_0} \leftarrow ... \leftarrow \Delta^{a_n}$ in Δ^{op} (so we have maps $f_{ij} : \Delta^{a_j} \longrightarrow \Delta^{a_i}$ for $i \leq j$);
- an n-simplex $s_\bullet : \Delta^n \longrightarrow S$;
- a choice of a_0-simplex $\sigma_0 \in C_{a_0}$;
- for $0 \leq i \leq n$, a map $\gamma_i : \Delta^i \longrightarrow \underline{\sigma_i}$, where $\sigma_i = \sigma_0 \circ f_{0i}$

such that for all $0 \leq i \leq j \leq n$, the diagram

$$\begin{array}{ccc} \Delta^i & \xrightarrow{\gamma_i} & \underline{\sigma_i} \\ {\scriptstyle \{0,\dots,i\}\subset[j]}\downarrow & & \downarrow {\scriptstyle f_{ij}^*} \\ \Delta^j & \xrightarrow{\gamma_j} & \underline{\sigma_j} \\ & {\scriptstyle (s_\bullet)|_{\{0,\dots,j\}}}\searrow & \downarrow \\ & & S \end{array}$$

commutes. Let $\overline{\gamma_i} : \Delta^i \times \Delta^{a_i} \times \Delta^1 \longrightarrow C$ denote the adjoint map.

We now define a map $A_n \longrightarrow C$ to be that uniquely specified by sending for all $0 \leq k \leq n$ the rectangle $\Delta^k \times \Delta^{n-k} \subset A_n$ given by $00 \mapsto 0k$ and $k(n-k) \mapsto kn$ to

$$\Delta^k \times \Delta^{n-k} \xrightarrow{\mathrm{id}\times(\lambda a)_k} \Delta^k \times \Delta^{a_k} \times \{1\} \xrightarrow{\overline{\gamma_i}|_{\{1\}}} C$$

where the maps $(\lambda a)_k$ are obtained from the first vertex section of $W \longrightarrow \Delta^{op}$ restricted to $a_\bullet$ as before. One may check that the composite $A_n \longrightarrow C \longrightarrow S$ factors as $A_n \longrightarrow \Delta^n \xrightarrow{s_\bullet} S$, so this defines a n-simplex of C''. This procedure is natural in $\Delta^n \in \Delta$, so yields a map $N^+_{\Delta^{op},S}(F_C) \longrightarrow C''$. Finally, postcomposition by $\delta \circ \sigma : C'' \longrightarrow C$ define our desired map $\upsilon_{C,S}$. By Proposition 12.8, $N^+_{\Delta^{op},S}(F_C) \xrightarrow{\pi'} S$ is an S-category with an edge π'-cocartesian if and only if it is degenerate when projected to Δ^{op}. These edges are evidently sent to π-cocartesian edges in C, so υ_C is a S-functor.

12.13. **Theorem.** *The S-first vertex map $\upsilon_{C,S} : N^+_{\Delta^{op},S}(F_C) \longrightarrow C$ is fiberwise a weak homotopy equivalence. Moreover, $\upsilon_{C,S}$ is S-final if either $C \longrightarrow S$ is a left fibration, or S is equivalent to the nerve of a 1-category.*

Proof. Let $t \in S$ be an object and $i_t : \{t\} \longrightarrow S$ the inclusion. Then $N^+_{\Delta^{op},S}(F_C)_t \cong N^+_{\Delta^{op}}(i_t^* F_C)$. We have a map

$$N^+_{\Delta^{op}}(i_t^* F_C) \longrightarrow \Delta^{op}_{/C} \cong N^+_{\Delta^{op}}(C_\bullet)$$

of left fibrations over Δ^{op} induced by the natural transformation $i_t^* F_C \longrightarrow C_\bullet$ which collapses each $\underline{\sigma} \times_S \{t\}$ to a point. Moreover, this natural transformation is objectwise a Kan fibration, so the map itself is a left fibration. Also define a map

$$N^+_{\Delta^{op}}(i_t^* F_C) \longrightarrow (S^{/t})^{op}$$

as follows: in the above notation, the γ_0 map in the data of an n-simplex $(a_\bullet, \gamma_i : \Delta^i \longrightarrow \underline{\sigma_i} \times_S \{t\})$ yields a map $\pi\gamma_0 : \Delta^{a_0} \longrightarrow \mathcal{O}(S) \times_S \{t\} = S^{/t}$, and we send the n-simplex to

$$\Delta^n \xrightarrow{(\lambda a^{rev})_0} (\Delta^{a_0})^{op} \xrightarrow{(\pi\gamma_0)^{op}} (S^{/t})^{op}$$

where $a_\bullet^{rev}$ is $(\Delta^{a_0})^{op} \leftarrow \ldots \leftarrow (\Delta^{a_n})^{op}$. Using these maps we obtain a commutative square

$$\begin{array}{ccc} N^+_{\Delta^{op}}(i_t^* F_C) & \longrightarrow & C^{op} \times_{S^{op}} (S^{/t})^{op} \\ \big\downarrow & & \big\downarrow \\ \Delta^{op}_{/C} & \xrightarrow{\mu_C^{op}} & C^{op}. \end{array}$$

We claim that the map

$$\theta_{C,t} : N^+_{\Delta^{op}}(i_t^* F_C) \longrightarrow (\Delta^{op}_{/C}) \times_{C^{op}} (C \times_S S^{/t})^{op}$$

is a categorical equivalence. Since $\theta_{C,t}$ is a map of left fibrations over $\Delta^{op}_{/C}$, it suffices to check that for every object $\sigma \in \Delta^{op}_{/C}$, the map on fibers

$$\underline{\sigma} \times_S \{t\} \longrightarrow (S^{op})^{t/} \times_{S^{op}} \{\pi\sigma(n)\} \simeq \{\pi\sigma(n)\} \times_S S^{/t}$$

is a homotopy equivalence. But this is the pullback of the trivial Kan fibration of Lemma 12.10 over $\{t\}$.

We next define a map $N^+_{\Delta^{op}}(i_t^* F_C) \longrightarrow S^{/t}$ by sending $(a_\bullet, \gamma_i)$ to $\pi\gamma_0 \circ (\lambda a)_0$. Then the outer rectangle

$$\begin{array}{ccccc} N^+_{\Delta^{op}}(i_t^* F_C) & \overset{\upsilon'_{C,t}}{\dashrightarrow} & C \times_S S^{/t} & \longrightarrow & S^{/t} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Delta^{op}_{/C} & \xrightarrow{\upsilon_C} & C & \xrightarrow{\pi} & S \end{array}$$

commutes so we obtain the dotted map $\upsilon'_{C,t}$.

Next, we choose a section P of the trivial Kan fibration $\mathcal{O}^{cocart}(C) \longrightarrow C \times_S \mathcal{O}(S)$ which restricts to the identity section on C. P restricts to a map $P_t : C \times_S S^{/t} \longrightarrow \mathcal{O}^{cocart}(C) \times_S \{t\}$, and it is tedious but straightforward to construct a homotopy between the composition $(\mathrm{ev}_1 P_t) \circ \upsilon'_{C,t}$ and $(\upsilon_{C,S})_t$. Finally, we define a map $\upsilon''_{C,t} : \Delta^{op}_{/C \times_S S^{/t}} \longrightarrow N^+_{\Delta^{op}}(i_t^* F_C)$ as follows: given an n-simplex

$$\begin{array}{ccccc} \Delta^{a_0} & \longleftarrow & \ldots & \longleftarrow & \Delta^{a_n} \\ \big\downarrow{\scriptstyle \tau_0} & & \swarrow{\scriptstyle \tau_n} & & \\ C \times_S S^{/t} & & & & \end{array}$$

let $\sigma_i = \mathrm{pr}_C \circ \tau_i$, and define $\gamma_i : \Delta^i \longrightarrow \underline{\sigma_i} \times_S \{t\}$ as the composition of the projection to Δ^0 and the adjoint of the map $P_t \circ \tau_i$. Then $(a_\bullet, \gamma_i)$ assembles to yield an n-simplex of $N^+_{\Delta^{op}}(i_t^* F_C)$.

Unwinding the definitions of the various maps, we identify the composition $\upsilon'_{C,t} \circ \upsilon''_{C,t}$ as given by $\upsilon_{C \times_S S^{/t}}$, and the composition $\theta_{C,t} \circ \upsilon''_{C,t}$ as given by the map $\Delta^{op}_{/\,\mathrm{pr}_C}$ to the factor $\Delta^{op}_{/C}$ and the map

$(\mu_{C\times_S S^{/t}})^{op}$ to the factor $(C \times_S S^{/t})^{op}$. By Proposition 12.2 and the fact that final maps pull back along cocartesian fibrations, we deduce that in

$$\Delta^{op}_{/C\times_S S^{/t}} \longrightarrow \Delta^{op}_{/C} \times_{C^{op}} (C \times_S S^{/t})^{op} \longrightarrow (C \times_S S^{/t})^{op}$$

the long composition and the second map are both final. Consequently, $\theta_{C,t} \circ \upsilon''_{C,t}$ is a weak homotopy equivalence. Moreover, if S is equivalent to the nerve of a 1-category then $\theta_{C,t} \circ \upsilon''_{C,t}$ is a categorical equivalence, as may be verified by checking that the map is a fiberwise equivalence over $\Delta^{op}_{/C}$. Since $\theta_{C,t}$ is a categorical equivalence, $\upsilon''_{C,t}$ is then a weak homotopy equivalence resp. a categorical equivalence. Since $\upsilon_{C\times_S S^{/t}}$ is final, $\upsilon'_{C,t}$ is then a weak homotopy equivalence resp. final.

For the last step, let $j_t : C_t \longrightarrow C \times_S S^{/t}$ denote the inclusion. As the inclusion of the fiber over a final object into a cocartesian fibration, j_t is final. $(\mathrm{ev}_1 P_t) \circ j_t = \mathrm{id}_{C_t}$, so by right cancellativity of final maps, $\mathrm{ev}_1 P_t$ is final. We conclude that $(\upsilon_{C,S})_t$ is a weak homotopy equivalence resp. final. In addition, if $C \longrightarrow S$ is a left fibration, $(\upsilon_{C,S})_t$ has target a Kan complex, so is final by [12, Lemma 2.3.4.6]. Invoking the S-cofinality Theorem 6.7, we conclude the proof. □

12.14. **Remark.** The above proof that the S-first vertex map $\upsilon_{C,S}$ is final in special cases hinges upon the finality of the map $\theta_{C,t} \circ \upsilon''_{C,t}$. We believe, but are currently unable to prove, that this map is always final.

We conclude this section with our main application to decomposing S-colimits.

12.15. **Corollary.** *Suppose that S^{op} admits multipullbacks. Then C is S-cocomplete if and only C admits all S-coproducts and geometric realizations.*

Proof. We prove the if direction, the only if direction being obvious. Let K be a $S^{s/}$-category and $p : K \longrightarrow C_{\underline{s}}$ a $S^{s/}$-diagram. First suppose that $K \longrightarrow S^{s/}$ is a left fibration. Consider the diagram

$$\begin{array}{ccc} N^+_{\Delta^{op},S^{s/}}(F_K) \xrightarrow{\upsilon_{K,S^{s/}}} K \xrightarrow{p} C_{\underline{s}} \\ \downarrow{\rho} \\ \Delta^{op} \times S^{s/}. \end{array}$$

By Theorem 12.13, the $S^{s/}$-colimit of p is equivalent to that of $p\circ \upsilon_{K,S^{s/}}$. Since ρ is S-cocartesian, by Theorem 9.15 the $S^{s/}$-left Kan extension of $p\circ\upsilon_{K,S^{s/}}$ along ρ exists provided that for all $n \in \Delta^{op}$ and $f : s \to t$, the $S^{t/}$-colimit exists for $(p\circ\upsilon_{K,S^{s/}})_{\underline{(n,f)}}$. To understand the domain of this map, note that because the pullback of ρ along $f^* : \Delta^{op} \times S^{t/} \longrightarrow \Delta^{op} \times S^{s/}$ is given by $N^+_{\Delta^{op},S^{t/}}(f^*F_K)$, the assumption that S^{op} admits multipullbacks ensures that the $\underline{(n,f)}$-fibers of ρ decompose as coproducts of representable left fibrations. Therefore, these colimits exist since C is assumed to admit S-coproducts. Now by transitivity of left $S^{s/}$-Kan extensions, the $S^{s/}$-colimit of $p \circ \upsilon_{K,S^{s/}}$ is equivalent to that of $\rho_!(p \circ \upsilon_{K,S^{s/}})$, and this exists since C is assumed to admit geometric realizations.

Now suppose that $K \longrightarrow S^{s/}$ is any cocartesian fibration. Consider the diagram

$$\begin{array}{ccc} \widetilde{\iota\mathrm{Fun}}_{\Delta^{op}\times S^{s/}}(W \times S^{s/}, \Delta^{op} \times K) \xrightarrow{\Upsilon'_{K,S^{s/}}} K \xrightarrow{p} C_{\underline{s}} \\ \downarrow{\rho'} \\ \Delta^{op} \times S^{s/}. \end{array}$$

By Theorem 12.6, the $S^{s/}$-colimit of p is equivalent to that of $p \circ \Upsilon'_{K,S^{s/}}$. By Proposition 9.7, the $\underline{(n,f)}$-fiber of ρ' is equivalent to $\iota\underline{\mathrm{Fun}}_{S^{t/}}(\Delta^n \times S^{t/}, K \times_{S^{s/}} S^{t/})$, which in any case remains a left fibration. We just showed that for all $t \in S$, $C_{\underline{t}}$ admits $S^{t/}$-colimits indexed by left fibrations. We are thereby able to repeat the above proof in order to show that the $S^{s/}$-colimit of p exists. □

13. Appendix: Fiberwise fibrant replacement

In this appendix, we formulate a result (Proposition 13.4) which will allow us to recognize a map as a cocartesian equivalence if it is a marked equivalence on the fibers. We begin by introducing a marked variant of Lurie's mapping simplex construction.

13.1. **Definition.** Suppose a functor $\phi : [n] \longrightarrow s\mathbf{Set}^+$, $A_0 \longrightarrow ... \longrightarrow A_n$. Define $M(\phi)$ to be the simplicial set which is the opposite of the mapping simplex construction of [10, §3.2.2], so that a m-simplex of $M(\phi)$ is given by the data of a map $\alpha : \Delta^m \longrightarrow \Delta^n$ together with a map $\beta : \Delta^m \longrightarrow A_{\alpha(0)}$. Endow $M(\phi)$ with a marking by declaring an edge $e = (\alpha, \beta)$ of $M(\phi)$ to be marked if and only if β is a marked edge of $A_{\alpha(0)}$. Note that if each A_i is given the degenerate marking, then the marking on $M(\phi)$ is that of [10, Notation 3.2.2.3].

13.2. **Lemma.** *Suppose $\eta : \phi \longrightarrow \psi$ is a natural transformation between functors $[n] \longrightarrow s\mathbf{Set}^+$ such that for all $0 \leq i \leq n$, $\eta_i : A_i \longrightarrow B_i$ is a cocartesian equivalence. Then $M(\eta) : M(\phi) \longrightarrow M(\psi)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$.*

Proof. Using the decomposition of $M(\phi)$ as the pushout

$$M(\phi') \cup_{A_0 \times \Delta^{n-1}} A_0 \times \Delta^n$$

for $\phi' : A_1 \longrightarrow ... \longrightarrow A_n$, this follows by an inductive argument in view of the left properness of $s\mathbf{Set}^+_{/\Delta^n}$. □

13.3. **Construction.** Let $X \longrightarrow \Delta^n$ be a cocartesian fibration, let σ be a section of the trivial Kan fibration $\mathcal{O}^{cocart}(X) \longrightarrow X \times_{\Delta^n} \mathcal{O}(\Delta^n)$ which restricts to the identity section on X, and let $P = \mathrm{ev}_1 \circ \sigma$ be the corresponding choice of pushforward functor. For $0 \leq i < n$, define $f_i : X_i \times \Delta^1 \longrightarrow X$ by $P \circ (\mathrm{id}_{X_i} \times f_i')$ where $f_i' : \Delta^1 \longrightarrow \mathcal{O}(\Delta^n)$ is the edge $(i = i) \longrightarrow (i \longrightarrow i+1)$, and let $\phi : X_0^\sim \longrightarrow ... \longrightarrow X_n^\sim$ be the sequence obtained from the $f_i \times \{1\}$. We will explain how to produce a map $M(\phi) \longrightarrow X$ over Δ^n via an inductive procedure. Begin by defining the map $M(\phi)_n = X_n \longrightarrow X_n$ to be the identity. Proceeding, observe that $M(\phi)$ is the pushout

$$\begin{array}{ccc} X_0 \times \Delta^{\{1,\dots,n\}} & \longrightarrow & X_0 \times \Delta^n \\ \downarrow{\scriptstyle \gamma} & & \downarrow \\ M(\phi') & \longrightarrow & M(\phi) \end{array}$$

with ϕ' the composable sequence $X_1 \longrightarrow ... \longrightarrow X_n$ and the map γ given by $X_0 \times \Delta^{n-1} \longrightarrow X_1 \times \Delta^{n-1} \longrightarrow M(\phi')$. Given a map $g' : M(\phi') \longrightarrow X$ over Δ^{n-1}, we have a commutative square

$$\begin{array}{ccc} X_0 \times \Delta^1 \cup_{X_0 \times \Delta^{\{1\}}} X_0 \times \Delta^{\{1,\dots,n\}} & \xrightarrow{(f_0,\, g' \circ \gamma)} & X \\ \downarrow & \nearrow & \downarrow \\ X_0 \times \Delta^n & \longrightarrow & \Delta^n, \end{array}$$

and the left vertical map is inner anodyne by [10, Lemma 2.1.2.3] and [10, Corollary 2.3.2.4]. Thus a dotted lift exists and we may extend g' to $g : M(\phi) \longrightarrow X$.

Note that g_i is the identity for all $0 \leq i \leq n$. Therefore, if we instead take the marking on $M(\phi)$ which arises from the degenerate marking on the X_i, then g is (the opposite of) a quasi-equivalence in the terminology of [10, Definition 3.2.2.6], hence a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$ by [10, Proposition 3.2.2.14]. Now by Lemma 13.2, g with the given marking is a cocartesian equivalence.

This construction of $M(\phi) \longrightarrow X$ enjoys a convenient functoriality property: given a cofibration $F : X \longrightarrow Y$ between cocartesian fibrations over Δ^n, we may first choose σ_X as above, and then define

σ_Y to be a lift in the diagram

$$\begin{array}{ccc} (X \times_{\Delta^n} \mathcal{O}(\Delta^n)) \cup_X Y & \xrightarrow{(F \circ \sigma_X, \iota)} & \mathcal{O}^{cocart}(Y) \\ \downarrow & \overset{\sigma_Y}{\nearrow} & \downarrow \sim \\ Y \times_{\Delta^n} \mathcal{O}(\Delta^n) & \xrightarrow{=} & Y \times_{\Delta^n} \mathcal{O}(\Delta^n). \end{array}$$

Consequently, we obtain compatible pushforward functors and a natural transformation $\eta : \phi_X \longrightarrow \phi_Y$, which yields, by a similar argument, a commutative square

$$\begin{array}{ccc} M(\phi_X) & \xrightarrow{M(\eta)} & M(\phi_Y) \\ \downarrow & & \downarrow \\ X & \xrightarrow{F} & Y. \end{array}$$

where the vertical maps are cocartesian equivalences in $s\mathbf{Set}^+_{/\Delta^n}$.

13.4. **Proposition.** *Let $p : X \longrightarrow S$ and $q : Y \longrightarrow S$ be cocartesian fibrations over S and let $F : X \longrightarrow Y$ be a S-functor. Suppose collections of edges $\mathcal{E}_X$, $\mathcal{E}_Y$ of X, Y such that*

(1) $\mathcal{E}_X$ resp. $\mathcal{E}_Y$ contains the p resp. q-cocartesian edges;
(2) For $\mathcal{E}^0_X \subset \mathcal{E}_X$ the subset of edges which are either p-cocartesian or lie in a fiber, we have that $(X, \mathcal{E}^0_X) \subset (X, \mathcal{E}_X)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, and ditto for Y;
(3) $F(\mathcal{E}_X) \subset \mathcal{E}_Y$;
(4) For all $s \in S$, $F_s : (X_s, (\mathcal{E}_X)_s) \longrightarrow (Y_s, (\mathcal{E}_Y)_s)$ is a cocartesian equivalence in $s\mathbf{Set}^+$.

Let $X' = (X, \mathcal{E}_X)$, $Y' = (Y, \mathcal{E}_Y)$, and $F' : X' \longrightarrow Y'$ be the map given on underlying simplicial sets by F. Then for all simplicial sets U and maps $U \longrightarrow S$, F'_U is a cocartesian equivalence in $s\mathbf{Set}^+_{/U}$.

Proof. Without loss of generality, we may assume that an edge e is in $\mathcal{E}_X$ if and only if either e is p-cocartesian or $p(e)$ is degenerate, and ditto for $\mathcal{E}_Y$. First suppose that F is a trivial fibration in $s\mathbf{Set}^+_{/S}$ and for all $s \in S$, F'_s reflects marked edges. Then F' is again a trivial fibration because F' has the right lifting property against all cofibrations. For the general case, factor F as $X \xrightarrow{G} Z \xrightarrow{H} Y$ where G is a cofibration and H is a trivial fibration, and let $Z' = (Z, \mathcal{E}_Z)$ for $\mathcal{E}_Z$ the collection of edges e where e is in $\mathcal{E}_Z$ if and only if $H(e)$ is in $\mathcal{E}_Y$. Then for all $s \in S$, $Z'_s \longrightarrow Y'_s$ is a trivial fibration in $s\mathbf{Set}^+$, so as we just showed $H' : Z' \longrightarrow Y'$ is a trivial fibration. We thereby reduce to the case that F is a cofibration.

Let $\mathcal{U}$ denote the collection of simplicial sets U such that for every map $U \longrightarrow S$, F'_U is a cocartesian equivalence in $s\mathbf{Set}^+_{/U}$. We need to prove that every simplicial set belongs to $\mathcal{U}$. For this, we will verify the hypotheses of [10, Lemma 2.2.3.5]. Conditions (i) and (ii) are obvious, condition (iv) follows from left properness of the cocartesian model structure and [12, Proposition B.2.9], and condition (v) follows from the stability of cocartesian equivalences under filtered colimits and [12, Proposition B.2.9]. It remains to check that every n-simplex belongs to $\mathcal{U}$, so suppose $S = \Delta^n$. Let

$$\begin{array}{ccc} M(\phi_X) & \xrightarrow{M(\eta)} & M(\phi_Y) \\ \downarrow & & \downarrow \\ X & \xrightarrow{F} & Y \end{array}$$

be as in Construction 13.3. Let ϕ'_X be the sequence $X'_0 \longrightarrow ... \longrightarrow X'_n$, where the maps are the same as in ϕ_X, and similarly define ϕ'_Y and η'. Then we have pushout squares

$$\begin{array}{ccc} M(\phi_X) & \longrightarrow & M(\phi'_X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & X'' \end{array}, \qquad \begin{array}{ccc} M(\phi_Y) & \longrightarrow & M(\phi'_Y) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y'' \end{array}$$

with all four vertical maps cocartesian equivalences in $s\mathbf{Set}^+_{/\Delta^n}$. Here we replace X' by X'', which has the same underlying simplicial set X but more edges marked with $X' \subset X''$ left marked anodyne, so that the vertical maps $M(\phi'_X) \longrightarrow X''$ are defined and the squares are pushout squares (again, ditto for Y''). Note that F defines a map $F'' : X'' \longrightarrow Y''$.

Finally, we have the commutative square

$$\begin{array}{ccc} M(\phi'_X) & \xrightarrow{M(\eta')} & M(\phi'_Y) \\ \downarrow & & \downarrow \\ X'' & \xrightarrow{F''} & Y''. \end{array}$$

By assumption, $\eta' : \phi'_X \longrightarrow \phi'_Y$ is a natural transformation through cocartesian equivalences in $s\mathbf{Set}^+$. By Lemma 13.2, $M(\eta')$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$. We deduce that F'', hence F', is as well. $\square$

13.5. **Remark.** By a simple modification of the above arguments, we may further prove that for any marked simplicial set $A \longrightarrow S$, F'_A is a cocartesian equivalence in $s\mathbf{Set}^+_{/A}$. We leave the details of this to the reader.

References

1. Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Advances in Mathematics **304** (2017), 646–727.
2. Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, *Parametrized higher category theory and parametrized higher algebra*, To appear.
3. ———, *Parametrized higher category theory and higher algebra: Exposé I – Elements of parametrized higher category theory*, Preprint `arXiv:1608.03657`, 2015.
4. Clark Barwick, Saul Glasman, and Denis Nardin, *Dualizing cartesian and cocartesian fibrations*, Theory and Applications of Categories **33** (2018), no. 4, 67–94.
5. Clark Barwick and Jay Shah, *Fibrations in ∞-category theory*, MATRIX Book Series, Springer International Publishing, 2018, pp. 17–42.
6. David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Documenta Mathematica **22** (2017), 1225–1266.
7. Bertrand Guillou and J. Peter May, *Models of G-spectra as presheaves of spectra*, Preprint `arXiv:1110.3571`, August 2017.
8. Mark Hovey, *Model categories*, no. 63, American Mathematical Soc., 1999.
9. André Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326. MR 2342834 (2008k:55037)
10. Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)
11. ———, *$(\infty, 2)$-categories and the Goodwillie calculus I*, Preprint from the web page of the author, October 2009.
12. ———, *Higher algebra*, Preprint from the web page of the author, September 2017.
13. Jacob Lurie, *Kerodon*, `https://kerodon.net`, 2021.
14. Aaron Mazel-Gee, *A user's guide to co/cartesian fibrations*, Preprint `arXiv:1510.02402`, 2015.
15. Aaron Mazel-Gee, *On the Grothendieck construction for ∞-categories*, Journal of Pure and Applied Algebra **223** (2019), no. 11, 4602–4651.
16. Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – stability with respect to an orbital ∞-category*, Preprint `arXiv:1608.07704`, 2016.
17. Emily Riehl and Dominic Verity, *Elements of ∞-category theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2022.
18. Jay Shah, *Parametrized higher category theory II: Universal constructions*, Preprint `arXiv:2109.11954`, 2021.

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PARAMETRIZED HIGHER CATEGORY THEORY II: UNIVERSAL CONSTRUCTIONS

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Abstract. We develop parametrized generalizations of a number of fundamental concepts in the theory of ∞-categories, including factorization systems, free fibrations, exponentiable fibrations, relative colimits and relative Kan extensions, filtered and sifted diagrams, and the universal constructions **Ind** and $\mathbf{P}^{\Sigma}$.

Contents

1. Introduction

In this paper, we continue our development of the foundations of *parametrized* (i.e., *indexed*) higher category theory from [Sha21]. Let $\mathcal{T}$ be an ∞-category.

1.1. **Definition.** A $\mathcal{T}$-∞-*category* is a cocartesian fibration $\mathcal{C} \longrightarrow \mathcal{T}^{\mathrm{op}}$. Given two $\mathcal{T}$-∞-categories $\mathcal{C}$ and $\mathcal{D}$, a $\mathcal{T}$-*functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a morphism of cocartesian fibrations, i.e., a functor over $\mathcal{T}^{\mathrm{op}}$ that preserves cocartesian edges.

1.2. **Warning.** In [Sha21], we set $S = \mathcal{T}^{\mathrm{op}}$ and instead spoke of S-∞-categories as cocartesian fibrations $\mathcal{C} \longrightarrow S$. As this is purely an issue of nomenclature, we will not hesitate in referring to results from [Sha21] with our opposite convention in force.

The basic idea of parametrized higher category theory is to develop a theory of ∞-categories internal to the $(\infty, 2)$-category of $\mathcal{T}$-∞-categories. The most fundamental new complication that arises is that of a broader notion of *point*; points should now be thought of as encompassing all the corepresentable left fibrations over $\mathcal{T}^{\mathrm{op}}$. For example, taking $\mathcal{T} = \mathbf{O}_G$ to be the orbit category of a finite group, the theory of G-colimits essentially amalgamates the usual theory of colimits together with that of coproducts indexed by G-orbits.[1] Our original motivation for this project lay in the necessity of having robust ∞-categorical foundations for equivariant homotopy theory – see [BDG+16] and the introduction of [Sha21] for more details on this. However, nothing in [Sha21] or this paper is specific to that application. In principle, the foundational work that we undertake here should prove useful wherever classical indexed category theory has found application, or for base ∞-categories $\mathcal{T}$ of algebro-geometric origin (e.g., in a motivic context). It

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[1]See Theorem B for a precise statement.

will also be essential for our development of the theory of *parametrized* ∞*-operads* in [NS], which underpins the work of Horev and his collaborators [Hor19, HHK$^+$20] on equivariant factorization homology.

Recall that in [Sha21] we accomplished the following primary objectives:

(1) We introduced the concepts of $\mathcal{T}$-(co)limits and $\mathcal{T}$-Kan extensions.[2] We also proved the basic existence and uniqueness theorem for $\mathcal{T}$-Kan extensions (cf. [Sha21, Thm. 10.3] and [Sha21, Thm. 10.5]).

(2) Say that $\mathcal{T}$ is *orbital* if its finite coproduct completion $\mathbf{F}_{\mathcal{T}}$ admits pullbacks. Supposing that $\mathcal{T}$ is orbital, we proved as [Sha21, Cor. 12.15] that a $\mathcal{T}$-∞-category $\mathcal{C}$ is $\mathcal{T}$-cocomplete [Sha21, Def. 5.13] if and only if $\mathcal{C}$ admits all $\mathcal{T}$-coproducts [Sha21, Def. 5.10], fiberwise geometric realizations, and the restriction functors preserve geometric realizations. This was done by a $\mathcal{T}$-colimit decomposition technique in the form of the parametrized Bousfield–Kan formula; cf. [Sha21, Thm. 12.6] and [Sha21, Thm. 12.13] coupled with the parametrized Quillen's Theorem A [Sha21, Thm. 6.7].

(3) We proved a parametrized Yoneda lemma [Sha21, Lem. 11.1] and subsequently established the universal property of the $\mathcal{T}$-∞-category of presheaves [Sha21, Thm. 11.5].

For more involved applications, we need to establish generalizations of all three of these results. Firstly, recall that Lurie in [Lur09, §4.3] set up a theory of *relative Kan extensions.* The idea is that given a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle \pi} \\ \mathcal{D} & \longrightarrow & \mathcal{B} \end{array}$$

of ∞-categories, one can give a pointwise criterion for the existence of an initial filler $i_! F$. If π is in addition a cocartesian fibration, then as a corollary one sees that $i_! F$ always exists if we suppose that for all objects $b \in \mathcal{B}$, the fiber $\mathcal{E}_b$ admits all colimits, and for all morphisms $f : b \longrightarrow b'$, the pushforward functor $f_! : \mathcal{E}_b \longrightarrow \mathcal{E}_{b'}$ preserves all colimits.

We will establish the theory of *relative* $\mathcal{T}$*-colimits* and *relative* $\mathcal{T}$*-left Kan extensions* and thereby obtain a generalization of Lurie's result in the parametrized setting in the form of Theorem 6.2.[3] Since the definitions of relative $\mathcal{T}$-colimit and relative $\mathcal{T}$-left Kan extension are technically involved (cf. Definition 5.1 and Definition 6.1), at this point we will only state a simplified corollary of our main existence result that nonetheless covers the case of most relevance. To formulate the analogous existence criterion in the parametrized context, we need the notion of a *parametrized fiber* of a $\mathcal{T}$-functor:

1.3. **Definition** ([Sha21, Notn. 2.29]). Let $\mathcal{B}$ be a $\mathcal{T}$-∞-category, $b \in \mathcal{B}_t$ an object, and let $\mathrm{Ar}^{cocart}(\mathcal{B})$ be the full subcategory of $\mathrm{Ar}(\mathcal{B})$ on the cocartesian edges in $\mathcal{B}$. We let $\underline{b} := \{b\} \times_{\mathcal{B},\mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{B})$. Note then that the functor $\underline{b} \longrightarrow (\mathcal{T}^{/t})^{\mathrm{op}} \cong \{t\} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}})$ induced by the structure map of $\mathcal{B}$ is a trivial fibration [Sha21, Lem. 12.10] and $\mathrm{ev}_1 : \underline{b} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-functor covering $(\mathcal{T}^{/t})^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$.

Now suppose $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-functor. The *parametrized fiber* of π over b is the $\mathcal{T}^{/t}$-∞-category

$$\mathcal{E}_{\underline{b}} := \underline{b} \times_{\mathrm{ev}_1, \mathcal{B}, \pi} \mathcal{E}.$$

Theorem A. *Suppose we have a commutative diagram of* $\mathcal{T}$*-*∞*-categories*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{E} \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle \pi} \\ \mathcal{D} & \longrightarrow & \mathcal{B} \end{array}$$

in which i *is the inclusion of a full* $\mathcal{T}$*-subcategory* [Sha21, Def. 2.2] *and* π *is in addition a cocartesian fibration. Consider the restriction functor*

$$i^* : \mathrm{Fun}_{/\mathcal{B},\mathcal{T}}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{E})$$

where $\mathrm{Fun}_{/\mathcal{B},\mathcal{T}}(-,-)$ *denotes the full subcategory of* $\mathrm{Fun}_{/\mathcal{B}}(-,-)$ *spanned by the* $\mathcal{T}$*-functors. Suppose that for all* $b \in \mathcal{B}_t$*, the parametrized fiber* $\mathcal{E}_{\underline{b}}$ *admits all* $\mathcal{T}^{/t}$*-colimits, and for all* $f : b \longrightarrow b' \in \mathcal{B}_t$*, the induced pushforward* $\mathcal{T}^{/t}$*-functor* $f_! : \mathcal{E}_{\underline{b}} \longrightarrow \mathcal{E}_{\underline{b'}}$ *preserves all* $\mathcal{T}^{/t}$*-colimits. Then* i^* *admits a left adjoint* $i_!$*. Moreover, the unit transformation* $\mathrm{id} \Rightarrow i^* i_!$ *is an equivalence, so* $i_!$ *is fully faithful.*

[2] We give a rapid review of these concepts in Section 2.

[3] Of course, one may dualize appropriately to obtain analogous results involving relative $\mathcal{T}$-limits and relative $\mathcal{T}$-right Kan extensions; cf. [Sha21, Cor. 5.25].

1.4. **Remark.** As with the ordinary theory of Kan extensions, the full faithfulness assertion in Theorem A is where the pointwise formula for $i_!$ comes into play. In particular, even if we assumed the relevant presentability hypotheses, it would not suffice to appeal to the adjoint functor theorem to verify this property.

Secondly, we develop the theory of $\mathcal{T}$-κ-small, $\mathcal{T}$-filtered, and $\mathcal{T}$-sifted $\mathcal{T}$-∞-categories. In order to speak of small and large simplicial sets and ∞-categories, we henceforth fix two strongly inaccessible cardinals $\delta_0 < \delta_1$.

1.5. **Convention.** For simplicity, we now also suppose throughout that the base ∞-category $\mathcal{T}$ is small.

Let **Cat** denote the (large) ∞-category of small ∞-categories and let $\mathbf{Cat}_{\mathcal{T}} := \mathbf{Cat}^{cocart}_{/\mathcal{T}^{\mathrm{op}}} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat})$ be the ∞-category of $\mathcal{T}$-small $\mathcal{T}$-∞-categories.[4]

1.6. **Definition** (Definition 8.2)**.** Let $\Delta_{\mathcal{T}} \subset \mathbf{Cat}_{\mathcal{T}}$ be the full subcategory spanned by the objects

$$\{\Delta^n \times \mathrm{Map}_{\mathcal{T}}(-, t)\}_{t \in \mathcal{T}, n \geq 0}.$$

Then for every regular cardinal κ, we define the full subcategory $\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}} \subset \mathbf{Cat}_{\mathcal{T}}$ to be the smallest full subcategory that contains $\Delta_{\mathcal{T}}$ and is closed under all colimits indexed by κ-small simplicial sets. We say that a $\mathcal{T}$-small $\mathcal{T}$-∞-category $\mathcal{C}$ is $\mathcal{T}$-*κ-small* if it belongs to $\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}}$. If $\kappa = \omega$, we also say that $\mathcal{C}$ is $\mathcal{T}$-*finite*.

1.7. **Remark.** Adopting the terminology of Definition 1.6 entails speaking of a host of seemingly redundant expressions such "$\mathcal{T}$-finite $\mathcal{T}$-∞-category". We avoid simply writing e.g. "finite $\mathcal{T}$-∞-category" because of the possible ambiguity as to whether, given some $\mathcal{T}$-∞-category $\mathcal{C}$, "finite" refers to $\mathcal{C}$ being finite as an ∞-category or as a $\mathcal{T}$-∞-category.

We then have the following generalization of [Sha21, Cor. 12.15], whose proof turns out to be far simpler than our earlier strategy of appealing to the parametrized Bousfield–Kan formula. We give the most useful formulation of this here; a slightly more general statement is recorded as Theorem 8.6.

Theorem B. *Suppose that $\mathcal{T}$ is orbital. Let $\mathcal{C}$ be an $\mathcal{T}$-∞-category and κ a regular cardinal. Then $\mathcal{C}$ strongly admits*[5] *all $\mathcal{T}$-κ-small $\mathcal{T}$-colimits if and only if*

(1) For every $t \in \mathcal{T}$, the fiber $\mathcal{C}_t$ admits all κ-small colimits, and for every $\alpha : s \longrightarrow t$, the restriction functor $\alpha^* : \mathcal{C}_t \longrightarrow \mathcal{C}_s$ preserves κ-small colimits.

(2) For every map $\alpha : U \longrightarrow V$ of finite $\mathcal{T}$-sets,[6] *the restriction functor $\alpha^* : \mathcal{C}_V \longrightarrow \mathcal{C}_U$ admits a left adjoint $\alpha_!$.*[7]

(3) $\mathcal{C}$ satisfies the Beck-Chevalley condition*, i.e., for every pullback square*

$$\begin{array}{ccc} U' & \xrightarrow{\beta'} & U \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ V' & \xrightarrow{\beta} & V \end{array}$$

in $\mathbf{F}_{\mathcal{T}}$, the mate

$$\alpha'_! \beta'^* \Rightarrow \beta^* \alpha_! : \mathcal{C}_U \longrightarrow \mathcal{C}_{V'}$$

is an equivalence.

1.8. **Remark.** Again supposing that $\mathcal{T}$ is orbital, note that by [Sha21, Prop. 5.12] $\mathcal{C}$ admits finite $\mathcal{T}$-coproducts if and only if conditions (2) and (3) in Theorem B hold. Moreover, the ordinary (∞-categorical) Bousfield–Kan formula shows that an ∞-category is cocomplete if and only if it admits coproducts and geometric realizations (cf. [Sha21, Cor. 12.3]). Taking κ to be our fixed inaccessible cardinal δ_0, we then see that the hypotheses of Theorem B are equivalent to those of [Sha21, Cor. 12.15].

[4]Since we suppose that $\mathcal{T}$ is small, a $\mathcal{T}$-∞-category $\mathcal{C}$ is $\mathcal{T}$-small if and only if $\mathcal{C}$ is small.

[5]We recall this notion as Definition 2.8.

[6]A finite $\mathcal{T}$-set is defined to be an object of the finite coproduct completion $\mathbf{F}_{\mathcal{T}}$ of $\mathcal{T}$.

[7]For a finite $\mathcal{T}$-set U with orbit decomposition $U_1 \sqcup ... \sqcup U_n$, we write $\mathcal{C}_U := \prod_{i=1}^n \mathcal{C}_{U_i}$, and the contravariant functoriality in the finite $\mathcal{T}$-set is inherited from that for orbits.

1.9. **Remark.** In [Nar16], Nardin implicitly *defines* a $\mathcal{T}$-∞-category to strongly admit $\mathcal{T}$-finite $\mathcal{T}$-(co)limits[8] if conditions (1) through (3) in Theorem B are satisfied. Moreover, using his formulation, a $\mathcal{T}$-stable $\mathcal{T}$-∞-category [Nar16, Def. 7.1] by definition strongly admits all $\mathcal{T}$-finite $\mathcal{T}$-colimits and $\mathcal{T}$-finite $\mathcal{T}$-limits. One practical consequence of Theorem B is that $\mathcal{T}$-stable $\mathcal{T}$-∞-categories then strongly admit $\mathcal{T}$-(co)limits indexed by an *a priori* larger class of $\mathcal{T}$-diagrams; for instance, when $\mathcal{T} = \mathbf{O}_G$ this includes those G-spaces that admit the structure of a finite G-CW complex.

Moving onto the theory of $\mathcal{T}$-κ-filtered and $\mathcal{T}$-sifted $\mathcal{T}$-∞-categories, we may make the following definitions as the evident parametrized generalizations of [Lur09, Def. 5.3.1.7] and [Lur09, Def. 5.5.8.1].

1.10. **Definition** (Definition 8.8)**.** Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category and let κ be a regular cardinal. We say that $\mathcal{J}$ is $\mathcal{T}$-*κ-filtered* if for all $t \in \mathcal{T}$ and $\mathcal{T}^{/t}$-κ-small $\mathcal{K}$, every $\mathcal{T}^{/t}$-functor $p : \mathcal{K} \longrightarrow \mathcal{J}_{\underline{t}}$ admits an extension to a $\mathcal{T}^{/t}$-functor $\overline{p} : \mathcal{K}^{\underline{\triangleright}} \longrightarrow \mathcal{J}_{\underline{t}}$.[9]

1.11. **Notation.** For a finite $\mathcal{T}$-set U with orbit decomposition $U_1 \sqcup ... \sqcup U_n$, we write

$$\underline{U} := \coprod_{i=1}^{n} (\mathcal{T}^{\mathrm{op}})^{U_i/}$$

for the $\mathcal{T}$-∞-category given by the coproduct of corepresentable left fibrations; this straightens to the presheaf $\mathrm{Map}_{\mathbf{F}_{\mathcal{T}}}(-, U)|_{\mathcal{T}^{\mathrm{op}}}$.

Recall from [Sha21] that we write $\underline{\mathrm{Fun}}_{\mathcal{T}}(-,-)$ for the internal hom for $\mathcal{T}$-∞-categories (defined at the level of marked simplicial sets as [Sha21, Def. 3.2]); for every $t \in \mathcal{T}$ we have that $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})_t \simeq \mathrm{Fun}_{\mathcal{T}^{/t}}(\mathcal{C}_{\underline{t}}, \mathcal{D}_{\underline{t}})$, and for every $\alpha : s \longrightarrow t$, the restriction functor α^* is given by restricting $\mathcal{T}^{/t}$-functors to $\mathcal{T}^{/s}$-functors.

1.12. **Definition** (Definition 8.14)**.** Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category. Then $\mathcal{J}$ is $\mathcal{T}$-*sifted* if for all $t \in \mathcal{T}$ and finite $\mathcal{T}^{/t}$-sets U, the diagonal $\mathcal{T}^{/t}$-functor $\delta : \mathcal{J}_{\underline{t}} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}^{/t}}(\underline{U}, \mathcal{J}_{\underline{t}})$ is $\mathcal{T}^{/t}$-cofinal in the sense of [Sha21, Def. 6.8], i.e. δ is fiberwise cofinal.

Our main theorems about these concepts should be read as confirming the following expectation: $\mathcal{T}$-filtered and $\mathcal{T}$-sifted $\mathcal{T}$-colimits are computed as ordinary filtered and ordinary sifted colimits in the fibers. To say this precisely, we need another definition.

1.13. **Definition** (Definition 9.5)**.** Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category. We say that $\mathcal{J}$ is *cofinal-constant* (cc) if for all morphisms $\alpha : s \longrightarrow t$ in $\mathcal{T}$, the restriction functor $\alpha^* : \mathcal{J}_t \longrightarrow \mathcal{J}_s$ is cofinal.

1.14. **Remark.** Let $\mathcal{J}$ be a cofinal-constant $\mathcal{T}$-∞-category and $p : \mathcal{J} \longrightarrow \mathcal{C}$ a $\mathcal{T}$-functor. Moreover, suppose that $\mathcal{T}$ has a terminal object t. Then by our hypothesis on $\mathcal{J}$ and [Sha21, Thm. 6.7], the $\mathcal{T}$-functor $\chi : \mathcal{J}_t \times \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{J}$ uniquely determined by the inclusion $\mathcal{J}_t \subset \mathcal{J}$ is $\mathcal{T}$-cofinal. Consequently, we obtain an equivalence

$$\mathrm{colim}^{\mathcal{T}}_{\mathcal{J}} p \simeq \mathrm{colim}^{\mathcal{T}}_{\mathcal{J}_t \times \mathcal{T}^{\mathrm{op}}} p \circ \chi$$

provided that either $\mathcal{T}$-colimit exists.

Now let **Spc** denote the (large) ∞-category of small spaces and let $\underline{\mathbf{Spc}}_{\mathcal{T}}$ be the $\mathcal{T}$-∞-category of small $\mathcal{T}$-spaces [Sha21, Exm. 3.12].

Theorem C (Theorem 8.11 and Theorem 8.13)**.** *Suppose that $\mathcal{T}$ is orbital. Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category and let κ be a regular cardinal. The following conditions are equivalent:*

(1) $\mathcal{J}$ is $\mathcal{T}$-κ-filtered.
(2) For all $t \in \mathcal{T}$, $\mathcal{J}_t$ is κ-filtered, and $\mathcal{J}$ is cofinal-constant.
(3) The $\mathcal{T}$-colimit $\mathcal{T}$-functor

$$\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{J}} : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{J}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \longrightarrow \underline{\mathbf{Spc}}_{\mathcal{T}}$$

strongly preserves $\mathcal{T}$-κ-small $\mathcal{T}$-limits.

[8]Note that Nardin writes instead "finite $\mathcal{T}$-(co)limits" for this notion and he also doesn't use the adjective "strongly"; see Remark 1.7.

[9]We recall the parametrized cone as Definition 2.2. Here, for the $\mathcal{T}^{/t}$-∞-category $\mathcal{K}$, $\mathcal{K}^{\underline{\triangleright}}$ is notation for $\mathcal{K} \star_{(\mathcal{T}^{/t})^{\mathrm{op}}} (\mathcal{T}^{/t})^{\mathrm{op}}$.

(4) For all $t \in \mathcal{T}$ *and* $\mathcal{T}^{/t}$*-*κ*-small* $\mathcal{K}$*, the diagonal* $\mathcal{T}^{/t}$*-functor*

$$\delta : \mathcal{J}_{\underline{t}} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}^{/t}}(\mathcal{K}, \mathcal{J}_{\underline{t}})$$

is $\mathcal{T}^{/t}$*-cofinal.*

Theorem D (Theorem 8.15)**.** *Suppose that* $\mathcal{T}$ *is orbital and let* $\mathcal{J}$ *be a* $\mathcal{T}$*-*∞*-category. The following conditions are equivalent:*

(1) $\mathcal{J}$ *is* $\mathcal{T}$*-sifted.*
(2) For all $t \in \mathcal{T}$*,* $\mathcal{J}_t$ *is sifted, and* $\mathcal{J}$ *is cofinal-constant.*
(3) The $\mathcal{T}$*-colimit* $\mathcal{T}$*-functor*

$$\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{J}} : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{J}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \longrightarrow \underline{\mathbf{Spc}}_{\mathcal{T}}$$

preserves finite $\mathcal{T}$*-products.*

Thirdly, building upon our earlier discussion of $\mathcal{T}$-presheaves, we introduce the universal constructions $\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C})$ and $\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C})$ that freely adjoin $\mathcal{T}$-κ-filtered $\mathcal{T}$-colimits and $\mathcal{T}$-sifted $\mathcal{T}$-colimits to $\mathcal{C}$, respectively (Definition 9.9). These are essentially defined to be the minimal full $\mathcal{T}$-subcategories of $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}) := \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}})$ closed under the relevant $\mathcal{T}$-colimits. However, in view of condition (2) in Theorem C and Theorem D, it turns out that $\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(-)$ and $\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(-)$ are obtained by fiberwise application of $\mathbf{Ind}^{\kappa}$ and $\mathbf{P}^{\Sigma}$ (cf. Variant 9.8).

Our main result identifies these constructions in terms of $\mathcal{T}$-presheaves that strongly preserve certain $\mathcal{T}$-limits if $\mathcal{C}$ admits sufficiently many $\mathcal{T}$-colimits.

1.15. **Notation.** Let $\mathcal{D}$ and $\mathcal{E}$ be $\mathcal{T}$-∞-categories, and suppose in the following that $\mathcal{D}, \mathcal{E}$ strongly admit the relevant $\mathcal{T}$-(co)limits. We introduce notation for certain full $\mathcal{T}$-subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})$, which may be specified by indicating over each $t \in \mathcal{T}$ what $\mathcal{T}^{/t}$-functors $\mathcal{D}_{\underline{t}} \longrightarrow \mathcal{E}_{\underline{t}}$ span the fiber:

(1) $\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})$: take those $\mathcal{T}^{/t}$-functors that strongly preserve all (small) $\mathcal{T}^{/t}$-colimits.
(2) $\underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})$: take those $\mathcal{T}^{/t}$-functors that preserve finite $\mathcal{T}^{/t}$-products.[10]
(3) $\underline{\mathrm{Fun}}^{\sqcup}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})$: take those $\mathcal{T}^{/t}$-functors that preserve finite $\mathcal{T}^{/t}$-coproducts.
(4) $\underline{\mathrm{Fun}}^{\kappa\text{-lex}}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})$: take those $\mathcal{T}^{/t}$-functors that strongly preserve $\mathcal{T}$-κ-small $\mathcal{T}^{/t}$-limits.
(5) $\underline{\mathrm{Fun}}^{\kappa\text{-rex}}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})$: take those $\mathcal{T}^{/t}$-functors that strongly preserve $\mathcal{T}$-κ-small $\mathcal{T}^{/t}$-colimits.

We only state the most important points here and refer the reader to the main body of the paper for the more comprehensive theorem.

Theorem E (Theorem 9.11)**.** *Suppose that* $\mathcal{T}$ *is orbital and let* $\mathcal{C}$ *be a* $\mathcal{T}$*-*∞*-category.*

(1) Suppose that $\mathcal{C}$ *admits finite* $\mathcal{T}$*-coproducts. We then have an equality*

$$\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C}) = \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}}).$$

Moreover, $\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C})$ *is* $\mathcal{T}$*-cocomplete, and given any* $\mathcal{T}$*-cocomplete* $\mathcal{T}$*-*∞*-category* $\mathcal{D}$*, restriction along the* $\mathcal{T}$*-Yoneda embedding* $j^{\Sigma}_{\mathcal{T}} : \mathcal{C} \hookrightarrow \underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C})$ *implements an equivalence*

$$\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\sqcup}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

with inverse given by $\mathcal{T}$*-left Kan extension.*

(2) Suppose that $\mathcal{C}$ *strongly admits* $\mathcal{T}$*-*κ*-small* $\mathcal{T}$*-colimits. We then have an equality*

$$\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C}) = \underline{\mathrm{Fun}}^{\kappa\text{-lex}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}}).$$

Moreover, $\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C})$ *is* $\mathcal{T}$*-cocomplete, and given any* $\mathcal{T}$*-cocomplete* $\mathcal{T}$*-*∞*-category* $\mathcal{D}$*, restriction along the* $\mathcal{T}$*-Yoneda embedding* $j^{\kappa}_{\mathcal{T}} : \mathcal{C} \hookrightarrow \underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C})$ *implements an equivalence*

$$\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\kappa\text{-rex}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

with inverse given by $\mathcal{T}$*-left Kan extension.*

[10] Note that there is no distinction between strongly preserving and preserving finite $\mathcal{T}^{/t}$-products, and likewise for finite $\mathcal{T}^{/t}$-coproducts.

Lastly, we also lay parametrized foundations for two other important concepts in the theory of ∞-categories: *factorization systems* and *exponentiable* (i.e., *flat*) *fibrations*.[11] We defer the statements of these results to their respective sections 3 and 4. Most notably, we use the theory of $\mathcal{T}$-factorization systems to establish the universal property of the free $\mathcal{T}$-cocartesian fibration (Example 3.8), while we use the theory of $\mathcal{T}$-flat fibrations and the associated $\mathcal{T}$-*pairing construction* (Theorem-Construction 4.2) to study $\mathcal{T}$-(co)limits in a $\mathcal{T}$-∞-category of sections (Theorem 4.16).

1.16. **Remark.** In the case $\mathcal{T} = *$, our main Theorem 3.6 on parametrized factorization systems applies to give a common generalization of the proof of the universal property of the usual free cocartesian fibration ([GHN17, Thm. 4.5]) with that of the universal property of the $\mathcal{O}$-monoidal envelope for an ∞-operad $\mathcal{O}$ ([Lur17, Prop. 2.2.4.9]).[12] In [NS], we will apply Theorem 3.6 to establish the theory of $\mathcal{O}$-monoidal envelopes for a $\mathcal{T}$-∞-operad $\mathcal{O}$.

1.17. **Remark.** Our main interest in Theorem 4.16 lies in using it in [NS] to study $\mathcal{T}$-(co)limits in a $\mathcal{T}$-∞-category of $\mathcal{O}$-algebras for a $\mathcal{T}$-∞-operad $\mathcal{O}$. Also see [BH21, Prop. 7.6] for a similar type of statement in the context of normed E_∞-algebras in motivic homotopy theory.

In the appendix, we take the opportunity to give the correct[13] definition of an exponentiable fibration of ∞-operads and then the construction of $\mathcal{O}$-*promonoidal Day convolution* with respect to a base ∞-operad $\mathcal{O}^\otimes$ (Theorem-Construction 10.6). This generalizes Lurie's construction in [Lur17, §2.2.6], which supposes that the source ∞-operad in question is $\mathcal{O}$-monoidal. We saw fit to include this material here because the main lemma behind it (Lemma 10.1) is also used to establish the theory of $\mathcal{T}$-flat fibrations.

1.18. **Remark.** Vladimir Hinich has informed us that our treatment of $\mathcal{O}$-promonoidal Day convolution is a slightly reorganized version of his discussion in [Hin20, §2.8]. In particular, our Theorem-Construction 10.6 is essentially his [Hin20, Prop. 2.8.3], and Lemma 10.1 when specialized to the context of ∞-operads is his [Hin20, Lem. 2.8.4].

Notation and terminology. We collect a few miscellaneous pieces of notation and terminology from [Sha21] that we have not introduced yet in our discussion.

1.19. **Convention.** Let $X, Y \longrightarrow Z$ be maps of simplicial sets. Unless otherwise indicated, when we write $X \times_Z \mathrm{Ar}(Z) \times_Z Y$ we mean $X \times_{Z,\mathrm{ev}_0} \mathrm{Ar}(Z) \times_{\mathrm{ev}_1,Z} Y$ (i.e., evaluation at the source goes to the left and evaluation at the target goes to the right).

We will need to use the theory of marked simplicial sets in various places in this paper; see [Sha21, §2] for a review.

1.20. **Notation.** (1) Given a simplicial set X, we let $X^\flat$ be the minimal marking on X and $X^\sharp$ the maximal marking on X.
(2) If $p : X \longrightarrow S$ is a cocartesian fibration, then we let ${}_\natural X$ denote X with its p-cocartesian edges marked.

1.21. **Notation.** We will generally write $\mathcal{T}^{\mathrm{op}}$ as $*_\mathcal{T}$ when we wish to think of it as the terminal $\mathcal{T}$-∞-category.

1.22. **Definition.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. We define the $\mathcal{T}$-∞-*category of arrows* in $\mathcal{C}$ to be

$$\mathrm{Ar}_\mathcal{T}(\mathcal{C}) := \mathcal{T}^{\mathrm{op}} \times_{\mathrm{Ar}(\mathcal{T}^{\mathrm{op}})} \mathrm{Ar}(\mathcal{C})$$

where the map $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ar}(\mathcal{T}^{\mathrm{op}})$ is the identity section.

1.23. **Recollection** ([Sha21, Def. 4.1])**.** Let S be a simplicial set, let $\iota : \partial\Delta^1 \times S \subset \Delta^1 \times S$ be the inclusion functor, and consider the right adjoint

$$\iota_* : \mathbf{sSet}_{/\partial\Delta^1 \times S} \longrightarrow \mathbf{sSet}_{/\Delta^1 \times S}$$

[11] Some authors (e.g., Ayala and Francis [AF20]) reserve the term *exponentiable* for the homotopy invariant definition, but we will elide this distinction in our narrative here.

[12] This line of reasoning is well-known to experts and has also appeared in the literature as [AMGR17, Prop. B.1]; we thank Rune Haugseng for the pointer.

[13] If $\mathcal{O}^\otimes$ is the commutative ∞-operad, then it turns out that our earlier definition of symmetric promonoidal given in [BGS20] was insufficiently general; see Example 10.4. We thank Yonatan Harpaz for alerting us to this issue.

to pullback along ι. Then for maps $p, q : X, Y \longrightarrow S$ of simplicial sets, we define the *S-join* $X \star_S Y$ to be $\iota_*(X, Y)$; this recovers the ordinary join if $S = *$. Note that if we let $\chi : S \times \Delta^1 \longrightarrow S \star S$ be the map adjoint to $\mathrm{id}_{S\times\partial\Delta^1}$ (using the universal property of the ordinary join), then we have a canonical isomorphism

$$X \star_S Y \cong S \times \Delta^1 \underset{\chi, S\star S, p\star q}{\times} X \star Y,$$

so the S-join is the *relative join* in the sense of Lurie [Lur21, Tag 0241]. In keeping with the terminology of [Sha21], however, we will prefer to generically call this the parametrized join.

Now if $S = \mathcal{T}^{\mathrm{op}}$ and we have $\mathcal{T}$-∞-categories $\mathcal{C}$ and $\mathcal{D}$, the $\mathcal{T}^{\mathrm{op}}$-join $\mathcal{C} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{D}$ is again a $\mathcal{T}$-∞-category, and in fact the structure map to $\mathcal{T}^{\mathrm{op}} \times \Delta^1$ is a $\mathcal{T}$-functor (cf. [Sha21, Prop. 4.3]).

1.24. **Recollection** ([Sha21, Def. 8.3]). Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{T}$-∞-categories and let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a relative adjunction with respect to $\mathcal{T}^{\mathrm{op}}$ [Lur17, Def. 7.3.2.2]. Then we say that $F \dashv G$ is a *$\mathcal{T}$-adjunction* if F and G are both $\mathcal{T}$-functors.

1.25. **Recollection** ([Sha21, Def. 7.1]). Let $p : \mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-functor. We say that p is a *$\mathcal{T}$-fibration* if p is a categorical fibration. In this case, p is *$\mathcal{T}$-cocartesian*, resp. *$\mathcal{T}$-cartesian* if

1. For every object $t \in \mathcal{T}$, $p_t : \mathcal{X}_t \longrightarrow \mathcal{B}_t$ is a cocartesian, resp. cartesian fibration.
2. For every morphism $\alpha : s \longrightarrow t$, the restriction functor $\alpha^* : \mathcal{X}_t \longrightarrow \mathcal{X}_s$ carries p_t-cocartesian, resp. p_t-cartesian edges to p_s-cocartesian, resp. p_s-cartesian edges.

If $p : \mathcal{X} \longrightarrow \mathcal{B}$ and $q : \mathcal{Y} \longrightarrow \mathcal{B}$ are two $\mathcal{T}$-cocartesian fibrations, we say that a $\mathcal{T}$-functor $F : \mathcal{X} \longrightarrow \mathcal{Y}$ over $\mathcal{B}$ is a *morphism of $\mathcal{T}$-cocartesian fibrations* if F preserves fiberwise (with respect to $\mathcal{T}$) cocartesian edges. Similarly, we have the analogous definition of a morphism of $\mathcal{T}$-cartesian fibrations.

Finally, note that p is $\mathcal{T}$-cocartesian if and only if p is a cocartesian fibration [Sha21, Rem. 7.4].

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2. Recollections on parametrized limits and colimits

In this section, we give a streamlined exposition of the concepts of parametrized (co)limits and Kan extensions introduced in [Sha21]. This is done primarily to fix notation and make this paper more self-contained. For the reader already familiar with [Sha21], the only points to bear in mind are our more concise notation for parametrized cones (Definition 2.2) and the notion of strongly admitting and preserving $\mathcal{K}$-indexed $\mathcal{T}$-colimits with respect to certain collections $\mathcal{K}$ of parametrized diagrams (Definition 2.8).

2.1. **Notation** ([Sha21, Notn. 3.5]). Let $p : \mathcal{K} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor. We then let

$$\sigma_p : *_{\mathcal{T}} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})$$

denote the cocartesian section given by adjointing the map $\mathrm{Ar}(\mathcal{T}^{\mathrm{op}})^\sharp \times_{\mathcal{T}^{\mathrm{op}}} {}_\natural\mathcal{K} \xrightarrow{\mathrm{pr}} {}_\natural\mathcal{K} \xrightarrow{p} {}_\natural\mathcal{C}$. This is an explicit choice of $\mathcal{T}$-functor corresponding to p under the equivalence

$$\mathrm{Fun}_{\mathcal{T}}(*_{\mathcal{T}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})$$

of [Sha21, Prop. 3.4].

2.2. **Definition** (Cones and slices). Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. We let

$$\mathcal{C}^{\underline{\triangleright}} := \mathcal{C} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{T}^{\mathrm{op}}\,, \qquad \mathcal{C}^{\underline{\triangleleft}} := \mathcal{T}^{\mathrm{op}} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{C}$$

denote the *$\mathcal{T}$-right* and *$\mathcal{T}$-left cones* on $\mathcal{C}$. We also write $v : *_{\mathcal{T}} \subset \mathcal{C}^{\underline{\triangleright}}$ or $\mathcal{C}^{\underline{\triangleleft}}$ for the inclusion of the cone $\mathcal{T}$-point.

For a $\mathcal{T}$-functor $p : \mathcal{K} \longrightarrow \mathcal{C}$, we then let

$$\mathcal{C}^{(p,\mathcal{T})/} := *_{\mathcal{T}} \times_{\sigma_p, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}^{\underline{\triangleright}}, \mathcal{C})\,, \qquad \mathcal{C}^{/(p,\mathcal{T})} := *_{\mathcal{T}} \times_{\sigma_p, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}^{\underline{\triangleleft}}, \mathcal{C})$$

denote the *slice* $\mathcal{T}$-∞-categories.

We will also need in a few places the following smaller model for slicing over and under a $\mathcal{T}$-object.

2.3. **Definition.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. For any object $x \in \mathcal{C}_t$, we write

$$\mathcal{C}^{/\underline{x}} := \mathrm{Ar}_{\mathcal{T}}(\mathcal{C}) \times_{\mathcal{C}} \underline{x}\,, \qquad \mathcal{C}^{\underline{x}/} := \underline{x} \times_{\mathcal{C}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{C})$$

and regard these as $\mathcal{T}^{/t}$-∞-categories via composition of the projection to $\underline{x}$ with the trivial fibration $\underline{x} \xrightarrow{\simeq} (\mathcal{T}^{/t})^{\mathrm{op}}$ ([Sha21, Lem. 12.10]).

2.4. **Observation** ([Sha21, Prop. 4.30])**.** In Definition 2.3, if we write $i_x : \underline{x} \longrightarrow \mathcal{C}_{\underline{t}}$ for the $\mathcal{T}^{/t}$-functor defined by x, then we have canonical equivalences

$$\mathcal{C}^{/\underline{x}} \simeq (\mathcal{C}_{\underline{t}})^{/(i_x, \mathcal{T}^{/t})}\,, \qquad \mathcal{C}^{\underline{x}/} \simeq (\mathcal{C}_{\underline{t}})^{(i_x, \mathcal{T}^{/t})/}$$

of $\mathcal{T}^{/t}$-∞-categories over $\mathcal{C}_{\underline{t}}$. Similarly, for a cocartesian section $\sigma : *_{\mathcal{T}} \longrightarrow \mathcal{C}$, we have canonical equivalences

$$\mathrm{Ar}_{\mathcal{T}}(\mathcal{C}) \times_{\mathcal{C},\sigma} *_{\mathcal{T}} \simeq \mathcal{C}^{/(\sigma,\mathcal{T})}\,, \qquad *_{\mathcal{T}} \times_{\sigma,\mathcal{C}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{C}) \simeq \mathcal{C}^{(\sigma,\mathcal{T})/}$$

of $\mathcal{T}$-∞-categories over $\mathcal{C}$.

We now proceed to our discussion on parametrized colimits; the case of parametrized limits is dual in view of [Sha21, Cor. 5.25] and hence will not be explicitly considered.

2.5. **Definition** ([Sha21, Def. 5.1-2])**.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. A $\mathcal{T}$-functor $\sigma : *_{\mathcal{T}} \longrightarrow \mathcal{C}$ is a $\mathcal{T}$-*initial object* if and only if $\sigma(t) \in \mathcal{C}_t$ is an initial object for all $t \in \mathcal{T}$. A $\mathcal{T}$-functor $\overline{p} : \mathcal{K}^{\underline{\triangleright}} \longrightarrow \mathcal{C}$ is then a $\mathcal{T}$-*colimit diagram* if and only if the $\mathcal{T}$-functor

$$(\mathrm{id}, \sigma_{\overline{p}}) : *_{\mathcal{T}} \longrightarrow \mathcal{C}^{(p,\mathcal{T})/} = *_{\mathcal{T}} \times_{\sigma_p, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}^{\underline{\triangleright}}, \mathcal{C})$$

is a $\mathcal{T}$-initial object. Lastly, we say that a $\mathcal{T}$-functor $p : \mathcal{K} \longrightarrow \mathcal{C}$ *admits a* $\mathcal{T}$-*colimit* if p admits an extension to a $\mathcal{T}$-colimit diagram $\overline{p}$, and we then write $\mathrm{colim}^{\mathcal{T}}_{\mathcal{K}} p = \overline{p}|_v$. If $\mathcal{T}$ moreover has a terminal object t, we will also identify the cocartesian section $\mathrm{colim}^{\mathcal{T}}_{\mathcal{K}} p$ with its value at t.[14]

2.6. **Notation.** Let $p : \mathcal{K} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor and let $\delta : \mathcal{C} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})$ be the constant $\mathcal{T}$-functor. We then write

$$\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{K}} : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C}) \dashrightarrow \mathcal{C}$$

for the partially-defined[15] $\mathcal{T}$-left adjoint of δ.

The next observation is the trivial case of [Sha21, Cor. 9.16] where we let $\mathcal{D} = \mathcal{T}^{\mathrm{op}}$ there.

2.7. **Observation.** For a $\mathcal{T}$-functor $p : \mathcal{K} \longrightarrow \mathcal{C}$, $\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{K}}$ is defined on an object $p : \mathcal{K}_{\underline{t}} \longrightarrow \mathcal{C}_{\underline{t}}$ in the fiber $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})_t \simeq \mathrm{Fun}_{\mathcal{T}^{/t}}(\mathcal{K}_{\underline{t}}, \mathcal{C}_{\underline{t}})$ if and only if p admits a $\mathcal{T}^{/t}$-colimit, in which case $\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{K}} p \simeq (\mathrm{colim}^{\mathcal{T}^{/t}}_{\mathcal{K}_{\underline{t}}} p)(t)$. In particular, if for each $t \in \mathcal{T}$ the parametrized fiber $\mathcal{C}_{\underline{t}}$ admits all $\mathcal{K}_{\underline{t}}$-indexed $\mathcal{T}^{/t}$-colimits, then $\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{K}}$ is defined on its entire domain.

Passing to cocartesian sections, we then see that

$$\mathrm{colim}^{\mathcal{T}}_{\mathcal{K}} : \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C}) \dashrightarrow \mathrm{Fun}_{\mathcal{T}}(*_{\mathcal{T}}, \mathcal{C})$$

is a partial left adjoint to the functor given by precomposing with the structure map of $\mathcal{K}$, and $\mathrm{colim}^{\mathcal{T}}_{\mathcal{K}}$ is defined on its entire domain if $\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{K}}$ is (but possibly not conversely).

This observation already highlights the need to systematically distinguish between $\mathcal{T}$-colimits in $\mathcal{C}$ and $\mathcal{T}^{/t}$-colimits in the parametrized fibers $\mathcal{C}_{\underline{t}}$. We do this as follows:

2.8. **Definition.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category.

(1) $\mathcal{C}$ *strongly admits* all $\mathcal{T}$-colimits, i.e., is $\mathcal{T}$-*cocomplete* [Sha21, Def. 5.13], if for each $t \in \mathcal{T}$, $\mathcal{C}_{\underline{t}}$ admits all $\mathcal{T}^{/t}$-colimits.
(2) If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{T}$-cocomplete $\mathcal{T}$-∞-categories, then a $\mathcal{T}$-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ *strongly preserves* all $\mathcal{T}$-colimits if for each $t \in \mathcal{T}$, $F_{\underline{t}} : \mathcal{C}_{\underline{t}} \longrightarrow \mathcal{D}_{\underline{t}}$ preserves all $\mathcal{T}^{/t}$-colimits [Sha21, Def. 11.2].

[14]If $t \in \mathcal{T}^{\mathrm{op}}$ is an initial object, then cocartesian sections are uniquely specified by their value at t.

[15]For a $\mathcal{T}$-functor $R : \mathcal{C} \longrightarrow \mathcal{D}$, the domain of its partial $\mathcal{T}$-left adjoint L is the largest full $\mathcal{T}$-subcategory $\mathcal{D}_0 \subset \mathcal{D}$ for which L_t is a partial left adjoint to R_t for all $t \in \mathcal{T}$ and $f^* L_t \xrightarrow{\simeq} L_s f^*$ for all $(f : s \longrightarrow t) \in \mathcal{T}$.

More generally, if we have a collection $\mathcal{K} = \{\mathcal{K}_t : t \in \mathcal{T}\}$ where $\mathcal{K}_t$ is a class of small $\mathcal{T}^{/t}$-∞-categories such that for each morphism $f : s \longrightarrow t$ in $\mathcal{T}$, $f^*(\mathcal{K}_t) \subset \mathcal{K}_s$, then we have analogous notions of strongly admitting and preserving $\mathcal{K}$-indexed $\mathcal{T}$-colimits. We will typically leave the collection $\{\mathcal{K}_t\}$ implicit when referring to $\mathcal{K}$. Abusing notation, we will also let $\mathcal{K}$ refer to the class of $\mathcal{T}$-∞-categories $\mathcal{K}$ such that $\mathcal{K}_{\underline{t}} \in \mathcal{K}_t$ for all $t \in \mathcal{T}$.

2.9. **Remark.** In this paper, all $\mathcal{T}$-colimits will be indexed by small $\mathcal{T}$-∞-categories, and we will typically suppress the adjective 'small' in this context (as was already done in Definition 2.8).

We next review the theory of $\mathcal{T}$-left Kan extensions along fully faithful $\mathcal{T}$-functors. We first need an auxiliary construction.

2.10. **Remark.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. By definition, the $\mathcal{T}$-right cone $\mathcal{C}^{\underline{\triangleright}}$ has a universal mapping property with respect to maps going *in*. In the following we will also need a universal mapping property of $\mathcal{C}^{\underline{\triangleright}}$ for maps going *out*. Namely, by [Sha21, Lem. 4.5] we have a homotopy pushout square of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc} \mathcal{C} \times \{1\} & \hookrightarrow & \mathcal{C} \times \Delta^1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ *_{\mathcal{T}} & \xhookrightarrow{v} & \mathcal{C}^{\underline{\triangleright}} \end{array}$$

where f is defined as the adjoint to $(\mathrm{id}_{\mathcal{C}}, p)$.

Now suppose that $\sigma : *_{\mathcal{T}} \longrightarrow \mathcal{C}$ is a $\mathcal{T}$-final object. Then we may construct a homotopy $h : \mathcal{C} \times \Delta^1 \longrightarrow \mathcal{C}$ from $\mathrm{id}_{\mathcal{C}}$ to σ, which yields a $\mathcal{T}$-functor

$$h' : \mathcal{C}^{\underline{\triangleright}} \longrightarrow \mathcal{C}$$

such that $h'|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$ and $h'|_v = \sigma$. Moreover, if one considers the bifibration (cf. [Sha21, Lem. 4.8])

$$(f, g) : \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\underline{\triangleright}}, \mathcal{C}) \longrightarrow \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{C}) \times \mathrm{Fun}_{\mathcal{T}}(*_{\mathcal{T}}, \mathcal{C})$$

then h' is obtained by taking a f-cartesian lift with target $[\mathcal{C}^{\underline{\triangleright}} \xrightarrow{\overline{p}} *_{\mathcal{T}} \xrightarrow{\sigma} \mathcal{C}]$ in $\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\underline{\triangleright}}, \mathcal{C})$ over the edge $\mathrm{id}_{\mathcal{C}} \longrightarrow \sigma \circ p$ in $\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{C})$ specified by h.

2.11. **Construction.** Let $\mathcal{D}$ be a $\mathcal{T}$-∞-category and let $x \in \mathcal{D}_t$. We then construct a $\mathcal{T}^{/t}$-functor

$$\theta_x : (\mathcal{D}^{/\underline{x}})^{\underline{\triangleright}} \longrightarrow \mathcal{D}_{\underline{t}}$$

as follows (where the parametrized right cone is formed with respect to the base $\mathcal{T}^{/t}$). First, we adjoint the projection $\mathcal{D}^{/\underline{x}} \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$ to obtain a $\mathcal{T}^{/t}$-functor $h_x : \mathcal{D}^{/\underline{x}} \times \Delta^1 \longrightarrow \mathcal{D}_{\underline{t}}$. We then let θ_x be the composite of h_x and the $\mathcal{T}^{/t}$-functor

$$(h', \pi) : (\mathcal{D}^{/\underline{x}})^{\underline{\triangleright}} \longrightarrow \mathcal{D}^{/\underline{x}} \times \Delta^1,$$

where π is the structure map to Δ^1 of the parametrized join and h' is as in Remark 2.10 (note that any choice of cocartesian section $j_x : *_{\mathcal{T}^{/t}} \longrightarrow \mathcal{D}^{/\underline{x}}$ determined by id_x is a $\mathcal{T}^{/t}$-final object).

Now suppose given a $\mathcal{T}$-functor $G : \mathcal{D} \longrightarrow \mathcal{E}$ and a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \mathcal{D}$. We let $F := G|_{\mathcal{C}}$, and for any $x \in \mathcal{D}_t$ we let $\mathcal{C}^{/\underline{x}} := \mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{/\underline{x}}$. We then write

$$G^x : (\mathcal{C}^{/\underline{x}})^{\underline{\triangleright}} \longrightarrow (\mathcal{D}^{/\underline{x}})^{\underline{\triangleright}} \xrightarrow{\theta_x} \mathcal{D}_{\underline{t}} \xrightarrow{G_{\underline{t}}} \mathcal{E}_{\underline{t}}$$

for the composite $\mathcal{T}^{/t}$-functor. Note that $G^x|_{\mathcal{C}^{/\underline{x}}}$ factors as $\mathcal{C}^{/\underline{x}} \longrightarrow \mathcal{C}_{\underline{t}} \xrightarrow{F_{\underline{t}}} \mathcal{E}_{\underline{t}}$; we write F^x for this $\mathcal{T}^{/t}$-functor.

The following is a simplification of [Sha21, Def. 10.1] in which we have chosen the datum of the natural transformation η present there to be the identity, which allows us to dispense with the auxiliary construction of $G'' : (\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})) \star_{\mathcal{D}} \mathcal{D} \longrightarrow \mathcal{E}$ in that definition.

2.12. **Definition.** Let $\mathcal{D}$ be a $\mathcal{T}$-∞-category and $\mathcal{C} \subset \mathcal{D}$ a full $\mathcal{T}$-subcategory. We say that a $\mathcal{T}$-functor $G : \mathcal{D} \longrightarrow \mathcal{E}$ is a $\mathcal{T}$-*left Kan extension* of its restriction $F = G|_{\mathcal{C}}$ if for all $x \in \mathcal{D}_t$, the $\mathcal{T}^{/t}$-functor G^x of Construction 2.11 is a $\mathcal{T}^{/t}$-colimit diagram.

A $\mathcal{T}$-functor $F : \mathcal{C} \longrightarrow \mathcal{E}$ then *admits a* $\mathcal{T}$-*left Kan extension to* $\mathcal{D}$ if there exists such a G, and we say that $\{F^x\}_{x \in \mathcal{D}}$ constitutes the set of *relevant diagrams* for the extension problem.

The author proved the following existence and uniqueness theorem for $\mathcal{T}$-left Kan extensions as [Sha21, Thm. 10.3], [Sha21, Thm. 10.5], and [Sha21, Prop. 10.6].

2.13. **Theorem.** *Let $\mathcal{D}$ be a $\mathcal{T}$-∞-category and $\mathcal{C} \subset \mathcal{D}$ a full $\mathcal{T}$-subcategory (with inclusion $\mathcal{T}$-functor ϕ).*

(1) A $\mathcal{T}$-functor $F : \mathcal{C} \longrightarrow \mathcal{E}$ admits a $\mathcal{T}$-left Kan extension G over $\mathcal{D}$ if and only if all the relevant diagrams for F admit parametrized colimits. Moreover, G is then uniquely specified up to contractible choice.

(2) The partial $\mathcal{T}$-left adjoint $\phi_!$ to the restriction $\mathcal{T}$-functor $\phi^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D}, \mathcal{E}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{E})$ is defined on all $F : \mathcal{C}_{\underline{t}} \longrightarrow \mathcal{E}_{\underline{t}}$ that admit a $\mathcal{T}^{/t}$-left Kan extension G, in which case $\phi_! F \simeq G$. In particular, $\phi_!$ is defined on its entire domain if for every $x \in \mathcal{D}_t$, the parametrized fiber $\mathcal{E}_{\underline{t}}$ admits all $\mathcal{C}^{/\underline{x}}$-indexed $\mathcal{T}^{/t}$-colimits.

In fact, constructing $\mathcal{T}$-left Kan extensions along fully faithful $\mathcal{T}$-functors suffices to handle the general case:

2.14. **Remark.** Let $\phi : \mathcal{C} \longrightarrow \mathcal{D}$ be a $\mathcal{T}$-functor and let $\pi : \mathcal{M} \longrightarrow \Delta^1 \times \mathcal{T}^{\mathrm{op}}$ be a cocartesian fibration classified by ϕ (so $\mathcal{M}$ is a $\mathcal{T}$-∞-category and $\mathcal{C} \simeq \mathcal{M}_0 \subset \mathcal{M}$ is the inclusion of a full $\mathcal{T}$-subcategory). Suppose that we have a $\mathcal{T}$-functor $\overline{G} : \mathcal{M} \longrightarrow \mathcal{E}$ that is a $\mathcal{T}$-left Kan extension of its restriction $F = \overline{G}|_{\mathcal{C}}$. Let $G = \overline{G}|_{\mathcal{D}}$. We then may construct a natural transformation $\eta : F \Rightarrow \phi^* G$ such that η exhibits G as the $\mathcal{T}$-left Kan extension of F along ϕ in the sense of [Sha21, Def. 10.1]. Indeed, consider the trivial fibration[16]

$$p = (\mathrm{ev}_0, \pi) : \mathrm{Ar}_{\mathcal{T}}^{cocart}(\mathcal{M}) \longrightarrow \mathcal{M} \times_{\Delta^1} \mathrm{Ar}(\Delta^1)$$

and let σ be a section that restricts to the identity on $\mathcal{M}$, given by a choice of dotted lift in the diagram

$$\begin{array}{ccc} \mathcal{M} & \xhookrightarrow{\iota} & \mathrm{Ar}_{\mathcal{T}}^{cocart}(\mathcal{M}) \\ \big\downarrow \iota & \overset{\sigma}{\nearrow} & \simeq \big\downarrow p \\ \mathcal{M} \times_{\Delta^1} \mathrm{Ar}(\Delta^1) & \xrightarrow{=} & \mathcal{M} \times_{\Delta^1} \mathrm{Ar}(\Delta^1) \end{array}$$

where ι generically denotes the identity section. Then let

$$\eta : \mathcal{C} \times \Delta^1 = \mathcal{C} \times \{[0=0] \to [0 \to 1]\} \subset \mathcal{M} \times_{\Delta^1} \mathrm{Ar}(\Delta^1) \xrightarrow{\sigma} \mathrm{Ar}_{\mathcal{T}}^{cocart}(\mathcal{M}) \xrightarrow{\mathrm{ev}_1} \mathcal{M} \xrightarrow{\overline{G}} \mathcal{E}$$

and note that $\eta|_{\mathcal{C} \times \{0\}} = F$ and $\eta|_{\mathcal{C} \times \{1\}} = G \circ \phi$. The assertion that $G \simeq \phi_! F$ then follows by examining the pointwise formula defining a $\mathcal{T}$-left Kan extension.

3. Parametrized factorization systems

Our goal in this section is to prove a theorem about parametrized factorization systems (Theorem 3.6) that will allow us to prove the universal property of the free $\mathcal{T}$-cocartesian fibration (Example 3.8) and subsequently that of the $\mathcal{O}$-monoidal envelope for a $\mathcal{T}$-∞-operad $\mathcal{O}$ in [NS].

3.1. **Definition.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. Then a $\mathcal{T}$-*factorization system* on $\mathcal{C}$ is the data of a factorization system $(\mathscr{L}_t, \mathscr{R}_t)$ on the fiber $\mathcal{C}_t$ for every $t \in \mathcal{T}$, subject to the condition that for every morphism $\alpha : s \longrightarrow t$ in $\mathcal{T}$, the restriction functor $\alpha^* : \mathcal{C}_t \longrightarrow \mathcal{C}_s$ sends $(\mathscr{L}_t, \mathscr{R}_t)$ into $(\mathscr{L}_s, \mathscr{R}_s)$.

3.2. **Remark.** In Definition 3.1, we could instead formulate the condition of compatibility of the fiberwise factorization systems with restriction in the following way. Let p denote the structure map of $\mathcal{C}$ and consider the collection of commutative squares in $\mathcal{C}$

$$\begin{array}{ccc} x & \xrightarrow{\alpha_x} & x' \\ f \big\downarrow & & \big\downarrow f' \\ y & \xrightarrow{\alpha_y} & y' \end{array}$$

such that f resp. f' lies in the fiber $\mathcal{C}_t$ resp. $\mathcal{C}_{t'}$, $p(\alpha_x) = p(\alpha_y)$, and α_x, α_y are p-cocartesian edges. Then we must have that if f is in $\mathscr{L}_t$ resp. $\mathscr{R}_t$, then f' is in $\mathscr{L}_{t'}$ resp. $\mathscr{R}_{t'}$.

3.3. **Definition.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category with structure map p. Given a $\mathcal{T}$-factorization system $(\mathscr{L}_t, \mathscr{R}_t)_{t \in \mathcal{T}}$ on $\mathcal{C}$, let $\mathscr{L}$ be the collection of edges $e : x \to y$ in $\mathcal{C}$ such that for any factorization $x \xrightarrow{e'} x' \xrightarrow{f} y$ of e by a p-cocartesian edge e' and a fiberwise edge f, f is in $\mathscr{L}_{p(y)}$. Let $\mathscr{R}$ be the closure of the union of the $\mathscr{R}_t$ under equivalences in $\mathcal{C}$.

[16] Cf. [Sha21, Lem. 2.23] applied to π and restrict arrows to be $\mathcal{T}$-fiberwise.

We have the following variant of [Lur17, Prop. 2.1.2.5].

3.4. **Lemma.** *$(\mathscr{L},\mathscr{R})$ is a factorization system on $\mathcal{C}$.*

Proof. We check the three conditions of a factorization system in turn.

(1) Using the stability of the classes $\{\mathscr{L}_t\}$ and $\{\mathscr{R}_t\}$ along with the p-cocartesian edges under retracts, we see that $\mathscr{L}$ and $\mathscr{R}$ are closed under retracts.
(2) Given an edge $e : x \to y$ in $\mathcal{C}$, factor e as $x \xrightarrow{e'} x' \xrightarrow{f} y$ for e' p-cocartesian and f in the fiber $\mathcal{C}_{p(y)}$. Using the factorization system $(\mathscr{L}_{p(y)},\mathscr{R}_{p(y)})$ on $\mathcal{C}_{p(y)}$, factor f as $x' \xrightarrow{f'} x'' \xrightarrow{f''} y$ where $f' \in \mathscr{L}_{p(y)}$ and $f'' \in \mathscr{R}_{p(y)}$. Then $x \xrightarrow{f'\circ e'} x'' \xrightarrow{f''} y$ is our desired factorization of e.
(3) Suppose we have a commutative square

$$\begin{array}{ccc} w & \longrightarrow & y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ x & \longrightarrow & z \end{array}$$

with $f \in \mathscr{L}$ and $g \in \mathscr{R}$; we want to produce an essentially unique filler $x \longrightarrow y$. Without loss of generality, we may suppose $p(y) = p(z)$ and $g \in \mathscr{R}_{p(y)}$. Choosing p-cocartesian edges we may factor the square as

$$\begin{array}{ccccccc} w & \longrightarrow & w' & \longrightarrow & w'' & \longrightarrow & y \\ & \searrow^{f} & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & \overset{h}{\nearrow} & \downarrow{\scriptstyle g} \\ & & x & \xrightarrow{\alpha} & x' & \longrightarrow & z \end{array}$$

where the edges which 'add a prime' are p-cocartesian, and vertical edges along with the rightmost square lie in a fiber. By definition, $f' \in \mathscr{L}_{p(x)}$, and since the $(\mathscr{L}_t,\mathscr{R}_t)_{t\in\mathcal{T}}$ constitute a $\mathcal{T}$-factorization system on $\mathcal{C}$, $f'' \in \mathscr{L}_{p(y)}$. Then we have an essentially unique filler h, and $h \circ \alpha : x \to y$ is our desired filler.

□

3.5. **Proposition.** *Suppose $\mathcal{D}$ is a $\mathcal{T}$-∞-category and $(\mathscr{L}_t,\mathscr{R}_t)_{t\in\mathcal{T}}$ is a $\mathcal{T}$-factorization system on $\mathcal{D}$. Let $(\mathscr{L},\mathscr{R})$ be the induced factorization system on $\mathcal{D}$ of Definition 3.3.*

(1) Let $\mathrm{Ar}^L_{\mathcal{T}}(\mathcal{D})$ resp. $\mathrm{Ar}^L(\mathcal{D})$ denote the full subcategory of $\mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$ resp. $\mathrm{Ar}(\mathcal{D})$ on the morphisms in $\mathscr{L}$. Then the source map

$$\mathrm{ev}_0 : \mathrm{Ar}^L_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathcal{D}$$

is a $\mathcal{T}$-cartesian fibration of $\mathcal{T}$-∞-categories, and the source map

$$\mathrm{ev}_0 : \mathrm{Ar}^L(\mathcal{D}) \longrightarrow \mathcal{D}$$

is a cartesian fibration (where here the domain is not generally a $\mathcal{T}$-∞-category).

(2) Let $\mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})$ denote the full subcategory of $\mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$ on the morphisms in $\mathscr{R}$. Suppose that $p : \mathcal{C} \longrightarrow \mathcal{D}$ is a $\mathcal{T}$-fibration which admits p-cocartesian lifts over all edges in $\mathscr{L}$. Then the target map

$$\mathrm{ev}_1 : \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathcal{D}$$

is a $\mathcal{T}$-cocartesian fibration.

Proof. (1): For the first assertion, since we have the factorization system $(\mathscr{L}_t,\mathscr{R}_t)$ on the fibers $\mathcal{D}_t$ for all $t \in \mathcal{T}$, ev_0 is fiberwise a cartesian fibration, with an edge in $\mathrm{Ar}^{L_t}(\mathcal{D}_t) = \mathrm{Ar}^L_{\mathcal{T}}(\mathcal{D})_t$

$$\begin{array}{ccc} x_0 & \xrightarrow{\alpha} & y_0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \beta} \\ x_1 & \xrightarrow{g} & y_1 \end{array}$$

$(\mathrm{ev}_0)_t$-cartesian if and only if g is in $\mathscr{R}_t$. Since the factorization systems in the fibers are compatible with restriction, it follows that ev_0 is in addition $\mathcal{T}$-cartesian.

For the second assertion, repeat the argument with the factorization system $(\mathscr{L},\mathscr{R})$ on $\mathcal{D}$ itself.

(2): The argument is dual to (1), except that now the ev_1-cocartesian edges are given by

$$(\alpha, (f, g)) : (c_0, x_0 \to y_0) \longrightarrow (c_1, x_1 \to y_1)$$

with $f \in \mathscr{L}$ and α a p-cocartesian edge. $\square$

3.6. **Theorem.** *Suppose we are in the setup of Proposition 3.5(2) so that we have a $\mathcal{T}$-fibration $p : \mathcal{C} \longrightarrow \mathcal{D}$ that admits p-cocartesian lifts over all edges in $\mathscr{L}$.*

(1) For every cocartesian fibration $q : \mathcal{E} \longrightarrow \mathcal{D}$, restriction along the inclusion $i : \mathcal{C} \longrightarrow \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})$ yields a trivial fibration

$$i^* : \mathrm{Fun}^{cocart}_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E}) \longrightarrow \mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C}, \mathcal{E})$$

where we define

$$\mathrm{Fun}^{cocart}_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E}) := \mathrm{Fun}_{/\mathcal{D}}({}_\natural(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})), {}_\natural\mathcal{E}),$$
$$\mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C}, \mathcal{E}) := \mathrm{Fun}_{/\mathcal{D}}((\mathcal{C}, M), {}_\natural\mathcal{E}),$$

and the marked edges M in $\mathcal{C}$ are the p-cocartesian edges of $\mathcal{C}$ over $\mathscr{L}$.[17] *In other words,*

$$i : (\mathcal{C}, M) \longrightarrow {}_\natural(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}))$$

is a cocartesian equivalence in $\mathbf{sSet}^+_{/\mathcal{D}}$.

(2) Let M' denote the ev_1-cocartesian edges in $\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})$ over $\mathscr{L}$ and define

$$\mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E}) := \mathrm{Fun}_{/\mathcal{D}}((\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), M'), {}_\natural\mathcal{E}).$$

Then we have an adjunction

$$i_! : \mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E}) : i^*$$

where $i_!$ is the fully faithful inclusion of the full subcategory $\mathrm{Fun}^{cocart}_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E})$ under the equivalence of (1).

Proof. (1): Given a monomorphism $A \longrightarrow B$ of simplicial sets, we need to solve the lifting problem

$$\begin{array}{ccc} A^\flat \times {}_\natural(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})) \bigcup_{A^\flat \times (\mathcal{C}, M)} B^\flat \times (\mathcal{C}, M) & \longrightarrow & {}_\natural\mathcal{E} \\ \downarrow & \nearrow & \downarrow q \\ B^\flat \times {}_\natural(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})) & \longrightarrow & \mathcal{D}^\sharp. \end{array}$$

Let us suppress markings for clarity. We can factor this square as

$$\begin{array}{ccccccc} A \times (\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})) \bigcup_{A \times \mathcal{C}} B \times \mathcal{C} & \longrightarrow & \mathcal{E} & \xrightarrow{\iota} & \mathrm{Ar}^{cocart}_{\mathcal{T}}(\mathcal{E}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{E} \\ \downarrow & & & \nearrow & \downarrow \simeq & & \downarrow q \\ B \times (\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})) & & \xrightarrow{\lambda} & & \mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{D} \end{array}$$

where the map λ is given by

$$B \times (\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})) \longrightarrow B \times \mathcal{C} \longrightarrow \mathcal{E},$$
$$B \times (\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})) \longrightarrow B \times \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}),$$

and the map $\mathrm{Ar}^{cocart}_{\mathcal{T}}(\mathcal{E}) \longrightarrow \mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$ is a pullback of the known trivial fibration $\mathrm{Ar}^{cocart}(\mathcal{E}) \longrightarrow \mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}(\mathcal{D})$ of [Sha21, Lem. 2.23] by the identity section $\mathcal{T} \longrightarrow \mathrm{Ar}(\mathcal{T})$, hence is a trivial fibration. Therefore, the dotted arrow exists, and postcomposing by ev_1 yields the desired lift.

(2): We need to show that $\mathrm{Fun}^{cocart}_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E}) \subset \mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E})$ is a coreflective subcategory. For this, it suffices to show that for every object $F \in \mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E})$, there exists a

[17] Note that objects of $\mathrm{Fun}^L_{/\mathcal{D}}(\mathcal{C}, \mathcal{E})$ are necessarily $\mathcal{T}$-functors.

colocalization $\epsilon_F : G \longrightarrow F$ relative to $\mathrm{Fun}^{cocart}_{/\mathcal{D}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}), \mathcal{E})$ in the sense of [Lur09, Def. 5.2.7.6] (after taking opposites there). We will construct this explicitly as follows. First define a homotopy

$$H : \Delta^1 \times \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$$

between the functors

$$H_0 = (F \circ i, \subset) : \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}), \qquad (c, x \to y) \mapsto (F(c, x = x), x \to y),$$
$$H_1 = (\mathrm{id}, \iota_{\mathcal{D}} q) \circ F : \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}), \qquad (c, x \to y) \mapsto (F(c, x \to y), y = y)$$

in the following way: let $\min, \max : \Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ be the min and max maps, form the functors

$$F' = \mathrm{Ar}_{\mathcal{T}}(F) \circ (\iota, \min^*) : \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D})) \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathcal{E}),$$
$$G' = \max^* \circ \mathrm{pr}_2 : \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathrm{Ar}_{\mathcal{T}}(\mathcal{D})),$$

and let H be the adjoint of the resulting map

$$(F', G') : \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})).$$

We then place H into the commutative diagram

$$\begin{array}{ccccccc}
\{1\} \times \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) & \xrightarrow{F} \mathcal{E} \xrightarrow{\iota} & \mathrm{Ar}^{cocart}_{\mathcal{T}}(\mathcal{E}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{E} \\
\downarrow & \overset{\epsilon'_F}{\nearrow} & \downarrow \simeq & & \downarrow q \\
\Delta^1 \times \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) & \xrightarrow{H} & \mathcal{E} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{D}.
\end{array}$$

Let ϵ'_F be any filler, and define $\epsilon_F = \mathrm{ev}_1 \circ \epsilon'_F$. Note that $\epsilon_F(0) \simeq i_! i^* F$ and $\epsilon_F(1) = F$. We now make the following simple observations, whose verification we leave to the reader:

(1) For every natural transformation $\theta : F \longrightarrow G$, the square

$$\begin{array}{ccc}
i_! i^* F & \xrightarrow{i_! i^* \theta} & i_! i^* G \\
\downarrow \epsilon_F & & \downarrow \epsilon_G \\
F & \xrightarrow{\theta} & G
\end{array}$$

is homotopy commutative.

(2) $i_! i^* \epsilon_F$ is an equivalence.

(3) $\epsilon_{i_! i^* F}$ is an equivalence.

Examining the part of the proof of [Lur09, Prop. 5.2.7.4] that establishes the implication 5.2.7.4(3) $\Rightarrow$ 5.2.7.4(1), we conclude that ϵ_F is indeed a colocalization, so we are done by [Lur09, Prop. 5.2.7.8]. ☐

3.7. **Remark.** Replacing the $\mathcal{T}$-factorization system $(\mathscr{L}_t, \mathscr{R}_t)_{t \in \mathcal{T}}$ by the factorization system $(\mathscr{L}, \mathscr{R})$ on $\mathcal{D}$ (Lemma 3.4), note that since edges in $\mathscr{R}$ map down to equivalences in $\mathcal{T}^{\mathrm{op}}$, we have that $\mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) \xrightarrow{\simeq} \mathrm{Ar}^R(\mathcal{D})$ where by the latter ∞-category we mean the full subcategory of $\mathrm{Ar}(\mathcal{D})$ on those edges in $\mathscr{R}$. Theorem 3.6 could thus be formulated entirely in 'non-parametrized' terms; this is related to the fact that a $\mathcal{T}$-functor $q : \mathcal{E} \longrightarrow \mathcal{D}$ is a $\mathcal{T}$-cocartesian fibration if and only if it is a cocartesian fibration [Sha21, Rem. 7.4]. In this form, Ayala–Mazel-Gee–Rozenblyum have also articulated Theorem 3.6(1) model-independently in terms of an adjunction of ∞-categories [AMGR17, Prop. B.1].

We end this section by giving two important applications of Theorem 3.6.

3.8. **Example.** Let $(\mathscr{L}_t, \mathscr{R}_t)_{t \in \mathcal{T}}$ be the $\mathcal{T}$-factorization system given by letting $\mathscr{L}_t$ be the equivalences and $\mathscr{R}_t$ be all morphisms for every $t \in \mathcal{T}$. Then $\mathrm{Ar}^R_{\mathcal{T}}(\mathcal{D}) = \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$, and $\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$ is the free $\mathcal{T}$-cocartesian fibration on $\mathcal{D}$ ([Sha21, Def. 7.6]). By Theorem 3.6(1), we see that $i : \mathcal{C} \longrightarrow \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$ has the expected universal property: for every $\mathcal{T}$-cocartesian fibration $\mathcal{E} \longrightarrow \mathcal{D}$,

$$i^* : \mathrm{Fun}^{cocart}_{/\mathcal{D}, \mathcal{T}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}), \mathcal{E}) \longrightarrow \mathrm{Fun}_{/\mathcal{D}, \mathcal{T}}(\mathcal{C}, \mathcal{E})$$

is an equivalence. This promotes to an adjunction

$$\mathrm{Fr}^{cocart} \colon (\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{D}} \rightleftarrows (\mathbf{Cat}_{\mathcal{T}})^{cocart}_{/\mathcal{D}} \simeq \mathbf{Cat}_{\mathcal{D}} : \mathrm{U}.$$

When $\mathcal{T} = *$, this recovers [GHN17, Thm. 4.5].

By Theorem 3.6(2), we also have an adjunction

$$i_! \colon \mathrm{Fun}_{/\mathcal{D},\mathcal{T}}(\mathcal{C},\mathcal{E}) \rightleftarrows \mathrm{Fun}_{/\mathcal{D},\mathcal{T}}(\mathcal{C}\times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}),\mathcal{E}) : i^*$$

in which $i_!$ is fully faithful.

3.9. **Example.** Suppose $\mathcal{T} = *$ and consider the inert-active factorization system on an ∞-operad $\mathcal{O}^{\otimes}$. Let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Then $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} := \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{act}(\mathcal{O}^{\otimes})$ is the $\mathcal{O}$-*monoidal envelope* of $\mathcal{C}^{\otimes}$ [Lur17, Constr. 2.2.4.1], and by Theorem 3.6(1) for any $\mathcal{O}$-monoidal ∞-category $\mathcal{D}^{\otimes}$ we have that

$$\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}),\mathcal{D}) \xrightarrow{\simeq} \mathbf{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}).$$

This recovers [Lur17, Prop. 2.2.4.9].

4. Parametrized pairing construction

In this section, we first introduce the concept of a $\mathcal{T}$-*flat fibration* $p : \mathcal{X} \longrightarrow \mathcal{B}$, which will amount to a condition on p that ensures that the pullback functor

$$p^* : (\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{B}} \longrightarrow (\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{X}}$$

admits a right adjoint p_* (Remark 4.10). Given another $\mathcal{T}$-fibration $q : \mathcal{Y} \longrightarrow \mathcal{B}$, we then recall from [Sha21, Constr. 9.1] the $\mathcal{B}$-*relative* $\mathcal{T}$-*pairing construction* $\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y})$ (Theorem-Construction 4.2) as a certain $\mathcal{T}$-fibration over $\mathcal{B}$. In our discussion in [Sha21, §9], we only established the properties of the $\mathcal{T}$-pairing construction needed for our application to proving the existence theorem for $\mathcal{T}$-left Kan extensions. We enter into a more systematic discussion here by first proving its base-change property (Proposition 4.5) and then its universal property internal to $\mathcal{T}$-∞-categories (Theorem 4.9), from which it follows that

$$\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y}) \simeq p_* p^*(\mathcal{Y} \xrightarrow{q} \mathcal{B})$$

at the level of the underlying ∞-category $(\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{B}}$ of $\mathbf{sSet}^+_{/\natural\mathcal{B}}$.[18] We then apply the $\mathcal{T}$-pairing construction to study $\mathcal{T}$-(co)limits in a $\mathcal{T}$-∞-category of sections (Theorem 4.16). This material will be used in [NS] to understand $\mathcal{T}$-(co)limits in $\mathcal{T}$-∞-categories of $\mathcal{O}$-algebras for a $\mathcal{T}$-∞-operad $\mathcal{O}$.

4.1. **Definition.** Let $p : \mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-fibration. We say that p is a $\mathcal{T}$-*flat fibration* if for every $t \in \mathcal{T}$, the pullback $p_t : \mathcal{X}_t \longrightarrow \mathcal{B}_t$ is flat.

In what follows, for a $\mathcal{T}$-∞-category $\mathcal{B}$ we let $\mathrm{Ar}^{cocart}(\mathcal{B}) \subset \mathrm{Ar}(\mathcal{B})$ denote the full subcategory on arrows that are cocartesian edges with respect to the structure map to $\mathcal{T}^{\mathrm{op}}$.

4.2. **Theorem-Construction** ([Sha21, Def. 9.1])**.** Let $p : \mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-flat fibration and consider the span of marked simplicial sets

$$\natural\mathcal{B} \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathrm{ev}_1,\mathcal{B},p} \mathcal{X}, \mathscr{E}) \xrightarrow{\mathrm{pr}_{\mathcal{X}}} \natural\mathcal{X}$$

in which $\mathcal{B}$ and $\mathcal{X}$ are given the cocartesian markings (with respect to the structure maps to $\mathcal{T}^{\mathrm{op}}$), and an edge e in $\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X}$ is marked if and only if $\mathrm{ev}_0(e)$ is marked and $\mathrm{pr}_{\mathcal{X}}(e)$ is marked. The functor

$$(\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^* : \mathbf{sSet}^+_{/\natural\mathcal{X}} \longrightarrow \mathbf{sSet}^+_{/\natural\mathcal{B}}$$

is then right Quillen with respect to the slice model structures induced from the cocartesian model structure on $\mathbf{sSet}^+_{/\mathcal{T}^{\mathrm{op}}}$. For a $\mathcal{T}$-fibration $q : \mathcal{Y} \longrightarrow \mathcal{B}$, we then define the $\mathcal{T}$-*pairing* of $(\mathcal{X}, p)$ and $(\mathcal{Y}, q)$ to be the $\mathcal{T}$-fibration over $\mathcal{B}$ given by

$$\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y}) := (\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^* q^*(\natural\mathcal{Y}),$$

where the marked edges are precisely the cocartesian edges with respect to the structure map to $\mathcal{T}^{\mathrm{op}}$.

Proof. The assertion that $(\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^*$ is right Quillen was proved under the assumption that p is a $\mathcal{T}$-cocartesian or $\mathcal{T}$-cartesian fibration in [Sha21, Thm. 9.3(1)]. However, that assumption was only used in the proof to show that ev_0 is flat. Using our weaker assumption that p is $\mathcal{T}$-flat, this follows instead from Lemma 10.1 applied to the factorization system $(\mathscr{L}, \mathscr{R})$ on $\mathcal{B}$ with $\mathscr{L}$ given by the cocartesian edges and $\mathscr{R}$ given by those edges lying over equivalences in $\mathcal{T}^{\mathrm{op}}$ [Lur09, Ex. 5.2.8.15]. □

[18]We equip $\mathbf{sSet}^+_{/\natural\mathcal{B}}$ with the slice model structure with respect to the cocartesian model structure on $\mathbf{sSet}^+_{/(\mathcal{T}^{\mathrm{op}})^\sharp}$. Since $\natural\mathcal{B} \longrightarrow (\mathcal{T}^{\mathrm{op}})^\sharp$ is fibrant, we may indeed identify the underlying ∞-category as $(\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{B}}$.

4.3. **Recollection.** If $p : \mathcal{X} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-cartesian fibration and $q : \mathcal{Y} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-cocartesian fibration, then we showed in [Sha21, Thm. 9.3] that $r : \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-cocartesian fibration. Moreover, we may produce $\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y})$ as a marked simplicial set with the r-cocartesian edges marked in the following way: let $\mathscr{E}' \subset (\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X})_1$ be the minimal collection of edges closed under composition that contains the class $\mathscr{E}$ in Theorem-Construction 4.2 and the ev_0-cartesian edges in $\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X}$, which are those edges

$$\left(\begin{array}{ccc} b_0 & \longrightarrow & b_1 \\ \downarrow & & \downarrow \\ c_0 & \xrightarrow{f} & c_1 \end{array}, \; x_0 \xrightarrow{g} x_1 \right)$$

such that f is sent to an equivalence in $\mathcal{T}^{\mathrm{op}}$ and g is a p-cartesian edge.[19] Then the span of marked simplicial sets

$$\mathcal{B}^\sharp \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X}, \mathscr{E}') \xrightarrow{\mathrm{ev}_1} \mathcal{B}^\sharp$$

defines via $(\mathrm{ev}_0)_*(\mathrm{ev}_1)^*(\mathcal{Y}, q\text{-cocart})$ the same underlying simplicial set $\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y})$ as before, but with the r-cocartesian edges marked. Unwinding the definitions, we thus see that $\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y})$ enjoys the following additional functoriality with respect to morphisms in $\mathcal{B}$: for every fiberwise morphism $f : b \longrightarrow b' \in \mathcal{B}_t$, we have a pushforward functor

$$f_! : \mathrm{Fun}_{\mathcal{T}^{/t}}(\mathcal{X}_{\underline{b}}, \mathcal{Y}_{\underline{b}}) \longrightarrow \mathrm{Fun}_{\mathcal{T}^{/t}}(\mathcal{X}_{\underline{b'}}, \mathcal{Y}_{\underline{b'}}), \; F \mapsto f_! \circ F \circ f^*$$

where $f^* : \mathcal{X}_{\underline{b'}} \longrightarrow \mathcal{X}_{\underline{b}}$ and $f_! : \mathcal{Y}_{\underline{b}} \longrightarrow \mathcal{Y}_{\underline{b'}}$ are the $\mathcal{T}^{/t}$-functors encoded by p and q.

We next establish the compatibility of the $\mathcal{T}$-pairing construction with base-change. First, we need a lemma.

4.4. **Lemma.** *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-functor. The functor*

$$\psi : \mathrm{Ar}^{cocart}(\mathcal{A}) \longrightarrow \mathcal{A} \times_{f,\mathcal{B},\mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{B})$$

induced by f is a homotopy equivalence of cartesian fibrations over $\mathcal{A}$ (with respect to ev_0 *on the source and projection to $\mathcal{A}$ on the target).*

Proof. By [Sha21, Lem. 9.2(1)], $\mathrm{ev}_0 : \mathrm{Ar}^{cocart}(\mathcal{A}) \longrightarrow \mathcal{A}$ is a cartesian fibration and an edge e is ev_0-cartesian if and only if the projection of $\mathrm{ev}_1(e)$ to $\mathcal{T}^{\mathrm{op}}$ is an equivalence. It follows that ψ preserves cartesian edges, so to show ψ is a homotopy equivalence it suffices to check that for every $a \in \mathcal{A}$, the map on fibers

$$\psi_a : \underline{a} = \{a\} \times_{\mathcal{A},\mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{A}) \longrightarrow \underline{f(a)} = \{f(a)\} \times_{\mathcal{B},\mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{B})$$

is an equivalence of ∞-categories. But if a lies over $t \in T^{\mathrm{op}}$, then the induced projections $\underline{a} \longrightarrow (\mathcal{T}^{/t})^{\mathrm{op}}$ and $\underline{f(a)} \longrightarrow (\mathcal{T}^{/t})^{\mathrm{op}}$ are equivalences (cf. [Sha21, Notn. 2.28]), so ψ_a is an equivalence. □

4.5. **Proposition.** *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-functor, let $\mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-flat fibration, and let $\mathcal{Y} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-fibration. We have a canonical and natural equivalence of $\mathcal{T}$-fibrations over $\mathcal{A}$*

$$\widetilde{\mathrm{Fun}}_{\mathcal{A},\mathcal{T}}(\mathcal{X} \times_{\mathcal{B}} \mathcal{A}, \mathcal{Y} \times_{\mathcal{B}} \mathcal{A}) \simeq \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y}) \times_{\mathcal{B}} \mathcal{A}.$$

Proof. Consider the morphism of spans

$$\begin{array}{ccccc} & & \mathrm{Ar}^{cocart}(\mathcal{A}) \times_{\mathcal{A}} (\mathcal{A} \times_{\mathcal{B}} \mathcal{X}) & & \\ & \swarrow^{\mathrm{ev}_0} & \downarrow \phi & \searrow^{\mathrm{pr}_{\mathcal{X}}} & \\ \mathcal{A} & \xleftarrow{\mathrm{pr}_{\mathcal{A}}} & \mathcal{A} \times_{\mathcal{B}} \mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X} & \xrightarrow{\mathrm{pr}_{\mathcal{X}}} & \mathcal{X} \end{array}$$

where ϕ is induced by f. Noting that $\mathcal{T}$-flat fibrations are stable under pullback, we see that ϕ induces a comparison functor (after marking as necessary)

$$\Phi : \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y}) \times_{\mathcal{B}} \mathcal{A} \longrightarrow \widetilde{\mathrm{Fun}}_{\mathcal{A},\mathcal{T}}(\mathcal{X} \times_{\mathcal{B}} \mathcal{A}, \mathcal{Y} \times_{\mathcal{B}} \mathcal{A}).$$

By Lemma 4.4, ϕ is a homotopy equivalence. Moreover, the homotopy inverse respects the projection to $\mathcal{X}$, so by the proof of [Sha21, Lem. 2.27], Φ is an equivalence of $\mathcal{T}$-fibrations over $\mathcal{A}$. □

[19] See [Sha21, Lem. 7.5] for a justification as to why we can take p-cartesian edges here as opposed to fiberwise cartesian.

We can use the base-change property of the $\mathcal{T}$-pairing construction explicated in Proposition 4.5 to give a more transparent identification of its parametrized fibers [Sha21, Prop. 9.7].

4.6. **Corollary.** *Let $\mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-flat fibration and $\mathcal{Y} \longrightarrow \mathcal{B}$ a $\mathcal{T}$-fibration. For every $b \in \mathcal{B}$ over $t \in \mathcal{T}^{op}$, we have an equivalence of $\mathcal{T}^{/t}$-∞-categories*

$$\widetilde{\operatorname{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y})_{\underline{b}} \simeq \underline{\operatorname{Fun}}_{\mathcal{T}^{/t}}(\mathcal{X}_{\underline{b}},\mathcal{Y}_{\underline{b}})$$

Proof. We may invoke Proposition 4.5 to replace $\mathcal{B}$ with $\mathcal{B}_{\underline{t}}$, and invoke Proposition 4.5 again with $\mathcal{A} = \underline{b} \longrightarrow \mathcal{B}$ to reduce to the case $\mathcal{B} = \mathcal{T}$, for which $\widetilde{\operatorname{Fun}}_{\mathcal{T},\mathcal{T}}(-,-) \cong \operatorname{Fun}_{\mathcal{T}}(-,-)$ as marked simplicial sets. □

We next proceed to articulate the universal property of the $\mathcal{T}$-pairing construction (Theorem 4.9) as a partially-defined internal hom for $\mathcal{T}$-fibrations over a fixed base $\mathcal{T}$-∞-category.

4.7. **Notation.** Let $\mathcal{B}$ be a $\mathcal{T}$-∞-category, let $p : \mathcal{X} \longrightarrow \mathcal{B}$, $q : \mathcal{Y} \longrightarrow \mathcal{B}$ be $\mathcal{T}$-fibrations over $\mathcal{B}$, and let $q_\circ : \underline{\operatorname{Fun}}_{\mathcal{T}}(\mathcal{X},\mathcal{Y}) \longrightarrow \underline{\operatorname{Fun}}_{\mathcal{T}}(\mathcal{X},\mathcal{B})$ be the $\mathcal{T}$-functor given by postcomposition by q. We then let

$$\underline{\operatorname{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y}) := *_{\mathcal{T}} \times_{\sigma_p,\underline{\operatorname{Fun}}_{\mathcal{T}}(\mathcal{X},\mathcal{B}),q_\circ} \underline{\operatorname{Fun}}_{\mathcal{T}}(\mathcal{X},\mathcal{Y})$$

denote the $\mathcal{T}$-∞-category of $\mathcal{T}$-functors $\mathcal{X} \longrightarrow \mathcal{Y}$ over $\mathcal{B}$.

4.8. **Lemma.** *Let $p : \mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-fibration and consider the span of marked simplicial sets*

$$(\mathcal{T}^{op})^\sharp \xleftarrow{\operatorname{ev}_0} \operatorname{Ar}(\mathcal{T}^{op})^\sharp \times_{\operatorname{ev}_1,(\mathcal{T}^{op})^\sharp} {}_\natural\mathcal{X} \xrightarrow{p \circ \operatorname{pr}_{\mathcal{X}}} {}_\natural\mathcal{B}.$$

For any $\mathcal{T}$-fibration $q : \mathcal{Y} \longrightarrow \mathcal{B}$, we then have an equivalence of $\mathcal{T}$-∞-categories

$$(\operatorname{ev}_0)_*(p \circ \operatorname{pr}_{\mathcal{X}})^*({}_\natural Y) \simeq \underline{\operatorname{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y})$$

which is an isomorphism at the level of marked simplicial sets.

Proof. By definition, given a map of marked simplicial sets $K \longrightarrow (\mathcal{T}^{op})^\sharp$, we have natural bijections

$$\begin{aligned}\operatorname{Hom}_{/(\mathcal{T}^{op})^\sharp}(K,(\operatorname{ev}_0)_*(p\circ\operatorname{pr}_{\mathcal{X}})^*({}_\natural\mathcal{Y})) &\cong \operatorname{Hom}_{/{}_\natural\mathcal{B}}(K \times_{\mathcal{T}^{op}} \operatorname{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} {}_\natural\mathcal{X}, {}_\natural\mathcal{Y}) \\ &\cong \operatorname{Hom}_{/(\mathcal{T}^{op})^\sharp}(K \times_{\mathcal{T}^{op}} \operatorname{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} {}_\natural\mathcal{X}, {}_\natural\mathcal{Y}) \times_{\operatorname{Hom}_{/(\mathcal{T}^{op})^\sharp}(K\times_{\mathcal{T}^{op}}\operatorname{Ar}(\mathcal{T}^{op})^\sharp\times_{\mathcal{T}^{op}}{}_\natural\mathcal{X},{}_\natural\mathcal{B})} \{\phi \circ \operatorname{pr}\}\end{aligned}$$

yielding an isomorphism of marked simplicial sets over $\mathcal{T}^{op}$

$$(\operatorname{ev}_0)_*(p \circ \operatorname{pr}_{\mathcal{X}})^*({}_\natural Y) \cong {}_\natural\underline{\operatorname{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y}).$$

□

4.9. **Theorem.** *We have a canonical equivalence of $\mathcal{T}$-∞-categories*

$$\underline{\operatorname{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},\widetilde{\operatorname{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y})) \simeq \underline{\operatorname{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C} \times_{\mathcal{B}} \mathcal{X},\mathcal{Y})$$

natural in $\mathcal{T}$-flat fibrations $p : \mathcal{X} \longrightarrow \mathcal{B}$ and $\mathcal{T}$-fibrations $\mathcal{C},\mathcal{Y} \longrightarrow \mathcal{B}$.

Proof. By Proposition 4.5, we may assume $\mathcal{C} = \mathcal{B}$ without loss of generality. We then adopt essentially the same strategy that we used to show [Sha21, Eqn. 9.12.1] (which is the equivalence when $\mathcal{Y} = \mathcal{B} \times_{\mathcal{T}^{op}} \mathcal{E}$) by considering the diagram of marked simplicial sets

$$\begin{array}{ccccc}\operatorname{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} {}_\natural\mathcal{X} & \xrightarrow{(j,\operatorname{id})} & (\operatorname{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X},\mathscr{E}) & \xrightarrow{p\circ\operatorname{pr}_{\mathcal{X}}} & {}_\natural\mathcal{B} \\ \downarrow{\scriptstyle(\operatorname{id},p)} & & \downarrow{\scriptstyle\operatorname{ev}_0} & & \\ \operatorname{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} {}_\natural\mathcal{B} & \xrightarrow{\operatorname{pr}_{\mathcal{B}}} & {}_\natural\mathcal{B} & & \\ \downarrow{\scriptstyle\operatorname{ev}_0} & & & & \\ (\mathcal{T}^{op})^\sharp & & & & \end{array}$$

where j is the composite $\operatorname{Ar}(\mathcal{T}^{op}) \times_{\mathcal{T}^{op}} \mathcal{X} \xrightarrow{\operatorname{pr}_{\mathcal{X}}} \mathcal{X} \xrightarrow{p} \mathcal{B} \xrightarrow{\iota} \operatorname{Ar}^{cocart}(\mathcal{B})$, ι being the identity section. Let

$$i : \operatorname{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} {}_\natural\mathcal{X} \longrightarrow (\operatorname{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} {}_\natural\mathcal{B}) \times_{{}_\natural\mathcal{B}} (\operatorname{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X},\mathscr{E})$$

denote the induced map to the pullback. By [Sha21, Lem. 9.12], i is a homotopy equivalence with respect to the projection to $\mathcal{X}$. By [Sha21, Lem. 2.27] and Lemma 4.8, we obtain an equivalence of $\mathcal{T}$-∞-categories

$$\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{B}, \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y})) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y}).$$

□

4.10. **Remark.** In the statement of Theorem 4.9, if we replace the span

$$\natural\mathcal{B} \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X}, \mathscr{E}) \xrightarrow{p \circ \mathrm{pr}_{\mathcal{X}}} \natural\mathcal{B}$$

with

$$\natural\mathcal{B} \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X}, \mathscr{E}) \xrightarrow{\mathrm{pr}_{\mathcal{X}}} \natural\mathcal{X}$$

then the same argument as in the proof of Theorem 4.9 shows that for all $\mathcal{T}$-fibrations $\mathcal{D} \longrightarrow \mathcal{X}$, we have a canonical and natural equivalence[20]

$$\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, (\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^*\mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{/\mathcal{X},\mathcal{T}}(\mathcal{C} \times_{\mathcal{B}} \mathcal{X}, \mathcal{D}).$$

Passing first to cocartesian sections and then to mapping spaces, this shows that at the level of underlying ∞-categories, $(\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^* : (\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{X}} \longrightarrow (\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{B}}$ computes the right adjoint to p^*. This justifies the terminology of "$\mathcal{T}$-flat fibration" since these are indeed exponentiable in the parametrized sense. Moreover, we then see that

$$\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, -) \simeq p_* p^*(-)$$

as endofunctors of $(\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{B}}$.

4.11. **Remark.** Suppose $\mathcal{X}, \mathcal{Y} \longrightarrow \mathcal{B}$ are $\mathcal{T}$-cocartesian fibrations and let $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{X},\mathcal{Y})$ denote the internal hom construction of [Sha21, §3],[21] so that $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{X},\mathcal{Y}) \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-cocartesian fibration. Consider the morphism of spans

$$\begin{array}{ccccc} & & (\mathrm{Ar}^{cocart}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X}, \mathscr{E}) & & \\ & \swarrow & \downarrow{\scriptstyle (i,\mathrm{id})} & \searrow & \\ \natural\mathcal{B} & \longleftarrow & \mathrm{Ar}(\mathcal{B})^\sharp \times_{\mathcal{B}} \natural\mathcal{X} & \longrightarrow & \natural\mathcal{B} \end{array}$$

in which i is the inclusion $\mathrm{Ar}^{cocart}(\mathcal{B}) \subset \mathrm{Ar}(\mathcal{B})$. Then this morphism induces a $\mathcal{T}$-functor over $\mathcal{B}$

$$\rho : \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{X},\mathcal{Y}) \longrightarrow \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y}).$$

that upon passage to cocartesian sections in the source, regarded as $\mathcal{T}$-sections in the target, induces the inclusion

$$\mathrm{Fun}_{\mathcal{B}}(\mathcal{X},\mathcal{Y}) \longrightarrow \mathrm{Fun}_{/\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y}).$$

Beware that even if $\mathcal{X}$ is in addition $\mathcal{T}$-cartesian so that $\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X},\mathcal{Y})$ is $\mathcal{T}$-cocartesian, ρ will not preserve cocartesian edges in general; indeed, one observes that the above functor (i, id) does not carry the marking $\mathscr{E}'$ of Recollection 4.3 to marked edges in the target. Nonetheless, we have the following proposition.

4.12. **Proposition.** *In the situation of Remark 4.11, if $\mathcal{X} \simeq \mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}$ for some $\mathcal{T}$-∞-category $\mathcal{K}$, then the comparison $\mathcal{T}$-functor ρ implements an equivalence*

$$\rho : \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{Y}) \xrightarrow{\simeq} \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{Y})$$

of $\mathcal{T}$-cocartesian fibrations over $\mathcal{B}$. Consequently, for every $\mathcal{T}$-cocartesian fibration $\mathcal{C} \longrightarrow \mathcal{B}$, the equivalence of Theorem 4.9 restricts to

$$\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C}, \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{Y})) \simeq \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{Y}).$$

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[20] Here, $(\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^* : \mathbf{sSet}^+_{/\natural\mathcal{X}} \longrightarrow \mathbf{sSet}^+_{/\natural\mathcal{B}}$ and $(\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^*\mathcal{D} := (\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{X}})^*(\natural\mathcal{D})$ regarded as a $\mathcal{T}$-fibration over $\mathcal{B}$.

[21] We mildly abuse our notational conventions by writing $\underline{\mathrm{Fun}}_{\mathcal{B}}(-,-)$ instead of $\underline{\mathrm{Fun}}_{\mathcal{B}^{\mathrm{op}}}(-,-)$ as we would do if $\mathcal{B} = *_{\mathcal{T}} = \mathcal{T}^{\mathrm{op}}$.

Proof. The consequence will follow immediately from the universal property of $\underline{\mathrm{Fun}}_{\mathcal{B}}(-,-)$ once we establish the first claim. For this, first observe that since $\mathcal{X} \simeq \mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}$, $\mathcal{X}$ is both a $\mathcal{T}$-cocartesian and $\mathcal{T}$-cartesian fibration over $\mathcal{B}$ such that the fiberwise cartesian edges, fiberwise cocartesian edges, and fiberwise equivalences all coincide in $\mathcal{X}$. In particular, the functor (i, id) in Remark 4.11 carries the class $\mathscr{E}'$ of Recollection 4.3 into the marked edges of $\mathrm{Ar}(\mathcal{B})^{\sharp} \times_{\mathcal{B}} {}_{\natural}\mathcal{X}$, so ρ is a morphism of $\mathcal{T}$-cocartesian fibrations. It therefore suffices to check the claimed equivalence fiberwise.

Given $b \in \mathcal{B}$ over $t \in \mathcal{T}^{\mathrm{op}}$, we may replace $\mathcal{B}$ by $\mathcal{B}^{b/}$ and $\mathcal{T}$ by $(\mathcal{T}^{/t})^{\mathrm{op}}$, so it further suffices to check that ρ induces an equivalence upon passage to cocartesian sections. But this is the map

$$\mathrm{Fun}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathcal{T}\text{-}cocart}_{/\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y}).$$

Indeed, for any $\mathcal{T}$-cartesian fibration $\mathcal{X}$ the cocartesian sections of $\widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y})$ is the full subcategory of $\mathrm{Fun}_{/\mathcal{B},\mathcal{T}}(\mathcal{X}, \mathcal{Y})$ spanned by those $\mathcal{T}$-functors $F : \mathcal{X} \longrightarrow \mathcal{Y}$ over $\mathcal{B}$ that carry fiberwise cartesian edges to fiberwise cocartesian edges, but in our case F equivalently preserves cocartesian edges. □

4.1. **Application: parametrized (co)limits in section $\mathcal{T}$-∞-categories.** We next use the $\mathcal{T}$-pairing construction to analyze $\mathcal{T}$-limits and $\mathcal{T}$-colimits in a $\mathcal{T}$-∞-category of sections. Actually, we work in somewhat greater generality: given a $\mathcal{T}$-cocartesian fibration $\mathcal{X} \longrightarrow \mathcal{B}$ and a $\mathcal{T}$-fibration $\mathcal{C} \longrightarrow \mathcal{B}$, we will study $\mathcal{T}$-(co)limits in $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X})$ (Theorem 4.16). First, we introduce some terminology concerning relative adjunctions, extending [Sha21, Def. 8.3].

4.13. **Definition.** Let $\mathcal{X}, \mathcal{Y} \longrightarrow \mathcal{B}$ be $\mathcal{T}$-fibrations and let

$$F \colon \mathcal{X} \rightleftarrows \mathcal{Y} : G$$

be a relative adjunction with respect to the structure maps to $\mathcal{B}$ (in the sense of [Lur17, Def. 7.3.2.2]). We then say that $F \dashv G$ is a $\mathcal{B}$-*relative* $\mathcal{T}$-*adjunction* if F and G are $\mathcal{T}$-functors.

4.14. **Remark** (Stability under base-change)**.** Suppose

$$F \colon \mathcal{X} \rightleftarrows \mathcal{Y} : G$$

is a $\mathcal{B}$-relative $\mathcal{T}$-adjunction and $\phi : \mathcal{C} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-functor. By [Lur17, Prop. 7.3.2.5], the pullback

$$F_{\mathcal{C}} \colon \mathcal{X} \times_{\mathcal{B}} \mathcal{C} \rightleftarrows \mathcal{Y} \times_{\mathcal{B}} \mathcal{C} : G_{\mathcal{C}}$$

is then a $\mathcal{C}$-relative $\mathcal{T}$-adjunction.

The following lemma illustrates the basic asymmetry between $\mathcal{B}$-relative left and right $\mathcal{T}$-adjoints that should already be familiar from the theory of relative adjunctions (compare [Lur17, Prop. 7.3.2.6] versus [Lur17, Prop. 7.3.2.11].)

4.15. **Lemma.** *Suppose $\mathcal{X}, \mathcal{Y} \longrightarrow \mathcal{B}$ are $\mathcal{T}$-cocartesian fibrations.*

(1) Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of $\mathcal{T}$-cocartesian fibrations over $\mathcal{B}$. Then F admits a $\mathcal{B}$-relative right $\mathcal{T}$-adjoint $R : \mathcal{Y} \longrightarrow \mathcal{X}$ if and only if for all $b \in \mathcal{B}_t$, the parametrized fiber $F_{\underline{b}} : \mathcal{X}_{\underline{b}} \longrightarrow \mathcal{Y}_{\underline{b}}$ admits a right $\mathcal{T}^{/t}$-adjoint $R_{\underline{b}}$.

(2) Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be a $\mathcal{T}$-functor over $\mathcal{B}$. Then F admits a $\mathcal{B}$-relative left $\mathcal{T}$-adjoint $L : \mathcal{Y} \longrightarrow \mathcal{X}$ if and only if for all $b \in \mathcal{B}_t$, the parametrized fiber $F_{\underline{b}} : \mathcal{X}_{\underline{b}} \longrightarrow \mathcal{Y}_{\underline{b}}$ admits a left $\mathcal{T}^{/t}$-adjoint $L_{\underline{b}}$, and for all fiberwise morphisms $f : b \longrightarrow b'$ in $\mathcal{B}_t$, the natural transformation

$$\begin{array}{ccc} \mathcal{Y}_b & \xrightarrow{L_b} & \mathcal{X}_b \\ {\scriptstyle f_!}\downarrow & \nearrow & \downarrow{\scriptstyle f_!} \\ \mathcal{Y}_{b'} & \xrightarrow[L_{b'}]{} & \mathcal{X}_{b'} \end{array} \quad \textit{adjoint to} \quad \begin{array}{ccc} \mathcal{Y}_b & \xleftarrow{F_b} & \mathcal{X}_b \\ {\scriptstyle f_!}\downarrow & \searrow & \downarrow{\scriptstyle f_!} \\ \mathcal{Y}_{b'} & \xleftarrow[F_{b'}]{} & \mathcal{X}_{b'} \end{array}$$

is an equivalence. Moreover, in this case L is a morphism of $\mathcal{T}$-cocartesian fibrations over $\mathcal{B}$.

Proof. (1): By the opposite of [Lur17, Prop. 7.3.2.6], F admits a $\mathcal{B}$-relative right adjoint R if and only if for all $b \in \mathcal{B}$, F_b admits a right adjoint R_b. But we are then reduced to showing that R is in addition a $\mathcal{T}$-functor if and only if $R_{\underline{b}}$ is in addition a $\mathcal{T}^{/t}$-functor for all $b \in \mathcal{B}$, which is clear.
(2): Since every morphism in $\mathcal{B}$ factors as the composite of a cocartesian edge and a fiberwise morphism, the claim follows directly from [Lur17, Prop. 7.3.2.11]. □

We may now state the main result of this subsection. Note that the cases of parametrized limits and colimits involve different hypotheses (as should be familiar from the theory of limits and colimits in ∞-categories of algebras).

4.16. **Theorem.** *Let $p : \mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-cocartesian fibration, let $\mathcal{K}$ be a $\mathcal{T}$-∞-category, and let $r : \mathcal{C} \longrightarrow \mathcal{B}$ be any $\mathcal{T}$-fibration.*

(1) There exists a $\mathcal{B}$-relative constant $\mathcal{K}$-indexed diagram $\mathcal{T}$-functor

$$\delta_p : \mathcal{X} \longrightarrow \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{X}),$$

which is a morphism of $\mathcal{T}$-cocartesian fibrations over $\mathcal{B}$, such that the constant $\mathcal{K}$-indexed diagram $\mathcal{T}$-functor

$$\delta_{r,p} : \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X}))$$

is given by $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \delta_p)$.

(2) Suppose that for every $b \in \mathcal{B}_t$, $\mathcal{X}_{\underline{b}}$ admits all $\mathcal{K}_{\underline{t}}$-indexed $\mathcal{T}^{/t}$-limits. Then δ_p admits a $\mathcal{B}$-relative $\mathcal{T}$-right adjoint

$$\lim{}^{\mathcal{B},\mathcal{T}} : \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{X}) \longrightarrow \mathcal{X}$$

which for all $b \in \mathcal{B}_t$ restricts to the $\mathcal{T}^{/t}$-limit $\mathcal{T}^{/t}$-functor

$$\lim{}^{\mathcal{T}^{/t}} : \underline{\mathrm{Fun}}_{\mathcal{T}^{/t}}(\mathcal{K}_{\underline{t}}, \mathcal{X}_{\underline{b}}) \longrightarrow \mathcal{X}_{\underline{b}}.$$

Consequently, $\delta_{r,p}$ admits a $\mathcal{T}$-right adjoint $\lim^{\mathcal{T}}$ given by $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \lim^{\mathcal{B},\mathcal{T}})$.

(3) A $\mathcal{T}^{/t}$-functor

$$\overline{f} : \mathcal{K}^{\triangleleft}_{\underline{t}} \longrightarrow \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X})_{\underline{t}}$$

is a $\mathcal{T}^{/t}$-limit diagram, resp. f admits a $\mathcal{T}^{/t}$-limit, if for all $\alpha : s \longrightarrow t$ in $\mathcal{T}$ and $c \in \mathcal{C}_s$ over $b \in \mathcal{B}_s$, the composite $\mathcal{T}^{/s}$-functor

$$\overline{f_c} : \mathcal{K}^{\triangleleft}_{\underline{s}} \xrightarrow{f_\alpha} \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X})_{\underline{s}} \xrightarrow{\mathrm{ev}_c} \underline{\mathrm{Fun}}_{/\mathcal{B}_{\underline{s}},\mathcal{T}^{/s}}(\underline{c}, \mathcal{X}_{\underline{s}}) \simeq \mathcal{X}_{\underline{b}}$$

is a $\mathcal{T}^{/s}$-limit diagram, resp. f_c admits a $\mathcal{T}^{/s}$-limit.

(4) Suppose that for every $b \in \mathcal{B}_t$, $\mathcal{X}_{\underline{b}}$ admits all $\mathcal{K}_{\underline{t}}$-indexed $\mathcal{T}^{/t}$-colimits, and for every morphism $f : b \longrightarrow b'$ in $\mathcal{B}_t$, the pushforward $\mathcal{T}^{/t}$-functor $f_! : \mathcal{X}_{\underline{b}} \longrightarrow \mathcal{X}_{\underline{b'}}$ preserves $\mathcal{K}_{\underline{t}}$-indexed $\mathcal{T}^{/t}$-colimits. Then δ_p admits a $\mathcal{B}$-relative $\mathcal{T}$-left adjoint

$$\mathrm{colim}^{\mathcal{B},\mathcal{T}} : \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{X}) \longrightarrow \mathcal{X}$$

which for all $b \in \mathcal{B}_t$ restricts to the $\mathcal{T}^{/t}$-colimit $\mathcal{T}^{/t}$-functor

$$\mathrm{colim}^{\mathcal{T}^{/t}} : \underline{\mathrm{Fun}}_{\mathcal{T}^{/t}}(\mathcal{K}_{\underline{t}}, \mathcal{X}_{\underline{b}}) \longrightarrow \mathcal{X}_{\underline{b}}.$$

Consequently, $\delta_{r,p}$ admits a $\mathcal{T}$-left adjoint $\mathrm{colim}^{\mathcal{T}}$ given by $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathrm{colim}^{\mathcal{B},\mathcal{T}})$.

(5) A $\mathcal{T}^{/t}$-functor

$$\overline{f} : \mathcal{K}^{\triangleright}_{\underline{t}} \longrightarrow \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X})_{\underline{t}}$$

is a $\mathcal{T}^{/t}$-colimit diagram, resp. f admits a $\mathcal{T}^{/t}$-colimit, if for all $\alpha : s \longrightarrow t$ in $\mathcal{T}$ and $c \in \mathcal{C}_s$ over $b \in \mathcal{B}_s$, the composite $\mathcal{T}^{/s}$-functor

$$\overline{f_c} : \mathcal{K}^{\triangleright}_{\underline{s}} \xrightarrow{f_\alpha} \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X})_{\underline{s}} \xrightarrow{\mathrm{ev}_c} \underline{\mathrm{Fun}}_{/\mathcal{B}_{\underline{s}},\mathcal{T}^{/s}}(\underline{c}, \mathcal{X}_{\underline{s}}) \simeq \mathcal{X}_{\underline{b}}$$

is a $\mathcal{T}^{/s}$-colimit diagram (resp. f_c admits a $\mathcal{T}^{/s}$-colimit), and for all $c \longrightarrow c' \in \mathcal{C}_s$ over $g : b \longrightarrow b' \in \mathcal{B}_s$, $g_! \circ \overline{f_c}$ is a $\mathcal{T}^{/s}$-colimit diagram (resp. $g_!$ preserves the $\mathcal{T}^{/s}$-colimit of f_c).

Proof. (1): Using Proposition 4.12, we may define δ_p as adjoint to the projection $\mathcal{X} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K} \longrightarrow \mathcal{X}$, since this is a morphism of $\mathcal{T}$-cocartesian fibrations over $\mathcal{B}$, and by construction this has the indicated property.
(2): Under our assumption, the existence of $\lim^{\mathcal{B},\mathcal{T}}$ follows immediately from Lemma 4.15(1). For the consequence, note that we have an equivalence of $\mathcal{T}$-∞-categories

$$\begin{aligned}\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{X})) &\simeq \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{X}) \\ &\simeq \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \widetilde{\mathrm{Fun}}_{\mathcal{B},\mathcal{T}}(\mathcal{B} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathcal{X}))\end{aligned}$$

where the first equivalence holds by the universal property of $\underline{\mathrm{Fun}}_{\mathcal{T}}(-,-)$ and the definition of $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},\mathcal{X})$ as a pullback, and the second equivalence holds by Theorem 4.9. It thus suffices to show that $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},-)$ covariantly transforms $\mathcal{B}$-relative $\mathcal{T}$-adjunctions into $\mathcal{T}$-adjunctions. For this, by Remark 4.14 we may suppose that $\mathcal{C}=\mathcal{B}$ without loss of generality, in which case the assertion is [Sha21, Cor. 8.5].
(3): This follows similarly to (2), but where we now consider how $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},-)$ transforms $\lim^{\mathcal{B},\mathcal{T}}$ as a partially defined $\mathcal{B}$-relative right $\mathcal{T}$-adjoint (under the given hypotheses on $\overline{f_c}$) to $\lim^{\mathcal{T}}$ as a partially defined right $\mathcal{T}$-adjoint.
(4) and (5): These are proven as for (2) and (3) but using Lemma 4.15(2) instead. □

4.17. **Corollary.** *Let $\mathcal{X}\longrightarrow\mathcal{B}$ be a $\mathcal{T}$-cocartesian fibration and let $\mathcal{C}\longrightarrow\mathcal{B}$ be a $\mathcal{T}$-fibration. Let $\mathcal{K}=\{\mathcal{K}_t : t\in\mathcal{T}\}$ be a collection of classes $\mathcal{K}_t$ of small $\mathcal{T}^{/t}$-∞-categories closed with respect to base-change in $\mathcal{T}$.*

1. *Suppose that for all $b\in\mathcal{B}_t$, $\mathcal{X}_{\underline{b}}$ admits all $\mathcal{K}_t$-indexed $\mathcal{T}^{/t}$-limits. Then $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},\mathcal{X})$ strongly admits all $\mathcal{K}$-indexed $\mathcal{T}$-limits.*
2. *Suppose that for all $b\in\mathcal{B}_t$, $\mathcal{X}_{\underline{b}}$ admits all $\mathcal{K}_t$-indexed $\mathcal{T}^{/t}$-colimits, and for all $g: b\longrightarrow b'\in\mathcal{B}_t$, the pushforward $\mathcal{T}^{/t}$-functor $g_! : \mathcal{X}_{\underline{b}}\longrightarrow\mathcal{X}_{\underline{b'}}$ preserves all $\mathcal{K}_t$-indexed $\mathcal{T}^{/t}$-colimits. Then $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},\mathcal{X})$ strongly admits all $\mathcal{K}$-indexed $\mathcal{T}$-colimits.*

Proof. We show how to deduce (1) from Theorem 4.16, the proof of (2) being similar. For this, the only additional point to note is that for any $\mathcal{K}\in\mathcal{K}_t$, by base-change of the given data to lie over $\mathcal{T}^{/t}$ we may apply Theorem 4.16 under our hypotheses to show that $\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},\mathcal{X})_{\underline{t}}$ admits all $\mathcal{K}$-indexed $\mathcal{T}^{/t}$-limits. □

5. Relative parametrized colimits

In this section and the next we work towards the proof of Theorem A.

5.1. **Definition.** Suppose we have a commutative diagram of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{p} & \mathcal{C} \\ {\scriptstyle i}\downarrow & \overset{\overline{p}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathcal{K}^{\trianglerighteq} & \xrightarrow{\overline{q}} & \mathcal{B} \end{array}$$

in which π is a $\mathcal{T}$-fibration. Let $q=\pi p=\overline{q}i$. We say that $\overline{p}$ is a *weak π-$\mathcal{T}$-colimit diagram* if the $\mathcal{T}$-functor

$$*_{\mathcal{T}}\longrightarrow *_{\mathcal{T}}\times_{\sigma_{\overline{q}},\mathcal{B}^{(q,\mathcal{T})/}}\mathcal{C}^{(p,\mathcal{T})/}$$

induced by $\sigma_{\overline{p}}$ is a $\mathcal{T}$-initial object.

We say that $\overline{p}$ is a *π-$\mathcal{T}$-colimit diagram* if the $\mathcal{T}$-functor

$$*_{\mathcal{T}}\longrightarrow \mathcal{B}^{(\overline{q},\mathcal{T})/}\times_{\mathcal{B}^{(q,\mathcal{T})/}}\mathcal{C}^{(p,\mathcal{T})/}$$

induced by $\sigma_{\overline{p}}$ is a $\mathcal{T}$-initial object. (Here, the projection to the first factor is induced by $\sigma_{\overline{q}'}$ for $\overline{q}'$ given by the composite $\mathcal{K}\star_{\mathcal{T}^{\mathrm{op}}}(\Delta^1\times\mathcal{T}^{\mathrm{op}})\xrightarrow{\mathrm{id}\star\mathrm{const}}\mathcal{K}^{\trianglerighteq}\xrightarrow{\overline{q}}\mathcal{B}$.)

5.2. **Example.** For $\mathcal{T}=\Delta^0$ and $\mathcal{K}=\Delta^0$, weak π-colimit diagrams are locally π-cocartesian edges, whereas π-colimit diagrams are π-cocartesian edges.

5.3. **Remark.** For Definition 5.1, $\overline{p}$ is a π-$\mathcal{T}$-colimit diagram if and only if the $\mathcal{T}$-functor

$$\mathcal{C}^{(\overline{p},\mathcal{T})/}\longrightarrow\mathcal{B}^{(\overline{q},\mathcal{T})/}\times_{\mathcal{B}^{(q,\mathcal{T})/}}\mathcal{C}^{(p,\mathcal{T})/}$$

is an equivalence of $\mathcal{T}$-∞-categories, or equivalently the commutative square

$$\begin{array}{ccc} \mathcal{C}^{(\overline{p},\mathcal{T})/} & \longrightarrow & \mathcal{C}^{(p,\mathcal{T})/} \\ \downarrow & & \downarrow \\ \mathcal{B}^{(\overline{q},\mathcal{T})/} & \longrightarrow & \mathcal{B}^{(q,\mathcal{T})/} \end{array}$$

is a homotopy pullback square (using Lemma 5.4(1)). This is ultimately because for any $\mathcal{T}$-category $\mathcal{E}$, a $\mathcal{T}$-functor $\sigma: *_{\mathcal{T}}\longrightarrow\mathcal{E}$ is a $\mathcal{T}$-initial object if and only if $\mathcal{E}^{(\sigma,\mathcal{T})/}\longrightarrow\mathcal{E}$ is an equivalence.

We now collect a few lemmas that will feature in the proof of our main result (Proposition 5.8) on the existence of π-$\mathcal{T}$-colimits. We first state a parametrized analogue of [Lur09, Prop. 4.2.1.6].

5.4. **Lemma.** *Suppose we have a commutative diagram of $\mathcal{T}$-∞-categories*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{p} & \mathcal{C} \\ {\scriptstyle i}\big\downarrow & \overset{\overline{p}}{\nearrow} & \big\downarrow{\scriptstyle \pi} \\ \mathcal{L} & \xrightarrow{\overline{q}} & \mathcal{B}. \end{array}$$

in which i is a monomorphism. Let $q = \pi \circ p = \overline{q} \circ i$ and let

$$\psi : \mathcal{C}^{(\overline{p},\mathcal{T})/} \longrightarrow \mathcal{C}^{(p,\mathcal{T})/} \times_{\mathcal{B}^{(q,\mathcal{T})/}} \mathcal{B}^{(\overline{q},\mathcal{T})/}$$

denote the induced $\mathcal{T}$-functor.

(1) If π is a categorical fibration, then

$$\phi : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{L}, \mathcal{C}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{B})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{L}, \mathcal{B})$$

is a categorical fibration, and ψ is a left fibration.

(2) Suppose that π is a $\mathcal{T}$-cocartesian fibration and let M_C denote the π-cocartesian edges in $\mathcal{C}$. Suppose that we have fiberwise markings $\{(M_K)_t\}$ on $\mathcal{K}$ and $\{(M_L)_t\}$ on $\mathcal{L}$ that are stable under base-change (i.e., for all $f : s \longrightarrow t \in \mathcal{T}$, $f^*(M_K)_t \subset (M_K)_s$). Let M_K be the minimal subset of the edges on $\mathcal{K}$ closed under composition that contains the cocartesian edges and $\{(M_K)_t\}$, and similarly define M_L. Let $\underline{\mathrm{Fun}}_{\mathcal{T}}((\mathcal{L}, M_L), (\mathcal{C}, M_C))$, etc. be the full $\mathcal{T}$-subcategories spanned by those $\mathcal{T}^{/t}$-functors that preserve the additional markings, and let

$$\phi' : \underline{\mathrm{Fun}}_{\mathcal{T}}((\mathcal{L}, M_L), (\mathcal{C}, M_C)) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}((\mathcal{K}, M_K), (\mathcal{C}, M_C)) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{B})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{L}, \mathcal{B})$$

be the restriction of ϕ. Then if $i : (\mathcal{K}, M_K) \longrightarrow (\mathcal{L}, M_L)$ is a cocartesian equivalence in $\mathbf{sSet}^+_{/\mathcal{T}^{\mathrm{op}}}$, ϕ' is a trivial fibration. Moreover, if $\overline{p}$ then sends M_L into M_C, ψ is a trivial fibration.

Proof. (1): It suffices to show that ϕ is a fibration in $\mathbf{sSet}^+_{/\mathcal{T}^{\mathrm{op}}}$ where we mark the cocartesian edges. This follows as in the proof of [Sha21, Lem. 3.5(1)]. Considering this categorical fibration for both i and $i^{\trianglerighteq}$ then shows ψ is a categorical fibration by a base-change argument. Since a functor between left fibrations over a common base that is a categorical fibration is necessarily a left fibration, to then show that ψ is a left fibration it suffices to show that $\mathcal{C}^{(f,\mathcal{T})/} \longrightarrow \mathcal{C}$ is a left fibration for any $\mathcal{T}$-functor $f : \mathcal{J} \longrightarrow \mathcal{C}$. But this is a consequence of the $\mathcal{T}$-functor

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{J}^{\trianglerighteq}, \mathcal{C}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{J}, \mathcal{C}) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{C}$$

being a $\mathcal{T}$-bifibration (cf. [Sha21, Ex. 7.10] and [Lur09, Rem. 2.4.7.4] for the stronger conclusion that $\mathcal{C}^{(f,\mathcal{T})/} \longrightarrow \mathcal{C}$ is a left fibration and not just a $\mathcal{T}$-cocartesian fibration).

(2): Since ϕ' is a categorical fibration by (1), it suffices to show that ϕ'_t is an equivalence for all $t \in \mathcal{T}$. After replacing $\mathcal{T}^{/t}$ with $\mathcal{T}$, we thus reduce to checking that

$$\widehat{\phi'} : \mathrm{Fun}_{\mathcal{T}}((\mathcal{L}, M_L), (\mathcal{C}, M_C)) \longrightarrow \mathrm{Fun}_{\mathcal{T}}((\mathcal{K}, M_K), (\mathcal{C}, M_C)) \times_{\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{B})} \mathrm{Fun}_{\mathcal{T}}(\mathcal{L}, \mathcal{B})$$

is a trivial fibration. Let $A \longrightarrow B$ be any cofibration of simplicial sets. The relevant lifting problem then transposes to

$$\begin{array}{ccc} A^\flat \times (\mathcal{L}, M_K) \bigcup_{A^\flat \times (\mathcal{K}, M_K)} B^\flat \times (\mathcal{K}, M_K) & \longrightarrow & (\mathcal{C}, M_C) \\ \big\downarrow & \nearrow & \big\downarrow \\ B^\flat \times (\mathcal{L}, M_L) & \longrightarrow & \mathcal{B}^\sharp. \end{array}$$

Now because trivial cofibrations in the cocartesian model structure on $\mathbf{sSet}^+_{/\mathcal{T}^{\mathrm{op}}}$ are stable under taking pushout-products with arbitrary cofibrations in $\mathbf{sSet}^+$ [Lur09, Cor. 3.1.4.3], there exists a dotted lift. The assertion about ψ then follows as in (1) by a base-change argument. □

We have an elementary observation about lifting $\mathcal{T}$-initial objects along a $\mathcal{T}$-cocartesian fibration.

5.5. **Lemma.** *Let $\pi : \mathcal{C} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-cocartesian fibration and suppose that $\sigma : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-initial object. Let $\widetilde{\sigma} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor lift of σ. Suppose that:*

(1) $\widetilde{\sigma}$ *is a* $\mathcal{T}$*-initial object in* $\mathcal{T}^{\mathrm{op}} \times_{\sigma,\mathcal{B}} \mathcal{C}$.
(2) For all $t \in \mathcal{T}$ *and morphisms* $f : \sigma(t) \longrightarrow y \in \mathcal{B}_t$*, the pushforward* $\mathcal{T}^{/t}$*-functor* $f_! : \mathcal{C}_{\underline{\sigma(t)}} \longrightarrow \mathcal{C}_{\underline{y}}$ *preserves* $\mathcal{T}^{/t}$*-initial objects.*

Then $\widetilde{\sigma}$ *is a* $\mathcal{T}$*-initial object.*

Proof. We may check for all $t \in \mathcal{T}$ that $\widetilde{\sigma}(t)$ is an initial object in $\mathcal{C}_t$, using the known assertion when $\mathcal{T} = \Delta^0$. □

We can then bootstrap from Lemma 5.5 to understand relative $\mathcal{T}$-colimits originating from $\mathcal{T}$-colimits in the parametrized fibers.

5.6. **Lemma.** *Suppose we have a commutative diagram of* $\mathcal{T}$*-*∞*-categories*

$$\begin{array}{ccccc} \mathcal{K} & \xrightarrow{p_0} & \mathcal{T}^{\mathrm{op}} \times_{\mathcal{B}} \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & \overset{\overline{p_0}}{\nearrow} & \downarrow & \overset{\overline{p}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathcal{K}^{\trianglerighteq} & \longrightarrow & \mathcal{T}^{\mathrm{op}} & \xrightarrow{\sigma} & \mathcal{B} \end{array} \qquad (p = \mathcal{K} \to \mathcal{C},\ \overline{q} = \mathcal{K}^{\trianglerighteq} \to \mathcal{B})$$

in which π *is a* $\mathcal{T}$*-cocartesian fibration and* $\overline{p_0}$ *is a* $\mathcal{T}$*-colimit diagram. Then* $\overline{p}$ *is a weak* π*-*$\mathcal{T}$*-colimit diagram.*

Suppose moreover that for every $t \in \mathcal{T}$ *and morphism* $f : \sigma(t) \longrightarrow y \in \mathcal{B}_t$*, the pushforward* $\mathcal{T}^{/t}$*-functor* $f_! : \mathcal{C}_{\underline{\sigma(t)}} \longrightarrow \mathcal{C}_{\underline{y}}$ *preserves the* $\mathcal{K}_{\underline{t}}$*-indexed* $\mathcal{T}^{/t}$*-colimit diagram given by* $(\overline{p_0})_{\underline{t}}$*. Then* $\overline{p}$ *is a* π*-*$\mathcal{T}^{/t}$*-colimit diagram.*

Proof. By Lemma 5.7, we have an equivalence $\mathcal{C}^{(p_0,\mathcal{T})/} \simeq \mathcal{T}^{\mathrm{op}} \times_{\mathcal{B}^{(q,\mathcal{T})/}} \mathcal{C}^{(p,\mathcal{T})/}$. Thus by definition, if $\overline{p_0}$ is a $\mathcal{T}$-colimit diagram, then $\overline{p}$ is a weak π-$\mathcal{C}$-colimit diagram.

For the second claim, we note that the fiberwise cofinal $\mathcal{T}$-functor $\mathcal{T}^{\mathrm{op}} \subset \mathcal{K}^{\trianglerighteq}$ given by inclusion of the $\mathcal{T}$-cone point induces an equivalence $\mathcal{B}^{(\overline{q},\mathcal{T})/} \xrightarrow{\simeq} \mathcal{B}^{(\sigma,\mathcal{T})/}$ by [Sha21, Thm. 6.7]. It thus suffices to check that the $\mathcal{T}$-functor

$$\begin{array}{ccc} \mathcal{T}^{\mathrm{op}} & \xrightarrow{\widetilde{\sigma}} & \mathcal{B}^{(\sigma,\mathcal{T})/} \times_{\mathcal{B}^{(\pi p,\mathcal{T})/}} \mathcal{C}^{(p,\mathcal{T})/} \\ & \underset{\sigma}{\searrow} & \downarrow{\scriptstyle \pi'} \\ & & \mathcal{B}^{(\sigma,\mathcal{T})/} \end{array}$$

induced by $\overline{p}$ is a $\mathcal{T}$-initial object (where we abuse notation and write σ also for the $\mathcal{T}$-initial object id_σ in $\mathcal{B}^{(\sigma,\mathcal{T})/}$). By Lemma 5.4(1), π' is a left fibration. Under the equivalence $\mathcal{B}^{(\sigma,\mathcal{T})/} \simeq \mathcal{T}^{\mathrm{op}} \times_{\sigma,\mathcal{B}} \mathrm{Ar}(\mathcal{B})$ of Observation 2.4, the objects of the base $\mathcal{B}^{(\sigma,\mathcal{T})/}$ are equivalently given by pairs $(t \in \mathcal{T}, f : \sigma(t) \longrightarrow y)$. A cocartesian section of the parameterized fiber $\pi'_{\underline{f}}$ is determined up to equivalence by a commutative diagram of $\mathcal{T}^{/t}$-∞-categories

$$\begin{array}{ccc} \mathcal{K}_{\underline{t}} & \xrightarrow{p_0'} & \mathcal{C}_{\underline{y}} \\ \downarrow & \overset{\overline{p_0}'}{\nearrow} & \downarrow \\ (\mathcal{K}_{\underline{t}})^{\trianglerighteq} & \longrightarrow & (\mathcal{T}^{/t})^{\mathrm{op}} \end{array}$$

and is a $\mathcal{T}^{/t}$-initial object if and only if $\overline{p_0}'$ is a $\mathcal{T}^{/t}$-colimit diagram. It is now clear that under our hypotheses, Lemma 5.5 applies to show that $\widetilde{\sigma}$ is a $\mathcal{T}$-initial object. □

5.7. **Lemma.** *Suppose we have a homotopy pullback square of* $\mathcal{T}$*-*∞*-categories*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \mathcal{Y} & \xrightarrow{k} & \mathcal{Z} \end{array}$$

and a $\mathcal{T}$-functor $p : \mathcal{K} \longrightarrow \mathcal{W}$. Then the commutative square of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc} \mathcal{W}^{(p,\mathcal{T})/} & \longrightarrow & \mathcal{X}^{(fp,\mathcal{T})/} \\ \downarrow & & \downarrow \\ \mathcal{Y}^{(gp,\mathcal{T})/} & \longrightarrow & \mathcal{Z}^{(hfp,\mathcal{T})/}. \end{array}$$

is a homotopy pullback square.

Proof. The proof is a straightforward diagram chase, starting from the known assertion that

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, -) : \mathbf{Cat}_{\mathcal{T}} \longrightarrow \mathbf{Cat}_{\mathcal{T}}$$

preserves limits. □

Finally, we arrive at our main existence result for relative $\mathcal{T}$-colimits.

5.8. **Proposition.** *Suppose we have a commutative diagram of $\mathcal{T}$-∞-categories*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{p} & \mathcal{C} \\ {\scriptstyle i}\downarrow & \overset{\overline{p}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathcal{K}^{\underline{\rhd}} & \xrightarrow{\overline{q}} & \mathcal{B} \end{array}$$

in which $\pi : \mathcal{C} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-cocartesian fibration. Let $\sigma = \overline{q}|_{\mathcal{T}^{\mathrm{op}}}$. If for all $t \in \mathcal{T}$, the parametrized fiber $\mathcal{C}_{\underline{\sigma(t)}}$ admits $\mathcal{K}_{\underline{t}}$-indexed $\mathcal{T}^{/t}$-colimits, then there exists a filler $\overline{p} : \mathcal{K}^{\underline{\rhd}} \longrightarrow \mathcal{C}$ which is a weak π-$\mathcal{T}$-colimit diagram.

Moreover, suppose that for all morphisms $f : \sigma(t) \longrightarrow y \in \mathcal{B}_t$, the induced pushforward $\mathcal{T}^{/t}$-functor $f_! : \mathcal{C}_{\underline{\sigma(t)}} \longrightarrow \mathcal{C}_{\underline{y}}$ preserves $\mathcal{K}_{\underline{t}}$-indexed $\mathcal{T}^{/t}$-colimits. Then $\overline{p}$ is a π-$\mathcal{T}$-colimit diagram.

Proof. We prove this by reducing to Lemma 5.6. Let M_C denote the π-cocartesian edges in $\mathcal{C}$. First consider the diagram

$$\begin{array}{ccccc} \mathcal{K} \times \{0\} & & \xrightarrow{p} & & \mathcal{C} \\ {\scriptstyle i_0}\downarrow & & \overset{h}{\nearrow} & & \downarrow{\scriptstyle \pi} \\ \mathcal{K} \times \Delta^1 & \xrightarrow{f} & \mathcal{K}^{\underline{\rhd}} & \xrightarrow{\overline{q}} & \mathcal{B} \end{array}$$

where the map f is adjoint to $(\mathcal{K} = \mathcal{K}, \mathcal{K} \longrightarrow \mathcal{T})$. Because $i_0 : {}_\natural\mathcal{K} \times \{0\} \longrightarrow {}_\natural\mathcal{K} \times (\Delta^1)^\sharp$ is left marked anodyne, the dotted map $h : {}_\natural\mathcal{K} \times (\Delta^1)^\sharp \longrightarrow (\mathcal{C}, M_C)$ exists. Consider the two commutative squares

$$\begin{array}{ccc} \mathcal{K} \times \Delta^1 & \xrightarrow{h} & \mathcal{C} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{K}^{\underline{\rhd}} & \xrightarrow{\overline{q}} & \mathcal{B} \end{array} \quad , \quad \begin{array}{ccccc} \mathcal{K} & & \xrightarrow{p' := h|_{\mathcal{K} \times \{1\}}} & & \mathcal{C} \\ {\scriptstyle i}\downarrow & & & & \downarrow{\scriptstyle \pi} \\ \mathcal{K}^{\underline{\rhd}} & \longrightarrow & \mathcal{T}^{\mathrm{op}} & \xrightarrow{\sigma} & \mathcal{B} \end{array}$$

where the bottom composite $\mathcal{K}^{\underline{\rhd}} \longrightarrow \mathcal{B}$ is $\overline{q}'$.

We obtain a zig-zag

$$\begin{array}{ccccc} \mathcal{B}^{(\overline{q},\mathcal{T})/} \times_{\mathcal{B}^{(\pi p,\mathcal{T})/}} \mathcal{C}^{(p,\mathcal{T})/} & \xleftarrow{\psi} & \mathcal{B}^{(\overline{q},\mathcal{T})/} \times_{\mathcal{B}^{(\pi h,\mathcal{T})/}} \mathcal{C}^{(h,\mathcal{T})/} & \xrightarrow{\phi} & \mathcal{B}^{(\overline{q}',\mathcal{T})/} \times_{\mathcal{B}^{(\pi p',\mathcal{T})/}} \mathcal{C}^{(p',\mathcal{T})/} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}^{(\overline{q},\mathcal{T})/} & \xleftarrow{=} & \mathcal{B}^{(\overline{q},\mathcal{T})/} & \xrightarrow{\chi} & \mathcal{B}^{(\overline{q}',\mathcal{T})/} \end{array}$$

where all the maps are obvious (except possibly χ, which is induced by precomposition by $\mathcal{K}^{\underline{\rhd}} \longrightarrow \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{K}^{\underline{\rhd}}$). We claim that ψ and ϕ are equivalences. For ψ, by Lemma 5.4(2),

$$\mathcal{C}^{(h,\mathcal{T})/} \longrightarrow \mathcal{B}^{(\pi h,\mathcal{T})/} \times_{\mathcal{B}^{(\pi p,\mathcal{T})/}} \mathcal{C}^{(p,\mathcal{T})/}$$

is a trivial fibration. But ψ is a pullback of this map, hence an equivalence. For ϕ, note that the $\mathcal{T}$-functors $\mathcal{K}^{\underline{\rhd}} \longrightarrow \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{K}^{\underline{\rhd}}$ and $\mathcal{K} \times \{1\} \longrightarrow \mathcal{K} \times \Delta^1$ are both fiberwise cofinal. Hence by [Sha21, Thm. 6.7], we deduce that ϕ is an equivalence.

Replacing p and $\overline{q}$ by p' and $\overline{q}'$, we find ourselves in the situation of Lemma 5.6, which immediately applies given our hypotheses. □

6. Relative parametrized left Kan extensions

6.1. **Definition.** Suppose we have a commutative diagram of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ {\scriptstyle i}\big\downarrow & \overset{G}{\nearrow} & \big\downarrow{\scriptstyle \pi} \\ \mathcal{D} & \longrightarrow & \mathcal{B} \end{array}$$

in which i is the inclusion of a full $\mathcal{T}$-subcategory. Then we say that G is a π-$\mathcal{T}$-*left Kan extension of* F if for every $x \in \mathcal{D}_t$, the commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{/\underline{x}} & \xrightarrow{F^x} & \mathcal{E}_{\underline{t}} \\ \big\downarrow & \overset{G^x}{\nearrow} & \big\downarrow{\scriptstyle \pi_{\underline{t}}} \\ (\mathcal{C}^{/\underline{x}})^{\underline{\triangleright}} & \longrightarrow & \mathcal{B}_{\underline{t}} \end{array}$$

exhibits G^x (defined in Construction 2.11) as a $\pi_{\underline{t}}$-$\mathcal{T}^{/t}$-colimit diagram. Here, the lower horizontal $\mathcal{T}^{/t}$-functor is the composite

$$(\mathcal{C}^{/\underline{x}})^{\underline{\triangleright}} \longrightarrow (\mathcal{D}^{/\underline{x}})^{\underline{\triangleright}} \xrightarrow{\theta_x} \mathcal{D}_{\underline{t}} \longrightarrow \mathcal{B}_{\underline{t}}.$$

We also say that G is a *weak* π-$\mathcal{T}$-*left Kan extension of* F if in the pulled-back diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F'} & \mathcal{E} \times_{\mathcal{B}} \mathcal{D} \\ {\scriptstyle i}\big\downarrow & \overset{G'}{\nearrow} & \big\downarrow{\scriptstyle \pi'} \\ \mathcal{D} & \xrightarrow{=} & \mathcal{D}, \end{array}$$

G' is a π'-$\mathcal{T}$-left Kan extension of F'.

We may now prove our main existence result on relative $\mathcal{T}$-left Kan extensions, from which Theorem A is an immediate corollary.

6.2. **Theorem.** *Let $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-cocartesian fibration*[22]*, let $\rho : \mathcal{D} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-functor, and let $i : \mathcal{C} \subset \mathcal{D}$ be the inclusion of a full $\mathcal{T}$-subcategory.*

(1) Let $F : \mathcal{C} \longrightarrow \mathcal{E}$ be a $\mathcal{T}$-functor over $\mathcal{B}$ and suppose that for all $x \in \mathcal{D}_t$, $F^x : \mathcal{C}^{/\underline{x}} \longrightarrow \mathcal{E}_{\underline{t}}$ admits a $\pi_{\underline{t}}$-$\mathcal{T}^{/t}$-colimit. Then F admits an essentially unique π-$\mathcal{T}$-left Kan extension $G : \mathcal{D} \longrightarrow \mathcal{E}$.

(2) The partial $\mathcal{T}$-left adjoint $i_!$ to the restriction $\mathcal{T}$-functor

$$i^* : \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{D}, \mathcal{E}) \longrightarrow \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C}, \mathcal{E})$$

is defined on all those $F : \mathcal{C}_{\underline{t}} \longrightarrow \mathcal{E}_{\underline{t}}$ that admit a weak $\pi_{\underline{t}}$-$\mathcal{T}^{/t}$-left Kan extension G, in which case $i_! F \simeq G$.

Proof. The overarching strategy is the same as in the proof of Theorem 2.13 given in [Sha21, §10]. The key idea is to factor i through the free $\mathcal{T}$-cocartesian fibration as

$$\mathcal{C} \xrightarrow{\iota} \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}) \xrightarrow{\mathrm{ev}_1} \mathcal{D}.$$

(1): Choose a section ξ of the trivial fibration $\mathrm{Ar}_{\mathcal{T}}^{cocart}(\mathcal{E}) \longrightarrow \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{B})$ that restricts to the identity section on $\mathcal{E}$ and let F' be the composite

$$\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}) \xrightarrow{F \times \mathrm{Ar}_{\mathcal{T}}(\rho)} \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{B}) \xrightarrow{\xi} \mathrm{Ar}_{\mathcal{T}}^{cocart}(\mathcal{E}) \xrightarrow{\mathrm{ev}_1} \mathcal{E}.$$

We then have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}) & \xrightarrow{F'} & \mathcal{E} \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ (\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})) \star_{\mathcal{D}} \mathcal{D} & \longrightarrow & \mathcal{B} \end{array}$$

[22]Because of this assumption, our theorem is slightly weaker than [Lur09, Thm. 4.3.2.15] in the case where $\mathcal{T} = \Delta^0$.

where $F = F'|_{\mathcal{C}}$. Note that if $\xi : \mathcal{M} \longrightarrow \mathcal{D}$ is any $\mathcal{T}$-cocartesian fibration, then for any $x \in \mathcal{D}_t$ we have a pullback square

$$\begin{array}{ccc} \mathcal{M}_{\underline{x}} & \longrightarrow & \mathcal{M}^{/\underline{x}} := \mathcal{M} \times_{\mathcal{M} \star_{\mathcal{D}} \mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{M} \star_{\mathcal{D}} \mathcal{D}) \times_{\mathcal{M} \star_{\mathcal{D}} \mathcal{D}} \underline{x} \\ \downarrow & & \downarrow \\ \underline{x} & \xrightarrow{\iota_x} & \mathcal{D}^{/\underline{x}} := \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}) \times_{\mathcal{D}} \underline{x} \end{array}$$

where the righthand vertical functor is a cocartesian fibration (induced by $\mathcal{M} \star_{\mathcal{D}} \mathcal{D} \longrightarrow \mathcal{D}$). Since cocartesian fibrations are smooth [Lur09, Prop. 4.1.2.15] and ι_x is fiberwise cofinal, it follows that $\mathcal{M}_{\underline{x}} \longrightarrow \mathcal{M}^{/\underline{x}}$ is fiberwise cofinal (with respect to the base $(\mathcal{T}^{/t})^{\mathrm{op}}$). In our situation, $\mathcal{M} = \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})$, $\mathcal{M}_{\underline{x}} \cong \mathcal{C}^{/\underline{x}}$, and $(F')^x$ restricts on $\mathcal{C}^{/\underline{x}}$ to F^x. By the proof of Proposition 5.8 together with [Sha21, Thm. 6.7], we see that F^x admits a π-$\mathcal{T}$-colimit if and only if $(F')^x$ admits a π-$\mathcal{T}$-colimit. We thereby reduce to the '$\mathcal{D}$-parametrized' situation of a $\mathcal{T}$-cocartesian fibration $\phi : \mathcal{M} \longrightarrow \mathcal{D}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{E} \\ {\scriptstyle j}\downarrow & \overset{G}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathcal{M} \star_{\mathcal{D}} \mathcal{D} & \longrightarrow & \mathcal{B} \end{array}$$

in which F sends ϕ-cocartesian edges to π-cocartesian edges. Pulling back along ρ, we may also suppose that $\mathcal{D} = \mathcal{B}$, noting that the discrepancy between strong and weak π-$\mathcal{T}$-left Kan extensions will lie only in the pointwise property of the eventual extension and does not feature in the constructive proof of existence.

We may solve the coherence problem of assembling the individual $\pi_{\underline{t}}$-$\mathcal{T}^{/t}$-colimits together into a π-$\mathcal{T}$-left Kan extension G by a similar method to the proof of [Sha21, Thm. 9.15]. Consider [Sha21, Constr. 9.8] applied to ϕ and F; this yields

$$\mathcal{W} = \mathcal{E}^{(\phi,F)/\mathcal{T}} := \mathcal{D} \times_{\widetilde{\mathrm{Fun}}_{\mathcal{D},\mathcal{T}}(\mathcal{M},\mathcal{E})} \widetilde{\mathrm{Fun}}_{\mathcal{D},\mathcal{T}}(\mathcal{M} \star_{\mathcal{D}} \mathcal{D}, \mathcal{E}),$$

such that for $x \in \mathcal{D}_t$, if we let $F|_x : \mathcal{M}_{\underline{x}} \longrightarrow \mathcal{E}_{\underline{x}}$ denote the $\mathcal{T}^{/t}$-functor given by restriction, then the parametrized fiber $\mathcal{W}_{\underline{x}}$ is equivalent to $(\mathcal{E}_{\underline{x}})^{(F|_x,\mathcal{T}^{/t})/}$ by Corollary 4.6 (or [Sha21, Cor. 9.9]). Let $\mathcal{W}' \subset \mathcal{W}$ be the full $\mathcal{T}$-subcategory spanned by the $\mathcal{T}^{/t}$-colimit diagrams $(\mathcal{M}_{\underline{x}})^{\underline{\triangleright}} \longrightarrow \mathcal{E}_{\underline{x}}$, so that for all $x \in \mathcal{D}$, the fiber $\mathcal{W}'_x$ is that spanned by the initial objects in $\mathcal{W}_x$, which are precisely weak $\pi_{\underline{t}}$-$\mathcal{T}^{/t}$-colimit diagrams extending $F|_x$. Note that the use of [Sha21, Prop. 9.10] and [Sha21, Lem. 9.11] in the proof of [Sha21, Thm. 9.15] won't apply here since $\mathcal{E} \longrightarrow \mathcal{D}$ isn't supposed to be a $\mathcal{T}$-cartesian fibration. However, we may argue directly that $\mathcal{W}' \longrightarrow \mathcal{D}$ is a trivial fibration by computing mapping spaces as follows:

(∗) For any $\alpha : x \longrightarrow y \in \mathcal{D}_t$ and extensions $\overline{F|_i}$ of $F|_i$ over $(\mathcal{M}_{\underline{i}})^{\underline{\triangleright}}$, $i \in \{x, y\}$, we have that maps $\overline{F|_x} \longrightarrow \overline{F|_y}$ in $\widetilde{\mathrm{Fun}}_{\mathcal{D},\mathcal{T}}(\mathcal{M} \star_{\mathcal{D}} \mathcal{D}, \mathcal{E})$ are defined by lax commutative squares of $\mathcal{T}^{/t}$-functors

$$\begin{array}{ccc} (\mathcal{M}_{\underline{x}})^{\underline{\triangleright}} & \xrightarrow{\overline{F|_x}} & \mathcal{E}_{\underline{x}} \\ {\scriptstyle \alpha_!}\downarrow & \swarrow & \downarrow{\scriptstyle \alpha_!} \\ (\mathcal{M}_{\underline{y}})^{\underline{\triangleright}} & \xrightarrow[\overline{F|_y}]{} & \mathcal{E}_{\underline{y}}, \end{array}$$

and hence we have

$$\mathrm{Map}_{\mathcal{W}}(\overline{F|_x}, \overline{F|_y}) \simeq \mathrm{Map}(\alpha_! \overline{F|_x}, \overline{F|_y} \alpha_!) \times_{\mathrm{Map}(\alpha_! F|_x, F|_y \alpha_!)} \{\mathrm{id}\}.$$

By assumption, if $\overline{F|_x}$ is a $\mathcal{T}^{/t}$-colimit diagram, then $\alpha_! \overline{F|_x}$ is as well. Therefore, $\mathrm{Map}_{\mathcal{W}'}(\overline{F|_x}, \overline{F|_y})$ is contractible for all $\mathcal{T}^{/t}$-colimit diagrams $\overline{F|_x}$ and $\overline{F|_y}$, and this suffices to show that $\mathcal{W}' \longrightarrow \mathcal{D}$ is a trivial fibration since we already have compatibility of these initial objects with restriction in the base $\mathcal{T}$. Furthermore, any section τ of this trivial fibration defines a relative left adjoint of $\mathcal{W} \longrightarrow \mathcal{D}$ with respect to the base $\mathcal{D}$.

Now by Theorem 4.9, applying $\underline{\mathrm{Fun}}_{/\mathcal{D},\mathcal{T}}(\mathcal{D}, -)$ to τ yields an extension $G : \mathcal{M} \star_{\mathcal{D}} \mathcal{D} \longrightarrow \mathcal{E}$ of F that is a $\mathcal{T}$-initial object of

$$\mathcal{T}^{\mathrm{op}} \times_{\sigma_F, \underline{\mathrm{Fun}}_{/\mathcal{D},\mathcal{T}}(\mathcal{M},\mathcal{E})} \underline{\mathrm{Fun}}_{/\mathcal{D},\mathcal{T}}(\mathcal{M} \star_{\mathcal{D}} \mathcal{D}, \mathcal{E}).$$

Taking cocartesian sections, we then see that G is an initial object in the space of such fillers and is in particular essentially unique.

(2): Let $\phi' : (\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})) \star_{\mathcal{D}} \mathcal{D} \longrightarrow \mathcal{D}$ be the structure map. Factor i^* as

$$\underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{D},\mathcal{E}) \xrightarrow{(\phi')^*} \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}((\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})) \star_{\mathcal{D}} \mathcal{D},\mathcal{E}) \xrightarrow{j^*} \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}),\mathcal{E}) \xrightarrow{\iota^*} \underline{\mathrm{Fun}}_{/\mathcal{B},\mathcal{T}}(\mathcal{C},\mathcal{E}).$$

Then since ϕ' is $\mathcal{T}$-left adjoint to $i_{\mathcal{D}}$, $(\phi')^*$ has $\mathcal{T}$-left adjoint $(i_{\mathcal{D}})^*$. Also, by Example 3.8 the procedure $F \mapsto F'$ of (1) defines a fully faithful $\mathcal{T}$-left adjoint to ι^* with essential image spanned by those $\mathcal{T}$-functors that send ϕ-cocartesian edges to π-cocartesian edges. To conclude, we observe that in the proof of (1) we showed that the partial left $\mathcal{T}$-adjoint $j_!$ is defined on F' if it is obtained from F satisfying the assumptions of (2). □

7. More on the parametrized Yoneda embedding

For a $\mathcal{T}$-∞-category $\mathcal{C}$, let $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}) \coloneqq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}})$ be the $\mathcal{T}$-∞-category of $\mathcal{T}$-presheaves and $j_{\mathcal{T}} : \mathcal{C} \longrightarrow \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ the $\mathcal{T}$-Yoneda embedding [Sha21, §11]. In this section, we record a generalization (Proposition 7.5) of our earlier result that $j_{\mathcal{T}}$ strongly preserves $\mathcal{T}$-limits [Sha21, Cor. 11.10] as well as some basic facts concerning $\mathcal{T}$-corepresentable $\mathcal{T}$-left fibrations (Lemma 7.8) that mirror the discussion in [Lur09, §4.4.4]. These results play a technical role in the remainder of the paper and so their proofs could be skipped on a first reading.

7.1. **Lemma.** *Let $\mathcal{K}$ and $\mathcal{C}$ be $\mathcal{T}$-∞-categories. Then we have a homotopy pullback square*

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}^{\trianglelefteq},\mathcal{C}) & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K} \times \Delta^1,\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}) & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}) \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}). \end{array}$$

Thus, for any $\mathcal{T}$-functor $p : \mathcal{K} \longrightarrow \mathcal{C}$ and $t \in \mathcal{T}$, we have an equivalence

$$(\mathcal{C}^{/(p,\mathcal{T})})_t \simeq \mathcal{C}_t \times_{\mathrm{Fun}_{\mathcal{T}^{/t}}(\mathcal{K}_{\underline{t}},\mathcal{C}_{\underline{t}})} \mathrm{Fun}_{\mathcal{T}^{/t}}(\mathcal{K}_{\underline{t}},\mathcal{C}_{\underline{t}})^{/\{p_{\underline{t}}\}}$$

of right fibrations over $\mathcal{C}_t$.

Proof. Consider the commutative square of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc} \mathcal{K} \times \partial\Delta^1 & \longrightarrow & \mathcal{T}^{\mathrm{op}} \coprod \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{K} \times \Delta^1 & \longrightarrow & \mathcal{K}^{\trianglelefteq} \end{array}$$

where the vertical maps are the inclusions and the horizontal maps are induced by the structure map $\mathcal{K} \longrightarrow \mathcal{T}^{\mathrm{op}}$ and the identity on $\mathcal{K}$. By application of [Lur09, Prop. 4.2.1.2] fiberwise, this is a homotopy pushout square, and the first claim follows by transforming the pushout to a pullback under $\underline{\mathrm{Fun}}_{\mathcal{T}}(-,\mathcal{C})$. For a $\mathcal{T}$-functor $p : \mathcal{K} \longrightarrow \mathcal{C}$, we thus obtain a commutative diagram of homotopy pullback squares

$$\begin{array}{ccccc} \mathcal{C}^{/(p,\mathcal{T})} & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})^{/(\sigma_p,\mathcal{T})} & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K} \times \Delta^1,\mathcal{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\delta} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}) & \xrightarrow{(\mathrm{id},\sigma_p)} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}) \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C}), \end{array}$$

where $\sigma_p : \mathcal{T}^{\mathrm{op}} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K},\mathcal{C})$ selects p. To identify $(\mathcal{C}^{/(p,\mathcal{T})})_t$, after replacing p by $p_{\underline{t}}$ we may suppose that $\mathcal{T}$ has a final object $*$. But then for any $\mathcal{T}$-∞-category $\mathcal{D}$ and cocartesian section $\sigma : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{D}$ that selects an object $x = \sigma(*) \in \mathcal{D}_*$, we have that $(\mathcal{D}^{/(\sigma,\mathcal{T})})_* \simeq (\mathcal{D}_*)^{/x}$ by [Sha21, Prop. 4.30]. □

The following lemma generalizes and supplies another proof of the fact that the Yoneda embedding preserves limits [Lur09, Prop. 5.1.3.2].

7.2. **Lemma.** *Let $p : \mathcal{K} \longrightarrow \mathcal{C}$ be functor of small ∞-categories. Then the commutative square of ∞-categories*

$$\begin{array}{ccc} \mathcal{C}^{/p} & \longrightarrow & \mathbf{P}(\mathcal{C})^{/jp} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{j} & \mathbf{P}(\mathcal{C}) \end{array}$$

is a homotopy pullback square. Consequently, the functor $\varphi : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Spc}$ classifying the right fibration $\mathcal{C}^{/p} \longrightarrow \mathcal{C}$ is canonically equivalent to $\lim_{\mathcal{K}} jp$.

Proof. It suffices to show that the induced functor $\psi : \mathcal{C}^{/p} \longrightarrow \mathcal{C} \times_{\mathbf{P}(\mathcal{C})} \mathbf{P}(\mathcal{C})^{/jp}$ is an equivalence of right fibrations over $\mathcal{C}$ by checking that for all $x \in \mathcal{C}$, ψ_x is an equivalence. First note that by the $\mathcal{T} = \Delta^0$ case of Lemma 7.1, we may identify ψ_x with the map

$$\psi'_x : \mathrm{Map}_{\mathcal{C}^{\mathcal{K}}}(\delta x, p) \longrightarrow \mathrm{Map}_{\mathbf{P}(\mathcal{C})^{\mathcal{K}}}(\delta j(x), jp) \simeq \mathrm{Map}_{\mathbf{P}(\mathcal{C})}(j(x), \lim_{\mathcal{K}} jp) \simeq (\lim_{\mathcal{K}} jp)(x)$$

induced by postcomposition by the Yoneda embedding. Using the end formula for mapping spaces in $\mathrm{Fun}(\mathcal{K}, \mathcal{C})$ [Gla16, Prop. 2.3], we have an equivalence

$$\mathrm{Map}_{\mathcal{C}^{\mathcal{K}}}(\delta x, p) \simeq \int_{\mathcal{K}} \mathrm{Map}_{\mathcal{C}}(x, p(-)) := \lim_{\mathrm{Tw}(\mathcal{K})} \mathrm{Map}_{\mathcal{C}}(x, p(-)).$$

where the limit is taken over the functor

$$\mathrm{Tw}(\mathcal{K}) \xrightarrow{\mathrm{ev}_1} \mathcal{K} \xrightarrow{p} \mathcal{C} \xrightarrow{\mathrm{Map}_{\mathcal{C}}(x,-)} \mathbf{Spc}.$$

Under this identification, ψ'_x is induced by restriction along ev_1 (using the contravariant functoriality of limits in the diagram). The claim then follows from Lemma 7.3.

Since representable right fibrations are classified by the corresponding representable functor [Lur09, Prop. 4.4.4.5], we then have that φ is equivalent to the composition

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{j^{\mathrm{op}}} \mathbf{P}(\mathcal{C})^{\mathrm{op}} \xrightarrow{\mathrm{Map}_{\mathbf{P}(\mathcal{C})}(-, \lim_{\mathcal{K}} jp)} \mathbf{Spc}.$$

An argument with the Yoneda lemma then shows φ is in turn equivalent to $\lim_{\mathcal{K}} jp$ (in more detail, see Remark 7.4). □

7.3. **Lemma.** *Let $\mathcal{K}$ be an ∞-category. Then the source and target functors*

$$\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Tw}(\mathcal{K}) \longrightarrow \mathcal{K}$$

are right cofinal.

Proof. We verify the hypotheses of Joyal's cofinality theorem [Lur09, Thm. 4.1.3.1] for ev_1 (in its opposite formulation). Let $y \in \mathcal{K}$ and consider the commutative diagram of homotopy pullbacks

$$\begin{array}{ccccc} (\mathcal{K}_{/y})^{\mathrm{op}} & \xrightarrow{\iota'} & \mathrm{Tw}(\mathcal{K}) \times_{\mathcal{K}} \mathcal{K}_{/y} & \longrightarrow & \mathrm{Tw}(\mathcal{K}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \{y\} & \xrightarrow{\iota} & \mathcal{K}_{/y} & \longrightarrow & \mathcal{K} \end{array}$$

Since ev_1 is a left fibration, ev_1 is a smooth map [Lur09, Prop. 4.1.2.15]. Therefore, the pullback ι' of the cofinal inclusion ι along ev_1 is again cofinal, so in particular a weak homotopy equivalence. We deduce that $\mathrm{Tw}(\mathcal{K}) \times_{\mathcal{K}} \mathcal{K}_{/y}$ is weakly contractible, which proves the claim. The proof for ev_0 is similar. □

7.4. **Remark.** Let $q \in \mathbf{P}(\mathcal{C})$ be a presheaf. By [Lur09, Lem. 5.1.5.2], we have an equivalence $q(c) \simeq \mathrm{Map}_{\mathbf{P}(\mathcal{C})}(j(c), q)$ of spaces for all $c \in \mathcal{C}$. We can promote this to an equivalence of presheaves

$$\mathrm{Map}_{\mathbf{P}(\mathcal{C})}(j(-), q) \simeq q(-)$$

as follows. Let $\pi : \mathcal{C}^{/q} = \mathcal{C} \times_{\mathbf{P}(\mathcal{C})} \mathbf{P}(\mathcal{C})^{/q} \longrightarrow \mathcal{C}$ denote the projection. We then have the sequence of equivalences in $\mathbf{P}(\mathcal{C})$

$$\begin{aligned} q(-) &\simeq \mathrm{colim}_{\mathcal{C}^{/q}} \mathrm{Map}_{\mathcal{C}}(-, \pi) \simeq \mathrm{colim}_{\mathcal{C}^{/q}} \mathrm{Map}_{\mathbf{P}(\mathcal{C})}(j(-), j\pi) \\ &\simeq \mathrm{Map}_{\mathbf{P}(\mathcal{C})}(j(-), \mathrm{colim}_{\mathcal{C}^{/q}} j\pi) \simeq \mathrm{Map}_{\mathbf{P}(\mathcal{C})}(j(-), q), \end{aligned}$$

where we use that $\mathrm{id}_{\mathbf{P}(\mathcal{C})}$ is the left Kan extension of j along itself [Lur09, Lem. 5.1.5.3] for the first and last equivalences, j is fully faithful [Lur09, Prop. 5.1.3.1] for the second equivalence, and $j(c)$ corepresents evaluation at c [Lur09, Lem. 5.1.5.2] (and is hence completely compact) to show the third equivalence, where the map in question is the canonical colimit interchange map.

Now suppose $\mathcal{C}$ is a $\mathcal{T}$-∞-category and let $q \in \mathbf{P}_{\mathcal{T}}(\mathcal{C}) \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}})$ be a $\mathcal{T}$-presheaf. We again have that the $\mathcal{T}$-Yoneda embedding $j_{\mathcal{T}} : \mathcal{C} \longrightarrow \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$, given fiberwise by $\mathcal{C}_t \subset \mathbf{P}(\mathcal{C}_t) \subset \mathbf{P}_{\mathcal{T}^{/t}}(\mathcal{C}) \simeq \mathbf{P}(\mathcal{C}_{\underline{t}}^{\mathrm{v}})$, is $\mathcal{T}$-fully faithful, $\mathrm{id}_{\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})}$ is the $\mathcal{T}$-left Kan extension of $j_{\mathcal{T}}$ along itself [Sha21, Lem. 11.1], and for any $c \in \mathcal{C}_t$, $j_{\mathcal{T}}(c)$ is completely compact as an object in $\mathbf{P}_{\mathcal{T}^{/t}}(\mathcal{C}_{\underline{t}}) \simeq \mathbf{P}(\mathcal{C}_{\underline{t}}^{\mathrm{v}})$, hence $\mathcal{T}^{/t}$-completely compact in $\underline{\mathbf{P}}_{\mathcal{T}^{/t}}(\mathcal{C}_{\underline{t}})$ since $\mathcal{T}$-colimits in $\underline{\mathbf{Spc}}_{\mathcal{T}}$ are computed as ordinary colimits under the correspondence of [Sha21, Prop. 5.5]. Repeating the above argument then shows that we have an equivalence of $\mathcal{T}$-presheaves

$$q(-) \simeq \underline{\mathrm{Map}}_{\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})}(j_{\mathcal{T}}(-), q) : \mathcal{C}^{\mathrm{vop}} \longrightarrow \mathbf{Spc},$$

where on the right we view q as a cocartesian section of $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$.

7.5. **Proposition.** *Let $p : \mathcal{K} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor of small $\mathcal{T}$-∞-categories. Then the commutative square of $\mathcal{T}$-∞-categories*

$$\begin{array}{ccc} \mathcal{C}^{/(p,\mathcal{T})} & \longrightarrow & \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})^{/(j_{\mathcal{T}}p,\mathcal{T})} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{j_{\mathcal{T}}} & \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}) \end{array}$$

is a homotopy pullback square. Consequently, the functor $\mathcal{C}^{\mathrm{vop}} \longrightarrow \mathbf{Spc}$ classifying the left fibration

$$(\mathcal{C}^{/(p,\mathcal{T})})^{\mathrm{vop}} \simeq (\mathcal{C}^{\mathrm{vop}})^{(p^{\mathrm{vop}},\mathcal{T})/} \longrightarrow \mathcal{C}^{\mathrm{vop}}$$

is canonically equivalent to $\lim^{\mathcal{T}}_{\mathcal{K}} j_{\mathcal{T}} p$.

Proof. The square in question is a homotopy pullback if and only if for all $t \in \mathcal{T}$, the square

$$\begin{array}{ccc} (\mathcal{C}^{/(p,\mathcal{T})})_t \simeq (\mathcal{C}_{\underline{t}}^{/(p_{\underline{t}},\mathcal{T}^{/t})})_{\mathrm{id}_t} & \longrightarrow & (\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})^{/(j_{\mathcal{T}}p,\mathcal{T})})_t \simeq (\mathbf{P}_{\mathcal{T}^{/t}}(\mathcal{C}_{\underline{t}})^{/(p_{\underline{t}},\mathcal{T}^{/t})})_{\mathrm{id}_t} \\ \downarrow & & \downarrow \\ \mathcal{C}_t \simeq (\mathcal{C}_{\underline{t}})_{\mathrm{id}_t} & \longrightarrow & \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_t \simeq \mathbf{P}_{\mathcal{T}^{/t}}(\mathcal{C}_{\underline{t}}) \end{array}$$

is a homotopy pullback of ∞-categories. Therefore, after replacing $\mathcal{T}$ by $\mathcal{T}^{/t}$, we may suppose that $\mathcal{T}$ has a final object $* \in \mathcal{T}$, and it suffices to check that the square of ∞-categories

$$\begin{array}{ccc} (\mathcal{C}^{/(p,\mathcal{T})})_* & \longrightarrow & (\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})^{/(j_{\mathcal{T}}p,\mathcal{T})})_* \\ \downarrow & & \downarrow \\ \mathcal{C}_* & \longrightarrow & \mathbf{P}_{\mathcal{T}}(\mathcal{C}) \end{array}$$

is a homotopy pullback. Let π generically denote all pullbacks of the structure map $\mathcal{K} \longrightarrow \mathcal{T}^{\mathrm{op}}$. By the universal property of $\underline{\mathbf{Spc}}_{\mathcal{T}}$ as a $\mathcal{T}$-∞-category of $\mathcal{T}$-objects in $\mathbf{Spc}$ [Sha21, Prop. 3.10], we have an identification of the constant $\mathcal{T}$-diagram functor

$$\mathbf{P}_{\mathcal{T}}(\mathcal{C}) = \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \longrightarrow \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}))$$

with the functor

$$\pi^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{vop}}, \mathbf{Spc}) \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathbf{Spc})$$

given by restriction along π. Abusing notation, let $j_{\mathcal{T}}p$ also denote the corresponding functor $\mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K} \longrightarrow \mathbf{Spc}$ under this equivalence. Then by Lemma 7.1, we have a homotopy pullback square

$$\begin{array}{ccc} (\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})^{/(j_{\mathcal{T}}p,\mathcal{T})})_* & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathbf{Spc})^{/j_{\mathcal{T}}p} \\ \downarrow & & \downarrow \\ \mathbf{P}_{\mathcal{T}}(\mathcal{C}) \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}}) & \xrightarrow{\pi^*} & \mathrm{Fun}(\mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathbf{Spc}). \end{array}$$

By the $S = \Delta^0$ case of [Sha21, Lem. 8.8] applied to the adjunction $\pi^* \dashv \pi_*$, we deduce an equivalence

$$(\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})^{/(j_{\mathcal{T}}p,\mathcal{T})})_* \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}})^{/\pi_*(j_{\mathcal{T}}p)}.$$

Next, consider the functor

$$P' = P \times \mathrm{id}_{\mathcal{K}} : \mathcal{C}_*^{\mathrm{op}} \times \mathcal{K} \cong (\mathcal{C}_*^{\mathrm{op}} \times \mathcal{T}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K} \longrightarrow \mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K},$$

defined to be the product of the unique $\mathcal{T}$-functor $P : \mathcal{C}_*^{\mathrm{op}} \times \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{vop}}$ extending the inclusion $\iota : \mathcal{C}_*^{\mathrm{op}} \subset \mathcal{C}^{\mathrm{vop}}$ on the first factor and the identity on $\mathcal{K}$ on the second factor; informally, $P'(c,k) = (\chi_t^*(c), k)$ for $k \in \mathcal{K}_t$ and $\chi_t : t \longrightarrow *$ the unique map. Using the canonical unit transformation $\iota \Rightarrow P$, P' fits into a lax commutative diagram

$$\begin{array}{ccc} \mathcal{C}_*^{\mathrm{op}} \times \mathcal{K} & \xrightarrow{P'} & \mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K} \\ \downarrow{\scriptstyle \mathrm{pr}} & \nearrow & \downarrow{\scriptstyle \pi} \\ \mathcal{C}_*^{\mathrm{op}} & \xrightarrow{\iota} & \mathcal{C}^{\mathrm{vop}}. \end{array}$$

We claim that the induced map $\theta : \iota^* \pi_*(j_{\mathcal{T}}p) \longrightarrow \mathrm{pr}_* P'^*(j_{\mathcal{T}}p)$ is an equivalence, which we may check objectwise at each $c \in \mathcal{C}_*$. Let $\underline{c} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{vop}}$ denote the unique $\mathcal{T}$-functor such that $\underline{c}(*) = c$. Since $\mathcal{T}$-limits in $\mathcal{T}$-functor categories are computed pointwise (by the dual of [Sha21, Prop. 9.17]), under the equivalences of [Sha21, Prop. 3.10] we have a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathbf{Spc}) & \xrightarrow{\underline{c}^*} & \mathrm{Fun}(\mathcal{K}, \mathbf{Spc}) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Fun}(\mathcal{C}^{\mathrm{vop}}, \mathbf{Spc}) & \xrightarrow{\underline{c}^*} & \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Spc}) \end{array}$$

where the vertical functors are given by right Kan extension and the horizontal functors by restriction, and after evaluation at $* \in \mathcal{T}^{\mathrm{op}}$ this equivalence identifies with $\theta(c)$, which proves the claim.

Abusing notation, let jp also denote its adjoint $\mathcal{K} \times \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Spc}$, and observe that $(P')^*(j_t p) \simeq (jp)|_{\mathcal{K} \times \mathcal{C}_*^{\mathrm{op}}}$, so that the equivalence θ yields an equivalence

$$\mathbf{P}(\mathcal{C}_*)^{/(\pi_*(j_{\mathcal{T}}p))|_{\mathcal{C}_*^{\mathrm{op}}}} \simeq \mathbf{P}(\mathcal{C}_*)^{/\mathrm{pr}_*((jp)|_{\mathcal{K} \times \mathcal{C}_*^{\mathrm{op}}})}.$$

Using that pr is a cartesian fibration, we have an equivalence $\mathrm{pr}_*((jp)|_{\mathcal{K} \times \mathcal{C}_*^{\mathrm{op}}}) \simeq (\mathrm{pr}_*(jp))|_{\mathcal{C}_*^{\mathrm{op}}}$. Note that the limit of $jp : \mathcal{K} \longrightarrow \mathbf{P}(\mathcal{C})$ is computed by $\mathrm{pr}_*(jp)$, so $\mathbf{P}(\mathcal{C})^{/jp} \simeq \mathbf{P}(\mathcal{C})^{/\mathrm{pr}_*(jp)}$. Using that $\mathcal{C}_* \longrightarrow \mathbf{P}_{\mathcal{T}}(\mathcal{C}) \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}})$ factors through $\mathbf{P}(\mathcal{C}_*)$ and invoking [Sha21, Lem. 8.8] with respect to the adjunctions $\mathbf{P}(\mathcal{C}_*) \rightleftarrows \mathbf{P}(\mathcal{C}^{\mathrm{v}})$ and $\mathbf{P}(\mathcal{C}_*) \rightleftarrows \mathbf{P}(\mathcal{C})$, we then reduce the claim to checking that the outer square

$$\begin{array}{ccccc} (\mathcal{C}^{/(p,\mathcal{T})})_* & \longrightarrow & \mathbf{P}(\mathcal{C}_*)^{/(jp)|_{\mathcal{C}_*^{\mathrm{op}}}} & \longrightarrow & \mathbf{P}(\mathcal{C})^{/jp} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_* & \longrightarrow & \mathbf{P}(\mathcal{C}_*) & \longrightarrow & \mathbf{P}(\mathcal{C}). \end{array}$$

is a homotopy pullback square. But this follows from Lemma 7.6 and Lemma 7.2.

The last statement then follows from [Sha21, Prop. 5.24], Lemma 7.8(3), and Remark 7.4. □

7.6. **Lemma.** *Suppose $\mathcal{T}$ has a final object $*$ and let $p : \mathcal{K} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor. Then we have a homotopy pullback square*

$$\begin{array}{ccc} (\mathcal{C}^{/(p,\mathcal{T})})_* & \longrightarrow & \mathcal{C}^{/p} \\ \downarrow & & \downarrow \\ \mathcal{C}_* & \longrightarrow & \mathcal{C}. \end{array}$$

Proof. Note that the inclusion of the initial object $\Delta^0 \longrightarrow (\mathcal{T}^{\mathrm{op}})^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\mathcal{T}^{\mathrm{op}}}$. By [Sha21, Thm. 4.16], $i : \Delta^0 \star_{\mathcal{T}^{\mathrm{op}}} {}_\natural\mathcal{K} \longrightarrow {}_\natural\mathcal{K}^{\trianglelefteq}$ is a cocartesian equivalence. Since $\mathrm{Fun}_{\mathcal{T}}(-,-)$ is a Quillen bifunctor, precomposition by i then yields a homotopy pullback square

$$\begin{array}{ccc} \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}^{\underline{\lhd}}, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathcal{K}^{\lhd}, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C}_* \times \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C}) & \longrightarrow & \mathcal{C} \times \mathrm{Fun}(\mathcal{K}, \mathcal{C}) \end{array}$$

where the bottom horizontal functor is the evident inclusion. Taking the pullback over $\{p\}$ then produces the desired homotopy pullback square. □

We have the evident parametrized analogue of a (co)representable fibration.

7.7. **Definition.** Let $f : \mathcal{D} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-left resp. $\mathcal{T}$-right fibration. If $\mathcal{D}$ admits a $\mathcal{T}$-initial object resp. $\mathcal{T}$-final object, then we say that f is $\mathcal{T}$-*corepresentable* resp. $\mathcal{T}$-*representable*.

We record some basic facts about $\mathcal{T}$-corepresentable $\mathcal{T}$-left fibrations and the $\mathcal{T}$-Yoneda embedding.

7.8. **Lemma.** *(1) Let $\mathcal{D}$ be a $\mathcal{T}$-∞-category and let $\sigma : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{D}$ be a cocartesian section. Then σ is a $\mathcal{T}$-initial object of $\mathcal{D}$ if and only if $\mathcal{D}^{(\sigma,\mathcal{T})/} \longrightarrow \mathcal{D}$ is a trivial fibration.*

(2) Let $f : \mathcal{D} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor. If f is a $\mathcal{T}$-corepresentable $\mathcal{T}$-left fibration with $\mathcal{T}$-initial object σ, then we have a canonical equivalence $\mathcal{D} \simeq \mathcal{C}^{(f\sigma,\mathcal{T})/}$ of $\mathcal{T}$-∞-categories over $\mathcal{C}$.

(3) Let $\sigma : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be any cocartesian section. The $\mathcal{T}$-left fibration $\mathcal{C}^{(\sigma,\mathcal{T})/} \longrightarrow \mathcal{C}$, as a left fibration, is classified by the functor

$$\underline{\mathrm{Map}}_{\mathcal{C}}(\sigma, -) : \mathcal{C} \xrightarrow{(\sigma^{\mathrm{vop}},\mathrm{id})} \mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{C} \xrightarrow{\underline{\mathrm{Map}}_{\mathcal{C}}(-,-)} \mathbf{Spc}$$

that sends $c \in \mathcal{C}_t$ to $\mathrm{Map}_{\mathcal{C}_t}(\sigma(t), c)$.

(4) The functor obtained by taking cocartesian sections of the $\mathcal{T}$-Yoneda embedding $j_{\mathcal{T}}$

$$\widehat{j_{\mathcal{T}}} : \mathrm{Fun}_{\mathcal{T}}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}) \longrightarrow \mathbf{P}_{\mathcal{T}}(\mathcal{C}) \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}}), \quad \widehat{j_{\mathcal{T}}}(\sigma) \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(-, \sigma) \simeq \underline{\mathrm{Map}}_{\mathcal{C}^{\mathrm{vop}}}(\sigma^{\mathrm{vop}}, -)$$

is fully faithful. Under the straightening equivalence $\mathbf{P}_{\mathcal{T}}(\mathcal{C}) \simeq \mathbf{LFib}(\mathcal{C}^{\mathrm{vop}})$, $\widehat{j_{\mathcal{T}}}$ has essential image spanned by the $\mathcal{T}$-corepresentable $\mathcal{T}$-left fibrations over $\mathcal{C}^{\mathrm{vop}}$.

(5) The composite $\mathcal{T}$-functor $\underline{\mathrm{colim}}^{\mathcal{T}} j_{\mathcal{T}} : \mathcal{C} \longrightarrow \underline{\mathbf{Spc}}_{\mathcal{T}}$ is constant with value the $\mathcal{T}$-final object of $\underline{\mathbf{Spc}}_{\mathcal{T}}$.

Proof. First note that we have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}) = \mathcal{T}^{\mathrm{op}} \times_{\mathrm{Ar}(\mathcal{T}^{\mathrm{op}})} \mathrm{Ar}(\mathcal{D}) & \xrightarrow{\simeq} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{T} \times \Delta^1, \mathcal{D}) \\ \downarrow{\scriptstyle \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathcal{D} & \xrightarrow{\simeq} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{T}, \mathcal{D}) \end{array}$$

in which the horizontal maps are equivalences of $\mathcal{T}$-∞-categories. Therefore, for all $t \in \mathcal{T}$, we have an equivalence $\mathcal{D}^{(\sigma,\mathcal{T})/} \times_{\mathcal{D}} \mathcal{D}_t \simeq \mathcal{D}_t^{\sigma(t)/}$, and (1) follows by checking fiberwise.

For (2), suppose f is a $\mathcal{T}$-left fibration and σ is a $\mathcal{T}$-initial object of $\mathcal{D}$. Using (1), let $\tau : \mathcal{D} \longrightarrow \mathcal{D}^{(\sigma,\mathcal{T})/}$ be a choice of section. We then claim that the composite map

$$\chi : \mathcal{D} \xrightarrow{\tau} \mathcal{D}^{(\sigma,\mathcal{T})/} \xrightarrow{f} \mathcal{C}^{(f\sigma,\mathcal{T})/}$$

is an equivalence. Since χ is a $\mathcal{T}$-functor between $\mathcal{T}$-left fibrations over $\mathcal{C}$, it suffices to check that χ_t is an equivalence of left fibrations over $\mathcal{C}_t$ for all $t \in \mathcal{T}$. But χ_t is a functor of corepresentable left fibrations that preserves the initial object, and is thus an equivalence.

For (3), recall that $\underline{\mathrm{Map}}_{\mathcal{C}}(-,-)$ was defined as the straightening of the $\mathcal{T}$-left fibration $\mathrm{Tw}_{\mathcal{T}}(\mathcal{C}) \longrightarrow \mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{C}$ given by the $\mathcal{T}$-fiberwise twisted arrow category (cf. the discussion right after [Sha21, Thm. 11.5]). By (2), it then suffices to show that the pullback $\mathcal{D} = \mathcal{C} \times_{(\mathcal{C}^{\mathrm{vop}} \times_{\mathcal{T}^{\mathrm{op}}} C)} \mathrm{Tw}_{\mathcal{T}}(\mathcal{C})$ has a $\mathcal{T}$-initial object that projects to σ. But this again reduces to the fiberwise assertion about $\mathcal{C}_t \times_{(\mathcal{C}_t^{\mathrm{op}} \times \mathcal{C}_t)} \mathrm{Tw}(\mathcal{C}_t) \simeq (\mathcal{C}_t)^{\sigma(x)/}$; note that the assumption that σ is a *cocartesian* section ensures that the collection of fiberwise initial objects in $\mathcal{D}$ is stable under cocartesian pushforward, so it indeed promotes to a $\mathcal{T}$-initial object.

For (4), since $j_{\mathcal{T}}$ is fiberwise fully faithful and the formation of cocartesian sections computes the limit of the corresponding functor into **Cat**, it follows that $\widehat{j_{\mathcal{T}}}$ is fully faithful. The assertion now follows from (3).

For (5), we may check the assertion fiberwise, so suppose that $\mathcal{T}$ has a final object $*$. Then we need to show that for any $x \in \mathcal{C}_*$, $\operatorname{colim}^{\mathcal{T}} j_{\mathcal{T}}(x) \simeq 1 \in \mathbf{Spc}_{\mathcal{T}}$. But note that for all $t \in \mathcal{T}$, if we let $\alpha_t : t \longrightarrow *$ denote the unique morphism then $j_{\mathcal{T}}(x)|_{\mathcal{C}_t^{\mathrm{op}}} \simeq \operatorname{Map}_{\mathcal{C}_t}(-, \alpha_t^* x)$, hence in view of [Sha21, Prop. 5.5] the assertion follows from its non-parametrized analogue $\operatorname{colim}_{\mathcal{C}_t^{\mathrm{op}}} j(\alpha_t^* x) \simeq 1$. □

8. Finite, filtered, and sifted diagrams

In this section, we develop the theory of $\mathcal{T}$-κ-small, $\mathcal{T}$-filtered, and $\mathcal{T}$-sifted $\mathcal{T}$-∞-categories (Definition 8.2, Definition 8.8, and Definition 8.14). To prepare for our discussion, we begin with the following proposition, which recovers and extends the homotopy colimit decomposition result of [Lur09, Cor. 4.2.3.10]. Its statement involves the "lower $\mathcal{T}$-slice" construction of [Sha21, Def. 4.17].

8.1. **Theorem.** *Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category.*

(1) The assignments $[K \xrightarrow{p} {}_\natural\mathcal{C}] \mapsto [{}_\natural\mathcal{C}_{(p,\mathcal{T})/} \longrightarrow \mathcal{C}]$ and $[L \xrightarrow{q} {}_\natural\mathcal{C}] \mapsto [{}_\natural\mathcal{C}_{/(p,\mathcal{T})} \longrightarrow {}_\natural\mathcal{C}]$ of marked simplicial sets assemble to a Quillen adjunction

$$\mathcal{C}_{(-,\mathcal{T})/} : s\mathbf{Set}^+_{/{}_\natural\mathcal{C}} \rightleftarrows (s\mathbf{Set}^+_{/{}_\natural\mathcal{C}})^{\mathrm{op}} : \mathcal{C}_{/(-,\mathcal{T})}$$

with respect to the slice model structure (and its opposite) induced from the cocartesian model structure on $s\mathbf{Set}^+_{/\mathcal{T}^{\mathrm{op}}}$. Consequently, we obtain an adjunction of ∞-categories

$$\mathcal{C}_{(-,\mathcal{T})/} : (\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{C}} \rightleftarrows (\mathbf{Cat}_{\mathcal{T}})^{\mathrm{op}}_{/\mathcal{C}} : \mathcal{C}_{/(-,\mathcal{T})}.$$

(2) Let $p_\bullet : \mathcal{I} \longrightarrow (\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{C}}$ be a functor with colimit $p : \mathcal{K} \longrightarrow \mathcal{C}$ and suppose that for every $i \in \mathcal{I}$, the $\mathcal{T}$-functor $p_i : \mathcal{K}_i \longrightarrow \mathcal{C}$ admits a $\mathcal{T}$-colimit σ_i. Then the σ_i assemble to a $\mathcal{T}$-functor $\sigma_\bullet : \mathcal{I} \times \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}$ such that if $\sigma_\bullet$ admits a $\mathcal{T}$-colimit σ, then p admits a $\mathcal{T}$-colimit given by σ.

Proof. (1): Let us suppress the markings on $\mathcal{C}$ and its relatives for clarity. The two displayed functors participate in an adjunction in view of the definitional isomorphisms of hom-sets

$$\operatorname{Hom}_{/\mathcal{C}}(L, \mathcal{C}_{(p,\mathcal{T})/}) \cong \operatorname{Hom}_{K \sqcup L//\mathcal{T}^{\mathrm{op}}}(K \star_{\mathcal{T}^{\mathrm{op}}} (L \times_{\mathcal{T}^{\mathrm{op}}} \operatorname{Ar}(\mathcal{T}^{\mathrm{op}})^\sharp), C) \cong \operatorname{Hom}_{/\mathcal{C}}(K, \mathcal{C}_{/(q,\mathcal{T})}).$$

The left adjoint preserves cofibrations and weak equivalences by [Sha21, Prop. 4.18], [Sha21, Prop. 4.19], and the discussion immediately proceeding it, so the adjunction is Quillen. Finally, even though this adjunction is not generally simplicial, we may descend to an adjunction on the underlying ∞-categories by [Lur17, Cor. 1.3.4.26].

(2): Let $\overline{p_\bullet} : \mathcal{I}^\triangleright \longrightarrow (\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{C}}$ be a colimit diagram extending $p_\bullet$ and let $\overline{\varphi} : (\mathcal{I}^{\mathrm{op}})^\triangleleft \longrightarrow (\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{C}}$ be the opposite of the postcomposition of $\overline{p_\bullet}$ with $\mathcal{C}_{(-,\mathcal{T})/}$. By (1), $\overline{\varphi}$ is a limit diagram. We wish to show that the value $\mathcal{C}_{(p,\mathcal{T})/} \longrightarrow \mathcal{C}$ on its cone point is a $\mathcal{T}$-corepresentable $\mathcal{T}$-left fibration with $\mathcal{T}$-initial object as indicated.

By assumption, φ factors through the subcategory of $\mathcal{T}$-corepresentable $\mathcal{T}$-left fibrations over $\mathcal{C}$, hence by Lemma 7.8(4) applied to $\mathcal{C}^{\mathrm{vop}}$ and using that $(-)^{\mathrm{op}}$ preserves limits, we may factor φ^{op} as

$$\psi : \mathcal{I} \longrightarrow \operatorname{Fun}_{\mathcal{T}}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}^{\mathrm{vop}})^{\mathrm{op}} \simeq \operatorname{Fun}_{\mathcal{T}}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}).$$

Let $\sigma_\bullet : \mathcal{I} \times \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be the adjoint $\mathcal{T}$-functor and suppose $\sigma_\bullet$ extends to a $\mathcal{T}$-colimit diagram

$$\overline{\sigma_\bullet} : (\mathcal{I} \times \mathcal{T}^{\mathrm{op}}) \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{T}^{\mathrm{op}} \simeq \mathcal{I}^\triangleright \times \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

(necessarily then adjoint to a colimit diagram $\overline{\psi} : \mathcal{I}^\triangleright \longrightarrow \operatorname{Fun}_{\mathcal{T}}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ extending ψ). Since the $\mathcal{T}$-Yoneda embedding $j_{\mathcal{T}}$ strongly preserves $\mathcal{T}$-limits [Sha21, Cor. 11.10], we obtain a $\mathcal{T}$-limit diagram $j_{\mathcal{T}} \circ \overline{\sigma_\bullet}^{\mathrm{vop}}$ and thus a limit diagram

$$\widehat{j_{\mathcal{T}}} \circ \overline{\psi}^{\mathrm{op}} : (\mathcal{I}^{\mathrm{op}})^\triangleleft \longrightarrow \operatorname{Fun}_{\mathcal{T}}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}^{\mathrm{vop}}) \longrightarrow \mathbf{P}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}) \simeq \mathbf{LFib}(\mathcal{C}).$$

Since $\mathbf{LFib}(\mathcal{C})$ is a subcategory of $(\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{C}}$ stable under limits and $\overline{\varphi}$ factors through this subcategory, we deduce that $\overline{\varphi} \simeq \widehat{j_{\mathcal{T}}} \circ \overline{\psi}^{\mathrm{op}}$ as limit diagrams extending φ, which proves the claim. □

8.2. **Definition.** Let $\Delta_{\mathcal{T}} \subset \mathbf{Cat}_{\mathcal{T}} \simeq \operatorname{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat})$ be the full subcategory spanned by the objects $\{\Delta^n \times \operatorname{Map}_{\mathcal{T}}(-, t) : t \in \mathcal{T}, n \geq 0\}$. Then for every regular cardinal κ, define the full subcategory $\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}} \subset \mathbf{Cat}_{\mathcal{T}}$ of *$\mathcal{T}$-κ-small $\mathcal{T}$-∞-categories* to be the smallest full subcategory that contains $\Delta_{\mathcal{T}}$ and is closed under all colimits indexed by κ-small simplicial sets.

If $\kappa = \omega$, then we will also use the terminology $\mathcal{T}$-*finite* in place of $\mathcal{T}$-ω-small.

8.3. **Remark.** Let $i : \Delta \subset \mathbf{Cat}$ be the usual inclusion of the simplex category, which extends to the adjunction $[L\colon \mathbf{P}(\Delta) \rightleftarrows \mathbf{Cat} : R]$ whose fully faithful right adjoint exhibits $\mathbf{Cat}$ as the full subcategory of complete Segal spaces in $\mathbf{P}(\Delta)$. Then applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to $L \dashv R$ yields an adjunction $[L_{\mathcal{T}}\colon \mathbf{P}(\Delta \times \mathcal{T}) \rightleftarrows \mathbf{Cat}_{\mathcal{T}} : R_{\mathcal{T}}]$, where $L_{\mathcal{T}}$ is the unique functor extending $i_{\mathcal{T}} : \Delta \times \mathcal{T} \longrightarrow \mathbf{Cat}_{\mathcal{T}}, ([n], t) \mapsto \Delta^n \times \mathrm{Map}_{\mathcal{T}}(-, t)$ and $R_{\mathcal{T}}$ is fully faithful. In particular, $i_{\mathcal{T}}$ is fully faithful with essential image $\Delta_{\mathcal{T}}$. Furthermore, using that $\mathrm{id}_{\mathbf{P}(\Delta \times \mathcal{T})}$ is the left Kan extension of the Yoneda embedding along itself, we deduce that $\mathrm{id}_{\mathbf{Cat}_{\mathcal{T}}}$ is the left Kan extension of $i_{\mathcal{T}}$ along itself; indeed, for any $\mathcal{T}$-∞-category $\mathcal{C}$, applying the colimit-preserving functor $L_{\mathcal{T}}$ to the equivalence

$$R_{\mathcal{T}}(\mathcal{C}) \simeq \mathrm{colim}[(\Delta \times \mathcal{T}) \times_{\mathbf{P}(\Delta \times \mathcal{T})} \mathbf{P}(\Delta \times \mathcal{T})^{/R_{\mathcal{T}}(\mathcal{C})} \longrightarrow \mathbf{P}(\Delta \times \mathcal{T})]$$

and using that $(\Delta \times \mathcal{T}) \times_{\mathbf{P}(\Delta \times \mathcal{T})} \mathbf{P}(\Delta \times \mathcal{T})^{/R_{\mathcal{T}}(\mathcal{C})} \simeq (\Delta \times \mathcal{T}) \times_{\mathbf{Cat}_{\mathcal{T}}} (\mathbf{Cat}_{\mathcal{T}})^{/\mathcal{C}}$ proves the claim.

As a corollary, if κ_0 is the strongly inaccessible cardinal that fixes our definition of small simplicial set, then $\mathbf{Cat}_{\mathcal{T}}^{\kappa_0\text{-small}} = \mathbf{Cat}_{\mathcal{T}}$. Our definition of small $\mathcal{T}$-∞-category is thus unambiguous.

8.4. **Remark.** In Definition 8.2, if $\mathcal{T} = \Delta^0$, then the notion of a $\mathcal{T}$-κ-small $\mathcal{T}$-∞-category $\mathcal{C}$ coincides with that of an *essentially* κ-small ∞-category [Lur09, Def. 5.4.1.3]: that is, there exists a κ-small simplicial set K and a categorical equivalence $K \longrightarrow \mathcal{C}$. To see this, note that if $\mathcal{C}$ is essentially κ-small then by [Lur09, Var. 4.2.3.15] if $\kappa > \omega$ or [Lur09, Var. 4.2.3.16] if $\kappa = \omega$, there exists a κ-small simplicial set L and a map $\phi : L \longrightarrow \Delta$ such that $\mathcal{C} \simeq \mathrm{colim}_L \phi$. Conversely, the full subcategory of essentially κ-small ∞-categories is closed under κ-small colimits by a direct argument if $\kappa = \omega$ and by the identification with κ-compact objects in $\mathbf{Cat}$ if $\kappa > \omega$ [Lur09, Prop. 5.4.1.2].

8.5. **Remark.** We may describe the ∞-category $\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}}$ more explicitly in the following manner. Define an increasing union of full subcategories $(\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\alpha \subset \mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}}$ for all ordinals $\alpha \leq \kappa$ inductively as follows:

(0) $(\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_0 = \Delta_{\mathcal{T}}$.
(1) For every successor ordinal $\beta = \alpha + 1$, let $(\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\beta$ consist of all colimits of diagrams

$$K \longrightarrow (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\alpha$$

indexed by κ-small simplicial sets K.
(2) For every limit ordinal $\lambda \leq \kappa$, let $(\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\lambda = \bigcup_{\alpha < \lambda} (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\alpha$.

Then since the ordinal κ is itself κ-filtered, we conclude that $\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}} = (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\kappa$.

We now apply Theorem 8.1 to prove a κ-$\mathcal{T}$-cocompleteness result that generalizes [Sha21, Cor. 12.15], with a different method of proof.

8.6. **Theorem.** *Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category and κ a regular cardinal. Then $\mathcal{C}$ strongly admits all $\mathcal{T}$-κ-small $\mathcal{T}$-colimits if and only if*

(1) For every $t \in \mathcal{T}$, $\underline{\mathcal{C}_t}$ admits $\mathcal{T}^{/t}$-colimits indexed by corepresentable left fibrations.
(2) For every $t \in \mathcal{T}$, $\mathcal{C}_t$ admits κ-small colimits, and for every $\alpha : s \longrightarrow t$, the restriction functor $\alpha^* : \mathcal{C}_t \longrightarrow \mathcal{C}_s$ preserves κ-small colimits.

Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{T}$-∞-categories that strongly admit $\mathcal{T}$-κ-small $\mathcal{T}$-colimits and $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a $\mathcal{T}$-functor, then F strongly preserves all $\mathcal{T}$-κ-small $\mathcal{T}$-colimits if and only if

a. For every $\alpha : s \longrightarrow t$, the mate $\alpha_! F_s \Rightarrow F_t \alpha_!$ is an equivalence.
b. For every $t \in \mathcal{T}$, F_t preserves all κ-small colimits.

Proof. For the 'only if' direction, suppose that $\mathcal{C}$ strongly admits $\mathcal{T}$-colimits. Then (1) holds since $\Delta_{\mathcal{T}^{/t}} \subset \mathbf{Cat}_{\mathcal{T}^{/t}}^{\kappa\text{-small}}$ by definition, and (2) holds since if $\mathcal{K}$ is an essentially κ-small ∞-category, then as noted in Remark 8.4 there exists a κ-small simplicial set L and a functor $\phi : L \longrightarrow \Delta \times \{\mathrm{id}_t\} \subset \Delta \times \mathcal{T}^{/t} \xrightarrow{\simeq} \Delta_{\mathcal{T}^{/t}}$ such that $\mathcal{K} \times (\mathcal{T}^{/t})^{\mathrm{op}} \simeq \mathrm{colim}_L \phi$ in $\mathbf{Cat}_{\mathcal{T}}$, hence $\mathcal{K} \times (\mathcal{T}^{/t})^{\mathrm{op}}$ is a $\mathcal{T}$-κ-small $\mathcal{T}$-∞-category.

Conversely, suppose $\mathcal{C}$ satisfies (1) and (2). Let $\mathcal{K}_t$ denote the full subcategory of $\mathbf{Cat}_{\mathcal{T}^{/t}}$ consisting of all $\mathcal{T}^{/t}$-∞-categories $\mathcal{K}$ such that all $\mathcal{K}$-indexed $\mathcal{T}^{/t}$-diagrams in $\underline{\mathcal{C}_t}$ admit $\mathcal{T}^{/t}$-colimits. We wish to show that $\mathcal{K}_t$ contains $\mathbf{Cat}_{\mathcal{T}^{/t}}^{\kappa\text{-small}}$. To ease notation, let us replace $\mathcal{T}^{/t}$ by $\mathcal{T}$ and suppose that $\mathcal{T}$ has a final object $*$. By our first assumption, $\mathcal{K}_*$ contains Δ_T. It thus suffices to show that $\mathcal{K}_*$ is closed under κ-small colimits in $\mathbf{Cat}_T$. So suppose we have a diagram $f : I \longrightarrow \mathcal{K}_*$ with I κ-small such that f has colimit $\mathcal{K}$ in $\mathbf{Cat}_{\mathcal{T}}$, and let $p : \mathcal{K} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor. Then since colimits in $(\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{C}}$ are created by the forgetful functor to $\mathbf{Cat}_{\mathcal{T}}$, we obtain a colimit diagram $p_\bullet : I^{\triangleright} \longrightarrow (\mathbf{Cat}_{\mathcal{T}})_{/\mathcal{C}}$ such that for each $i \in I$, the $\mathcal{T}$-functor p_i admits a $\mathcal{T}$-colimit $x_i \in \mathcal{C}_*$. By our second assumption combined with [Sha21, Cor. 5.9] and Theorem 8.1(2), we deduce that p admits a $\mathcal{T}$-colimit, which proves the claim. Repeating the same type of argument establishes the assertion about the $\mathcal{T}$-functor F. □

8.7. **Remark.** If we suppose that $\mathcal{T}$ is orbital in Theorem 8.6, then by [Sha21, Prop. 5.12] we may replace (1) with the assumption that for all $\alpha : s \longrightarrow t$, α^* admits a left adjoint $\alpha_!$, and for all pullback squares in $\mathbf{F}_{\mathcal{T}}$

$$\begin{array}{ccc} U' & \xrightarrow{\beta'} & U \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ V' & \xrightarrow{\beta} & V, \end{array}$$

the mate $\alpha'_! \beta'^* \Rightarrow \beta^* \alpha_!$ is an equivalence.

8.8. **Definition.** Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category and let κ be a regular cardinal. Then $\mathcal{J}$ is $\mathcal{T}$-*κ-filtered* if for all $t \in \mathcal{T}$ and $\mathcal{T}^{/t}$-κ-small $\mathcal{K}$, every $\mathcal{T}^{/t}$-functor $p : \mathcal{K} \longrightarrow \mathcal{J}_{\underline{t}}$ admits an extension to a $\mathcal{T}^{/t}$-functor $\overline{p} : \mathcal{K}^{\underline{\triangleright}} \longrightarrow \mathcal{J}_{\underline{t}}$.

8.9. **Lemma.** *Suppose that $\mathcal{T}$ is orbital and has a final object $*$. Let κ be a regular cardinal and suppose that $\mathcal{K}$ and $\mathcal{L}$ are $\mathcal{T}$-κ-small $\mathcal{T}$-∞-categories. Then $\mathcal{K} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{L}$ is $\mathcal{T}$-κ-small.*

Proof. Let $(\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\alpha$ be as in Remark 8.5. First observe that if $\mathcal{K} = \Delta^n \times (\mathcal{T}^{/t})^{\mathrm{op}}$ and $\mathcal{L} = \Delta^m \times (\mathcal{T}^{/s})^{\mathrm{op}}$, then $\mathcal{K} \times \mathcal{L} \in (\mathbf{Cat}_{\mathcal{T}}^{\omega\text{-small}})_1$ using that $\mathcal{T}$ is orbital. Because

$$\mathcal{K} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{L} \simeq (\mathcal{K} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{L}) \times \Delta^1 \coprod_{(\mathcal{K} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{L}) \times \partial\Delta^1} \mathcal{K} \sqcup \mathcal{L} \,,$$

we deduce that the $\mathcal{T}$-join restricts to a functor $\Delta_{\mathcal{T}} \times \Delta_{\mathcal{T}} \longrightarrow (\mathbf{Cat}_{\mathcal{T}}^{\omega\text{-small}})_1 \subset (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_1$. We then formulate the following inductive hypothesis:

(1) If $\beta = \alpha + 1 < \kappa$ is a successor ordinal, then for all $\mathcal{K}, \mathcal{L} \in (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\alpha$, $\mathcal{K} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{L} \in (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\beta$.
(2) If $\lambda \leq \kappa$ is a limit ordinal, then for all $\mathcal{K}, \mathcal{L} \in (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\lambda$, $\mathcal{K} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{L} \in (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\lambda$.

Because the $\mathcal{T}$-join preserves colimits separately in each variable (as may be checked fiberwise) and κ is a regular cardinal, we may proceed by induction to confirm the inductive hypothesis for all $\lambda \leq \kappa$; this proves the claim since $\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}} = (\mathbf{Cat}_{\mathcal{T}}^{\kappa\text{-small}})_\kappa$. □

8.10. **Lemma.** *Suppose that $\mathcal{T}$ is orbital and $\mathcal{J}$ is $\mathcal{T}$-κ-filtered. Then for any $\mathcal{T}^{/t}$-κ-small $\mathcal{K}$ and $\mathcal{T}^{/t}$-functor $p : \mathcal{K} \longrightarrow \mathcal{J}_{\underline{t}}$, $(\mathcal{J}_{\underline{t}})^{(p,\mathcal{T}^{/t})/}$ is $\mathcal{T}^{/t}$-κ-filtered.*

Proof. Replace $\mathcal{T}^{/t}$ by $\mathcal{T}$ and suppose $\mathcal{K}$ is $\mathcal{T}$-κ-small and $p : \mathcal{K} \longrightarrow \mathcal{J}$ is a $\mathcal{T}$-functor. To check that $\mathcal{J}^{(p,\mathcal{T})/}$ is $\mathcal{T}$-filtered, after replacing $\mathcal{T}^{/t}$ by $\mathcal{T}$ once more it suffices to show that for any $\mathcal{T}$-functor $\phi : \mathcal{L} \longrightarrow \mathcal{J}^{(p,\mathcal{T})/}$ with $\mathcal{T}$-κ-small $\mathcal{L}$, ϕ extends over $\mathcal{L}^{\underline{\triangleright}}$. Using that $\mathcal{J}^{(p,\mathcal{T})/} \simeq \mathcal{J}_{(p,\mathcal{T})/}$, by adjunction it suffices to extend the $\mathcal{T}$-functor $\psi : \mathcal{K} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{L} \longrightarrow \mathcal{J}$ (under $p \sqcup q$) over $(\mathcal{K} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{L})^{\underline{\triangleright}}$. By our assumption that $\mathcal{J}$ is $\mathcal{T}$-filtered, this is possible since $\mathcal{K} \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{L}$ is $\mathcal{T}$-κ-small by Lemma 8.9. □

8.11. **Theorem.** *Suppose that $\mathcal{T}$ is orbital. Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category and let κ be a regular cardinal. The following conditions are equivalent:*

(1) $\mathcal{J}$ is $\mathcal{T}$-κ-filtered.
(2) For all $t \in \mathcal{T}$, $\mathcal{J}_t$ is κ-filtered, and $\mathcal{J}$ is cofinal-constant (Definition 9.5).
(3) The $\mathcal{T}$-colimit $\mathcal{T}$-functor $\underline{\operatorname{colim}}^{\mathcal{T}}_{\mathcal{J}} : \underline{\operatorname{Fun}}_{\mathcal{T}}(\mathcal{J}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \longrightarrow \underline{\mathbf{Spc}}_{\mathcal{T}}$ strongly preserves $\mathcal{T}$-κ-small $\mathcal{T}$-limits.

Proof. First suppose (1). Then for any essentially κ-small ∞-category $\mathcal{K}$ and $t \in \mathcal{T}$, our assumption ensures that every $\mathcal{T}^{/t}$-functor $\mathcal{K} \times (\mathcal{T}^{/t})^{\mathrm{op}} \longrightarrow \mathcal{J}_{\underline{t}}$ extends over $\mathcal{K}^{\triangleright} \times (\mathcal{T}^{/t})^{\mathrm{op}}$, which shows that $\mathcal{J}_t$ is κ-filtered. Now suppose $\alpha : s \longrightarrow t$ is a morphism in $\mathcal{T}$. We want to show that $\alpha^* : \mathcal{J}_t \longrightarrow \mathcal{J}_s$ is cofinal. Let $x \in \mathcal{J}_s$ and $\sigma : (\mathcal{T}^{/s})^{\mathrm{op}} \longrightarrow \mathcal{J}_{\underline{t}}$ be the unique $\mathcal{T}^{/t}$-functor that selects x, and note that by Lemma 7.1

$$\mathcal{J}_t \times_{\mathcal{J}_s} (\mathcal{J}_s)^{x/} \simeq ((\mathcal{J}_{\underline{t}})^{(\sigma,\mathcal{T}^{/t})/})_t.$$

By Lemma 8.10, $(\mathcal{J}_{\underline{t}})^{(\sigma,\mathcal{T}^{/t})/}$ is $\mathcal{T}^{/t}$-κ-filtered, so $((\mathcal{J}_{\underline{t}})^{(\sigma,\mathcal{T}^{/t})/})_t$ is κ-filtered by what was previously shown, and hence weakly contractible.[23] The claim now follows by Joyal's cofinality theorem. We conclude that $\mathcal{J}$ is cofinal-constant, so $\mathcal{J}$ satisfies (2).

Next suppose (2). To show (3), by the dual of Theorem 8.6 it suffices to show that $\mathrm{colim}^{\mathcal{T}}_{\mathcal{J}}$ preserves κ-small limits fiberwise and intertwines with the 'coinduction' right adjoints. By [Sha21, Prop. 5.5], under the equivalence $\mathrm{Fun}_{\mathcal{T}^{/t}}(\mathcal{J}_{\underline{t}}, \underline{\mathbf{Spc}}_{\mathcal{T}^{/t}}) \simeq \mathrm{Fun}(\mathcal{J}_{\underline{t}}, \mathbf{Spc})$, the functor $\mathrm{colim}^{\mathcal{T}^{/t}}_{\mathcal{J}_{\underline{t}}}$ identifies with left Kan extension along the structure map $\mathcal{J}_{\underline{t}} \longrightarrow (\mathcal{T}^{/t})^{\mathrm{op}}$. In view of our assumption that the fibers of $\mathcal{J}$ are κ-filtered, by [Lur09, Prop. 5.3.3.3] and [Lur09, Prop. 4.3.3.10] we conclude that $\mathrm{colim}^{\mathcal{T}^{/t}}_{\mathcal{J}_{\underline{t}}}$ preserves κ-small limits. Next, let $\alpha : s \longrightarrow t$ be a morphism in $\mathcal{T}$ and consider the resulting pullback square

$$\begin{array}{ccc} \mathcal{J}_{\underline{s}} & \xrightarrow{\phi} & \mathcal{J}_{\underline{t}} \\ \downarrow \pi & & \downarrow \pi \\ (\mathcal{T}^{/s})^{\mathrm{op}} & \xrightarrow{\phi} & (\mathcal{T}^{/t})^{\mathrm{op}} \end{array} \quad \text{yielding} \quad \begin{array}{ccc} \mathrm{Fun}(\mathcal{J}_{\underline{t}}, \mathbf{Spc}) & \xrightarrow{\pi_!} & \mathrm{Fun}((\mathcal{T}^{/t})^{\mathrm{op}}, \mathbf{Spc}) \\ \phi_* \uparrow & \Searrow & \phi_* \uparrow \\ \mathrm{Fun}(\mathcal{J}_{\underline{s}}, \mathbf{Spc}) & \xrightarrow{\pi_!} & \mathrm{Fun}((\mathcal{T}^{/s})^{\mathrm{op}}, \mathbf{Spc}). \end{array}$$

We need to show that the mate $\chi : \pi_! \phi_* \Rightarrow \phi_* \pi_!$ is an equivalence. To ease notation, let us replace $\mathcal{T}^{/t}$ by $\mathcal{T}$ and t by $*$. Let $p : \mathcal{J}_* \times \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{J}$ be the unique $\mathcal{T}$-functor extending the inclusion $\mathcal{J}_* \subset \mathcal{J}$ and note that our assumption that p is $\mathcal{T}$-cofinal yields a factorization of $\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{J}}$ as

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{J}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \xrightarrow{p^*} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{J}_* \times \mathcal{T}^{\mathrm{op}}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \xrightarrow{\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{J}_* \times \mathcal{T}^{\mathrm{op}}}} \underline{\mathbf{Spc}}_{\mathcal{T}}.$$

Since the $\mathcal{T}$-functor p^* admits a $\mathcal{T}$-left adjoint given by $\mathcal{T}$-left Kan extension, p^* commutes with all $\mathcal{T}$-limits [Sha21, Cor. 8.9]. We may thus replace $\mathcal{J}$ by the constant diagram $\mathcal{J}_* \times \mathcal{T}^{\mathrm{op}}$ in the proof.

Now let $F : \mathcal{J}_* \times (\mathcal{T}^{/s})^{\mathrm{op}} \longrightarrow \mathbf{Spc}$ be a functor and $u \in \mathcal{T}$; we will show that $\chi_F(u)$ is an equivalence. For every $\gamma : v \longrightarrow s$, let $F_\gamma = F|_{\mathcal{J}_* \times \{\gamma\}}$. Invoking our assumption that $\mathcal{T}$ is orbital, let $\{v_i \in \mathcal{T} : i \in I\}$ be a finite collection such that $\mathcal{T}^{/s} \times_{\mathcal{T}} \mathcal{T}^{/u} \simeq \coprod_{i \in I} \mathcal{T}^{/v_i}$ and let $\gamma_i : v_i \longrightarrow s$ be the structure maps. Then

$$(\phi_* \pi_! F)(u) \simeq \lim \left(\coprod_{i \in I} (\mathcal{T}^{\mathrm{op}})^{v_i/} \longrightarrow (\mathcal{T}^{\mathrm{op}})^{s/} \xrightarrow{\pi_! F} \mathbf{Spc} \right) \simeq \prod_{i \in I} (\pi_! F)(\gamma_i) \simeq \prod_{i \in I} \mathrm{colim}_{\mathcal{J}_*} F_{\gamma_i}$$

On the other hand, using that the upper ϕ is a map of cartesian fibrations over $\mathcal{J}_*$, we get that $\phi_* F$ is computed fiberwise over $\mathcal{J}_*$ by [Lur09, Prop. 4.3.3.10]. Thus for all $x \in \mathcal{J}_*$,

$$(\phi_* F)(x, u) \simeq \lim \left(\coprod_{i \in I} (\mathcal{T}^{\mathrm{op}})^{v_i/} \longrightarrow \{x\} \times (\mathcal{T}^{\mathrm{op}})^{s/} \xrightarrow{F} \mathbf{Spc} \right) \simeq \prod_{i \in I} F(x, \gamma_i),$$

and using that κ-filtered colimits commute with finite products in $\mathbf{Spc}$, we deduce that $\chi_F(u)$ is an equivalence.

Finally, suppose (3). To show (1), after the usual reduction it suffices to prove that if $\mathcal{T}$ admits a final object $*$ and $q : \mathcal{K} \longrightarrow \mathcal{J}$ is a $\mathcal{T}$-functor with $\mathcal{K}$ $\mathcal{T}$-κ-small, then $(\mathcal{C}^{(q,\mathcal{T})/})_*$ is nonempty. Let $\varphi = \lim^{\mathcal{T}}_{\mathcal{K}^{\mathrm{vop}}} (j_{\mathcal{T}} q^{\mathrm{vop}}) \in \mathbf{P}_{\mathcal{T}}(\mathcal{J}^{\mathrm{vop}}) \simeq \mathrm{Fun}(\mathcal{J}, \mathbf{Spc})$. Then

$$\mathrm{colim}^{\mathcal{T}}_{\mathcal{J}}(\varphi) \simeq \lim^{\mathcal{T}}_{\mathcal{K}^{\mathrm{vop}}} (\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{J}} j_{\mathcal{T}} q^{\mathrm{vop}}) \simeq \lim^{\mathcal{T}}_{\mathcal{K}^{\mathrm{vop}}} (1_{\mathcal{T}}) \simeq 1_{\mathcal{T}},$$

using that $\underline{\mathrm{colim}}^{\mathcal{T}} j_{\mathcal{T}}$ factors through the $\mathcal{T}$-final object $1_{\mathcal{T}}$ of $\underline{\mathbf{Spc}}_{\mathcal{T}}$ by Lemma 7.8(5) and any $\mathcal{T}$-limit of $\mathcal{T}$-final objects is again $\mathcal{T}$-final. By Proposition 7.5 applied to $p = q^{\mathrm{vop}}$ and Lemma 8.12, we deduce that $(\mathcal{C}^{(q,\mathcal{T})/})_*$ is weakly contractible, so in particular nonempty. □

8.12. **Lemma.** *Let $\pi : \mathcal{C} \longrightarrow \mathcal{T}^{\mathrm{op}}$ be a $\mathcal{T}$-∞-category, $p : \mathcal{D} \longrightarrow \mathcal{C}$ a $\mathcal{T}$-left fibration, and $F : \mathcal{C} \longrightarrow \underline{\mathbf{Spc}}_{\mathcal{T}}$ a $\mathcal{T}$-functor that classifies p. Let $|\mathcal{D}|_{\mathcal{T}}$ denote the $\mathcal{T}$-space obtained by inverting all morphisms in the fibers $\mathcal{D}_t$ for all $t \in \mathcal{T}$. Then $|\mathcal{D}|_{\mathcal{T}} \simeq \mathrm{colim}^{\mathcal{T}} F$.*

[23] In fact, we don't need to invoke Lemma 8.9 as in the proof of Lemma 8.10 because we are only interested in the extension property for constant $\mathcal{T}^{/t}$-diagrams; in particular, the assumption that $\mathcal{T}$ is orbital there is not necessary in this instance.

Proof. Recall again that under the equivalence $\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathcal{C}, \mathbf{Spc})$ and the identification of $\mathcal{T}$-left fibrations with left fibrations, p is classified as a left fibration by $F^\dagger : \mathcal{C} \longrightarrow \mathbf{Spc}$ and $\mathrm{colim}^{\mathcal{T}} F \simeq \pi_! F^\dagger$. Denote the classifying space adjunction by $|-|: \mathbf{Cat} \rightleftarrows \mathbf{Spc} : \iota$. In view of the functoriality of the straightening equivalence [Lur09, Prop. 3.2.1.4], we have that the functor $\mathbf{LFib}(\mathcal{C}) \longrightarrow \mathbf{LFib}(\mathcal{T}^{\mathrm{op}})$ defined by $\mathcal{D} \mapsto |\mathcal{D}|_{\mathcal{T}}$ identifies with the composite

$$L : \mathrm{Fun}(\mathcal{C}, \mathbf{Spc}) \xrightarrow{\iota} \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}) \xrightarrow{\pi_!} \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}) \xrightarrow{|-|} \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Spc}).$$

Since $\pi^* \iota \simeq \iota \pi^*$ the right adjoint of L identifies with π^*, so L is canonically equivalent to $\pi_!$. □

8.13. **Theorem.** *Suppose that $\mathcal{T}$ is orbital. Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category and κ a regular cardinal. Then $\mathcal{J}$ is $\mathcal{T}$-κ-filtered if and only if for all $t \in \mathcal{T}$ and $\mathcal{T}^{/t}$-κ-small $\mathcal{K}$, the diagonal $\mathcal{T}^{/t}$-functor*

$$\delta : \mathcal{J}_{\underline{t}} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}^{/t}}(\mathcal{K}, \mathcal{J}_{\underline{t}})$$

is $\mathcal{T}^{/t}$-cofinal.

Proof. In the proof, let us replace $\mathcal{T}^{/t}$ with $\mathcal{T}$ and suppose that $\mathcal{T}$ has a final object $*$. For the 'if' direction, after replacing $\mathcal{T}^{/t}$ by $\mathcal{T}$ once more it suffices to show that for any $\mathcal{T}$-κ-small $\mathcal{K}$ and $\mathcal{T}$-functor $p : \mathcal{K} \longrightarrow \mathcal{J}$, p extends over $\mathcal{K}^{\underline{\triangleright}}$, i.e., $(\mathcal{J}^{(p,\mathcal{T})/})_*$ is nonempty. But since $(\mathcal{J}^{(p,\mathcal{T})/})_* \simeq \mathcal{J}_* \times_{\mathrm{Fun}_{\mathcal{T}}(\mathcal{K},\mathcal{J})} \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})^{\{p\}/}$ by the dual of Lemma 7.1, this follows by our assumption that $\mathcal{J}_* \longrightarrow \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ is cofinal. Conversely, for the 'only if' direction suppose that $\mathcal{J}$ is $\mathcal{T}$-κ-filtered and let $\mathcal{K}$ be $\mathcal{T}$-κ-small. After replacing $\mathcal{T}^{/t}$ by $\mathcal{T}$, it suffices to show that δ_* is cofinal. For this, by Joyal's cofinality theorem and Lemma 7.1 again, it suffices to show that $(\mathcal{J}^{(p,\mathcal{T})/})_*$ is weakly contractible for all $p : \mathcal{K} \longrightarrow \mathcal{J}$. But this follows by Lemma 8.10 and the (1)⇒(2) implication of Theorem 8.11. □

To formulate the notion of a $\mathcal{T}$-sifted $\mathcal{T}$-∞-category, we adopt the viewpoint of the alternative characterization of Theorem 8.13, but over a more restrictive class of diagrams.

8.14. **Definition.** Let $\mathcal{J}$ be a $\mathcal{T}$-∞-category. Then $\mathcal{J}$ is $\mathcal{T}$-*sifted* if for all $t \in \mathcal{T}$ and finite $\mathcal{T}^{/t}$-sets U, the diagonal $\mathcal{T}^{/t}$-functor $\delta : \mathcal{J}_{\underline{t}} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}^{/t}}(\underline{U}, \mathcal{J}_{\underline{t}})$ is $\mathcal{T}^{/t}$-cofinal.

8.15. **Theorem.** *Suppose that $\mathcal{T}$ is orbital and let $\mathcal{J}$ be a $\mathcal{T}$-∞-category. The following conditions are equivalent:*

(1) $\mathcal{J}$ is $\mathcal{T}$-sifted.
(2) For all $t \in \mathcal{T}$, $\mathcal{J}_t$ is sifted, and $\mathcal{J}$ is cofinal-constant (Definition 9.5).
(3) The $\mathcal{T}$-colimit $\mathcal{T}$-functor $\underline{\mathrm{colim}}^{\mathcal{T}}_{\mathcal{J}} : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{J}, \underline{\mathbf{Spc}}_{\mathcal{T}}) \longrightarrow \underline{\mathbf{Spc}}_{\mathcal{T}}$ preserves finite $\mathcal{T}$-products.

Proof. First suppose (1). Then for $t \in \mathcal{T}$ and $U = \mathrm{id}_t \sqcup \mathrm{id}_t$, the $\mathcal{T}^{/t}$-cofinality of $\delta : \mathcal{J}_{\underline{t}} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{U}, \mathcal{J}_{\underline{t}})$ ensures that $\mathcal{J}_t$ is sifted, whereas if we let $U = [\alpha : s \longrightarrow t]$, then because $\delta_{\mathrm{id}_t} \simeq \alpha^* : \mathcal{J}_t \longrightarrow \mathcal{J}_s$ we deduce that α^* is cofinal, hence $\mathcal{J}$ is cofinal-constant. This shows that $\mathcal{J}$ satisfies (2).

The implication (2)⇒(3) follows by the same proof as (2)⇒(3) in Theorem 8.11. Finally, suppose (3). By Joyal's cofinality theorem and Lemma 7.1, it suffices to show that for all $t \in \mathcal{T}$, finite $\mathcal{T}^{/t}$-set U, and $\mathcal{T}^{/t}$-functor $p : \underline{U} \longrightarrow \mathcal{J}_{\underline{t}}$, the ∞-category $((\mathcal{J}_{\underline{t}})^{(p,\mathcal{T}^{/t})/})_{\mathrm{id}_t}$ is weakly contractible. But this follows by the same proof as (3)⇒(1) in Theorem 8.11. □

We end this section by explaining a parametrized generalization of the following fact: if $F : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{D}$ is a functor that commutes with colimits separately in each variable, then F preserves sifted colimits. To do this, we need to recall the appropriate parametrized notion of distributive functor, whose definition is due to Nardin. We first fix some local notation.

8.16. **Definition.** Let U be a finite $\mathcal{T}$-set. A $\underline{U}$-∞-*category* is a cocartesian fibration over $\underline{U}$.[24]

8.17. **Notation.** Let $f : U \longrightarrow V$ be a map of finite $\mathcal{T}$-sets. Then we have the adjunction

$$f^* : \mathbf{Cat}_{\underline{V}} \rightleftarrows \mathbf{Cat}_{\underline{U}} : f_*$$

where f^* is pullback along $\underline{U} \longrightarrow \underline{V}$. Note also that some authors also prefer to write f_* as $\prod_f$ to emphasize its interpretation as an indexed product.

[24]Perhaps confusingly, this is reversing our convention that a $\mathcal{T}$-∞-category is a cocartesian fibration over $\mathcal{T}^{\mathrm{op}}$. However, we don't want to write "$\underline{U}^{\mathrm{op}}$-$\infty$-category".

To understand the following definition, the reader should convince themselves that it reduces to "preserving colimits separately in each variable" when $\mathcal{T} = *$.

8.18. **Definition** ([Nar17, Def. 3.15])**.** Suppose that $\mathcal{T}$ is orbital, let $f : U \longrightarrow V$ be a map of finite $\mathcal{T}$-sets, let $\mathcal{C}$ be a $\underline{U}$-∞-category, and let $\mathcal{D}$ be a $\underline{V}$-∞-category. Let $F : f_* \mathcal{C} \longrightarrow \mathcal{D}$ be a $\underline{V}$-functor. Then we say that F is *$\underline{V}$-distributive* if for every pullback square

$$\begin{array}{ccc} U' & \xrightarrow{f'} & V' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

of finite $\mathcal{T}$-sets and $\underline{U'}$-colimit diagram $\overline{p} : \mathcal{K}^{\underline{\triangleright}} \longrightarrow g'^* \mathcal{C}$, the $\underline{V'}$-functor

$$(f'_* \mathcal{K})^{\underline{\triangleright}} \xrightarrow{\text{can}} f'_*(\mathcal{K}^{\underline{\triangleright}}) \xrightarrow{f'_* \overline{p}} f'_* g'^* \mathcal{C} \simeq g^* f_* \mathcal{C} \xrightarrow{g^* F} g^* \mathcal{D}$$

is a $\underline{V'}$-colimit diagram.[25]

8.19. **Proposition.** *Suppose that $\mathcal{T}$ is orbital, let $f : U \longrightarrow V$ be a map of finite $\mathcal{T}$-sets, and let $\mathcal{C}$ resp. $\mathcal{D}$ be a $\underline{U}$ resp. $\underline{V}$-∞-category. Suppose that $F : f_* \mathcal{C} \longrightarrow \mathcal{D}$ is a $\underline{V}$-distributive $\underline{V}$-functor. Then F strongly preserves $\underline{V}$-sifted $\underline{V}$-colimits.*

Proof. Since the property of parametrized distributivity is stable under base-change, it suffices to show that F preserves $\underline{V}$-sifted $\underline{V}$-colimits. Without loss of generality, we may also suppose that V is an orbit. Let $\mathcal{K}$ be a $\underline{V}$-sifted $\underline{V}$-∞-category and suppose that $\overline{p} : \mathcal{K}^{\underline{\triangleright}} \longrightarrow f_* \mathcal{C}$ is a $\underline{V}$-colimit diagram. Then because the counit map $f^* f_* \mathcal{C} \simeq \underline{\mathrm{Fun}}_{\underline{U}}(\underline{U} \times_{\underline{V}} \underline{U}, \mathcal{C}) \longrightarrow \mathcal{C}$ is given by restriction along the diagonal $\underline{U} \longrightarrow \underline{U} \times_{\underline{V}} \underline{U}$, the adjoint map $(\mathcal{K}_{\underline{U}})^{\underline{\triangleright}} \longrightarrow \mathcal{C}$ is a $\underline{U}$-colimit diagram. Since F is $\underline{V}$-distributive, the $\underline{V}$-functor

$$\overline{\psi} : (f_* f^* \mathcal{K})^{\underline{\triangleright}} \longrightarrow f_*(f^* \mathcal{K}^{\underline{\triangleright}}) \longrightarrow f_* \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a $\underline{V}$-colimit diagram. Since $\mathcal{K}$ is $\underline{V}$-sifted, the unit $\underline{V}$-functor $\delta : \mathcal{K} \longrightarrow f_* f^* \mathcal{K} \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\underline{U}, \mathcal{K})$ is $\underline{V}$-cofinal, so $\overline{\psi \circ \delta}$ is also a $\underline{V}$-colimit diagram. But this composite is homotopic to $F \circ \overline{p}$ via the triangle identity for $f^* \dashv f_*$, proving the claim. ☐

We will use Proposition 8.19 together with Theorem 4.16 in [NS] to show e.g. that given a $\mathcal{T}$-distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category $\mathcal{C}$ (such as the G-∞-category of G-spectra equipped with the Hill–Hopkins–Ravenel norms when $\mathcal{T} = \mathbf{O}_G$), the forgetful $\mathcal{T}$-functor from the $\mathcal{T}$-∞-category of $\mathcal{T}$-commutative algebras in $\mathcal{C}$ to $\mathcal{C}$ creates all $\mathcal{T}$-sifted $\mathcal{T}$-colimits.

9. Universal constructions

In this section, we introduce a few more universal constructions in addition to that of $\mathcal{T}$-presheaves that formally adjoin smaller classes of $\mathcal{T}$-colimits. We begin with the following lemma.

9.1. **Lemma.** *Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of small ∞-categories and let $F : \mathbf{P}(\mathcal{C}) \longrightarrow \mathbf{P}(\mathcal{D})$ be the unique colimit-preserving functor that extends $j \circ f$. Then for every presheaf $\varphi \in \mathbf{P}(\mathcal{C})$, the induced functor*

$$\mathcal{C} \times_{\mathbf{P}(\mathcal{C})} \mathbf{P}(\mathcal{C})^{/\varphi} \longrightarrow \mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)}$$

is cofinal.

Proof. We verify the hypotheses of Joyal's cofinality theorem. Let $\tau = [d, \gamma : j(d) \longrightarrow F(\varphi)]$ be an object of $\mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)}$. We want to show that the ∞-category

$$\mathcal{E} := (\mathcal{C} \times_{\mathbf{P}(\mathcal{C})} \mathbf{P}(\mathcal{C})^{/\varphi}) \times_{(\mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)})} (\mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)})_{\tau/}$$

[25] Using the compatibility of the parametrized join with restriction [Sha21, Lem. 4.4], the canonical map $(f'_* \mathcal{K})^{\underline{\triangleright}} \xrightarrow{\text{can}} f'_*(\mathcal{K}^{\underline{\triangleright}})$ is defined to be the adjoint to $\epsilon^{\underline{\triangleright}} : (f'^* f'_* \mathcal{K})^{\underline{\triangleright}} \longrightarrow \mathcal{K}^{\underline{\triangleright}}$ for ϵ the counit of the adjunction $f'^* \dashv f'_*$.

is weakly contractible. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{E} & \xrightarrow{G'} & (\mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)})_{\tau/} & & \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \\
\mathcal{D}_{d/} \times_{\mathcal{D}} (\mathcal{C} \times_{\mathbf{P}(\mathcal{C})} \mathbf{P}(\mathcal{C})^{/\varphi}) & \xrightarrow{G} & \mathcal{D}_{d/} \times_{\mathcal{D}} (\mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)}) & \longrightarrow & \mathcal{D}_{d/} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C} \times_{\mathbf{P}(\mathcal{C})} \mathbf{P}(\mathcal{C})^{/\varphi} & \longrightarrow & \mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)} & \longrightarrow & \mathcal{D}.
\end{array}$$

Observe that since $F(\varphi) \simeq \operatorname{colim}(\mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)} \longrightarrow \mathbf{P}(\mathcal{D}))$ and $\operatorname{Map}_{\mathbf{P}(\mathcal{D}))}(j(d), -) : \mathbf{P}(\mathcal{D}) \longrightarrow \mathbf{Spc}$ preserves colimits, we have that $\mathcal{D}_{d/} \times_{\mathcal{D}} (\mathcal{D} \times_{\mathbf{P}(\mathcal{D})} \mathbf{P}(\mathcal{D})^{/F(\varphi)})$ is weakly homotopy equivalent to $F(\varphi)(d)$. Likewise, since F preserves colimits, the functor G is a weak homotopy equivalence. By Lemma 9.2, the functor π is a Kan fibration. By right properness of the Quillen model structure on simplicial sets, we deduce that G' is a weak homotopy equivalence, hence W is weakly contractible. □

9.2. **Lemma.** *Let $\psi : X \longrightarrow Y$ be a right fibration and let $p : K \longrightarrow X$ be a functor. Then the induced functor*

$$X_{p/} \longrightarrow Y_{\psi p/} \times_Y X$$

is a Kan fibration.

Proof. Let $n > 0$ and $\iota : A = \Lambda^n_i \longrightarrow B = \Delta^n$ be a horn inclusion. We need to solve the lifting problem

$$\begin{array}{ccc}
A & \longrightarrow & X_{p/} \\
\downarrow{\scriptstyle \iota} & \nearrow & \downarrow \\
B & \longrightarrow & Y_{\psi p/} \times_Y X
\end{array}
\quad \text{which transposes to} \quad
\begin{array}{ccc}
K \star A \bigcup_A B & \longrightarrow & X \\
\downarrow{\scriptstyle \iota'} & \nearrow & \downarrow{\scriptstyle \psi} \\
K \star B & \longrightarrow & Y.
\end{array}$$

If $i < n$ so that ι is left anodyne, then by [Lur09, Lem. 2.1.2.3], ι' is inner anodyne, and if $i > 0$ so that ι is right anodyne, then by the opposite of [Lur09, Lem. 2.1.2.4], ι' is right anodyne. Therefore, the dotted lift exists. □

9.3. **Definition.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. We define the *fiberwise $\mathcal{T}$-∞-category of presheaves of* $\mathcal{C}$ to be the full subcategory

$$\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C}) \subset \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$$

whose fiber over each object $t \in \mathcal{T}$ is the full subcategory $\mathbf{P}(\mathcal{C}_t)$ of $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_t \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}}_{\underline{t}})$, embedded via left Kan extension along the fully faithful inclusion $\mathcal{C}^{\mathrm{op}}_t \subset \mathcal{C}^{\mathrm{vop}}_{\underline{t}}$.

9.4. **Remark.** In Definition 9.3, we note that $\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})$ is a full $\mathcal{T}$-subcategory of $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$, i.e., it is a sub-cocartesian fibration over $\mathcal{T}^{\mathrm{op}}$. Indeed, the existence of the $\mathcal{T}$-Yoneda embedding $j_{\mathcal{T}}$ as a $\mathcal{T}$-functor implies that for any morphism $\alpha : s \longrightarrow t$ in $\mathcal{T}$, the diagram

$$\begin{array}{ccccc}
\mathcal{C}_t & \longrightarrow & \mathcal{C}^{\mathrm{v}}_{\underline{t}} & \xrightarrow{j} & \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_t \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}}_{\underline{t}}) \\
\downarrow{\scriptstyle \alpha^*} & & & & \downarrow{\scriptstyle \overline{\alpha}^*} \\
\mathcal{C}_s & \longrightarrow & \mathcal{C}^{\mathrm{v}}_{\underline{s}} & \xrightarrow{j} & \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_s \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}}_{\underline{s}})
\end{array}$$

commutes, where $\overline{\alpha}^*$ is given by restriction along $\mathcal{C}^{\mathrm{vop}}_{\underline{s}} \longrightarrow \mathcal{C}^{\mathrm{vop}}_{\underline{t}}$. Since the inclusions $\mathbf{P}(\mathcal{C}_t) \subset \mathbf{P}(\mathcal{C}^{\mathrm{v}}_{\underline{t}})$ and $\mathbf{P}(\mathcal{C}_s) \subset \mathbf{P}(\mathcal{C}^{\mathrm{v}}_{\underline{s}})$ along with $\overline{\alpha}^*$ are colimit-preserving, we have a factorization of the outer rectangle as

$$\begin{array}{ccccc}
\mathcal{C}_t & \xrightarrow{j} & \mathbf{P}(\mathcal{C}_t) & \longrightarrow & \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_t \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}}_{\underline{t}}) \\
\downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \overline{\alpha}^*|_{\mathbf{P}(\mathcal{C}_t)}} & & \downarrow{\scriptstyle \overline{\alpha}^*} \\
\mathcal{C}_s & \xrightarrow{j} & \mathbf{P}(\mathcal{C}_s) & \longrightarrow & \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_s \simeq \mathbf{P}(\mathcal{C}^{\mathrm{v}}_{\underline{s}}),
\end{array}$$

which both establishes the claim and also identifies $\overline{\alpha}^*|_{\mathbf{P}(\mathcal{C}_t)}$ with the prolongation of $j \circ \alpha^*$ obtained via the universal property of $\mathbf{P}(\mathcal{C}_t)$. The $\mathcal{T}$-Yoneda embedding then restricts to a $\mathcal{T}$-functor $j^{\mathrm{fb}}_{\mathcal{T}} : \mathcal{C} \longrightarrow \underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})$.

9.5. **Definition.** Let $\mathcal{K}$ be a $\mathcal{T}$-∞-category. Then $\mathcal{K}$ is *cofinal-constant* (*cc*) if for all morphisms $\alpha : s \longrightarrow t$ in $\mathcal{T}$, the restriction functor $\alpha^* : \mathcal{T}_t \longrightarrow \mathcal{T}_s$ is cofinal.

We say that a $\mathcal{T}$-∞-category $\mathcal{C}$ is *cc* $\mathcal{T}$-*cocomplete* if $\mathcal{C}$ strongly admits all cc $\mathcal{T}$-colimits. If $\mathcal{C}$ and $\mathcal{D}$ are cc $\mathcal{T}$-cocomplete, we will let $\underline{\mathrm{Fun}}^{cc}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ denote the full $\mathcal{T}$-subcategory of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ whose fiber over each $t \in \mathcal{T}$ is spanned by those $\mathcal{T}^{/t}$-functors that strongly preserve all cc $\mathcal{T}^{/t}$-colimits.

More generally, if $\mathcal{K}$ is a collection of small simplicial sets, then we have analogous definitions of $\mathcal{K}$-cc $\mathcal{T}$-∞-categories, $\mathcal{K}$-cc $\mathcal{T}$-cocompleteness and $\underline{\mathrm{Fun}}^{\mathcal{K}\text{-cc}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$, where we suppose that the collection of $\mathcal{T}$-diagrams in question are cofinal-constant and have fibers in $\mathcal{K}$.

9.6. **Proposition.** *Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. Then $\mathcal{C}$ is cc $\mathcal{T}$-cocomplete if and only if $\mathcal{C}$ strongly admits all constant $\mathcal{T}$-colimits. Similarly, if $\mathcal{C}$ and $\mathcal{D}$ are cc $\mathcal{T}$-cocomplete, then a $\mathcal{T}$-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ strongly preserves all cc $\mathcal{T}$-colimits if and only if F_t preserves all colimits for all $t \in \mathcal{T}$.*

Proof. We prove the first assertion about $\mathcal{C}$; the second assertion about F will then follow immediately. The 'only if' implication is obvious. Conversely, suppose $\mathcal{C}$ strongly admits all constant $\mathcal{T}$-colimits. Let $t \in \mathcal{T}$ and let $\mathcal{K}$ be a cc $\mathcal{T}^{/t}$-∞-category. We have the $\mathcal{T}^{/t}$-functor

$$\psi : \mathcal{K}_t \times (\mathcal{T}^{/t})^{\mathrm{op}} \longrightarrow \mathcal{K}$$

given as the cocartesian extension of the inclusion of the fiber $\mathcal{K}_t \subset \mathcal{K}$ over the initial object $\mathrm{id}_t \in (\mathcal{T}^{/t})^{\mathrm{op}}$. By assumption, for all morphisms $\alpha : s \longrightarrow t$ in $\mathcal{T}$, the functor $\psi_\alpha \simeq \alpha^* : \mathcal{K}_t \longrightarrow \mathcal{K}_s$ is cofinal, so by [Sha21, Thm. 6.7], for each $\mathcal{T}^{/t}$-functor $f : \mathcal{K} \longrightarrow \mathcal{C}_{\underline{t}}$, the induced $\mathcal{T}$-functor $\psi^* : \mathcal{C}^{(f, \mathcal{T}^{/t})/} \longrightarrow \mathcal{C}^{(f\psi, \mathcal{T}^{/t})/}$ is an equivalence. In particular, $\mathcal{C}^{(f, \mathcal{T}^{/t})/}$ admits a $\mathcal{T}^{/t}$-initial object if and only if $\mathcal{C}^{(f\psi, \mathcal{T}^{/t})/}$ does. Therefore, f extends to a $\mathcal{T}^{/t}$-colimit diagram if and only if $f\psi$ does, which completes the proof. □

Recall that $\mathcal{C}$ strongly admits all constant $\mathcal{T}$-colimits if and only if its fibers admit all colimits and its pushforward functors preserve all colimits. For example, $\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})$ strongly admits all constant $\mathcal{T}$-colimits, so by Proposition 9.6, $\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})$ is cc $\mathcal{T}$-cocomplete.

9.7. **Proposition.** *Let $\mathcal{C}$ be a small $\mathcal{T}$-∞-category and let $\mathcal{D}$ be cc $\mathcal{T}$-cocomplete. Then for any $\mathcal{T}$-functor $f : \mathcal{C} \longrightarrow \mathcal{D}$, the $\mathcal{T}$-left Kan extension F of f along $j^{\mathrm{fb}}_{\mathcal{T}}$ exists. Moreover, restriction along $j^{\mathrm{fb}}_{\mathcal{T}}$*

$$(j^{\mathrm{fb}}_{\mathcal{T}})^* : \underline{\mathrm{Fun}}^{cc}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

implements an equivalence of $\mathcal{T}$-∞-categories, with inverse given by $\mathcal{T}$-left Kan extension.

Proof. For any $\varphi \in \underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})_t \simeq \mathbf{P}(\mathcal{C}_t)$, note that by Lemma 9.1

$$\mathcal{C} \times_{\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})} \underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})^{/\underline{\varphi}} = \mathcal{C} \times_{\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})} \mathrm{Ar}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})) \times_{\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})} \underline{\varphi} \longrightarrow \underline{\varphi} \xrightarrow{\simeq} (\mathcal{T}^{/t})^{\mathrm{op}}$$

is a cofinal-constant $\mathcal{T}^{/t}$-∞-category. Therefore, by the pointwise formula for $\mathcal{T}$-left Kan extensions [Sha21, Thm. 10.3], $F = (j^{\mathrm{fb}}_{\mathcal{T}})_! f$ exists and is computed by $F_t \simeq j_! f_t$, so F_t preserves all colimits. Furthermore, given a $\mathcal{T}$-functor $G : \underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C}) \longrightarrow \mathcal{D}$ such that G_t preserves colimits for all $t \in \mathcal{T}$, since $j_! j^* G_t \xrightarrow{\simeq} G_t$ it follows that $(j^{\mathrm{fb}}_{\mathcal{T}})_!(j^{\mathrm{fb}}_{\mathcal{T}})^* G \xrightarrow{\simeq} G$ from the pointwise formula. By the same logic as [Sha21, Cor. 10.7], we thus obtain a $\mathcal{T}$-adjunction

$$(j^{\mathrm{fb}}_{\mathcal{T}})_! : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) : (j^{\mathrm{fb}}_{\mathcal{T}})^*$$

in which $(j^{\mathrm{fb}}_{\mathcal{T}})_!$ is $\mathcal{T}$-fully faithful with essential image $\underline{\mathrm{Fun}}^{cc}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D})$. □

9.8. **Variant.** Let $\mathcal{C}$ be a small $\mathcal{T}$-∞-category. For a collection $\mathcal{K}$ of small simplicial sets, let $\underline{\mathbf{P}}^{\mathcal{K}\text{-fb}}_{\mathcal{T}}(\mathcal{C}) \subset \underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})$ be the full subcategory whose fiber over each $t \in \mathcal{T}$ is given by $\mathbf{P}^{\mathcal{K}}(\mathcal{C}_t) \subset \mathbf{P}(\mathcal{C}_t)$ [Lur09, Prop. 5.3.6.2], and let $j^{\mathcal{K}\text{-fb}}_{\mathcal{T}}$ denote the factorization of the $\mathcal{T}$-Yoneda embedding through $\underline{\mathbf{P}}^{\mathcal{K}\text{-fb}}_{\mathcal{T}}(\mathcal{C})$. Then by the universal property of $\mathbf{P}^{\mathcal{K}}(-)$, $\underline{\mathbf{P}}^{\mathcal{K}\text{-fb}}_{\mathcal{T}}(\mathcal{C})$ is a sub-cocartesian fibration of $\underline{\mathbf{P}}^{\mathrm{fb}}_{\mathcal{T}}(\mathcal{C})$ and hence a $\mathcal{T}$-∞-category. Note that the proof of [Lur09, Prop. 5.3.6.2] shows that for a $\mathcal{K}$-cocomplete ∞-category $\mathcal{D}$, the equivalence $\mathrm{Fun}^{\mathcal{K}}(\mathbf{P}^{\mathcal{K}}(\mathcal{C}_t), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}_t, \mathcal{D})$ implemented by restriction has inverse given by left Kan extension. Thus, by the same proof as in Proposition 9.7, we see that if $\mathcal{D}$ is a $\mathcal{T}$-∞-category that is $\mathcal{K}$-cc $\mathcal{T}$-cocomplete, we have an $\mathcal{T}$-adjunction

$$(j^{\mathcal{K}\text{-fb}}_{\mathcal{T}})_! : \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathcal{K}\text{-fb}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) : (j^{\mathcal{K}\text{-fb}}_{\mathcal{T}})^*$$

in which $(j^{\mathcal{K}\text{-fb}}_{\mathcal{T}})_!$ is $\mathcal{T}$-fully faithful with essential image $\underline{\mathrm{Fun}}^{\mathcal{K}\text{-cc}}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathcal{K}\text{-fb}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D})$.

9.9. **Definition.** If $\mathcal{K}$ is the collection of sifted resp. κ-filtered simplicial sets, we will write $\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C})$ and $j^{\Sigma}_{\mathcal{T}}$ resp. $\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C})$ and $j^{\kappa}_{\mathcal{T}}$ for $\underline{\mathbf{P}}^{\mathcal{K}\text{-fb}}_{\mathcal{T}}(\mathcal{C})$ and $j^{\mathcal{K}\text{-fb}}_{\mathcal{T}}$. If $\kappa = \omega$, we will also write $\underline{\mathbf{Ind}}_{\mathcal{T}}(\mathcal{C})$.

For the following lemma, note that if $\mathcal{T}$ is orbital and $\mathcal{E}$ is an ∞-category that admits finite products, then $\underline{\mathcal{E}}_T$ admits finite $\mathcal{T}$-products in view of [Sha21, Prop. 5.6] and the pointwise formula for right Kan extension.

9.10. **Lemma.** *Suppose that $\mathcal{T}$ is an orbital ∞-category. Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category that admits finite $\mathcal{T}$-products and let $\mathcal{E}$ be an ∞-category that admits finite products. Then under the equivalence*

$$(-)^{\dagger} : \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \underline{\mathcal{E}}_{\mathcal{T}}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

of [Sha21, Prop. 3.10], *a $\mathcal{T}$-functor $F : \mathcal{C} \longrightarrow \underline{\mathcal{E}}_{\mathcal{T}}$ preserves finite $\mathcal{T}$-products if and only if $F^{\dagger} : \mathcal{C} \longrightarrow \mathcal{E}$ sends cartesian edges to equivalences and $F^{\dagger}|_{\mathcal{C}_t}$ preserves finite products for all $t \in \mathcal{T}$. Moreover, if $\mathcal{T}$ admits a final object $*$, then $(-)^{\dagger}$ restricts to an equivalence $\mathrm{Fun}^{\times}_{\mathcal{T}}(\mathcal{C}, \underline{\mathcal{E}}_{\mathcal{T}}) \xrightarrow{\simeq} \mathrm{Fun}^{\times}(\mathcal{C}_*, \mathcal{E})$.*

Proof. For the first statement, first note that $F : \mathcal{C} \longrightarrow \underline{\mathcal{E}}_{\mathcal{T}}$ preserves finite products fiberwise if and only if for all $\alpha : s \longrightarrow t$ in $\mathcal{T}$, $\mathrm{ev}_{\alpha} F_t : \mathcal{C}_t \longrightarrow \mathcal{E}_{\mathcal{T}/t} \longrightarrow \mathcal{E}$ preserves finite products. But since $\mathrm{ev}_{\alpha} F_t \simeq \mathrm{ev}_{\mathrm{id}_s} F_s \alpha^*$, this occurs if and only if $F^{\dagger}|_{\mathcal{C}_t}$ preserves finite products for all $t \in \mathcal{T}$. Furthermore, by definition $F^{\dagger}$ inverts cartesian edges if and only if for all $\alpha : s \longrightarrow t$ in $\mathcal{T}$ and $x \in \mathcal{C}_s$, the natural map $(F_t \alpha_* x)(\mathrm{id}_t) \xrightarrow{\simeq} (\alpha_* F_s x)(\mathrm{id}_t)$ is an equivalence. This shows the 'only if' implication. Now let $\beta : u \longrightarrow t$ be any morphism and write $s \times_t u \simeq \coprod_{i \in I} o_i$ for $o_i \in \mathcal{T}$ and a finite set I. For each o_i, let $\alpha_i : o_i \longrightarrow u$ and $\beta_i : o_i \longrightarrow s$ denote the implicit maps, so that $\beta^* \alpha_* x \simeq \prod_{i \in I} \alpha_{i*} {\beta_i}^* x$ by our assumption that $\mathcal{C}$ admits finite $\mathcal{T}$-products. If we suppose that F preserves finite products fiberwise and $F^{\dagger}$ inverts cartesian edges, we then have

$$\begin{aligned}(F_t \alpha_* x)(\beta) &\simeq (F_u \beta^* \alpha_* x)(\mathrm{id}_u) \simeq \prod_{i \in I} (F_u \alpha_{i*} \beta_i^* x)(\mathrm{id}_u) \simeq \prod_{i \in I} (\alpha_{i*} \beta_i^* F_s)(\mathrm{id}_u) \\ &\simeq (\beta^* \alpha_* F_s x)(\mathrm{id}_u) \simeq (\alpha_* F_s x)(\beta),\end{aligned}$$

which shows the 'if' implication.

To prove the second statement, suppose now that $\mathcal{T}$ has a final object $*$. First note that if we let W denote the set of cartesian edges in $\mathcal{C}$, then the composite $\mathcal{C}_* \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ is an equivalence of ∞-categories in view of [Lur09, Cor. 3.3.4.3]. Now let $G : \mathcal{C} \longrightarrow \mathcal{E}$ be a functor that inverts W and suppose $G|_{\mathcal{C}_*}$ preserves finite products. For any $t \in \mathcal{T}$, let $\alpha_t : t \longrightarrow *$ denote the unique morphism. If $\prod_{i \in I} x_i$ is a finite product in $\mathcal{C}_t$, then we have a cartesian edge $\prod_{i \in I} \alpha_{t*} x_i \longrightarrow \prod_{i \in I} x_i$ in $\mathcal{C}$ lifting α_t since α_{t*} is a right adjoint, hence $G(\prod_{i \in I} x_i) \simeq \prod_{i \in I} G(x_i)$ and the claim is proven. □

9.11. **Theorem.** *Suppose that $\mathcal{T}$ is an orbital ∞-category and let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. Suppose that $\mathcal{C}$ admits finite $\mathcal{T}$-coproducts. Then the following statements obtain:*

(1) We have an equality

$$\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C}) = \underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}})$$

as full $\mathcal{T}$-subcategories of $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$.

(2) The inclusion $\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C}) \subset \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ strongly preserves $\mathcal{T}$-sifted $\mathcal{T}$-colimits and admits a $\mathcal{T}$-left adjoint L such that $j^{\Sigma}_{\mathcal{T}} \simeq L \circ j_{\mathcal{T}}$.

(3) $\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C})$ is $\mathcal{T}$-cocomplete, $j^{\Sigma}_{\mathcal{T}}$ preserves finite $\mathcal{T}$-coproducts, and if $\mathcal{D}$ is any $\mathcal{T}$-cocomplete $\mathcal{T}$-∞-category, then restriction along $j^{\Sigma}_{\mathcal{T}}$ implements an equivalence

$$\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathbf{P}}^{\Sigma}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\sqcup}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

with inverse given by $\mathcal{T}$-left Kan extension.

Similarly, if $\mathcal{C}$ strongly admits $\mathcal{T}$-κ-small $\mathcal{T}$-colimits, then:

(1) $\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C})$ equals the full $\mathcal{T}$-subcategory $\underline{\mathrm{Fun}}^{\kappa\text{-lex}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}})$ of $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ whose fiber over $t \in \mathcal{T}$ is spanned by those $\mathcal{T}^{/t}$-presheaves that strongly preserve $\mathcal{T}^{/t}$-κ-small $\mathcal{T}^{/t}$-limits.

(2) The inclusion $\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C}) \subset \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ strongly preserves $\mathcal{T}$-κ-filtered $\mathcal{T}$-colimits and admits a $\mathcal{T}$-left adjoint L such that $j^{\kappa}_{\mathcal{T}} \simeq L \circ j_{\mathcal{T}}$.

(3) $\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C})$ is $\mathcal{T}$-cocomplete, $j^{\kappa}_{\mathcal{T}}$ strongly preserves $\mathcal{T}$-κ-small $\mathcal{T}$-colimits, and if $\mathcal{D}$ is any $\mathcal{T}$-cocomplete $\mathcal{T}$-∞-category, then restriction along $j^{\kappa}_{\mathcal{T}}$ implements an equivalence

$$\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathbf{Ind}}^{\kappa}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\kappa\text{-rex}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

with inverse given by $\mathcal{T}$-left Kan extension.

Proof. In both cases, (1) is an immediate consequence of Lemma 9.10 (together with the dual of Theorem 8.6 in the second instance). Given Theorem 8.15, Theorem 8.11, and Variant 9.8, the rest of the statements then follow formally as in the proof of [Lur09, Prop. 5.5.8.10] and [Lur09, Prop. 5.5.8.15]. ☐

10. Appendix: Promonoidal Day convolution

We record the following important technical lemma on flat fibrations and apply it to construct $\mathcal{O}$-*promonoidal Day convolution* with respect to a base ∞-operad $\mathcal{O}$.

10.1. **Lemma.** *Let $\mathcal{B}$ be an ∞-category with a factorization system $(\mathscr{L},\mathscr{R})$ and let $p : \mathcal{X} \longrightarrow \mathcal{B}$ be a categorical fibration. Let $\mathrm{Ar}^L(\mathcal{B})$ denote the full subcategory of $\mathrm{Ar}(\mathcal{B})$ on those arrows in $\mathscr{L}$ and consider the functor*

$$\pi = \mathrm{ev}_0 \circ \mathrm{pr}_1 : \mathrm{Ar}^L(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X} \longrightarrow \mathcal{B}.$$

Suppose that

1. *For every edge $e : a \longrightarrow b$ in $\mathscr{L}$ and $x \in \mathcal{X}$ such that $p(x) = a$, there exists a p-cocartesian edge $x \longrightarrow y$ covering e.*
2. *The pullback $\mathcal{X}_R = \mathcal{X} \times_{\mathcal{B}} \mathcal{B}_R \longrightarrow \mathcal{B}_R$ is a flat categorical fibration, where $\mathcal{B}_R \subset \mathcal{B}$ denotes the wide subcategory on those morphisms in $\mathscr{R}$.*

Then π is a flat categorical fibration.

Proof. We apply the criterion of [Lur17, Prop. B.3.2] to show flatness. In other words, if we let $\sigma_0 = [a_0 \to b_0 \to c_0]$ be a 2-simplex in $\mathcal{B}$ and let

$$\left(\begin{array}{ccc} a_0 & \longrightarrow & c_0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ a_1 & \longrightarrow & c_1 \end{array} , \; x \longrightarrow z \right)$$

be an edge in $\mathrm{Ar}^L(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X}$ covering $a_0 \longrightarrow c_0$ via π, then we need to show that

$$\begin{aligned}(\mathrm{Ar}^L(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X})^{(\alpha,x)//(\gamma,z)}_{b_0} &:= \{\sigma_0\} \times_{\mathcal{B}^{a_0//c_0}} (\mathrm{Ar}^L(\mathcal{B}) \times_{\mathcal{B}} \mathcal{X})^{(\alpha,x)//(\gamma,z)} \\ &\simeq \mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0} \times_{\mathcal{B}^{a_1//c_1}} \mathcal{X}^{x//z}\end{aligned}$$

is weakly contractible.

As we noted in Proposition 3.5(1), the functor $\mathrm{ev}_0 : \mathrm{Ar}^L(\mathcal{B}) \longrightarrow \mathcal{B}$ is a cartesian fibration, with ev_0-cartesian edges given by morphisms $f \longrightarrow g$ such that the edge $f(1) \longrightarrow g(1)$ is in $\mathscr{R}$. Therefore, we may identify the full subcategory of $\mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0}$ spanned by the final objects with that spanned by objects of the form

$$\begin{array}{ccccc} a_0 & \longrightarrow & b_0 & \longrightarrow & c_0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ a_1 & \longrightarrow & b_1 & \longrightarrow & c_1 \end{array}$$

in which $b_1 \longrightarrow c_1$ is in $\mathscr{R}$. Fix such a choice of final object $\sigma_\bullet$, and let

$$\theta : \Delta^1 \times \mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0} \longrightarrow \mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0}$$

be the natural transformation recording the essentially unique homotopy of the identity functor to the constant functor at $\sigma_\bullet$ (i.e., the unit transformation of the associated localization functor). Also let

$$\theta' : \mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0} \longrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0}) \longrightarrow \mathrm{Fun}'(\Delta^1, \mathcal{B}^{a_1//c_1})$$

be the composite of the adjoint to θ and evaluation at the target. Here, Fun' denotes the full subcategory on objects $\tau = [a_1 \to b_1' \to b_1 \to c_1]$ with $d_1 \tau = \sigma_1$ and such that $b_1' \longrightarrow b_1$ is in $\mathscr{L}$; in other words, $d_0 \tau$ is the essentially unique factorization of $b_1' \longrightarrow c_1$ furnished by $(\mathscr{L},\mathscr{R})$. We then define a natural transformation

$$\eta : \Delta^1 \times \mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0} \times_{\mathcal{B}^{a_1//c_1}} \mathcal{X}^{x//z} \longrightarrow \mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0} \times_{\mathcal{B}^{a_1//c_1}} \mathcal{X}^{x//z}$$

as θ on the first factor and the adjoint to

$$\mathrm{Ar}^L(\mathcal{B})^{\alpha//\gamma}_{b_0} \times_{\mathcal{B}^{a_1//c_1}} \mathcal{X}^{x//z} \xrightarrow{(\theta',\mathrm{id})} \mathrm{Fun}'(\Delta^1, \mathcal{B}^{a_1//c_1}) \times_{\mathrm{ev}_0, \mathcal{B}^{a_1//c_1}} \mathcal{X}^{x//z} \xrightarrow{P} \mathrm{Fun}(\Delta^1, \mathcal{X}^{x//z})$$

on the second factor, where P is the cocartesian pushforward functor that on objects is given by

$$([a_1 \to b_1' \xrightarrow{e} b_1 \to c_1],\, x \to y \to z) \rightsquigarrow [x \to e_! y \to z]$$

and rigorously defined as in [Sha21, Lem. 2.23].

Let $L = \eta_1$ and observe that the essential image of L is $\mathcal{X}^{x//z}_{b_1} := \{\sigma_1\} \times_{\mathcal{B}^{a_1//c_1}} \mathcal{X}^{x//z}$. It is then straightforward to show that η satisfies condition (3) of [Lur09, Prop 5.2.7.4] so that L is a localization functor. In particular, it suffices to show that $\mathcal{X}^{x//z}_{b_1}$ is weakly contractible. Moreover, after choosing a $(\mathscr{L}, \mathscr{R})$-factorization $[a_1 \xrightarrow{e} a_1' \to b_1]$, we have a p-cocartesian lift $\overline{e} : x \longrightarrow x'$ of e by assumption (1), and by the universal property of $\overline{e}$ we have an equivalence

$$\mathcal{X}^{x//z}_{b_1} \simeq (\mathcal{X}_R)^{x'//z}_{a_1'}.$$

But hypothesis (2) then ensures that $(\mathcal{X}_R)^{x'//z}_{a_1'}$ is weakly contractible, which completes the proof. □

Now let $\mathcal{O}^\otimes$ be an ∞-operad and consider the factorization system given by the inert and active edges (cf. [Lur17, Def. 2.1.2.3] and [Lur17, Prop. 2.1.2.4]). Let $\mathcal{O}^\otimes_{act} \subset \mathcal{O}^\otimes$ be the wide subcategory on the active edges.

10.2. **Definition.** Let $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ be a fibration of ∞-operads. We say that p *exhibits* $\mathcal{C}^\otimes$ *as a* $\mathcal{O}$-*promonoidal* ∞-*category* if the restricted functor $p_{act} : \mathcal{C}^\otimes_{act} \longrightarrow \mathcal{O}^\otimes_{act}$ is flat.

10.3. **Example.** Suppose that $\mathcal{C}^\otimes$ is a $\mathcal{O}$-monoidal ∞-category, so that its structure map p is a cocartesian fibration. $\mathcal{C}^\otimes$ is then $\mathcal{O}$-promonoidal since cocartesian fibrations are flat [Lur17, Exm. B.3.4].

The following example was pointed out to us by Harpaz and shows that our earlier definition of symmetric promonoidal given as [BGS20, Def. 1.4] was too restrictive.

10.4. **Example.** There exists examples of $\mathcal{O}$-promonoidal ∞-categories $(\mathcal{C}^\otimes, p)$ such that p itself is not flat. For instance, consider the ∞-operad $\mathrm{MCom}^\otimes$ that parametrizes modules over commutative algebras (cf. [HNP19, 4.3]). Then $\mathrm{MCom}^\otimes$ is symmetric promonoidal, but $p : \mathrm{MCom}^\otimes \longrightarrow \mathbf{F}_*$ is not flat. Indeed, let $\langle n \rangle = \{1, ..., n, +\}$ and consider the composition of maps of pointed finite sets

$$h : \langle 3 \rangle \xrightarrow{f} \langle 2 \rangle \xrightarrow{g} \langle 1 \rangle$$

where $f(1) = 1, f(2) = 2, f(3) = 2$ and $g(1) = 1, g(2) = +$. Let m be the object of MCom representing the module factor and consider the inert edge $e : (m, m, m) \longrightarrow m$ over h. Then e doesn't factor over $h = g \circ f$, so p is not flat.

10.5. **Remark.** Let $(\mathcal{C}^\otimes, p)$ be a $\mathcal{O}$-promonoidal ∞-category. Then if $\mathcal{C}^\otimes$ is moreover *corepresentable* in the sense that p is locally cocartesian, we claim that p is cocartesian so that $\mathcal{C}^\otimes$ is $\mathcal{O}$-monoidal. Indeed, by [BGS20, Prop. 1.5] we see that p_{act} is cocartesian, and using the inert-active factorization system on $\mathcal{C}^\otimes$ together with the decomposition of mapping spaces in $\mathcal{C}^\otimes$ ensured by the definition of an ∞-operad [Lur17, Def. 2.1.1.10(2)], it is not difficult to check that p itself is cocartesian.

We now generalize Lurie's construction of Day convolution [Lur17, Thm. 2.2.6.2], which assumed that $\mathcal{C}^\otimes$ was $\mathcal{O}$-monoidal. Let $\mathrm{Ar}^{ne}(\mathcal{O}^\otimes)$ denote the full subcategory of $\mathrm{Ar}(\mathcal{O}^\otimes)$ on the inert edges.

10.6. **Theorem-Construction.** Let $(\mathcal{C}^\otimes, p)$ be a $\mathcal{O}$-promonoidal ∞-category. Consider the span of marked simplicial sets

$$(\mathcal{O}^\otimes, \mathrm{Ne}) \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathrm{ev}_1, \mathcal{O}^\otimes, p} \mathcal{C}^\otimes, \mathrm{Ne}) \xrightarrow{\mathrm{pr}_{\mathcal{C}^\otimes}} (\mathcal{C}^\otimes, \mathrm{Ne})$$

where the middle marking consists of those edges in $\mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes$ whose source in $\mathcal{O}^\otimes$ is inert and whose projection to $\mathcal{C}^\otimes$ is inert. Then the functor

$$(\mathrm{ev}_0)_* \circ (\mathrm{pr}_{\mathcal{C}^\otimes})^* : \mathbf{sSet}^+_{/(\mathcal{C}^\otimes, \mathrm{Ne})} \longrightarrow \mathbf{sSet}^+_{/(\mathcal{O}^\otimes, \mathrm{Ne})}$$

is right Quillen with respect to the operadic model structures of [Lur17, Prop. 2.1.4.6]. For a fibration $\mathcal{D}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ of ∞-operads, we then define the *p-operadic coinduction* of $\mathcal{D}^{\otimes}$ to be

$$(\mathrm{Nm}_p \mathcal{D})^{\otimes} := (\mathrm{ev}_0)_* (\mathrm{pr}_{\mathcal{C}^{\otimes}})^* (\mathcal{D}^{\otimes}, \mathrm{Ne}).$$

For a fibration $\mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ of ∞-operads, we define the *Day convolution* (of $\mathcal{C}^{\otimes}$ with $\mathcal{D}^{\otimes}$ over $\mathcal{O}^{\otimes}$) to be

$$\widetilde{\mathrm{Fun}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes} := (\mathrm{Nm}_p p^* \mathcal{D})^{\otimes}.$$

Proof. It suffices to verify the hypotheses of [Lur17, Thm. B.4.2]. For (1), ev_0 is flat by Lemma 10.1. The remainder of the proof is now identical to that of [Lur17, Prop. 2.2.6.20(a)]; the only additional point to note is that the verification of (5) only uses that $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ admits p-cocartesian lifts over *inert* edges in the base. □

10.7. **Remark.** It follows readily from the definition that the underlying ∞-category of the Day convolution $\widetilde{\mathrm{Fun}}_{\mathcal{O}}(\mathcal{C}, \mathcal{E})^{\otimes}$ is equivalent to the pairing construction $\widetilde{\mathrm{Fun}}_{\mathcal{O}}(\mathcal{C}, \mathcal{E})$ (Theorem-Construction 4.2 with $\mathcal{T} = *$).

Given Theorem-Construction 10.6, all the usual properties of Day convolution with this extra generality in the source variable then hold; we will give a comprehensive treatment of the parametrized theory in [NS]. In particular, we have that for any fibration $\mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ of ∞-operads, the identity section

$$\iota : (\mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathrm{Ne}) \hookrightarrow (\mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathrm{Ne})$$

is a homotopy equivalence in $\mathbf{sSet}^+_{/(\mathcal{C}^{\otimes}, \mathrm{Ne})}$, so for all fibrations $\mathcal{E}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ of ∞-operads, restriction along ι induces an equivalence of ∞-categories

$$\mathbf{Alg}_{\mathcal{D}/\mathcal{O}}(\mathrm{Nm}_p \mathcal{E}) \xrightarrow{\simeq} \mathbf{Alg}_{\mathcal{D} \times_{\mathcal{O}} \mathcal{C}/\mathcal{C}}(\mathcal{E}).$$

Thus, we may think of the class of $\mathcal{O}$-promonoidal ∞-categories as singling out the *exponentiable* fibrations of ∞-operads over $\mathcal{O}^{\otimes}$.

References

[AF20] David Ayala and John Francis, *Fibrations of ∞-categories*, Higher Structures **4** (2020), no. 1.

[AMGR17] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum, *Factorization homology of enriched ∞-categories*, arXiv preprint arXiv:1710.06414 (2017).

[BDG+16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, *Parametrized higher category theory and higher algebra: A general introduction*, arXiv:1608.03654, 2016.

[BGS20] Clark Barwick, Saul Glasman, and Jay Shah, *Spectral Mackey functors and equivariant algebraic K-theory, II*, Tunisian J. Math. **2** (2020), no. 1, 97–146.

[BH21] Tom Bachmann and Marc Hoyois, *Norms in motivic homotopy theory*, Astérisque **425** (2021).

[GHN17] David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Documenta Mathematica **22** (2017), 1225–1266.

[Gla16] Saul Glasman, *A spectrum-level Hodge filtration on topological Hochschild homology*, Selecta Mathematica **22** (2016), no. 3, 1583–1612.

[HHK+20] Jeremy Hahn, Asaf Horev, Inbar Klang, Dylan Wilson, and Foling Zou, *Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra*, arXiv preprint arXiv:2006.13348 (2020).

[Hin20] Vladimir Hinich, *Yoneda lemma for enriched ∞-categories*, Advances in Mathematics **367** (2020), 107129.

[HNP19] Yonatan Harpaz, Joost Nuiten, and Matan Prasma, *Tangent categories of algebras over operads*, Israel Journal of Mathematics **234** (2019), no. 2, 691–742.

[Hor19] Asaf Horev, *Genuine equivariant factorization homology*, arXiv preprint arXiv:1910.07226 (2019).

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)

[Lur17] ———, *Higher algebra*, Preprint from the web page of the author, May 2017.

[Lur21] Jacob Lurie, *Kerodon*, https://kerodon.net, 2021.

[Nar16] Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category*, arXiv:1608.07704, 2016.

[Nar17] Denis Nardin, *Stability and distributivity over orbital ∞-categories*, Ph.D. thesis, 2017.

[NS] Denis Nardin and Jay Shah, *Parametrized and equivariant higher algebra*, In preparation.

[Sha21] Jay Shah, *Parametrized higher category theory*, to appear in Algebraic & Geometric Topology (2021), arXiv:1809.05892.

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PARAMETRIZED HIGHER CATEGORY THEORY AND HIGHER ALGEBRA: EXPOSÉ IV – STABILITY WITH RESPECT TO AN ORBITAL ∞-CATEGORY

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ABSTRACT. In this paper we develop a theory of stability for G-categories (presheaf of categories on the orbit category of G), where G is a finite group. We give a description of Mackey functors as G-commutative monoids exploit it to characterize G-spectra as the G-stabilization of G-spaces. As an application of this we provide an alternative proof of a theorem by Guillou and May. The theory here is developed in the more general setting of orbital categories.

CONTENTS

1. INTRODUCTION

It is often said that spectra are the same as homology theories. This is strictly speaking wrong when homology theories are interpreted as valued in graded abelian groups, due to the presence of phantom maps. Luckily, Goodwillie calculus provides us with an equivalence between spectra and linear functors from finite pointed spaces to spaces (that is space-valued homology theories). This allows us to state a universal property for the category of spectra: it is the universal source of a linear functor to spaces ([21, Pr. 1.4.2.22]).

One would imagine that a similar statement should be true for G-spectra, where G is a finite group. The category of G-spectra is not, however, the universal source of linear functors to G-spaces (that would be spectral presheaves over the orbit category of G). It has been an important insight in the solution to the Kervaire invariant one problem by Hill, Hopkins, and Ravenel ([18]) that in G-spectra one should ask for a stronger form of additivity: they should not only turn coproducts into products, but also coproducts indexed by a finite G-set into the corresponding product. This is merely a form of Atiyah duality for finite G-sets, but a highly suggestive one.

In order to speak of indexed products and coproducts it is necessary to be able to remember the notion of objects with an H-action for every subgroup H of G. So we need to move from the notion of ∞-category to the notion of G-∞-category, which is a presheaf of categories over $\mathbf{O}_G$, the orbit category of G. This sends G/H to the ∞-category of objects corresponding to the subgroup H (e.g. H-spaces, H-spectra etc.) and encodes all the functoriality of restriction to subgroups (corresponding to the map $G/H \longrightarrow G/K$ for $H \subseteq K$) and of twisting the action by conjugation (corresponding to the isomorphism of G/H with G/gHg^{-1} in $\mathbf{O}_G$). The general theory of (co)limits indexed by a G-∞-category has been developed in [26]. We will briefly summarize the necessary results in section 2.

Once the notion of G-(co)limit has been set up, one can try to mimic the whole theory of additive and stable ∞-categories in this equivariant setting. This works nicely and provides us with a universal property for the G-∞-category of G-spectra: it is the universal recipient of a G-linear functor from the G-∞-category of finite G-spaces (cf. theorem 7.4).

1.1. **Theorem.** *For any G-category with finite G-colimits C the G-functor Ω^∞ : $\mathbf{Sp}^G \longrightarrow \mathbf{Top}^G$ induces an equivalence*

$$\mathrm{Fun}_G^{G-\mathrm{rex}}(C, \underline{\mathbf{Sp}}^G) \longrightarrow \mathrm{Lin}_G(C, \underline{\mathbf{Top}}_G)$$

between the category of G-functors $C \longrightarrow \underline{\mathbf{Sp}}^G$ preserving finite G-colimits and the category of G-linear G-functors $C \longrightarrow \underline{\mathbf{Top}}_G$.

Another important result in the same spirit is the identification of connective spectra with group-like commutative monoids in spaces, as done in [25]. This too has an equivariant analogue (cf. corollary A.4.1). In fact it turns out that G-commutative monoids are the same thing as product-preserving functors from the effective Burnside category of [2]. This explains the ubiquity of Mackey functors in equivariant homotopy theory and allows us to give an alternative proof of [17, Th. 0.1], identifying orthogonal G-spectra with spectral Mackey functors (see appendix A).

Two important predecessors of this paper are [14] and [15]. In the first a description of G-spectra as enriched functors from G-spaces to G-spaces is provided for a general compact Lie group, while the second contains a characterization of G-spectra as functors in term of an excisivity condition for a finite group G. While the approach taken here is different, the intuition behind it is very similar.

In this paper we will work in the general setting of atomic orbital categories (see section 2). Examples of atomic orbital categories beyond the orbit category of a (pro)finite group are an ∞-groupoid (thus recovering the theory of [23]), the cyclonic orbit category ([8, Df. 1.10]) and the global orbit category for finite groups (the full subcategory of $\mathbf{O}_{gl}$ defined in [24, Cn. 8.32] spanned by completely universal finite subgroups of $\mathcal{L}$). One important non-example is the orbit ∞-category of a compact Lie group. This is due to the lack of a good notion of finite G-set stable under restriction to subgroups when G is compact Lie. The reader uninterested in such generality can safely substitute $\mathbf{O}_G$ every time T appears in this paper.

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2. Preliminaries on equivariant (co)limits

2.1. We will be using extensively the theory of T-∞-categories for a general base category T, developed in [3] and [26]. In this section we will recall the most important results.

Motivated by the discussion of G-categories in the introduction, we want to study presheaves of ∞-categories over T. However, a different model for those turns out to be more convenient (e.g. allowing us to state results like theorem 2.8). To describe it we will make use of the following foundational result of the theory of ∞-categories (cfr. [20, Th. 3.2.0.1] and [20, Sec. 3.3.2]):

2.2. **Theorem.** *There is a cocartesian fibration $\mathcal{Z} \longrightarrow \mathbf{Cat}_\infty$ such that for every ∞-category S there is an equivalence between $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ and the ∞-category of cocartesian fibrations over S, sending $F : S \longrightarrow \mathbf{Cat}_\infty$ to the pullback of $\mathcal{Z} \longrightarrow \mathbf{Cat}_\infty$ along F.*

2.3. **Definition.** Motivated by the previous result, we let a T-∞-*category* to be a cocartesian fibration over T^{op}. A T-*functor* between two T-∞-categories is simply a map of cocartesian fibrations (that is a map of simplicial sets over T^{op} that sends cocartesian arrows to cocartesian arrows). Using the simplicial nerve of [20, Df. 1.1.5.5] we can form the ∞-category of T-∞-categories.

2.4. **Notation.** If C is a T-∞-category and $e : t \longrightarrow t'$ is an edge of T, we denote the pushforward functor $C_{t'} \longrightarrow C_t$ by δ_e or $\delta_{t/t'}$.

2.5. **Definition.** For any ∞-category T, the ∞-category $\mathbf{F}_T$ of *finite T-sets* is the full subcategory of the category of presheaves on T spanned by finite coproducts of representables. It satisfies the following universal property: for any ∞-category D with all finite coproducts the forgetful functor

$$\mathrm{Fun}^{\amalg}(\mathbf{F}_T, D) \longrightarrow \mathrm{Fun}(T, D)$$

is an equivalence, where the left hand side is the category of functors preserving finite coproducts. There is a functor $\mathrm{Orbit} : \mathbf{F}_T \longrightarrow \mathbf{F}$ (where $\mathbf{F} = \mathbf{F}_{\Delta^0}$ is the category of finite sets) sending every finite T-set to the set of summands.

For any finite T-set U there is a T-∞-category $\mathbf{U}$, called the *category of points* of U, which as a simplicial set over T^{op} is defined by

$$\mathbf{U} = T^{op} \times_{\mathbf{F}_T^{op}} (\mathbf{F}_T^{op})_{U/} \ .$$

This is the left fibration classified by the functor sending V to the space of arrows $[V \longrightarrow U]$.

2.6. **Construction.** If C, D are two T-∞-categories, there exists a T-∞-category $\underline{\mathrm{Fun}}_T(C, D)$ classified by the functor

$$b \mapsto \mathrm{Fun}_{T_{/b}}(C|_{T_{b/}}, D|_{T_{b/}})$$

There is an obvious evaluation T-functor

$$C \times_{T^{op}} \underline{\mathrm{Fun}}_T(C, D) \longrightarrow D$$

2.7. **Definition.** For every ∞-category C we want to construct a T-∞-category $\underline{C}_T$ classified by the functor $b \mapsto \mathrm{Fun}(T^{op}_{/b}, C)$. This is the *$T$-$\infty$-category of T-objects in C*. As a simplicial set over T^{op} it is given by

$$\mathrm{Mor}_{/T^{op}}(K, \underline{C}_T) = \mathrm{Mor}(K \times_{T^{op}} \mathrm{Fun}(\Delta^1, T^{op}), C)$$

where $\mathrm{Fun}(\Delta^1, T^{op})$ lies above T^{op} with the evaluation at 0. This is a cocartesian fibration thanks to [20, Co. 3.2.2.13].

When $T = \mathbf{O}_G$ and $C = \mathbf{Top}$, this is the cocartesian fibration classified by the functor sending G/H to the ∞-category of genuine H-spaces (that is presheaves of spaces over $\mathbf{O}_H$). One of the pleasant features of the model of T-∞-categories we are using is that the T-∞-category of T-objects has a simple universal property:

2.8. **Theorem.** *Suppose T an ∞-category, C a T-∞-category, and D an ∞-category. Then there is a natural equivalence*

$$\mathrm{Fun}_T(C, \underline{D}_T) \simeq \mathrm{Fun}(C, D)\,.$$

In particular, by [20, Th. 3.2.0.1]*, the ∞-category $\mathrm{Fun}_T(C, \underline{\mathbf{Cat}_\infty}_T)$ is equivalent to the ∞-category associated to the simplicial category of cocartesian fibrations over C and under this equivalence left fibrations correspond to functors whose image lies in $\underline{\mathbf{Top}}_T$.*

2.9. **Definition.** A T-adjunction between two T-∞-categories C and D is an adjunction $FG : C \leftrightarrows D$ between the two total categories such that F and G are T-functors (that is they send cocartesian arrows to cocartesian arrows) and unit and counit lie above the identity natural transformation of the identity functor on T. This is the same thing as a relative adjunction in the sense of [21, Sec. 7.3.2] such that both functors are T-functors. Note that the left adjoint in a relative adjunction is is automatically a T-functor, but this is not true for the right adjoint.

2.10. **Definition.** Precomposition with the structure map $C \longrightarrow T^{op}$ induces a diagonal T-functor

$$\Delta : D \cong \underline{\mathrm{Fun}}_T(T, D) \longrightarrow \underline{\mathrm{Fun}}_T(C, D)\,.$$

When this T-functor has a left T-adjoint we say that D has all C-indexed T-colimits. Similarly, if it has a right T-adjoint we say that D has all C-indexed T-limits. If a T-∞-category D has all C-indexed T-colimits (respectively T-limits) for every small T-category C we say that D is T-cocomplete (respectively T-complete).

A T-colimit indexed by a T-category of the form $\mathrm{pr}_2 : K \times T^{op} \longrightarrow T^{op}$ for K an ∞-category is called a fiberwise T-colimit. A T-colimit indexed by the category of points of a finite T-set is called a finite T-coproduct. A T-∞-category is said to be *pointed* if it has both a T-initial and a T-terminal object (that are cocartesian sections of the structure map that fiberwise select the initial and the terminal object respectively) and the canonical comparison map is an equivalence.

The following proposition summarizes the results on T-(co)limits from [26] that will be needed in this paper.

2.11. **Proposition.** *Let C be a T-∞-category.*

- *C has all T-colimits indexed by $K \times T^{op}$ if and only if for every $b \in T$ the fiber C_b has all colimits indexed by K and for every edge $e : b \longrightarrow b'$ in T the pushforward functor $\delta_e : C_{b'} \longrightarrow C_b$ preserves colimits indexed by K.*

- *Suppose $\mathbf{F}_T$ has all fiber products (that is T is orbital, cf. Df. 4.1). Then C has all (finite) T-coproducts if and only if the following two conditions are satisfied*
  (1) *for every $b \in T$ the fiber C_b has all (finite) coproducts and for every edge $e : b \longrightarrow b'$ the pushforward δ_e preserves (finite) coproducts;*
  (2) *For every edge $e : b \longrightarrow b'$ the pushforward δ_e has a left adjoint $\coprod_e$ satisfying the Beck-Chevalley condition: for every pair of edges $e : b \longrightarrow b'$ and $e' : b'' \longrightarrow b'$ the canonical base change natural transformation of functors from $C_{b''}$ to C_b*
  $$\delta_e \coprod_{e'} \longrightarrow \coprod_{o \in \mathrm{Orbit}(b \times_{b'} b'')} \coprod_{pr_1} \delta_{pr_2}$$
  is an equivalence, where $pr_1 : o \longrightarrow b$ and $pr_2 : o \longrightarrow b'$ are the restrictions to o of the two projections from $b \times_{b'} b''$.
- *C has all T-colimits if and only if it has all fiberwise colimits and all finite T-coproducts.*

Similar statements hold for T-limits. When T-products exist the right adjoint of δ_e will be denoted by $\prod_e$.

2.12. **Definition.** There is also a notion of T-Kan extension, defined exactly as for the T-(co)limit: if we have a T-functor $j : I \longrightarrow J$ there is a T-functor induced by precomposition with j:

$$j^* : \underline{\mathrm{Fun}}_T(J, D) \longrightarrow \underline{\mathrm{Fun}}_T(I, D)\,.$$

If j^* has a left T-adjoint we denote it by $j_!$ and call it the *left T-Kan extension* along j. Similarly, when j^* has a right T-adjoint we call it the *right T-Kan extension j_**.

The proof of the following proposition can be found in [26].

2.13. **Proposition.** *Let C be a T-∞-category with all T-colimits. Then for every map of small T-∞-categories $j : I \longrightarrow I'$ the left Kan extension along j*

$$j_! : \underline{\mathrm{Fun}}_T(I, C) \longrightarrow \underline{\mathrm{Fun}}_T(I', C)$$

exists. Similarly for T-limits and the right Kan extension j_*.

2.14. **Notation.** A T-functor is said to be *fiberwise left exact*, *T-left exact*, *fiberwise right exact*, *T-right exact* if it preserves finite fiberwise limits, finite T-limits, finite fiberwise colimits and finite T-colimits respectively. We will denote the full T-∞-subcategories of $\underline{\mathrm{Fun}}_T(C, D)$ preserving certain (co)limits will be denoted as in the following list:

- $\underline{\mathrm{Fun}}_T^{T-\mathrm{lex}}(C, D)$: finite T-colimits;
- $\underline{\mathrm{Fun}}_T^{fb-\mathrm{lex}}(C, D)$: finite fiberwise colimits;
- $\underline{\mathrm{Fun}}_T^{\amalg}(C, D)$: finite T-coproducts;
- $\underline{\mathrm{Fun}}_T^{T-\mathrm{rex}}(C, D)$: finite T-limits;
- $\underline{\mathrm{Fun}}_T^{fb-\mathrm{rex}}(C, D)$: finite fiberwise limits;
- $\underline{\mathrm{Fun}}_T^{\times}(C, D)$: finite T-products.

3. Fiberwise stability

3.1. **Recollection.** If C is an ∞-category with finite colimits and D is an ∞-category with finite limits a functor $F : C \longrightarrow D$ is called *linear* if it sends the initial object of C to the terminal object of D and pushout squares in C to pullback squares in D (this functors are called *pointed excisive* in [21]). In [21, Pr. 1.4.2.13] it is proven that a pointed functor is linear if and only if the natural transformation $F \longrightarrow \Omega F \Sigma$ is an equivalence. The full subcategory of $\mathrm{Fun}(C, D)$ spanned by linear functors is denoted $\mathrm{Lin}(C, D)$.

3.2. **Definition.** Let C, D T-∞-categories and assume that C has all finite fiberwise colimits and D has all finite fiberwise limits. We say that a T-functor $F : C \longrightarrow D$ is *fiberwise linear* if the restriction on the fiber $F_b : C_b \longrightarrow D_b$ is linear for every $b \in T$. We denote the full T-subcategory of $\underline{\mathrm{Fun}}_T(C, D)$ spanned by fiberwise linear functors with $\underline{\mathrm{Lin}}_T(C, D)$.

3.3. First we want to show that, if C is T-pointed, $\mathrm{Lin}_T(C, D)$ is a localization of the subcategory $\mathrm{Fun}_{T,*}(C, D)$ of functors sending the zero object to the terminal object in each fiber. To do so we introduce two additional functors $\Sigma_T : C \longrightarrow C$ and $\Omega_T : D \longrightarrow D$ which are the pushout (respectively pullback) of the diagrams

$$\begin{array}{ccc} id_C & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & id_D \end{array} .$$

Since fiberwise linearity can be checked fiberwise it is clear that a functor $F \in \mathrm{Fun}_{T,*}(C, D)$ is in $\mathrm{Lin}(C, D)$ if and only if the canonical map $F \longrightarrow \Omega_T F \Sigma_T$ is an equivalence.

3.4. **Lemma.** *Suppose that C is a pointed T-category. Then the ∞-category $\mathrm{Lin}_T(C, D)$ is stable.*

Proof. It is clear that $\mathrm{Lin}_T(C, D)$ has finite limits and that it is pointed. If we show that Ω is an equivalence we are done by proposition 1.4.2.24 of [21]. But Ω is just postcomposition with $\Omega_T : D \longrightarrow D$ and then it is obvious that precomposition with $\Sigma_T : C \longrightarrow C$ is an inverse. □

3.5. **Definition.** We say that a T-∞-category D with all finite fiberwise limits and colimits is fiberwise stable if all fibers D_b are stable.

3.6. **Construction.** If C is a T-∞-category we want to construct a fiberwise stabilization, that is the universal source of a fiberwise linear T-functor to C. Let $\mathcal{E}(D)$ be the simplicial set over T^{op} such that

$$\mathrm{Mor}_{/T^{op}}(K, \mathcal{E}(D)) \cong \mathrm{Mor}_{/T^{op}}(K \times \mathbf{Top}_*^{fin}, D)$$

(this is an instance of the pairing construction of [20, Cor. 3.2.2.13]). The fiber over $b \in T^{op}$ is the category $\mathrm{Fun}(\mathbf{Top}_*^{fin}, D_b)$.

We let $\underline{\mathbf{Sp}}_T(D)$ be the simplicial subset of $\mathcal{E}(D)$ consisting of all simplices whose vertices are linear functors $\mathbf{Top}_*^{fin} \longrightarrow D_b$. This is the same simplicial set denoted by $Stab(D)$ in [21, Cn. 6.2.2.2]. It comes equipped with a natural map of simplicial sets $\Omega^\infty : \underline{\mathbf{Sp}}_T(D) \longrightarrow D$ over T^{op} that on vertices is evaluation at S^0.

3.7. **Proposition.** *The map $\underline{\mathbf{Sp}}_T(D) \longrightarrow T^{op}$ is a fiberwise stable T-∞-category.*

Moreover the natural functor $\Omega^\infty \colon \underline{\mathbf{Sp}}_T(D) \longrightarrow D$ is a fiberwise left exact T-functor and for every pointed T-∞-category C with finite T-colimits the induced map

$$\underline{\mathrm{Fun}}_T^{fb-\mathrm{rex}}(C, \underline{\mathbf{Sp}}_T(D)) \longrightarrow \underline{\mathrm{Lin}}_T(C, D)$$

is an equivalence of categories.

Proof. From [20, Cor. 3.2.2.13] it follows immediately that $\mathcal{E}(D)$ is a cocartesian fibration whose cocartesian edges are those maps $(\Delta^1)^\sharp \times (\mathbf{Top}_*^{fin})^\flat \longrightarrow D^\natural$ that are marked. So to prove that $\underline{\mathbf{Sp}}_T(D) \longrightarrow T^{op}$ is a cocartesian fibration we need only to prove that it contains all cocartesian edges whose source is in it (that is, that $\underline{\mathbf{Sp}}_T(D)$ is closed under pushforward). But, by our description of cocartesian edges, the pushforward functor along an edge $e : b \longrightarrow b'$ of $\mathcal{E}$ is given by

$$(\delta_e)_* : \mathcal{E}_{b'} = \mathrm{Fun}(\mathbf{Top}_*^{fin}, D_{b'}) \longrightarrow \mathcal{E}_b = \mathrm{Fun}(\mathbf{Top}_*^{fin}, D_b)\,,$$

that is postcomposition with the pushforward in D. Since the pushforward in D preserves finite limits by definition, $(\delta_e)_*$ preserves linear functors and so $\underline{\mathbf{Sp}}_T(D)$ is a T-∞-category.

Note that the fiber of $\underline{\mathbf{Sp}}_T(D)$ over $b \in T$ is exactly the stabilization of the fiber D_b and that the pushforward functors between fibers of $\underline{\mathbf{Sp}}_T(D)$ are the functors induced by the pushforward between the fibers of D. So the cocartesian fibration $\underline{\mathbf{Sp}}_T(D)$ has all finite fiberwise limits and colimits and is fiberwise stable. Moreover the functor $\underline{\mathbf{Sp}}_T(D) \longrightarrow D$ is a T-functor preserving T-limits.

Finally let us prove the universal property. Since the fibers of

$$\underline{\mathrm{Fun}}_T^{fb-\mathrm{lex}}(C, \underline{\mathbf{Sp}}_T(D)) \text{ and } \underline{\mathrm{Lin}}_T(C, D)$$

over $b \in T$ are

$$\mathrm{Fun}_{T_{/b}}^{fb-\mathrm{lex}}(C \times_{T^{op}} T_{b/}^{op}, \underline{\mathbf{Sp}}_{T_{/b}}(D \times_{T^{op}} T_{b/}^{op})) \text{ and } \mathrm{Lin}_{T_{/b}}(C \times_{T^{op}} T_{b/}^{op}, D \times_{T^{op}} T_{b/}^{op})$$

respectively, up to replacing T by its slice $T_{/b}$ it is enough to prove that the functor

$$(\Omega^\infty)_* : \mathrm{Fun}_T^{fb-\mathrm{lex}}(C, \underline{\mathbf{Sp}}_T(D)) \longrightarrow \mathrm{Lin}_T(C, D)$$

is an equivalence of categories (since being an equivalence can be checked on every fiber). Observe that $\mathrm{Fun}_T^{fb-\mathrm{lex}}(C, \underline{\mathbf{Sp}}_T(D))$ and $\mathbf{Sp}(\mathrm{Lin}_T(C, D))$ are the same subcategory of $\mathrm{Fun}_T(C \times \mathbf{Top}^{fin}, D)$, because both are spanned by the functors $F : C \times \mathbf{Top}_T^{fin} \longrightarrow D$ whose restriction to $C_b \times \mathbf{Top}^{fin}$ lie in $\mathrm{Fun}^{lex}(C_b, \mathbf{Sp}(D_b)) = \mathbf{Sp}(\mathrm{Lin}(C_b, D_b))$ for any $b \in T$. Then the thesis is obvious because $\mathrm{Lin}(C_b, D_b)$ is stable. □

4. Categories of finite T-sets

4.1. **Definition.** A small ∞-category T is said to be *orbital* if the category $\mathbf{F}_T$ of definition 2.5 has all pullbacks. An orbital category T is *atomic* if there are no nontrivial retracts, that is if every map with a left inverse is an equivalence.

A more in depth treatment of orbital ∞-categories can be found in [7].

4.2. **Example.** The following are examples of atomic orbital categories:

- The orbit category $\mathbf{O}_G$ of a (pro)finite group is atomic orbital;
- The category of finite sets and surjections is atomic orbital;
- In general any epiorbital category ([16, Df. 2.1]) is atomic orbital;
- All ∞-groupoids are atomic orbital categories;
- Every cosieve of an atomic orbital category is atomic orbital;
- More generally, the total category of every right fibration over an atomic orbital ∞-category is atomic orbital;
- The cyclonic orbit category of [8, Df. 1.10] is atomic orbital;
- The category of connected groupoids (that is groupoids of the form BG for a finite group G) and covering maps is atomic orbital. This is the full subcategory of the global orbit category of [24, Cn. 8.32] spanned by the completely universal finite subgroups of $\mathcal{L}$.

4.3. For the remainder of this paper, T will be a fixed atomic orbital category. We will now construct T-∞-categories of finite T-sets that will be used to parametrize the various multiplications composing the structure of a T-commutative monoid.

4.4. **Definition.** We want to construct the T-category classified by the functor $T \longrightarrow \mathbf{Cat}_\infty$ sending V to $\mathbf{F}_{T_{/V}}$. We contemplate the arrow ∞-category $\mathrm{Fun}(\Delta^1, \mathbf{F}_T)$ of the ∞-category $\mathbf{F}_T$ of finite T-sets. Since $\mathbf{F}_T$ admits all pullbacks, the target functor

$$\mathrm{Fun}(\Delta^1, \mathbf{F}_T) \longrightarrow \mathrm{Fun}(\{1\}, \mathbf{F}_T) \cong \mathbf{F}_T$$

is a cartesian fibration. We may pull it back along the fully faithful inclusion $T \hookrightarrow \mathbf{F}_T$ to obtain a cartesian fibration

$$\tau\colon \mathrm{Fun}(\Delta^1, \mathbf{F}_T) \times_{\mathrm{Fun}(\{1\}, \mathbf{F}_T)} T \longrightarrow T.$$

It is classified by the functor $T^{op} \longrightarrow \mathbf{Cat}_\infty$ that carries an orbit V to the ∞-category $\mathbf{F}_{T_{/V}}$.

We now write

$$p\colon \underline{\mathbf{F}}_T \longrightarrow T^{op}$$

for the dual cocartesian fibration $\tau^\vee$ (constructed in [9]) to the cartesian fibration τ. This is now a T-∞-category, called the T-∞-category of finite T-sets, and once again it is classified by the functor $T^{op} \longrightarrow \mathbf{Cat}_\infty$ that carries an orbit V to the nerve of the ∞-category $\mathbf{F}_{T_{/V}}$. Its objects are arrows $I = [U \longrightarrow V]$ with $U \in \mathbf{F}_T$ and $V \in T$ and an arrow $[U \longrightarrow V] \longrightarrow [U' \longrightarrow V']$ is a diagram

$$\begin{array}{ccccc} U & \longleftarrow & W & \longrightarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & V' & = & V' \end{array}$$

where the left square is cartesian. Composition is then defined by forming suitable pullbacks. The target functor

$$[U \longrightarrow V] \rightsquigarrow V$$

is the structure map $p\colon \underline{\mathbf{F}}_T \longrightarrow T^{op}$.

4.5. **Example.** If $T = \mathbf{O}_G$ is the orbit category of a profinite group G, then $\underline{\mathbf{F}}_G \longrightarrow \mathbf{O}_G^{op}$ is the cocartesian fibration classified by the functor sending an orbit G/H to the category of finite H-sets (under the canonical identification that sends a finite G-set over G/H to the fiber over eH).

4.6. **Example.** Suppose V is an object of T. Then there is an object

$$I(V) = [id\colon V \longrightarrow V]$$

of $\underline{\mathbf{F}}_T$, which enjoys the following property. For any object $J = [g\colon X \longrightarrow Y]$ of $\underline{\mathbf{F}}_T$, one has an equivalence

$$\mathrm{Map}_{\underline{\mathbf{F}}_T}(J, I(V)) \simeq \mathrm{Map}_T(V, Y).$$

In particular, the assignment $V \rightsquigarrow I(V)$ defines a fully faithful right adjoint to the structure map $p\colon \underline{\mathbf{F}}_T \longrightarrow T^{op}$.

4.7**.** In what follows it will be convenient to have at our disposal more general categories whose objects are finite T-sets. They will be all more easily manipulated as subcategories of the Burnside category of finite T-sets. The latter is the dual version of the main construction in [6], but we will repeat it here both because of its simplicity and because we will need to use some details in our main results.

4.8. **Construction.** Let again us consider the cartesian fibration

$$\tau\colon S_T := \mathrm{Fun}(\Delta^1, \mathbf{F}_T) \times_{\mathrm{Fun}(\{1\}, \mathbf{F}_T)} T \longrightarrow T.$$

It is also, for much easier reasons, a cocartesian fibration.

With this in mind, we now proceed to define triple structures on these ∞-categories. Denote by $\iota T \subset T$ the subcategory consisting of the equivalences of T. Then we can contemplate the triple structures

$$(T, \iota T, T) \quad \text{and} \quad (S_T, S_T \times_T \iota T, S_T).$$

It is a simple matter to see that these triple structures are adequate in the sense of [2, Df. 5.2]. We may therefore construct their effective Burnside ∞-categories, and the projection induces a functor

$$t'\colon A^{\mathit{eff}}(S_T, S_T \times_T \iota T, S_T) \longrightarrow A^{\mathit{eff}}(T, \iota T, T).$$

An object of $A^{\mathit{eff}}(S_T, S_T \times_T \iota T, S_T)$ is a morphism $[U \longrightarrow V]$ of finite T-sets in which $V \in T$. If

$$I = [U \longrightarrow V] \quad \text{and} \quad J = [X \longrightarrow Y]$$

are two objects, then a morphism $I \longrightarrow J$ of $A^{\mathit{eff}}(S_T, S_T \times_T \iota T, S_T)$ is a commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & W & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Z & \xrightarrow{\sim} & Y \end{array}$$

in which the morphism $Z \xrightarrow{\sim} Y$ is an equivalence in T.

4.9. **Lemma.** *The functor t' above is both a cartesian and a cocartesian fibration. Furthermore, any morphism of $A^{\mathit{eff}}(S_T, S_T \times_T \iota T, S_T)$ represented as a commutative diagram*

$$\begin{array}{ccccc} U & \longleftarrow & W & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Z & \xrightarrow{\sim} & Y \end{array}$$

is

- *t'-cartesian if the morphisms $W \xrightarrow{\sim} U$ and $W \xrightarrow{\sim} X$ are equivalences;*
- *t'-cocartesian if the left square is cartesian and $\equiv WX$ is an equivalence.*

Proof. This follows immediately from (the opposite of) the "omnibus theorem" for effective Burnside ∞-categories [2, Th. 12.2]. ☐

4.10. **Definition.** We have an inclusion $T^{op} \hookrightarrow A^{eff}(T, \iota T, T)$, which is a weak equivalence. Write

$$\underline{\mathbf{A}}^{eff}(T) := A^{eff}(S_T, S_T \times_T \iota T, S_T) \times_{A^{eff}(T,\iota T,T)} T^{op};$$

the projection $\underline{\mathbf{A}}^{eff}(T) \longrightarrow A^{eff}(S_T, S_T \times_T \iota T, S_T)$ is thus an equivalence, and the projection

$$t\colon \underline{\mathbf{A}}^{eff}(T) \longrightarrow T^{op}$$

is a cartesian and cocartesian fibration, so it is a T-∞-category.

It is classified by the functor sending $V \in T$ to the Burnside category $A^{eff}(\mathbf{F}_{T_{/V}})$.

4.11. Note that $\underline{\mathbf{F}}_T$ is naturally a T-subcategory of $\underline{\mathbf{A}}^{eff}(T)$, consisting of all objects and all morphisms such that the left square is cartesian. This is the analogue of the classical inclusion of the category $\mathbf{F}$ of finite sets inside the Burnside category of finite sets $A^{eff}(\mathbf{F})$ by considering only the egressive maps.

This can be extended to an inclusion of *pointed* finite sets $\mathbf{F}_*$ inside $A^{eff}(\mathbf{F})$ as the subcategory containing all objects and as maps the spans $[I \longleftarrow \tilde{I} \longrightarrow I']$ such that the "left leg" is an inclusion ($\tilde{I}$ under this identification corresponds to the preimage of I' under the map $I_+ \longrightarrow I'_+$, so that we can identify $\mathbf{F}_*$ with the category of finite sets and partially defined maps. It will be convenient for us to turn this into the definition of finite pointed T-sets.

4.12. **Definition.** We'll say that a map $U \longrightarrow U'$ of finite T-sets is a *summand inclusion* if there is $U'' \longrightarrow U'$ such that the map $U \amalg U'' \longrightarrow U'$ is an equivalence.

Consider the subcategory of $\underline{\mathbf{A}}^{eff}(T)$ containing all objects and whose morphisms are those diagrams

$$\begin{array}{ccccc} U & \longleftarrow & \tilde{U} & \longrightarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & V' & = & V' \end{array}$$

such that the arrow $\tilde{U} \longrightarrow U \times_V V'$ is a summand inclusion (this is a condition of the left square of the diagram and does not depend on the particular choice of pullback $U \times_V V'$). This subcategory contains all cocartesian morphisms of $\underline{\mathbf{A}}^{eff}(T) \longrightarrow T^{op}$ and so it is a T-subcategory. We will call it the *T-∞-category of finite pointed T-sets* and denote it by $\underline{\mathbf{F}}_{*T}$.

4.13. **Notation.** We will often decorate an object $I = [U \longrightarrow V]$ of $\underline{\mathbf{F}}_{*T}$ with a subscript $+$, to remind ourselves that we see it as living in $\underline{\mathbf{F}}_{*T}$ rather than of $\underline{\mathbf{F}}_T$ or $\underline{\mathbf{A}}(T)$. The $+$ does not have any real meaning (in our construction there are no "basepoints") and it is only a mnemonic aid. The canonical inclusion $\mathbf{F}^T \longrightarrow \mathbf{F}_*^T$ will be indicated by $(-)_+ : I \mapsto I_+$.

4.14. **Lemma.** *The cocartesian fibration $\underline{\mathbf{F}}_{*T} \longrightarrow T^{op}$ is classified by the functor sending V to the category of pointed objects in $(\mathbf{F}_T)_{V/}$.*

Moreover the canonical inclusion $(-)_+ : \underline{\mathbf{F}}_T \longrightarrow \underline{\mathbf{F}}_{*T}$ has a right T-adjoint sending $[U \longrightarrow V]$ to $[U \amalg V \longrightarrow V]$.

Proof. The fiber of $\underline{\mathbf{F}}_{*T}$ over V consists in the Burnside category of $\mathbf{F}_{T/V}$ where the egressive morphisms are the summand-inclusions. We can identify this with the category of pointed objects by sending a span

$$U \longleftarrow \tilde{U} \longrightarrow U'$$

where $U = \tilde{U} \amalg W$ to the map

$$U \amalg V \longrightarrow U' \amalg V$$

where the central map is $U \amalg V = \tilde{U} \amalg (W \amalg V) \longrightarrow U' \amalg V$, given by $\tilde{U} \longrightarrow U'$ on the first component and the structure map to V on the second component.

Now it is clear that the functor

$$[U \amalg V \longrightarrow V] \mapsto [U \amalg V \longrightarrow V]$$

is the right adjoint to the inclusion of $\mathbf{F}_{T/V}$ into pointed objects. So in order to have a T-adjunction we need only to verify that the right adjoint provided by [21, Pr. 7.3.2.6] is a T-functor, but this follows from the universality of finite coproducts in $\mathbf{F}_T$ and the fact that coproducts therein are disjoint. □

5. T-semiadditive functors and T-semiadditive categories

5.1. **Notation.** Let $I = [U \longrightarrow V] \in \underline{\mathbf{F}}_T$ and $W \in \mathrm{Orbit}(U)$, then the canonical map $W \longrightarrow U \times_V W$ must be a summand-inclusion, since it factors through an unique orbit, of which W is a retract. So we can define the *characteristic map*

$$\chi_{[W \subseteq U]} \colon I_+ \longrightarrow I(W)_+$$

(where $I(W)$ is the construction of example 4.6) as the map of pointed finite T-sets described by the following diagram

$$\begin{array}{ccccc} U & \longleftarrow & W & = & W \\ \downarrow & & \| & & \| \\ V & \longleftarrow & W & = & W \end{array}.$$

Note that this map is in $\underline{\mathbf{F}}_{*T}$ due to the fact that, thanks to the atomicity of T, the map $W \longrightarrow U \times_V W$ is a summand inclusion, since the orbit it factors through retract onto W..

5.2. **Construction.** Let C be a pointed T-∞-category with all finite T-coproducts, $I = [U \longrightarrow V] \in \underline{\mathbf{F}}_T$ and $X \in \mathrm{Fun}_T(\mathbf{U}, C)$ be a diagram. Then there is a map induced on the colimits

$$(\chi_{[W \subseteq U]})_* : \delta_{W/V} \coprod_I X \longrightarrow X_{[W \subseteq U]} .$$

above $\chi_{[W \subseteq U]}$, where $\coprod_I$ is the left adjoint to $\delta_I : C_V \cong \mathrm{Fun}_T(\mathbf{V}, C) \longrightarrow \mathrm{Fun}_T(\mathbf{U}, C)$ and $X_{[W \subseteq U]}$ is the value of X at $[W \subseteq U] \in \mathbf{U}$. We can describe it as follows: by the base change condition in proposition 2.11 we have

$$\delta_{W/V} \coprod_I X \cong \coprod_{[U \times_V W/W]} \delta_{[U \times_V W/U]} X$$

As before, the atomicity of T implies that we can write

$$U \times_V W \cong W \amalg \tilde{U}$$

where W on the right hand side is the diagonal copy. Hence

$$\delta_{W/V} \coprod_I X \cong X_{[W \longrightarrow U]} \amalg \coprod_{[\tilde{U} \longrightarrow U]} X \,.$$

So we can define a map

$$(\chi_{[W \subseteq U]})_* : \delta_{W/V} \coprod_I X \longrightarrow X_{[W \subseteq U]}$$

which is the identity on the first summand and the zero map on the other.

If D is a T-∞-category with finite T-products and $F : C \longrightarrow D$ a T-functor we will denote by $(\chi_{[W \subseteq U]})_*$ also the natural transformation

$$F\left(\coprod_I X\right) \longrightarrow \prod_{W/V} F(X)$$

obtained by adjunction on $F(\chi_{[W \subseteq U]})_*$, where $\prod_{W/V}$ is the right adjoint of $\delta_{W/V}$.

5.3. **Definition.** Let C be a pointed T-∞-category with all finite T-coproducts and D a T-∞-category with all finite T-products. Then a T-functor $F : C \longrightarrow D$ is said to be *T-semiadditive* if for every $I = [U \longrightarrow V] \in \underline{\mathbf{F}}_T$ and $X \in \mathrm{Fun}_T(\mathbf{U}, C)$ the map
(5.3.1)

$$\prod_{W \in \mathrm{Orbit}(U)} (\chi_{[W \subseteq U]})_* : F\left(\coprod_I X\right) \longrightarrow \prod_{W \in \mathrm{Orbit}(U)} \prod_{W/V} F(X_{[W \subseteq U]}) \cong \prod_I F(X)$$

is an equivalence. We will denote the T-∞-category of all T-semiadditive T-functors with $\underline{\mathrm{Fun}}_T^{\oplus}(C, D)$.

We say that a pointed T-∞-category with all finite T-products and T-coproducts is *T-semiadditive* if the identity functor is T-semiadditive. That is, if the map

$$\coprod_I X \longrightarrow \prod_I X \tag{5.3.2}$$

is an equivalence for every I-uple X.

It is clear that if $F : C \longrightarrow D$ preserves finite T-coproducts and $G : D \longrightarrow E$ is T-semiadditive then the composition GF is T-semiadditive. Similarly if F is T-semiadditive and G preserves finite T-products.

5.4. **Example.** Let C be a pointed T-∞-category with all finite T-coproducts and D an ∞-category with all finite products. Then the category of T-objects $\underline{D}_T$ has all finite T-products and a T-functor $C \longrightarrow \underline{D}_T$ is T-semiadditive if and only if the associated functor $F : C \longrightarrow D$ is such that for every $[U \longrightarrow V] \in \underline{\mathbf{F}}_T$ the map

$$F\left(\coprod_I X\right) \longrightarrow \prod_{W \in \mathrm{Orbit}(U)} F(X_{[W \subseteq U]})$$

is an equivalence. In particular if D is semiadditive (i.e. it has biproducts), then $\underline{D}_T$ is T-semiadditive.

5.5. **Example.** The T-∞-category $\mathbf{A}^{\mathit{eff}}(T)$ is T-semiadditive. In fact every fiber is semiadditive by proposition 4.3 of [2], so it is sufficient to observe that for any arrow $W \longrightarrow V$ in T the functor

$$\delta_{W/V} \colon A^{\mathit{eff}}(T_{/V}) \longrightarrow A^{\mathit{eff}}(T_{/W})$$

has well behaved left and right adjoints and the canonical comparison map is an equivalence.

5.6. **Construction.** Let C be a pointed T-∞-category with finite T-products. Then if $I = [U \longrightarrow V] \in \mathbf{F}^T$ and $X : \mathbf{U} \longrightarrow C$, let us consider for every $W \in \mathrm{Orbit}(U)$ the map of C_V

$$\eta_{[W\subseteq U]} : \coprod_{W/V} X_{[W \longrightarrow U]} \hookrightarrow \coprod_I X \longrightarrow \prod_I X .$$

This can be described as the adjoint to the map in the fiber over W

$$X_{[W\subseteq U]} \longrightarrow \delta_{W/V} \prod_{I_W} X \cong \prod_{W'\in \mathrm{Orbit}(U\times_V W)} \prod_{W'/W} X_{[W' \longrightarrow U]} ,$$

given by the identity map $X_W \longrightarrow \prod_{W'/W} X_{W'}$ when W' is the diagonal copy W in $U \times_V W$ and the zero map on the other components.

Then C being T-semiadditive is equivalent to the fact that $\{\eta_W\}_{W\in\mathrm{Orbit}(U)}$ assemble to an equivalence

$$\coprod_I X \cong \coprod_{W\in\mathrm{Orbit}(U)} \coprod_{W/V} X_{[W\subseteq U]} \xrightarrow{\coprod \eta_{[W\subseteq U]}} \prod_I X .$$

The previous remark immediately yields the following criterion for determining when a category is T-semiadditive

5.7. **Lemma.** *Let C be a pointed T-∞-category with finite products and suppose that for every $I = [U \longrightarrow V] \in \mathbf{F}^T$ there is a natural transformation*

$$\mu^I : \prod_I \Delta X \longrightarrow X$$

of functors $C_V \longrightarrow C_V$, where $\Delta : C_V \longrightarrow \mathrm{Fun}_T(\mathbf{U}, C)$ is the functor of definition 2.10, such that for every $W \in \mathrm{Orbit}(U)$ the composition

$$\mu^I \circ \eta_{[W\subseteq U]} : \coprod_{W/V} \delta_{W/V} X \longrightarrow X$$

is homotopic to the counit of the adjunction $\coprod_{W/V} \dashv \delta_{W/V}$. Then C is T-semiadditive.

Proof. We need to prove that for every $[U \longrightarrow V] \in \underline{\mathbf{F}}_T$, $X \in \mathrm{Fun}_T(\mathbf{U}, C)$ and $Y \in C_V$, the map

$$\prod_{W\in\mathrm{Orbit}(U)} (\eta_{[W\subseteq U]})^* : \mathrm{Map}_{C_V}\left(\prod_I X, Y\right) \longrightarrow \prod_{W\in\mathrm{Orbit}(U)} \mathrm{Map}_{C_V}\left(\coprod_{W/V} X_{[W\subseteq U]}, Y\right)$$

is an equivalence. But using the μ^I we can construct an inverse

$$(\mu^I_Y)_* \circ \prod_I : \prod_{W\in\mathrm{Orbit}(U)} \mathrm{Map}_{C_V}\left(\coprod_{W/V} X_{[W\subseteq U]}, Y\right) \cong \mathrm{Map}_{\mathrm{Fun}(\mathbf{U},C)}(X, \Delta Y) \longrightarrow$$

$$\longrightarrow \mathrm{Map}_{C_V}\left(\prod_I X, \prod_I \Delta I\right) \longrightarrow \mathrm{Map}_{C_V}\left(\prod_I X, Y\right) .$$

□

5.8. **Proposition.** *If C is a pointed T-∞-category with finite T-coproducts and D is a T-∞-category with finite T-products then $\underline{\mathrm{Fun}}^{\oplus}(C, D)$ is T-semiadditive*

Proof. First let us note that by I to be empty in 5.3.1 every T-semiadditive functor must send the zero object of C to the terminal object of D. Then for any additive functor F the left T-Kan extension of the restriction to the zero object of C is the constant functor at the terminal object, since the T-colimit of a constant functor at the zero object is the zero object. So, if $i : \{0\} \subseteq C$ is the inclusion of the zero object

$$\mathrm{Map}_{\mathrm{Fun}_T^{\oplus}}(*, F) = \mathrm{Map}(i_!(F|_0), F) = \mathrm{Map}(F|_0, F|_0) = *$$

Hence the constant functor at the terminal object is the zero object of $\underline{\mathrm{Fun}}_T^{\oplus}(C, D)$.

Then we need to prove that $\underline{\mathrm{Fun}}_T^{\oplus}(C, D)$ satisfies the hypothesis of the previous lemma. But this is easy: for F a T-semiadditive T-functor remember that $\left(\prod_I F\right)(-) \cong F\left(\coprod_I -\right)$ so we can choose

$$\mu^I : \left(\prod_I F\right)(-) \cong F\left(\coprod_I -\right) \longrightarrow F(-)$$

given by precomposition with the canonical map $id_C \longrightarrow \coprod_I$ provided by the universal property of the coproduct of C. Since the required identites are easily verified we are done. □

5.9. **Definition.** Let C be a T-∞-category with finite products. Then a T-commutative monoid is a T-semiadditive functor $\underline{\mathbf{F}_{*}}_T \longrightarrow C$. We will indicate the T-∞-category of T-commutative monoids in C with $\underline{\mathbf{CMon}}_T(C)$. Precomposition with the cocartesian section $I(-)_+ : T^{op} \longrightarrow \underline{\mathbf{F}_{*}}_T$ induces a T-functor

$$\underline{\mathbf{CMon}}_T(C) \longrightarrow C\,.$$

In order to prove the universal property of $\underline{\mathbf{CMon}}_T(C)$ we will need the following lemma

5.10. **Lemma.** *Let C be a pointed T-∞-category with finite T-coproducts. Then the map*

$$\mathrm{Fun}_T^{\amalg}(\underline{\mathbf{F}_{*}}_T, C) \longrightarrow C$$

given by precomposition with $I(-)_+$ is an equivalence

Proof. We can construct an inverse by sending every $c \in C_V$ to the left Kan extension of its cocartesian section $\mathbf{V} \longrightarrow C$ along $I(-)_+ : T^{op} \longrightarrow \underline{\mathbf{F}_{*}}_T$. □

5.11. **Proposition.** *Let C be a T-∞-category with finite T-products. The functor*

$$\underline{\mathbf{CMon}}_T(C) \longrightarrow C\,.$$

induced by precomposition with the cocartesian section $I(-)_+ : T \longrightarrow \mathbf{F}_*^T$ is an equivalence if and only if C is T-semiadditive.

Proof. If the map is an equivalence then C is T-semiadditive, since $\underline{\mathbf{CMon}}_T(C)$ is. Vice versa if C is T-semiadditive then

$$\underline{\mathbf{CMon}}_T(C) = \underline{\mathrm{Fun}}_T^{\oplus}(\underline{\mathbf{F}_{*}}_T, C) = \underline{\mathrm{Fun}}_T^{\amalg}(\underline{\mathbf{F}_{*}}_T, C) \longrightarrow C$$

is an equivalence by lemma 5.10. □

5.11.1. **Corollary.** *Let C be a T-∞-category with finite T-products and D a pointed T-∞-category with finite T-coproducts. Then the map*

$$\mathrm{Fun}_T^{\amalg}(D, \underline{\mathbf{CMon}}_T(C)) \longrightarrow \mathrm{Fun}_T^{\oplus}(D, C)$$

is an equivalence of categories.

Proof. Observe that

$$\mathrm{Fun}_T^{\amalg}(D, \underline{\mathbf{CMon}}_T(C)) \cong \mathrm{Fun}_T^{\oplus}(D, \underline{\mathbf{CMon}}_T(C)) \cong \underline{\mathbf{CMon}}_T(\mathrm{Fun}_T^{\oplus}(D, C))$$

where the first equivalence comes from the T-semiadditivity of $\underline{\mathbf{CMon}}_T(C)$. Since $\mathrm{Fun}_T^{\oplus}(D, C)$ is T-semiadditive the thesis follows by the previous proposition. □

6. T-commutative monoids and Mackey functors

6.1. The notion of T-commutative monoid, while being the natural generalization of Γ-space to the parametrized setting, might seem abstract and difficult to work with. The aim of this section is that in fact T-commutative monoids are just objects very familiar in equivariant homotopy theory: Mackey functors.

6.2. **Lemma.** *Let $\underline{\mathbf{F}_*^{in}}_T$ be the T-subcategory of $\underline{\mathbf{F}_*}_T$ containing all objects and all the maps represented by spans in S_T*

$$I \longleftarrow \tilde{I} \longrightarrow I'$$

where the right arrow is an equivalence. Then a functor $M : \underline{\mathbf{F}_*}_T \longrightarrow \mathbf{Top}$ is a T-commutative monoid if and only if its restriction to $\underline{\mathbf{F}_*^{in}}_T$ is a right Kan extension along $I(-)_+ : T^{op} \longrightarrow \mathbf{F}_*^T$.

Proof. Obvious from the limit description of right Kan extensions (see [20, Pr. 4.3.2.15]). □

6.3. **Lemma.** *The inclusion $j : \underline{\mathbf{F}_*}_T \longrightarrow \underline{\mathbf{A}}^{\mathit{eff}}(T)$ is a T-commutative monoid.*

Proof. Clear from the fact that $\underline{\mathbf{F}_*}_T$ contains all T-coproduct diagrams of $\underline{\mathbf{A}}^{\mathit{eff}}(T)$ and example 5.5. □

6.4. **Lemma.** *Let C be an ∞-category and let E, F, D subcategories of C such that (C, E, D) and (C, E, F) are adequate triples in the sense of* [2]*. Consider the diagram of categories*

$$\begin{array}{ccc} A^{\mathit{eff}}(C, \iota E, D) & \xrightarrow{f} & A^{\mathit{eff}}(C, E, D) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ A^{\mathit{eff}}(C, \iota E, F) & \xrightarrow{f'} & A^{\mathit{eff}}(C, E, F) \end{array}$$

Then if a right Kan extension along g' exists so does the right Kan extension along g' and the natural transformation

$$f'^* g_* \longrightarrow g'_* f^*$$

is an equivalence.

Proof. Let us fix $I \in C$. We need to prove that the functor

$$A^{\text{eff}}(C, \iota E, D) \times_{A^{\text{eff}}(C,\iota E,F)} A^{\text{eff}}(C, \iota E, F)_{I/} \longrightarrow A^{\text{eff}}(C, E, D) \times_{A^{\text{eff}}(C,E,F)} A^{\text{eff}}(C, E, F)_{I/}$$

is coinitial. This is equivalent to the fact that for every $J \in C$ with a map from I the category

$$\left(A^{\text{eff}}(C, \iota E, D) \times_{A^{\text{eff}}(C,\iota E,F)} A^{\text{eff}}(C, \iota E, F)_{I/}\right) \times_{\left(A^{\text{eff}}(C,E,D) \times_{A^{\text{eff}}(C,E,F)} A^{\text{eff}}(C,E,F)_{I/}\right)} \left(A^{\text{eff}}(C, E, D) \times_{A^{\text{eff}}(C,E,F)} A^{\text{eff}}(C, E, F)_{I/}\right)_{/J}$$

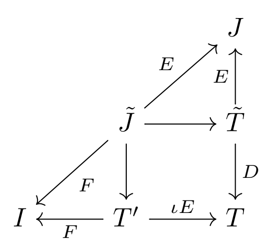
is weakly contractible. Let us start naming names. We have a fixed map $I \longrightarrow J$ in $A^{\text{eff}}(C, E, F)$. This correspond to a span

$$I \xleftarrow{F} \tilde{J} \xrightarrow{E} J\,,$$

where we decorate every arrow with the subcategory it lives in. Now an object of our category is $T \in T$ together with an arrow in $A^{\text{eff}}(C, \iota E, F)$ from I and an arrow in $A^{\text{eff}}(C, E, D)$ to J. These correspond to spans

$$I \xleftarrow{F} T' \xrightarrow{\iota E} T \qquad \text{and} \qquad T \xleftarrow{D} \tilde{T} \xrightarrow{E} J\,.$$

The last piece of data needed is an homotopy of their composition with the given map $I \longrightarrow J$, that is a diagram



where the central square is cartesian. But this is equivalent to the map $\tilde{J} \longrightarrow \tilde{T}$ being an equivalence. Summing up, an object of our category is a factorization of $\tilde{J} \longrightarrow I$

$$\tilde{J} \xrightarrow{D} T \xrightarrow{F} I\,.$$

Moreover a similar analysis on higher simplices shows that this is the opposite of the category of factorizations. It is easy to see that $\tilde{J} \xrightarrow{=} \tilde{J} \longrightarrow I$ is a terminal object for this category, which is then weakly contractible. ☐

6.5. **Theorem.** *Let C be a T-∞-category with finite T-limits. Precomposition with the inclusion $j : \underline{\mathbf{F}}_{*T} \longrightarrow \underline{\mathbf{A}}^{\text{eff}}(T)$ induces an equivalence*

$$\underline{\text{Fun}}_T^{\times}(\underline{\mathbf{A}}^{\text{eff}}(T), C) \longrightarrow \underline{\mathbf{CMon}}_T(C)\,.$$

We denote the category on the left hand side by $\underline{\mathbf{Mack}}^T(C)$ and call it the T-∞-category of T-Mackey functors valued in C.

Proof. Let $\underline{\mathrm{Psh}}_T(C)$ be the T-presheaf T-∞-category of C. Then, by the full faithfulness of the T-Yoneda embedding (see [3, Th. 10.4]) we have a pullback square

$$\begin{array}{ccc} \underline{\mathrm{Fun}}^{\times}_T(\underline{\mathbf{A}}^{eff}(T), C) & \longrightarrow & \underline{\mathbf{CMon}}_T(C) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Fun}}^{\times}_T(\underline{\mathbf{A}}^{eff}(T), \mathrm{Psh}_T(C)) & \longrightarrow & \underline{\mathbf{CMon}}_T(\underline{\mathrm{Psh}}_T C) \\ \| & & \| \\ \underline{\mathrm{Fun}}_T\left(C^{\vee op}, \underline{\mathrm{Fun}}^{\times}_T(\underline{\mathbf{A}}^{eff}(T), \underline{\mathbf{Top}}_T)\right) & \longrightarrow & \underline{\mathrm{Fun}}_T\left(C^{\vee op}, \underline{\mathbf{CMon}}_T(\underline{\mathbf{Top}}_T)\right) \end{array},$$

where $C^{\vee op}$ is the fiberwise opposite of [9], so it is enough to show the thesis for $C = \underline{\mathbf{Top}}_T$.

We claim that sending every T-commutative monoid to its right Kan extension is the inverse of the restriction map. The first step is showing that the natural map

$$j_* M \circ j \longrightarrow M$$

is an equivalence. By applying 6.4 with $C = F = S_T$, E the subcategory of fiberwise arrows and D the category of summand inclusions (that is the egressive maps in the definition of $\underline{\mathbf{F}_*}_T$) we see that it is enough to prove that the map

$$k_*(M|_{\underline{\mathbf{F}^{in}_*}_T}) \circ k \longrightarrow M|_{\underline{\mathbf{F}^{in}_*}_T}$$

is an equivalence, where k is the inclusion of E^{op} in F^{op}. But this follows immediately from the fact that $M|_{\underline{\mathbf{F}^{in}_*}_T}$ is the right Kan extension of its restriction to T^{op}.

Hence $j_* M$ must be a product preserving functor (since the image of j contains all product diagrams in $\underline{\mathbf{A}}^{eff}(T)$. Viceversa let suppose that N is a product preserving functor from $\underline{\mathbf{A}}^{eff}(T)$ to C. Then there is a natural map

$$N \longrightarrow j_*(N \circ j)\,.$$

But since j is essentially surjective we can check that this is an equivalence after precomposing with j, which follows immediately from the previous case. □

6.6**.** With a similar proof it is possible to prove that if D is an ∞-category with finite products there is an equivalence

$$\mathrm{Fun}^{\times}(A^{eff}(T), D) \cong \mathbf{CMon}_T(D^T)\,.$$

7. T-linear functors and T-stability

Recall the definition of fiberwise linear functor and fiberwise stable cocartesian fibration from section 3.

The following definition is inspired to hypothesis (A) of [14].

7.1. **Definition.** Let C be a pointed T-∞-category with finite T-colimits and let D be a T-∞-category with finite T-limits. Then a T-functor $F : C \longrightarrow D$ is T-linear if it is fiberwise linear and T-semiadditive. A T-∞-category with all finite T-limits and T-colimits is T-stable if it is fiberwise stable and T-semiadditive.

We will denote the T-subcategory of $\underline{\mathrm{Fun}}_T(C, D)$ which on the fiber above V is spanned by $T_{/V}$-linear functors from $C \times_{T^{op}} (T_{/V})^{op}$ to $D \times_{T^{op}} (T_{/V})^{op}$ with $\underline{\mathrm{Lin}}^T(C, D)$.

7.2. **Lemma.** *Let D be a T-semiadditive T-∞-category. Then $\underline{\mathbf{Sp}}_T(D)$ is T-semiadditive (and hence T-stable) and the functor $\Omega^\infty : \underline{\mathbf{Sp}}_T(D) \longrightarrow D$ preserves T-products (and so all T-limits).*

Proof. Recall that $\underline{\mathbf{Sp}}_T(D)$ is the cocartesian fibration classified by the functor $V \mapsto \mathbf{Sp}(D_V)$ so it is clearly fiberwise semiadditive and we just need to show that for every arrow $W \longrightarrow V$ in T the pushforward functor

$$\mathbf{Sp}(D_V) \longrightarrow \mathbf{Sp}(D_W)$$

has a coinciding left and right adjoints that satisfies the Beck-Chevalley condition. But since the left and right adjoint are clearly given by postcomposition of those of $D_V \longrightarrow D_W$ the thesis follows. □

7.3. **Definition.** Let D be a T-∞-category with all finite T-limits. Then the T-∞-category of T-spectra is

$$\underline{\mathbf{Sp}}^T(D) = \underline{\mathbf{Sp}}_T(\underline{\mathbf{CMon}}_T(D)) .$$

By the previous lemma the latter category is T-stable

Note that there is a natural T-functor $\Omega^\infty : \underline{\mathbf{Sp}}^T(D) \longrightarrow D$ given by the composition

$$\underline{\mathbf{Sp}}^T(D) = \underline{\mathbf{Sp}}_T(\underline{\mathbf{CMon}}_T(D)) \xrightarrow{\Omega^\infty} \underline{\mathbf{CMon}}_T(D) \xrightarrow{I(-)_+^*} D .$$

It is immediate by the previous lemma that it preserves all T-limits.

7.4. **Theorem** (Universal property of T-spectra)**.** *Let C be a pointed T-∞-category with finite T-colimits and D be a T-∞-category with finite T-limits. Then the functor*

$$(\Omega^\infty)_* \colon \underline{\mathrm{Fun}}_T^{T-\mathrm{rex}}(C, \underline{\mathbf{Sp}}^T(D)) \longrightarrow \underline{\mathrm{Lin}}^T(C, D)$$

is an equivalence of T-∞-categories, where the source categories is the full subcategory of those functors preserving finite T-limits. In particular

$$\underline{\mathbf{Sp}}^T(D) \cong \underline{\mathrm{Lin}}_T(\underline{\mathbf{Top}_*}_T^{fin}, D) ,$$

and the functor Ω^∞ is given by evaluation at the cocartesian section $I(-)_+ : T^{op} \longrightarrow \underline{\mathbf{F}}_{*T}$.

Proof. Since the map is clearly a T-functor we just need to check that it is an equivalence fiberwise. But, remembering that finite T-colimits are generated by T-coproducts and finite fiberwise colimits, we can apply 5.11.1 and 3.7 and conclude

$$\mathrm{Fun}_T^{T-\mathrm{rex}}(C, \underline{\mathbf{Sp}}_T(\underline{\mathbf{CMon}}_T(D))) = \mathrm{Lin}_T^{\amalg}(C, \mathbf{CMon}_T(D)) = \mathrm{Lin}^T(C, D) .$$

□

7.4.1. **Corollary.** *Let D be an ∞-category with all finite limits. Then there is an equivalence*

$$\mathrm{Lin}^T(\underline{\mathbf{Top}_*}_T^{fin}, \underline{D}_T) \cong \mathrm{Fun}^{\oplus}(A^{eff}(T), \mathbf{Sp}(D)) .$$

Proof. Both of those are equivalent to the global sections of $\underline{\mathbf{Sp}}^T(\underline{D}_T)$. □

Appendix A. Comparison with orthogonal spectra

A.1. Let G be a finite group. In this appendix we will prove that our notion of G-spectra coincides with the orthogonal G-spectra developed in [22], thus reproving a theorem by Guillou and May ([17]). Fix once and for all a complete G-universe $\mathcal{U}$ (that is an isometric G-action on $\mathbb{R}^\infty$ such that every finite-dimensional representation can be isometrically embedded in $\mathbb{R}^\infty$ countably many times) and note that its restriction to a subgroup H of G is an H-universe, which we will take with indexing set given by the G-invariant subspaces. The category of orthogonal H-spectra with respect to U ([22, Df. II.2.6]) will be denoted by $\mathrm{Sp}^H_{(1)}$.

A.2. **Definition.** Let $(\mathbf{O}_G)_*$ be the category of G-orbits with a distinguished basepoint but with any possible map.[1], We will write an element of $(\mathbf{O}_G)_*$ as G/H where the distinguished basepoint is eH. It is clear that the functor $(\mathbf{O}_G)_* \longrightarrow \mathbf{O}_G$ that forgets the basepoint is an equivalence of categories. A map $G/H \longrightarrow G/K$ is the datum of $gK \in G/K$ such that $g^{-1}Hg \subseteq K$. We have a functor from $(\mathbf{O}_G)_*^{op}$ to categories sending

- A pointed orbit G/H to the category $\mathrm{Sp}^H_{(1)}$ of orthogonal H-spectra with respect to U;
- A map $G/H \longrightarrow G/K$ the composition of the functors

$$\mathrm{Sp}^K_{(1)} \longrightarrow \mathrm{Sp}^{g^{-1}Hg}_{(1)} \longrightarrow \mathrm{Sp}^H_{(1)}$$

  where the first functor is the restriction along the inclusion $g^{-1}Hg \subseteq K$ and the second functor is induced by the isomorphism $g^{-1}Hg \cong H$ given by conjugating by g^{-1}.

If we equip every category $\mathrm{Sp}^H_{(1)}$ with the family of π_*-isomorphisms ([22, Df. III.3.2]) this becomes a functor from $(\mathbf{O}_G)_*^{op}$ to the category of relative categories (since [22, Lm. V.2.2] implies that change of groups preserve π_*-isomorphisms). By precomposing with the equivalence $\mathbf{O}_G^{op} \cong (\mathbf{O}_G)_*^{op}$ and postcomposing with the localization functor from relative categories to ∞-categories we finally obtain a functor

$$\mathbf{O}_G^{op} \longrightarrow \mathbf{Cat}_\infty$$

that classifies a cocartesian fibration $\underline{\mathbf{Sp}}^G_{orth} \longrightarrow \mathbf{O}_G^{op}$. We call this cocartesian fibration the G-∞-category of orthogonal G-spectra. It comes equipped with a natural G-functor

$$\Omega^\infty : \underline{\mathbf{Sp}}^G_{orth} \longrightarrow \underline{\mathbf{Top}}_G$$

induced by the natural transformation obtained by sending every orthogonal H-spectrum to its 0-th space.

A.3. **Lemma.** *The G-∞-category $\underline{\mathbf{Sp}}^G_{orth}$ is G-stable*

Proof. Since the fibers are obtained by localizing a stable model category at the weak equivalences $\mathbf{Sp}^G_{orth}$ is fiberwise stable. So we just need to check G-semiadditivity. But after unwrapping the definitions this is equivalent to the Wirthmüller isomorphism ([19, Th. II.6.2], which holds for orthogonal G-spectra by [22, Th. III.4.16]). ∎

[1] Another way of thinking of this category is as the category of G-orbits together with an explicit isomorphism with an orbit of the form G/H. This is of course purely bookkeeping and has nothing to do with the use of basepoints when defining G-spectra.

We can now give a very simple proof of [17, Th. 0.1] along the outline in section 11 of [10].

A.4. **Theorem** (Guillou-May)**.** *The functor* $\Omega^\infty : \mathbf{Sp}^G_{orth} \longrightarrow \underline{\mathbf{Top}}_G$ *lifts to an equivalence of* G*-*∞*-categories* $\underline{\mathbf{Sp}}^G_{orth} \cong \underline{\mathbf{Sp}}^G$*.*

Proof. Since the functor $\underline{\mathbf{Sp}}^G_{orth} \longrightarrow \underline{\mathbf{Top}}_G$ preserves all finite G-limits (it has a left G-adjoint by proposition [21, Pr. 7.3.2.1]) it lifts uniquely to a functor Ξ : $\mathbf{Sp}^G_{orth} \longrightarrow \mathbf{Sp}^G$.

For every orbit V the fibers $(\mathbf{Sp}^G_{orth})_V$ and $(\mathbf{Sp}^G)_V$ are both generated by suspension spectra of orbits. Moreover Ξ sends suspension spectra of orbits to suspension spectra of orbits and is fully faithful when restricted to those subcategories by [10, Th. 10.6] and [22, Th. V.11.1], since in both settings $\mathrm{Map}(\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K)$ is just $\Omega^\infty \left(\Sigma^\infty_+ (G/H \times G/K)\right)^G$. Hence it is an equivalence by the Schwede-Shipley theorem [21, Th. 7.1.2.1]. □

From this description of G-spectra we immediately obtain a recognition principle for G-connective G-spectra

A.4.1. **Corollary.** *There is an adjuction*

$$B \dashv \Omega^\infty : \mathbf{CMon}_G(\underline{\mathbf{Top}}_G) \leftrightarrows \mathbf{Sp}^G$$

such that

- *the unit* $X \longrightarrow \Omega^\infty BX$ *is an equivalence if and only* X^H *is a group-like monoid for every subgroup* $H < G$*;*
- *the counit* $B\Omega^\infty E \longrightarrow E$ *is an equivalence if and only if* E^H *is connective for every subgroup* $H < G$*.*

Proof. After our identifications this is just the adjunction

$$\mathrm{Fun}^\times(A^{\mathit{eff}}(G), \mathbf{Top}) \cong \mathrm{Fun}^\oplus(A^{\mathit{eff}}(G), \mathbf{CMon}(\mathbf{Top})) \leftrightarrows \mathrm{Fun}^\oplus(A^{\mathit{eff}}(G), \mathbf{Sp})$$

given by postcomposition with the adjunction for ordinary spectra, and the thesis follows from the classical recognition theorem. □

References

1. C. Barwick, *Algebraic K-theory of parametrized ∞-categories*, To appear.
2. ———, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Preprint `arXiv:1404.0108`, April 2014.
3. C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah, *Elements of parametrized higher category theory*, To appear.
4. ———, *A general introduction*, To appear.
5. ———, *Parametrized ∞-operads and parametrized symmetric monoidal ∞-categories*, To appear.
6. C. Barwick and S. Glasman, *On the fibrewise effective Burnside ∞-category*, Preprint available from *arXiv:1607.02786.*
7. ———, *Orbital ∞-categories*, To appear.
8. ———, *Cyclonic spectra, cyclotomic spectra, and a conjecture of Kaledin*, Preprint `arXiv:1602.02163`, February 2016.
9. C. Barwick, S. Glasman, and D. Nardin, *Dualizing cartesian and cocartesian fibrations*, Preprint from the web pages of the authors.
10. C. Barwick, S. Glasman, and J. Shah, *Spectral Mackey functors and equivariant algebraic K-theory (II)*, Preprint `arXiv:1505.03098`, May 2015.
11. C. Barwick and D. Nardin, *Norms in algebraic K-theory*, To appear.

12. C. Barwick, D. Nardin, and J. Shah, *Module theory for parametrized e_∞ ring spectra*, To appear.
13. ———, *Parametrized presentable ∞-categories and distributive functors*, To appear.
14. A. J. Blumberg, *Continuous functors as a model for the equivariant stable homotopy category*, Algebr. Geom. Topol. **6** (2006), 2257–2295. MR 2286026 (2008a:55006)
15. E. Dotto and K. Moi, *Homotopy theory of G-diagrams and equivariant excision*, Algebr. Geom. Topol. **16** (2016), no. 1, 325–395. MR 3470703
16. S. Glasman, *Stratified categories, geometric fixed points and a generalized Arone-Ching theorem*, ArXiv e-prints (2015).
17. B. Guillou and J. P. May, *Models of G-spectra as presheaves of spectra*, Preprint arXiv:1110.3571v3, July 2013.
18. M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the non-existence of elements of Kervaire invariant one*, Preprint arXiv:0908.3724v2, November 2010.
19. L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986, With contributions by J. E. McClure. MR 866482 (88e:55002)
20. J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)
21. ———, *Higher algebra*, Preprint from the web page of the author, August 2012.
22. M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108. MR 1922205 (2003i:55012)
23. J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs, vol. 132, American Mathematical Society, Providence, RI, 2006. MR 2271789
24. S. Schwede, *Global homotopy theory*, Preprint from the web page of the author.
25. G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR 50 #5782
26. J. Shah, *Parametrized homotopy limits and colimits*, To appear.

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Part III

Equivariant operads and normed algebras

PARAMETRIZED AND EQUIVARIANT HIGHER ALGEBRA

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Abstract. We develop the rudiments of a theory of parametrized ∞-operads, including parametrized generalizations of monoidal envelopes, Day convolution, operadic left Kan extensions, results on limits and colimits of algebras, and the symmetric monoidal Yoneda embedding.

Contents

1. Introduction

The goal of this paper is to lay foundations for a theory of *parametrized ∞-operads*. To explain the concept, suppose G is a finite group and let us first recall the concept of a G-symmetric monoidal ∞-category, after Hill–Hopkins [HH16], Blumberg–Hill [BH20], and Bachmann–Hoyois [BH21]. Let $\mathbf{F}_G$ be the category of finite G-sets, $\mathrm{Span}(\mathbf{F}_G)$ the $(2,1)$-category of spans of finite G-sets, and $\widehat{\mathbf{Cat}}$ the (huge) ∞-category of (large) ∞-categories.

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1.0.1. **Definition.** A *G-symmetric monoidal ∞-category* is a product-preserving functor

$$\mathcal{C}^{\otimes} : \mathrm{Span}(\mathbf{F}_G) \longrightarrow \widehat{\mathbf{Cat}}.$$

For example, the ∞-category $\mathbf{Sp}^G$ of genuine G-spectra extends to a G-symmetric monoidal ∞-category $(\mathbf{Sp}^G)^{\otimes}$ whose value on G/H is equivalent to $\mathbf{Sp}^H$ and whose covariant functoriality encodes the symmetric monoidal structures on $\{\mathbf{Sp}^H\}_{H \leq G}$ as well as the *Hill–Hopkins–Ravenel norm functors* $f_{\otimes} : \mathbf{Sp}^H \longrightarrow \mathbf{Sp}^K$ associated to maps of G-orbits $f : G/H \longrightarrow G/K$ (cf. [BH21, §9]). More generally, one can substitute other base ∞-categories apart from $\mathbf{F}_G$ as needed for other applications; in particular, in the motivic context Bachmann and Hoyois work with spans over certain categories of schemes and have extensively investigated the properties of such normed symmetric monoidal ∞-categories and their algebras in [BH21].

Just as the theory of symmetric monoidal ∞-categories admits a generalization to a theory of ∞-operads, we will see that the theory of G-symmetric monoidal ∞-categories admits a corresponding sort of generalization. Roughly speaking, a *simplicial G-operad* should consist of the data of a space of multimorphisms associated to every map of finite G-sets, with a composition law then associated to every composite of maps of finite G-sets.[1] In fact, just as an ∞-operad is really the ∞-categorical counterpart of a simplicial colored operad (i.e, a simplicial multicategory), our theory of G-∞-operads will encompass both G-symmetric monoidal ∞-categories and simplicial colored G-operads via a suitably defined coherent nerve construction. Abstracting away from the equivariant situation, we will be able to make this idea work under the following hypotheses on our base ∞-category, which were first articulated in the first author's work [Nar16] on parametrized stability.

1.0.2. **Definition** ([Nar16, Def. 4.1])**.** Let $\mathcal{T}$ be a small ∞-category. We say that $\mathcal{T}$ is *orbital* if its finite coproduct completion admits all pullbacks. We say that $\mathcal{T}$ is *atomic* if it has no non-trivial retracts, so that every map with a left inverse is an equivalence.

1.0.3. **Example.** The orbit category $\mathbf{O}_G$ of a finite group is atomic orbital. Some other examples are enumerated in [Nar16, Ex. 4.2].

1.0.4. **Remark.** The condition for an ∞-category to be atomic orbital is a highly restrictive one; for example, if $\mathcal{T}$ is atomic orbital and admits a terminal object, then $\mathcal{T}$ is equivalent to the nerve of a 1-category T (Proposition 2.5.1).

At this point, the reader should examine the definition of a simplicial colored T-operad (Definition 2.5.4) to get a conceptual handle on the forthcoming definition of a $\mathcal{T}$-∞-operad.

1.1. **Summary of results.** After some preliminaries on the $\mathcal{T}$-∞-category $\underline{\mathbf{F}}_{\mathcal{T},*}$ of pointed finite $\mathcal{T}$-sets (Definition 2.1.2), we give the definition of $\mathcal{T}$-∞-operad as Definition 2.1.7 and algebras therein as Definition 2.2.1. We explicate the parametrized Segal condition (Theorem 2.3.3) and show how the definition of a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category recovers Definition 1.0.1 (Theorem 2.3.9). We then study parametrized generalizations of three essential constructions in the theory of ∞-operads: monoidal envelopes (Definition 2.8.4), Day convolution (Definition 3.1.6), and operadic left Kan extension (Definition 4.3.5). Finally, we study $\mathcal{T}$-(co)limits in $\mathcal{T}$-∞-categories of $\mathcal{O}$-algebras, first in the context of a general $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$ (Theorem 5.1.3 and Theorem 5.1.4) and then in the special case of the $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$ (Theorem 5.3.7), and we establish a $\mathcal{T}$-symmetric monoidal refinement of the universal property of $\mathcal{T}$-presheaves (Corollary 6.0.12).

1.2. **Related work.** This paper is part of a larger body of work on parametrized higher category theory and higher algebra [BDG+16a, BDG+16b, Sha21a, Sha21b, Nar16, Nar17]. In particular, all of the conventions, terminology, and notation from [Sha21b] are in force in this paper, and the reader should at least skim the introduction and §2 of [Sha21b] before reading this work. Furthermore, the definition of a $\mathcal{T}$-∞-operad was developed in joint work with Barwick, Dotto, and Glasman circa 2016 and has previously appeared in the first author's thesis [Nar17, §3.1]. On the other hand, this paper doesn't otherwise expand on [Nar17, §3]; for instance, we will not recapitulate the first author's work on tensor products of $\mathcal{T}$-presentable $\mathcal{T}$-∞-categories.

As this paper is intended to play a foundational and supporting role in the literature, we don't discuss many interesting examples or applications here. Horev [Hor19] has used these results in his development of a theory of genuine equivariant factorization homology (see also [HHK+20]), and he in particular discusses the example of the G-∞-operad $\mathbf{E}_V$ associated to a finite-dimensional real G-representation V. The second

[1] Beware that this isn't the notion of G-operad that appears in the work of Blumberg-Hill [BH15].

author and Quigley have applied these results in their study of the parametrized Tate construction [QS21a] and real cyclotomic spectra [QS21b]. Hilman has introduced similar ideas in his study of parametrized noncommutative motives and equivariant algebraic K-theory [Hil22a, Hil22b].

In a different direction, the theory of G-operads in their various guises has a long history that we don't attempt to summarize here; some recent references are [BH15, GW18, Rub21, BP21, MMO21, GMMO18]. In terms of the relationship to the N_∞-operads of Blumberg–Hill, we discuss $\mathcal{T}$-indexing systems $\mathcal{I}$ in our framework in Definition 2.4.8, the corresponding commutative $\mathcal{T}$-∞-operad $\mathrm{Com}^{\otimes}_{\mathcal{I}}$ in Definition 2.4.10, and how they identify with G-indexing systems in the sense of Blumberg–Hill when $\mathcal{T} = \mathbf{O}_G$ in Remark 2.4.12. It should be possible to adapt ideas of Hinich from [Hin15] to establish a formal comparison between the ∞-category of $\mathrm{Com}_{\mathcal{I}}$-algebras in our sense and those in the sense of [BH15], but we do not attempt to do this now.

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2. Parametrized ∞-operads

2.1. **First definitions.** We begin by introducing the basic definitions of parametrized higher algebra in parallel to Lurie's development of the foundations of ∞-operads [Lur17, §2.1]. Let $\mathcal{T}$ be an atomic orbital ∞-category, whose objects we refer to as *orbits*, and let $\mathbf{F}_{\mathcal{T}}$ be its finite coproduct completion[2], which we refer to as the ∞-category of *finite* $\mathcal{T}$*-sets*.

2.1.1. **Definition.** For every orbit $V \in \mathcal{T}$, let

$$\mathbf{F}_{\mathcal{T}/V} := (\mathbf{F}_{\mathcal{T}})^{/V} = \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}) \times_{\mathbf{F}_{\mathcal{T}}} \{V\}$$

(thus fixing a preferred choice of finite coproduct completion of $\mathcal{T}^{/V}$), and let

$$\mathbf{F}_{\mathcal{T}/V,*} := (\mathbf{F}_{\mathcal{T}/V})^{\mathrm{id}_V/} = (\mathbf{F}_{\mathcal{T}})^{V//V}$$

be the ∞-category of *finite pointed* $\mathcal{T}^{/V}$*-sets*.

Using that $\mathcal{T}$ is orbital, we could then define the $\mathcal{T}$-∞*-category of finite* $\mathcal{T}$*-sets* $\underline{\mathbf{F}}_{\mathcal{T}}$ as the full $\mathcal{T}$-subcategory of $\underline{\mathbf{Spc}}_{\mathcal{T}}$ spanned by the finite $\mathcal{T}^{/V}$-sets in each fiber $(\underline{\mathbf{Spc}}_{\mathcal{T}})_V$ over an orbit V, so that as a cocartesian fibration, $\underline{\mathbf{F}}_{\mathcal{T}}$ is classified by the assignment $V \rightsquigarrow (\mathbf{F}_{\mathcal{T}})^{/V}$ with functoriality that given by pullback. Similarly, we could define a pointed variant $\underline{\mathbf{F}}_{\mathcal{T},*}$ as the full $\mathcal{T}$-subcategory of $\underline{\mathbf{Spc}}_{\mathcal{T},*}$ classified by the assignment $V \rightsquigarrow (\mathbf{F}_{\mathcal{T}})^{V//V}$.

However, although conceptually transparent, these definitions of $\underline{\mathbf{F}}_{\mathcal{T}}$ and $\underline{\mathbf{F}}_{\mathcal{T},*}$ are ill-suited to writing down arbitrary morphisms that may interpolate between different fibers. Instead, we will follow the first author's work in [Nar16, §4] and instead define $\underline{\mathbf{F}}_{\mathcal{T}}$ and $\underline{\mathbf{F}}_{\mathcal{T},*}$ as certain ∞-categories of spans, along the lines of the construction of the dual cocartesian fibration in [BGN18] as well as the span description of finite pointed sets in terms of finite sets and partially defined maps (cf. [Nar16, 4.11]).

2.1.2. **Definition.** Let

$$\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}} := \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}) \times_{\mathbf{F}_{\mathcal{T}}} \mathcal{T},$$

so that the functor $\mathrm{ev}_1 : \underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}} \longrightarrow \mathcal{T}$ given by evaluation at the target is a cartesian fibration classified by $V \rightsquigarrow (\mathbf{F}_{\mathcal{T}})^{/V}$. Labeling an arbitrary morphism $[\phi : f \longrightarrow g]$ of $\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}}$ as

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{k} & Y, \end{array}$$

we define wide subcategories

$$(\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{tdeg},\ (\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{si},\ (\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{cart} \subset \underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}}$$

[2] Explicitly, we could take $\mathbf{F}_{\mathcal{T}} \subset \mathbf{P}(\mathcal{T})$ to be the full subcategory spanned by finite coproducts of representables. However, any equivalent choice will suffice.

as containing those morphisms ϕ such that k is degenerate, $U \longrightarrow X \times_Y V$ is a summand inclusion, and $U \longrightarrow X \times_Y V$ is an equivalence, respectively.[3] Then the triples

$$(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}; (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{cart}, (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{tdeg}) \quad \text{and} \quad (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}; (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{si}, (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{tdeg})$$

are adequate in the sense of [Bar17, 5.2].[4] Consequently, we may form the associated span ∞-categories[5]

$$\underline{\mathbf{F}}_{\mathcal{T}} \coloneqq \mathrm{Span}(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}; (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{cart}, (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{tdeg}) \quad \text{and} \quad \underline{\mathbf{F}}_{\mathcal{T},*} \coloneqq \mathrm{Span}(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}; (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{si}, (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{tdeg}).$$

We regard $\underline{\mathbf{F}}_{\mathcal{T}}$ and $\underline{\mathbf{F}}_{\mathcal{T},*}$ as $\mathcal{T}$-∞-categories via the structure map ev_1 given by evaluation at the target, so that a morphism ψ (for either ∞-category)

$$\begin{array}{ccccc} U & \longleftarrow & Z & \xrightarrow{m} & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Y & \xrightarrow{=} & Y. \end{array}$$

is ev_1-cocartesian if and only if $m : Z \longrightarrow X$ is an equivalence and $Z \longrightarrow U \times_V Y$ is an equivalence (cf. [Nar16, Lem. 4.9 and Def. 4.12]). The canonical inclusion $\underline{\mathbf{F}}_{\mathcal{T}} \subset \underline{\mathbf{F}}_{\mathcal{T},*}$ of span ∞-categories is thus the inclusion of a $\mathcal{T}$-subcategory. We also have an 'identity' cocartesian section $I : \mathcal{T}^{\mathrm{op}} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T}}$ that sends V to $[V = V]$.

2.1.3. **Definition.** In the notation of Definition 2.1.2, we declare a morphism ψ in $\underline{\mathbf{F}}_{\mathcal{T},*}$ to be *inert* if $m : Z \longrightarrow X$ is an equivalence and *active* if $Z \longrightarrow U \times_V Y$ is an equivalence. Note that a morphism ψ in $\underline{\mathbf{F}}_{\mathcal{T},*}$ is both inert and active if and only if ψ is ev_1-cocartesian.

2.1.4. **Remark.** Note that $\underline{\mathbf{F}}_{\mathcal{T}}$ is by definition the dual cocartesian fibration to $\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}$ in the sense of [BGN18, Def. 3.4]. As such, for any orbit V we have an equivalence

$$(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})_V = \mathbf{F}_{\mathcal{T}/V} \xrightarrow{\simeq} \mathrm{Span}(\mathbf{F}_{\mathcal{T}/V}; (\mathbf{F}_{\mathcal{T}/V})^{\simeq}, \mathbf{F}_{\mathcal{T}/V}) = (\underline{\mathbf{F}}_{\mathcal{T}})_V$$

implemented by inclusion.

We next describe $\underline{\mathbf{F}}_{\mathcal{T},*}$. Let $\mathbf{F}^{si}_{\mathcal{T}}$ denote the wide subcategory on the summand inclusions in $\mathbf{F}_{\mathcal{T}}$, so $(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{si}_V = \mathbf{F}^{si}_{\mathcal{T}/V}$. As was noted in [Nar16, Lem. 4.14], for any orbit V we have an equivalence

$$(\underline{\mathbf{F}}_{\mathcal{T},*})_V = \mathrm{Span}(\mathbf{F}_{\mathcal{T}/V}; \mathbf{F}^{si}_{\mathcal{T}/V}, \mathbf{F}_{\mathcal{T}/V}) \xrightarrow{\simeq} \mathbf{F}_{\mathcal{T}/V,*},$$

under which an object $[U \xrightarrow{f} V]$ is sent to $[U \sqcup V \xrightarrow{f \sqcup \mathrm{id}} V]$ pointed at V, and a span

$$U \xleftarrow{\alpha} W \xrightarrow{\beta} U',$$

with α given by the summand inclusion $W \subset W \sqcup W' \simeq U$, is sent to the pointed map

$$U \sqcup V \xrightarrow{\gamma} U' \sqcup V$$

with $\gamma|_W = \beta$ and $\gamma|_{W'} = \mathrm{const}_V$. Consequently, we will often refer to an object $f = [U \longrightarrow V]$ of $\underline{\mathbf{F}}_{\mathcal{T},*}$ as $f_+ = [U_+ \longrightarrow V]$ to emphasize the implicit presence of the basepoint. We will also denote the canonical inclusion $\underline{\mathbf{F}}_{\mathcal{T}} \subset \underline{\mathbf{F}}_{\mathcal{T},*}$ of span ∞-categories by

$$(-)_+ : \underline{\mathbf{F}}_{\mathcal{T}} \hookrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$$

and refer to this as the *pointing* T-functor. By [Nar16, Lem. 4.14], $(-)_+$ has a 'forgetful' right T-adjoint which sends $[U_+ \longrightarrow V]$ to $[U \sqcup V \longrightarrow V]$. Note also that a morphism ψ in $\underline{\mathbf{F}}_{\mathcal{T},*}$ is active if and only if it is in the image of $(-)_+$.

2.1.5. **Remark.** For an orbit $V \in \mathcal{T}$, we obtain from Definition 2.1.3 a definition for inert and active edges in $(\underline{\mathbf{F}}_{\mathcal{T},*})_V$ by restriction to the fiber. Under the equivalence $(\underline{\mathbf{F}}_{\mathcal{T},*})_V \simeq \mathbf{F}_{\mathcal{T}/V,*}$ of Remark 2.1.4, a pointed map $f : U \sqcup V \longrightarrow U' \sqcup V$ is then inert if and only if its pullback along $U' \subset U' \sqcup V$ is an equivalence, and is active if and only if $f \simeq g_+$ for some $g : U \longrightarrow U'$ in $\mathbf{F}_{\mathcal{T}/V}$.

[3] Note that the "target degenerate" morphisms are a subclass of ev_1-cocartesian edges (for which more generally the map k is an equivalence) and the "cartesian" morphisms are exactly the ev_1-cartesian edges.

[4] Note that we have swapped the order of the wide subcategories from that of [Bar17], so that the first subcategory will indicate the backward facing arrows and the second will indicate the forward facing arrows when forming the span ∞-category.

[5] In [Bar17] and [Nar16], the term "effective Burnside ∞-category" A^{eff} is used as a synonym for "span ∞-category".

2.1.6. **Definition.** Suppose $f_+ = [U_+ \longrightarrow V]$ is an object of $\underline{\mathbf{F}}_{\mathcal{T},*}$. Let $\mathrm{Orbit}(U)$ be the set of orbits of U, so that we have an equivalence

$$U \simeq \coprod_{W \in \mathrm{Orbit}(U)} W$$

in $\mathbf{F}_{\mathcal{T}}$ with each W an object in $\mathcal{T}$. Given $W \in \mathrm{Orbit}(U)$, the *characteristic morphism*

$$\chi_{[W \subset U]} : f_+ \longrightarrow I(W)_+$$

is defined to be

$$\begin{array}{ccccc} U & \longleftarrow & W & \xrightarrow{=} & W \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ V & \longleftarrow & W & \xrightarrow{=} & W. \end{array}$$

Here we make essential use of our assumption that $\mathcal{T}$ is atomic to ensure that $W \longrightarrow U \times_V W$ is a summand inclusion. Clearly, $\chi_{[W \subset U]}$ is inert.

2.1.7. **Definition.** A $\mathcal{T}$-∞-*operad* is a pair $(\mathcal{C}^{\otimes}, p)$ consisting of a $\mathcal{T}$-∞-category $\mathcal{C}^{\otimes}$ along with a $\mathcal{T}$-functor $p : \mathcal{C}^{\otimes} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$, which is a categorical fibration and satisfies the following additional conditions:

(1) For every inert morphism $\psi : f_+ \longrightarrow g_+$ of $\underline{\mathbf{F}}_{\mathcal{T},*}$ and every object $x \in \mathcal{C}^{\otimes}_{f_+}$, there is a p-cocartesian edge $x \longrightarrow y$ in $\mathcal{C}^{\otimes}$ covering ψ.

(2) For any object $f_+ = [U_+ \longrightarrow V]$ of $\underline{\mathbf{F}}_{\mathcal{T},*}$, the p-cocartesian edges lying over the characteristic morphisms

$$\left\{ \chi_{[W \subset U]} : f_+ \longrightarrow I(W)_+ \ \middle|\ W \in \mathrm{Orbit}(U) \right\}$$

together induce an equivalence

$$\prod_{W \in \mathrm{Orbit}(U)} (\chi_{[W \subset U]})_! : \mathcal{C}^{\otimes}_{f_+} \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U)} \mathcal{C}^{\otimes}_{I(W)_+}.$$

(3) For any morphism

$$\psi : f_+ = [U_+ \longrightarrow V] \longrightarrow g_+ = \left[U'_+ \longrightarrow V'\right]$$

of $\underline{\mathbf{F}}_{\mathcal{T},*}$, objects $x \in \mathcal{C}^{\otimes}_{f_+}$ and $y \in \mathcal{C}^{\otimes}_{g_+}$, and any choice of p-cocartesian edges

$$\{y \longrightarrow y_W \ \mid\ W \in \mathrm{Orbit}(U')\}$$

lying over the characteristic morphisms

$$\left\{ \chi_{[W \subset U']} : g_+ \longrightarrow I(W)_+ \ \middle|\ W \in \mathrm{Orbit}(U') \right\},$$

the induced map

$$\mathrm{Map}^{\psi}_{\mathcal{C}^{\otimes}}(x, y) \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U')} \mathrm{Map}^{\chi_{[W \subset U']} \circ \psi}_{\mathcal{C}^{\otimes}}(x, y_W)$$

is an equivalence.

We will typically omit the structure map p and simply refer to $\mathcal{C}^{\otimes}$ as a $\mathcal{T}$-∞-operad. Given a $\mathcal{T}$-∞-operad $\mathcal{C}^{\otimes}$, its *underlying* $\mathcal{T}$-∞-*category* is the fiber product

$$\mathcal{C} := \mathcal{T}^{\mathrm{op}} \times_{I(-)_+, \underline{\mathbf{F}}_{\mathcal{T},*}} \mathcal{C}^{\otimes}.$$

2.1.8. **Definition.** Suppose $(\mathcal{C}^{\otimes}, p)$ is a $\mathcal{T}$-∞-operad. Then an edge of $\mathcal{C}^{\otimes}$ is *inert* if it is p-cocartesian over an inert edge of $\underline{\mathbf{F}}_{\mathcal{T},*}$, and is *active* if it factors as a p-cocartesian edge followed by an edge lying over a fiberwise active edge in $\underline{\mathbf{F}}_{\mathcal{T},*}$. We let $\mathcal{C}^{\otimes}_{ne}$ be the wide $\mathcal{T}$-subcategory of $\mathcal{C}^{\otimes}$ on the inert edges, and $\mathcal{C}^{\otimes}_{act}$ the wide $\mathcal{T}$-subcategory of $\mathcal{C}^{\otimes}$ on the active edges.

2.1.9. **Remark.** Let $\mathcal{C}^{\otimes}$ be a $\mathcal{T}$-∞-operad and $U \in \mathbf{F}_{\mathcal{T}}$. Note that for *any* orbit V and morphism $f : U \longrightarrow V$, we have an equivalence

$$\mathcal{C}^{\otimes}_{f_+} \simeq \mathcal{C}_U = \prod_{W \in \mathrm{Orbit}(U)} \mathcal{C}_W.$$

We will often write objects $x \in \mathcal{C}^{\otimes}_{f_+}$ as tuples (x_W).

2.1.10. **Remark** (Simplified condition on mapping spaces)**.** In Definition 2.1.7, in view of the inert-fiberwise active factorization system on a $\mathcal{T}$-∞-operad (Example 2.8.1) we may replace (3) by the following apparently weaker condition:

(3') Let $\alpha : [U \xrightarrow{f} V] \longrightarrow [U' \xrightarrow{g} V]$ be a morphism in $\mathbf{F}_{\mathcal{T}/V}$, which defines an active edge α_+ in $(\underline{\mathbf{F}}_{\mathcal{T},*})_V$. Let $x \in \mathcal{C}^{\otimes}_{f_+}$, $y \in \mathcal{C}^{\otimes}_{g_+}$ be objects, and for each $W \in \mathrm{Orbit}(U')$ let $y \longrightarrow y_W$ be a p-cocartesian edge lifting the characteristic morphism $\chi_{[W \subset U']}$. For every $W \in \mathrm{Orbit}(U')$ we have a commutative square

$$\begin{array}{ccc} [U_+ \longrightarrow V] & \xrightarrow{\alpha_+} & [U'_+ \longrightarrow V] \\ \big\downarrow{\scriptstyle \rho_W} & & \big\downarrow{\scriptstyle \chi_{[W \subset U']}} \\ [(U \times_{U'} W)_+ \longrightarrow W] & \xrightarrow{(\alpha_W)_+} & [W_+ \longrightarrow W] \end{array}$$

where ρ_W is the inert edge corresponding to the summand inclusion $U \times_{U'} W \longrightarrow U \times_V W$ and $\alpha_W : U \times_{U'} W \longrightarrow W$ is the pullback of $\alpha : U \longrightarrow U'$ along $W \subset U'$. (Note that the lower composition is the inert-fiberwise active factorization of the upper composition $\chi_{[W \subset U']} \circ \alpha_+$.) Let

$$\{x \longrightarrow x_W \mid W \in \mathrm{Orbit}(U')\}$$

be any choice of p-cocartesian edges lying over the morphisms ρ_W. Then the induced map

$$\mathrm{Map}^{\alpha_+}_{\mathcal{C}^{\otimes}}(x, y) \xrightarrow{\simeq} \prod_{W \in \mathrm{Orbit}(U')} \mathrm{Map}^{(\alpha_W)_+}_{\mathcal{C}^{\otimes}}(x_W, y_W) \tag{2.1.1}$$

is an equivalence.

2.1.11. **Remark** (Spaces of multimorphisms and operadic composition)**.** Suppose $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-∞-operad, $\alpha : U \longrightarrow U'$ is a morphism in $\mathbf{F}_{\mathcal{T}}$, and $x \in \mathcal{C}_U$, $y \in \mathcal{C}_{U'}$ are tuples of objects in $\mathcal{C}$. For every $W \in \mathrm{Orbit}(U')$, let

$$\alpha_W : U_W = U \times_{U'} W \longrightarrow W$$

be the pullback of α along the summand inclusion $W \subset U'$. Consider the component y_W of y as an object in $\mathcal{C}^{\otimes}_{I(W)_+}$ and the sub-tuple $x_W \in \mathcal{C}_{U_W}$ of x as an object in $\mathcal{C}^{\otimes}_{(\alpha_W)_+}$. Let

$$\mathrm{Mul}^{\alpha}_{\mathcal{C}}(x, y) := \prod_{W \in \mathrm{Orbit}(U')} \mathrm{Map}^{(\alpha_W)_+}_{\mathcal{C}^{\otimes}}(x_W, y_W)$$

be the space of $(\alpha; x, y)$-*multimorphisms* encoded by $\mathcal{C}^{\otimes}$. Then for any choice of map $U' \longrightarrow V$ in $\mathbf{F}_{\mathcal{T}}$ down to an orbit V, we have the canonical equivalence

$$\mathrm{Map}^{\alpha_+}_{\mathcal{C}^{\otimes}}(x, y) \simeq \mathrm{Mul}^{\alpha}_{\mathcal{C}}(x, y)$$

of (2.1.1) (compare Remark 2.1.9).

These spaces of multimorphisms are interrelated by the structure of $\mathcal{C}^{\otimes}$. For instance, for every composite morphism $U_0 \xrightarrow{\alpha} U_1 \xrightarrow{\beta} U_2$ in $\mathbf{F}_{\mathcal{T}}$ and $x_i \in \mathcal{C}_{U_i}$, $i \in \{0, 1, 2\}$, any choice of map $\rho : U_2 \longrightarrow V$ to an orbit V yields a map

$$\circ : \mathrm{Mul}^{\alpha}_{\mathcal{C}}(x_0, x_1) \times \mathrm{Mul}^{\beta}_{\mathcal{C}}(x_1, x_2) \longrightarrow \mathrm{Mul}^{\beta \circ \alpha}_{\mathcal{C}}(x_0, x_2)$$

defined by the composition in $\mathcal{C}^{\otimes}$, and one may check that this map is independent of the choice of ρ. Likewise, for every composition of pullback squares in $\mathbf{F}_{\mathcal{T}}$

$$\begin{array}{ccccc} X & \xrightarrow{f^*\alpha} & X' & \longrightarrow & W \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\ U & \xrightarrow{\alpha} & U' & \longrightarrow & V \end{array}$$

with V, W orbits, and objects $x \in \mathcal{C}_U$, $y \in \mathcal{C}_{U'}$, one has a base-change map

$$f^* : \mathrm{Mul}^{\alpha}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Mul}^{f^*\alpha}_{\mathcal{C}}(f^*x, f^*y)$$

induced by the cocartesian pushforward in $\mathcal{C}^{\otimes}$ along f in $\mathcal{T}^{\mathrm{op}}$. Note that these maps extend the functoriality on the underlying $\mathcal{T}$-∞-category $\mathcal{C}$.

Altogether, these maps satisfy homotopy coherent unitality, associativity, and base-change compatibility constraints as encapsulated by $\mathcal{C}^{\otimes}$.

2.2. **Morphisms of operads.** We next introduce morphisms of $\mathcal{T}$-∞-operads and algebras over $\mathcal{T}$-∞-operads.

2.2.1. **Definition.** Suppose $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are two $\mathcal{T}$-∞-operads. A *morphism of* $\mathcal{T}$-∞-*operads* is a $\mathcal{T}$-functor

$$A : \mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes}$$

over $\underline{\mathbf{F}}_{\mathcal{T},*}$ that carries inert morphisms to inert morphisms. Conceptually, A is a $\mathcal{C}$-*algebra valued in* $\mathcal{D}$.

If A is moreover a categorical fibration, then we call A a *fibration of* $\mathcal{T}$-∞-*operads*. Given fibrations of $\mathcal{T}$-∞-operads $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, we let

$$\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

denote the full subcategory of $\mathrm{Fun}_{/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the morphisms of $\mathcal{T}$-∞-operads, and

$$\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

the corresponding full $\mathcal{T}$-subcategory of the $\mathcal{T}$-∞-category $\underline{\mathrm{Fun}}_{/\mathcal{O}^{\otimes},\mathcal{T}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ [Sha21b, Notn. 4.7].[6]

If p is the identity on $\mathcal{O}^{\otimes}$, then we will also denote $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$ as $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{D})$. If $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, then we will also denote $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$ as $\mathbf{Alg}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$. Combining these two cases, if $\mathcal{C}^{\otimes} = \mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, then we will also denote $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$ as $\mathbf{CAlg}_{\mathcal{T}}(\mathcal{D})$, the ∞-category of $\mathcal{T}$-*commutative algebras* in $\mathcal{D}$.

2.2.2. **Warning.** In the case $\mathcal{T} = *$, our notation for ∞-categories of algebras conflicts with that of Lurie in [Lur17, Def. 2.1.3.1].

2.2.3. **Definition.** Suppose $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a fibration of $\mathcal{T}$-∞-operads in which p is moreover a cocartesian fibration. In this case, we call $\mathcal{C}^{\otimes}$ a $\mathcal{O}$-*monoidal* $\mathcal{T}$-∞-category. If $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, we also call $\mathcal{C}^{\otimes}$ a $\mathcal{T}$-*symmetric monoidal* $\mathcal{T}$-∞-category.[7] We also refer to $\mathcal{C}$ as $\mathcal{O}$-monoidal if the additional structure $(\mathcal{C}^{\otimes}, p)$ is understood from context.

2.2.4. **Notation.** Let $\mathcal{C}^{\otimes}$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category. For an active morphism $f : x \longrightarrow y$ in $\mathcal{O}^{\otimes}$, we typically denote the cocartesian pushforward functor associated to f by $f_{\otimes} : \mathcal{C}^{\otimes}_x \longrightarrow \mathcal{C}^{\otimes}_y$ and refer to it as the *norm functor* for f. If $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, then for any morphism $f : U \longrightarrow V$ of finite $\mathcal{T}$-sets with V an orbit, we have a norm functor $f_{\otimes} : \mathcal{C}_U \longrightarrow \mathcal{C}_V$ associated to $f_+ : [U_+ \to V] \longrightarrow [V_+ \to V]$. More generally, if V is a finite $\mathcal{T}$-set with orbit decomposition $\coprod_{i=1}^{n} V_i$ so that $f = \coprod_{i=1}^{n}(f_i : U_i \longrightarrow V_i)$, then we let $f_{\otimes}$ be the product of the functors $\{(f_i)_{\otimes}\}_{i=1}^{n}$. (We will also describe in Section 2.7 how to dispense with the orbit restriction in the formalism by passing to 'big' $\mathcal{T}$-∞-operads.)

2.2.5. **Definition.** Given two $\mathcal{O}$-monoidal $\mathcal{T}$-∞-categories $p, q : \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, a $\mathcal{T}$-functor $F : \mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes}$ is *lax* $\mathcal{O}$-*monoidal* if it is a morphism of $\mathcal{T}$-∞-operads, and is *(strict)* $\mathcal{O}$-*monoidal* if it carries p-cocartesian edges to q-cocartesian edges. We let

$$\mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

denote the subcategory of $\mathrm{Fun}_{/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the $\mathcal{O}$-monoidal $\mathcal{T}$-functors, and

$$\underline{\mathrm{Fun}}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

the corresponding $\mathcal{T}$-subcategory of $\underline{\mathrm{Fun}}_{/\mathcal{O}^{\otimes},\mathcal{T}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$. We will also drop $\mathcal{O}$ from the notation if $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$ and speak of lax and strict $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-functors.

In the situation of a cocartesian fibration $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ over a $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, we have the following simplification of the conditions for $\mathcal{C}^{\otimes}$ to be a $\mathcal{T}$-∞-operad (and hence $\mathcal{O}$-monoidal).

2.2.6. **Proposition.** *Let* $(\mathcal{O}^{\otimes}, q)$ *be a* $\mathcal{T}$-∞-*operad and let* $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ *be a cocartesian fibration of* $\mathcal{T}$-∞-*categories. Then* $(\mathcal{C}^{\otimes}, q \circ p)$ *is a* $\mathcal{T}$-∞-*operad if and only if for every* $f_+ = [U_+ \longrightarrow V] \in \underline{\mathbf{F}}_{\mathcal{T},*}$ *and* $x \in \mathcal{O}^{\otimes}_{f_+}$, *the inert edges* $\{x \longrightarrow x_W \mid W \in \mathrm{Orbit}(U)\}$ *in* $\mathcal{O}^{\otimes}$ *together induce an equivalence*

$$\mathcal{C}^{\otimes}_x \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U)} \mathcal{C}^{\otimes}_{x_W}.$$

[6] For this definition to be sensible, note that for any map $V \longrightarrow W$ in $\mathcal{T}$, the pullback of a $\mathcal{T}^{/W}$-∞-operad along the induced functor $(\mathcal{T}^{/V})^{\mathrm{op}} \longrightarrow (\mathcal{T}^{/W})^{\mathrm{op}}$ is again a $\mathcal{T}^{/V}$-∞-operad, and likewise for morphisms of $\mathcal{T}^{/W}$-∞-operads.

[7] We may also write "$\mathcal{T}$-symmetric monoidal ∞-category" for this notion since there is no potential for ambiguity.

Proof. The proof is exactly analogous to that of [Lur17, Prop. 2.1.2.12], so we will omit it. □

Lastly, we state the evident notions of $\mathcal{T}$-suboperad and $\mathcal{O}$-monoidal $\mathcal{T}$-subcategory.

2.2.7. **Definition.** Let $(\mathcal{C}^{\otimes}, p)$ be a $\mathcal{T}$-∞-operad and let $\mathcal{D}^{\otimes}$ be a $\mathcal{T}$-subcategory of $\mathcal{C}^{\otimes}$ with inclusion $\mathcal{T}$-functor i. We say that $\mathcal{D}^{\otimes}$ is a $\mathcal{T}$-*suboperad* of $\mathcal{C}^{\otimes}$ if $p \circ i$ exhibits $\mathcal{D}^{\otimes}$ as a $\mathcal{T}$-∞-operad and i is a morphism of $\mathcal{T}$-∞-operads. If $\mathcal{C}^{\otimes}$ is moreover an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category via $q : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, then $\mathcal{D}^{\otimes}$ is a $\mathcal{O}$-*monoidal* $\mathcal{T}$-*subcategory* of $\mathcal{C}^{\otimes}$ if $\mathcal{D}^{\otimes} \subset \mathcal{C}^{\otimes}$ is stable under q-cocartesian edges, so that $q \circ i$ exhibits $\mathcal{D}^{\otimes}$ as an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category and i is an $\mathcal{O}$-monoidal functor.

2.3. **Parametrized Segal condition.** We next want to interpret condition (2) of Definition 2.1.7 as an equivalence of $\mathcal{T}^{/V}$-∞-categories (i.e., a $\mathcal{T}$-*Segal condition*). First, we extend our notation for the $\mathcal{T}$-fibers of a $\mathcal{T}$-functor.

2.3.1. **Notation.** Let $F : \mathcal{X} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-functor and let $\sigma : \Delta^n \longrightarrow \mathcal{C}$ be a n-simplex of $\mathcal{C}$. Define the $\mathcal{T}$-*fiber of* $\mathcal{X}$ *over* σ to be

$$\mathcal{X}_{\underline{\sigma}} := \Delta^n \times_{\sigma, \mathcal{C}, \mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{C}) \times_{\mathrm{ev}_1, \mathcal{C}, F} \mathcal{X}.$$

2.3.2. **Construction.** For any $\mathcal{T}$-∞-category $\mathcal{X}$ and edge $f : x \longrightarrow y$ in $\mathcal{X}$, we can construct a $\mathcal{T}$-functor

$$\phi : \Delta^1 \times \underline{y} = \Delta^1 \times (\Delta^0 \times_{y, \mathcal{X}, \mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{X})) \longrightarrow \Delta^1 \times_{f, \mathcal{X}, \mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{X})$$

which fits into the commutative diagram

$$\begin{array}{ccc}
\{0\} \times \underline{y} & \xrightarrow{\quad f^* \quad} & \underline{x} \\
\downarrow & & \downarrow \\
\Delta^1 \times \underline{y} & \xrightarrow{\phi} & \Delta^1 \times_{f, \mathcal{X}, \mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{X}) \\
\uparrow & & \uparrow \\
\{1\} \times \underline{y} & \xrightarrow{\quad = \quad} & \underline{y},
\end{array}$$

where $f^* : \underline{y} \longrightarrow \underline{x}$ is the $\mathcal{T}$-functor defined in [Sha21a, Rem. 12.11], which sends a cocartesian edge $[e : y \to z]$ to the cocartesian edge $[f^*(e) : x \to z']$ given by the factorization of $[e \circ f]$ as the composite of $f^*(e)$ and a fiberwise edge $\phi(e)_1$.

Explicitly, let $h : \Delta^1 \times \Delta^1 \longrightarrow \mathcal{X}$ be given by

$$\begin{array}{ccc}
x & \xrightarrow{f} & y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle =} \\
y & \xrightarrow{=} & y
\end{array}$$

and let

$$\mathcal{M} = \Delta^1 \times_{h, \widetilde{\mathrm{Fun}}_{\Delta^1}(\Delta^1 \times \Delta^1, \mathcal{X} \times \Delta^1), \mathrm{ev}_0} \widetilde{\mathrm{Fun}}_{\Delta^1}(\Delta^1 \times \Delta^1, \mathrm{Ar}^{cocart}(\mathcal{X}) \times \Delta^1) \times_{\mathrm{ev}_1, \widetilde{\mathrm{Fun}}_{\Delta^1}(\Delta^1 \times \Delta^1, \mathcal{T}^{\mathrm{op}} \times \Delta^1), I} (\mathcal{T}^{\mathrm{op}} \times \Delta^1),$$

where I denotes the identity section, so that $\mathcal{M} \longrightarrow \Delta^1 \times \mathcal{T}^{\mathrm{op}}$ is a $\mathcal{T}$-correspondence with

$$\mathcal{M}_0 = \{0\} \times_{f, \mathrm{Ar}(\mathcal{X}), \mathrm{ev}_0} \mathrm{Ar}(\mathrm{Ar}^{cocart}(\mathcal{X})) \times_{\mathrm{ev}_1, \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}), I} \mathcal{T}^{\mathrm{op}},$$
$$\mathcal{M}_1 = \{0\} \times_{\mathrm{id}_y, \mathrm{Ar}(\mathcal{X}), \mathrm{ev}_0} \mathrm{Ar}(\mathrm{Ar}^{cocart}(\mathcal{X})) \times_{\mathrm{ev}_1, \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}), I} \mathcal{T}^{\mathrm{op}}.$$

We have a zig-zag of $\mathcal{T}$-functors over $\Delta^1 \times \mathcal{T}^{\mathrm{op}}$

$$\underline{y} \times \Delta^1 \overset{\tau}{\underset{\pi}{\leftrightarrows}} \mathcal{M} \xrightarrow{\rho} \Delta^1 \times_{f, \mathcal{X}, \mathrm{ev}_0} \mathrm{Ar}^{cocart}(\mathcal{X})$$

where π restricts to the trivial fibrations $\mathcal{M}_0 \longrightarrow \underline{y} \times \{0\}$, $\mathcal{M}_1 \longrightarrow \underline{y} \times \{1\}$ of [Sha21a, Lem. 12.10] and ρ restricts to $\mathcal{M}_0 \longrightarrow \underline{x}$, $\mathcal{M}_1 \longrightarrow \underline{y}$. Thus π is a trivial fibration and we may choose a section τ which fixes $\underline{y} \times \{1\} \subset \mathcal{M}_1$. Then we let $\phi = \rho \circ \tau$.

2.3.3. **Theorem.** *Let $\mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ be a fibration of $\mathcal{T}$-∞-operads. Let $x \in \mathcal{O}^\otimes$ be an object over $[f_+ : U_+ \to V] \in \underline{\mathbf{F}}_{\mathcal{T},*}$. Let $U \simeq U_1 \coprod ... \coprod U_n$ be an orbit decomposition, let $f_i : U_i \longrightarrow V$ denote the induced morphisms, and let $e_i : x \longrightarrow x_i$ be inert edges in $\mathcal{O}^\otimes$ lifting the characteristic morphisms $\chi_{[U_i \subset U]}$ in $\underline{\mathbf{F}}_{\mathcal{T},*}$. Then we have an equivalence of $\mathcal{T}^{/V}$-∞-categories*

$$\mathcal{C}^\otimes_{\underline{x}} \simeq \prod_f \left(\coprod_{1 \leq i \leq n} \mathcal{C}_{\underline{x_i}} \right) \simeq \prod_{1 \leq i \leq n} \left(\prod_{f_i} \mathcal{C}_{\underline{x_i}} \right).^{8}$$

Proof. Let

$$h_i : \Delta^1 \times \underline{x_i} \longrightarrow \Delta^1 \times_{e_i, \mathcal{O}^\otimes} \mathrm{Ar}^{cocart}(\mathcal{O}^\otimes) \longrightarrow \mathcal{O}^\otimes$$

be the homotopy associated to the edge e_i as defined as in Construction 2.3.2. Because h_i lands in $\mathcal{O}^\otimes_{ne}$, the pullback $\mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} (\Delta^1 \times \underline{x_i}) \longrightarrow \Delta^1 \times \underline{U_i}$ is a cocartesian fibration. This corresponds to a $\mathcal{T}^{/U_i}$-functor $\rho^i : f_i^*(\mathcal{C}^\otimes_{\underline{x}}) \longrightarrow \mathcal{C}_{\underline{x_i}}$. Taking the coproduct of the ρ^i and taking the adjoint of that, we get a comparison $\mathcal{T}^{/V}$-functor

$$\rho : \mathcal{C}^\otimes_{\underline{x}} \longrightarrow \prod_f \left(\coprod_{1 \leq i \leq n} \mathcal{C}_{\underline{x_i}} \right).$$

We claim that ρ is a equivalence of $\mathcal{T}^{/V}$-∞-categories. We will check that for every object $[g : V' \to V]$, the fiber ρ_g is an equivalence. Consider the pullback square

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ V' & \xrightarrow{g} & V. \end{array}$$

Let $[U_+ \to V] \longrightarrow [U'_+ \to V']$ be the corresponding inert morphism in $\underline{\mathbf{F}}_{\mathcal{T},*}$ and let $x \longrightarrow x'$ be an inert lift of that morphism to $\mathcal{O}^\otimes$. Also let $U' \simeq U'_1 \coprod ... \coprod U'_m$ be an orbit decomposition and let $e'_j : x' \longrightarrow x'_j$ be inert morphisms lifting the characteristic morphisms $\chi_{[U'_j \subset U']}$.

Note that

$$(\mathcal{C}^\otimes_{\underline{x}})_g \simeq (\mathcal{C}^\otimes_{\underline{x'}})_{\mathrm{id}_{V'}} \simeq \mathcal{C}^\otimes_{x'}$$

and

$$\begin{aligned} \left(\prod_f \left(\coprod_{1 \leq i \leq n} \mathcal{C}_{\underline{x_i}} \right) \right)_g &\simeq \left(\prod_{f'} \left((g')^* \left(\coprod_{1 \leq i \leq n} \mathcal{C}_{\underline{x_i}} \right) \right) \right)_{\mathrm{id}_{V'}} \\ &\simeq \left((g')^* \left(\coprod_{1 \leq i \leq n} \mathcal{C}_{\underline{x_i}} \right) \right)_{\mathrm{id}_{U'}} \\ &\simeq \left(\coprod_{1 \leq j \leq m} \mathcal{C}_{\underline{x'_j}} \right)_{\mathrm{id}_{U'}} \\ &\simeq \coprod_{1 \leq j \leq m} \mathcal{C}_{x'_j} \end{aligned}$$

A diagram chase then shows that the functor $\rho_g : \mathcal{C}^\otimes_{x'} \longrightarrow \coprod_{1 \leq j \leq m} \mathcal{C}_{x'_j}$ implements the equivalence of condition (2) in Definition 2.1.7. □

[8] In this expression, each $\mathcal{C}_{\underline{x_i}}$ is a $\mathcal{T}^{/U_i}$-∞-category, their coproduct is a cocartesian fibration over $(\mathcal{T}^{/U})^{\mathrm{op}} = \underline{U} \simeq \coprod_{1 \leq i \leq n} \underline{U_i}$, the righthand product is taken in $\mathcal{T}^{/V}$-∞-categories, and the indexed product $\prod_f$ denotes the right adjoint to pullback along the induced functor $\underline{U} \longrightarrow \underline{V}$.

2.3.4. **Corollary** ($\mathcal{T}$-Segal condition). *Let $\mathcal{C}^{\otimes} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ be a $\mathcal{T}$-∞-operad. Then for every object $[U_+ \xrightarrow{f_+} V]$ in $\underline{\mathbf{F}}_{\mathcal{T},*}$, we have an equivalence of $\mathcal{T}^{/V}$-∞-categories*

$$\mathcal{C}^{\otimes}_{\underline{f_+}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}^{/V}}(\underline{U}, \mathcal{C}_{\underline{V}}).$$

Proof. In view of Theorem 2.3.3, we only need to note that $\underline{\mathrm{Fun}}_{\mathcal{T}^{/V}}(\underline{U}, -) \simeq \prod_f f^*$ as endofunctors of $\mathbf{Cat}_{\mathcal{T}^{/V}}$ and that for an orbit decomposition $U \simeq U_1 \coprod \dots \coprod U_n$, $\coprod_{1 \leq i \leq n} \mathcal{C}_{\underline{U_i}} \simeq \mathcal{C}_{\underline{U}} \simeq \underline{U} \times_{\underline{V}} \mathcal{C}_{\underline{V}}$. □

2.3.5. **Example.** Let $\mathcal{C}^{\otimes}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. Then for every morphism $f : U \longrightarrow V$ of finite $\mathcal{T}$-sets, the norm functor $f_{\otimes} : \mathcal{C}_U \longrightarrow \mathcal{C}_V$ of Notation 2.2.4 canonically refines to a norm $\mathcal{T}^{/V}$-functor $\underline{\mathrm{Fun}}_{\mathcal{T}^{/V}}(\underline{U}, \mathcal{C}_{\underline{V}}) \longrightarrow \mathcal{C}_{\underline{V}}$. Indeed, if V is an orbit this is encoded by the cocartesian fibration $\mathcal{C}^{\otimes} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ in view of Corollary 2.3.4, and one extends to general V by taking coproducts.

We also have a reformulation of condition (3) in Definition 2.1.7 (or rather (3') in Remark 2.1.10), whose proof is the same as that of Theorem 2.3.3. Recall the notion of $\mathcal{T}$-mapping spaces from [Sha21a, §11].

2.3.6. **Notation.** Let $p : \mathcal{X} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-fibration, let $\alpha : a \longrightarrow b$ be a morphism in a fiber $\mathcal{B}_V$, and let $x, y \in \mathcal{X}$ so that $p(x) = a$ and $p(y) = b$. Then we define as a pullback of $\mathcal{T}^{/V}$-spaces

$$\underline{\mathrm{Map}}^{\alpha}_{\mathcal{X}}(x, y) := \underline{\alpha} \times_{\underline{\mathrm{Map}}_{\mathcal{B}}(a,b)} \underline{\mathrm{Map}}_{\mathcal{X}}(x, y).$$

2.3.7. **Proposition.** *Let $\mathcal{C}^{\otimes}$ be a $\mathcal{T}$-∞-operad and let notation be as in Remark 2.1.10. Then we have an equivalence of $\mathcal{T}^{/V}$-spaces*

$$\underline{\mathrm{Map}}^{\alpha_+}_{\mathcal{C}^{\otimes}}(x, y) \xrightarrow{\sim} \prod_g \left(\coprod_{W \in \mathrm{Orbit}(U')} \underline{\mathrm{Map}}^{(\alpha_W)_+}_{\mathcal{C}^{\otimes}}(x_W, y_W) \right).$$

Let us now apply the $\mathcal{T}$-Segal condition to characterize $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-categories as $\mathcal{T}$-commutative monoids in $\underline{\mathbf{Cat}}_{\mathcal{T}}$.

2.3.8. **Remark.** Under the equivalences (given by straightening and [Sha21a, Prop. 3.10], respectively)

$$\mathbf{Cat}^{cocart}_{/\underline{\mathbf{F}}_{\mathcal{T},*}} \simeq \mathrm{Fun}(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathbf{Cat}) \simeq \mathrm{Fun}_T(\underline{\mathbf{F}}_{\mathcal{T},*}, \underline{\mathbf{Cat}}_{\mathcal{T}})$$

we see that $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-categories $\mathcal{C}^{\otimes}$ correspond to $\mathcal{T}$-commutative monoids M, i.e., $\mathcal{T}$-semiadditive $\mathcal{T}$-functors [Nar16, Def. 5.3], since by Corollary 2.3.4, M transforms

$$[U_+ \longrightarrow V] \simeq \coprod_{W \in \mathrm{Orbit}(U)} \coprod_{W \to V} I(W)_+ \in \mathbf{F}_{\mathcal{T}^{/V},*}$$

into

$$\prod_{W \in \mathrm{Orbit}(U)} \prod_{W \to V} \mathcal{C}_{\underline{W}} \simeq \prod_{W \in \mathrm{Orbit}(U)} \underline{\mathrm{Fun}}_{\mathcal{T}^{/V}}(\underline{W}, \mathcal{C}_{\underline{V}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}^{/V}}(\underline{U}, \mathcal{C}_{\underline{V}}) \in \mathbf{Cat}_{\mathcal{T}^{/V}}.$$

Furthermore, in [Nar16, Thm. 6.5] the first author identified $\mathcal{T}$-commutative monoids with $\mathcal{T}$-Mackey functors. Using this, we can relate our notion of $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category with that which appears in [BH21], wherein the norm functors of Notation 2.2.4 appear as the covariant part of categorical Mackey functor.

2.3.9. **Theorem.** *We have a canonical equivalence of ∞-categories*

$$\mathbf{Cat}^{\otimes}_{\mathcal{T}} \simeq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbf{F}_{\mathcal{T}}), \mathbf{Cat})$$

between the ∞-category $\mathbf{Cat}^{\otimes}_{\mathcal{T}}$ of $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-categories and the ∞-category of product-preserving functors from $\mathrm{Span}(\mathbf{F}_{\mathcal{T}})$ *to* $\mathbf{Cat}$.

Proof. In [Nar16, Thm. 6.5], the first author proved that given a $\mathcal{T}$-∞-category $\mathcal{D}$ with finite $\mathcal{T}$-limits, precomposition by the inclusion $\underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}) := \mathrm{Span}(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}; (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}), (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{tdeg})$ induces an equivalence of $\mathcal{T}$-∞-categories

$$\underline{\mathrm{Fun}}^{\times}_{\mathcal{T}}(\underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}), \mathcal{D}) \xrightarrow{\sim} \underline{\mathbf{CMon}}_{\mathcal{T}}(\mathcal{D}).$$

In particular, if $\mathcal{D} = \mathbf{Cat}_{\mathcal{T}}$, then as we just observed $\mathbf{CMon}_{\mathcal{T}}(\mathcal{D}) \simeq \mathbf{Cat}^{\otimes}_{\mathcal{T}}$, and passing to cocartesian sections we obtain

$$\mathrm{Fun}^{\times}_{\mathcal{T}}(\underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}), \underline{\mathbf{Cat}}_{\mathcal{T}}) \simeq \mathbf{Cat}^{\otimes}_{\mathcal{T}}.$$

Under the equivalence

$$\mathrm{Fun}_T(\underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}), \underline{\mathbf{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}), \mathbf{Cat})$$

let $\mathrm{Fun}'(\underline{\mathrm{Span}}(\mathbf{F}_T), \mathbf{Cat})$ denote the image of $\mathrm{Fun}^{\times}_T(\underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}), \mathbf{Cat}_{\mathcal{T}})$. Explicitly, functors in Fun' send cartesian edges to equivalences and fiberwise products to products, where cartesian edges in $\underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}})$ are given by

$$\begin{array}{ccccc} U & \xleftarrow{=} & U & \xrightarrow{=} & U \\ \downarrow{=} & & \downarrow & & \downarrow \\ W & \xleftarrow{f} & V & \xrightarrow{=} & V \end{array}$$

in view of the adjunction

$$f^* : \mathrm{Span}(\mathbf{F}_{\mathcal{T}/W}) \rightleftarrows \mathrm{Span}(\mathbf{F}_{\mathcal{T}/V}) : f_!.$$

Let

$$s : \underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}) \longrightarrow \mathrm{Span}(\mathbf{F}_{\mathcal{T}})$$

denote the source map. Then we claim that precomposition by s induces an equivalence

$$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathbf{F}_{\mathcal{T}}), \mathbf{Cat}) \xrightarrow{\simeq} \mathrm{Fun}'(\underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}), \mathbf{Cat})$$

with inverse given by right Kan extension.

First note that given a product preserving functor $G : \mathrm{Span}(\mathbf{F}_{\mathcal{T}}) \longrightarrow \mathbf{Cat}$, s^*G evidently sends cartesian edges to equivalences and fiberwise products to products. Conversely, suppose we have a functor $F : \underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}) \longrightarrow \mathbf{Cat}$ in Fun'. Let $X \in \mathrm{Span}(\mathbf{F}_{\mathcal{T}})$ be an object. Note that $(\mathbf{F}_{\mathcal{T}}^{\mathrm{op}})^{X/} \longrightarrow \mathrm{Span}(\mathbf{F}_{\mathcal{T}})^{X/}$ is an initial functor as it admits a right adjoint which sends a span $X \longleftarrow Z \longrightarrow Y$ to $X \longleftarrow Z$. Pulling back, we thereby obtain an initial functor

$$(\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{\mathrm{op}} \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} (\mathbf{F}_{\mathcal{T}}^{\mathrm{op}})^{X/} \longrightarrow \underline{\mathrm{Span}}(\mathbf{F}_{\mathcal{T}}) \times_{\mathrm{Span}(\mathbf{F}_{\mathcal{T}})} \mathrm{Span}(\mathbf{F}_{\mathcal{T}})^{X/}$$

and we are interested in computing the limit of the functor

$$F' = F \circ \mathrm{pr} : (\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{\mathrm{op}} \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} (\mathbf{F}_{\mathcal{T}}^{\mathrm{op}})^{X/} \longrightarrow (\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{\mathrm{op}} \longrightarrow \mathbf{Cat}.$$

By our assumption on F, F' is the right Kan extension of its restriction to $\mathcal{T}^{\mathrm{op}} \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} (\mathbf{F}_{\mathcal{T}}^{\mathrm{op}})^{X/}$ (where $\mathcal{T}^{\mathrm{op}} \longrightarrow (\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{\mathrm{op}}$ is the identity section). Indeed, given an object $I = [V \longleftarrow U \longrightarrow X]$ and an orbit decomposition $U \simeq \coprod_{i=1}^n U_i$, the n projection maps $I \longrightarrow I_i = [U_i = U_i \longrightarrow X]$ induce an equivalence

$$F'(I) = F([U \longrightarrow V]) \xrightarrow{\simeq} \prod_{i=1}^{n} F'(I_i) = \prod_{i=1}^{n} F([U_i = U_i]).$$

We conclude that the limit of F' is $\prod_{i \in I} F([X_i = X_i])$ for some orbit decomposition $X \simeq \coprod_{i \in I} X_i$, so $s_* F(X) \simeq \prod_{i \in I} F(\mathrm{id}_{X_i})$. Using this pointwise formula and a simple argument regarding the morphisms, we see that $s_* F$ preserves products and the counit and unit maps are equivalences. □

2.4. **Examples.** In this subsection, we discuss some basic examples of $\mathcal{T}$-∞-operads.

2.4.1. **Example** ($\mathcal{T}$-(co)cartesian $\mathcal{T}$-symmetric monoidal structures)**.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category and let $\pi : \mathcal{C}^{\times} \longrightarrow \mathbf{F}_{\mathcal{T}}$ be the cartesian fibration defined as in [Sha21a, Prop. 5.12], so $(\mathcal{C}^{\times})_U \simeq \prod_{W \in \mathrm{Orbit}(U)} \mathcal{C}_W$ and the functoriality is given by restriction. Suppose $\mathcal{C}$ admits finite $\mathcal{T}$-coproducts. Then by [Sha21a, Prop. 5.12], π is a Beck–Chevalley fibration with cocartesian functoriality given by the coinduction functors, and by Barwick's unfurling construction [Bar17, §11], π straightens to a product-preserving functor $\mathrm{Span}(\mathbf{F}_{\mathcal{T}}) \longrightarrow \mathbf{Cat}$. Let

$$\mathcal{C}^{\amalg} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$$

denote the resulting $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category under the equivalence of Theorem 2.3.9. We call $\mathcal{C}^{\amalg}$ the $\mathcal{T}$-*cocartesian* $\mathcal{T}$-*symmetric monoidal structure* on $\mathcal{C}$.

Dually, suppose that $\mathcal{C}$ admits finite $\mathcal{T}$-products. Then by the dual of [Sha21a, Prop. 5.12], the vertical opposite $\pi^{\mathrm{vop}} : (\mathcal{C}^{\times})^{\mathrm{vop}} \longrightarrow \mathbf{F}_{\mathcal{T}}$ is a Beck–Chevalley fibration with cocartesian functoriality given by the (opposite of the) induction functors, and thus we obtain a product-preserving functor $\mathrm{Span}(\mathbf{F}_{\mathcal{T}}) \longrightarrow \mathbf{Cat}$. After postcomposing by the opposite automorphism of $\mathbf{Cat}$, let

$$\mathcal{C}^{\Pi} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$$

denote the resulting $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category under the equivalence of Theorem 2.3.9. We call $\mathcal{C}^{\Pi}$ the $\mathcal{T}$-*cartesian* $\mathcal{T}$-*symmetric monoidal structure* on $\mathcal{C}$.

2.4.2. **Example** (G-spectra)**.** Let $\mathbf{Gpd}_{\mathrm{fin}}$ be the $(2,1)$-category of finite groupoids and let

$$\mathbf{SH}^{\otimes} : \mathrm{Span}(\mathbf{Gpd}_{\mathrm{fin}}) \longrightarrow \mathbf{CAlg}(\mathbf{Cat}^{\mathrm{sift}})$$

denote the (restriction of the) functor of [BH21, §9.2]. Let G be a finite group and let

$$\omega_G : \mathrm{Span}(\mathbf{F}_G) \longrightarrow \mathrm{Span}(\mathbf{Gpd}_{\mathrm{fin}})$$

be the action groupoid functor. Let $(\underline{\mathbf{Sp}}^G)^{\otimes}$ be the G-symmetric monoidal G-∞-category associated to $\mathbf{SH}^{\otimes} \circ \omega_G$ under Theorem 2.3.9. Then $(\underline{\mathbf{Sp}}^G)^{\otimes}$ is the G-symmetric monoidal structure on $\underline{\mathbf{Sp}}^G$ that encodes the Hill–Hopkins–Ravenel norm functors.

2.4.3. **Example** (Trivial $\mathcal{T}$-∞-operad)**.** Let $\mathrm{Triv}^{\otimes}_{\mathcal{T}} \subset \underline{\mathbf{F}}_{\mathcal{T},*}$ be the wide subcategory on the inert edges. Then $\mathrm{Triv}^{\otimes}_{\mathcal{T}}$ is a $\mathcal{T}$-suboperad of $\underline{\mathbf{F}}_{\mathcal{T},*}$ such that the identity cocartesian section $I_+ : \mathcal{T}^{\mathrm{op}} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ restricts to a fully faithful functor into $\mathrm{Triv}^{\otimes}_{\mathcal{T}}$ and an equivalence onto $\mathrm{Triv}_{\mathcal{T}}$. We call $\mathrm{Triv}^{\otimes}_{\mathcal{T}}$ the *trivial* $\mathcal{T}$-∞-operad.

We claim that given any $\mathcal{T}$-∞-operad $\mathcal{C}^{\otimes}$, we have an equivalence

$$\underline{\mathbf{Alg}}_{\mathrm{Triv}_{\mathcal{T}},\mathcal{T}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}$$

implemented by restriction along I_+. To show this, we need the following lemma.

2.4.4. **Lemma.** *Let* $(\mathcal{O}^{\otimes}, p)$ *be a* $\mathcal{T}$-∞-*operad and let*

$$\mathcal{O}^{\otimes}_{ne} := \mathrm{Triv}^{\otimes}_{\mathcal{T}} \times_{\underline{\mathbf{F}}_{\mathcal{T},*}} \mathcal{O}^{\otimes}$$

be the wide subcategory on the inert edges. Let $F^{ne}_{\mathcal{O}} : \mathrm{Triv}^{\otimes}_{\mathcal{T}} \longrightarrow \mathbf{Cat}$ *be the functor classifying the cocartesian fibration* $p|_{\mathcal{O}^{\otimes}_{ne}}$. *Then* $F^{ne}_{\mathcal{O}}$ *is the right Kan extension of its restriction* $F_{\mathcal{O}}$ *along* $I_+ : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Triv}^{\otimes}_{\mathcal{T}}$ *(which classifies the underlying* $\mathcal{T}$-∞-*category* $\mathcal{O}$*).*

Proof. Let $f_+ = [U_+ \longrightarrow V]$ be any object in $\mathrm{Triv}^{\otimes}_{\mathcal{T}}$ and let $\mathcal{J}^{\mathrm{op}} = \mathcal{T}^{\mathrm{op}} \times_{\mathrm{Triv}^{\otimes}_{\mathcal{T}}} (\mathrm{Triv}^{\otimes}_{\mathcal{T}})^{f_+/}$. We need to show that the natural map

$$F^{ne}_{\mathcal{O}}(f_+) \longrightarrow \lim(\theta : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}} \xrightarrow{F_{\mathcal{O}}} \mathbf{Cat})$$

is an equivalence. If we view $\mathrm{Orbit}(U)$ as a discrete category, then we have a functor $\phi : \mathrm{Orbit}(U) \longrightarrow \mathcal{J}^{\mathrm{op}}$ that sends W to $(W, \chi_{[W \subset U]})$, and by definition we have an equivalence

$$F^{ne}_{\mathcal{O}}(f_+) \xrightarrow{\simeq} \lim(\theta \circ \phi).$$

Consequently, it suffices to show that ϕ is right cofinal. Since $\mathrm{Triv}^{\otimes}_{\mathcal{T}} \simeq ((\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{si})^{\mathrm{op}}$ under the inclusion into $\underline{\mathbf{F}}_{\mathcal{T},*} = \mathrm{Span}(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}}; (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{si}, (\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{tdeg})$, we may equivalently show that

$$\phi^{\mathrm{op}} : \mathrm{Orbit}(U)^{\mathrm{op}} = \mathrm{Orbit}(U) \longrightarrow \mathcal{J} \simeq \mathcal{T} \times_{(\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{si}} ((\underline{\mathbf{F}}^{\vee}_{\mathcal{T}})^{si})^{/f}$$

is left cofinal. For this, we will apply Quillen's Theorem A. Let

$$\overline{\alpha} = \left(\begin{array}{ccc} X & \xrightarrow{\alpha} & U \\ \downarrow{=} & & \downarrow{f} \\ X & \longrightarrow & V \end{array} \right)$$

be any object in $\mathcal{J}$. Then since $\mathrm{Orbit}(U)$ is discrete, we have an equivalence

$$\mathrm{Orbit}(U) \times_{\mathcal{J}} \mathcal{J}^{\overline{\alpha}/} \simeq \coprod_{W \in \mathrm{Orbit}(U)} \mathrm{Map}_{\mathcal{J}}(\overline{\alpha}, \chi_{[W \subset U]}).$$

If we let W be the orbit in U that α factors through, then for all other $W' \in \mathrm{Orbit}(X)$, we have

$$\mathrm{Map}_{\mathcal{J}}(\overline{\alpha}, \chi_{[W' \subset U]}) = \emptyset.$$

To compute the remaining mapping space, observe that we have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{J}}(\overline{\alpha}, \chi_{[W \subset U]}) & \longrightarrow & \mathrm{Map}_{(\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{si}}(\mathrm{id}_X, \mathrm{id}_W) \simeq \mathrm{Map}_{\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}}}(\mathrm{id}_X, \mathrm{id}_W) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\quad \overline{\alpha} \quad} & \mathrm{Map}_{(\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{si}}(\mathrm{id}_X, f) \simeq \mathrm{Map}_{\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}}}(\mathrm{id}_X, f) \end{array}$$

where for the righthand equivalences we use our assumption that $\mathcal{T}$ is orbital. But the righthand vertical map identifies with

$$\mathrm{Map}_{\mathbf{F}_{\mathcal{T}}}(X, W) \longrightarrow \mathrm{Map}_{\mathbf{F}_{\mathcal{T}}}(X, U) \times_{\mathrm{Map}_{\mathbf{F}_{\mathcal{T}}}(X,V)} \mathrm{Map}_{\mathbf{F}_{\mathcal{T}}}(X, V) \simeq \mathrm{Map}_{\mathbf{F}_{\mathcal{T}}}(X, U)$$

and is hence an equivalence. We conclude that $\mathrm{Orbit}(U) \times_{\mathcal{J}} \mathcal{J}^{\overline{\alpha}/}$ is contractible, so ϕ^{op} is left cofinal. □

2.4.5. **Corollary.** *Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category, let $F_{\mathcal{C}} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ be the functor that classifies $\mathcal{C}$, and let $q : \mathrm{Triv}_{\mathcal{T}}(\mathcal{C})^{\otimes} \longrightarrow \mathrm{Triv}^{\otimes}_{\mathcal{T}}$ be the cocartesian fibration classified by the right Kan extension of $F_{\mathcal{C}}$ along $I_+ : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Triv}^{\otimes}_{\mathcal{T}}$.*[9] *Then $\mathrm{Triv}_{\mathcal{T}}(\mathcal{C})^{\otimes}$ is a $\mathcal{T}$-∞-operad and for any $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, we have an equivalence of $\mathcal{T}$-∞-categories*

$$\underline{\mathbf{Alg}}_{\mathcal{T}}(\mathrm{Triv}_{\mathcal{T}}(\mathcal{C}), \mathcal{O}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{O})$$

implemented by restriction along I_+.

Proof. By Lemma 2.4.4, we have an equivalence of ∞-categories between $\mathrm{Triv}_{\mathcal{T}}$-monoidal $\mathcal{T}$-∞-categories and the full subcategory of $\mathrm{Fun}(\mathrm{Triv}^{\otimes}_{\mathcal{T}}, \mathbf{Cat})$ spanned by those functors right Kan extended from $\mathcal{T}^{\mathrm{op}}$ along I_+, so in particular $\mathrm{Triv}_{\mathcal{T}}(\mathcal{C})^{\otimes}$ is a $\mathcal{T}$-∞-operad. For the second statement, since restriction along I_+ is a $\mathcal{T}$-functor, it suffices to show an equivalence of fibers over every orbit $V \in \mathcal{T}$, so after replacing $\mathcal{T}$ by $\mathcal{T}^{/V}$ we reduce to the claim that

$$\mathbf{Alg}_{\mathcal{T}}(\mathrm{Triv}_{\mathcal{T}}(\mathcal{C}), \mathcal{O}) \simeq \mathrm{Fun}^{cocart}_{/\mathrm{Triv}^{\otimes}_{\mathcal{T}}}(\mathrm{Triv}_{\mathcal{T}}(\mathcal{C})^{\otimes}, \mathcal{O}^{\otimes}_{ne}) \longrightarrow \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{O})$$

is an equivalence of ∞-categories. But this follows immediately from Lemma 2.4.4. □

2.4.6. **Example.** By Corollary 2.4.5, the $\mathcal{T}$-suboperads of the trivial $\mathcal{T}$-∞-operad are in bijective correspondence with sieves of $\mathcal{T}$. For instance, we have that the initial $\mathcal{T}$-∞-operad $\mathrm{Triv}_{\mathcal{T}}(\emptyset)^{\otimes}$ corresponds to $\emptyset \subset \mathcal{T}$ and identifies with the full subcategory of $\underline{\mathbf{F}}_{\mathcal{T},*}$ on objects $[\emptyset_+ \longrightarrow V]$.

2.4.7. **Example** (Indexing systems and the family of commutative $\mathcal{T}$-∞-operads)**.** Clearly $\underline{\mathbf{F}}_{\mathcal{T},*}$ is itself a $\mathcal{T}$-∞-operad, which deserves to be called the *$\mathcal{T}$-commutative $\mathcal{T}$-∞-operad* $\mathrm{Com}^{\otimes}_{\mathcal{T}}$. Note that the identity cocartesian section $I_+ : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Com}^{\otimes}_{\mathcal{T}}$ restricts to an equivalence $\mathcal{T}^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Com}_{\mathcal{T}}$.

In the parametrized setting, we may further define a family of $\mathcal{T}$-suboperads of $\mathrm{Com}^{\otimes}_{\mathcal{T}}$ so as to encode different flavors of parametrized commutativity. First, define the *minimal* $\mathcal{T}$-commutative $\mathcal{T}$-∞-operad $\mathrm{Com}^{\otimes}_{\mathcal{T}\simeq}$ to be the wide subcategory of $\underline{\mathbf{F}}_{\mathcal{T},*}$ containing all morphisms

$$\begin{array}{ccccc} U & \longleftarrow & Z & \xrightarrow{m} & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Y & \xrightarrow{=} & Y. \end{array}$$

where m is a coproduct of fold maps (including possibly empty fold maps). In other words, if we let ∇ denote the collection of fold maps and $(\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{\nabla:tdeg} \subset \underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}}$ the wide subcategory on morphisms with source in ∇ and target degenerate, then

$$\mathrm{Com}^{\otimes}_{\mathcal{T}\simeq} = \mathrm{Span}(\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}}; (\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{si}, (\underline{\mathbf{F}}^{\mathrm{v}}_{\mathcal{T}})^{\nabla:tdeg}),$$

where we use that ∇ is stable under pullback by summand inclusions, and $\mathrm{Com}^{\otimes}_{\mathcal{T}\simeq}$ is classified by the functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ that sends an orbit V to $\mathrm{Span}(\mathbf{F}_{\mathcal{T}/V}; \mathbf{F}^{si}_{\mathcal{T}/V}, \mathbf{F}^{\nabla}_{\mathcal{T}/V})$. Since $\mathrm{Com}^{\otimes}_{\mathcal{T}\simeq}$ contains all inert edges in $\mathrm{Com}^{\otimes}_{\mathcal{T}}$ and $\mathrm{Com}_{\mathcal{T}\simeq} \simeq \mathcal{T}^{\mathrm{op}}$, to verify that $\mathrm{Com}^{\otimes}_{\mathcal{T}\simeq}$ is a $\mathcal{T}$-suboperad it only remains to check condition (3') of Remark 2.1.10. But this condition is satisfied since a map $\alpha : U \longrightarrow U'$ of finite $\mathcal{T}$-sets is a fold map if and only if for all $W \in \mathrm{Orbit}(U')$, the pullback $\alpha_W : U \times_{U'} W \longrightarrow W$ is a fold map.

[9] Using [Sha21a, Ex. 2.26], we could give a definition of $\mathrm{Triv}_{\mathcal{T}}(\mathcal{C})^{\otimes}$ at the level of marked simplicial sets, without passing through straightening and unstraightening.

Now suppose that we want to define a $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$ such that we have $\mathcal{T}$-operadic inclusions

$$\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}} \subset \mathcal{O}^{\otimes} \subset \mathrm{Com}^{\otimes}_{\mathcal{T}}.$$

Since $\mathrm{Com}_{\mathcal{T}^{\simeq}} = \mathrm{Com}_{\mathcal{T}} \simeq \mathcal{T}^{\mathrm{op}}$, the only constraint on $\mathcal{O}^{\otimes}$ to be a $\mathcal{T}$-suboperad arises from condition (3'). In other words, to specify $\mathcal{O}^{\otimes}$ we may as well specify the morphisms $\alpha : U \longrightarrow V$ with V an orbit that we wish to be active. We already have that all fold maps are active, so in particular all summand inclusions are active. Furthermore, as observed in Remark 2.1.11, for any orbit $W \subset U$, the composite map $W \subset U \xrightarrow{\alpha} V$ yields an operadic composition map

$$\mathrm{Mul}_{\mathcal{O}}^{W \subset U} \times \mathrm{Mul}_{\mathcal{O}}^{\alpha} \simeq * \times \mathrm{Mul}_{\mathcal{O}}^{\alpha} \longrightarrow \mathrm{Mul}_{\mathcal{O}}^{\alpha|_W},$$

so if α is active, we must have that $\alpha|_W$ is active for all $W \in \mathrm{Orbit}(U)$. The converse holds by a similar argument since α factors as

$$U \xrightarrow{\simeq} \coprod_{W \in \mathrm{Orbit}(U)} W \xrightarrow{\coprod \alpha|_W} \coprod_{W \in \mathrm{Orbit}(U)} V \xrightarrow{\nabla} V .$$

We thereby reduce to specifying whether or not morphisms $\alpha : W \longrightarrow V$ are active for both W and V orbits. Moreover, by examining Remark 2.1.11 again we see that the only constraints are:

(1) The active morphisms contain all equivalences and are closed under composition, so assemble to a subcategory $\mathcal{I} \subset \mathcal{T}$ such that $\mathcal{I}$ contains the maximal subgroupoid $\mathcal{T}^{\simeq}$ of $\mathcal{T}$.
(2) The active morphisms are closed under base-change, in the sense that for any commutative square

$$\begin{array}{ccc} W' & \longrightarrow & W \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ V' & \longrightarrow & V \end{array}$$

such that the map $W' \longrightarrow V' \times_V W$ is a summand inclusion, if α is in $\mathcal{I}$ then α' is in $\mathcal{I}$.

2.4.8. **Definition.** A $\mathcal{T}$-*indexing system* is a subcategory $\mathcal{I}$ of $\mathcal{T}$ that satisfies the above conditions (1) and (2).

2.4.9. **Remark.** An indexing system $\mathcal{I}$ is the same data as a subcategory $\overline{\mathcal{I}} \subset \mathbf{F}_{\mathcal{T}}$ such that:

(1) $\overline{\mathcal{I}}$ contains the maximal subgroupoid $\mathbf{F}_{\mathcal{T}}^{\simeq}$.
(2) Morphisms in $\overline{\mathcal{I}}$ are closed under base-change and binary coproducts.
(3) $\overline{\mathcal{I}}$ contains all fold maps (and hence all summand inclusions).

Indeed, the assignment $\overline{\mathcal{I}} \rightsquigarrow \mathcal{I} = \overline{\mathcal{I}} \times_{\mathbf{F}_{\mathcal{T}}} \mathcal{T}$ is seen to identify the two notions, with inverse given by taking the finite coproduct completion of $\mathcal{I}$.

2.4.10. **Definition.** Let $\mathcal{I}$ be a $\mathcal{T}$-indexing system and let $(\underline{\mathbf{F}}_{\mathcal{T}}^{\mathrm{v}})^{\mathcal{I}:tdeg} \subset \underline{\mathbf{F}}_{\mathcal{T}}^{\mathrm{v}}$ be the wide subcategory on morphisms with source in $\overline{\mathcal{I}}$ and target degenerate. We then define the $\mathcal{I}$-*commutative* $\mathcal{T}$-∞-*operad* to be

$$\mathrm{Com}^{\otimes}_{\mathcal{I}} := \mathrm{Span}(\underline{\mathbf{F}}_{\mathcal{T}}^{\mathrm{v}}; (\underline{\mathbf{F}}_{\mathcal{T}}^{\mathrm{v}})^{si}, (\underline{\mathbf{F}}_{\mathcal{T}}^{\mathrm{v}})^{\mathcal{I}:tdeg}).$$

More generally, we define a *commutative* $\mathcal{T}$-∞-operad to be any $\mathcal{T}$-suboperad of $\mathrm{Com}^{\otimes}_{\mathcal{T}}$ containing $\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}}$.

The above analysis confirms the following proposition.

2.4.11. **Proposition.** *The assignment $\mathcal{I} \rightsquigarrow \mathrm{Com}^{\otimes}_{\mathcal{I}}$ implements an inclusion-preserving bijection between $\mathcal{T}$-indexing systems and commutative $\mathcal{T}$-∞-operads.*

2.4.12. **Remark.** Let $\mathcal{T} = \mathbf{O}_G$. A G-indexing system I in the sense of Blumberg-Hill [BH15, Def. 3.22] as reformulated by Rubin [Rub21, Def. 2.12] is a collection $\{I(H)\}$ of finite H-sets for every subgroup $H \leq G$ such that $I(H)$ contains all finite H-sets with trivial H-action and satisfies the following closure properties:

(1) If $U \in I(H)$ and $U' \cong U$, then $U' \in I(H)$.
(2) For every subgroup $K \leq H$, if $U \in I(H)$ then $\mathrm{res}^H_K U \in I(K)$.
(3) For every conjugate $H' = gHg^{-1}$ of H, if $U \in I(H)$ then its conjugate $U' \in I(H')$.
(4) If $U \in I(H)$ and $U' \subset U$, then $U' \in I(H)$.
(5) If $U, U' \in I(H)$, then $U \coprod U' \in I(H)$.
(6) For any subgroup $K \leq H$, if $U \in I(K)$ and $H/K \in I(H)$ then $\mathrm{ind}^H_K U \in I(H)$.

In [BH15, Thm. 3.17], it was shown that G-indexing systems are in bijective correspondence with subcategories of $\mathbf{F}_G$ satisfying the conditions of Remark 2.4.9 (and thus with the $\mathbf{O}_G$-indexing systems of Definition 2.4.8). For the convenience of the reader, we review this correspondence. Note that under the equivalences $\mathbf{F}_H \simeq \mathbf{F}_G^{/(G/H)}$, given a subgroup $K \leq H$, induction corresponds to postcomposition by $G/K \longrightarrow G/H$. Therefore, given a G-indexing system I, we may define a wide subcategory $\mathcal{I} \subset \mathbf{O}_G$ to be the subcategory whose morphisms $f : V \longrightarrow W \cong G/H$ are such that $f \in I(H)$. The enumerated conditions then imply that $\mathcal{I}$ is a $\mathbf{O}_G$-indexing system, using closure under restriction and inclusion to validate the base-change condition.

Conversely, suppose $\mathcal{I}$ is a $\mathbf{O}_G$-indexing system and let $\overline{\mathcal{I}} \subset \mathbf{F}_G$ be the subcategory generated by $\mathcal{I}$ as in Remark 2.4.9. Let $I(H)$ be the subset of objects of $\mathbf{F}_H$ given by morphisms in $\overline{\mathcal{I}}$ with target G/H under $\mathbf{F}_H \simeq \mathbf{F}_G^{/(G/H)}$. Then one sees that I is a G-indexing system and these assignments are mutually inverse – note that condition (5) holds since given $U, U' \longrightarrow G/H$, the coproduct in $\mathbf{F}_H$ is given by the composition $U \coprod U' \longrightarrow G/H \coprod G/H \longrightarrow G/H$.

Consequently, by the work of Bonventre–Pereira [BP21], Gutiérrez–White [GW18], and Rubin [Rub21], we see that the commutative G-∞-operads are in bijection with the N_∞-operads of Blumberg–Hill.

A straightforward adaptation of the proof of Theorem 2.3.9 shows the following.

2.4.13. **Definition.** Given a $\mathcal{T}$-indexing system $\mathcal{I}$, let $\mathbf{Cat}_{\mathcal{I}}^{\otimes}$ be the ∞-category of $\mathcal{I}$-symmetric monoidal $\mathcal{T}$-∞-categories and $\mathcal{I}$-symmetric monoidal $\mathcal{T}$-functors thereof.

2.4.14. **Theorem.** *Let $\mathcal{I}$ be a $\mathcal{T}$-indexing system. We then have a canonical equivalence*

$$\mathbf{Cat}_{\mathcal{I}}^{\otimes} \simeq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbf{F}_{\mathcal{T}}; \mathbf{F}_{\mathcal{T}}, \overline{\mathcal{I}}), \mathbf{Cat}).$$

2.4.15. **Corollary.** *For the minimal indexing system $\mathcal{I} = \mathcal{T}^{\simeq}$, we have a canonical identification of $\mathcal{T}^{\simeq}$-symmetric monoidal $\mathcal{T}$-∞-categories with $\mathcal{T}^{\mathrm{op}}$-cocartesian families of symmetric monoidal ∞-categories (*[Lur17, Def. 4.8.3.1]*).*

2.5. **T-operadic nerve.** In this subsection, we suppose that $\mathcal{T}$ is equivalent to the nerve of a 1-category T. For example, by the following proposition we could take $\mathcal{T}$ to be any atomic orbital ∞-category that admits a final object.

2.5.1. **Proposition.** *Suppose $\mathcal{T}$ is an atomic orbital ∞-category that admits a final object $*$. Then $\mathcal{T}$ is equivalent to the nerve of a 1-category.*

Proof. By [Lur09, Prop. 2.3.4.18] it suffices to show that the mapping spaces of $\mathcal{T}$ are 0-truncated, or equivalently that the essentially unique maps $V \longrightarrow *$ are 0-truncated for all $V \in \mathcal{T}$. By [Lur09, Lem. 5.5.6.15], this occurs if and only if the diagonal $\delta : V \longrightarrow V \times V$ in $\mathbf{F}_{\mathcal{T}}$ is (-1)-truncated. But since $\mathcal{T}$ is atomic and δ is split by either projection to V, it follows that δ is a summand inclusion and hence a monomorphism. □

Correspondingly, let $\mathbf{F}_T$ denote the subcategory of the category of **Set**-valued presheaves on T spanned by the finite coproducts of representables, so that $\mathbf{F}_{\mathcal{T}} \simeq N(\mathbf{F}_T)$.

2.5.2. **Remark.** If T is a 1-category, then the *a priori* $(2,1)$-category

$$\underline{\mathbf{F}}_{T,*} := \mathrm{Span}(\mathbf{F}_T^{\Delta^1} \times_{\mathbf{F}_T} T, (\mathbf{F}_T^{\Delta^1})^{tdeg}, (\mathbf{F}_T^{\Delta^1})^{si})$$

is enriched in setoids and therefore equivalent to a 1-category. Indeed, the only automorphisms of spans of the form

$$\begin{array}{ccccc} U & \longleftarrow & \tilde{U} & \longrightarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & V' & = & V' \end{array}$$

where the left square is in $(\mathbf{F}_T^{\Delta^1})^{si}$ are identities. In what follows, we will implicitly make a choice of 1-categorical model for $\underline{\mathbf{F}}_{T,*}$ by picking a representative for every equivalence class of morphisms.

Our goal is to indicate how to prescribe the data of a $\mathcal{T}$-∞-operad in terms of the stricter data of a *simplicial colored T-operad*, which will be defined along the lines suggested by Remark 2.1.11. To concisely state its definition, we first need to introduce some notation.

2.5.3. **Notation.** Let $A\mathbf{F}_T \subseteq \mathrm{Ar}(\mathbf{F}_T)$ denote the wide subcategory of the arrow category whose morphisms are cartesian squares. Suppose that $\mathrm{ob}O \longrightarrow \mathbf{F}_T$ is a Grothendieck fibration fibered in sets, and let $\mathrm{ob}O(U)$ denote the fiber of $\mathrm{ob}O$ over $U \in \mathbf{F}_T$. We then write

$$A^O\mathbf{F}_T \coloneqq A\mathbf{F}_T \times_{(\mathbf{F}_T\times\mathbf{F}_T)} (\mathrm{ob}O \times \mathrm{ob}O)$$

for the category whose objects are triples $(f : U \longrightarrow V,\, x \in \mathrm{ob}O(U),\, y \in \mathrm{ob}O(V))$ and whose morphisms are cartesian squares

$$\begin{array}{ccc} U' & \xrightarrow{\phi} & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ V' & \xrightarrow{\psi} & V \end{array}$$

such that $x' = \phi^*x$ and $y' = \psi^*y$. We have functors

$$1 : \mathrm{ob}O \longrightarrow A^O\mathbf{F}_T, \quad (x \in \mathrm{ob}O(U)) \mapsto (\mathrm{id}_U, x, x),$$

$$\circ : A^O\mathbf{F}_T \times_{\mathrm{ob}O} A^O\mathbf{F}_T \longrightarrow A^O\mathbf{F}_T, \quad (f : U \longrightarrow V, g : V \longrightarrow W, x, y, z) \mapsto (gf : U \longrightarrow W, x, z).$$

2.5.4. **Definition.** A (fibrant) *simplicial colored T-operad O* is the data of

(1) A 'T-set of colors', given as a Grothendieck fibration fibered in sets

$$\mathrm{ob}O \longrightarrow \mathbf{F}_T$$

classified by a functor $\mathbf{F}_T^{\mathrm{op}} \longrightarrow \mathbf{Set}$ preserving finite products.

(2) A collection of spaces of multimorphisms, packaged into a functor

$$\mathrm{Mul}_O : (A^O\mathbf{F}_T)^{\mathrm{op}} \longrightarrow \mathbf{sSet}, \quad (f : U \longrightarrow V,\, x \in \mathrm{ob}O(U),\, y \in \mathrm{ob}O(V)) \mapsto \mathrm{Mul}_O^f(x,y)$$

that preserves finite products and is valued in Kan complexes.[10]

(3) A distinguished 'identity' for Mul_O, given by a natural transformation

$$1 : * \longrightarrow \mathrm{Mul}_O^{\mathrm{id}_U}(x,x)$$

of functors $(\mathrm{ob}O)^{\mathrm{op}} \longrightarrow \mathbf{sSet}$, where the right hand side is the composition

$$(\mathrm{ob}O)^{\mathrm{op}} \xrightarrow{1} A^O\mathbf{F}_T^{\mathrm{op}} \xrightarrow{\mathrm{Mul}} \mathbf{sSet}\,.$$

(4) A 'composition law' for Mul_O, given by a natural transformation

$$\circ : \mathrm{Mul}_O^f(x,y) \times \mathrm{Mul}_O^g(y,z) \longrightarrow \mathrm{Mul}_O^{gf}(x,z)$$

of functors $(A^O\mathbf{F}_T \times_{\mathrm{ob}O} A^O\mathbf{F}_T)^{\mathrm{op}} \longrightarrow \mathbf{sSet}$, where the left hand side is the composition

$$(A^O\mathbf{F}_T \times_{\mathrm{ob}O} A^O\mathbf{F}_T)^{\mathrm{op}} \longrightarrow (A^O\mathbf{F}_T \times A^O\mathbf{F}_T)^{\mathrm{op}} \xrightarrow{\mathrm{Mul}\times\mathrm{Mul}} \mathbf{sSet} \times \mathbf{sSet} \xrightarrow{\times} \mathbf{sSet}\,,$$

and the right hand side is the composition

$$(A^O\mathbf{F}_T \times_{\mathrm{ob}O} A^O\mathbf{F}_T)^{\mathrm{op}} \xrightarrow{\circ} A^O\mathbf{F}_T^{\mathrm{op}} \xrightarrow{\mathrm{Mul}} \mathbf{sSet}\,.$$

These data are required to satisfy the following compatibilities:

- **Unitality**: The compositions

$$\mathrm{Mul}_O^f(x,y) \xrightarrow{(\mathrm{id},1_y)} \mathrm{Mul}_O^f(x,y) \times \mathrm{Mul}_O^{\mathrm{id}_V}(y,y) \xrightarrow{\circ} \mathrm{Mul}_O^f(x,y)$$

and

$$\mathrm{Mul}_O^f(x,y) \xrightarrow{(1_x,\mathrm{id})} \mathrm{Mul}_O^{\mathrm{id}_U}(x,x) \times \mathrm{Mul}_O^f(x,y) \xrightarrow{\circ} \mathrm{Mul}_O^f(x,y)$$

are the identity natural transformation.

- **Associativity** The following diagram is commutative

$$\begin{array}{ccc} \mathrm{Mul}_O^f(x,y) \times \mathrm{Mul}_O^g(y,z) \times \mathrm{Mul}_O^h(z,t) & \xrightarrow{(\circ,\mathrm{id})} & \mathrm{Mul}_O^{gf}(x,z) \times \mathrm{Mul}_O^h(z,t) \\ \downarrow{\scriptstyle (\mathrm{id},\circ)} & & \downarrow{\scriptstyle \circ} \\ \mathrm{Mul}_O^f(x,y) \times \mathrm{Mul}_O^{hg}(y,t) & \xrightarrow{\circ} & \mathrm{Mul}_O^{hgf}(x,t) \end{array}.$$

[10] If V is an orbit, we thus obtain a $T^{/V}$-space $\underline{\mathrm{Mul}}_O^f(x,y) : (T^{/V})^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ by precomposing Mul_O with the functor $T^{/V} \longrightarrow A^O\mathbf{F}_T$ over $\mathbf{F}_T$ determined by (f,x,y) – indeed, one has an equivalence $(A^O\mathbf{F}_T)^{/(f,x,y)} \simeq (\mathbf{F}_T)^{/V}$.

2.5.5. **Construction.** From the data of Definition 2.5.4 we will build a simplicial category $O^{\otimes}$ over $\underline{\mathbf{F}}_{T,*}$ as follows. We let the objects of $O^{\otimes}$ be the pairs $([U_+ \longrightarrow V], x)$, where $[U_+ \longrightarrow V]$ is an object of $\underline{\mathbf{F}}_{T,*}$ and $x \in \mathrm{ob}O(U)$, and we define as mapping simplicial sets

$$\mathrm{Map}_{O^{\otimes}}(([U_+ \longrightarrow V], x), ([U'_+ \longrightarrow V'], x')) \coloneqq \coprod_{U \overset{i}{\longleftarrow} U_0 \overset{f}{\longrightarrow} U'} \mathrm{Mul}^f(i^*x, x')$$

where the coproduct is indexed by the set of all maps $[U_+ \longrightarrow V] \longrightarrow [U'_+ \longrightarrow V']$ in $\underline{\mathbf{F}}_{T,*}$. The identity of $([U_+ \longrightarrow V], x)$ is given by $1_{(U,x)} \in \mathrm{Mul}^{\mathrm{id}_U}(x, x)$. If

$$\begin{array}{ccccccc} & & & U_1 & & & \\ & & \overset{j'}{\swarrow} & & \overset{f'}{\searrow} & & \\ & U_0 & & & & U_0' & \\ \overset{i}{\swarrow} & & \overset{f}{\searrow} & & \overset{j}{\swarrow} & & \overset{g}{\searrow} \\ U & & & U' & & & U'' \end{array}$$

is a diagram representing a composition in $\underline{\mathbf{F}}_{T,*}$, composition over it is given by

$$\mathrm{Mul}^f(i^*x, x') \times \mathrm{Mul}^g(j^*x', x'') \longrightarrow \mathrm{Mul}^{f'}((j')^*i^*x, j^*x') \times \mathrm{Mul}^g(j^*x', x'') \longrightarrow \mathrm{Mul}^{gf'}((ij')^*x, x'').$$

Verifying that this satisfies associativity and unitality is left as an exercise for the reader.

2.5.6. **Proposition.** *The map $N(O^{\otimes}) \longrightarrow N(\underline{\mathbf{F}}_{T,*}) \simeq \underline{\mathbf{F}}_{\mathcal{T},*}$ is a $\mathcal{T}$-∞-operad.*

Proof. Using [Lur09, Prop. 2.4.1.10] we see that the above map is an inner fibration and that its restriction over the subcategory of inert edges is a cocartesian fibration. Then remaining properties are true because $\mathrm{ob}O$ and Mul preserve finite products. ☐

2.6. **Model structures.** In this subsection, we introduce a model structure for $\mathcal{T}$-∞-operads by means of Lurie's theory of categorical patterns ([Lur17, Def. B.0.19]). For a $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, let $\mathrm{Ne} \subset (\mathcal{O}^{\otimes})_1$ denote the subset of inert morphisms.

2.6.1. **Definition.** We define categorical patterns $\mathfrak{P}_{\mathcal{T}}$ and $\mathfrak{P}_{\mathcal{T}}^{\otimes}$ on $\underline{\mathbf{F}}_{\mathcal{T},*}$ as follows. For each collection of morphisms $\underline{\alpha} = \{\alpha_i : U_i \longrightarrow V\}_{i=1}^n$ in $\mathcal{T}$, let $\alpha : U = \coprod_{i=1}^n U_i \xrightarrow{(\alpha_i)} V$ and define a morphism

$$f_\alpha : (n^{\lhd})^{\sharp} \longrightarrow (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})$$

(for $n = \{1, ..., n\}$ regarded as a discrete category) that sends the cone point v to $[U_+ \to V]$, $i \in n$ to $[U_{i+} \to U_i]$, and $v \longrightarrow i$ to the characteristic morphism $\chi_{[U_i \subset U]}$. Then let A be the set of the $\underline{\alpha}$ and let

$$\mathfrak{P}_{\mathcal{T}} = (\mathrm{Ne}, \mathcal{A}ll, \{f_\alpha : n^{\lhd} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}\}_{\underline{\alpha} \in A}),$$
$$\mathfrak{P}_{\mathcal{T}}^{\otimes} = (\mathcal{A}ll, \mathcal{A}ll, \{f_\alpha : n^{\lhd} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}\}_{\underline{\alpha} \in A}).$$

Furthermore, for any $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, we define categorical patterns $\mathfrak{P}_{\mathcal{O}}$ and $\mathfrak{P}_{\mathcal{O}}^{\otimes}$ on $\mathcal{O}^{\otimes}$ by the construction of [Lur17, Rem. B.1.5]. In other words, let B denote the set of pairs $(\underline{\alpha}, \overline{f_\alpha})$ where $\overline{f_\alpha} : (n^{\lhd})^{\sharp} \longrightarrow (\mathcal{O}^{\otimes}, \mathrm{Ne})$ is any lift of f_α, and let

$$\mathfrak{P}_{\mathcal{O}} = (\mathrm{Ne}, \mathcal{A}ll, \{\overline{f_\alpha} : n^{\lhd} \longrightarrow \mathcal{O}^{\otimes}\}_{(\underline{\alpha}, \overline{f_\alpha}) \in B}),$$
$$\mathfrak{P}_{\mathcal{O}}^{\otimes} = (\mathrm{Ne}, \mathcal{A}ll, \{\overline{f_\alpha} : n^{\lhd} \longrightarrow \mathcal{O}^{\otimes}\}_{(\underline{\alpha}, \overline{f_\alpha}) \in B}).$$

2.6.2. **Theorem-Construction.** The $\mathcal{T}$-*operadic model structure* on the category $\mathbf{sSet}^+_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})}$ is that defined by the categorical pattern $\mathfrak{P}_{\mathcal{T}}$ of Definition 2.6.1 according to [Lur17, Thm. B.0.20]. The $\mathcal{T}$-operadic model structure is left proper, combinatorial, simplicial, and has the following properties:

(1) The cofibrations are precisely the monomorphisms.
(2) A marked map $X \longrightarrow Y$ over $(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})$ is a weak equivalence if for any $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, the induced map
$$\mathrm{Map}_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})}(Y, (\mathcal{O}^{\otimes}, \mathrm{Ne})) \longrightarrow \mathrm{Map}_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})}(X, (\mathcal{O}^{\otimes}, \mathrm{Ne}))$$
is a weak equivalence.

(3) An object is fibrant if it is of the form $(\mathcal{O}^{\otimes}, \mathrm{Ne})$ for some $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$.
(4) The fibrations between fibrant objects in $\mathbf{sSet}^+_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})}$ are exactly given by the fibrations of $\mathcal{T}$-∞-operads.

Furthermore, for any $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, we define the $\mathcal{T}$-*operadic model structure* on the category $\mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})}$ via the categorical pattern $\mathfrak{P}_{\mathcal{O}}$, and this coincides with the model structure induced from the $\mathcal{T}$-operadic model structure on $\mathbf{sSet}^+_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})}$ by slicing over $(\mathcal{O}^{\otimes}, \mathrm{Ne})$.

Finally, we also define the $\mathcal{T}$-*monoidal model structure* on the category $\mathbf{sSet}^+_{/\mathcal{O}^{\otimes}}$ via the categorical pattern $\mathfrak{P}^{\otimes}_{\mathcal{O}}$. This construction has the same formal properties as with the $\mathcal{T}$-operadic model structure, but where the fibrant objects are precisely the $\mathcal{O}$-monoidal $\mathcal{T}$-∞-categories with the cocartesian edges marked.

Proof. The construction of the $\mathcal{T}$-operadic model structure on $\mathbf{sSet}^+_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})}$ and the first three claims about it follows immediately from [Lur17, Thm. B.0.20] and the definition of a $\mathcal{T}$-∞-operad. The fourth claim and the assertion about the model structure on $\mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})}$ follow from [Lur17, Prop. B.2.7]. Finally, the analogous assertions about the $\mathcal{T}$-monoidal model structure all follow in the same way (cf. [Lur17, Var. 2.1.4.13]). □

2.6.3. **Definition.** We define the ∞-*category of (small)* $\mathcal{T}$-∞-*operads*

$$\mathbf{Op}_{\mathcal{T}} := N((\mathbf{sSet}^+_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})})^f)$$

to be the simplicial nerve of the full simplicial subcategory of $\mathbf{sSet}^+_{/(\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne})}$ spanned by the fibrant objects in the $\mathcal{T}$-operadic model structure. We further let $\mathbf{Cat}^{\otimes}_{\mathcal{T}}$ denote the subcategory of $\mathbf{Op}_{\mathcal{T}}$ spanned by the (small) $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-categories and $\mathcal{T}$-symmetric monoidal functors thereof, or equivalently, the simplicial nerve $N((\mathbf{sSet}^+_{/\underline{\mathbf{F}}_{\mathcal{T},*}})^f)$ taken with respect to the $\mathcal{T}$-monoidal model structure.

For a small $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, we then let

$$\mathbf{Op}_{\mathcal{O},\mathcal{T}} := N((\mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})})^f), \quad \mathbf{Cat}^{\otimes}_{\mathcal{O},\mathcal{T}} := N((\mathbf{sSet}^+_{/\mathcal{O}^{\otimes}})^f).$$

Note that $\mathbf{Op}_{\mathcal{O},\mathcal{T}} \simeq (\mathbf{Op}_{\mathcal{T}})^{/\mathcal{O}^{\otimes}}$ and $\mathbf{Cat}^{\otimes}_{\mathcal{O},\mathcal{T}}$ includes as the subcategory of $\mathbf{Op}_{\mathcal{O},\mathcal{T}}$ on the $\mathcal{O}$-monoidal $\mathcal{T}$-∞-categories and morphisms thereof.

2.6.4. **Corollary.** *For any small* $\mathcal{T}$-∞-*operad* $\mathcal{O}^{\otimes}$*, the* ∞-*categories* $\mathbf{Op}_{\mathcal{O},\mathcal{T}}$ *and* $\mathbf{Cat}^{\otimes}_{\mathcal{O},\mathcal{T}}$ *are presentable.*

Proof. Since the $\mathcal{T}$-operadic model structure on $\mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})}$ and the $\mathcal{T}$-monoidal model structure on $\mathbf{sSet}^+_{/\mathcal{O}^{\otimes}}$ are combinatorial and simplicial by Theorem-Construction 2.6.2, it follows from [Lur09, Prop. A.3.7.6] that $\mathbf{Op}_{\mathcal{O},\mathcal{T}}$ and $\mathbf{Cat}^{\otimes}_{\mathcal{O},\mathcal{T}}$ are presentable. □

Using the theory of categorical patterns, it is easy to construct *cotensors* in the ∞-category of $\mathcal{T}$-∞-operads fibered over a given base.

2.6.5. **Construction.** Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad and let K be a marked simplicial set. By [Lur17, Prop. B.1.9] applied to the trivial categorical pattern on $\mathbf{sSet}^+$ and $\mathcal{P}_{\mathcal{O}}$ on $\mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})}$, the functor

$$(- \times K) : \mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})} \longrightarrow \mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})}, \quad A \mapsto A \times K$$

is left Quillen. We denote its right adjoint on fibrant objects by $(\mathcal{C}^{\otimes}, p) \mapsto ((\mathcal{C}^{\otimes}, p)^K, p^K)$ and the underlying $\mathcal{T}$-∞-category by $(\mathcal{C}, p)^K$. Since this adjunction is also simplicial, for $(\mathcal{C}^{\otimes}, p)$ and $(\mathcal{D}^{\otimes}, q)$ fibrations of $\mathcal{T}$-∞-operads over $\mathcal{O}^{\otimes}$ we obtain equivalences of ∞-categories

$$\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, (\mathcal{D}, q)^K) \simeq \mathrm{Fun}_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})}(K \times (\mathcal{C}^{\otimes}, \mathrm{Ne}), (\mathcal{D}^{\otimes}, \mathrm{Ne})) \simeq \mathrm{Fun}(K, \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})^{\sim})$$

where $(-)^{\sim}$ means we take the marking given by the equivalences. In other words, we have constructed the cotensor of $\mathbf{Op}_{\mathcal{O},\mathcal{T}}$ over $\mathbf{Cat}$ at the level of marked simplicial sets. Note also that a fibrant replacement of $K \times (\mathcal{C}^{\otimes}, \mathrm{Ne})$ computes the tensor. Repeating this analysis with the categorical pattern $\mathcal{P}^{\otimes}_{\mathcal{O}}$, we see that if $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are $\mathcal{O}$-monoidal, then $(\mathcal{D}^{\otimes}, p)^K$ is moreover $\mathcal{O}$-monoidal, and we have equivalences of ∞-categories

$$\mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, (\mathcal{D}, q)^K) \simeq \mathrm{Fun}_{/\mathcal{O}^{\otimes}}(K \times (\mathcal{C}^{\otimes})^{\sharp}, (\mathcal{D}^{\otimes})^{\sharp}) \simeq \mathrm{Fun}(K, \mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})^{\sim}).$$

Moreover, note that if $F : \mathcal{O} \longrightarrow \mathbf{Cat}$ denotes the functor classifying $\mathcal{D} \longrightarrow \mathcal{O}$, then $(\mathcal{D}, q)^K \longrightarrow \mathcal{O}$ is classified by the functor $\mathcal{O} \longrightarrow \mathbf{Cat}$ given by applying $\mathrm{Fun}(K, (-)^{\sim})$ fiberwise to F. We may thus consider $(\mathcal{D}^{\otimes}, p)^K$ to be a construction of the *pointwise* $\mathcal{O}$-monoidal structure on $(\mathcal{D}, q)^K$.

2.7. **Big $\mathcal{T}$-∞-operads.** For certain arguments, it is technically inconvenient that the base $\mathcal{T}$ does not admit pullbacks. We will thus need to consider an equivalent definition of a $\mathcal{T}$-∞-operad.

2.7.1. **Definition.** Let $\mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{tdeg}, \mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{si}$ denote the wide subcategories of $\mathrm{Ar}(\mathbf{F}_{\mathcal{T}})$ on morphisms

$$\sigma = \begin{pmatrix} U & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ V & \longrightarrow & Y \end{pmatrix}$$

such that $\mathrm{ev}_1(\sigma) : V \longrightarrow Y$ is a degenerate edge, resp. the induced morphism $U \longrightarrow V \times_Y X$ is a summand inclusion. Then $(\mathrm{Ar}(\mathbf{F}_{\mathcal{T}}); \mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{si}, \mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{tdeg})$ is a disjunctive triple, and we define

$$\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*} := \mathrm{Span}(\mathrm{Ar}(\mathbf{F}_{\mathcal{T}}); \mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{si}, \mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{tdeg}).$$

Evaluation at the target defines a structure map $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*} \longrightarrow \mathbf{F}^{\mathrm{op}}_{\mathcal{T}}$, which is a cocartesian fibration.

2.7.2. **Definition.** We define a categorical pattern $\widetilde{\mathfrak{P}}_{\mathcal{T}}$ on $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ as follows. Let $\phi : U \longrightarrow V$ be a morphism in $\mathbf{F}_{\mathcal{T}}$ and $\Sigma = \{\sigma_1, ..., \sigma_n\}$ be a collection of commutative squares in $\mathbf{F}_{\mathcal{T}}$

$$\sigma_i = \begin{pmatrix} U_i & \xrightarrow{\alpha_i} & U \\ \big\downarrow{\scriptstyle \phi_i} & & \big\downarrow{\scriptstyle \phi} \\ V_i & \xrightarrow{\beta_i} & V \end{pmatrix}$$

such that α_i is a summand inclusion and the induced map $U_i \longrightarrow V_i \times_V U$ is also a summand inclusion. Let $\chi_{\sigma_i} : \Delta^1 \longrightarrow (\mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{si})^{\mathrm{op}} \subset \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ be the morphism corresponding to σ_i.

Suppose moreover that the summand inclusions α_i combine to yield an equivalence $\coprod_{1 \leq i \leq n} U_i \simeq U$. Let

$$f_{\phi,\Sigma} : n^{\lhd} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$$

denote the functor which selects the n morphisms χ_{σ_1}, ..., χ_{σ_n}. We then let

$$\widetilde{\mathfrak{P}}_{\mathcal{T}} = (\mathrm{Ne}, \mathcal{A}ll, \{f_{\phi,\Sigma}\})$$

where ϕ and Σ range over all possible choices.

2.7.3. **Definition.** The $\mathcal{T}$-*operadic model structure* on the category $\mathbf{sSet}^+_{/(\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}, \mathrm{Ne})}$ is that defined by the categorical pattern $\widetilde{\mathfrak{P}}_{\mathcal{T}}$ of Definition 2.7.2 according to [Lur17, Thm. B.0.20]. We call the fibrant objects *big* $\mathcal{T}$-∞-*operads*.

For any big $\mathcal{T}$-∞-operad $\widetilde{\mathcal{O}}^{\otimes} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$, we then define the $\mathcal{T}$-*operadic model structure* on $\mathbf{sSet}^+_{/(\widetilde{\mathcal{O}}^{\otimes}, \mathrm{Ne})}$ via the categorical pattern

$$\widetilde{\mathfrak{P}}_{\mathcal{O}} = (\mathrm{Ne}, \mathcal{A}ll, \{f_{x,\phi,\Sigma}\}),$$

where the $f_{x,\phi,\Sigma} : n^{\lhd} \longrightarrow \widetilde{\mathcal{O}}^{\otimes}$ range over all cocartesian sections of the $f_{\phi,\Sigma}$, and the x in the notation denotes the value of $f_{x,\phi,\Sigma}$ on the cone point v.

Given a big $\mathcal{T}$-∞-operad $\widetilde{\mathcal{O}}^{\otimes} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*} \longrightarrow \mathbf{F}^{\mathrm{op}}_{\mathcal{T}}$, we will let $\mathcal{O}^{\otimes} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \mathcal{T}^{\mathrm{op}}$ denote its pullback along the inclusion $\mathcal{T}^{\mathrm{op}} \subset \mathbf{F}^{\mathrm{op}}_{\mathcal{T}}$. Clearly, $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-∞-operad.

2.7.4. **Proposition.** *Let $\widetilde{\mathcal{O}}^{\otimes}$ be a big $\mathcal{T}$-∞-operad over $\underline{\mathbf{F}}^{big}_{\mathcal{T},*}$ and consider the span*

$$(\widetilde{\mathcal{O}}^{\otimes}, \mathrm{Ne}) \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes}) \times_{\widetilde{\mathcal{O}}^{\otimes}} O^{\otimes}, \mathrm{Ne}) \xrightarrow{\mathrm{pr}_{\mathcal{O}^{\otimes}}} (\mathcal{O}^{\otimes}, \mathrm{Ne}).$$

Then the adjunction

$$(\mathrm{pr}_{\mathcal{O}^{\otimes}})_!(\mathrm{ev}_0)^* : \mathbf{sSet}^+_{/(\widetilde{\mathcal{O}}^{\otimes}, \mathrm{Ne})} \rightleftarrows \mathbf{sSet}^+_{/(\mathcal{O}^{\otimes}, \mathrm{Ne})} : (\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{O}^{\otimes}})^*$$

is a Quillen equivalence.

Proof. We first show that $(\mathrm{pr}_{\mathcal{O}^\otimes})_!(\mathrm{ev}_0)^* \dashv (\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{O}^\otimes})^*$ is a Quillen adjunction. For this, it suffices to show that $(\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{O}^\otimes})^*$ preserves fibrant objects. Examining the proof of [Sha21b, Prop. 3.5(1)], we see that it implies

$$\mathrm{ev}_0 : \mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^\otimes) \times_{\widetilde{\mathcal{O}}^\otimes} \mathcal{O}^\otimes \longrightarrow \widetilde{\mathcal{O}}^\otimes$$

is a cartesian fibration, because any fiberwise active edge with target in $\mathcal{O}^\otimes$ necessarily has source in $\mathcal{O}^\otimes$. Therefore, the hypotheses of [Lur17, Thm. B.4.2] are satisfied *excluding those involving the maps* $f_{x,\phi,\Sigma}$. We deduce that $(\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{O}^\otimes})^*$ sends fibrant objects to fibrations over $\widetilde{\mathcal{O}}^\otimes$ which are cocartesian over the inert edges. Given a $\mathcal{T}$-∞-operad $\mathcal{C}^\otimes$, let $\widetilde{\mathcal{C}}^\otimes = (\mathrm{ev}_0)_*(\mathrm{pr}_{\mathcal{O}^\otimes})^*(\mathcal{C}^\otimes)$. It remains to show that for every $f_{x,\phi,\Sigma} : n^\lhd \longrightarrow \widetilde{\mathcal{O}}^\otimes$,

(i) the functor $n^\lhd \longrightarrow \mathbf{Cat}$ classifying the cocartesian fibration $n^\lhd \times_{\widetilde{\mathcal{O}}^\otimes} \widetilde{\mathcal{C}}^\otimes$ is a limit diagram.
(ii) For every cocartesian section $n^\lhd \longrightarrow n^\lhd \times_{\widetilde{\mathcal{O}}^\otimes} \widetilde{\mathcal{C}}^\otimes$, the composite $n^\lhd \longrightarrow \widetilde{\mathcal{C}}^\otimes$ is a f-limit diagram for $f : \widetilde{\mathcal{C}}^\otimes \longrightarrow \widetilde{\mathcal{O}}^\otimes$.

In fact, we only need to consider $f_{x,\phi,\Sigma}$ where Σ is given by squares

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \phi} \\ U_i & \longrightarrow & V \end{array}$$

with U_i an orbit, so we will suppose this in the remainder of the argument.

For (i), recall from [Sha21a, Ex. 2.26] that the right Kan extension of a functor $\mathcal{C} \longrightarrow \mathbf{Cat}$ along $\mathcal{C} \longrightarrow \mathcal{D}$ is modeled at the level of cocartesian fibrations by the push-pull construction involving the span

$$\mathcal{D} \longleftarrow \mathrm{Ar}(\mathcal{D}) \times_{\mathcal{D}} \mathcal{C} \longrightarrow \mathcal{C}.$$

Therefore, $\widetilde{\mathcal{C}}^\otimes_{ne}$ is the right Kan extension of $\mathcal{C}^\otimes_{ne}$ along $\mathcal{O}^\otimes_{ne} \subset \widetilde{\mathcal{O}}^\otimes_{ne}$. In particular, given $x \in \widetilde{\mathcal{O}}^\otimes$ over $[f_+ : U_+ \to V] \in \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ and an orbit decomposition $V \simeq V_1 \coprod ... \coprod V_m$, let $x \longrightarrow x_j$ be inert morphisms lifting the cocartesian morphisms $[U_+ \to V] \longrightarrow [(U \times_V V_j)_+ \times V_j]$. Then we have an equivalence $\widetilde{\mathcal{C}}^\otimes_x \simeq \prod_{1 \leq j \leq m} \mathcal{C}^\otimes_{x_j}$, and postcomposing with the further decompositions of $\mathcal{C}^\otimes_{x_j}$ given by orbit decompositions of $U \times_V V_j$ verifies (i).

For (ii), it suffices to prove that for every fiberwise active edge $\alpha : x \longrightarrow y \in \widetilde{\mathcal{O}}^\otimes$ over $[U'_+ \to V] \longrightarrow [U_+ \to V]$, objects $\overline{x}, \overline{y} \in \widetilde{\mathcal{C}}^\otimes$ over x, y, and identification $\overline{y} \simeq (\overline{y}_1, ..., \overline{y}_n)$ induced by $f_{y,\phi,\Sigma}$,

$$\mathrm{Map}^\alpha_{\widetilde{\mathcal{C}}^\otimes}(\overline{x}, \overline{y}) \simeq \prod_{1 \leq i \leq n} \mathrm{Map}^{\alpha_i}_{\widetilde{\mathcal{C}}^\otimes}(\overline{x}, \overline{y}_i).$$

where α_i is the composition $x \longrightarrow y \longrightarrow y_i$.

However, for an orbit decomposition $V \simeq V_1 \coprod ... \coprod V_m$ and corresponding inert morphisms $\overline{x} \longrightarrow \overline{x}_j$, $\overline{y} \longrightarrow \overline{y}_j$, we have that

$$\mathrm{Map}^\alpha_{\widetilde{\mathcal{C}}^\otimes}(\overline{x}, \overline{y}) \simeq \prod_{1 \leq j \leq m} \mathrm{Map}^{\alpha^j}_{C^\otimes}(\overline{x}_j, \overline{y}_j)$$

where $\alpha \simeq (\alpha^j : x_j \longrightarrow y_j)$ under the same decomposition for mapping spaces in $\widetilde{\mathcal{O}}^\otimes$. Using the known decompositions of mapping spaces in $\mathcal{C}^\otimes$ then yields the claim.

Finally, it is easy to see that the induced adjunction of ∞-categories

$$(\mathbf{Op}^{\mathrm{big}}_{\mathcal{T}})_{/\widetilde{\mathcal{O}}^\otimes} \rightleftarrows (\mathbf{Op}_{\mathcal{T}})_{/\mathcal{O}^\otimes}$$

is an equivalence because the unit and counit transformations are equivalences. Hence the Quillen adjunction is a Quillen equivalence. ☐

2.7.5. **Corollary.** *Let $\widetilde{\mathcal{O}}^\otimes$ be a big $\mathcal{T}$-∞-operad over $\underline{\mathbf{F}}^{big}_{\mathcal{T},*}$ and let $i : \mathcal{O}^\otimes \longrightarrow \widetilde{\mathcal{O}}^\otimes$ denote the inclusion. Then we have a Quillen equivalence*

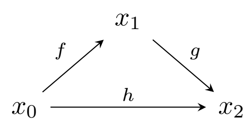
$$i_! : \mathbf{sSet}^+_{/(\mathcal{O}^\otimes, \mathrm{Ne})} \rightleftarrows \mathbf{sSet}^+_{/(\widetilde{\mathcal{O}}^\otimes, \mathrm{Ne})} : i^*$$

Proof. i is obviously compatible with the categorical patterns defining the model structures in the sense of [Lur17, B.2.8], so $i_! \dashv i^*$ is a Quillen adjunction. Moreover, at the level of the underlying ∞-categories i^* is left adjoint to the right Quillen functor of Proposition 2.7.4. Hence the Quillen adjunction is a Quillen equivalence. ☐

2.8. **Monoidal envelopes.** In this subsection, we apply the theory of parametrized factorization systems [Sha21b, §3] to construct the $\mathcal{O}$-monoidal envelope of any fibration of $\mathcal{T}$-∞-operads $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$. First recall the notion of $\mathcal{T}$-factorization system from [Sha21b, Def. 3.1] and the associated 'total' factorization system of [Sha21b, Def. 3.2].

2.8.1. **Example.** We have the inert-active $\mathcal{T}$-factorization system on $\underline{\mathbf{F}}_{\mathcal{T},*}$ given fiberwise by the inert-active factorization system on $\mathbf{F}_{\mathcal{T}^{/V},*}$ as in Remark 2.1.5. Note that the definition of (possibly non-fiberwise) inert and active edges in $\underline{\mathbf{F}}_{\mathcal{T},*}$ given initially in Definition 2.1.3 then matches that of [Sha21b, Def. 3.2]. More generally, we have the inert-active $\mathcal{T}$-factorization system on any $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$. From this, we obtain the inert-fiberwise active factorization system on $\mathcal{O}^{\otimes}$ itself.

2.8.2. **Remark.** The inert edges in $\underline{\mathbf{F}}_{\mathcal{T},*}$ and $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ satisfy the following right cancellation property: if we have a 2-simplex



such that f is inert, then g is inert if and only if h is inert. The 'only if' direction is clear. To see the converse, note that by factoring f as a cocartesian edge followed by a fiberwise inert edge, we may suppose that f is of either form. Then by examining the composition of spans, we see that the claimed assertion reduces to the two-out-of-three property for equivalences in $(\mathbf{F}_{\mathcal{T}})^{/U}$, where x_2 lies over U.

In contrast, the active edges *do not* satisfy the left cancellation property, because cocartesian edges do not. However, fiberwise active edges *do* satisfy the left cancellation property, just as they do in the theory of ∞-operads.

2.8.3. **Notation.** Given a $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, let $\mathrm{Ar}^{act}(\mathcal{O}^{\otimes})$ denote the full $\mathcal{T}$-subcategory of $\mathrm{Ar}(\mathcal{O}^{\otimes})$ on the active morphisms, and let

$$\mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}) = \mathcal{T}^{\mathrm{op}} \times_{\mathrm{Ar}(\mathcal{T}^{\mathrm{op}})} \mathrm{Ar}^{act}(\mathcal{O}^{\otimes}).$$

2.8.4. **Definition.** Given a fibration of $\mathcal{T}$-∞-operads $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, the $\mathcal{O}$-*monoidal envelope* of p is

$$\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes} := \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}) \longrightarrow \mathcal{O}^{\otimes}.$$

If $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, we will abbreviate $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}$ as $\mathrm{Env}_{\mathcal{T}}(\mathcal{C})^{\otimes}$ and refer to it as the $\mathcal{T}$-*symmetric monoidal envelope* of $\mathcal{C}^{\otimes}$.

2.8.5. **Remark.** For a $\mathcal{T}$-∞-operad $\mathcal{C}^{\otimes}$, the underlying $\mathcal{T}$-∞-category of $\mathrm{Env}_{\mathcal{T}}(\mathcal{C})^{\otimes}$ is $\mathcal{C}^{\otimes}_{act}$.

2.8.6. **Proposition.** *Let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads. Then $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}$ is a $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category.*

Proof. We need to show that

$$\mathrm{ev}_1 : \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}) \longrightarrow \mathcal{O}^{\otimes}$$

is a cocartesian fibration of $\mathcal{T}$-∞-operads. By [Sha21b, Prop. 3.5(2)], ev_1 is a cocartesian fibration. We now seek to verify the criterion of Proposition 2.2.6 to finish the proof. Because $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-∞-operad, for any object $y \in \mathcal{O}^{\otimes}_{[U_+ \to V]}$, orbit decomposition $U \simeq U_1 \coprod ... \coprod U_n$, and inert edges $\rho^i : y \longrightarrow y_i$ lifting the characteristic morphisms $\chi_{[U_i \subset U]}$, the ρ^i induce an equivalence

$$((\mathcal{O}^{\otimes}_{act})_V)^{/y} \simeq \prod_{1 \leq i \leq n} ((\mathcal{O}^{\otimes}_{act})_{U_i})^{/y_i}.$$

Using that $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a fibration of $\mathcal{T}$-∞-operads, we have the further equivalence

$$(\mathcal{C}^{\otimes}_{act})_V \times_{(\mathcal{O}^{\otimes}_{act})_V} ((\mathcal{O}^{\otimes}_{act})_V)^{/y} \simeq \prod_{1 \leq i \leq n} (\mathcal{C}^{\otimes}_{act})_{U_i} \times_{(\mathcal{O}^{\otimes}_{act})_{U_i}} ((\mathcal{O}^{\otimes}_{act})_{U_i})^{/y_i}.$$

Using that the fiberwise active edges are left cancellative, we identify the lefthand side with $(\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}))_y$, and similarly for the righthand side. The stated equivalence is then the one induced by the Segal maps, and we conclude that $\mathrm{ev}_1 : \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}) \longrightarrow \mathcal{O}^{\otimes}$ is a cocartesian fibration of $\mathcal{T}$-∞-operads. □

2.8.7. **Proposition.** *Let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads and let $q : \mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a cocartesian fibration of $\mathcal{T}$-∞-operads. Let $i : \mathcal{C}^{\otimes} \subset \mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}$ denote the inclusion of $\mathcal{C}^{\otimes}$ into its $\mathcal{O}$-monoidal envelope.*

(1) Precomposition by i yields an equivalence

$$i^* : \mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D}).$$

(2) We have an adjunction

$$i_! : \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C}), \mathcal{D}) : i^*$$

where $i_!$ is the fully faithful inclusion of $\mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C}), \mathcal{D})$ under the equivalence of (1).

Proof. This follows immediately from [Sha21b, Thm. 3.6], using the inert-active $\mathcal{T}$-factorization system on $\mathcal{O}^{\otimes}$. □

2.8.8. **Corollary.** *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad. We have an adjunction*

$$\mathrm{Env}^{\otimes}_{\mathcal{O},\mathcal{T}} : \mathbf{Op}_{\mathcal{O},\mathcal{T}} \rightleftarrows \mathbf{Cat}^{\otimes}_{\mathcal{O},\mathcal{T}} : \mathrm{U}.$$

2.9. **Subcategories and localization.** Let $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads. Given a full $\mathcal{T}$-subcategory $\mathcal{D} \subset \mathcal{C}$, let $\mathcal{D}^{\otimes}$ be the full $\mathcal{T}$-subcategory of $\mathcal{C}^{\otimes}$ on the objects of $\mathcal{D}$ (using the Segal decompositions of the fibers of $\mathcal{C}^{\otimes}$). Clearly, $\mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a fibration of $\mathcal{T}$-∞-operads, and the inclusion $\mathcal{D}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ is a morphism of $\mathcal{T}$-∞-operads over $\mathcal{O}^{\otimes}$. In this subsection, we state conditions under which $\mathcal{D}^{\otimes}$ inherits an $\mathcal{O}$-monoidal structure from $\mathcal{C}^{\otimes}$. Our presentation of these results parallels and extends those of [Lur17, §2.2.1].

2.9.1. **Proposition.** *Let $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads and let $\mathcal{D} \subset \mathcal{C}$ be a full $\mathcal{T}$-subcategory. Suppose that for every fiberwise active edge $\alpha : x \longrightarrow y$ in $\mathcal{O}^{\otimes}_V$ with $y \in \mathcal{O}_V$, the pushforward functor $\otimes_{\alpha} : \mathcal{C}^{\otimes}_x \longrightarrow \mathcal{C}_y$ restricts to a functor $\mathcal{D}^{\otimes}_x \longrightarrow \mathcal{D}_y$. Then $\mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a cocartesian fibration and the inclusion $\mathcal{D}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal $\mathcal{T}$-functor.*

Proof. This is immediate in light of the inert-fiberwise active factorization system on $\mathcal{C}^{\otimes}$ (Example 2.8.1). □

We say that a $\mathcal{T}$-functor $L : \mathcal{C} \longrightarrow \mathcal{C}$ is a *$\mathcal{T}$-localization* if for every object $V \in \mathcal{T}$, L_V is a localization functor. If we let $\mathcal{D}$ denote the essential image of L, then by [Lur17, 7.3.2.6] we have a $\mathcal{T}$-adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

where $R : \mathcal{D} \longrightarrow \mathcal{C}$ is the inclusion. Given a $\mathcal{T}$-localization $L : \mathcal{C} \longrightarrow \mathcal{C}$, a morphism in $\mathcal{C}$ is an *L-equivalence* if it lies in a fiber $\mathcal{C}_V$ and is a L_V-equivalence. Similarly, given a $\mathcal{T}$-∞-operad $\mathcal{C}^{\otimes}$, a morphism in $\mathcal{C}^{\otimes}$ is an *L-equivalence* if it lies entirely in a fiber $\mathcal{C}^{\otimes}_{[U_+ \to V]}$ and is a product of L-equivalences under the Segal decomposition of that fiber.

2.9.2. **Theorem.** *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad and $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category. Let $L : \mathcal{C} \longrightarrow \mathcal{C}$ be a $\mathcal{T}$-localization and let $\mathcal{D} \subset \mathcal{C}$ be its essential image. Suppose that for every fiberwise active edge $\alpha : x \longrightarrow y \in \mathcal{O}^{\otimes}_V$ with $y \in \mathcal{O}_V$, the pushforward functor $\otimes_{\alpha} : \mathcal{C}^{\otimes}_x \longrightarrow \mathcal{C}_y$ preserves L-equivalences. Then we have a relative adjunction over $\mathcal{O}^{\otimes}$*

$$L^{\otimes} : \mathcal{C}^{\otimes} \rightleftarrows \mathcal{D}^{\otimes} : R^{\otimes}$$

with $L^{\otimes}$ an $\mathcal{O}$-monoidal functor (i.e., preserving cocartesian edges over $\mathcal{O}^{\otimes}$) and $R^{\otimes}$ a lax $\mathcal{O}$-monoidal functor (i.e., a morphism of T-∞-operads), which prolongs the $\mathcal{T}$-adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$.

Proof. This is immediate from the inert-fiberwise active factorization system on $\mathcal{C}^{\otimes}$ (Example 2.8.1) together with the criterion of [BH21, Prop. D.7]. □

2.9.3. **Remark.** In the case where $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, the criterion of Theorem 2.9.2 amounts to

(1) For every object $V \in \mathcal{T}$ and $Z \in \mathcal{C}_V$, the functor

$$- \otimes Z : \mathcal{C}_V \longrightarrow \mathcal{C}_V$$

preserves L_V-equivalences.

(2) For every morphism $f : V \longrightarrow W$ in $\mathcal{T}$, the norm functor
$$f_\otimes : \mathcal{C}_V \longrightarrow \mathcal{C}_W$$
sends L_V-equivalences to L_W-equivalences.

3. Parametrized Day convolution

In this section, we construct a (partially-defined) internal hom for $\mathcal{T}$-∞-operads fibered over an arbitrary base $\mathcal{T}$-∞-operad $\mathcal{O}^\otimes$: the *$\mathcal{T}$-Day convolution*. We first introduce the notion of an *$\mathcal{O}$-promonoidal $\mathcal{T}$-∞-category* $\mathcal{C}^\otimes$, which is the analogue of a flat categorical fibration[11] in the context of $\mathcal{T}$-∞-operads. The $\mathcal{O}$-promonoidal condition ensures the existence of the *p-operadic coinduction* functor (Corollary 3.1.5), and given a $\mathcal{O}$-promonoidal $\mathcal{T}$-∞-category $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ and any fibration $\mathcal{E}^\otimes \longrightarrow \mathcal{O}^\otimes$ of $\mathcal{T}$-∞-operads, we may then use the p-operadic coinduction on the pullback of $\mathcal{E}^\otimes$ over $\mathcal{C}^\otimes$ to construct the *$\mathcal{T}$-Day convolution* (Definition 3.1.6)
$$\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^\otimes \longrightarrow \mathcal{O}^\otimes.$$

We then state conditions on $\mathcal{E}^\otimes$ under which $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^\otimes$ is $\mathcal{O}$-monoidal – these amount to the existence of certain $\mathcal{T}$-colimits as well as $\mathcal{T}$-distributivity of the tensor product (Theorem 3.2.6).

3.1. $\mathcal{O}$-promonoidal $\mathcal{T}$-∞-categories and $\mathcal{T}$-Day convolution.

3.1.1. **Definition.** Let $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ be a fibration of $\mathcal{T}$-∞-operads. We say that $\mathcal{C}^\otimes$ is *$\mathcal{O}$-promonoidal* if for every $V \in \mathcal{T}$, the functor $p_V : (\mathcal{C}^\otimes_V)_{act} \longrightarrow (\mathcal{O}^\otimes_V)_{act}$ is a flat categorical fibration.

3.1.2. **Example.** Suppose $(\mathcal{C}^\otimes, p)$ is a $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category, so that p is a cocartesian fibration. $\mathcal{C}^\otimes$ is then $\mathcal{O}$-promonoidal since cocartesian fibrations are flat [Lur17, Ex. B.3.4].

3.1.3. **Remark.** To understand the relevance of the $\mathcal{O}$-promonoidal condition, the reader may find it useful to first review the nonparametrized story from [Sha21b, §10]. To our knowledge, the correct definition of an $\mathcal{O}$-promonoidal ∞-category first appeared in Hinich's work [Hin20] (and was misstated in [BGS20]).

For technical reasons, to construct the $\mathcal{T}$-Day convolution we will first work in the setting of big $\mathcal{T}$-∞-operads (i.e., so that the base is $\mathbf{F}^{\mathrm{op}}_{\mathcal{T}}$ in place of $\mathcal{T}^{\mathrm{op}}$). Let $\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^\otimes)$ be notation for the full subcategory of $\mathrm{Ar}(\widetilde{\mathcal{O}}^\otimes)$ on the inert edges.

3.1.4. **Theorem.** *Let $\widetilde{\mathcal{O}}^\otimes$ and $\widetilde{\mathcal{C}}^\otimes$ be big $\mathcal{T}$-∞-operads over $\underline{\mathbf{F}}^{big}_{\mathcal{T},*}$ and let $p : \widetilde{\mathcal{C}}^\otimes \longrightarrow \widetilde{\mathcal{O}}^\otimes$ be a fibration of big $\mathcal{T}$-∞-operads such that the restriction $p' : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ is $\mathcal{O}$-promonoidal. Consider the span of marked simplicial sets*
$$(\widetilde{\mathcal{O}}^\otimes, \mathrm{Ne}) \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^\otimes) \times_{\widetilde{\mathcal{O}}^\otimes} \widetilde{\mathcal{C}}^\otimes, \mathrm{Ne}) \xrightarrow{\mathrm{pr}_{\widetilde{\mathcal{C}}^\otimes}} (\widetilde{\mathcal{C}}^\otimes, \mathrm{Ne})$$
where by the middle marking Ne *we mean those edges in* $\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^\otimes) \times_{\widetilde{\mathcal{O}}^\otimes} \widetilde{\mathcal{C}}^\otimes$ *whose source in $\widetilde{\mathcal{O}}^\otimes$ is inert and whose projection to $\widetilde{\mathcal{C}}^\otimes$ is inert. Then the functor*
$$(\mathrm{ev}_0)_* \circ (\mathrm{pr}_{\widetilde{\mathcal{C}}^\otimes})^* : \mathbf{sSet}^+_{/(\widetilde{\mathcal{C}}^\otimes,\mathrm{Ne})} \longrightarrow \mathbf{sSet}^+_{/(\widetilde{\mathcal{O}}^\otimes,\mathrm{Ne})}$$
is right Quillen with respect to the $\mathcal{T}$-operadic model structures of Definition 2.7.3.

Proof. We verify the hypotheses of [Lur17, Thm. B.4.2].

(1) ev_0 is flat by [Sha21b, Lem. 10.1] applied to the inert-fiberwise active factorization system on $\widetilde{\mathcal{O}}^\otimes$ (Example 2.8.1), noting that products of flat fibrations are flat in order to promote the flatness condition on $(p')_{f.act}$ to $p_{f.act}$.
(2) It is obvious that inert edges are closed under composition and contain the equivalences.
(3) Vacuously true since the categorical patterns we are looking at contain all 2-simplices.

[11] Some authors also call this an exponentiable fibration to highlight its key property: a categorical fibration $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be *flat* if the right adjoint π_* to the pullback functor $\pi^* : \mathbf{Cat}_{/\mathcal{D}} \longrightarrow \mathbf{Cat}_{/\mathcal{C}}$ exists.

(4) Suppose $e : x_0 \to y_0$ is an inert edge in $\widetilde{\mathcal{O}}^{\otimes}$. Then as we saw in [Sha21b, Prop. 3.5(1)], given an inert edge $y_0 \to y_1$ the cartesian lift of e to an edge $e' : [x_0 \to x_1] \to [y_0 \to y_1]$ in $\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes})$ has $\mathrm{ev}_1(e') : x_1 \to y_1$ an equivalence. It follows that given a further lift of $y_0 \to y_1$ to an object $(y_0 \to y_1, c \in \widetilde{\mathcal{C}}^{\otimes}_{y_1})$ in $\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes}) \times_{\widetilde{\mathcal{O}}^{\otimes}} \widetilde{\mathcal{C}}^{\otimes}$, e admits a cartesian lift to an edge $e'' : (x_0 \to x_1, c') \to (y_0 \to y_1, c)$ (with $c' \to c$ an equivalence).

(5) Let $f_{x,\phi,\Sigma} : n^{\lhd} \longrightarrow \widetilde{\mathcal{O}}^{\otimes}$ be in the categorical pattern defining the $\mathcal{T}$-operadic model structure on $\mathbf{sSet}^{+}_{/(\widetilde{\mathcal{O}}^{\otimes},\mathrm{Ne})}$. We claim that the pullback

$$\pi : n^{\lhd} \times_{\widetilde{\mathcal{O}}^{\otimes}} \mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes}) \times_{\widetilde{\mathcal{O}}^{\otimes}} \widetilde{\mathcal{C}}^{\otimes} \longrightarrow n^{\lhd}$$

is a cocartesian fibration. Because inert edges are right cancellative (Remark 2.8.2) and $\widetilde{\mathcal{C}}^{\otimes}$ is cocartesian over the inert edges of $\widetilde{\mathcal{O}}^{\otimes}$, it suffices to show that

$$n^{\lhd} \times_{\widetilde{\mathcal{O}}^{\otimes}_{ne}} \mathrm{Ar}(\widetilde{\mathcal{O}}^{\otimes}_{ne}) \longrightarrow n^{\lhd}$$

is cocartesian, where $\widetilde{\mathcal{O}}^{\otimes}_{ne} \subset \widetilde{\mathcal{O}}^{\otimes}$ is the wide subcategory with morphisms restricted to the inert edges. In fact, we will prove the stronger assertion that

$$\mathrm{ev}_0 : \mathrm{Ar}(\widetilde{\mathcal{O}}^{\otimes}_{ne}) \longrightarrow \widetilde{\mathcal{O}}^{\otimes}_{ne}$$

is cocartesian. For this, by [Lur09, 6.1.1.1], it suffices to show that $\widetilde{\mathcal{O}}^{\otimes}_{ne}$ admits pushouts. Since the inert edges in $\widetilde{\mathcal{O}}^{\otimes}$ are defined to be the cocartesian lifts of inert edges in $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$, we may reduce to the case $\widetilde{\mathcal{O}}^{\otimes} = \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$. It then suffices to show that $\mathrm{Ar}(\mathbf{F}_{\mathcal{T}})^{\dagger}$ (as in Definition 2.7.1) admits pullbacks. So suppose we have a commutative cube

$$\begin{array}{ccccccc}
 & & W \times_U X & \longrightarrow & & & X \\
 & \swarrow & & & & \swarrow & \\
W & & \longrightarrow & & U & & \downarrow \\
 & & \downarrow & & & & \\
\downarrow & & Z \times_V Y & \longrightarrow & \downarrow & & Y \\
 & \swarrow & & & & \swarrow & \\
Z & & \longrightarrow & & V & &
\end{array}$$

We want to show that if $W \longrightarrow Z \times_V U$ is a summand inclusion, then $W \times_U X \longrightarrow Z \times_Y X$ is a summand inclusion. To see this, consider the diagram

$$\begin{array}{ccccc}
W \times_U X & \longrightarrow & Z \times_Y X & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
W & \longrightarrow & Z \times_V U & \longrightarrow & U
\end{array}$$

The right square and outer rectangle are both pullback squares, so the left square is as well. Since summand inclusions are stable under pullback, the desired conclusion follows.

(6) Let $s : n^{\lhd} \longrightarrow n^{\lhd} \times_{\widetilde{\mathcal{O}}^{\otimes}} \mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes}) \times_{\widetilde{\mathcal{O}}^{\otimes}} \widetilde{\mathcal{C}}^{\otimes}$ be a cocartesian section of π defined as above. Suppose that $s(\{v\}) = (x \xrightarrow{e} y \in \widetilde{\mathcal{O}}^{\otimes}_{ne},\ c \in \widetilde{\mathcal{C}}^{\otimes}_{y})$ and e is a cocartesian lift of the inert morphism in $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ defined by the square

$$\begin{array}{ccc}
U' & \longrightarrow & U \\
\downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\
V' & \longrightarrow & V
\end{array}$$

in $\mathbf{F}_{\mathcal{T}}$. If $\Sigma = \{\sigma_1, ..., \sigma_n\}$ is given by squares σ_i

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ \downarrow & & \downarrow \phi \\ V_i & \longrightarrow & V \end{array}$$

then let $\Sigma' = \{\sigma'_1, ..., \sigma'_n\}$ be given by the collection of squares σ'_i

$$\begin{array}{ccc} U_i \times_U U' & \longrightarrow & U' \\ \downarrow & & \downarrow \phi' \\ V_i \times_V V' & \longrightarrow & V'. \end{array}$$

Because summand inclusions are stable under pullback, the morphisms $U_i \times_U U' \longrightarrow U'$ are summand inclusions, and clearly induce an equivalence $\coprod_{1 \leq i \leq n} U_i \times_U U' \simeq U'$. Therefore, the data of (c, ϕ', Σ') defines a morphism $f_{c,\phi',\Sigma'} : n^{\lhd} \longrightarrow \widetilde{\mathcal{C}}^{\otimes}$ which is part of the categorical pattern defining the $\mathcal{T}$-operadic model structure on $\mathbf{sSet}^+_{/(\widetilde{\mathcal{C}}^{\otimes},\mathrm{Ne})}$. Moreover, by the analysis done in (5) we may identify the composite map

$$n^{\lhd} \longrightarrow n^{\lhd} \times_{\widetilde{\mathcal{O}}^{\otimes}} \mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes}) \times_{\widetilde{\mathcal{O}}^{\otimes}} \widetilde{\mathcal{C}}^{\otimes} \longrightarrow \widetilde{\mathcal{C}}^{\otimes}$$

with $f_{c,\phi',\Sigma'}$.

(7) We check the following: suppose we have a commutative diagram

$$\begin{array}{ccccc} x_0 & \longrightarrow & x_1 & \longrightarrow & x_2 \\ \downarrow & & \downarrow & & \downarrow \\ y_0 & \longrightarrow & y_1 & \longrightarrow & y_2 \end{array}$$

in $\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes})$ and $c_0 \longrightarrow c_1 \longrightarrow c_2$ in $\widetilde{\mathcal{C}}^{\otimes}$ that covers $y_0 \longrightarrow y_1 \longrightarrow y_2$, such that $c_1 \longrightarrow c_2$ is an equivalence (so $y_1 \longrightarrow y_2$ is an equivalence), $x_1 \longrightarrow x_2$ is inert, $x_0 \longrightarrow x_1$ is an equivalence. Then $c_0 \longrightarrow c_1$ is inert if and only if $c_0 \longrightarrow c_2$ is inert. But this is clear from the definitions.

(8) It suffices to check the following: suppose we have a commutative diagram

$$\begin{array}{ccccc} x_0 & \longrightarrow & x_1 & \longrightarrow & x_2 \\ \downarrow & & \downarrow & & \downarrow \\ y_0 & \longrightarrow & y_1 & \longrightarrow & y_2 \end{array}$$

in $\mathrm{Ar}^{ne}(\widetilde{\mathcal{O}}^{\otimes})$ and $c_0 \longrightarrow c_1 \longrightarrow c_2$ in $\widetilde{\mathcal{C}}^{\otimes}$ that covers $y_0 \longrightarrow y_1 \longrightarrow y_2$, such that $x_0 \longrightarrow x_1$, $y_0 \longrightarrow y_1$, and $c_0 \longrightarrow c_1$ are inert. Then $\{x_1 \longrightarrow x_2,\ y_1 \longrightarrow y_2,\ c_1 \longrightarrow c_2\}$ are inert if and only if $\{x_0 \longrightarrow x_2,\ y_0 \longrightarrow y_2,\ c_0 \longrightarrow c_2\}$ are inert. But this follows from the right cancellativity of inert morphisms (Remark 2.8.2).

□

Having passed to the big $\mathcal{T}$-∞-operads to construct the $\mathcal{T}$-operadic coinduction and verify its properties, we now pass back to the usual formulation of $\mathcal{T}$-∞-operads.

3.1.5. **Corollary.** *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad and let $(\mathcal{C}^{\otimes}, p)$ be an $\mathcal{O}$-promonoidal $\mathcal{T}$-∞-category. Consider the span diagram of marked simplicial sets*

$$(\mathcal{O}^{\otimes}, \mathrm{Ne}) \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathrm{Ne}) \xrightarrow{\mathrm{pr}_{\mathcal{C}^{\otimes}}} (\mathcal{C}^{\otimes}, \mathrm{Ne}).$$

Then the functor

$$(\mathrm{ev}_0)_* \circ (\mathrm{pr}_{\mathcal{C}^{\otimes}})^* : \mathbf{sSet}^+_{/(\mathcal{C}^{\otimes},\mathrm{Ne})} \longrightarrow \mathbf{sSet}^+_{/(\mathcal{O}^{\otimes},\mathrm{Ne})}$$

is right Quillen with respect to the $\mathcal{T}$-operadic model structures.

Proof. Combine Theorem 3.1.4, Proposition 2.7.4, and Corollary 2.7.5. □

3.1.6. **Definition.** In the situation of Corollary 3.1.5, given a fibration of $\mathcal{T}$-∞-operads $\mathcal{E}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$, define the *p-operadic coinduction* or *p-norm* of $\mathcal{E}^{\otimes}$ to be the $\mathcal{T}$-∞-operad

$$(\mathrm{Norm}_p \mathcal{E})^{\otimes} := (\mathrm{ev}_0)_* (\mathrm{pr}_{\mathcal{C}^{\otimes}})^* (\mathcal{E}^{\otimes}, \mathrm{Ne}),$$

given as a marked simplicial set fibered over $(\mathcal{O}^{\otimes}, \mathrm{Ne})$.

Given a fibration of $\mathcal{T}$-∞-operads $\mathcal{E}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, define the *Day convolution* $\mathcal{T}$-∞-operad to be

$$\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes} := (\mathrm{Norm}_p p^* \mathcal{E})^{\otimes}.$$

If $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, we will also denote $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}$ as $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}$.

3.1.7. **Proposition.** *Let $p, q : \mathcal{D}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be fibrations of $\mathcal{T}$-∞-operads. Then the functor*

$$\iota : (\mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathrm{Ne}) \longrightarrow (\mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathrm{Ne})$$

induced by the identity section is a homotopy equivalence in $\mathbf{sSet}^+_{/(\mathcal{C}^{\otimes},\mathrm{Ne})}$. Consequently, for $\mathcal{O}$-promonoidal $(\mathcal{C}^{\otimes}, p)$, ι^* induces an equivalence of $\mathcal{T}$-∞-categories

$$\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{D}, \mathrm{Norm}_p \mathcal{E}) \xrightarrow{\simeq} \underline{\mathbf{Alg}}_{\mathcal{C},\mathcal{T}}(\mathcal{D} \times_{\mathcal{O}} \mathcal{C}, \mathcal{E})$$

and an equivalence of ∞-categories

$$\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{D}, \mathrm{Norm}_p \mathcal{E}) \xrightarrow{\simeq} \mathbf{Alg}_{\mathcal{C},\mathcal{T}}(\mathcal{D} \times_{\mathcal{O}} \mathcal{C}, \mathcal{E}).$$

Proof. Let $P : \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \longrightarrow \mathcal{D}^{\otimes}$ be a cocartesian pushforward chosen so that $P|_{\mathcal{D}^{\otimes}} = \mathrm{id}$, and let

$$P' = P \times \mathrm{id}_{\mathcal{C}^{\otimes}} : \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}.$$

Then P' respects the given markings, and as in the proof of [Sha21a, Lem. 3.2(2)] we may construct an explicit homotopy $h : \mathrm{id} \longrightarrow \iota \circ P'$ such that h sends objects to fiberwise marked edges. This shows that P is a marked homotopy inverse to ι and hence ι is a homotopy equivalence in $\mathbf{sSet}^+_{/(\mathcal{C}^{\otimes},\mathrm{Ne})}$. The consequences then follow from the definition of the left adjoint to Norm_p. □

3.1.8. **Corollary.** *The Quillen adjunction of Corollary 3.1.5 descends to an adjunction of ∞-categories*

$$p^* \colon (\mathbf{Op}_{\mathcal{T}})_{/\mathcal{O}^{\otimes}} \rightleftarrows (\mathbf{Op}_{\mathcal{T}})_{/\mathcal{C}^{\otimes}} : \mathrm{Norm}_p.$$

In particular, if $(\mathcal{C}^{\otimes}, p)$ is $\mathcal{O}$-promonoidal, then the right adjoint to p^* exists and is computed by Norm_p.

3.1.9. **Proposition.** *The underlying $\mathcal{T}$-∞-category of $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}$ is the $\mathcal{T}$-pairing construction $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})$ of* [Sha21a, Constr. 9.1]. *In particular, if $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, the underlying $\mathcal{T}$-∞-category of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}$ is $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{E})$.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{Ar}^{cocart}(\mathcal{O}) \times_{\mathcal{O}} \mathcal{C} & \longrightarrow & \mathcal{O} & & \\
\downarrow \iota & & & \searrow & \\
\mathcal{O} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} & \longrightarrow & \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \\
\downarrow & & \downarrow & & \\
\mathcal{O} & \longrightarrow & \mathcal{O}^{\otimes} & &
\end{array}$$

where ι is defined using the inclusions $\mathcal{O} \subset \mathcal{O}^{\otimes}$ and $\mathcal{C} \subset \mathcal{C}^{\otimes}$. Unwinding the definitions, to prove the claim it suffices to show that the map

$$\mathrm{Ar}^{cocart}(\mathcal{O}) \times_{\mathcal{O}} \mathcal{C} \longrightarrow \mathcal{O} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{O} \times_{\mathcal{O}} \mathcal{C}$$

is a homotopy equivalence (in $\mathbf{sSet}^+_{/_\natural \mathcal{O}}$ via the target map). But this is clear, since the inert edges in $\mathcal{O}^{\otimes}$ with source and target in $\mathcal{O}$ are, up to equivalence, precisely the cocartesian edges in $\mathcal{O}$. □

3.2. **$\mathcal{O}$-monoidality of the $\mathcal{T}$-Day convolution.** We now establish $\mathcal{O}$-monoidality of the $\mathcal{T}$-Day convolution $\mathcal{T}$-∞-operad given appropriate conditions on the input $\mathcal{T}$-∞-operads. For this, we will use repeatedly the following criterion for when a fibration of $\mathcal{T}$-∞-operads is locally cocartesian or cocartesian.

3.2.1. **Lemma.** *Let $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ be a fibration of $\mathcal{T}$-∞-operads. Suppose that for every fiberwise active edge $e : x \longrightarrow y$ in $\mathcal{O}^\otimes$ with $y \in \mathcal{O}$, and $c \in \mathcal{O}^\otimes$ with $p(c) = x$, there exists a locally cocartesian edge $f : c \longrightarrow c'$ over e. Then p is a locally cocartesian fibration.*

Furthermore, suppose that for every composition of fiberwise active edges $x \xrightarrow{e} y \xrightarrow{e'} z$ in $\mathcal{O}^\otimes$ with $z \in \mathcal{O}$, and $c \in \mathcal{O}^\otimes$ with $p(c) = x$, locally cocartesian lifts of e to $f : c \longrightarrow c'$ and e' to $f' : c' \longrightarrow c''$ compose to yield a locally cocartesian edge $f'' : c \longrightarrow c''$. Then p is a cocartesian fibration.

Proof. For the first assertion, using the inert-fiberwise active factorization system on $\mathcal{O}^\otimes$ and that p admits cocartesian lifts over inert edges, we reduce to checking that p admits locally cocartesian lifts over fiberwise active edges. Then given any fiberwise active edge $e : x \longrightarrow y$ in $\mathcal{O}^\otimes$, we may use that $\mathcal{O}^\otimes$ is a $\mathcal{T}$-∞-operad to obtain a decomposition of e as $(e_i : x_i \longrightarrow y_i)_{i\in I}$ for $y_i \in \mathcal{O}$ under the identifications $\mathcal{O}^\otimes_y \simeq \prod_{i\in I} \mathcal{O}^\otimes_{y_i}$, $\mathcal{O}^\otimes_x \simeq \prod_{i\in I} \mathcal{O}^\otimes_{x_i}$, and $\mathrm{Map}_{\mathcal{O}^\otimes_{act}}(x, y) \simeq \prod_{i\in I} \mathrm{Map}_{\mathcal{O}^\otimes_{act}}(x_i, y_i)$. Suppose $c \in \mathcal{C}^\otimes$ over x. Because p is a fibration of $\mathcal{T}$-∞-operads, we get $c_i \in \mathcal{C}^\otimes$ over x_i, and we may take the product of locally cocartesian lifts $c_i \longrightarrow c'_i$ over the e_i to obtain the desired locally cocartesian lift $c \longrightarrow c'$ of e.

For the second assertion, we need to check that locally cocartesian edges compose to yield again a locally cocartesian edge. Note that for any commutative diagram in $\mathcal{C}^\otimes$

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ c' & \xrightarrow{f'} & d' \end{array}$$

with f locally cocartesian over a fiberwise active edge and g, g' inert, then f' is necessarily locally cocartesian. Using this, we can reduce to checking that locally cocartesian edges over fiberwise active edges compose. Then as before, we can further reduce to supposing that the last object lies in $\mathcal{O}$. □

The next proposition allows us to understand $\mathcal{T}$-Day convolution as a *locally* cocartesian fibration over the base $\mathcal{T}$-∞-operad $\mathcal{O}^\otimes$.

3.2.2. **Proposition.** *Let $(\mathcal{C}^\otimes, p)$ be an $\mathcal{O}$-promonoidal $\mathcal{T}$-∞-category and let $(\mathcal{E}^\otimes, q)$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category in which for every object $x \in \mathcal{O}_V$, the parametrized fiber $\mathcal{E}_{\underline{x}}$ is $\mathcal{T}^{/V}$-cocomplete.*

(1) The $\mathcal{T}$-Day convolution $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes \longrightarrow \mathcal{O}^\otimes$ is a locally cocartesian fibration, and for every $x \in \mathcal{O}_V$, its parametrized fiber $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})_{\underline{x}}$ is $\mathcal{T}^{/V}$-cocomplete.

(2) Suppose $\overline{\alpha}$ is a fiberwise active edge in $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes_{\underline{V}}$ lifting $\alpha : x \longrightarrow y$ in $\mathcal{O}^\otimes_{\underline{V}}$ with $y \in \mathcal{O}_V$, which in turn lifts $f : U \longrightarrow V$ in $\mathbf{F}_\mathcal{T}$ (identified with the fiberwise active edge $[U_+ \to V] \longrightarrow [V_+ \to V]$ in $\underline{\mathbf{F}}_{\mathcal{T},*}$). The data of $\overline{\alpha}$ is given by a commutative diagram of $\mathcal{T}^{/V}$-∞-categories

$$\begin{array}{ccc} \{x\} \times_{\mathcal{O}^\otimes} \mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes & \xrightarrow{F} & \mathcal{E}^\otimes_{\underline{V}} \\ \downarrow{\scriptstyle i} & \overset{H}{\nearrow} & \downarrow{\scriptstyle q_{\underline{V}}} \\ \Delta^1 \times_{\alpha,\mathcal{O}^\otimes} \mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes & \longrightarrow & \mathcal{O}^\otimes_{\underline{V}} \end{array}$$

where we use the projections of the lefthand ∞-categories to $\{V\} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}) \cong (\mathcal{T}^{/V})^{\mathrm{op}}$ in order to pullback to the base $(\mathcal{T}^{/V})^{\mathrm{op}}$.

Then $\overline{\alpha}$ is a locally cocartesian edge if and only if H is a weak $q_{\underline{V}}$-$\mathcal{T}^{/V}$-left Kan extension of F in the sense of [Sha21b, Def. 6.1].

(3) Note that we have inclusions of full $\mathcal{T}$-subcategories

$$\mathcal{C}^\otimes_{\underline{x}} := \{x\} \times_{\mathcal{O}^\otimes} \mathrm{Ar}^{cocart}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes \subset \{x\} \times_{\mathcal{O}^\otimes} \mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes$$
$$\mathcal{C}^\otimes_{\underline{\alpha}} := \Delta^1 \times_{\alpha,\mathcal{O}^\otimes} \mathrm{Ar}^{cocart}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes \subset \Delta^1 \times_{\alpha,\mathcal{O}^\otimes} \mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes.$$

Choose a cocartesian pushforward $P_\alpha : \mathcal{E}^{\otimes}_{\underline{\alpha}} \longrightarrow \mathcal{E}_{\underline{y}}$ and consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}^{\otimes}_{\underline{x}} & \xrightarrow{F} & \mathcal{E}^{\otimes}_{\underline{\alpha}} & \xrightarrow{P_\alpha} & \mathcal{E}_{\underline{y}} \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle H} & & & \downarrow \\ \mathcal{C}^{\otimes}_{\underline{\alpha}} & & \longrightarrow & & (\mathcal{T}^{/V})^{\mathrm{op}} \end{array}$$

(where we abusively continue to write H and F for the canonical lifts of those functors to have codomains $\mathcal{E}^{\otimes}_{\underline{\alpha}}$ and $\mathcal{E}^{\otimes}_{\underline{x}} \subset \mathcal{E}^{\otimes}_{\underline{\alpha}}$ respectively). Then $\overline{\alpha}$ is a locally cocartesian edge if and only if $P_\alpha \circ H$ is a $\mathcal{T}^{/V}$-left Kan extension of $P_\alpha \circ F$.

Furthermore, if we let $G = (P_\alpha \circ H)|_{\mathcal{C}_{\underline{y}}} : \mathcal{C}_{\underline{y}} \longrightarrow \mathcal{E}_{\underline{y}}$, then we have the data of a diagram

$$\begin{array}{ccccc} \mathcal{C}^{\otimes}_{\underline{x}} & \xrightarrow{F} & \mathcal{E}^{\otimes}_{\underline{x}} & \xrightarrow{\alpha_\otimes} & \mathcal{E}_{\underline{y}} \\ {\scriptstyle \alpha_\otimes}\downarrow & \Downarrow\eta & & \nearrow & \\ \mathcal{C}_{\underline{y}} & & \xrightarrow{G} & & \end{array}$$

in which the natural transformation η exhibits G as a $\mathcal{T}^{/V}$-left Kan extension of $\alpha_\otimes \circ F$ along $\alpha_\otimes$.

Proof. (2): By definition, $\overline{\alpha}$ is a locally cocartesian edge if and only if H is initial in the space of all such fillers. But if H is a weak $q_{\underline{V}}$-$\mathcal{T}^{/V}$-left Kan extension of F, then it is in particular initial. Conversely, provided that we know a weak $q_{\underline{V}}$-$\mathcal{T}^{/V}$-left Kan extension of F exists, then it necessarily coincides with the filler defined by $\overline{\alpha}$.

(1): We now show existence of these weak $q_{\underline{V}}$-$\mathcal{T}^{/V}$-left Kan extensions. Let

$$K := \{x\} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}$$
$$L := \Delta^1 \times_{\alpha, \mathcal{O}^{\otimes}} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}.$$

Note that any object in L that is not in K is either of the form $(y \to z = p(c) \in \mathrm{Ar}^{cocart}(\mathcal{O}),\ c \in \mathcal{C})$ or $(y \to z = p(c) \in \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}),\ c \in \mathcal{C}^{\otimes})$ with z over $[\emptyset_+ \to W] \in \underline{\mathbf{F}}_{\mathcal{T},*}$. Let $L_0 \subset L$ denote the full $\mathcal{T}$-subcategory excluding the second type of objects. Because the fiber of $\mathcal{E}^{\otimes}$ over any object $[\emptyset_+ \to W] \in \underline{\mathbf{F}}_{\mathcal{T},*}$ is contractible, we may replace L with L_0 and instead consider fillers $H_0 : L_0 \longrightarrow \mathcal{E}^{\otimes}$. Moreover, note that there are no inert edges in L_0 not either cocartesian over $\mathcal{T}^{\mathrm{op}}$ or in K, so any extension H_0 necessarily defines an edge of $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}$.

Because both the additional objects in L_0 and morphisms between those objects lie over $\underline{y} \longrightarrow \mathcal{O} \subset \mathcal{O}^{\otimes}$, by [Sha21b, Thm. 6.2] in conjunction with [Sha21b, Prop. 5.8] it suffices to have $\mathcal{E}_{\underline{y}}$ be $\mathcal{T}^{/V}$-cocomplete for a weak $q_{\underline{V}}$-$\mathcal{T}^{/V}$-left Kan extension of F to exist. This is ensured by our hypothesis.

By Lemma 3.2.1, we see that the case we just considered suffices to show that $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a locally cocartesian fibration. In addition, by Proposition 3.1.9 and [Sha21a, Prop. 9.7], for every $x \in \mathcal{O}_V$ we have an equivalence

$$\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})_{\underline{x}} \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\mathcal{C}_{\underline{x}}, \mathcal{E}_{\underline{x}}),$$

and the latter $\mathcal{T}^{/V}$-∞-category is $\mathcal{T}^{/V}$-cocomplete by the pointwise computation of $\mathcal{T}^{/V}$-colimits in $\mathcal{T}^{/V}$-functor categories.

(3): Observe that for $l = (y \to z, c) \in \mathcal{C}^{\otimes}_{\underline{\alpha}} \subset L$, $K \times_L L^{/\underline{l}} \simeq \mathcal{C}^{\otimes}_{\underline{x}} \times_{\mathcal{C}^{\otimes}_{\underline{\alpha}}} (\mathcal{C}^{\otimes}_{\underline{\alpha}})^{/\underline{l}}$. By the pointwise formula for $\mathcal{T}^{/V}$-left Kan extensions, the first part of the claim follows. The second part then follows by [Sha21b, Rem. 2.14]. □

As with ordinary Day convolution, we need an additional distributivity hypothesis on the target for $\mathcal{T}$-Day convolution to be $\mathcal{O}$-monoidal. We first recall the definition of a distributive functor (as originally formulated by the first author).

3.2.3. **Definition** ([Sha21b, Def. 8.18]). Let $f : U \longrightarrow V$ be a map of finite $\mathcal{T}$-sets, let $\mathcal{C}$ be a $\mathcal{T}^{/U}$-∞-category, and let $\mathcal{D}$ be a $\mathcal{T}^{/V}$-∞-category. Let $F : \prod_f \mathcal{C} = f_* \mathcal{C} \longrightarrow \mathcal{D}$ be a $\mathcal{T}^{/V}$-functor. Then we say that

F is *distributive* if for every pullback square

$$\begin{array}{ccc} U' & \xrightarrow{f'} & V' \\ \downarrow{g'} & & \downarrow{g} \\ U & \xrightarrow{f} & V \end{array}$$

of finite $\mathcal{T}$-sets and $\mathcal{T}^{/U'}$-colimit diagram $\overline{p} : \mathcal{K}^{\underline{\triangleright}} \longrightarrow g'^*\mathcal{C}$, the $\mathcal{T}^{/V'}$-functor

$$(f'_*\mathcal{K})^{\underline{\triangleright}} \xrightarrow{\text{can}} f'_*(\mathcal{K}^{\underline{\triangleright}}) \xrightarrow{f'_*\overline{p}} f'_*g'^*\mathcal{C} \simeq g^*f_*\mathcal{C} \xrightarrow{g^*F} g^*\mathcal{D}$$

is a $\mathcal{T}^{/V'}$-colimit diagram.

3.2.4. **Definition.** Let $\mathcal{C}^{\otimes}$ be a $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category and suppose that for all $y \in \mathcal{O}_V$, $\mathcal{C}_{\underline{y}}$ is $\mathcal{T}^{/V}$-cocomplete. We say that $\mathcal{C}^{\otimes}$ is *distributive* if for every fiberwise active edge $\alpha : x \longrightarrow y$ lifting $[U_+ \to V] \longrightarrow [V_+ \to V]$ (corresponding to $f : U \longrightarrow V$ in $\mathbf{F}_{\mathcal{T}}$), the associated pushforward $\mathcal{T}^{/V}$-functor

$$\alpha_{\otimes} : \mathcal{C}^{\otimes}_{\underline{x}} \longrightarrow \mathcal{C}_{\underline{y}}$$

is distributive. Here, for this condition to be sensible, we use that for an orbit decomposition $U \simeq U_1 \coprod ... \coprod U_n$ and n cocartesian morphisms $x \longrightarrow x_i$ lifting the characteristic maps $\chi_{[U_i \subset U]} : [U_+ \to V] \longrightarrow [(U_i)_+ \to U_i]$, the parametrized fiber $\mathcal{C}^{\otimes}_{\underline{x}}$ is identified with $\prod_f \left(\mathcal{C}_{\underline{x_1}} \coprod ... \coprod \mathcal{C}_{\underline{x_n}}\right)$ by the $\mathcal{T}$-Segal condition (cf. Example 2.3.5).

Also note that in particular, for each morphism $\alpha : x \longrightarrow y$ in $\mathcal{O}_V$, the condition that the pushforward $\mathcal{T}^{/V}$-functor $\alpha_{\otimes} : \mathcal{C}_{\underline{x}} \longrightarrow \mathcal{C}_{\underline{y}}$ is distributive is equivalent to $\alpha_{\otimes}$ strongly preserving all small $\mathcal{T}^{/V}$-colimits.

The following proposition furnishes some examples of distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-categories.

3.2.5. **Proposition.** *Let $\mathcal{C}$ be a cocomplete ∞-category with finite products such that the products commute with colimits separately in each variable. Let $f : U \longrightarrow U'$ be a morphism of finite $\mathcal{T}$-sets. Then the product $\mathcal{T}^{/U'}$-functor*

$$\mu : f_*f^*\underline{\mathcal{C}}_{\mathcal{T}/U'} \longrightarrow \underline{\mathcal{C}}_{\mathcal{T}/U'}$$

is distributive. Consequently, the $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathcal{C}}_{\mathcal{T}}$ is $\mathcal{T}$-distributive.

Proof. By the universal property of the category of $\mathcal{T}^{/U'}$-objects ([Sha21a, Prop. 3.10]), μ can be identified with a functor

$$\mu^{\dagger} : f_*f^*\underline{\mathcal{C}}_{\mathcal{T}/U'} \longrightarrow \mathcal{C}$$

such that its restriction to the fiber over $[W \to U'] \in \mathcal{T}^{/U'}$ is the functor

$$\prod_{V \in \mathrm{Orbit}(U \times_{U'} W)} \mathrm{Fun}(\underline{V}, \mathcal{C}) \xrightarrow{\prod \mathrm{ev}_V} \prod_{V \in \mathrm{Orbit}(U \times_{U'} W)} \mathcal{C} \xrightarrow{\times} \mathcal{C}\,.$$

For the proof of distributivity we can assume without loss of generality that U' is an orbit. Suppose $p : \mathcal{K}^{\underline{\triangleright}} \longrightarrow \underline{\mathcal{C}}_{\mathcal{T}/U}$ is a $\mathcal{T}^{/U}$-colimit diagram and let $p^{\dagger} : \mathcal{K}^{\underline{\triangleright}} \longrightarrow \mathcal{C}$ be the functor under the equivalence of [Sha21a, Prop. 3.10]. By [Sha21a, Prop. 5.5], any such p is $\mathcal{T}^{/U}$-colimit diagram if and only if for every $[V \to U] \in \mathcal{T}^{/U}$ the functor $p^{\dagger}_V : \mathcal{K}^{\triangleright}_{[V \to U]} \longrightarrow \mathcal{C}$ given by restriction to the fiber is a colimit diagram. By [Sha21a, Prop. 5.5] again, it suffices to prove that the diagram

$$(f_*\mathcal{K})^{\underline{\triangleright}} \longrightarrow f_*\underline{\mathcal{C}}_{\mathcal{T}/U} \xrightarrow{\mu^{\dagger}} \mathcal{C}$$

is a colimit diagram when restricted to each fiber. But the restriction to the fiber above $[W \to U'] \in \mathcal{T}^{/U'}$ is

$$\left(\prod_{V \in \mathrm{Orbit}(U \times_{U'} W)} \mathcal{K}_V\right)^{\triangleright} \longrightarrow \prod_{V \in \mathrm{Orbit}(U \times_{U'} W)} \mathcal{K}_V^{\triangleright} \xrightarrow{\prod p_V} \prod_{V \in \mathrm{Orbit}(U \times_{U'} W)} \mathcal{C} \xrightarrow{\times} \mathcal{C},$$

which is a colimit diagram since the cartesian product in $\mathcal{C}$ commutes with colimits separately in each variable. □

We now return to $\mathcal{T}$-Day convolution and prove the main result of this subsection.

3.2.6. **Theorem.** *In the situation of Proposition 3.2.2, suppose moreover that* $q : \mathcal{E}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ *is distributive. Then* $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ *is a cocartesian fibration and* $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^{\otimes}$ *is distributive.*

Proof. Let

$$\begin{array}{ccccc} & & y & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\searrow} & \\ x & & \xrightarrow{\gamma} & & z \end{array}$$

be a 2-simplex σ of fiberwise active edges in $\mathcal{O}^{\otimes}_{W}$ with $z \in \mathcal{O}_W$, which covers

$$\begin{array}{ccccc} & & V & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ U & & \xrightarrow{h} & & W \end{array}$$

in $\mathbf{F}_{\mathcal{T}}$ (viewed as a 2-simplex in $(\underline{\mathbf{F}}_{\mathcal{T},*})^{act}_{W}$). We need to verify that given a lift of σ to a 2-simplex $\overline{\sigma}$

$$\begin{array}{ccccc} & & \overline{y} & & \\ & \overset{\overline{\alpha}}{\nearrow} & & \overset{\overline{\beta}}{\searrow} & \\ \overline{x} & & \xrightarrow{\overline{\gamma}} & & \overline{z} \end{array}$$

in $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^{\otimes}$, if $\overline{\alpha}$ and $\overline{\beta}$ are locally cocartesian edges then $\overline{\gamma}$ is a locally cocartesian edge.

Suppose that $V \simeq \coprod_{1 \leq i \leq n} V_i$ is an orbit decomposition of V with respect to which α decomposes as $\{\alpha_i : x_i \longrightarrow y_i\}_{1 \leq i \leq n}$. Then the locally cocartesian edge $\overline{\alpha}$ corresponds to n commutative diagrams of $\mathcal{T}^{/V_i}$-∞-categories

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\underline{x_i}} & \xrightarrow{F_{x_i}} & \mathcal{E}_{\underline{y_i}} \\ \downarrow & \overset{F_{\alpha_i}}{\nearrow} & \\ \mathcal{C}^{\otimes}_{\underline{\alpha_i}} & & \end{array}$$

in which F_{α_i} is a $\mathcal{T}^{/V_i}$-left Kan extension of F_{x_i}. From this, we obtain the commutative diagram of $\mathcal{T}^{/W}$-∞-categories

$$\begin{array}{ccccc} \prod_g(\coprod_{1 \leq i \leq n} \mathcal{C}^{\otimes}_{\underline{x_i}}) \simeq \mathcal{C}^{\otimes}_{\underline{x}} & \xrightarrow{F_x} & \prod_g(\coprod_{1 \leq i \leq n} \mathcal{E}_{\underline{y_i}}) \simeq \mathcal{E}^{\otimes}_{\underline{y}} & \xrightarrow{\beta_{\otimes}} & \mathcal{E}_{\underline{z}} \\ \downarrow & \overset{F_{\alpha}}{\nearrow} & & \overset{\beta_{\otimes} \circ F_{\alpha}}{\nearrow} & \\ \prod_g(\coprod_{1 \leq i \leq n} \mathcal{C}^{\otimes}_{\underline{\alpha_i}}) \simeq \mathcal{C}^{\otimes}_{\underline{\alpha}} & & & & \end{array}$$

in which $\beta_{\otimes} \circ F_{\alpha}$ is a $\mathcal{T}^{/W}$-left Kan extension of $\beta_{\otimes} \circ F_x$, invoking the hypothesis that $\mathcal{E}^{\otimes}$ is distributive.

On the other hand, the locally cocartesian edge $\overline{\beta}$ corresponds to a commutative diagram of $\mathcal{T}^{/W}$-∞-categories

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\underline{y}} & \xrightarrow{F_{y}} & \mathcal{E}_{\underline{z}} \\ \downarrow & \overset{F_{\beta}}{\nearrow} & \\ \mathcal{C}^{\otimes}_{\underline{\beta}} & & \end{array}$$

in which F_y is the restriction of $\beta_\otimes \circ F_\alpha$ along the inclusion $\mathcal{C}^\otimes_{\underline{y}} \subset \mathcal{C}^\otimes_{\underline{\alpha}}$ and F_β is a $\mathcal{T}^{/W}$-left Kan extension of F_y. Combining these two diagrams, we obtain a commutative diagram of $\mathcal{T}^{/W}$-∞-categories

$$\begin{array}{ccccc}
 & & \mathcal{C}^\otimes_{\underline{x}} & \xrightarrow{F_x} & \mathcal{E}^\otimes_{\underline{y}} \\
 & & \downarrow & & \downarrow{\scriptstyle \beta_\otimes} \\
\mathcal{C}^\otimes_{\underline{y}} & \longrightarrow & \mathcal{C}^\otimes_{\underline{\alpha}} & \xrightarrow{\beta_\otimes \circ F_\alpha} & \mathcal{E}_{\underline{z}} \\
\downarrow & \nearrow{\scriptstyle F_\beta} & \downarrow & \nearrow{\scriptstyle F_\sigma}\,(\text{dotted}) & \\
\mathcal{C}^\otimes_{\underline{\beta}} & \longrightarrow & \mathcal{C}^\otimes_{\underline{\sigma}} & &
\end{array}$$

where $\mathcal{C}^\otimes_{\underline{\sigma}} := \Delta^2 \times_{\sigma, \mathcal{O}^\otimes} \mathrm{Ar}^{cocart}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes$. Because $\mathcal{C}^\otimes$ is $\mathcal{O}$-promonoidal and σ is a 2-simplex of fiberwise active edges, the lefthand square is a pushout square of $\mathcal{T}^{/W}$-∞-categories, and the dotted $\mathcal{T}^{/W}$-functor F_σ is obtained from gluing together $\beta_\otimes \circ F_\alpha$ and F_β. Indeed, F_σ corresponds to the 2-simplex $\overline{\sigma}$ in $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes$. By Lemma 3.2.7, F_σ is a $\mathcal{T}^{/W}$-left Kan extension of $\beta_\otimes \circ F_\alpha$. By transitivity of $\mathcal{T}^{/W}$-left Kan extensions, F_σ is a $\mathcal{T}^{/W}$-left Kan extension of $\beta_\otimes \circ F_x$, and the restriction F_γ of F_σ to $\mathcal{C}^\otimes_{\underline{\gamma}} \subset \mathcal{C}^\otimes_{\underline{\sigma}}$ is also a $\mathcal{T}^{/W}$-left Kan extension of $\beta_\otimes \circ F_x$. But this exactly means that $\overline{\gamma}$ is a locally cocartesian edge. Finally, by Lemma 3.2.1 this suffices to show that $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes \longrightarrow \mathcal{O}^\otimes$ is a cocartesian fibration.

To show that $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes$ is distributive, we check the definition. So suppose that $\alpha : x \longrightarrow y$ is a fiberwise active edge in $\mathcal{O}^\otimes_V$ with $y \in \mathcal{O}_V$, which lifts $f : U \longrightarrow V$ in $\mathbf{F}_\mathcal{T}$. Let $U \simeq U_1 \coprod ... \coprod U_n$ be an orbit decomposition and suppose we have $\mathcal{T}^{/U_i}$-colimits

$$\begin{array}{ccc}
\mathcal{K}_i & \xrightarrow{p_i} & \underline{\mathrm{Fun}}_{\mathcal{T}^{/U_i}}(\mathcal{C}_{\underline{x_i}}, \mathcal{E}_{\underline{x_i}}) \\
\downarrow & \nearrow{\scriptstyle q_i} & \\
(\mathcal{T}^{/U_i})^{\mathrm{op}} & &
\end{array}$$

(Here and throughout we suppress the data of the natural transformations.) By [Sha21a, Prop. 9.17] and its proof, these are adjoint to $\mathcal{T}^{/U_i}$-left Kan extensions

$$\begin{array}{ccc}
\mathcal{K}_i \times_{(\mathcal{T}^{/U_i})^{\mathrm{op}}} \mathcal{C}_{\underline{x_i}} & \xrightarrow{p'_i} & \mathcal{E}_{\underline{x_i}} \\
\downarrow & \nearrow{\scriptstyle q'_i} & \\
\mathcal{C}_{\underline{x_i}} & &
\end{array}$$

Let $\mathcal{K} = \coprod_{1 \le i \le n} \mathcal{K}_i$, $p = \coprod_{1 \le i \le n} p_i$, and $q = \coprod_{1 \le i \le n} q_i$, and the same for $\mathcal{K}', p', q'$. We need to show that

$$\begin{array}{ccccc}
\prod_f \mathcal{K} & \xrightarrow{\prod_f p} & \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes_{\underline{x}} & \xrightarrow{\alpha_\otimes} & \underline{\mathrm{Fun}}_{\mathcal{T}^{/V}}(\mathcal{C}_{\underline{y}}, \mathcal{E}_{\underline{y}}) \\
\downarrow & & & \nearrow{\scriptstyle \alpha_\otimes \circ \prod_f q} & \\
(\mathcal{T}^{/V})^{\mathrm{op}} & & & &
\end{array}$$

is a $\mathcal{T}^{/V}$-colimit. Equivalently, we need to show that in the diagram

$$\begin{array}{ccccc}
\prod_f \mathcal{K} \times_{(\mathcal{T}^{/V})^{\mathrm{op}}} \mathcal{C}^{\otimes}_{\underline{x}} & \xrightarrow{(\prod_f p)'} & \mathcal{E}^{\otimes}_{\underline{x}} & \xrightarrow{\alpha_\otimes} & \mathcal{E}_{\underline{y}} \\
\downarrow & & & \nearrow_{(\alpha_\otimes \circ \prod_f p)'} & \\
\prod_f \mathcal{K} \times_{(\mathcal{T}^{/V})^{\mathrm{op}}} \mathcal{C}^{\otimes}_{\underline{\alpha}} & & & & \\
\downarrow & & \nearrow_{(\alpha_\otimes \circ \prod_f q)'} & & \\
\mathcal{C}^{\otimes}_{\underline{\alpha}} & & & &
\end{array}$$

the $\mathcal{T}^{/V}$-functor $(\alpha_\otimes \circ \prod_f q)'$ is a $\mathcal{T}^{/V}$-left Kan extension of $(\alpha_\otimes \circ \prod_f p)'$. Here we change the domain from $\mathcal{C}^{\otimes}_{\underline{y}}$ to $\mathcal{C}^{\otimes}_{\underline{\alpha}}$ because we are not supposing that $\mathcal{C}^{\otimes}$ is $\mathcal{O}$-monoidal. Note that $(\alpha_\otimes \circ \prod_f p)'$ is by definition a $\mathcal{T}^{/V}$-left Kan extension of $\alpha_\otimes \circ (\prod_f p)'$, so it suffices to show that $(\alpha_\otimes \circ \prod_f q)'$ is a $\mathcal{T}^{/V}$-left Kan extension of $\alpha_\otimes \circ (\prod_f p)'$.

Because $\mathcal{E}^{\otimes}$ is distributive, we have that

$$\begin{array}{ccccc}
\prod_f \mathcal{K} \times_{(\mathcal{T}^{/V})^{\mathrm{op}}} \mathcal{C}^{\otimes}_{\underline{x}} & \xrightarrow{\prod_f (p')} & \mathcal{E}^{\otimes}_{\underline{x}} & \xrightarrow{\alpha_\otimes} & \mathcal{E}_{\underline{y}} \\
\downarrow & & & \nearrow_{\alpha_\otimes \circ \prod_f (q')} & \\
\mathcal{C}^{\otimes}_{\underline{x}} & & & &
\end{array}$$

is a $\mathcal{T}^{/V}$-left Kan extension. By definition, $(\alpha_\otimes \circ \prod_f q)'$ is a $\mathcal{T}^{/V}$-left Kan extension of $\alpha_\otimes \circ \prod_f (q')$ along the inclusion $\mathcal{C}^{\otimes}_{\underline{x}} \subset \mathcal{C}^{\otimes}_{\underline{\alpha}}$. By transitivity of $\mathcal{T}^{/V}$-left Kan extensions, we are done. □

3.2.7. **Lemma.** *Suppose we have a diagram of $\mathcal{T}$-∞-categories*

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{i} & \mathcal{B} & \xrightarrow{F} & \mathcal{E} \\
\downarrow \phi & & \downarrow \phi & \nearrow_{H} & \\
\mathcal{C} & \xrightarrow{i} & \mathcal{D} & &
\end{array}$$

such that the lefthand square is a pushout square in which every $\mathcal{T}$-functor is an inclusion. Further suppose that the relevant $\mathcal{T}$-colimits exists for the $\mathcal{T}$-left Kan extensions of F along ϕ and of $F \circ i$ along ϕ to exist. Then H is a $\mathcal{T}$-left Kan extension of F if and only if $H \circ i$ is a $\mathcal{T}$-left Kan extension of $F \circ i$.

Proof. Consider the commutative diagram of $\mathcal{T}$-∞-categories

$$\begin{array}{ccccc}
\mathcal{P} & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D}, \mathcal{E}) & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{E}) \\
\sigma \uparrow\downarrow & & \downarrow & & \downarrow \\
\mathcal{T}^{\mathrm{op}} & \xrightarrow{\sigma_F} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{B}, \mathcal{E}) & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{A}, \mathcal{E}) \\
 & & \underset{\sigma_{Fi}}{\curvearrowright} & &
\end{array}$$

in which every square is a pullback square. Then $\mathcal{P}$ is identified with

$$\mathcal{T}^{\mathrm{op}} \times_{\sigma_{Fi}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{A}, \mathcal{E})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{E}) \simeq \mathcal{T}^{\mathrm{op}} \times_{\sigma_F, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{B}, \mathcal{E})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D}, \mathcal{E})$$

and the cocartesian section σ is equivalently determined by a $\mathcal{T}$-functor H extending F or a $\mathcal{T}$-functor G extending $F \circ i$. Moreover, because of our hypothesis on the existence of the relevant $\mathcal{T}$-colimits, σ is an $\mathcal{T}$-initial object in $\mathcal{P}$ if and only if H is a $\mathcal{T}$-left Kan extension of F along ϕ if and only if G is a $\mathcal{T}$-left Kan extension of $F \circ i$ along ϕ. □

3.2.8. **Example** (Smash product)**.** Define a functor

$$(\underline{\Delta^1})^{\otimes} : \mathrm{Span}(\mathbf{F}_{\mathcal{T}}) \longrightarrow \mathrm{hoSpan}(\mathbf{F}_{\mathcal{T}}) \longrightarrow \mathbf{Cat}_1$$

by sending an object $U = \coprod_I U_i$ (decomposed as a disjoint union of orbits) to $\prod_I \Delta^1$, and a morphism $f : \coprod_I U_i \longrightarrow \coprod_J V_j$, $\phi : I \longrightarrow J$ contravariantly to the restriction functor $\phi^* : \prod_J \Delta^1 \longrightarrow \prod_I \Delta^1$ and covariantly to the product over J of

$$\min : \prod_{I_j} \Delta^1 \longrightarrow \Delta^1,\ (x_i) \mapsto \min(x_i)$$

if I_j is nonempty, and $1 : \Delta^0 \longrightarrow \Delta^1$ otherwise. (One easily verifies the base-change condition, so this indeeds defines a functor.)

Let $(\Delta^1 \times \mathcal{T}^{\mathrm{op}})^{\otimes}$ denote the resulting $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category under the equivalence of Theorem 2.3.9. Let $\mathcal{C}^{\otimes}$ be a distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. Then by Theorem 3.2.6, we have that $\underline{\mathrm{Fun}}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C})^{\otimes}$ is a distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. Moreover, the fiberwise tensor products admit a simple description because the underlying $\mathcal{T}$-category of the source is constant. Namely, for the fold map $\nabla : U \coprod U \longrightarrow U$, the tensor product

$$\otimes : \mathrm{Fun}(\Delta^1, \mathcal{C}_U) \times \mathrm{Fun}(\Delta^1, \mathcal{C}_U) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}_U)$$

is given by taking the cartesian product into $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}_U \times \mathcal{C}_U)$, postcomposition by $\otimes : \mathcal{C}_U \times \mathcal{C}_U \longrightarrow \mathcal{C}_U$, and then left Kan extension along $\min : \Delta^1 \times \Delta^1 \longrightarrow \Delta^1$ (which is computed by taking colimits fiberwise because min is a cocartesian fibration).

Now suppose in addition that $\mathcal{C}^{\otimes}$ is $\mathcal{T}$-*cartesian* $\mathcal{T}$-symmetric monoidal, so the tensor product on the fibers of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C})$ is the pushout product. Let $\mathcal{C}_*$ denote the full $\mathcal{T}$-subcategory of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C})$ given over an object $U \in \mathcal{T}$ by those functors $\Delta^1 \longrightarrow \mathcal{C}_U$ which take 0 to a final object $*$ of $\mathcal{C}_U$. The inclusion $\mathcal{C}_* \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\Delta^1 \times \mathcal{T}^{\mathrm{op}}, \mathcal{C})$ admits a left adjoint L, which over an object $U \in \mathcal{T}$ takes a functor $F : \Delta^1 \longrightarrow \mathcal{C}_U$ to

$$L(F) : \Delta^1 \longrightarrow \mathcal{C}_U,\ [0 \to 1] \rightsquigarrow [* \to F(1)/F(0)].$$

L is a $\mathcal{T}$-localization functor, so to descend the cartesian $\mathcal{T}$-symmetric monoidal structure on $\mathcal{C}$ to the *smash product* on $\mathcal{C}_*$, we can check the criterion of Theorem 2.9.2 (or rather, Remark 2.9.3):

(1) For the fold map $\nabla : U \coprod U \longrightarrow U$ and an object $[z_0 \longrightarrow z_1] \in \mathrm{Fun}(\Delta^1, \mathcal{C}_U)$,

$$- \otimes [z_0 \longrightarrow z_1] : \mathrm{Fun}(\Delta^1, \mathcal{C}_U) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}_U)$$

preserves L_U-equivalences. Indeed, let

$$\begin{array}{ccc} x_0 & \longrightarrow & y_0 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & y_1 \end{array}$$

be an L_U-equivalence, i.e. $x_1/x_0 \longrightarrow y_1/y_0$ is an equivalence in $\mathcal{C}_U$. We have an equivalence

$$(x_1 \times z_1) \,/\, (x_0 \times z_1 \cup_{x_0 \times z_0} x_1 \times z_0) \simeq \left(\frac{x_1 \times z_1}{x_0 \times z_1}\right) / \left(\frac{x_1 \times z_0}{x_0 \times z_0}\right).$$

Using that we have a diagram of pushout squares

$$\begin{array}{ccccc} x_0 \times z_j & \longrightarrow & z_j & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ x_1 \times z_j & \longrightarrow & x_1/x_0 \times z_j & \longrightarrow & (x_1 \times z_j)/(x_0 \times z_j) \end{array}$$

(and similarly for y) by cartesian closedness, we deduce that

$$\left(\frac{x_1 \times z_1}{x_0 \times z_1}\right) / \left(\frac{x_1 \times z_0}{x_0 \times z_0}\right) \longrightarrow \left(\frac{y_1 \times z_1}{y_0 \times z_1}\right) / \left(\frac{y_1 \times z_0}{y_0 \times z_0}\right)$$

is an equivalence, as desired.

(2) For a map $f : U \longrightarrow V$ in $\mathcal{T}$,

$$f_{\otimes} : \mathrm{Fun}(\Delta^1, \mathcal{C}_U) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}_V)$$

sends L_U-equivalences to L_V-equivalences: to show this, suppose

$$\theta : [x_0 \to x_1] \longrightarrow [y_0 \to y_1]$$

is a L_U-equivalence in $\mathrm{Fun}(\Delta^1, \mathcal{C}_U)$. Equivalently, we have a left Kan extension

$$\begin{array}{ccc} \Lambda^2_0 \times \Delta^1 & \longrightarrow & \mathcal{C}_U \\ \downarrow & \nearrow & \\ (\Lambda^2_0)^{\triangleright} \times \Delta^1 & & \end{array}$$

where restriction to the cone point is sent to an equivalence $x_1/x_0 \xrightarrow{\sim} y_1/y_0$. Using distributivity, we get a $\mathcal{T}^{/V}$-left Kan extension diagram

$$\begin{array}{ccc} \prod_f ((\Lambda^2_0 \times \Delta^1) \times (\mathcal{T}^{/U})^{\mathrm{op}}) & \longrightarrow & \prod_f \mathcal{C}_{\underline{U}} \xrightarrow{\prod_f} \mathcal{C}_{\underline{V}} \\ \downarrow & \nearrow & \\ ((\Lambda^2_0)^{\triangleright} \times \Delta^1) \times (\mathcal{T}^{/V})^{\mathrm{op}} & & \end{array}$$

The vertical arrow factors as

$$\prod_f ((\Lambda^2_0 \times \Delta^1) \times (\mathcal{T}^{/U})^{\mathrm{op}}) \longrightarrow (\Lambda^2_0 \times \Delta^1) \times (\mathcal{T}^{/V})^{\mathrm{op}} \longrightarrow ((\Lambda^2_0)^{\triangleright} \times \Delta^1) \times (\mathcal{T}^{/V})^{\mathrm{op}}$$

where the first arrow is induced by the symmetric monoidal structure on $\Delta^1 \times \mathcal{T}^{\mathrm{op}}$. Therefore, the left Kan extension corresponding to $f_\otimes(\theta)$

$$\begin{array}{ccc} \Lambda^2_0 \times \Delta^1 & \longrightarrow & \mathcal{C}_V \\ \downarrow & \nearrow & \\ (\Lambda^2_0)^{\triangleright} \times \Delta^1 & & \end{array}$$

restricts on the cone point to an equivalence, so $f_\otimes(\theta)$ is a L_V-equivalence.

In particular, suppose that $\mathcal{T} = \mathbf{O}_G$ and $\mathcal{C} = \underline{\mathbf{Spc}}_G$. Then we obtain the smash product G-symmetric monoidal structure on pointed G-spaces $\underline{\mathbf{Spc}}_{G,*}$, and given a map of orbits $f : G/H \longrightarrow G/K$ corresponding to an inclusion of subgroups $K \subset H$ and a real K-representation R, the norm functor $f_\otimes$ sends the representation sphere S^R to the representation sphere $S^{\mathbf{Ind}^H_K R}$.

3.3. **Pointwise $\mathcal{O}$-monoidal structure.** In this brief subsection, we indicate how to adapt the construct of the $\mathcal{T}$-Day convolution so as to construct the cotensor of $\mathbf{Op}_{\mathcal{O},\mathcal{T}}$ over $\mathbf{Cat}_{\mathcal{T}}$. In other words, given a fibration of $\mathcal{T}$-∞-operads $p : \mathcal{D}^\otimes \longrightarrow \mathcal{O}^\otimes$ and a $\mathcal{T}$-∞-category $\mathcal{K}$, we have a $\mathcal{T}$-∞-operad $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{K} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{O}, \mathcal{D})^\otimes$ that satisfies the universal mapping property

$$\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{K} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{O}, \mathcal{D})) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})) \tag{3.3.1}$$

for all fibrations of ∞-operads $q : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$.

For the following construction, observe the isomorphism

$$\mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathrm{ev}_1, \mathcal{O}^\otimes, \mathrm{pr}} (\mathcal{O}^\otimes \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}) \cong \mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}.$$

3.3.1. **Theorem-Construction.** Let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-∞-operad and $\mathcal{K}$ a $\mathcal{T}$-∞-category. Consider the span diagram of marked simplicial sets

$$(\mathcal{O}^\otimes, \mathrm{Ne}) \xleftarrow{\mathrm{ev}_0} (\mathrm{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{K}, \mathrm{Ne}) \xrightarrow{\mathrm{ev}_1} (\mathcal{O}^\otimes, \mathrm{Ne}).$$

where by the middle marking Ne we mean those edges whose source and target in $\mathcal{O}^\otimes$ are inert and whose projection to $\mathcal{K}$ is a cocartesian edge. Then the functor

$$(\mathrm{ev}_0)_* \circ (\mathrm{ev}_1)^* : \mathbf{sSet}^+_{/(\mathcal{O}^\otimes, \mathrm{Ne})} \longrightarrow \mathbf{sSet}^+_{/(\mathcal{O}^\otimes, \mathrm{Ne})}$$

is right Quillen with respect to the $\mathcal{T}$-operadic model structures. For a fibration of ∞-operads $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$, we let

$$\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{K} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{O}, \mathcal{C})^\otimes := (\mathrm{ev}_0)_*(\mathrm{ev}_1)^*(\mathcal{C}^\otimes, \mathrm{Ne}).$$

If $\mathcal{O} \simeq \mathcal{T}^{\mathrm{op}}$, then we more simply write

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})^\otimes := (\mathrm{ev}_0)_*(\mathrm{ev}_1)^*(\mathcal{C}^\otimes, \mathrm{Ne}).$$

This construction satisfies the universal property (3.3.1) and its underlying $\mathcal{T}$-∞-category is as the notation indicates.

Proof. This follows along the same lines as our proofs of the analogous results for $\mathcal{T}$-Day convolution in Section 3.1. ☐

3.3.2. **Remark.** In the situation of Theorem-Construction 3.3.1, a fibrant replacement of $(\mathcal{O}^{\otimes}, \mathrm{Ne}) \times_{\mathcal{T}^{op}} {}_{\natural}\mathcal{K}$ in the $\mathcal{T}$-operadic model structure on $\mathbf{sSet}^{+}_{/(\mathcal{O}^{\otimes},\mathrm{Ne})}$ computes the *tensor* of $\mathbf{Op}_{\mathcal{O},\mathcal{T}}$ over $\mathbf{Cat}_{\mathcal{T}}$.

If we then suppose that $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category, we obtain the *pointwise* $\mathcal{O}$-monoidal structure on $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})$. In contrast to the $\mathcal{T}$-Day convolution, we don't need to impose any further hypotheses on $\mathcal{C}^{\otimes}$ for this to exist.

3.3.3. **Theorem-Construction.** Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad and $\mathcal{K}$ a $\mathcal{T}$-∞-category. Consider the span diagram of marked simplicial sets

$$(\mathcal{O}^{\otimes})^{\sharp} \xleftarrow{\mathrm{ev}_0} \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes})^{\sharp} \times_{\mathcal{T}^{op}} {}_{\natural}\mathcal{K} \xrightarrow{\mathrm{ev}_1} (\mathcal{O}^{\otimes})^{\sharp}.$$

Then the functor

$$(\mathrm{ev}_0)_* \circ (\mathrm{ev}_1)^* : \mathbf{sSet}^{+}_{/\mathcal{O}^{\otimes}} \longrightarrow \mathbf{sSet}^{+}_{/\mathcal{O}^{\otimes}}$$

agrees with the construction of Theorem-Construction 3.3.1 on underlying simplicial sets and is right Quillen with respect to the $\mathcal{T}$-monoidal model structures. Given any $\mathcal{O}$-monoidal $\mathcal{T}$-∞-categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$, we then have a natural equivalence

$$\mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{K} \times_{\mathcal{T}^{op}} \mathcal{O}, \mathcal{D})) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{K}, \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})).$$

Proof. That $(\mathrm{ev}_0)_* \circ (\mathrm{ev}_1)^*$ is right Quillen follows by [Lur17, Thm. B.4.2] as in the proof of Theorem 3.1.4. The only differences to note are regarding conditions (4) and (7). For (4), by [Sha21b, Prop. 3.5(1)] the ev_0-cartesian edges in $\mathrm{Ar}^{ne}(\mathcal{O}^{\otimes})$ are fiberwise active in the target, hence $\mathrm{ev}_0 : \mathrm{Ar}^{ne}(\mathcal{O}^{\otimes}) \times_{\mathcal{T}^{op}} \mathcal{K} \longrightarrow \mathcal{O}^{\otimes}$ is a cartesian fibration whose cartesian edges project to equivalences in $\mathcal{K}$. (7) then follows by inspection.

The universal mapping property then follows by restricting the equivalence (3.3.1). ☐

3.3.4. **Example.** Suppose $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, let $\mathcal{C}^{\otimes}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category, and let $\mathcal{K}$ be a $\mathcal{T}$-∞-category. Then we may unwind the pointwise $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})$ as follows:

(∗) Let $f : U \longrightarrow V$ be a map of finite $\mathcal{T}$-sets. Then the norm functor

$$f_{\otimes} : \mathrm{Fun}_{\mathcal{T}^{/U}}(\mathcal{K}_{\underline{U}}, \mathcal{C}_{\underline{U}}) \longrightarrow \mathrm{Fun}_{\mathcal{T}^{/V}}(\mathcal{K}_{\underline{V}}, \mathcal{C}_{\underline{V}})$$

sends a $\mathcal{T}^{/U}$-functor $F : \mathcal{K}_{\underline{U}} \longrightarrow \mathcal{C}_{\underline{U}}$ to the $\mathcal{T}^{/V}$-functor

$$\mathcal{K}_{\underline{V}} \xrightarrow{\eta} \prod_f \mathcal{K}_{\underline{U}} \simeq \mathrm{Fun}_{\mathcal{T}^{/V}}(\underline{U}, \mathcal{K}_{\underline{V}}) \xrightarrow{\prod_f F} \prod_f \mathcal{C}_{\underline{U}} \simeq \mathrm{Fun}_{\mathcal{T}^{/V}}(\underline{U}, \mathcal{C}_{\underline{V}}) \xrightarrow{f_{\otimes}} \mathcal{C}_{\underline{V}}$$

where η is the diagonal $\mathcal{T}^{/V}$-functor (i.e., the unit of the restriction-coinduction adjunction) and the norm $\mathcal{T}^{/V}$-functor $f_{\otimes}$ is defined as in Example 2.3.5.

In particular, note that if $\mathcal{C}^{\otimes}$ is distributive, then $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})^{\otimes}$ is also distributive.

4. Parametrized operadic left Kan extensions

In this section, we construct $\mathcal{T}$-operadic left Kan extensions, implementing in the operadic context the strategy that the second author used to construct $\mathcal{T}$-left Kan extensions in [Sha21a, §§9-10].

4.0.1. **Remark.** The strategy of our construction of $\mathcal{T}$-operadic left Kan extensions will be to first show that the $\mathcal{T}$-colimit of a lax $\mathcal{O}$-monoidal functor canonically inherits a $\mathcal{O}$-algebra structure, and to then reduce to this case via the $\mathcal{O}$-monoidal envelope. If we let $\mathcal{T} = \Delta^0$, this gives a new construction of Lurie's operadic left Kan extension ([Lur17, §3.1.2]). See also [CH21].

We first begin with some necessary preliminaries on generalized $\mathcal{T}$-∞-operads and the $\mathcal{T}$-operadic join.

4.1. **Generalized $\mathcal{T}$-∞-operads.** Given a map of finite $\mathcal{T}$-sets $\phi : U \longrightarrow V$, let σ_1, σ_2 be two squares in $\mathbf{F}_{\mathcal{T}}$

$$\begin{array}{ccc} U_i & \xrightarrow{\alpha_i} & U \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi} \\ V_i & \xrightarrow{\beta_i} & V \end{array}$$

such that α_i is a summand inclusion, the induced map $U_i \longrightarrow V_i \times_V U$ is also a summand inclusion, and moreover $(\alpha_1, \alpha_2) : U_1 \sqcup U_2 \longrightarrow U$ is an epimorphism. Let $U_{12} = U_1 \times_U U_2$ and $V_{12} = V_1 \times_V V_2$. Then we have an induced map

$$g_{\phi,\sigma_1,\sigma_2} : (\Lambda^2_2)^{\lhd} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$$

which selects the square

$$\begin{array}{ccc} [U_+ \to V] & \longrightarrow & [(U_1)_+ \to V_1] \\ \downarrow & & \downarrow \\ [(U_2)_+ \to V_2] & \longrightarrow & [(U_{12})_+ \to V_{12}] \end{array}$$

in which every morphism is inert.

We define the *generalized* $\mathcal{T}$*-operadic model structure* on $\mathbf{sSet}^+_{/(\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*},\mathrm{Ne})}$ to be the model structure defined using the categorical pattern $(\mathrm{Ne}, \mathrm{All}, \{g_{\phi,\sigma_1,\sigma_2}\})$ on $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$, letting the ϕ and $\{\sigma_1, \sigma_2\}$ range over all possible choices. We call the fibrant objects for this model structure *generalized* $\mathcal{T}$-∞*-operads*. Note that $\mathcal{T}$-∞-operads are fibrant in this model structure, so are examples of generalized $\mathcal{T}$-∞-operads. However, the converse is not true. Indeed, let $\sigma_0 : \mathbf{F}^{\mathrm{op}}_{\mathcal{T}} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ be the cocartesian section which selects $[\emptyset_+ \to V]$ in each fiber and define $\widetilde{\mathcal{C}}_0 := \mathbf{F}^{\mathrm{op}}_{\mathcal{T}} \times_{\sigma_0, \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}} \widetilde{\mathcal{C}}^{\otimes}$. Then if $\widetilde{\mathcal{C}}^{\otimes} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ is a generalized $\mathcal{T}$-∞-operad, $\widetilde{\mathcal{C}}_0$ is not necessarily the terminal $\mathcal{T}$-∞-category.

We then say that $\mathcal{C}^{\otimes} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ is a *generalized* $\mathcal{T}$-∞*-operad* if it is the pullback of a generalized $\mathcal{T}$-∞-operad $\widetilde{\mathcal{C}}^{\otimes} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ under the inclusion $\underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$. Let $\mathcal{C}_0$ be the corresponding pullback of $\widetilde{\mathcal{C}}_0$. The $\mathcal{T}$-functor $\mathcal{T}^{\mathrm{op}} \times \Delta^1 \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ which selects the inert edge $[V_+ \to V] \longrightarrow [\emptyset_+ \to V]$ in each fiber induces a $\mathcal{T}$-functor $\mathcal{C} \longrightarrow \mathcal{C}_0$.

4.1.1. **Lemma.** *Suppose $\mathcal{C}^{\otimes} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ is a generalized $\mathcal{T}$-∞-operad. Let $f : U \longrightarrow V$ be a morphism in $\mathbf{F}_{\mathcal{T}}$ and suppose that we have an orbit decomposition $U \simeq \sqcup_{1 \leq i \leq n} U_i$. Abbreviate $U_i \times_V U_j$ as U_{ij} and let $f_i : U_i \longrightarrow V$, $f_{ij} : U_{ij} \longrightarrow V$ denote the induced maps. Then we have an equivalence of $\mathcal{T}^{/V}$-∞-categories*

$$\underline{\mathcal{C}^{\otimes}_{[U_+ \to V]}} \longrightarrow \prod_{f_1} \mathcal{C}_{\underline{U_1}} \underset{\prod_{f_{12}} (\mathcal{C}_0)_{\underline{U_{12}}}}{\times} \prod_{f_2} \mathcal{C}_{\underline{U_2}} \underset{\prod_{f_{23}} (\mathcal{C}_0)_{\underline{U_{23}}}}{\times} \cdots \underset{\prod_{f_{(n-1)n}} (\mathcal{C}_0)_{\underline{U_{(n-1)n}}}}{\times} \prod_{f_n} \mathcal{C}_{\underline{U_n}}$$

Proof. Note that U_{ij} need not be an orbit, so the notation $(\mathcal{C}_0)_{\underline{U_{ij}}}$ means $\coprod_{\mathcal{O} \subset U_{ij}} (\mathcal{C}_0)_{\underline{\mathcal{O}}} \longrightarrow \underline{U_{ij}}$ for any orbit decomposition of U_{ij}. Without loss of generality we may suppose $n = 2$. We have a pullback square in $\mathbf{F}_{\mathcal{T}}$

$$\begin{array}{ccc} U_{12} & \xrightarrow{g_2} & U_1 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle f_1} \\ U_2 & \xrightarrow{f_2} & V. \end{array}$$

The unit maps for the restriction-coinduction adjunction yield $\mathcal{T}^{/V}$-functors

$$\prod_{f_1} \mathcal{C}_{\underline{U_1}} \longrightarrow \prod_{f_{12}} \mathcal{C}_{\underline{U_{12}}} \longleftarrow \prod_{f_1} \mathcal{C}_{\underline{U_1}}$$

and postcomposing with

$$\prod_{f_{12}} \mathcal{C}_{\underline{U_{12}}} \longrightarrow \prod_{f_{12}} (\mathcal{C}_0)_{\underline{U_{12}}}$$

we obtain the $\mathcal{T}^{/V}$-functors which define the pullback. Using the 'fiberwise' definition of generalized $\mathcal{T}$-∞-operad, we can use the same argument as in the proof of Theorem 2.3.3 to accomplish the proof. $\square$

4.2. $\mathcal{T}$-operadic join.

4.2.1. **Definition.** Let $\widetilde{\mathcal{C}}^\otimes, \widetilde{\mathcal{D}}^\otimes, \widetilde{\mathcal{O}}^\otimes$ be generalized $\mathcal{T}$-∞-operads over $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ and let $p, q : \widetilde{\mathcal{C}}^\otimes, \widetilde{\mathcal{D}}^\otimes \longrightarrow \widetilde{\mathcal{O}}^\otimes$ be categorical fibrations preserving the inert edges. Then the $\mathcal{T}$-*operadic join* of p and q is defined to be

$$\widetilde{(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})}^\otimes := \widetilde{\mathcal{C}}^\otimes \star_{\widetilde{\mathcal{O}}^\otimes} \widetilde{\mathcal{D}}^\otimes \longrightarrow \widetilde{\mathcal{O}}^\otimes.$$

Similarly, given $\mathcal{C}^\otimes, \mathcal{D}^\otimes \longrightarrow \mathcal{O}^\otimes$ fibrations of generalized $\mathcal{T}$-∞-operads over $\underline{\mathbf{F}}_{\mathcal{T},*}$, we define the $\mathcal{T}$-operadic join $(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})^\otimes$ to be $\mathcal{C}^\otimes \star_{\mathcal{O}^\otimes} \mathcal{D}^\otimes \longrightarrow \mathcal{O}^\otimes$.

Note that the underlying $\mathcal{T}$-∞-category of a $\mathcal{T}$-operadic join $(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})^\otimes$ is the $\mathcal{O}$-join of the underlying $\mathcal{T}$-∞-categories of the factors (by the base-change property [Sha21a, Lem. 4.4]).

4.2.2. **Proposition.** $\widetilde{(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})}^\otimes$ *is a generalized* $\mathcal{T}$-∞-*operad and the structure map to* $\widetilde{\mathcal{O}}^\otimes$ *is a fibration of generalized* $\mathcal{T}$-∞-*operads.*

Proof. By the proof of [Sha21a, Prop. 4.7(2)], $\pi : \widetilde{(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})}^\otimes \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ admits cocartesian lifts over the inert edges. Moreover, an edge $\Delta^1 \longrightarrow \widetilde{(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})}^\otimes$ is π-cocartesian if and only if it factors through either $\widetilde{\mathcal{C}}^\otimes$ or $\widetilde{\mathcal{D}}^\otimes$ and is inert there.

The fiber of $\widetilde{(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})}^\otimes$ over an object $[U_+ \to V]$ of $\underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$ is given by $\widetilde{\mathcal{C}}^\otimes_{[U_+ \to V]} \star_{\widetilde{\mathcal{O}}^\otimes_{[U_+ \to V]}} \widetilde{\mathcal{D}}^\otimes_{[U_+ \to V]}$. Moreover, the relative join is functorial in the following sense: given commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}, \quad \begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

we have an induced map $X \star_B Y \longrightarrow X' \star_{B'} Y'$ covering $B \times \Delta^1 \longrightarrow B' \times \Delta^1$. From our explicit description of the π-cocartesian edges, we see that the Segal maps for $\widetilde{(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})}^\otimes$ are obtained in this way. Consequently, it is clear that they are equivalences.

It remains to check that for all of the defining maps $g_{\phi,\sigma_1,\sigma_2} : (\Lambda^2_2)^\triangleleft \longrightarrow \underline{\mathbf{F}}^{\mathrm{big}}_{\mathcal{T},*}$, we have that for every lift $G : (\Lambda^2_2)^\triangleleft \longrightarrow \widetilde{(\mathcal{C} \star_{\mathcal{O}} \mathcal{D})}^\otimes$ where all edges are sent to π-cocartesian edges, G is a π-limit diagram. For this, there are two cases to consider. Either G factors through $\widetilde{\mathcal{C}}^\otimes$, in which case the assertion obviously follows from G being a $\pi|_{\widetilde{\mathcal{C}}^\otimes}$-limit diagram, or G factors through $\widetilde{\mathcal{D}}^\otimes$, in which case the assertion is a consequence of Lemma 4.2.3.

Finally, the second assertion is obvious given the first. ☐

4.2.3. **Lemma.** *Let K, X, Y and B be ∞-categories, let $f_1, f_2 : X, Y \longrightarrow B$ be categorical fibrations and let $p : K \longrightarrow Y$ be a functor. Also let p denote the composition $K \xrightarrow{p} Y \subset X \star_B Y$. Then we have an equivalence of ∞-categories over $B^{/f_2 p} \times \Delta^1$*

$$(X \star_B Y)^{/p} \simeq (X \times_B B^{/f_2 p}) \star_{B^{/f_2 p}} Y^{/p}.$$

Consequently, if $\overline{p} : K^\triangleleft \longrightarrow Y$ is a f_2-limit diagram, then $\overline{p} : K^\triangleleft \longrightarrow Y \subset X \star_B Y$ is a f-limit diagram (where f denotes the structure map $X \star_B Y \longrightarrow B$).

Proof. Let q denote $K \longrightarrow X \star_B Y \longrightarrow B \times \Delta^1$ and note that q factors as $K \xrightarrow{f_2 p} B \times \{1\} \subset B \times \Delta^1$. We first place $(X \star_B Y)^{/p}$ into the diagram of pullback squares

$$\begin{array}{ccccc} (X \star_B Y)^{/p} & \longrightarrow & Z & \longrightarrow & \mathrm{Fun}(K^\triangleleft, X \star_B Y) \\ \downarrow & & \downarrow & & \downarrow \\ \{p\} & \longrightarrow & \mathrm{Fun}_{/B}(K, Y) & \longrightarrow & \mathrm{Fun}(K, X \star_B Y) \\ & & \downarrow & & \downarrow \\ & & \{q\} & \longrightarrow & \mathrm{Fun}(K, B \times \Delta^1). \end{array}$$

In addition, using that the composition $\mathrm{Fun}(K^{\triangleleft}, X \star_B Y) \longrightarrow \mathrm{Fun}(K, X \star_B Y) \longrightarrow \mathrm{Fun}(K, B \times \Delta^1)$ agrees with $\mathrm{Fun}(K^{\triangleleft}, X \star_B Y) \longrightarrow \mathrm{Fun}(K^{\triangleleft}, B \times \Delta^1) \longrightarrow \mathrm{Fun}(K, B \times \Delta^1)$, Z fits into the diagram of pullback squares

$$\begin{array}{ccc}
Z & \longrightarrow & \mathrm{Fun}(K^{\triangleleft}, X \star_B Y) \\
\downarrow & & \downarrow \\
(B \times \Delta^1)^{/q} & \longrightarrow & \mathrm{Fun}(K^{\triangleleft}, B \times \Delta^1) \cong \mathrm{Fun}(K^{\triangleleft}, B) \times \mathrm{Fun}(K^{\triangleleft}, \Delta^1) \\
\downarrow & & \downarrow \\
\{q\} = \{f_2 p\} \times \{\mathrm{const}_1\} & \longrightarrow & \mathrm{Fun}(K, B \times \Delta^1) \cong \mathrm{Fun}(K, B) \times \mathrm{Fun}(K, \Delta^1).
\end{array}$$

Because $(\Delta^1)^{/\mathrm{const}_1} \simeq \Delta^1$, we get that $(B \times \Delta^1)^{/q} \simeq B^{/f_2 p} \times \Delta^1$. Consequently,

$$(X \star_B Y)^{/p} \simeq \{p\} \times_{\mathrm{Fun}_{/B}(K,Y)} \left(\mathrm{Fun}(K^{\triangleleft}, X \star_B Y) \times_{\mathrm{Fun}(K^{\triangleleft}, B \times \Delta^1)} B^{/f_2 p} \times \Delta^1 \right).$$

Let $A \longrightarrow B^{/f_2 p} \times \Delta^1$ be any functor. We will identify $\mathrm{Fun}_{/(B^{/f_2 p} \times \Delta^1)}(A, (X \star_B Y)^{/p})$ with $\mathrm{Fun}_{/B}(A_0, X) \times \mathrm{Fun}_{/(B^{/f_2 p})}(A_1, Y^{/p})$ and thereby prove the claim. Let

$$A \times K^{\triangleleft} \longrightarrow B^{/f_2 p} \times \Delta^1 \times K^{\triangleleft} \longrightarrow B \times \Delta^1$$

be the composite of the given map and the map adjoint to $B^{/f_2 p} \times \Delta^1 \longrightarrow \mathrm{Fun}(K^{\triangleleft}, B \times \Delta^1)$. Note that $(A \times K^{\triangleleft})_0 \cong A_0$ and $(A \times K^{\triangleleft})_1 \cong (A \times K) \cup_{A_1 \times K} A_1 \times K^{\triangleleft}$. We then have the chain of equivalences

$$\begin{aligned}
\mathrm{Fun}_{/(B^{/f_2 p} \times \Delta^1)}(A, (X \star_B Y)^{/p}) &\simeq \{p \circ \mathrm{pr}_K\} \times_{\mathrm{Fun}(A \times K, X \star_B Y)} \mathrm{Fun}_{/(B \times \Delta^1)}(A \times K^{\triangleleft}, X \star_B Y) \\
&\simeq \{p \circ \mathrm{pr}_K\} \times_{\mathrm{Fun}(A \times K, X \star_B Y)} (\mathrm{Fun}_{/B}(A_0, X) \times \mathrm{Fun}_{/B}((A \times K) \cup_{A_1 \times K} A_1 \times K^{\triangleleft}, Y)) \\
&\simeq \mathrm{Fun}_{/B}(A_0, X) \times \left(\{p \circ \mathrm{pr}_K\} \underset{\mathrm{Fun}_{/B}(A \times K, Y)}{\times} \mathrm{Fun}_{/B}(A \times K, Y) \underset{\mathrm{Fun}_{/B}(A_1 \times K, Y)}{\times} \mathrm{Fun}_{/B}(A_1 \times K^{\triangleleft}, Y) \right) \\
&\simeq \mathrm{Fun}_{/B}(A_0, X) \times \left(\{p \circ \mathrm{pr}_K\} \underset{\mathrm{Fun}_{/B}(A_1 \times K, Y)}{\times} \mathrm{Fun}_{/B}(A_1 \times K^{\triangleleft}, Y) \right)
\end{aligned}$$

and finally

$$\{p \circ \mathrm{pr}_K\} \underset{\mathrm{Fun}_{/B}(A_1 \times K, Y)}{\times} \mathrm{Fun}_{/B}(A_1 \times K^{\triangleleft}, Y) \simeq \mathrm{Fun}_{/(B^{/f_2 p})}(A_1, Y^{/p})$$

because both sides compute the total fiber of the punctured cube

$$\begin{array}{ccccc}
 & & & & \mathrm{Fun}(A_1 \times K^{\triangleleft}, Y) \\
 & & & \swarrow & \downarrow \\
\Delta^0 & \longrightarrow & \mathrm{Fun}(A_1 \times K, Y) & & \\
\downarrow & & \downarrow & & \\
 & \Delta^0 & \longrightarrow & & \mathrm{Fun}(A_1 \times K^{\triangleleft}, B) \\
 & \swarrow & \downarrow & \swarrow & \\
\Delta^0 & \longrightarrow & \mathrm{Fun}(A_1 \times K, B) & &
\end{array}$$

For the last assertion, we need to show that

$$\begin{array}{ccc}
(X \star_B Y)^{/\overline{p}} & \longrightarrow & (X \star_B Y)^{/p} \\
\downarrow & & \downarrow \\
B^{/f\overline{p}} & \longrightarrow & B^{/fp}
\end{array}$$

is a homotopy pullback square. But by the first part of the lemma this is equivalent to

$$\begin{array}{ccc} (X \times_B B^{/f\overline{p}}) \star_{B^{/f\overline{p}}} Y^{/\overline{p}} & \longrightarrow & (X \times_B B^{/fp}) \star_{B^{/fp}} Y^{/p} \\ \downarrow & & \downarrow \\ B^{/f\overline{p}} & \longrightarrow & B^{/fp}. \end{array}$$

One easily deduces that this is a homotopy pullback square as a consequence of the two squares

$$\begin{array}{ccc} X \times_B B^{/f\overline{p}} & \longrightarrow & X \times_B B^{/fp} \\ \downarrow & & \downarrow \\ B^{/f\overline{p}} & \longrightarrow & B^{/fp} \end{array} \quad , \quad \begin{array}{ccc} Y^{/\overline{p}} & \longrightarrow & Y^{/p} \\ \downarrow & & \downarrow \\ B^{/f\overline{p}} & \longrightarrow & B^{/fp} \end{array}$$

being homotopy pullback squares (the second by our hypothesis). □

4.3. **Construction of $\mathcal{T}$-operadic left Kan extensions.** We now turn towards constructing $\mathcal{T}$-operadic left Kan extensions. We will need a variant of the $\mathcal{T}$-Day convolution for our proofs, where we allow the source $\mathcal{T}$-∞-operad to be generalized.

4.3.1. **Variant.** Suppose that $\mathcal{C}^\otimes$ is a generalized $\mathcal{T}$-∞-operad. Then the proofs of Theorem 3.1.4 and Corollary 3.1.5 still go through to show that

$$\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^\otimes \longrightarrow \mathcal{O}^\otimes$$

is a (non-generalized) $\mathcal{T}$-∞-operad. Moreover, if $\mathcal{C}^\otimes, \mathcal{E}^\otimes \longrightarrow \mathcal{O}^\otimes$ are cocartesian fibrations, then the proof of Proposition 3.2.2 goes through to show that $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^\otimes \longrightarrow \mathcal{O}^\otimes$ is a locally cocartesian fibration. However, the proof of Theorem 3.2.6 doesn't directly apply because if $\mathcal{C}^\otimes$ is generalized, we have a different formula for $\mathcal{C}^\otimes_{\underline{x}}$ involving fiber products instead of products (cf. Lemma 4.1.1).

For the following results, let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-∞-operad, $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category and $q : \mathcal{E}^\otimes \longrightarrow \mathcal{O}^\otimes$ a distributive $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category.

4.3.2. **Lemma.** *Consider the commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C} \star_{\mathcal{O}} \mathcal{O}, \mathcal{E})^\otimes & \xrightarrow{\rho} & \mathcal{E}^\otimes \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle q} \\ \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^\otimes & \xrightarrow{\pi} & \mathcal{O}^\otimes. \end{array}$$

where λ is given by restriction along $\mathcal{C}^\otimes \subset (\mathcal{C} \star_{\mathcal{O}} \mathcal{O})^\otimes$, ρ is given by restriction along $\mathcal{O}^\otimes \subset (\mathcal{C} \star_{\mathcal{O}} \mathcal{O})^\otimes$ followed by the equivalence $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{O},\mathcal{E})^\otimes \xrightarrow{\simeq} \mathcal{E}^\otimes$ induced by precomposition with the identity section $\iota : \mathcal{O}^\otimes \longrightarrow \mathrm{Ar}^{ne}(\mathcal{O}^\otimes)$ (cf. Proposition 3.1.7), and π is the structure map.

1. *An edge e in $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C} \star_{\mathcal{O}} \mathcal{O}, \mathcal{E})^\otimes$ is locally $\pi\lambda$-cocartesian if and only if $\lambda(e)$ is locally π-cocartesian and $\rho(e)$ is locally q-cocartesian. Consequently, $\pi\lambda$ is cocartesian.*
2. *λ is $\mathcal{O}^\otimes$-cartesian and ρ is cocartesian.*

Proof. (1): ρ and λ are fibrations of $\mathcal{T}$-∞-operads by the functoriality of the Day convolution, hence preserve inert edges. By Lemma 3.2.1, it suffices to consider a locally $\pi\lambda$-cocartesian edge e over a fiberwise active edge $f : U \longrightarrow V \in \mathbf{F}_{\mathcal{T}}$ (identified with $[U_+ \to V] \longrightarrow [V_+ \to V]$ in $\underline{\mathbf{F}}_{\mathcal{T},*}$). Suppose e covers $\alpha : x \longrightarrow y$ in $\mathcal{O}^\otimes_V$. Then by Proposition 3.2.2(3), e corresponds to a $\mathcal{T}^{/V}$-left Kan extension

$$\begin{array}{ccccc} (\mathcal{C}^\otimes_{\underline{x}})^{\trianglerighteq} & \xrightarrow{F} & \mathcal{E}^\otimes_{\underline{x}} & \xrightarrow{\alpha_\otimes} & \mathcal{E}_{\underline{y}} \\ \downarrow{\scriptstyle (\alpha_\otimes)^{\trianglerighteq}} & \Downarrow & & \nearrow{\scriptstyle G} & \downarrow \\ (\mathcal{C}_{\underline{y}})^{\trianglerighteq} & & \longrightarrow & & (\mathcal{T}^{/V})^{\mathrm{op}}. \end{array}$$

Examining the pointwise formula defining $\mathcal{T}^{/V}$-left Kan extensions, we see that G is a $\mathcal{T}^{/V}$-left Kan extension of $\alpha_\otimes \circ F$ along $\alpha_\otimes \star \mathrm{id}$ if and only if $G|_{\mathcal{C}_{\underline{y}}}$ is a $\mathcal{T}^{/V}$-left Kan extension of $\alpha_\otimes \circ F|_{\mathcal{C}^\otimes_{\underline{x}}}$ along $\alpha_\otimes$ and $G|_{\underline{V}} \simeq \alpha_\otimes \circ F|_{\underline{V}}$

(for the latter, using that the inclusion of the right $\mathcal{T}$-cone point is fiberwise cofinal and [Sha21a, Thm. 6.7]). This implies the first part of the claim. For the consequence, we only need to check that the composition of locally cocartesian edges is again locally cocartesian, and this is clear using the claim and Theorem 3.2.6.

(2): Taking the fiber over an object $y \in \mathcal{O}_V$, we get a bifibration

$$\mathrm{Fun}_{\mathcal{T}^{/V}}((\mathcal{C}_{\underline{y}})^{\underline{\triangleright}}, \mathcal{E}_{\underline{y}}) \longrightarrow \mathrm{Fun}_{\mathcal{T}^{/V}}(\mathcal{C}_{\underline{y}}, \mathcal{E}_{\underline{y}}) \times \mathcal{E}_{\underline{y}}.$$

Combining this observation with the product decomposition over a general object $x \in \mathcal{O}^{\otimes}$ obtained by the $\mathcal{T}$-Segal condition, we deduce that λ is fiberwise cartesian and ρ is fiberwise cocartesian (over $\mathcal{O}^{\otimes}$). It remains to show that for λ the cocartesian pushforward of fiberwise cartesian edges remain fiberwise cartesian, and the dual statement for ρ. This is obvious over inert edges, so it suffices to consider a fiberwise active edge $\alpha : x \longrightarrow y$ in $\mathcal{O}^{\otimes}_V$. Also, without loss of generality suppose $y \in \mathcal{O}_V$. Let $\theta : F_0 \longrightarrow F_1$ be an edge in $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}_x$, which corresponds to a natural transformation

$$\theta : \Delta^1 \times \mathcal{C}^{\otimes}_{\underline{x}} \longrightarrow \mathcal{E}^{\otimes}_{\underline{x}}.$$

A fiberwise cartesian edge in $\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C} \star_{\mathcal{O}} \mathcal{O}, \mathcal{E})^{\otimes}_x$ over θ is given by

$$\theta' : \Delta^1 \times (\mathcal{C}^{\otimes}_{\underline{x}})^{\underline{\triangleright}} \longrightarrow \mathcal{E}^{\otimes}_{\underline{x}}$$

which restricts to θ and is a constant natural transformation when restricted to the right $\mathcal{T}$-cone point. The cocartesian pushforward $\alpha_! \theta'$ is given by the $\mathcal{T}^{/V}$-left Kan extension of $\alpha_{\otimes} \circ \theta'$ along $\mathrm{id} \times \alpha_{\otimes}$. Clearly, this is still a constant natural transformation when restricted to the right $\mathcal{T}$-cone point, which proves that $\alpha_! \theta'$ is a fiberwise cartesian edge lifting $\alpha_! \theta$. A similar argument handles the fiberwise cocartesian edges. □

The next proposition is a very general and parametrized form of the following observation: the colimit of a lax symmetric monoidal functor canonically inherits the structure of a commutative algebra.

4.3.3. **Proposition.** *Let $F : \mathcal{C}^{\otimes} \longrightarrow \mathcal{E}^{\otimes}$ be a lax $\mathcal{O}$-monoidal $\mathcal{T}$-functor and let $\sigma_F : \mathcal{O}^{\otimes} \longrightarrow \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}$ be the associated section (which is an $\mathcal{O}$-algebra map). Then there exists a $\mathcal{O}$-algebra lift of σ_F to*

$$\sigma_{\overline{F}} : \mathcal{O}^{\otimes} \longrightarrow \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C} \star_{\mathcal{O}} \mathcal{O}, \mathcal{E})^{\otimes}$$

such that the resulting $\mathcal{O}$-algebra

$$A = \overline{F}|_{\mathcal{O}^{\otimes}} : \mathcal{O}^{\otimes} \longrightarrow \mathcal{E}^{\otimes}$$

has underlying section $A|_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathcal{E}$ computed as the $q|_{\mathcal{E}}$-$\mathcal{T}$-left Kan extension of $F|_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{E}$ along the inclusion $\mathcal{C} \longrightarrow \mathcal{C} \star_{\mathcal{O}} \mathcal{O}$.

Proof. Factoring σ_F through the $\mathcal{O}$-monoidal envelope $\mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes})$ of $\mathrm{id} : \mathcal{O}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, we obtain a pullback square of $\mathcal{O}$-monoidal $\mathcal{T}$-∞-categories and (strong) $\mathcal{O}$-monoidal $\mathcal{T}$-functors

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C} \star_{\mathcal{O}} \mathcal{O}, \mathcal{E})^{\otimes} \\ \downarrow{\scriptstyle \lambda_F} & & \downarrow{\scriptstyle \lambda} \\ \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}) & \xrightarrow{\tau_F} & \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{E})^{\otimes}. \end{array}$$

We proceed to identify the fibers of λ_F. By definition, for any object $x \in \mathcal{O}^{\otimes}$, $\sigma_F(x)$ is given by the functor $F_{\underline{x}} : \mathcal{C}^{\otimes}_{\underline{x}} \longrightarrow \mathcal{E}^{\otimes}_{\underline{x}}$, and for any fiberwise active edge $\alpha : x \longrightarrow y$, $\tau_F(\alpha)$ is given by the cocartesian pushforward $\alpha_! F_{\underline{x}} : \mathcal{C}^{\otimes}_{\underline{y}} \longrightarrow \mathcal{E}^{\otimes}_{\underline{y}}$. If α decomposes via the $\mathcal{T}$-∞-operad axioms as $(\alpha_i : x_i \longrightarrow y_i)_{1 \leq i \leq n}$ for $y_i \in \mathcal{O}_{V_i}$ (induced by an orbit decomposition $V \simeq V_1 \sqcup ... \sqcup V_n$ if y covers $[V_+ \longrightarrow W]$ in $\underline{\mathbf{F}}_{\mathcal{T},*}$), then $\alpha_! F_{\underline{x}}$ may be explicitly identified as the collection of $\mathcal{T}^{/V_i}$-left Kan extensions $(\alpha_i)_! F_{\underline{x_i}}$ of $(\alpha_i)_{\otimes} \circ F_{\underline{x_i}} : \mathcal{C}^{\otimes}_{\underline{x_i}} \longrightarrow \mathcal{E}_{\underline{y_i}}$ along $(\alpha_i)_{\otimes} : \mathcal{C}^{\otimes}_{\underline{x_i}} \longrightarrow \mathcal{C}_{\underline{y_i}}$. Therefore, the fiber of λ_F over $\{\alpha\}$ is given by

$$\prod_{1 \leq i \leq n} \mathcal{E}_{\underline{y_i}}^{((\alpha_i)_! F_{\underline{x_i}}, \mathcal{T}^{/V_i})/}.$$

Because each $\mathcal{E}_{\underline{y_i}}$ is $\mathcal{T}^{/V_i}$-cocomplete by assumption, these fibers all have initial objects, which are moreover preserved by the cocartesian edges over $\mathcal{T}^{\mathrm{op}}$ (i.e., by restriction). We claim that restricting λ_F to the full $\mathcal{T}$-subcategory X_0 on these initial objects yields a trivial Kan fibration $\lambda'_F : X_0 \longrightarrow \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes})$. By (2) of

Lemma 4.3.2, λ is $\mathcal{O}^{\otimes}$-cartesian, hence the pulled back map λ_F is $\mathcal{O}^{\otimes}$-cartesian. Therefore, it suffices to check that the cocartesian edges in X over $\mathcal{O}^{\otimes}$ preserve initial objects. This is obvious for cocartesian edges over inert edges in $\mathcal{O}^{\otimes}$, so it suffices to consider the case of a fiberwise active edge $\beta : y \longrightarrow z$ in $\mathcal{O}^{\otimes}_W$ with $z \in \mathcal{O}_W$. Then for our fiberwise active edge $\alpha : x \longrightarrow y \in \mathcal{O}^{\otimes}_W$ above, $\beta_!(\alpha) = \beta \circ \alpha : x \longrightarrow z$ computes the cocartesian pushforward in $\mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes})$. As we have seen, an initial object in X covering α corresponds to a collection of $\mathcal{T}^{/V_i}$-colimits of $(\alpha_i)_! F_{\underline{x_i}}$

$$\begin{array}{ccc} \mathcal{C}_{\underline{y_i}} & \xrightarrow{(\alpha_i)_! F_{\underline{x_i}}} & \mathcal{E}_{\underline{y_i}} \\ \downarrow & & \downarrow \\ (\mathcal{T}^{/V_i})^{\mathrm{op}} & \xrightarrow{=} & (\mathcal{T}^{/V_i})^{\mathrm{op}} \end{array}$$

By our assumption that $\mathcal{E}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is distributive, applying $\prod_\beta$ and postcomposing with $\otimes_\beta : \mathcal{E}^{\otimes}_{\underline{y}} \longrightarrow \mathcal{E}_{\underline{z}}$ yields a $\mathcal{T}^{/W}$-colimit diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\underline{y}} & \longrightarrow & \mathcal{E}_{\underline{z}} \\ \downarrow & & \downarrow \\ (\mathcal{T}^{/W})^{\mathrm{op}} & \xrightarrow{=} & (\mathcal{T}^{/W})^{\mathrm{op}} \end{array}$$

Factoring $\mathcal{C}^{\otimes}_{\underline{y}} \longrightarrow (\mathcal{T}^{/W})^{\mathrm{op}}$ through $\mathcal{C}_{\underline{z}}$ and using the transitivity of $\mathcal{T}^{/W}$-left Kan extensions, this further implies that the diagram

$$\begin{array}{ccc} \mathcal{C}_{\underline{z}} & \xrightarrow{(\beta\circ\alpha)_! F_{\underline{x}}} & \mathcal{E}_{\underline{z}} \\ \downarrow & & \downarrow \\ (\mathcal{T}^{/W})^{\mathrm{op}} & \xrightarrow{=} & (\mathcal{T}^{/W})^{\mathrm{op}} \end{array}$$

is a $\mathcal{T}^{/W}$-colimit diagram, where inspecting the definitions reveals that the top horizontal arrow may be identified with $(\beta \circ \alpha)_! F_{\underline{x}}$. This is an initial object covering $\beta \circ \alpha$, as desired.

Choosing a section of λ'_F and postcomposing by the map $X_0 \longrightarrow \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C} \star_{\mathcal{O}} \mathcal{O}, \mathcal{E})^{\otimes}$, we obtain the desired extension $\sigma_{\overline{F}}$. Finally, the assertion about $A|_{\mathcal{O}}$ is clear from the construction if we consider only those objects x, $\alpha = \mathrm{id}_x$, and edges β entirely in $\mathcal{O}$. ☐

We can then promote Proposition 4.3.3 to a global existence result.

4.3.4. **Theorem.** *We have $\mathcal{O}^{\otimes}$-adjunctions*

$$\widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})^{\otimes} \underset{\mathrm{ev}_{\mathcal{C}}}{\rightleftarrows} \widetilde{\mathrm{Fun}}_{\mathcal{O},\mathcal{T}}(\mathcal{C} \star_{\mathcal{O}} \mathcal{O},\mathcal{E})^{\otimes} \underset{\delta}{\overset{\mathrm{ev}_{\mathcal{O}}}{\rightleftarrows}} \mathcal{E}^{\otimes}.$$

Consequently, on passing to ∞-categories of $\mathcal{O}$-algebras, we obtain the adjunction

$$p_! \colon \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E}) \rightleftarrows \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{E}) : p^*$$

where $p_!$ is computed as in Proposition 4.3.3 and p^* is restriction along p.

Proof. The only subtlety involves the first $\mathcal{O}^{\otimes}$-adjunction. We may invoke [Lur17, 7.3.2.12] because the second condition there is ensured by distributivity in $\mathcal{E}^{\otimes}$, using the same argument as in the proof of Proposition 4.3.3. We can then extract the adjunction involving ∞-categories of $\mathcal{O}$-algebra maps by pulling back along the structure map $\mathrm{ev}_1 : \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}) \longrightarrow \mathcal{O}^{\otimes}$ of the $\mathcal{O}$-monoidal envelope and taking cocartesian sections. ☐

Now suppose that $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is only a fibration of $\mathcal{T}$-∞-operads and consider the factorization of p through its $\mathcal{O}$-monoidal envelope $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}$. In view of Proposition 2.8.7 and Theorem 4.3.4, we have the composite adjunction

$$p_! : \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E}) \rightleftarrows \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C}),\mathcal{E}) \rightleftarrows \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{E}) : p^*.$$

4.3.5. **Definition.** Given a $\mathcal{O}$-algebra map $F : \mathcal{C}^{\otimes} \longrightarrow \mathcal{E}^{\otimes}$, we define the $\mathcal{T}$-*operadic left Kan extension* of F to be $p_! F$.

4.3.6. **Remark.** Given a fibration of $\mathcal{T}$-∞-operads $\mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ and $\mathcal{E}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ a distributive $\mathcal{P}$-monoidal $\mathcal{T}$-∞-category, we will also speak of the $\mathcal{T}$-operadic left Kan extension of a $\mathcal{P}$-algebra map $F : \mathcal{C}^{\otimes} \longrightarrow \mathcal{E}^{\otimes}$ along p in the obvious way (note that distributivity is stable under pullback). In other words, we also have an adjunction

$$p_! \colon \mathbf{Alg}_{\mathcal{P},\mathcal{T}}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathbf{Alg}_{\mathcal{P},\mathcal{T}}(\mathcal{O}, \mathcal{E}) : p^*.$$

Note that the underlying $\mathcal{T}$-functor $p_!(F)|_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathcal{E}$ is computed by first extending F to

$$i_! F : \mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes} \longrightarrow \mathcal{E}^{\otimes}$$

and then taking the $\mathcal{T}$-left Kan extension of $i_! F|_{\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})}$ along the structure map to $\mathcal{O}$.

4.3.7. **Example.** Suppose that $\mathcal{C}^{\otimes} = \mathrm{Triv}^{\otimes} = (\underline{\mathbf{F}}_{\mathcal{T},*})_{ne}$ and $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$. Then the $\mathcal{T}$-symmetric monoidal envelope of $\mathrm{Triv}^{\otimes}$ is $(\mathrm{Triv}^{\otimes})_{act} = (\underline{\mathbf{F}}_{\mathcal{T},*})_{cocart}$, the maximal sub-left fibration of $\underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \mathcal{T}^{\mathrm{op}}$ obtained by taking the wide subcategory spanned by the cocartesian edges.

In the case that $\mathcal{T} = \mathbf{O}_G$ is the orbit category of a finite group, we can identify this with something familiar. Namely, let Σ_n be the symmetric group on n letters, and let $\mathbf{O}_{G\times\Sigma_n,\Gamma_n}$ be the full subcategory of the orbit category of $G \times \Sigma_n$ on the Σ_n-free transitive $G \times \Sigma_n$-sets. (Recall that every object in this subcategory is isomorphic to an orbit $G \times \Sigma_n/\Gamma_\phi$, where Γ_ϕ is the graph of a homomorphism $\phi : H \longrightarrow \Sigma_n$ for some subgroup H of G.) Define a functor

$$-/\Sigma_n : \mathbf{O}_{G\times\Sigma_n,\Gamma_n} \longrightarrow \mathbf{O}_G,\ U \rightsquigarrow U/\Sigma_n.$$

This is left adjoint to restriction along the projection $G \times \Sigma_n \longrightarrow G$ and sends $(G \times \Sigma_n)/\Gamma_\phi$ to G/H. Then $(-/\Sigma_n)^{\mathrm{op}}$ is a left fibration and exhibits $\mathbf{O}^{\mathrm{op}}_{G\times\Sigma_n,\Gamma_n}$ as a G-∞-category. In fact, $\mathbf{O}^{\mathrm{op}}_{G\times\Sigma_n,\Gamma_n}$ is a model for the G-space $B_G\Sigma_n$ which classifies G-equivariant principal Σ_n-bundles ([QS21a, Rem. 3.17]).

Now define a G-functor $F : \mathbf{O}^{\mathrm{op}}_{G\times\Sigma_n,\Gamma_n} \longrightarrow (\underline{\mathbf{F}}_G)_{cocart}$ which sends an object U to the morphism of G-sets $(U \times n)/\Sigma_n \longrightarrow U/\Sigma_n$ and a morphism $U \longleftarrow V$ to

$$\begin{array}{ccccc} (U\times n)/\Sigma_n & \longleftarrow & (V\times n)/\Sigma_n & \xrightarrow{=} & (U\times n)/\Sigma_n \\ \downarrow & & \downarrow & & \downarrow \\ U/\Sigma_n & \longleftarrow & V/\Sigma_n & \xrightarrow{=} & V/\Sigma_n \end{array}$$

where we note that the left square is a pullback square of G-sets. (Note that this suffices to define a functor since $\underline{\mathbf{F}}_G$ is equivalent to a 1-category.) It follows from Σ_n-freeness and an elementary argument that F is fully faithful. Moreover, taking the disjoint union over all $n \geq 0$ and postcomposing with $(-)_+$, we obtain an equivalence of G-∞-categories

$$\coprod_{n\geq 0} \mathbf{O}^{\mathrm{op}}_{G\times\Sigma_n,\Gamma_n} \xrightarrow{\sim} (\underline{\mathbf{F}}_G)_{cocart} \xrightarrow{\sim} (\underline{\mathbf{F}}_{G,*})_{cocart}.$$

Therefore, for a G-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the free G-commutative algebra on an object $x : \mathbf{O}^{\mathrm{op}}_G \longrightarrow \mathcal{C}$ is computed by the G-colimit of the induced functor

$$\coprod_{n\geq 0} \mathbf{O}^{\mathrm{op}}_{G\times\Sigma_n,\Gamma_n} \longrightarrow \mathcal{C}.$$

4.3.8. **Warning.** In the proofs of Lemma 4.3.2, Proposition 4.3.3 and Theorem 4.3.4, the results would fail if we replaced $(\mathcal{C} \star_{\mathcal{O}} \mathcal{O})^{\otimes}$ by $(\mathcal{O} \star_{\mathcal{O}} \mathcal{C})^{\otimes}$. This corresponds to there generally being no $\mathcal{T}$-operadic *right* Kan extension.

5. $\mathcal{T}$-∞-categories of $\mathcal{O}$-algebras

Let $\mathcal{O}$ be a $\mathcal{T}$-∞-operad. In this section, we study $\mathcal{T}$-limits and $\mathcal{T}$-colimits in the $\mathcal{T}$-∞-category of $\mathcal{O}$-algebras within an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category. Our results are straightforward generalizations of Lurie's results in [Lur17, §3.2] and overlap with similar work of Bachmann–Hoyois undertaken in [BH21, §7] (in particular, compare [BH21, Prop. 7.6]). Before reading this section, the reader should first review [Sha21b, Thm. 4.16 and Cor. 4.17] on how to compute $\mathcal{T}$-(co)limits in a $\mathcal{T}$-∞-category of sections.

5.1. **Parametrized (co)limits in general.** In this section, let $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category, let $\mathcal{P}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads, and let $\mathcal{K} = \{\mathcal{K}_V : V \in \mathcal{T}\}$ be a collection of classes $\mathcal{K}_V$ of small $\mathcal{T}^{/V}$-∞-categories closed with respect to base-change in $\mathcal{T}$. We are interested in criteria for when the $\mathcal{T}$-∞-category $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})$ of algebras strongly admits $\mathcal{K}$-indexed $\mathcal{T}$-limits and colimits. To solve this problem, we will first need to understand how to compute $\mathcal{T}$-limits and $\mathcal{T}$-colimits in an indexed product.

5.1.1. **Lemma.** *Let $f : \mathcal{T}_0 \longrightarrow \mathcal{T}_1$ be a categorical fibration of ∞-categories, let $\mathcal{C}$ be a $\mathcal{T}_0$-∞-category, and let $\mathcal{K}$ be a $\mathcal{T}_1$-∞-category. Let*

$$f^* \colon \mathbf{Cat}_{\mathcal{T}_1} \rightleftarrows \mathbf{Cat}_{\mathcal{T}_0} : f_*$$

denote the restriction-coinduction adjunction.

(1) Let $(F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G)$ be a $\mathcal{T}_0$-adjunction. Then $(f_* F \colon f_*\mathcal{C} \rightleftarrows f_*\mathcal{D} : f_* G)$ is a $\mathcal{T}_1$-adjunction.

(2) We have a canonical equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{T}_1}(\mathcal{K}, f_*\mathcal{C}) \simeq f_* \underline{\mathrm{Fun}}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C})$$

under which $\delta_{\mathcal{K}} \simeq f_*(\delta_{f^*\mathcal{K}})$ as $\mathcal{T}_1$-functors with common domain $f_*\mathcal{C}$, where $\delta_{\mathcal{K}}$, resp. $\delta_{f^*\mathcal{K}}$ is the constant $\mathcal{K}$-diagram $\mathcal{T}_1$-functor, resp. constant $f^*\mathcal{K}$-diagram $\mathcal{T}_0$-functor.

(3) Let $p : \mathcal{K} \longrightarrow f_*\mathcal{C}$ be a $\mathcal{T}_1$-functor and let $q : f^*\mathcal{K} \longrightarrow \mathcal{C}$ be its adjoint $\mathcal{T}_0$-functor. Then p admits a $\mathcal{T}_1$-limit if and only if q admits a $\mathcal{T}_0$-limit, and moreover an extension $\overline{p} : \mathcal{K}^{\trianglelefteq} \longrightarrow f_*\mathcal{C}$ is a $\mathcal{T}_1$-limit diagram if and only if the adjoint extension $\overline{q} : (f^*\mathcal{K})^{\trianglelefteq} \simeq f^*(\mathcal{K}^{\trianglelefteq}) \longrightarrow \mathcal{C}$ is a $\mathcal{T}_0$-limit diagram. The analogous statements also hold for parametrized colimits.

Proof. (1): It suffices to show that for all $t \in \mathcal{T}_1$, $(f_*F)_t \dashv (f_*G)_t$ is an adjunction. In fact, since $(f_*\mathcal{C})_t \simeq \mathrm{Fun}_{\mathcal{T}_1}((\mathcal{T}_1^{t/})^{\mathrm{op}}, f_*\mathcal{C})$, we can more generally verify that $\mathrm{Fun}_{\mathcal{T}_1}(\mathcal{K}, f_*F) \dashv \mathrm{Fun}_{\mathcal{T}_1}(\mathcal{K}, f_*G)$ is an adjunction for every $\mathcal{T}_1$-∞-category $\mathcal{K}$. But this holds since

$$\mathrm{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, F) \colon \mathrm{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C}) \rightleftarrows \mathrm{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{D}) : \mathrm{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, G)$$

is an adjunction by [Sha21a, Prop. 8.2].
(2): We check the claimed equivalence at the level of representable functors:

$$\begin{aligned}\mathrm{Fun}_{\mathcal{T}_1}(\mathcal{L}, \underline{\mathrm{Fun}}_{\mathcal{T}_1}(\mathcal{K}, f_*\mathcal{C})) &\simeq \mathrm{Fun}_{\mathcal{T}_1}(\mathcal{L} \times_{\mathcal{T}_1^{\mathrm{op}}} \mathcal{K}, f_*\mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{T}_0}(f^*\mathcal{L} \times_{\mathcal{T}_0^{\mathrm{op}}} f^*\mathcal{K}, \mathcal{C}) \\ &\simeq \mathrm{Fun}_{\mathcal{T}_0}(f^*\mathcal{L}, \underline{\mathrm{Fun}}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{T}_0}(\mathcal{L}, f_*\underline{\mathrm{Fun}}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C}))\end{aligned}$$

The assertion about constant functors is shown in a similar manner.
(3): This follows from combining (1) and (2). ☐

5.1.2. **Corollary.** *Let $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads, let $f : U \simeq \coprod_{1 \leq i \leq n} U_i \longrightarrow V$ be a morphism in $\mathbf{F}_{\mathcal{T}}$ with U_i and V orbits, let $x \in \mathcal{O}^{\otimes}_{f_+}$, and let $e_i : x \longrightarrow x_i$ be inert edges in $\mathcal{O}^{\otimes}$ lifting the characteristic morphisms $\chi_{[U_i \subset U]}$ in $\underline{\mathbf{F}}_{\mathcal{T},*}$. Suppose $\mathcal{C}_{\underline{x_i}}$ admits all $\mathcal{K}_{U_i}$-indexed $\mathcal{T}^{U_i/}$-(co)limits for each $1 \leq i \leq n$. Then $\mathcal{C}^{\otimes}_{\underline{x}}$ admits all $\mathcal{K}_V$-indexed $\mathcal{T}^{V/}$-(co)limits.*

Proof. Combine Lemma 5.1.1(3) and the Segal equivalence $\mathcal{C}^{\otimes}_{\underline{x}} \simeq \prod_{1 \leq i \leq n} \prod_{f_i} \mathcal{C}_{\underline{x_i}}$ of Theorem 2.3.3. ☐

We may now prove our main result on parametrized limits.

5.1.3. **Theorem.** *Let $\mathcal{C}$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category and let $\mathcal{P}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads. Let $\mathcal{K} = \{\mathcal{K}_V : V \in \mathcal{T}\}$ be a collection of classes $\mathcal{K}_V$ of small $\mathcal{T}^{/V}$-∞-categories closed with respect to base-change in $\mathcal{T}$ (e.g., we could take $\mathcal{K}_V$ to be all small $\mathcal{T}^{/V}$-∞-categories for each $V \in \mathcal{T}$). Suppose for all $x \in \mathcal{O}_V$ that $\mathcal{C}_{\underline{x}}$ admits all $\mathcal{K}_V$-indexed $\mathcal{T}^{/V}$-limits. Then:*

(1) Both $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})$ and $\underline{\mathrm{Fun}}_{/\mathcal{O}^{\otimes},\mathcal{T}}(\mathcal{P}^{\otimes}, \mathcal{C}^{\otimes})$ strongly admit all $\mathcal{K}$-indexed $\mathcal{T}$-limits, and the inclusion

$$\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C}) \subset \underline{\mathrm{Fun}}_{/\mathcal{O}^{\otimes},\mathcal{T}}(\mathcal{P}^{\otimes}, \mathcal{C}^{\otimes})$$

strongly preserves all $\mathcal{K}$-indexed $\mathcal{T}$-limits.

(2) $\underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})$ strongly admits all $\mathcal{K}$-indexed $\mathcal{T}$-limits, and the forgetful $\mathcal{T}$-functor

$$\mathrm{U} : \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C}) \longrightarrow \underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})$$

strongly creates all $\mathcal{K}$-indexed $\mathcal{T}$-limits.

Proof. By [Sha21b, Cor. 4.17] and Corollary 5.1.2, $\underline{\mathrm{Fun}}_{/\mathcal{O}^\otimes,\mathcal{T}}(\mathcal{P}^\otimes,\mathcal{C}^\otimes)$ strongly admits all $\mathcal{K}$-indexed $\mathcal{T}$-limits. Moreover, by the explicit formula for parametrized limits in $\underline{\mathrm{Fun}}_{/\mathcal{O}^\otimes,\mathcal{T}}(\mathcal{P}^\otimes,\mathcal{C}^\otimes)$ given in [Sha21b, 4.16(3)], we see the remaining claims follow from the observation that for every fiberwise inert edge $e : x \longrightarrow x'$ in $\mathcal{O}^\otimes_V$, the associated pushforward functor $e_! : \mathcal{C}^\otimes_{\underline{x}} \longrightarrow \mathcal{C}^\otimes_{\underline{x'}}$ is identified with projection to a subset of factors in a fiber product under the Segal equivalence of Theorem 2.3.3, so in particular preserves all $\mathcal{K}_V$-indexed $\mathcal{T}^{/V}$-limits. This proves (1). (2) then follows by invoking [Sha21b, 4.16(3)] once more. ☐

Next, we handle the case of parametrized colimits, for which we will need an additional distributivity assumption on $\mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$. Let $\mathcal{K}_V$ be the class $\mathcal{K}^{\mathrm{sift}}_V$ of $\mathcal{T}^{/V}$-sifted $\mathcal{T}^{/V}$-colimits and write $\mathcal{K}^{\mathrm{sift}} = \mathcal{K}$.

5.1.4. **Theorem.** *Suppose $\mathcal{C}$ is a distributive $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category (Definition 3.2.4), and let $\mathcal{P}^\otimes \longrightarrow \mathcal{O}^\otimes$ be a fibration of $\mathcal{T}$-∞-operads. Then:*

1. *Both $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ and $\underline{\mathrm{Fun}}_{/\mathcal{O}^\otimes,\mathcal{T}}(\mathcal{P}^\otimes,\mathcal{C}^\otimes)$ strongly admit all $\mathcal{K}^{\mathrm{sift}}$-indexed $\mathcal{T}$-colimits, and the inclusion*
$$\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C}) \subset \underline{\mathrm{Fun}}_{/\mathcal{O}^\otimes,\mathcal{T}}(\mathcal{P}^\otimes,\mathcal{C}^\otimes)$$
strongly preserves all $\mathcal{K}^{\mathrm{sift}}$-indexed $\mathcal{T}$-colimits.
2. *$\underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ is $\mathcal{T}$-cocomplete and the forgetful $\mathcal{T}$-functor*
$$\mathrm{U} : \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C}) \longrightarrow \underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$$
strongly creates all $\mathcal{K}^{\mathrm{sift}}$-indexed $\mathcal{T}$-colimits.
3. *$\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ is $\mathcal{T}$-cocomplete.*
4. *Suppose in addition that $\mathcal{C}$ is fiberwise presentable. Then $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ is fiberwise presentable.*

Proof. (1) and (2): To show the claim for $\underline{\mathrm{Fun}}_{/\mathcal{O}^\otimes,\mathcal{T}}(\mathcal{P}^\otimes,\mathcal{C}^\otimes)$, we verify the criterion of [Sha21b, Thm. 4.16(4)]. Since a fiberwise morphism $\alpha : x \longrightarrow y$ in $\mathcal{O}^\otimes_V$ factors as the composite of a fiberwise inert edge and a fiberwise active edge, and the pushforward functor associated to a fiberwise inert edge is a projection, we may suppose α is fiberwise active. Moreover, using again the Segal equivalence of Theorem 2.3.3, we may suppose that α covers a fiberwise active edge $f_+ : [U_+ \to V] \longrightarrow [V_+ \to V]$ in $\underline{\mathbf{F}}_{\mathcal{T},*}$ defined by a map $f : U \longrightarrow V$ of finite $\mathcal{T}$-sets. Then using the distributive hypothesis on $\mathcal{C}^\otimes$ together with [Sha21b, Prop. 8.19], we have that the pushforward $\mathcal{T}^{/V}$-functor $\alpha_! = \otimes_\alpha : \mathcal{C}^\otimes_{\underline{x}} \longrightarrow \mathcal{C}_{\underline{y}}$ preserves all $\mathcal{K}^{\mathrm{sift}}_V$-indexed colimits, so $\underline{\mathrm{Fun}}_{/\mathcal{O}^\otimes,\mathcal{T}}(\mathcal{P}^\otimes,\mathcal{C}^\otimes)$ strongly admits all $\mathcal{K}^{\mathrm{sift}}$-indexed $\mathcal{T}$-colimits. Similarly, we see that $\underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ is $\mathcal{T}$-cocomplete. The remaining claims then follow as in the proof of Theorem 5.1.3, now using [Sha21b, Thm. 4.16(4)].

(3): By part (1) and [Sha21a, Cor. 12.15] (or [Sha21b, Thm. 8.6]), it suffices to check that $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ admits finite $\mathcal{T}$-coproducts. For this, we employ the same strategy as in the proof of [Lur17, Cor. 3.2.3.3]. Pulling back $\mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ via $\mathcal{P}^\otimes \longrightarrow \mathcal{O}^\otimes$, we may suppose $\mathcal{P}^\otimes = \mathcal{O}^\otimes$ without loss of generality. Now let $\mathcal{O}^\otimes_{ne} = \mathcal{O}^\otimes \times_{\underline{\mathbf{F}}_{\mathcal{T},*}} \mathrm{Triv}^\otimes_{\mathcal{T}}$ and note that by Lemma 2.4.4 and Corollary 2.4.5, we have that $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_{ne},\mathcal{C}) \simeq \underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{O},\mathcal{C})$. By Theorem 4.3.4, we thus obtain the free-forgetful $\mathcal{T}$-adjunction
$$\mathrm{F} \colon \underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{O},\mathcal{C}) \rightleftarrows \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}) : \mathrm{U}$$
as an instance of $\mathcal{T}$-operadic left Kan extension along $\mathcal{O}^\otimes_{ne} \longrightarrow \mathcal{O}^\otimes$. This implies that each fiber $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C})_V$ admits finite coproducts of free objects, and for each morphism $\alpha : V \longrightarrow W$ in $\mathcal{T}$, the putative left adjoint $\alpha_!$ to the restriction functor $\alpha^* : \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C})_W \longrightarrow \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C})_V$ is at least defined on the full subcategory of free objects, using the pointwise criterion for the existence of an adjoint. By part (2) and the observation that U is fiberwise conservative, the assumptions of [Lur17, Prop. 4.7.3.14] are satisfied for each of the adjunctions $\mathrm{F}_V \dashv \mathrm{U}_V$, so for each object $A \in \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C})_V$, there exists a simplicial object $A_\bullet$ such that each A_n is free and $A \simeq |A_\bullet|$. It follows that the requisite finite coproducts and left adjoints exist for $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C})$. Finally, verification of the base-change condition also reduces to free objects in the same way.

(4): Upon replacing $\mathcal{T}$ by $\mathcal{T}^{/V}$ this amounts to showing that $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ is presentable. Given (3), it remains to show that $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P},\mathcal{C})$ is accessible. But since all the pushforward functors $\alpha_! : \mathcal{C}^\otimes_x \longrightarrow \mathcal{C}^\otimes_{x'}$ indexed by morphisms $\alpha : x \longrightarrow x' \in \mathcal{O}^\otimes$ preserve sifted colimits (which reduces to the aforementioned assertion for fiberwise active α given the inert-fiberwise active factorization on $\mathcal{O}^\otimes$), this follows from [Lur09, Prop. 5.4.7.11] in exactly the same way as in the proof of [Lur17, Cor. 3.2.3.5]. ☐

5.1.5. **Corollary.** *Suppose $\mathcal{C}$ is a distributive $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category. Then the free-forgetful $\mathcal{T}$-adjunction*

$$\mathrm{F}\colon \underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{O},\mathcal{C}) \rightleftarrows \underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}) : \mathrm{U}$$

of Theorem 4.3.4 applied to $\mathcal{O}^{\otimes}_{ne} \subset \mathcal{O}^{\otimes}$ is fiberwise monadic.

Proof. We verify the hypotheses of the Barr–Beck–Lurie Theorem [Lur17, Thm. 4.7.3.5]. After Theorem 5.1.4(2), it only remains to note that U is fiberwise conservative, which is immediate from the definitions. □

5.2. **Units and initial objects.** In this subsection, we identify $\mathcal{T}$-initial objects in $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C})$ in the case where $\mathcal{O}^{\otimes}$ is a *unital* $\mathcal{T}$-∞-operad and $\mathcal{C}^{\otimes}$ is any $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category.

5.2.1. **Definition.** Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad. We say that $\mathcal{O}^{\otimes}$ is *unital* if for all orbits $V \in \mathcal{T}$ and objects $x \in \mathcal{O}_V$, the space of multimorphisms $\mathrm{Mul}_{\mathcal{O}}(\emptyset, x)$ is contractible.

For example, $\underline{\mathbf{F}}_{\mathcal{T},*}$ is unital and $\mathrm{Triv}^{\otimes}_{\mathcal{T}}$ is not unital. We next introduce the minimal $\mathcal{T}$-suboperad of $\underline{\mathbf{F}}_{\mathcal{T},*}$ which remains unital.

5.2.2. **Definition.** Let $\mathbf{E}^{\otimes}_{0,\mathcal{T}} \subset \underline{\mathbf{F}}_{\mathcal{T},*}$ be the $\mathcal{T}$-suboperad given by the wide subcategory on those morphisms

$$\begin{array}{ccccc} U & \longleftarrow & Z & \xrightarrow{m} & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Y & \xrightarrow{=} & Y \end{array}$$

for which m is a summand inclusion.

Given any $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, we will then write $\mathcal{O}^{\otimes}_0$ for the pullback $\mathbf{E}^{\otimes}_{0,\mathcal{T}} \times_{\underline{\mathbf{F}}_{\mathcal{T},*}} \mathcal{O}^{\otimes}$ in this subsection. Note that the inclusion $\mathbf{E}^{\otimes}_{0,\mathcal{T}} \subset \underline{\mathbf{F}}_{\mathcal{T},*}$ is stable under equivalences and is thus a fibration of $\mathcal{T}$-∞-operads, and the same is true for the pullback $\mathcal{O}^{\otimes}_0 \subset \mathcal{O}^{\otimes}$.

5.2.3. **Remark.** Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad and for each orbit $V \in \mathcal{T}$, let $*_V$ be a choice of object in the fiber $\mathcal{O}^{\otimes}_{[\emptyset_+ \to V]} \simeq *$, which is unique up to contractible choice. We note that $*_V$ is a final object in $\mathcal{O}^{\otimes}_V$. Indeed, if $\mathcal{O}^{\otimes} = \underline{\mathbf{F}}_{\mathcal{T},*}$, then $[\emptyset_+ \to V]$ is a zero object in $(\underline{\mathbf{F}}_{\mathcal{T},*})_V \simeq \mathbf{F}_{\mathcal{T}/V,*}$, and the general case follows by the definition of a $\mathcal{T}$-∞-operad. Since for each $\alpha : V \longrightarrow W \in \mathcal{T}$ we have that $\alpha^*(*_W) \simeq *_V$, the $*_V$ assemble to define a $\mathcal{T}$-final object $* : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{O}^{\otimes}$.

Now suppose $\mathcal{O}^{\otimes}$ is unital. Then by the same reasoning, we see that $*_V$ is a zero object in $\mathcal{O}^{\otimes}_V$ and hence $*$ is a $\mathcal{T}$-zero object. In this case, we will also write 0_V for $*_V$ and 0 for $*$.

5.2.4. **Definition.** We define the $\mathcal{T}$-functor $\omega : \Delta^1 \times \mathcal{T}^{\mathrm{op}} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ to be the unique homotopy from 0 to $I(-)_+$. For a unital $\mathcal{T}$-∞-operad $\mathcal{O}^{\otimes}$, we then define the $\mathcal{T}$-functor $\omega_{\mathcal{O}}$ lying in the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{\trianglelefteq} & \xrightarrow{\omega_{\mathcal{O}}} & \mathcal{O}^{\otimes}_0 & \longrightarrow & \mathcal{O}^{\otimes} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^1 \times \mathcal{T}^{\mathrm{op}} & \xrightarrow{\omega} & \mathbf{E}^{\otimes}_{0,\mathcal{T}} & \longrightarrow & \underline{\mathbf{F}}_{\mathcal{T},*} \end{array}$$

to be the unique $\mathcal{T}$-functor extending the inclusion $\mathcal{O} \subset \mathcal{O}^{\otimes}_0$ of the underlying ∞-category, whose restriction along the cone inclusion $\mathcal{T}^{\mathrm{op}} \subset \mathcal{O}^{\trianglelefteq}$ is 0.

For an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, we define the $\mathcal{T}$-functor $\widetilde{\omega}_{\mathcal{C}}$ lying in the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{\trianglelefteq} & \xrightarrow{\widetilde{\omega}_{\mathcal{C}}} & \mathcal{C}^{\otimes}_0 & \longrightarrow & \mathcal{C}^{\otimes} \\ & \searrow^{\omega_{\mathcal{O}}} & \downarrow & & \downarrow p \\ & & \mathcal{O}^{\otimes}_0 & \longrightarrow & \mathcal{O}^{\otimes} \end{array}$$

to be the unique lift of $\omega_{\mathcal{O}}$ so that $\widetilde{\omega}_{\mathcal{C}}$ sends the edges $(0_V \longrightarrow x) \in \mathcal{O}^{\triangleleft}_V$ to p-cocartesian edges (and hence all edges to p-cocartesian edges).

We then define the *unit* of $(\mathcal{C}^{\otimes}, p)$ to be the $\mathcal{T}$-functor $1 := \widetilde{\omega}_{\mathcal{C}}|_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathcal{C}$.

5.2.5. **Remark.** The $\mathcal{T}$-functors ω, $\omega_{\mathcal{O}}$, and $\widetilde{\omega}_{\mathcal{C}}$ of Definition 5.2.4 may be defined rigorously as follows. For ω, choose a section σ of the trivial fibration $\mathcal{T}^{\mathrm{op}} \times_{0,\mathbf{E}^{\otimes}_{0,\mathcal{T}},\mathrm{ev}_0} \mathrm{Ar}_{\mathcal{T}}(\mathbf{E}^{\otimes}_{0,\mathcal{T}}) \xrightarrow{\mathrm{ev}_1} \mathbf{E}^{\otimes}_{0,\mathcal{T}}$ of Lemma 5.2.6 and define ω to be the adjoint to the composite $\omega^{\perp} = \mathrm{pr} \circ \sigma \circ I_+ : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathbf{E}^{\otimes}_{0,\mathcal{T}})$.

Then for $\omega_{\mathcal{O}}$, choose a section $\sigma_{\mathcal{O}}$ of the trivial fibration ψ of Lemma 5.2.6 applied to $\mathcal{O}^{\otimes}_0 \longrightarrow \mathbf{E}^{\otimes}_{0,\mathcal{T}}$, let $\rho : \mathcal{O} \longrightarrow \mathcal{T}^{\mathrm{op}}$ and $i : \mathcal{O} \subset \mathcal{O}^{\otimes}_0$ denote the structure map and inclusion, and define the $\mathcal{T}$-functor

$$\omega^{\perp}_{\mathcal{O}} = \sigma \circ (\omega^{\perp}\rho, i) : \mathcal{O} \longrightarrow (\mathcal{T}^{\mathrm{op}} \times_{\mathbf{E}^{\otimes}_{0,\mathcal{T}}} \mathrm{Ar}_{\mathcal{T}}(\mathbf{E}^{\otimes}_{0,\mathcal{T}})) \times_{\mathbf{E}^{\otimes}_{0,\mathcal{T}}} \mathcal{O}^{\otimes}_0 \xrightarrow{\simeq} \mathcal{T}^{\mathrm{op}} \times_{0,\mathcal{O}^{\otimes}_0,\mathrm{ev}_0} \mathrm{Ar}_{\mathcal{T}}(\mathcal{O}^{\otimes}_0).$$

By [Sha21a, Cor. 4.27] and [Sha21a, Prop. 4.30], for any $\mathcal{T}$-∞-category $\mathcal{D}$ and cocartesian section $\phi : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{D}$ we have natural equivalences

$$\mathcal{D}_{(\phi,\mathcal{T})/} \xrightarrow{\simeq} \mathcal{D}^{(\phi,\mathcal{T})/} \xrightarrow{\simeq} \mathcal{T}^{\mathrm{op}} \times_{\phi,\mathcal{D}} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D}),$$

and by [Sha21a, Prop. 4.25] for any $\mathcal{T}$-∞-category $\mathcal{A}$ we have the natural $\mathcal{T}$-join and slice equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{A}, \mathcal{D}_{(\phi,\mathcal{T})/}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}//\mathcal{T}}(\mathcal{A}^{\underline{\triangleleft}}, \mathcal{D}).$$

We may thus adjoin $\omega^{\perp}_{\mathcal{O}}$ to define $\omega_{\mathcal{O}}$ so that it fits into the indicated commutative diagram over ω.

Finally, to define $\widetilde{\omega}_{\mathcal{C}}$, first let $* : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\otimes}_0$ be a lift of 0 to the $\mathcal{T}$-final object of Remark 5.2.3, and let $\sigma_{\mathcal{C}} : \mathcal{C}^{\otimes}_0 \times_{\mathcal{O}^{\otimes}_0} \mathrm{Ar}(\mathcal{O}^{\otimes}_0) \xrightarrow{\simeq} \mathrm{Ar}^{cocart}(\mathcal{C}^{\otimes}_0)$ be a choice of section for the trivial fibration. We then have the composite

$$\mathcal{T}^{\mathrm{op}} \times_{0,\mathcal{O}^{\otimes}_0} \mathrm{Ar}(\mathcal{O}^{\otimes}_0) \xrightarrow{*\times\mathrm{id}} \mathcal{C}^{\otimes}_0 \times_{\mathcal{O}^{\otimes}_0} \mathrm{Ar}(\mathcal{O}^{\otimes}_0) \xrightarrow{\sigma_{\mathcal{C}}} \mathrm{Ar}^{cocart}(\mathcal{C}^{\otimes}_0) \subset \mathrm{Ar}(\mathcal{C}^{\otimes}_0),$$

which restricts to

$$f : \mathcal{T}^{\mathrm{op}} \times_{0,\mathcal{O}^{\otimes}_0} \mathrm{Ar}_{\mathcal{T}}(\mathcal{O}^{\otimes}_0) \longrightarrow \mathcal{T}^{\mathrm{op}} \times_{*,\mathcal{C}^{\otimes}_0} \mathrm{Ar}_{\mathcal{T}}(\mathcal{C}^{\otimes}_0).$$

The adjoint of $f \circ \omega^{\perp}_{\mathcal{O}}$ then defines the lift $\widetilde{\omega}_{\mathcal{C}}$ of $\omega_{\mathcal{O}}$.

We also verify the uniqueness assertion for $\widetilde{\omega}_{\mathcal{C}}$ and leave that for $\omega_{\mathcal{O}}$ as an exercise for the reader. By Lemma 5.2.7, the functor given by restriction along the $\mathcal{T}$-cone point

$$\mathrm{Fun}^{cocart}_{/\mathcal{O}^{\otimes}_0}(\mathcal{O}^{\underline{\triangleleft}}, \mathcal{C}^{\otimes}_0) \longrightarrow \mathrm{Fun}_{\mathcal{T}}(\mathcal{T}^{\mathrm{op}}, \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes},0} \mathcal{T}^{\mathrm{op}})$$

is an equivalence, and since $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes},0} \mathcal{T}^{\mathrm{op}} \simeq \mathcal{T}^{\mathrm{op}}$, we see that the righthand side is contractible, which shows the claim (and gives another construction of $\widetilde{\omega}_{\mathcal{C}}$).

5.2.6. **Lemma.** *Suppose $q : \mathcal{D} \longrightarrow \mathcal{B}$ is a $\mathcal{T}$-fibration and $0 : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{D}$ is a $\mathcal{T}$-functor such that 0 and $q \circ 0$ are $\mathcal{T}$-initial objects. Then the $\mathcal{T}$-functor*

$$\psi : (\mathcal{T}^{\mathrm{op}} \times_{0,\mathcal{D},\mathrm{ev}_0} \mathrm{Ar}_{\mathcal{T}}(\mathcal{D})) \longrightarrow (\mathcal{T}^{\mathrm{op}} \times_{0,\mathcal{B},\mathrm{ev}_0} \mathrm{Ar}_{\mathcal{T}}(\mathcal{B})) \times_{\mathrm{ev}_1,\mathcal{B},q} \mathcal{D}$$

is a trivial fibration.

Proof. Since q is a categorical fibration, it follows that ψ is as well. It thus suffices to prove that ψ is a categorical equivalence, for which we may suppose that $\mathcal{B} = \mathcal{T}^{\mathrm{op}}$ by the two-out-of-three property of equivalences. The claim then follows by our hypothesis that 0_V is initial for all $V \in \mathcal{T}$. ☐

5.2.7. **Lemma.** *Let $q : \mathcal{D} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-cocartesian fibration and let $i : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{B}$ be a $\mathcal{T}$-initial object. Then the $\mathcal{T}$-functor*

$$i^* : \underline{\mathrm{Fun}}^{cocart}_{/\mathcal{B},\mathcal{T}}(\mathcal{B}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{T}^{\mathrm{op}} \times_{\mathcal{B}} \mathcal{D}$$

is an equivalence.

Proof. It suffices to check the assertion fiberwise. Replacing $\mathcal{T}$ by $\mathcal{T}^{/V}$, we may further suppose that $\mathcal{T}$ has a final object $*$, and we reduce to showing that $i^* : \mathrm{Fun}^{cocart}_{/\mathcal{B}}(\mathcal{B}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{cocart}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{T}^{\mathrm{op}}, \mathcal{T}^{\mathrm{op}} \times_{\mathcal{B}} \mathcal{D})$ is an equivalence of ∞-categories, or equivalently, that $(\mathcal{T}^{\mathrm{op}})^{\sharp} \longrightarrow \mathcal{B}^{\sharp}$ is a cocartesian equivalence in $\mathbf{sSet}^+_{/\mathcal{B}}$. But the inclusion of the initial object $* \in \mathcal{T}^{\mathrm{op}}$ is a cocartesian equivalence to both $(\mathcal{T}^{\mathrm{op}})^{\sharp}$ and $\mathcal{B}^{\sharp}$, so by the two-out-of-three property of the cocartesian equivalences we are done. ☐

We may canonically endow the unit of an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category with the structure of an $\mathcal{O}$-algebra in the following way.

5.2.8. **Proposition.** *Let $\mathcal{O}^{\otimes}$ be a unital $\mathcal{T}$-∞-operad and let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category. Then there is a unique cocartesian section $1^{\otimes}$ of p such that $1^{\otimes}$ extends the unit $1 : \mathcal{O} \longrightarrow \mathcal{C}$.*

Proof. Since $\mathcal{O}^{\otimes}$ has the $\mathcal{T}$-initial object 0 by assumption, the claim follows from Lemma 5.2.7. □

We next identify $\mathcal{O}_0^{\otimes}$-monoidal $\mathcal{T}$-∞-categories and $\mathcal{O}_0$-algebras therein in more familiar terms.

5.2.9. **Proposition.** *Let $\mathcal{O}^{\otimes}$ be a unital $\mathcal{T}$-∞-operad.*

1. *Suppose $(\mathcal{C}^{\otimes}, p)$ is a $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category and let $F^{un}_{\mathcal{C}} : \mathcal{O}_0^{\otimes} \longrightarrow \mathbf{Cat}$ be the functor classifying the cocartesian fibration $p|_{\mathcal{O}_0^{\otimes}}$. Then $F^{un}_{\mathcal{C}}$ is the right Kan extension of its restriction $f^{un}_{\mathcal{C}}$ along $\omega_{\mathcal{O}}$.*
2. *Let $f : \mathcal{O}^{\trianglelefteq} \longrightarrow \mathbf{Cat}$ be a functor whose restriction along the cone inclusion $\mathcal{T}^{\mathrm{op}} \subset \mathcal{O}^{\trianglelefteq}$ is constant at the final object, and let F be the right Kan extension of f along $\omega_{\mathcal{O}}$. The cocartesian fibration $q : \mathcal{D} \longrightarrow \mathcal{O}_0^{\otimes}$ classifed by F is then an $\mathcal{O}_0$-monoidal $\mathcal{T}$-∞-category.*
3. *The assignment $(\mathcal{C}^{\otimes}, p) \mapsto (* \to f_{\mathcal{C}})$ (where $f_{\mathcal{C}}$ denotes the restriction of $f^{un}_{\mathcal{C}}$ to $\mathcal{C}$, regarded as pointed in $\mathrm{Fun}(\mathcal{O}, \mathbf{Cat})$ via the unit $1 : \mathcal{O} \longrightarrow \mathcal{C}$) implements an equivalence of ∞-categories*
$$\mu : \mathbf{Cat}^{\otimes}_{\mathcal{O}_0} \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{O}, \mathbf{Cat})^{*/}$$
and an equivalence of $\mathcal{T}$-∞-categories
$$\underline{\mathbf{Cat}}^{\otimes}_{\mathcal{O}_0} \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O}, (\underline{\mathbf{Cat}}_{\mathcal{T}})^{(*,\mathcal{T})/}).$$
4. *For any two $\mathcal{O}$-monoidal $\mathcal{T}$-∞-categories $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$, μ induces an equivalence of ∞-categories*
$$\mathrm{Fun}^{\otimes}_{\mathcal{O}_0,\mathcal{T}}(\mathcal{C}_0, \mathcal{D}_0) \xrightarrow{\simeq} \mathrm{Nat}_*(f_{\mathcal{C}}, f_{\mathcal{D}}).$$

Proof. We first analyze right Kan extension along $\omega_{\mathcal{O}}$ in general. Let $x \in \mathcal{O}^{\otimes}_{[U_+ \to V]}$, let $U \simeq \coprod_{i=1}^n U_i$ be an orbit decomposition, and let $\rho_i : x \longrightarrow x_i$ be cocartesian edges lifting the characteristic morphisms $\chi_{[U_i \subset U]}$. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{O}^{\trianglelefteq} \times_{\mathcal{O}_0^{\otimes}} (\mathcal{O}_0^{\otimes})^{x/}$ be the full subcategories on objects over
$$\begin{pmatrix} U & \longleftarrow & Z & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Y & \xrightarrow{=} & Y \end{pmatrix} \in (\Delta^1 \times \mathcal{T}^{\mathrm{op}}) \times_{\mathbf{E}^{\otimes}_{0,\mathcal{T}}} (\mathbf{E}^{\otimes}_{0,\mathcal{T}})^{[U_+ \to V]/}$$
such that $Z = \emptyset$ and $Z \neq \emptyset$, respectively, and note that $\mathcal{O}^{\trianglelefteq} \times_{\mathcal{O}_0^{\otimes}} (\mathcal{O}_0^{\otimes})^{x/}$ decomposes as the disjoint union of $\mathcal{A}$ and $\mathcal{B}$. Let $\phi = \phi_A \sqcup \phi_B : \{a\} \sqcup \mathrm{Orbit}(U) \longrightarrow \mathcal{A} \sqcup \mathcal{B}$ be the functor that sends a to $(x \to 0_V)$ and U_i to ρ_i.

We claim that ϕ is right cofinal. By the same argument as in the proof of Lemma 2.4.4, ϕ_B is right cofinal, so it only remains to show $(x \to 0_V)$ is an initial object in $\mathcal{A}$. Since $\mathcal{O}^{\trianglelefteq} \times_{\mathcal{O}_0^{\otimes}} (\mathcal{O}_0^{\otimes})^{x/} \longrightarrow \mathcal{O}^{\trianglelefteq} \longrightarrow \mathcal{T}^{\mathrm{op}}$ is a cocartesian fibration, it suffices to show that for all morphisms $\alpha : W \longrightarrow V \in \mathcal{T}$, $\alpha^*(x \to 0_V) \simeq (x \to 0_W)$ is an initial object in $\mathcal{B}_W$. We first check that for objects $(x \to y) \in \mathcal{B}$ covering $\gamma : [U_+ \to V] \longrightarrow I(W)_+$, the mapping space $\mathrm{Map}_{\mathcal{B}_W}(x \to 0_W, x \to y)$ is contractible. This mapping space fits into the commutative diagram
$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{B}_W}(x \to 0_W, x \to y) & \longrightarrow & \mathrm{Map}_{(\mathcal{O}_0^{\otimes})^{x/}}(x \to 0_W, x \to y) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (x \to y)} \\ * \simeq \mathrm{Map}_{\mathcal{O}^{\trianglelefteq}_W}(0_W, y) & \longrightarrow & \mathrm{Map}_{\mathcal{O}_0^{\otimes}}(0_W, y) & \xrightarrow{(x \to 0_W)^*} & \mathrm{Map}_{\mathcal{O}_0^{\otimes}}(x, y), \end{array}$$
where the lower horizontal composite selects the inert-fiberwise active factorization of $x \longrightarrow y$ in the connected component $\mathrm{Map}^{\gamma}_{\mathcal{O}_0^{\otimes}}(x, y)$. In fact, since $\mathrm{Mul}_{\mathcal{O}}(\emptyset, y) \simeq *$, this map is an equivalence onto that connected component, and the contractibility of $\mathrm{Map}_{\mathcal{B}_W}(x \to 0_W, x \to y)$ follows. Finally, the argument for $(x \to 0_W)$ itself proceeds the same way.

We conclude that given a functor $f : \mathcal{O}^{\trianglelefteq} \longrightarrow \mathbf{Cat}$, the right Kan extension $(\omega_{\mathcal{O}})_* f$ sends x to $f(0_V) \times \prod_{i=1}^n f(x_i)$. This shows (1) – more precisely, the unit map $F^{un}_{\mathcal{C}} \xrightarrow{\simeq} (\omega_{\mathcal{O}})_*(\omega_{\mathcal{O}})^* F^{un}_{\mathcal{C}}$ is seen to be an equivalence. By Proposition 2.2.6, this also shows (2). Moreover, we see that $(\omega_{\mathcal{O}})^*(\omega_{\mathcal{O}})_* f \xrightarrow{\simeq} f$ if and only if $f|_{\mathcal{T}^{\mathrm{op}}}$ is constant at $* \in \mathbf{Cat}$. We deduce that the adjunction $(\omega_{\mathcal{O}})^* \dashv (\omega_{\mathcal{O}})_*$ restricts to an adjoint equivalence
$$(\omega_{\mathcal{O}})^* : \mathbf{Cat}^{\otimes}_{\mathcal{O}_0} \rightleftarrows \mathrm{Fun}'(\mathcal{O}^{\trianglelefteq}, \mathbf{Cat}) : (\omega_{\mathcal{O}})_*$$
where we take the righthand side to consist of the full subcategory on functors $\mathcal{O}^{\trianglelefteq} \longrightarrow \mathbf{Cat}$ that restrict to $*$ on $\mathcal{T}^{\mathrm{op}}$. Note that since the inclusion of a final object is fully faithful, we have an equivalence
$$\mathrm{Fun}'(\mathcal{O}^{\trianglelefteq}, \mathbf{Cat}) \simeq \Delta^0 \times_{*,\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat})} \mathrm{Fun}(\mathcal{O}^{\trianglelefteq}, \mathbf{Cat}),$$

and under the equivalence $\mathrm{Fun}_{\mathcal{T}}(\mathcal{O}^{\trianglelefteq}, \mathbf{Cat}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}^{\trianglelefteq}, \underline{\mathbf{Cat}}_{\mathcal{T}})$ of [Sha21a, Prop. 3.9], this yields an equivalence

$$\mathrm{Fun}'(\mathcal{O}^{\trianglelefteq}, \mathbf{Cat}) \simeq \mathrm{Fun}_{\mathcal{T}//\mathcal{T}}(\mathcal{O}^{\trianglelefteq}, \underline{\mathbf{Cat}}_{\mathcal{T}})$$

to the ∞-category of $\mathcal{T}$-functors $\mathcal{O}^{\trianglelefteq} \longrightarrow \underline{\mathbf{Cat}}_{\mathcal{T}}$ that restrict on $\mathcal{T}^{\mathrm{op}}$ to the $\mathcal{T}$-final object of $\underline{\mathbf{Cat}}_{\mathcal{T}}$. By the $\mathcal{T}$-join and slice adjunction of [Sha21a, Prop. 4.25] (together with [Sha21a, Cor. 4.27]), we have an equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{T}//\mathcal{T}}(\mathcal{O}^{\trianglelefteq}, \underline{\mathbf{Cat}}_{\mathcal{T}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{O}, (\underline{\mathbf{Cat}}_{\mathcal{T}})^{(*,\mathcal{T})/}),$$

which we claim yields an equivalence $\mathrm{Fun}_{\mathcal{T}//\mathcal{T}}(\mathcal{O}^{\trianglelefteq}, \underline{\mathbf{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathcal{O}, \mathbf{Cat})^{*/}$ upon passage to cocartesian sections – for this, just examine the pullback square of ∞-categories

$$\begin{array}{ccc} \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, (\underline{\mathbf{Cat}}_{\mathcal{T}})^{(*,\mathcal{T})/}) & \longrightarrow & \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{T}^{\mathrm{op}} \times \Delta^1, \underline{\mathbf{Cat}}_{\mathcal{T}})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{O}, \mathbf{Cat})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \mathcal{T}^{\mathrm{op}}) \simeq \Delta^0 & \xrightarrow{\quad * \quad} & \mathrm{Fun}_{\mathcal{T}}(\mathcal{O}, \underline{\mathbf{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathcal{O}, \mathbf{Cat}). \end{array}$$

This shows the first part of (3), and the second follows since we have a comparison $\mathcal{T}$-functor that we just proved is an equivalence fiberwise. The claim of (4) (i.e., that μ promotes to an equivalence of $(\infty, 2)$-categories) then follows since μ clearly respects cotensors by ∞-categories (cf. Construction 2.6.5). ☐

5.2.10. **Theorem.** *Suppose $\mathcal{O}^{\otimes}$ is a unital $\mathcal{T}$-∞-operad and $\mathcal{C}^{\otimes}$ is a $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category. Then we have a canonical equivalence of $\mathcal{T}$-∞-categories*

$$\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0, \mathcal{C}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{/\mathcal{O},\mathcal{T}}(\mathcal{O}, \mathcal{C})^{(1,\mathcal{T})/}.$$

Proof. By the usual reduction, it will suffice to prove the statement without the 'underlining'. Our strategy is to replace $\mathcal{O}_0^{\otimes}$ by its $\mathcal{O}_0$-monoidal envelope and then invoke Proposition 5.2.9(4). We first identify $\mathrm{Env}_{\mathcal{O}_0,\mathcal{T}}(\mathcal{O}_0)^{\otimes} = \mathrm{Ar}_{\mathcal{T}}^{act}(\mathcal{O}_0^{\otimes})$ in simpler terms. Let

$$\lambda : \mathcal{O} \times \Delta^1 \longrightarrow \mathrm{Ar}_{\mathcal{T}}^{act}(\mathcal{O}_0^{\otimes}) \times_{\mathcal{O}_0^{\otimes}} \mathcal{O}$$

be the $\mathcal{T}$-functor which for $x \in \mathcal{O}_V$ sends $(x, 0) \longrightarrow (x, 1)$ to the evident morphism $[0_V \to x] \longrightarrow \mathrm{id}_x$ of active arrows. More precisely, we may define the adjoint of λ projecting to $\mathrm{Ar}_{\mathcal{T}}^{act}(\mathcal{O}_0^{\otimes})$ as the composite

$$\mathcal{O} \times \Delta^1 \times \Delta^1 \xrightarrow{h} \mathcal{O}^{\trianglelefteq} \xrightarrow{\omega_{\mathcal{O}}} \mathcal{O}_0^{\otimes},$$

where to define h, we regard $\mathcal{O} \times (\Delta^1 \times \Delta^1)$ as fibered over $\mathcal{T}^{\mathrm{op}} \times \Delta^1$ via the structure map π for $\mathcal{O}$ and $(i, j) \mapsto \max(i, j)$, and let h be given by $(\pi, \mathrm{pr}_{\mathcal{O}})$ under the defining universal property of the $\mathcal{T}$-join (cf. [Sha21a, Prop. 4.3]). We then let

$$\overline{\lambda} : \mathrm{Ar}_{\mathcal{T}}(\mathcal{O}) \times \Delta^1 \longrightarrow \mathrm{Ar}_{\mathcal{T}}^{act}(\mathcal{O}_0^{\otimes}) \times_{\mathcal{O}_0^{\otimes}} \mathcal{O}$$

be the induced morphism of $\mathcal{T}$-cocartesian fibrations over $\mathcal{O}$ extending λ under the equivalence of [Sha21b, Ex. 3.8]. We claim that $\overline{\lambda}$ is an equivalence, for which it suffices to check fiberwise. But for every $x \in \mathcal{O}_V$, we have that λ_x is essentially surjective in view of the unital assumption on $\mathcal{O}^{\otimes}$ (since the active edges must be of the form $[f : x' \to x]$ in $\mathcal{O}_V$ or $[0_V \to x]$ factoring through some f), and an easy computation of mapping spaces shows that λ_x is also fully faithful.

By similar reasoning, we also see that the composition

$$\mathcal{O} \times \{0\} \subset \mathcal{O} \times \Delta^1 \longrightarrow \mathrm{Ar}_{\mathcal{T}}(\mathcal{O}) \times \Delta^1 \xrightarrow{\simeq} \mathrm{Ar}_{\mathcal{T}}^{act}(\mathcal{O}_0^{\otimes}) \times_{\mathcal{O}_0^{\otimes}} \mathcal{O}$$

identifies with the unit map for $\mathrm{Env}_{\mathcal{O}_0,\mathcal{T}}(\mathcal{O}_0)^{\otimes}$. Now let $\mathcal{E} : \mathcal{O} \longrightarrow \mathbf{Cat}$ be the functor obtained by straightening $\mathrm{ev}_1 : \mathrm{Ar}_{\mathcal{T}}(\mathcal{O}) \longrightarrow \mathcal{O}$. We have shown that under the correspondence of Proposition 5.2.9, $\mathrm{Env}_{\mathcal{O}_0,\mathcal{T}}(\mathcal{O}_0)^{\otimes}$ straightens to $\Delta^1 \times \mathcal{E}$. In the notation of that proposition, consider the pullback square

$$\begin{array}{ccc} \mathrm{Nat}_*(\Delta^1 \times \mathcal{E}, f_{\mathcal{C}}) & \longrightarrow & \mathrm{Nat}(\Delta^1 \times \mathcal{E}, f_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & \mathrm{Nat}(\mathcal{E}, f_{\mathcal{C}}). \end{array}$$

Using the universal property of the free $\mathcal{T}$-cocartesian fibration, we deduce an equivalence

$$\mathrm{Nat}_*(\Delta^1 \times \mathcal{E}, f_{\mathcal{C}}) \simeq \mathrm{Fun}_{/\mathcal{O},\mathcal{T}}(\mathcal{O}, \mathcal{C})^{(1,\mathcal{T})/}.$$

We may now conclude using Proposition 5.2.9(4). □

5.2.11. **Theorem.** *Let $\mathcal{O}^{\otimes}$ be a unital $\mathcal{T}$-∞-operad and let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-∞-category.*

(1) The $\mathcal{O}$-algebra $1^{\otimes}$ of Proposition 5.2.8 is an initial object of $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C})$.
(2) $\underline{\mathbf{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C})$ admits a $\mathcal{T}$-initial object given fiberwise by $(1^{\otimes})_{\underline{V}}$.

Proof. Since units are stable under base-change, it suffices to prove the first assertion. Let $1^{\otimes}_{un} : \mathcal{O}^{\otimes}_0 \longrightarrow \mathcal{C}^{\otimes}$ be the unique morphism of $\mathcal{T}$-∞-operads over $\mathcal{O}^{\otimes}$ that sends all edges in $\mathcal{O}^{\otimes}_0$ to p-cocartesian edges in $\mathcal{C}^{\otimes}$ (so $1^{\otimes}_{un}$ extends $* : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\otimes}$ under the equivalence of Lemma 5.2.7). Then $1^{\otimes}_{un}$ corresponds to $1 \in \mathrm{Fun}_{/\mathcal{O},\mathcal{T}}(\mathcal{O},\mathcal{C})^{(1,\mathcal{T})/}$ under the equivalence of Theorem 5.2.10 and is hence an initial object of $\mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0,\mathcal{C})$.

Let $i : \mathcal{O}^{\otimes}_0 \longrightarrow \mathcal{O}^{\otimes}$ denote the inclusion. It will suffice to show that the left adjoint $i_!$ to the the forgetful functor $i^* : \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}) \longrightarrow \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0,\mathcal{C})$ is defined on $1^{\otimes}_{un}$ and sends $1^{\otimes}_{un}$ to $1^{\otimes}$. Consider the factorization of i through its $\mathcal{O}$-monoidal envelope

$$\mathcal{O}^{\otimes}_0 \xrightarrow{\iota} \mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0)^{\otimes} = \mathcal{O}^{\otimes}_0 \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^{\otimes}) \xrightarrow{\mathrm{ev}_1} \mathcal{O}^{\otimes}$$

and the resulting sequence of adjunctions

$$\begin{array}{ccc} \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0,\mathcal{C}) \simeq \mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0),\mathcal{C}) & \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} & \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0),\mathcal{C}) \\ & {\scriptstyle i_!}\searrow \ \nwarrow{\scriptstyle i^*} & (\mathrm{ev}_1)^* \uparrow \ \downarrow (\mathrm{ev}_1)_! \\ & & \mathbf{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}) \end{array}$$

where the dotted left adjoints are not necessarily defined. Observe that for every orbit $V \in \mathcal{T}$, the fiber $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0)^{\otimes}_V$ admits an initial object given by id_{0_V}, and these assemble to define a $\mathcal{T}$-initial object of $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0)^{\otimes}$. By Lemma 5.2.7, the $\mathcal{O}$-monoidal $\mathcal{T}$-functor $1^{\otimes} \circ \mathrm{ev}_1$ restricts to $1^{\otimes}_{un}$ as they both send *all* edges to p-cocartesian edges and extend $*$, so $\iota_!(1^{\otimes}_{un}) \simeq 1^{\otimes} \circ \mathrm{ev}_1$. Now observe that for every $x \in \mathcal{O}^{\otimes}_V$, the unique map $0_V \to x$ is active and is an initial object in the fiber $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{O}_0)^{\otimes} \times_{\mathcal{O}^{\otimes}} \{x\}$. Consequently, the ordinary left Kan extension of $1^{\otimes} \circ \mathrm{ev}_1$ along ev_1 exists and is computed by $1^{\otimes}$ itself. Since the ordinary left Kan extension is an $\mathcal{O}$-algebra in this case, we conclude that $(\mathrm{ev}_1)_!(1^{\otimes} \circ \mathrm{ev}_1) \simeq 1^{\otimes}$ and hence $i_!(1^{\otimes}_{un}) \simeq 1^{\otimes}$. □

5.3. **Indexed coproducts in the $\mathcal{T}$-symmetric monoidal case.** We identify finite $\mathcal{T}$-indexed coproducts with tensor products in the case of a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category $\mathcal{C}$, following the strategy of [Lur17, §3.2.4]. To precisely articulate this identification, we first discuss how to equip $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})$ with a $\mathcal{T}$-symmetric monoidal structure.

5.3.1. **Construction.** We define the smash product $\mathcal{T}$-functor

$$\wedge : \underline{\mathbf{F}}_{\mathcal{T},*} \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}, \quad ([U_+ \to V],[U'_+ \to V]) \mapsto [(U \times_V U')_+ \mapsto V]$$

as follows. Recall that given an ∞-category $\mathcal{D}$ with finite products, we can define the smash product functor $\wedge : \mathcal{D}_* \times \mathcal{D}_* \longrightarrow \mathcal{D}_*$ as the composition of functors

$$\mathcal{D}_* \times \mathcal{D}_* \subset \mathcal{D}^{\Delta^1} \times \mathcal{D}^{\Delta^1} \xrightarrow{\times} \mathcal{D}^{\Delta^1 \times \Delta^1} \xrightarrow{\min_!} \mathcal{D}^{\Delta^1} \xrightarrow{(d^2)_*} \mathcal{D}^{\Lambda^2_0} \xrightarrow{\mathrm{colim}} \mathcal{D}^{(\Lambda^2_0)^{\triangleright}} \xrightarrow{\mathrm{ev}_{\{2\}^{\triangleright}}} \mathcal{D}$$

provided the functors $\min_!$ and colim exist pointwise on objects $(* \to x, * \to y)$ and $(* \leftarrow x \vee y \to x \times y)$, as they then extend to partially defined left adjoints so that the composition exists. If we then have a presheaf $\mathcal{D}_{\bullet} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ such that for every $\alpha : V \longrightarrow W$ in $\mathcal{T}$, $\alpha^* : \mathcal{D}_W \longrightarrow \mathcal{D}_V$ preserves finite products, wedge sums, and cofibers $(x \times y)/(x \vee y)$, it follows from the existence theorem for relative adjunctions ([Lur17, Prop. 7.3.2.6] and [Lur17, Prop. 7.3.2.11]) that we obtain a $\mathcal{T}$-functor

$$\wedge : \underline{\mathcal{D}_*} \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathcal{D}_*} \longrightarrow \underline{\mathcal{D}_*}$$

given fiberwise by formation of smash products, where $\underline{\mathcal{D}_*}$ is the unstraightening of the pointed presheaf $\mathcal{D}_{\bullet,*}$.

5.3.2. **Definition.** Let $\mathcal{O}^\otimes, \mathcal{P}^\otimes, \mathcal{Q}^\otimes$ be $\mathcal{T}$-∞-operads. We say that a $\mathcal{T}$-functor $F : \mathcal{O}^\otimes \times_{\mathcal{T}^{op}} \mathcal{P}^\otimes \longrightarrow \mathcal{Q}^\otimes$ is a *bifunctor of* $\mathcal{T}$-∞*-operads* if it sends pairs of inert edges to inert edges and the diagram

$$\begin{array}{ccc} \mathcal{O}^\otimes \times_{\mathcal{T}^{op}} \mathcal{P}^\otimes & \xrightarrow{F} & \mathcal{Q}^\otimes \\ \downarrow & & \downarrow \\ \underline{\mathbf{F}}_{\mathcal{T},*} \times_{\mathcal{T}^{op}} \underline{\mathbf{F}}_{\mathcal{T},*} & \xrightarrow{\wedge} & \underline{\mathbf{F}}_{\mathcal{T},*} \end{array}$$

is homotopy commutative.[12]

5.3.3. **Variant.** By the same construction as in Construction 5.3.1, we have a smash product $\mathbf{F}_{\mathcal{T}}^{op}$-functor

$$\wedge : \underline{\mathbf{F}}_{\mathcal{T},*}^{\mathrm{big}} \times_{\mathbf{F}_{\mathcal{T}}^{op}} \underline{\mathbf{F}}_{\mathcal{T},*}^{\mathrm{big}} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}^{\mathrm{big}}$$

that can be chosen to extend the smash product on $\underline{\mathbf{F}}_{\mathcal{T},*}$. Likewise, we have the analogous definition of a bifunctor of big $\mathcal{T}$-∞-operads, and the datum of one determines the other essentially uniquely under the correspondence of Corollary 2.7.5.

5.3.4. **Theorem-Construction.** Suppose $F : \mathcal{O}^\otimes \times_{\mathcal{T}^{op}} \mathcal{P}^\otimes \longrightarrow \mathcal{Q}^\otimes$ is a bifunctor of $\mathcal{T}$-∞-operads and let the $\mathcal{T}$-functor

$$P : \mathcal{O}^\otimes \times_{\mathcal{T}^{op}} \mathrm{Ar}(\mathcal{T}^{op}) \longrightarrow \mathcal{O}^\otimes, \quad (x, V \xleftarrow{\beta} W) \mapsto \beta^*(x)$$

be a choice of cocartesian pushforward. Consider the spans of marked simplicial sets

$$(\mathcal{O}^\otimes, \mathrm{Ne}) \xleftarrow{\pi} (\mathcal{O}^\otimes, \mathrm{Ne}) \times_{\mathcal{T}^{op}} \mathrm{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} (\mathcal{P}^\otimes, \mathrm{Ne}) \xrightarrow{G} (\mathcal{Q}^\otimes, \mathrm{Ne}),$$

$$(\mathcal{O}^\otimes)^\sharp \xleftarrow{\pi} (\mathcal{O}^\otimes)^\sharp \times_{\mathcal{T}^{op}} \mathrm{Ar}(\mathcal{T}^{op})^\sharp \times_{\mathcal{T}^{op}} (\mathcal{P}^\otimes, \mathrm{Ne}) \xrightarrow{G} (\mathcal{Q}^\otimes)^\sharp,$$

where $G = F \circ (P \times \mathrm{id}_{\mathcal{P}^\otimes})$ and π is the projection to $\mathcal{O}^\otimes$. Then these spans determine Quillen adjunctions

$$G_! \pi^* : \mathbf{sSet}^+_{/(\mathcal{O}^\otimes, \mathrm{Ne})} \rightleftarrows \mathbf{sSet}^+_{/(\mathcal{Q}^\otimes, \mathrm{Ne})} : \pi_* G^*, \qquad G_! \pi^* : \mathbf{sSet}^+_{/\mathcal{O}^\otimes} \rightleftarrows \mathbf{sSet}^+_{/\mathcal{Q}^\otimes} : \pi_* G^*$$

with respect to the $\mathcal{T}$-operadic model structures and $\mathcal{T}$-monoidal model structures. Given a fibration $\mathcal{C}^\otimes \longrightarrow \mathcal{Q}^\otimes$ of $\mathcal{T}$-∞-operads, we then let

$$\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P}, \mathcal{C})^\otimes \longrightarrow \mathcal{O}^\otimes$$

denote the resulting fibration of $\mathcal{T}$-∞-operads given by $\pi_* G^*(\mathcal{C}^\otimes, \mathrm{Ne})$.[13]

If $\mathcal{C}^\otimes$ is $\mathcal{Q}$-monoidal, then $\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P}, \mathcal{C})^\otimes$ is $\mathcal{O}$-monoidal, and has cocartesian edges marked as in $\pi_* G^*({}_\natural \mathcal{C}^\otimes)$.

Proof. Note that the underlying simplicial sets of $\pi_* G^*(-)$ are the same regardless of whether we work over $(\mathcal{Q}^\otimes, \mathrm{Ne})$ or $(\mathcal{Q}^\otimes)^\sharp$. We first establish the assertion on the Quillen adjunction between $\mathcal{T}$-operadic model structures. For the proof, it will be convenient to pass to big $\mathcal{T}$-∞-operads (cf. Variant 5.3.3). Let

$$\widetilde{F} : \widetilde{\mathcal{O}}^\otimes \times_{\mathbf{F}_{\mathcal{T}}^{op}} \widetilde{\mathcal{P}}^\otimes \longrightarrow \widetilde{\mathcal{Q}}^\otimes$$

be the bifunctor of big $\mathcal{T}$-∞-operads extending F, let the $\mathbf{F}_{\mathcal{T}}$-functor

$$\widetilde{P} : \widetilde{\mathcal{O}}^\otimes \times_{\mathbf{F}_{\mathcal{T}}^{op}} \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}^{op}) \longrightarrow \widetilde{\mathcal{O}}^\otimes$$

be a choice of cocartesian pushforward over $\mathbf{F}_{\mathcal{T}}^{op}$ extending P, let $\widetilde{G} = \widetilde{F} \circ (\widetilde{P} \times \mathrm{id}_{\widetilde{\mathcal{P}}^\otimes})$, and consider the span of marked simplicial sets

$$(\widetilde{\mathcal{O}}^\otimes, \mathrm{Ne}) \xleftarrow{\widetilde{\pi}} (\widetilde{\mathcal{O}}^\otimes, \mathrm{Ne}) \times_{\mathbf{F}_{\mathcal{T}}^{op}} \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}^{op})^\sharp \times_{\mathbf{F}_{\mathcal{T}}^{op}} (\widetilde{\mathcal{P}}^\otimes, \mathrm{Ne}) \xrightarrow{\widetilde{G}} (\widetilde{\mathcal{Q}}^\otimes, \mathrm{Ne}).$$

We claim that this span satisfies the hypotheses of [Lur17, Thm. B.4.2] with respect to the categorical patterns $\widetilde{\mathfrak{P}}_{\mathcal{O}}$ and $\widetilde{\mathfrak{P}}_{\mathcal{Q}}$ of Definition 2.7.3. (2) is clear and (3) is vacuous. By [Lur09, Cor. 2.4.7.17], the source functor $\mathrm{ev}_0 : \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}^{op}) \times_{\mathbf{F}_{\mathcal{T}}^{op}} \widetilde{P}^\otimes \longrightarrow \mathbf{F}_{\mathcal{T}}^{op}$ is a cartesian fibration, so the pullback $\widetilde{\pi}$ is a cartesian fibration.

[12] If $\mathcal{T} = *$, then the smash product for $\mathbf{Fin}_*$ can be defined without the ambiguity of a contractible space of choices. Therefore, one can choose the square to strictly commute in $\mathbf{sSet}^+_{/(\mathbf{Fin}_*, \mathrm{Ne})}$ for the non-parametrized definition of a bifunctor of ∞-operads as in [Lur17, Def. 2.2.5.3].

[13] Beware that the notation $\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P}, \mathcal{C})^\otimes$ hides the dependence of the structure map $\mathcal{P}^\otimes \longrightarrow \mathcal{Q}^\otimes$ on the choice of parameter in $\mathcal{O}^\otimes$.

This proves (1) and (4). Moreover, an edge in $\widetilde{\mathcal{O}}^{\otimes} \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}) \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} \widetilde{\mathcal{P}}^{\otimes}$ is $\widetilde{\pi}$-cartesian if and only if its projection to $\widetilde{\mathcal{P}}^{\otimes}$ is an equivalence, which implies (7).

Now let $f_{x,\phi,\Sigma} : n^{\lhd} \longrightarrow \widetilde{\mathcal{O}}^{\otimes}$ be as in Definition 2.7.3, so that $\phi : U \longrightarrow V$ is a morphism in $\mathbf{F}_{\mathcal{T}}$, $\Sigma = \{\sigma_1, ..., \sigma_n\}$ is a collection of commutative squares

$$\begin{array}{ccc} U_i & \xrightarrow{\alpha_i} & U \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi} \\ V_i & \xrightarrow{\beta_i} & V \end{array}$$

such that α_i and $U_i \longrightarrow V_i \times_V U$ are summand inclusions and $U \simeq \coprod_{1 \leq i \leq n} U_i$, $f(v) = x$, and each morphism $\overline{\chi}_i := f(v \longrightarrow i) : x \longrightarrow x_i := f(i)$ is an inert edge covering $\chi_{\sigma_i} : [U_+ \to V] \mapsto [(U_i)_+ \to V_i]$ in $\underline{\mathbf{F}}_{\mathcal{T},*}^{\mathrm{big}}$. We first prove (5) by showing that the restriction

$$n^{\lhd} \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}) \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} \widetilde{\mathcal{P}}^{\otimes} \longrightarrow n^{\lhd}$$

of π along $f_{x,\phi,\Sigma}$ is a cocartesian fibration. In fact, by [Lur17, Lem. 6.1.1.1], for any cocartesian fibration $\mathcal{X} \longrightarrow \mathbf{F}_{\mathcal{T}}^{\mathrm{op}}$, the source functor $\mathrm{ev}_0 : \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}) \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} \mathcal{X} \longrightarrow \mathbf{F}_{\mathcal{T}}^{\mathrm{op}}$ is a cocartesian fibration, with an edge

$$\left(\begin{array}{ccc} V & \longleftarrow & V' \\ \uparrow & & \uparrow \\ W & \longleftarrow & W' \end{array} , \; x \longrightarrow y \right)$$

cocartesian if and only if the square in $\mathbf{F}_{\mathcal{T}}$ is a pullback square and $x \longrightarrow y$ is a cocartesian edge. Next, let

$$s : n^{\lhd} \longrightarrow n^{\lhd} \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} \mathrm{Ar}(\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}) \times_{\mathbf{F}_{\mathcal{T}}^{\mathrm{op}}} \widetilde{\mathcal{P}}^{\otimes}$$

be a cocartesian section determined by $s(v) = (V \xleftarrow{\gamma} W, y \in \widetilde{\mathcal{P}}^{\otimes}_{[U'_+ \to W]})$, so that $s(i) = (V_i \xleftarrow{\gamma_i} W_i := W \times_V V_i, y_i)$ for $y \longrightarrow y_i$ an inert edge lifting $W_i \longrightarrow W$. Let $z = \widetilde{F}(\gamma^* x, y)$, let σ'_i be

$$\begin{array}{ccc} U_i \times_V U' & \longrightarrow & U \times_V U' \\ \downarrow & & \downarrow{\scriptstyle \phi'} \\ W_i & \longrightarrow & W, \end{array}$$

and let $\Sigma' = \{\sigma'_1, ..., \sigma'_n\}$. Then a chase of the definitions shows that the composition $\widetilde{F} \circ \widetilde{P} \circ f_{x,\phi,\Sigma}$ is of the form $f_{z,\phi',\Sigma'}$, which shows (6). Finally, (8) follows from the right cancellation property of inert edges. This completes the verification of the hypotheses of [Lur17, Thm. B.4.2]. We then deduce the theorem in question by means of Proposition 2.7.4 and Corollary 2.7.5. Finally, repeating this analysis for the other span proves the assertions regarding the monoidality of the construction. □

5.3.5. **Proposition.** *Let $F : \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ be a bifunctor of $\mathcal{T}$-∞-operads and let $\mathcal{C}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads. We then have the following properties of the construction $\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$:*

(1) *For every object $x \in \mathcal{O}$ over $V \in \mathcal{T}^{\mathrm{op}}$, the parametrized restriction*

$$F_{\underline{x}} : \underline{x} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} \longrightarrow (\mathcal{Q}^{\otimes})_{\underline{V}}$$

is a morphism of $\mathcal{T}^{/V}$-∞-operads, and we obtain a canonical equivalence of $\mathcal{T}^{/V}$-∞-categories

$$\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes} \times_{\mathcal{O}^{\otimes}} \underline{x} \simeq \underline{\mathbf{Alg}}_{\mathcal{Q}_{\underline{V}},\mathcal{T}^{/V}}(\mathcal{P}_{\underline{V}}, \mathcal{C}_{\underline{V}}).$$

Similarly, for every cocartesian section $\tau : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{O}$, we have a canonical equivalence of $\mathcal{T}$-∞-categories

$$\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes} \times_{\mathcal{O}^{\otimes},\tau} \mathcal{T}^{\mathrm{op}} \simeq \underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P}, \mathcal{C}).$$

(2) *For every object $y \in \mathcal{P}$ over $V \in \mathcal{T}^{\mathrm{op}}$, the parametrized restriction*

$$F_{\underline{y}} : \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \underline{y} \longrightarrow (\mathcal{Q}^{\otimes})_{\underline{V}}$$

is a morphism of $\mathcal{T}^{/V}$-∞-operads, and 'evaluation at y' furnishes a commutative square of $\mathcal{T}^{/V}$-∞-operads

$$\begin{array}{ccc}
(\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P},\mathcal{C})^{\otimes})_{\underline{V}} & \xrightarrow{\mathrm{ev}_y} & (\mathcal{C}^{\otimes})_{\underline{V}} \\
\downarrow & & \downarrow \\
(\mathcal{O}^{\otimes})_{\underline{V}} \simeq \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \underline{y} & \xrightarrow{F_{\underline{y}}} & (\mathcal{Q}^{\otimes})_{\underline{V}}.
\end{array}$$

Similarly, for every cocartesian section $\tau : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{P}$, we have a morphism of $\mathcal{T}$-∞-operads

$$\mathrm{ev}_{\tau} : \underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P},\mathcal{C})^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$$

covering $F_{\tau} : \mathcal{O}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ and given fiberwise by evaluation at $\tau(V)$.

(3) If $\mathcal{C}^{\otimes}$ is $\mathcal{Q}$-monoidal (so that $\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P},\mathcal{C})^{\otimes}$ is $\mathcal{O}$-monoidal), then ev_y *and* ev_{τ} *preserve cocartesian edges.*

Proof. (1): We prove the assertion about x – that for τ will hold by the same reasoning. We have a commutative diagram

$$\begin{array}{ccccc}
\underline{x} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \xrightarrow{F} & \mathcal{Q}^{\otimes} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{T}^{/V})^{\mathrm{op}} \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathbf{F}}_{\mathcal{T},*} & \longrightarrow & \underline{\mathbf{F}}_{\mathcal{T},*} \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathbf{F}}_{\mathcal{T},*} & \xrightarrow{\wedge} & \underline{\mathbf{F}}_{\mathcal{T},*}
\end{array}$$

where the outer square induces the $\mathcal{T}^{/V}$-functor $F_{\underline{x}}$. The definition of a bifunctor of $\mathcal{T}$-∞-operads then immediately shows that $F_{\underline{x}}$ is a morphism of $\mathcal{T}^{/V}$-∞-operads. Next, consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Ar}(\underline{x}) \times_{\underline{x}} (\underline{x} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes}) & & & & \\
 & \swarrow^{\mathrm{ev}_0} & \downarrow{\scriptstyle \rho}\,\simeq & \searrow^{\mathrm{pr}} & & & \\
\underline{x} & \xleftarrow{\pi} & \underline{x} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \xrightarrow{P \times \mathrm{id}} & \underline{x} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \xrightarrow{F_{\underline{x}}} & \mathcal{Q}^{\otimes}_{\underline{V}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}^{\otimes} & \xleftarrow{\pi} & \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \xrightarrow{P \times \mathrm{id}} & \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \xrightarrow{F} & \mathcal{Q}^{\otimes}
\end{array}$$

where the functor ρ is the trivial fibration used in the definition of the cocartesian pushforward. Using the variant of [Sha21b, Lem. 4.8] for algebra maps, after marking appropriately this diagram induces a comparison $\mathcal{T}^{/V}$-functor

$$\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P},\mathcal{C})^{\otimes} \times_{\mathcal{O}^{\otimes}} \underline{x} \longrightarrow \underline{\mathbf{Alg}}_{\mathcal{Q}_{\underline{V}},\mathcal{T}^{/V}}(\mathcal{P}_{\underline{V}},\mathcal{C}_{\underline{V}}),$$

which by [Sha21a, Lem. 2.27] is an equivalence.

(2): By the same logic as in (1), $F_{\underline{y}}$ is a morphism of $\mathcal{T}^{/V}$-∞-operads. Using the compatibility of the construction $\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P},\mathcal{C})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ with base-change in $\mathcal{T}$, without loss of generality we may replace $\mathcal{T}^{/V}$ with $\mathcal{T}$ and suppose $y \in \mathcal{P}$ lies over a final object in $\mathcal{T}$. Choosing a section σ of the trivial Kan fibration $\underline{y} \xrightarrow{\simeq} \mathcal{T}^{\mathrm{op}}$, let $j = (\mathrm{id}, \iota, \sigma) : \mathcal{O}^{\otimes} \longrightarrow \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes}$ and consider the morphism of spans

$$\begin{array}{ccccccc}
 & & \mathcal{O}^{\otimes} & & & & \\
 & \swarrow^{=} & \downarrow{\scriptstyle j} & & \searrow^{F_{\underline{y}}\sigma} & & \\
\mathcal{O}^{\otimes} & \xleftarrow{\pi} & \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \xrightarrow{P \times \mathrm{id}} & \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{P}^{\otimes} & \xrightarrow{F} & \mathcal{Q}^{\otimes}.
\end{array}$$

Noting that j respects markings for the first span in Theorem-Construction 5.3.4, we then see that j induces the desired morphism of $\mathcal{T}$-∞-operads $\underline{\mathbf{Alg}}_{\mathcal{Q},\mathcal{T}}(\mathcal{P},\mathcal{C})^{\otimes} \longrightarrow \mathcal{O}^{\otimes} \times_{\mathcal{Q}^{\otimes}} \mathcal{C}^{\otimes}$.

(3): This follows from the proof of (2) since the functor j also respects markings for the second span in Theorem-Construction 5.3.4. □

We now specialize to the bifunctor $\wedge : \underline{\mathbf{F}}_{\mathcal{T},*} \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$. Fix a choice of cocartesian pushforward $P : \underline{\mathbf{F}}_{\mathcal{T},*} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}) \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ and also write

$$\wedge : \underline{\mathbf{F}}_{\mathcal{T},*} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} \underline{\mathbf{F}}_{\mathcal{T},*} := \underline{\mathbf{F}}_{\mathcal{T},*} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Ar}(\mathcal{T}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$$

for the composition $\wedge \circ (P \times \mathrm{id})$.

5.3.6. **Construction.** Let $\mathcal{C}^{\otimes}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. We construct a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-functor

$$(-)^{\mathrm{can}} : \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})^{\otimes} \longrightarrow \underline{\mathbf{CAlg}}_{\mathcal{T}}(\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C}))^{\otimes}$$

that is split by the 'forgetful' evaluation $\mathcal{T}$-functor U of Proposition 5.3.5(2). First observe that we have a commutative diagram of marked simplicial sets

$$\begin{array}{ccccc}
\underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) & \xrightarrow{\wedge \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} \mathrm{id}} & \underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) & \xrightarrow{\wedge} & \underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} \\
\downarrow \mathrm{pr}_{12} & & \downarrow \mathrm{pr}_1 & & \\
\underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) & \xrightarrow{\wedge} & \underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} & & \\
\downarrow \mathrm{pr}_1 & & & & \\
\underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} & & & &
\end{array}$$

in which the square is a pullback (here, pr_{12} denotes projection away from the third factor). Also write $\wedge$ for the upper horizontal composite. Then we have that (cf. [Sha21a, Lem. 2.26])

$$(\mathrm{pr}_1)_* \wedge^* \cong (\mathrm{pr}_1)_* \wedge^* (\mathrm{pr}_1)_* \wedge^* : \mathbf{sSet}^+_{/\underline{\mathbf{F}}_{\mathcal{T},*}} \longrightarrow \mathbf{sSet}^+_{/\underline{\mathbf{F}}_{\mathcal{T},*}}, \quad \natural\mathcal{C}^{\otimes} \mapsto \natural\underline{\mathbf{CAlg}}_{\mathcal{T}}(\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C}))^{\otimes}.$$

Now consider the morphisms of spans

$$\begin{array}{ccccc}
 & \xleftarrow{\mathrm{pr}_1} & \underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) & \xrightarrow{\wedge} & \\
 & & \downarrow \mathrm{id} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} \mathrm{id} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} \iota & & \\
\underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} & \xleftarrow{\mathrm{pr}_1} & \underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) & \xrightarrow{\wedge} & \underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp}. \\
 & & \downarrow \mathrm{id} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} \wedge & & \\
 & \xleftarrow{\mathrm{pr}_1} & \underline{\mathbf{F}}_{\mathcal{T},*}{}^{\sharp} \times^{\mathrm{lax}}_{\mathcal{T}^{\mathrm{op}}} (\underline{\mathbf{F}}_{\mathcal{T},*}, \mathrm{Ne}) & \xrightarrow{\wedge} &
\end{array}$$

Then the lower vertical arrow defines $(-)^{\mathrm{can}}$, and since the upper vertical arrow induces U and the composite is homotopic to the identity, we see that $\mathrm{U} \circ (-)^{\mathrm{can}} \simeq \mathrm{id}$.

5.3.7. **Theorem.** *Let $\mathcal{C}^{\otimes}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. Then $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})$ has all finite $\mathcal{T}$-coproducts. Moreover, for any map of finite $\mathcal{T}$-sets $f : U \longrightarrow V$, we have a canonical equivalence*

$$\coprod_f \simeq f_{\otimes} : \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_U \longrightarrow \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_V,$$

where $f_{\otimes}$ is furnished by the $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})$ of Theorem-Construction 5.3.4.

Proof. Since the base-change condition for left adjoints to the restriction functors $\{f^*\}$ of $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})$ to furnish finite $\mathcal{T}$-coproducts is already satisfied by the maps $\{f_{\otimes}\}$, it will suffice to construct unit and counit maps exhibiting $f_{\otimes}$ as left adjoint to f^*. Without loss of generality, we may suppose V is an orbit, and after replacing $\mathcal{T}$ by $\mathcal{T}^{/V}$, we may suppose $V = *$ is the final object of $\mathcal{T}$ so that $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_V \simeq \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})$. In addition, by Theorem 5.2.11, $\mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})$ admits an initial object given by the unit 1, which is $f_{\otimes}(*)$ for $f : \emptyset \longrightarrow *$. We may thus suppose that U is nonempty. Using Construction 5.3.6, given a $\mathcal{T}$-commutative algebra A we get

$$A^{\mathrm{can}} : \underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})^{\otimes},$$

and we then get a map $f_{\otimes} f^* A \longrightarrow A$ by applying A^{can} to $f_+ : [U_+ \to *] \longrightarrow [*_+ \to *]$ and factoring the resulting map $f^* A \longrightarrow A$ through a cocartesian lift over f_+ in the base. Using the naturality of this procedure in A, we then obtain our candidate for the counit transformation $\epsilon : f_{\otimes} f^* \longrightarrow \mathrm{id}$. To define the unit transformation $\eta : \mathrm{id} \longrightarrow f^* f_\otimes$, consider the pullback square

$$\begin{array}{ccc} U \times U & \xrightarrow{\mathrm{pr}_2} & U \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{f} & * \end{array}$$

and the associated equivalence $f^* f_\otimes \simeq (\mathrm{pr}_1)_\otimes (\mathrm{pr}_2)^*$. The summand inclusion $\delta : U \longrightarrow U \times U$ yields a natural transformation

$$\epsilon_{\mathrm{pr}_2^*(-)} : \delta_\otimes \simeq \delta_\otimes(\delta^* \mathrm{pr}_2^*) = (\delta_\otimes \delta^* \mathrm{pr}_2^*) \longrightarrow \mathrm{pr}_2^*,$$

which on objects $B \in \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_U$ may be described as follows: if we write $U \times U \simeq U \coprod U'$ and $g = \mathrm{pr}_2|_{U'}$, then in terms of the decomposition $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_{U \times U} \simeq \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_U \times \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_{U'}$, we have that

$$\epsilon_{\mathrm{pr}_2^* B} : \delta_\otimes(B) = (B, 1_{U'}) \longrightarrow \mathrm{pr}_2^*(B) = (B, g^* B)$$

is given by the identity on the first factor and the unique map out of the initial object on the second factor. We then let

$$\eta = (\mathrm{pr}_1)_\otimes(\epsilon_{\mathrm{pr}_2^*(-)}) : \mathrm{id} \simeq (\mathrm{pr}_1)_\otimes \delta_\otimes \longrightarrow (\mathrm{pr}_1)_\otimes \mathrm{pr}_2^* \simeq f^* f_\otimes.$$

It remains to verify the triangle identities. Let $U \simeq \coprod_{i=1}^n U_i$ be an orbit decomposition of U, let $\iota_i : U_i \longrightarrow U$ denote the inclusion, and let $f_i = f \circ \iota_i$. Observe that after pullback to U_i, the map f acquires a canonical section, i.e., for all $1 \leq i \leq n$ we have a factorization of the identity map

$$\mathrm{id} : U_i \xrightarrow{(\mathrm{id}, \iota_i)} U_i \times U \xrightarrow{p^i} U_i.$$

where p^i denote the projection. Using $(-)^{\mathrm{can}}$, this furnishes a factorization

$$\mathrm{id} : f_i^* A \longrightarrow (p^i)_\otimes (p^i)^* (f_i^* A) \simeq (p^i)_\otimes (\mathrm{pr}_2)^* f^* A \longrightarrow f_i^* A,$$

where we use the commutative square

$$\begin{array}{ccc} U_i \times U & \xrightarrow{\mathrm{pr}_2} & U \\ \downarrow{\scriptstyle p^i} & & \downarrow{\scriptstyle f} \\ U_i & \xrightarrow{f_i} & * \end{array}$$

for the middle equivalence. To express this in more familiar terms, note that if we write $U_i \times U \simeq U_i \coprod U_i'$ and $q^i = p^i|_{U_i'}$, then we may identify this as

$$f_i^* A \simeq f_i^* A \otimes 1_{U_i} \xrightarrow{\mathrm{id} \otimes 1} f_i^* A \otimes (q^i)_\otimes (q^i)^* A \xrightarrow{\mathrm{id} \otimes \epsilon} f_i^* A \otimes f_i^* A \xrightarrow{\epsilon} f_i^* A,$$

where $\otimes$ denotes the fiberwise tensor product on $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_{U_i}$ induced by the fold map $\nabla : U_i \coprod U_i \longrightarrow U_i$.

Now regarding the composition $f^* A \xrightarrow{\eta f^*} f^* f_\otimes f^* A \xrightarrow{f^* \epsilon} f^* A$, by an elementary diagram chase we see that after pullback along ι_i this identifies with the factorization of $f_i^* A$ given above, which validates this half of the triangle identities.

Finally, we consider the composition $f_\otimes B \xrightarrow{f_\otimes \eta} f_\otimes f^* f_\otimes B \xrightarrow{\epsilon f^*} f_\otimes B$. By the $\mathcal{T}$-symmetric monoidality of $(-)^{\mathrm{can}}$, we get a canonical equivalence $f_\otimes(B^{\mathrm{can}}) \simeq (f_\otimes B)^{\mathrm{can}}$ in $\mathbf{CAlg}_{\mathcal{T}}(\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C}))$. If we then write $B = (A_1, ..., A_n)$ under the decomposition $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_U \simeq \prod_{i=1}^n \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_{U_i}$, it follows that we obtain an equivalence

$$f_\otimes f^* f_\otimes B \simeq f_\otimes((p^1)_\otimes (p^1)^* A_1, ..., (p^n)_\otimes (p^n)^* A_n)$$

under which $\epsilon_{f_\otimes B} \simeq f_\otimes(\epsilon_{A_1}, ..., \epsilon_{A_n})$ and the composite $\epsilon_{f_\otimes B} \circ f_\otimes \eta_B$ identifies with $f_\otimes$ of the composite defined factorwise by the map $A_i \longrightarrow (p^i)_\otimes (p^i)^* A_i \longrightarrow A_i$ induced from $\mathrm{id} : U_i \xrightarrow{(\mathrm{id}, \iota_i)} U_i \times U \xrightarrow{p^i} U_i$. Since these all compose as identities, we deduce the other half of the triangle identities. □

5.3.8. **Corollary.** *Let $\mathcal{C}^\otimes$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. Then the $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})$ of Theorem-Construction 5.3.4 is cocartesian.*

Next, we consider the more general case of an arbitrary $\mathcal{T}$-indexing system $\mathcal{I}$ (Definition 2.4.8).

5.3.9. **Theorem.** *Let $\mathcal{C}^{\otimes}$ be a $\mathcal{I}$-symmetric monoidal $\mathcal{T}$-∞-category. Then for all orbits $V \in \mathcal{T}$, the fiber $\underline{\mathbf{CAlg}}_{\mathcal{I}}(\mathcal{C})_V$ admits finite coproducts, and for all morphisms $f : V \longrightarrow W \in \mathcal{T}$, the restriction functor $f^* : \underline{\mathbf{CAlg}}_{\mathcal{I}}(\mathcal{C})_W \longrightarrow \underline{\mathbf{CAlg}}_{\mathcal{I}}(\mathcal{C})_V$ preserves finite coproducts. Moreover, finite coproducts are computed as tensor products in terms of the symmetric monoidal structures on the fibers $\underline{\mathbf{CAlg}}_{\mathcal{I}}(\mathcal{C})_V$ constructed via Theorem-Construction 5.3.4 applied to the bifunctor*

$$\wedge_{\mathcal{I}} : \mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Com}^{\otimes}_{\mathcal{I}} \longrightarrow \mathrm{Com}^{\otimes}_{\mathcal{I}}$$

obtained by restriction of $\wedge : \underline{\mathbf{F}}_{\mathcal{T},*} \times_{\mathcal{T}^{\mathrm{op}}} \underline{\mathbf{F}}_{\mathcal{T},*} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$.

Proof. The proof is exactly analogous to that of Theorem 5.3.7, where in place of Construction 5.3.6 we instead use the composition of the span involving $\wedge_{\mathcal{I}}$ with that involving $\wedge_{\mathcal{T}^{\simeq}}$ to define the $\mathcal{T}^{\simeq}$-symmetric monoidal $\mathcal{T}$-functor

$$\underline{\mathbf{CAlg}}_{\mathcal{I}}(\mathcal{C})^{\otimes} \longrightarrow \underline{\mathbf{CAlg}}_{\mathcal{T}^{\simeq}}(\underline{\mathbf{CAlg}}_{\mathcal{I}}(\mathcal{C}))^{\otimes},$$

and also the identification of Corollary 2.4.15. □

5.3.10. **Corollary.** *Let $\mathcal{C}^{\otimes}$ be a distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category such that $\mathcal{C}$ is fiberwise presentable, let $\mathcal{I}$ be a $\mathcal{T}$-indexing system, and write $\mathcal{C}^{\otimes}_{\mathcal{I}} = \mathrm{Com}^{\otimes}_{\mathcal{I}} \times_{\underline{\mathbf{F}}_{\mathcal{T},*}} \mathcal{C}^{\otimes}$. Let*

$$\mathrm{U} : \underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C}) \longrightarrow \underline{\mathbf{CAlg}}_{\mathcal{I}}(\mathcal{C}_{\mathcal{I}})$$

be the forgetful $\mathcal{T}$-functor implemented by restriction along $\mathrm{Com}^{\otimes}_{\mathcal{I}} \subset \underline{\mathbf{F}}_{\mathcal{T},*}$. Then for all $V \in \mathcal{T}$, U_V is a conservative functor of presentable ∞-categories that preserves all small limits and colimits, and is hence comonadic. In particular, $\underline{\mathbf{CAlg}}_{\mathcal{T}}(\mathcal{C})_V$ is comonadic over $\mathbf{CAlg}(\mathcal{C}_V)$.

Proof. To reduce notational clutter, let us replace $\mathcal{T}$ by $\mathcal{T}^{/V}$ so that $V = *$ is a terminal object of $\mathcal{T}$. By Theorem 5.1.4(4), both $\mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})$ and $\mathbf{CAlg}_{\mathcal{I}}(\mathcal{C}_{\mathcal{I}})$ are presentable. Since the forgetful functor to $\mathcal{C}_*$ is conservative for a reduced $\mathcal{T}$-∞-operad, U_* is conservative as well. By Theorem 5.1.3(2), U_* preserves all small limits. By Theorem 5.1.4(2), U_* preserves all sifted colimits. By Theorem 5.3.7 and Theorem 5.3.9, U_* preserves all finite coproducts. It follows that U_* preserves all small colimits and hence admits a right adjoint by the adjoint functor theorem, so we are entitled to ask about the comonadicity of U_*. The conclusion then follows from the Barr–Beck–Lurie Theorem [Lur09, Thm. 4.7.3.5]. □

6. $\mathcal{T}$-symmetric monoidal structure on $\mathcal{T}$-presheaves

Let $\mathcal{C}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. In this section, we construct a $\mathcal{T}$-symmetric monoidal structure on the $\mathcal{T}$-∞-category of $\mathcal{T}$-presheaves $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ such that for any $\mathcal{T}$-distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category $\mathcal{D}$, the universal mapping property

$$\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

of [Sha21a, Thm. 11.5] refines to an equivalence

$$\underline{\mathrm{Fun}}^{L,\otimes}_{\mathcal{T}}(\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{T}}(\mathcal{C}, \mathcal{D});$$

cf. Corollary 6.0.12. This result has also been achieved by Hilman as [Hil22a, Thm. 2.3.6].

We first extend our discussion of universal constructions from [Sha21b] by proving the parametrized analogue of [Lur09, Prop. 5.3.6.2].[14] Let $\mathcal{K} = \{\mathcal{K}_U : U \in \mathcal{T}\}$ be a collection of classes $\mathcal{K}_U$ of small $\mathcal{T}^{/U}$-∞-categories such that for each morphism $f : U \longrightarrow V$ in $\mathcal{T}$, $f^*(\mathcal{K}_V) \subset \mathcal{K}_U$; we call such a collection *closed*. For each $V \in \mathcal{T}$, let $\mathcal{K}|_V = \{\mathcal{K}_U : [f : U \to V] \in \mathcal{T}\}$. Recall the following definition from [Sha21b, Def. 2.8]

6.0.1. **Definition.** Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. We say that $\mathcal{C}$ *strongly admits* $\mathcal{K}$-indexed $\mathcal{T}$-colimits if for all $U \in \mathcal{T}$, $\mathcal{C}_{\underline{U}}$ admits $\mathcal{K}_U$-indexed $\mathcal{T}^{/U}$-colimits. Likewise, for any $V \in \mathcal{T}$, we may refer to $\mathcal{C}_{\underline{V}}$ strongly admitting $\mathcal{K}|_V$-indexed $\mathcal{T}^{/V}$-colimits.

Given a $\mathcal{T}$-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, we say that F *strongly preserves* $\mathcal{K}$-indexed $\mathcal{T}$-colimits if for all $U \in \mathcal{T}$, the $\mathcal{T}^{/U}$-functor $F_{\underline{U}} : \mathcal{C}_{\underline{U}} \longrightarrow \mathcal{D}_{\underline{U}}$ preserves all $\mathcal{K}_U$-indexed $\mathcal{T}^{/U}$-colimits. Likewise, for any $V \in \mathcal{T}$, we may refer to $F_{\underline{V}}$ strongly preserving $\mathcal{K}|_V$-indexed $\mathcal{T}^{/V}$-colimits. We then let

$$\underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

[14] For this, we need not suppose that $\mathcal{T}$ is atomic.

be the full $\mathcal{T}$-subcategory spanned in each fiber over $V \in \mathcal{T}$ by those $\mathcal{T}^{/V}$-functors that strongly preserve $\mathcal{K}|_V$-indexed $\mathcal{T}^{/V}$-colimits. Note that the global sections $\mathrm{Fun}^{\mathcal{K}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ of $\underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ is then the full subcategory of $\mathrm{Fun}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ spanned by those $\mathcal{T}$-functors that strongly preserve $\mathcal{K}$-indexed $\mathcal{T}$-colimits.

Suppose given a collection $\mathcal{R} = \{\mathcal{R}_U : U \in \mathcal{T}\}$ of classes $\mathcal{R}_U = \{\overline{p_\alpha} : \mathcal{K}_\alpha^{\triangleright} \longrightarrow \mathcal{C}_{\underline{U}}\}$ of $\mathcal{T}^{/U}$-diagrams in $\mathcal{C}_{\underline{U}}$ (which are not necessarily $\mathcal{T}^{/U}$-colimit diagrams), such that for each morphism $f : U \longrightarrow V$ in $\mathcal{T}$, $f^*(\mathcal{R}_V) \subset \mathcal{R}_U$. Then a $\mathcal{T}$-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ *strongly preserves* $\mathcal{R}$-indexed $\mathcal{T}$-colimits if for all $U \in \mathcal{T}$, $F_{\underline{U}} : \mathcal{C}_{\underline{U}} \longrightarrow \mathcal{D}_{\underline{U}}$ sends each $\overline{p_\alpha}$ to $\mathcal{T}^{/U}$-colimit diagram in $\mathcal{D}_{\underline{U}}$. Likewise, for any $V \in \mathcal{T}$ we have the same notion for $F_{\underline{V}}$ with respect to $\mathcal{R}|_V$. We then let

$$\underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\mathcal{C},\mathcal{D}) \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$$

be defined as before.

6.0.2. **Proposition.** *Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category and let $\mathcal{R}$ be as in Definition 6.0.1, such that each $\mathcal{K}_\alpha$ lies in $\mathcal{K}_U$.*

Then there exists a $\mathcal{T}$-∞-category $\underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ and a $\mathcal{T}$-functor $j : \mathcal{C} \longrightarrow \underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ such that:

(1) $\underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ strongly admits $\mathcal{K}$-indexed $\mathcal{T}$-colimits.
(2) For all $\mathcal{T}$-∞-categories $\mathcal{D}$ such that $\mathcal{D}$ strongly admits $\mathcal{K}$-indexed $\mathcal{T}$-colimits, precomposition with j induces an equivalence of $\mathcal{T}$-∞-categories

$$j^* : \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}),\mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$$

which upon passage to global sections yields an equivalence of ∞-categories

$$j^* : \mathrm{Fun}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}),\mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathcal{R}}_{\mathcal{T}}(\mathcal{C},\mathcal{D}).$$

(3) Suppose that all $\mathcal{T}^{/U}$-functors $\overline{p_\alpha} : \mathcal{K}_\alpha^{\triangleright} \longrightarrow \mathcal{C}_{\underline{U}}$ in $\mathcal{R}_U$ are $\mathcal{T}^{/U}$-colimit diagrams. Then j is fully faithful.

Proof. The proof is essentially identical to that of [Lur09, Prop. 5.3.6.2]; we spell out a few details for the reader's benefit. After enlarging universes, we may suppose $\mathcal{C}$ is small.[15] Let $j_0 : \mathcal{C} \longrightarrow \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ be the $\mathcal{T}$-Yoneda embedding. For every $\overline{p_\alpha} \in \mathcal{R}_U$, let $p_\alpha = \overline{p_\alpha}|_{\mathcal{K}_\alpha}$, let $Y_\alpha \in \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_U$ be the image of the cone point (in the fiber over U) under $\overline{p_\alpha}$, let $X_\alpha \in \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_U$ be a $\mathcal{T}^{/U}$-colimit for $(j_0)_{\underline{U}} \circ p_\alpha : \mathcal{K}_\alpha \longrightarrow \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_{\underline{U}}$, let $s_\alpha : X_\alpha \longrightarrow j_0(Y_\alpha)$ be the induced map, and let $S_U = \{s_\alpha\}$.

Note that for all morphisms $f : U \longrightarrow V$ in $\mathcal{T}$, $f^* S_V \subset S_U$ by our closure hypothesis on $\mathcal{R}$. Thus, we may form the full $\mathcal{T}$-subcategory $S^{-1}\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}) \subset \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ given on each fiber over $U \in \mathcal{T}$ by the full subcategory of S_U-local objects of $\mathbf{P}_{\mathcal{T}^{/U}}(\mathcal{C}_{\underline{U}}) = \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})_U$. We then have a $\mathcal{T}$-left adjoint $L : \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}) \longrightarrow S^{-1}\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ given fiberwise by the usual localization. Finally, we define $\underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ to be the smallest full $\mathcal{T}$-subcategory of $S^{-1}\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ which contains the essential image of $L \circ j_0$ and such that for each $U \in \mathcal{T}$, $\underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ is closed under $\mathcal{K}_U$-indexed colimits, and we let $j = L \circ j_0$ be the induced map.

Given this construction, the verification of properties (1)–(3) proceeds exactly as in the proof of [Lur09, Prop. 5.3.6.2] (with parametrized analogues of non-parametrized statements involving colimits, left Kan extensions, etc. substituted as appropriate). □

6.0.3. **Remark.** If $\mathcal{K} = \mathcal{A}ll$ then we may also write $\underline{\mathbf{P}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ as $\underline{\mathbf{P}}_{\mathcal{R}}(\mathcal{C})$.

Now let $\underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T}} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*}$ be the $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category given by the $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathbf{Cat}}_{\mathcal{T}}$ (Example 2.4.1). Objects of $\underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T}}$ are then given by tuples

$$([f : U \to V] \in \underline{\mathbf{F}}_{\mathcal{T},*},\ \mathcal{C}_1 \in \mathbf{Cat}_{\mathcal{T}^{/U_1}},\ \dots,\ \mathcal{C}_n \in \mathbf{Cat}_{\mathcal{T}^{/U_n}})$$

where $U \simeq U_1 \sqcup \dots \sqcup U_n$ is an orbit decomposition.

To describe morphisms in $\underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T}}$, consider a morphism

$$\begin{array}{ccccc} U & \longleftarrow & Z & \xrightarrow{m} & X \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle g} \\ V & \longleftarrow & Y & \xrightarrow{=} & Y \end{array}$$

[15] Since we suppose $\mathcal{T}$ is small, a $\mathcal{T}$-∞-category $\mathcal{C}$ is small if and only if it is fiberwise small.

in $\underline{\mathbf{F}}_{\mathcal{T},*}$, let $U \simeq \coprod_{i=1}^n U_i$, $X \simeq \coprod_{j=1}^m X_j$ be orbit decompositions, and let $m_j : Z_j \simeq X_j \times_Y Z \longrightarrow X_j$ be the restriction over the summand X_j. Let $\{\mathcal{C}_i \in \mathbf{Cat}_{\mathcal{T}/U_i}\}$ and $\{\mathcal{D}_j \in \mathbf{Cat}_{\mathcal{T}/X_j}\}$. Let $\mathcal{C} = \coprod_{i=1}^n \mathcal{C}_i$ denote the given $\mathcal{T}^{/U}$-∞-category and let $\mathrm{pr}^*\mathcal{C}$ denote the pullback of $\mathcal{C}$ to a $\mathcal{T}^{/U\times_V Y}$-$\infty$-category along the projection $U \times_V Y \longrightarrow U$. By definition, the map $Z \longrightarrow U \times_V Y$ is a summand inclusion; let $\mathcal{C}'$ denote the corresponding summand of $\mathcal{C}$ regarded as a $\mathcal{T}^{/Z}$-∞-category, and also let $\mathcal{C}'_j$ denote the $\mathcal{T}^{/Z_j}$-∞-category given by the further summand of $\mathcal{C}'$. Then a morphism

$$(f, \{\mathcal{C}_i\}) \longrightarrow (g, \{\mathcal{D}_j\})$$

is given by a collection of $\mathcal{T}^{/X_j}$-functors $F_j : (m_j)_*(\mathcal{C}'_j) \longrightarrow \mathcal{D}_j$, or written more concisely, a $\mathcal{T}^{/X}$-functor $F : m_*(\mathcal{C}') \longrightarrow \mathcal{D}$.

6.0.4. **Definition.** Let $\mathcal{M} \subset \underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T}} \times \Delta^1$ be the subcategory defined as follows:

- $\mathcal{M}_0 = \underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T}}$.
- An object $([f : U \to V], \mathcal{C}_1 \in \mathbf{Cat}_{\mathcal{T}/U_1}, ..., \mathcal{C}_n \in \mathbf{Cat}_{\mathcal{T}/U_n})$ belongs to $\underline{\mathbf{Cat}}^{\otimes}_{\mathcal{T}} \times \{1\}$ if and only if each $\mathcal{C}_i$ strongly admits $\mathcal{T}^{/U_i}$-colimits.
- All morphisms $(f, \{\mathcal{C}_i\}, 0) \longrightarrow (g, \{\mathcal{D}_j\}, 1)$ belong to $\mathcal{M}$.
- A morphism $(f, \{\mathcal{C}_i\}, 1) \longrightarrow (g, \{\mathcal{D}_j\}, 1)$ belongs to $\mathcal{M}$ if and only if each F_j is $\mathcal{T}^{/X_i}$-distributive.

Let $p : \mathcal{M} \longrightarrow \underline{\mathbf{F}}_{\mathcal{T},*} \times \Delta^1$ denote the composite of the inclusion and the structure map.

For the following, we have implicitly extended the notion of (closed) collection $\mathcal{K} = \{\mathcal{K}_U\}$ to be indexed over all finite $\mathcal{T}$-sets U.

6.0.5. **Notation.** Let $f : U \longrightarrow V$ be a morphism in $\mathbf{F}_{\mathcal{T}}$ and let $\mathcal{C}$ be a $\mathcal{T}^{/U}$-∞-category. Let $\mathcal{R} = \{\mathcal{R}_\alpha : [U' \xrightarrow{\alpha} U] \in \mathbf{F}_{\mathcal{T}}\}$ be a closed collection of diagrams in $\mathcal{C}$, $\mathcal{R}_\alpha = \{\overline{p_\alpha} : \mathcal{K}_\alpha^{\triangleright} \longrightarrow \mathcal{C}_{\underline{U'}}\}$. We let $f_*\mathcal{R}$ denote the closed collection of diagrams in $f_*\mathcal{C}$ specified at each morphism $\gamma : V' \longrightarrow V$ as follows:

(∗) Let $U' = U \times_V V'$ and let $f' : U' \longrightarrow V'$ be the pullback of f. For every $\mathcal{T}^{/U'}$-diagram $\overline{p_\alpha} : \mathcal{K}_\alpha^{\triangleright} \longrightarrow \mathcal{C}_{\underline{U'}}$ in $\mathcal{R}_{U'}$, we may form the $\mathcal{T}^{/V'}$ diagram

$$f'_*(\mathcal{K}_\alpha)^{\triangleright} \xrightarrow{\mathrm{can}} f'_*(\mathcal{K}_\alpha^{\triangleright}) \xrightarrow{f'_*(\overline{p_\alpha})} f'_*(\mathcal{C}_{\underline{U'}}) \simeq (f_*\mathcal{C})_{\underline{V'}}.$$

Let $(f_*\mathcal{C})_\gamma$ be the set of these diagrams.

6.0.6. **Notation.** Given a finite $\mathcal{T}$-set U with orbit decomposition $U \simeq \coprod_{i=1}^n U_i$, let $\underline{\mathbf{P}}_{\mathcal{T}/U}(-)$ be given by the coproduct of the $\underline{\mathbf{P}}_{\mathcal{T}/U_i}(-)$.

6.0.7. **Lemma.** *Let $f : U \longrightarrow V$ be a morphism of finite $\mathcal{T}$-sets and let $\mathcal{C}$ be a $\mathcal{T}^{/U}$-∞-category. Consider the $\mathcal{T}^{/V}$-functor*

$$\phi : f_*\underline{\mathbf{P}}_{\mathcal{T}/U}(\mathcal{C}) \longrightarrow \underline{\mathbf{P}}_{\mathcal{T}/V}(f_*\mathcal{C})$$

given by the composite of the $\mathcal{T}^{/U}$-Yoneda embedding (for $f_*\underline{\mathbf{P}}_{\mathcal{T}/U}(\mathcal{C})$) and restriction along f_* of the $\mathcal{T}^{/V}$-Yoneda embedding (for $\mathcal{C}$). Then ϕ is $\mathcal{T}^{/V}$-distributive.

Proof. Suppose $p : \mathcal{K} \longrightarrow \underline{\mathbf{P}}_{\mathcal{T}/U}(\mathcal{C})$ is a $\mathcal{T}^{/U}$-diagram. We need to show that the $\mathcal{T}^{/V}$-colimit of the composite

$$f_*\mathcal{K} \xrightarrow{f_*(p)} f_*\underline{\mathbf{P}}_{\mathcal{T}/U}(\mathcal{C}) \xrightarrow{\phi} \underline{\mathbf{P}}_{\mathcal{T}/V}(f_*\mathcal{C})$$

evaluates to $f_*(\mathrm{colim}^{\mathcal{T}^{/U}} p)$. It suffices to check this after evaluation at all objects of $f_*\mathcal{C}$, and without loss of generality it suffices to consider $x \in (f_*\mathcal{C})_V \simeq \mathcal{C}_U$ (by the usual base-change argument). A diagram chase then shows that the composite

$$f_*\underline{\mathbf{P}}_{\mathcal{T}/U}(\mathcal{C}) \xrightarrow{\phi} \underline{\mathbf{P}}_{\mathcal{T}/V}(f_*\mathcal{C}) \xrightarrow{\mathrm{ev}_x} \underline{\mathbf{Spc}}_{\mathcal{T}/V}$$

identifies with the composite

$$f_*\underline{\mathbf{P}}_{\mathcal{T}/U}(\mathcal{C}) \xrightarrow{f_*\mathrm{ev}_x} f_*\underline{\mathbf{Spc}}_{\mathcal{T}/U} = f_*f^*\underline{\mathbf{Spc}}_{\mathcal{T}/V} \xrightarrow{m} \underline{\mathbf{Spc}}_{\mathcal{T}/V}$$

where m is the multiplication given by the $\mathcal{T}$-distributive $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathbf{Spc}}_{\mathcal{T}}$. Thus, m is $\mathcal{T}^{/V}$-distributive, which implies that $\mathrm{ev}_x\phi$ is distributive. □

6.0.8. **Lemma.** *Let $f : U \longrightarrow V$ be a morphism of finite $\mathcal{T}$-sets, let $\mathcal{C}$ be a $\mathcal{T}^{/U}$-∞-category, and let $\mathcal{D}$ be a $\mathcal{T}^{/U}$-cocomplete $\mathcal{T}^{/U}$-∞-category. Then a $\mathcal{T}^{/V}$-functor $F : f_* \underline{\mathbf{P}}_{\mathcal{T}^{/U}}(\mathcal{C}) \longrightarrow \mathcal{D}$ is $\mathcal{T}^{/V}$-distributive if and only if it is the $\mathcal{T}^{/V}$-left Kan extension of its restriction along $f_*(j) : f_*\mathcal{C} \subset f_*\underline{\mathbf{P}}_{\mathcal{T}^{/U}}(\mathcal{C})$.*

Proof. For the 'if' direction, it suffices to consider the universal example given by the $\mathcal{T}^{/V}$-functor ϕ of Lemma 6.0.7, which we showed to be $\mathcal{T}^{/V}$-distributive.[16] For the 'only if' direction, let $F' = (f_*j)_!(f_*j)^*F$ and consider the comparison map $\theta : F' \Rightarrow F$. Without loss of generality, it suffices to show that for all $x \in (f_*\underline{\mathbf{P}}_{\mathcal{T}^{/U}}(\mathcal{C}))_V \simeq (\underline{\mathbf{P}}_{\mathcal{T}^{/U}}(\mathcal{C}))_U$, θ_x is an equivalence. On the one hand, by the pointwise formula for $\mathcal{T}^{/V}$-left Kan extensions, we have that $F'(x)$ is given by the $\mathcal{T}^{/V}$-colimit of $p : (f_*\mathcal{C})^{/\underline{x}} \longrightarrow f_*\mathcal{C} \longrightarrow \mathcal{D}$. On the other hand, since $f_* : \mathbf{Cat}_{\mathcal{T}^{/U}} \longrightarrow \mathbf{Cat}_{\mathcal{T}^{/V}}$ preserves cotensors and limits, we have a natural equivalence $(f_*\mathcal{C})^{/\underline{x}} \simeq f_*(\mathcal{C}^{/\underline{x}})$ such that $[(f_*\mathcal{C})^{/\underline{x}} \longrightarrow f_*\mathcal{C}] \simeq [f_*(\mathcal{C}^{/\underline{x}} \longrightarrow \mathcal{C})]$, and using that F is $\mathcal{T}^{/V}$-distributive, we get that $F(x)$ is also given by the $\mathcal{T}^{/V}$-colimit of p. The naturality of all operations considered shows further that θ_x implements this equivalence. □

In the following corollary, we disambiguate our terminology for distributive functors by referring to the morphism of finite $\mathcal{T}$-sets and not just the target.

6.0.9. **Corollary.** *Let $U \xrightarrow{f} V \xrightarrow{g} W$ be a composite of morphisms of finite $\mathcal{T}$-sets, let $\mathcal{C}$ be a $\mathcal{T}^{/U}$-cocomplete $\mathcal{T}^{/U}$-∞-category, and let $\mathcal{D}$ be a $\mathcal{T}^{/W}$-cocomplete $\mathcal{T}^{/W}$-∞-category. Consider the composite $\mathcal{T}^{/W}$-functor*

$$j : g_*f_*\mathcal{C} \xrightarrow{j_0} g_*\underline{\mathbf{P}}_{\mathcal{T}^{/V}}(f_*\mathcal{C}) \longrightarrow g_*\underline{\mathbf{P}}_{f_*\mathcal{A}ll}(f_*\mathcal{C})$$

Then a $\mathcal{T}^{/W}$-functor $F : g_*\underline{\mathbf{P}}_{f_*\mathcal{A}ll}(f_*\mathcal{C}) \longrightarrow \mathcal{D}$ is g-distributive if and only if j^*F is gf-distributive. Consequently, we have an equivalence

$$\underline{\mathbf{P}}_{(gf)_*\mathcal{A}ll}((gf)_*\mathcal{C}) \simeq \underline{\mathbf{P}}_{g_*\mathcal{A}ll}(g_*\underline{\mathbf{P}}_{f_*\mathcal{A}ll}(f_*\mathcal{C})).$$

Proof. The first claim follows immediately by restricting the equivalence of Lemma 6.0.8, noting that by definition $g_*\underline{\mathbf{P}}_{f_*\mathcal{A}ll}(f_*\mathcal{C})$ is a localization of $g_*\underline{\mathbf{P}}_{\mathcal{T}^{/V}}(f_*\mathcal{C})$ at the relevant class of morphisms. The equivalence then follows by universal property. □

6.0.10. **Proposition.** *The map p is a cocartesian fibration.*

Proof. We adapt the proof of [Lur17, Prop. 4.8.1.3] to the parametrized context. We first show that p is a locally cocartesian fibration. This is clear if we restrict to the fiber over 0. For the other cases, first suppose $(f : U \to V, \{\mathcal{C}_i\}) \in \mathcal{M}_0$ and let $(f, 0) \longrightarrow (g, 1)$ be a morphism in $\underline{\mathbf{F}}_{\mathcal{T},*} \times \Delta^1$, with notation as above. Let $\mathcal{D}_j = \underline{\mathbf{P}}_{\mathcal{T}^{/X_j}}((m_j)_*(\mathcal{C}'_j))$ and take F_j to be the identity. We then have a morphism $(f, \{\mathcal{C}_i\}, 0) \longrightarrow (g, \{\mathcal{D}_j\}, 1)$ which is a locally cocartesian edge by the universal property of the $\mathcal{T}^{/X_j}$-presheaves.

Next, suppose $(f : U \to V, \{\mathcal{C}_i\}) \in \mathcal{M}_1$ and let $(f, 1) \longrightarrow (g, 1)$ be a morphism in $\underline{\mathbf{F}}_{\mathcal{T},*} \times \Delta^1$. Let $\mathcal{C}'$ be the $\mathcal{T}^{/Z}$-∞-category as above. Let $\mathcal{R}$ be the closed collection of parametrized colimit diagrams in $\mathcal{C}'$, i.e., for each morphism $\alpha : Z' \longrightarrow Z$ in $\mathbf{F}_{\mathcal{T}}$, $\mathcal{R}_\alpha$ is the collection of $\mathcal{T}^{/Z'}$-colimit diagrams in $\mathcal{C}'_{\underline{Z'}}$. We let $\mathcal{D} = \underline{\mathbf{P}}^{\mathcal{A}ll}_{m_*\mathcal{R}}(m_*\mathcal{C}')$ and $F = j : m_*\mathcal{C}' \longrightarrow \mathcal{D}$ be the $\mathcal{T}^{/X}$-functor as in Proposition 6.0.2. We then have that the morphism $j : (f, \mathcal{C}, 1) \longrightarrow (g, \mathcal{D}, 1)$ lies in $\mathcal{M}$ by definition. Moreover, it is a locally cocartesian edge in view of the universal property supplied by Proposition 6.0.2.

To then see that p is a cocartesian fibration, we need to see that the composite of locally cocartesian edges is again locally cocartesian. We already know the restriction over 0 is a cocartesian fibration. If the first edge lies over $[0 \to 1]$, we may apply the parametrized analogue of [Lur09, Prop. 5.3.6.11]; since this step is straightforward we leave the details to the reader. If both edges lie over 1, then without loss of generality we may suppose both edges are fiberwise active as edges over $\underline{\mathbf{F}}_{\mathcal{T},*}$, in which case the claim follows from the transitivity property established in Corollary 6.0.9. □

Let p_0 and p_1 denote the two fibers of p over $0, 1 \in \Delta^1$.

6.0.11. **Corollary.** *The maps p_0 and p_1 exhibit $\mathcal{M}_0$ and $\mathcal{M}_1$ as $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-categories.*

[16] cf. https://math.stackexchange.com/questions/4094599/when-left-kan-extension-preserve-colimits for a description of this standard reduction, which also works in the parametrized context.

Proof. We already have that the map p_0 is the structure map of $\underline{\mathbf{Cat}}_{\mathcal{T}}^{\otimes}$. As for p_1, since it is a cocartesian fibration it remains to verify the parametrized Segal condition. But in the definition of $\mathcal{M}$, all the inert morphisms in $\underline{\mathbf{Cat}}_{\mathcal{T}}^{\otimes}$ continue to lie in $\mathcal{M}$, so we see that $\mathcal{M}_1$ inherits the parametrized Segal condition from $\underline{\mathbf{Cat}}_{\mathcal{T}}^{\otimes}$. □

Now write $(\underline{\mathbf{Cat}}_{\mathcal{T}}^{L})^{\otimes} = \mathcal{M}_1$. The cocartesian fibration p classifies the $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-functor

$$\underline{\mathbf{P}}_{\mathcal{T}/-}^{\otimes} : \underline{\mathbf{Cat}}_{\mathcal{T}}^{\otimes} \longrightarrow (\underline{\mathbf{Cat}}_{\mathcal{T}}^{L})^{\otimes}$$

whose underlying $\mathcal{T}$-functor is given by the usual $\mathcal{T}$-presheaf construction $\underline{\mathbf{P}}_{\mathcal{T}/-}$. Since $\underline{\mathbf{P}}_{\mathcal{T}/-}$ admits a right $\mathcal{T}$-adjoint given by the forgetful $\mathcal{T}$-functor U, U canonically inherits a lax $\mathcal{T}$-symmetric monoidal structure. Passing to $\mathcal{T}$-commutative algebra objects and considering the unit of the adjunction, we obtain:

6.0.12. **Corollary.** *Let $\mathcal{C}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. Then $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ and the $\mathcal{T}$-Yoneda embedding $j : \mathcal{C} \longrightarrow \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ inherit a $\mathcal{T}$-symmetric monoidal structure such that:*

(1) $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ is $\mathcal{T}$-distributive.
(2) For every $\mathcal{T}$-distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category $\mathcal{D}$, restriction along j yields an equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{T}}^{L,\otimes}(\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{\otimes}(\mathcal{C}, \mathcal{D}).$$

6.0.13. **Remark.** Let $\mathcal{C}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-∞-category. Consider the $\mathcal{T}$-distributive $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C}) = \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{vop}}, \underline{\mathbf{Spc}}_{\mathcal{T}})$ given by $\mathcal{T}$-Day convolution (Theorem 3.2.6), where we have the $\mathcal{T}$-symmetric monoidal structure on $\mathcal{C}^{\mathrm{vop}}$ induced by the opposite automorphism on **Cat** under the equivalence of Theorem 2.3.9 and the $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathbf{Spc}}_{\mathcal{T}}$. Then one may show directly that the full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbf{P}}_{\mathcal{T}}(\mathcal{C})$ is closed under this $\mathcal{T}$-symmetric monoidal structure. Using Corollary 6.0.12(2), it then follows that the $\mathcal{T}$-Day convolution $\mathcal{T}$-symmetric monoidal structure agrees with that defined above.

References

[Bar17] Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Advances in Mathematics **304** (2017), 646 – 727.

[BDG+16a] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, *Parametrized higher category theory and higher algebra: A general introduction*, arXiv:1608.03654, 2016.

[BDG+16b] ———, *Parametrized higher category theory and higher algebra: Exposé I - Elements of parametrized higher category theory*, arXiv:1608.03657, 2016.

[BGN18] Clark Barwick, Saul Glasman, and Denis Nardin, *Dualizing cartesian and cocartesian fibrations*, Theory and Applications of Categories **33** (2018), no. 4, 67–94.

[BGS20] Clark Barwick, Saul Glasman, and Jay Shah, *Spectral Mackey functors and equivariant algebraic K-theory, II*, Tunisian J. Math. **2** (2020), no. 1, 97–146.

[BH15] Andrew J. Blumberg and Michael A. Hill, *Operadic multiplications in equivariant spectra, norms, and transfers*, Adv. Math. **285** (2015), 658–708. MR 3406512

[BH20] Andrew J. Blumberg and Michael A. Hill, *G-symmetric monoidal categories of modules over equivariant commutative ring spectra*, Tunisian Journal of Mathematics **2** (2020), no. 2, 237–286.

[BH21] Tom Bachmann and Marc Hoyois, *Norms in motivic homotopy theory*, Astérisque **425** (2021).

[BP21] Peter Bonventre and Luís A. Pereira, *Genuine equivariant operads*, Advances in Mathematics **381** (2021), 107502.

[CH21] Hongyi Chu and Rune Haugseng, *Free algebras through day convolution*, arXiv:2006.08269, 2021.

[GMMO18] Bertrand Guillou, J. Peter May, Mona Merling, and Angélica M. Osorno, *A symmetric monoidal and equivariant segal infinite loop space machine*, 2018, arXiv:1711.09183.

[GW18] Javier J Gutiérrez and David White, *Encoding equivariant commutativity via operads*, Algebraic & Geometric Topology **18** (2018), no. 5, 2919–2962.

[HH16] M. A. Hill and M. J. Hopkins, *Equivariant symmetric monoidal structures*, arXiv:1610.03114, 2016.

[HHK+20] Jeremy Hahn, Asaf Horev, Inbar Klang, Dylan Wilson, and Foling Zou, *Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra*, arXiv preprint arXiv:2006.13348 (2020).

[Hil22a] Kaif Hilman, *Parametrised noncommutative motives and equivariant algebraic K-theory*, arXiv:2202.02591, 2022.

[Hil22b] Kaif Hilman, *Parametrised presentability over orbital categories*, 2022.

[Hin15] Vladimir Hinich, *Rectification of algebras and modules*, Documenta Mathematica (2015), no. 20, 879–926.

[Hin20] Vladimir Hinich, *Yoneda lemma for enriched ∞-categories*, Advances in Mathematics **367** (2020), 107129.

[Hor19] Asaf Horev, *Genuine equivariant factorization homology*, arXiv preprint arXiv:1910.07226 (2019).

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)

[Lur17] ———, *Higher algebra*, Preprint from the web page of the author, May 2017.

[MMO21] J. Peter May, Mona Merling, and Angélica M. Osorno, *Equivariant infinite loop space theory, the space level story*, 2021, arXiv:1704.03413.

[Nar16] Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category*, arXiv:1608.07704, 2016.

[Nar17] Denis Nardin, *Stability and distributivity over orbital ∞-categories*, Ph.D. thesis, 2017.

[QS21a] J. D. Quigley and Jay Shah, *On the parametrized Tate construction*, arXiv:2110.07707, 2021.

[QS21b] J.D. Quigley and Jay Shah, *On the equivalence of two theories of real cyclotomic spectra*, 2021.

[Rub21] Jonathan Rubin, *Combinatorial N_∞ operads*, Algebraic & Geometric Topology **21** (2021), no. 7, 3513–3568.

[Sha21a] Jay Shah, *Parametrized higher category theory*, to appear in Algebraic & Geometric Topology (2021), arXiv:1809.05892.

[Sha21b] ———, *Parametrized higher category theory II: Universal constructions*, arXiv:2109.11954, 2021.

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On normed $\mathbb{E}_\infty$-rings in genuine equivariant C_p-spectra

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Abstract

Genuine equivariant homotopy theory is equipped with a multitude of coherently commutative multiplication structures generalizing the classical notion of an $\mathbb{E}_\infty$-algebra. In this paper we study the C_p-$\mathbb{E}_\infty$-algebras of Nardin–Shah with respect to a cyclic group C_p of prime power order. We show that many of the higher coherences inherent to the definition of parametrized algebras collapse; in particular, they may be described more simply and conceptually in terms of ordinary $\mathbb{E}_\infty$-algebras as a diagram category which we call *normed algebras*. Our main result provides a relatively straightforward criterion for identifying C_p-$\mathbb{E}_\infty$-algebra structures. We visit some applications of our result to real motivic invariants.

Contents

1 Introduction

1.1 Motivation

Algebraic invariants such as integral cohomology $H^*(-;\mathbb{Z})$ detect information about spaces; identifying and applying such tools form the basic premise of algebraic topology. Moreover, considering more structured algebraic objects leads to more refined invariants: Cochains with integral coefficients $X \mapsto C^*(X;\mathbb{Z})$ considered as a functor from spaces to $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras is *better* at detecting information about spaces than integral cohomology. For instance, $C^*(X;\mathbb{Z})$ inherits power operations as a consequence of its $\mathbb{E}_\infty$-structure, and two (non-equivalent) spaces X and Y may have isomorphic cohomology rings but different power operations. In this way, we see that the study of structured multiplications and their operations is foundational to homotopy theory.

In parallel, the study of structured multiplications is essential to the study of genuine equivariant homotopy types. Our line of inquiry naturally leads to algebraic structures whose operations are inherently genuine equivariant. To motivate the particular equivariant multiplicative structure we will focus on, recall that an ordinary $\mathbb{E}_\infty$-algebra in spectra may be modeled by a functor satisfying certain conditions defined on the category of finite pointed sets [Seg74]. In particular, the smash product $A^{\otimes \ell}$ parametrizes formal sums of ℓ-tuples in A, and said functor takes the collapse map $\langle 2 \rangle \to \langle 1 \rangle$ to a morphism $A^{\otimes 2} \to A$. In particular, $\mathbb{E}_\infty$-algebra structures are governed by the category of finite pointed sets.

In genuine equivariant homotopy theory with respect to the finite cyclic group C_p of order a prime p, the role of finite sets is supplanted by *finite pointed sets with C_p-action* [HH14, §4; HH16, §3.3]. The Hill–Hopkins–Ravenel norm $N_e^{C_p} A =: A^{\otimes C_p}$ parametrizes $|C_p|$-tuples in A indexed by a free C_p-set. In [BH15], Blumberg and Hill introduced a genuine equivariant operads encoding multiplications indexed by G-sets; the algebras they give rise to are called N_∞-algebras. In [NS22, Definition 2.2.2], Nardin and Shah defined an ∞-categorical analogue of the N_∞-algebras of Blumberg–Hill[1]; we shall refer to the latter as C_p-$\mathbb{E}_\infty$-algebras (Definition 2.32, Example 2.33). Their structure of operations is governed by the category of finite pointed C_p-sets $\mathrm{Fin}_{C_p,*}$.

Unravelling definitions, a C_p-$\mathbb{E}_\infty$-algebra is the data of

(1) An underlying C_p-genuine equivariant spectrum R.

(2) For each morphism of finite C_p-sets $S \to T$, a morphism of C_p-spectra $N_e^S R \to N_e^T R$. In particular, the collapse map $C_p \twoheadrightarrow C_p/C_p$ indexes a morphism of $\mathbb{E}_\infty$ rings $n_R \colon N_e^{C_p} R \to R$ called the *norm*.

(3) higher coherences...

To exhibit a C_p-$\mathbb{E}_\infty$-algebra structure on a genuine C_p-spectrum R is no small task. In the literature, one often resorts to simplifying assumptions such as requiring R to be Borel, e.g. [Hil22, Proposition 3.3.6]. We set out to provide a relatively straightforward criterion for identifying C_p-$\mathbb{E}_\infty$-algebra structures.

When $p = 2$, by [QS22, Definition 5.2] the category of C_2-$\mathbb{E}_\infty$-algebras is the natural domain of definition for real (i.e. C_2-equivariant) topological Hochschild homology and other real motivic invariants. This work grew out of the author's interest in real motivic invariants and will be used in upcoming work on a real version of the Hochschild–Kostant–Rosenberg theorem.

[1] These notions are expected to agree.

1.2 Main result

To motivate our main result, note that any morphism of C_p-sets $S \to C_p/C_p$ can be expressed as the composite of 'collapse the free C_p-orbits' followed by a (non-equivariant) map of finite sets. Thus a C_p-$\mathbb{E}_\infty$ algebra, regarded as functor defined on C_p-sets, determines two pieces of data: its restriction to sets on which C_p acts trivially, and its value on collapse maps. The former specifies an $\mathbb{E}_\infty$-algebra structure, while the latter specifies a norm map n. To assert that these data are *enough* to specify a C_p-$\mathbb{E}_\infty$-algebra structure means that any higher coherence conditions on the norm map n collapse. We might hope that this is indeed the case, since the category $\mathrm{Fin}_{C_p,*}$ is *freely* generated by the C_p-set C_p/C_p.

Let $A \in \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})$ be an $\mathbb{E}_\infty$-algebra with naïve C_p-action and $\sigma \in C_p$ a generator.

Observation 1.1 (Observation 3.7)**.** Write $A^{\otimes^\Delta p}$ for the object in $\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})$ with the *diagonal* action, i.e. such that σ acts by $\sigma(a_1 \otimes \cdots \otimes a_p) = \sigma(a_p) \otimes \sigma(a_1) \otimes \cdots \otimes \sigma(a_{p-1})$. Write $A^{\otimes^\tau p}$ for the object in $\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})$ with the *transposition* action, i.e. such that σ acts by $\sigma(a_1 \otimes \cdots \otimes a_p) = a_p \otimes a_1 \otimes \cdots \otimes a_{p-1}$. Then the endomorphism $\mathrm{id}_A \otimes \sigma \otimes \cdots \otimes \sigma^{p-1}$ of $A^{\otimes p} \in \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ promotes to an equivalence $A^{\otimes^\Delta p} \to A^{\otimes^\tau p}$ in $\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})^{BC_p}$–in particular it is C_p-equivariant.

Definition 1.2 (Definition 3.11)**.** Write $\mathcal{O}_{C_p}$ for the category of finite sets with transitive C_p-action, and let $\sigma \in C_p$ be a generator. A *normed $\mathbb{E}_\infty$-algebra* in C_p-spectra is the data of an $\mathbb{E}_\infty$-algebra A in Sp^{C_p}, a morphism of $\mathbb{E}_\infty$-rings $n_A \colon N^{C_p}(A^e_{hC_p}) \to A$, and a homotopy making the following diagram $\mathcal{O}_{C_p} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})^{BC_p}$

$$\begin{array}{ccc} (A^e)^{\otimes^\tau p} & \xrightarrow{n^e_A} & \\ {\scriptstyle \mathrm{id}\otimes\sigma\otimes\cdots\otimes\sigma^{p-1}}\downarrow\wr & & A^e \\ (A^e)^{\otimes^\Delta p} & \xrightarrow{m_A} & \end{array}$$

commute, where the C_p-action on $(A^e)^{\otimes^\Delta p}$ corresponding to the inclusion $BC_p \subseteq \mathcal{O}_{C_p}$ is the transposition.

The main result of this paper both formalizes and confirms the aforementioned intuitive picture.

Theorem 1.3 (Corollary 4.7 & Theorem 4.23)**.** *The canonical forgetful functor from the category of C_p-$\mathbb{E}_\infty$ algebras in C_p-spectra (Example 2.33) to the category of normed $\mathbb{E}_\infty$-algebras in C_p-spectra is an equivalence.*

A key input to the proof of Theorem 4.23 is an explicit description of the free C_p-$\mathbb{E}_\infty$ algebra in Sp^{C_p} on an $\mathbb{E}_\infty$-algebra A in Sp^{C_p}. By Theorem 4.14 and Proposition 2.15, the underlying C_p-spectrum of the free C_p-$\mathbb{E}_\infty$ algebra $F(A)$ on A is given by

$$F(A) \simeq \begin{array}{ccc} & & A^{\varphi C_p} \otimes A^e_{hC_p} \\ & & \downarrow{\scriptstyle s_A \otimes \nu_A} \\ A^e & \longmapsto & A^{tC_p} \end{array}$$

where u is the unit, $s_A : A^{\varphi C_p} \to A^{tC_p}$ is the structure map, and ν_A is the twisted Tate-valued norm (Definition 3.8).

Remark 1.4. There is an analogous statement (Theorem 5.15), proved in essentially the same way, for relative normed algebras, i.e. C_p-$\mathbb{E}_\infty$-algebras *over* a fixed C_p-$\mathbb{E}_\infty$-algebra A.

One expects an analogous result to hold for arbitrary G, but we stick with C_p here because the author's motivating example is the case $G = C_2$, and because the complexity of (3.12) seemingly grows exponentially in the subgroup lattice of G.

1.3 Applications & Examples

The power of Theorem 4.23 is that, in many cases, it is easier to identify objects in the diagram category Definition 3.11 than to produce a C_p-$\mathbb{E}_\infty$-algebra in the sense of Definition 2.32, which requires exhibiting an infinite amount of coherence data. In particular, a normed ring is the data of an $\mathbb{E}_\infty$-ring in Sp^{C_p} plus the additional datum of a commutative diagram (3.18). As an application, in §5 we show that various $\mathbb{E}_\infty$-rings in Sp^{C_p} admit natural lifts to C_p-$\mathbb{E}_\infty$-rings in Sp^{C_p}.

Corollary 1.5 (Theorem 5.1)**.** *Let k be a discrete commutative ring. The constant C_p-Mackey functor $\underline{k}$ on k acquires an essentially unique structure of a C_p-$\mathbb{E}_\infty$-ring.*

Using our main theorem, we are able to give an alternative proof of a special case of a result [Hil22, Proposition 3.3.6] of Kaif Hilman. In view of the expected correspondence between N_∞-algebras and C_p-$\mathbb{E}_\infty$-algebras, the following result should also be compared to Theorem 6.26 of [BH15].

Corollary 1.6 (Proposition 5.3)**.** *Every Borel $\mathbb{E}_\infty$-algebra in C_p-spectra admits an essentially unique refinement to a C_p-$\mathbb{E}_\infty$-algebra.*

Many examples arise in the case $p = 2$ because involutions are ubiquitous in topology. Natural examples of $\mathbb{E}_\infty$-rings in Sp^{C_2} include real topological and algebraic K-theories ([Ati66] & 5.8).

Corollary 1.7 (Corollary 5.10)**.**

- *Real topological K-theory $KU_\mathbb{R}$ admits a unique refinement to a C_2-$\mathbb{E}_\infty$ ring spectrum.*
- *If A satisfies the* homotopy limit problem*, then $K_\mathbb{R}(A)$ admits a unique refinement to a C_2-$\mathbb{E}_\infty$ ring spectrum.*

A slightly less trivial class of examples are provided by the following

Corollary 1.8 (Proposition 5.5)**.** *Let $B \in \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ be an $\mathbb{E}_\infty$-algebra. Then $N^{C_p}B$ admits a canonical structure of a C_p-$\mathbb{E}_\infty$-algebra.*

Remark 1.9. Real motivic invariants and their associated real trace theories provided the impetus for this work. In particular, Theorem 5.1 will be used in the author's upcoming work on real trace theories.

1.4 Outline

Despite the nice intuitive picture outlined in §1.1, handling the higher coherence conditions associated to n gets complicated quickly. Thus our proof strategy does not appeal directly to understanding the operadic indexing category (although we will need some understanding of this to write down a comparison functor).

In §2, we collect background on genuine equivariant homotopy theory, as well as the parametrized ∞-categorical perspective on equivariant algebras. In §3, we define normed rings. In §4.1, we define a comparison functor from parametrized algebras to normed rings. In §4.2, we exhibit a formula for the free C_p-$\mathbb{E}_\infty$-algebra on an $\mathbb{E}_\infty$-algebra. In §4.3, we show that the free C_p-$\mathbb{E}_\infty$-algebra on an $\mathbb{E}_\infty$-algebra also computes the free *normed algebra*, and conclude by the Barr–Beck–Lurie theorem. In §5, we look into a few examples and applications.

1.5 Background, Notation, & Conventions

We assume some familiarity with the language of ∞-categories (in the form of quasi-categories) as introduced by Joyal [Joy08] and developed in [Lur09]. All categories are understood to be ∞-categories unless otherwise specified. We do a cursory review of the theory of parametrized ∞-categories as developed by Barwick, Dotto, Glasman, Nardin, and Shah [Bar+16a; Bar+16b; Bar+17; Nar17; Sha18], but the reader should consult the former references for more details. We will assume some familiarity with the ∞-operads of [Lur17, Chapters 2 & 3], which we will compare to the parametrized algebras of [NS22].

To reduce visual clutter, we regularly drop subscripts such as a prime p or a (C_p-)$\mathbb{E}_\infty$-algebra A when they are understood to be fixed (e.g. within the proof of a particular proposition).

1.6 Acknowledgements

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2 Background

We collect some background on genuine equivariant homotopy theory and parametrized ∞-categories here. In §2.1 and §2.3, we recall the parametrized ∞-categorical language and parametrized algebras, resp. of Barwick–Dotto–Glasman–Nardin–Shah. In §2.2, we collect background and structural results on the C_p-genuine equivariant category.

2.1 Parametrized ∞-categories

Let G be a finite group.

Recollection 2.1. The orbit category $\mathcal{O}_G$ is the category with objects finite transitive G-sets and morphisms G-equivariant maps. We let Fin_G denote the finite coproduct completion of $\mathcal{O}_G$, i.e. the category of finite G-sets and G-equivariant maps. We recall that $\mathcal{O}_G^{\mathrm{op}}$ is an *orbital* ∞-category in the sense of Definition 1.2 of [Nar17].

Definition 2.2. ([Nar17, between Examples 1.3 & 1.4; Bar+16b, Definition 1.3]) A *G-∞-category* is a cocartesian fibration $\mathcal{C} \to \mathcal{O}_G^{\mathrm{op}}$.

[Nar17, beginning of §1.2] A morphism of G-∞-categories is a functor F of ∞-categories over $\mathcal{O}_G^{\mathrm{op}}$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{O}_G^{\mathrm{op}} & \end{array}$$

which takes p-cocartesian arrows in $\mathcal{C}$ to q-cocartesian arrows in $\mathcal{D}$. We denote the category of G-functors by $\mathrm{Fun}_G(\mathcal{C}, \mathcal{D})$.

Remark 2.3. [Lur09, §3.2.2; Sha18, Example 2.5] Let $\mathcal{C}\mathrm{at}_\infty$ denote the large ∞-category of small ∞-categories. There is a universal cocartesian fibration $\mathcal{U} \to \mathcal{C}\mathrm{at}_\infty$ such that pullback induces an equivalence

$$\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \simeq \mathcal{C}\mathrm{at}^{\mathrm{cocart}}_{\infty/\mathcal{O}_G^{\mathrm{op}}}.$$

Unraveling definitions and taking $G = C_p$, a C_p-∞-category is the data of

- an ∞-category $\mathcal{C}^{C_p}$,
- an ∞-category with C_p-action $\mathcal{C}^e$, and
- a functor $\mathcal{C}^{C_p} \to \mathcal{C}^e$ which lifts along the C_p homotopy fixed points $(\mathcal{C}^e)^{hC_p} \to \mathcal{C}^e$. In particular, if $\mathcal{C}^e$ is endowed with the trivial C_p-action, then $(\mathcal{C}^e)^{hC_p} \simeq (\mathcal{C}^e)^{BC_p} \simeq \mathrm{Fun}(BC_p, \mathcal{C}^e)$ comprises objects in $\mathcal{C}^e$ with (naïve) C_p-action.

In particular, we see that a cocartesian section $\sigma\colon \mathcal{O}_{C_p}^{\mathrm{op}} \to \mathcal{C}$ is determined by its value on $\sigma(C_p/C_p)$. Informally, we regard the category of cocartesian sections of $\mathcal{C}$ as the category of objects in $\mathcal{C}$.

Notation 2.4. Going forward, we use the notation $\mathcal{T}$ for $\mathcal{O}_{C_p}^{\mathrm{op}}$ to reduce notational clutter. While most of the general theory in §2.1 and §2.3 applies to $\mathcal{T}$ a general atomic orbital ∞-category, we will not need this level of generality to formulate our main results.

There is an (internal to $\mathcal{T}$-parametrized categories) version of functor categories. The notion of *parametrized functor categories* of [Sha18, §3] will be necessary to our investigation of parametrized colimits.

Proposition 2.5. *[Sha18, Proposition 3.1; Bar+16b, Construction 5.2] Let $\mathcal{C} \to \mathcal{T}^{op}$, $\mathcal{D} \to \mathcal{T}^{op}$ be cocartesian fibrations. Then there exists a cocartesian fibration $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D}) \to \mathcal{T}^{op}$ such that under the straightening-unstraightening equivalence of Remark 2.3, $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$ represents the presheaf*

$$\mathcal{E} \mapsto \hom_{\mathcal{T}^{op}}(\mathcal{E} \times_{\mathcal{T}^{op}} \mathcal{C}, \mathcal{D}).$$

Notice that an object of $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$ over $t \in \mathcal{T}$ is a $(\mathcal{T}^{\mathrm{op}})_{t/}$-functor

$$(\mathcal{T}^{\mathrm{op}})_{t/} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{C} \to (\mathcal{T}^{\mathrm{op}})_{t/} \times_{\mathcal{T}^{\mathrm{op}}} \mathcal{D}.$$

Construction 2.6 ($\mathcal{T}$-category of objects)**.** [Bar+16b, Definition 7.4] Let E be a (non-parametrized) ∞-category. The product $E \times \mathcal{T}^{\mathrm{op}}$ may be regarded as a $\mathcal{T}$-∞-category via projection onto the second factor. Evaluation at the source exhibits the (non-parametrized) functor category $\mathrm{Fun}(\Delta^1, \mathcal{T}^{\mathrm{op}}) \xrightarrow{\mathrm{ev}_0} \mathcal{T}^{\mathrm{op}}$ as a cartesian fibration. The parametrized functor category of [Bar+16b, Recollection 5.1]

$$\underline{E}_{\mathcal{T}} := \underline{\mathrm{Fun}}_{\mathcal{T}^{\mathrm{op}}} \left(\mathrm{Fun}(\Delta^1, \mathcal{T}^{\mathrm{op}}), E \times \mathcal{T}^{\mathrm{op}} \right)$$

is the *$\mathcal{T}$-∞-category of $\mathcal{T}$-objects in E.*

Theorem 2.7. *[Bar+16b, Theorem 7.8] Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. Let $\mathcal{D}$ be an ∞-category. Then the $\mathcal{T}$-category of objects of Construction 2.6 satisfies*

$$\mathrm{Fun}_{\mathcal{T}^{op}}(\mathcal{C}, \underline{\mathcal{D}}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Example 2.8. Taking $E = \mathrm{Spc}$ and $\mathcal{C} = \mathcal{T}^{\mathrm{op}}$ in Theorem 2.7, we see that cocartesian sections of $\underline{\mathrm{Spc}}_{\mathcal{T}}$ correspond exactly to $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Spc})$.

We will need to know what a G-left Kan extension is. In service of keeping the background section brief, we take Remark 10.2(3) of [Sha18], which is equivalent to Definition 10.1 of *loc.cit*.

Notation 2.9. [Sha18, Notation 2.29] Let $p : \mathcal{D} \to \mathcal{T}^{\mathrm{op}}$ be a $\mathcal{T}$-∞-category. Given an object $x \in \mathcal{D}$, define

$$\underline{x} := \{x\} \times_{\mathcal{D}} \mathrm{Ar}^{cocart}(\mathcal{D}).$$

Given a $\mathcal{T}$-functor $\psi : \mathcal{C} \to \mathcal{D}$, define the *parametrized fiber of ψ over $x \in \mathcal{D}$* to be

$$\mathcal{C}_{\underline{x}} := \underline{x} \underset{\mathcal{D}, \psi}{\times} \mathcal{C}.$$

Observe that $\mathcal{C}_{\underline{x}}$ may be naturally regarded as a $(\mathcal{T}^{\mathrm{op}})^{p(x)/-}$-category.

Definition 2.10. [Sha18, Remark 10.2(3)] Suppose given a diagram of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc} C & \xrightarrow{F} & E \\ {\scriptstyle\psi}\downarrow & \overset{\eta}{\Searrow} & \nearrow_{G} \\ D & & \end{array} .$$

We say that G is a *left $\mathcal{T}$-Kan extension of F along ψ* if for all $t \in \mathcal{T}$ and all $x \in D_t$, $G|_{\underline{x}}$ is a left $(\mathcal{T}^{\mathrm{op}})^{t/}$-Kan extension of $F|_{\underline{x}} : C_{\underline{x}} \to E_{\underline{t}}$ along $\psi_{\underline{x}}$.

2.2 Genuine equivariant homotopy theory

In this section, we introduce the stable C_p-genuine equivariant category, discuss a parametrized lift (Example 2.14) and give an alternative presentation (Proposition 2.15) which will be useful to our study of algebras. Finally, we recall the Hill–Hopkins–Ravenel norms.

Proposition 2.11. *Let G be a finite group. Then there exists an ∞-category* $\mathrm{Span}(\mathrm{Fin}_G)$ *having*

- *the same objects as* Fin_G
- *homotopy classes of morphisms from V to U in* $\mathrm{Span}(\mathrm{Fin}_G)$ *are in bijection with diagrams $V \leftarrow T \to U$ up to isomorphism of diagrams fixing V and U.*

- *The composite of $V \leftarrow T \to U$ and $U \leftarrow S \to W$ is equivalent to the diagram $V \leftarrow T \times_U S \to W$.*

Moreover, $\mathrm{Span}(\mathrm{Fin}_G)$ *is semiadditive, i.e. finite coproducts and products are isomorphic, and are given on underlying G-sets by the disjoint union.*

Proof. The construction of $\mathrm{Span}(\mathrm{Fin}_G)$ is [Bar17, Proposition 5.6] applied to [Bar17, Example 5.4]. The (0-)semiadditivity of $\mathrm{Span}(\mathrm{Fin}_G)$ follows from noticing that $\mathrm{Span}(\mathrm{Fin}_G)$ is a module over $\mathrm{Span}(\mathrm{Fin})$ and [Har20, Corollary 3.19]. □

The notion of a Mackey functor first appeared in [Dre71] in algebra and in [May96] in homotopy theory; the following ∞-categorical version of the definition is taken from [Nar17, §2.3].

Definition 2.12. Let G be a finite group and let $\mathrm{Span}(\mathrm{Fin}_G)$ be the span category of Proposition 2.11. Let $\mathcal{C}$ be a category which admits finite products. Then the category of $\mathcal{C}$-valued *G-Mackey functors* is given by

$$\mathrm{Mack}_G(\mathcal{C}) := \mathrm{Fun}^{\Sigma}(\mathrm{Span}(\mathrm{Fin}_G), \mathcal{C})$$

where the right-hand side denotes the full subcategory on functors which take direct sums in $\mathrm{Span}(\mathrm{Fin}_G)$ to products in $\mathcal{C}$. We will denote the category of *genuine equivariant G-spectra* by $\mathrm{Sp}^G = \mathrm{Mack}_G(\mathrm{Sp})$.

We identify the theory of orthogonal G-spectra (where weak equivalences are detected levelwise) with G-spectral Mackey functors via the equivalence established in [GM17, §3].

Recollection 2.13 (Smash product of G-Mackey functors)**.** The category $\mathrm{Span}(\mathrm{Fin}_G)$ inherits a symmetric monoidal structure from Fin_G given on underlying objects by cartesian product of finite G-sets [BGS16, Proposition 2.9]. Suppose that $\mathcal{C}$ has a presentably symmetric monoidal structure[2] $\otimes$. Then we can equip $\mathrm{Mack}_G(\mathcal{C}) = \mathrm{Fun}^{\Sigma}(\mathrm{Span}(\mathrm{Fin}_G), \mathcal{C})$ with a symmetric monoidal structure given by Day convolution [Gla16, Proposition 2.11]. When we take $\mathcal{C} = \mathrm{Sp}$ and the symmetric monoidal structure to be the smash product on spectra, this recovers the usual smash product of G-spectra.

The ∞-category of G-Mackey functors in spectra is equivalent to the category of cocartesian sections of a G-parametrized ∞-category.

Example 2.14. The G-∞-category of G-spectra $\underline{\mathrm{Sp}}^G$ is [Nar16, Definition 7.3 & Corollary 7.4.1] applied to $D = \underline{\mathrm{Spc}}^G$.

There is an alternative way of understanding $\mathrm{Mack}_{C_p}(\mathcal{C})$ as a recollement *when $\mathcal{C}$ is stable* and admits BC_p-shaped colimits. The following is [MNN17, Theorem 6.24].

Proposition 2.15. *There is an equivalence of stable ∞-categories*

$$\mathrm{Sp}^{C_p} = \mathrm{Mack}_{C_p}(\mathrm{Sp}) \to \mathrm{Sp}^{BC_p} \times_{\mathrm{Sp}} Ar(\mathrm{Sp})$$
$$X \mapsto \left(X^e, \mathrm{cofib}\left((X^e)_{hC_p} \xrightarrow{\mathrm{tr}} X^{C_p}\right) \to (X^e)^{tC_p}\right)$$

where the map $Ar(\mathrm{Sp}) \to \mathrm{Sp}$ is evaluation at the target. We call $X^{\varphi C_p} := \mathrm{cofib}\left((X^e)_{hC_p} \xrightarrow{\mathrm{tr}} X^{C_p}\right)$ the C_p-geometric fixed points *of X.*

[2] That is, the tensor product commutes with (small) colimits separately in each variable.

Notation 2.16. We will denote the projection $\mathrm{Sp}^{C_p} \to \mathrm{Ar}(\mathrm{Sp})$ by $s_{(-)}$, i.e. for any C_p-spectrum A we have a map $s_A \colon A^{\varphi C_p} \to A^{tC_p}$.

It will be convenient to know that the recollement of Proposition 2.15 is compatible with symmetric monoidal structures.

Proposition 2.17. *Let C_p be a cyclic group of prime power order. Then the recollement of Notation 2.15 is a symmetric monoidal recollement in the sense of [Sha21, Definition 2.20].*

Corollary 2.18. *Let C_p be a cyclic group of prime power order. Then there is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathbb{E}_\infty}\mathrm{Sp}^{C_p} \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_\infty}\mathrm{Sp}^{BC_p} \times_{\mathrm{Alg}_{\mathbb{E}_\infty}\mathrm{Sp}} Ar(\mathrm{Alg}_{\mathbb{E}_\infty}\mathrm{Sp})$$

such that applying forgetful functors recovers the equivalence of Proposition 2.15.

Proof. The corollary follows from [Sha21, Theorem 1.2] and the definition of $\mathbb{E}_\infty$-algebras. □

Observation 2.19. Now suppose $A, B \in \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$. Then the morphism space is computed as

$$\hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(A,B) \simeq \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})}(A^e,B^e) \underset{\hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})}(A^{tC_p},B^{tC_p})}{\times} \hom_{\mathrm{Ar}(\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}))}\begin{pmatrix} A^{\varphi C_p} & & B^{\varphi C_p} \\ \downarrow & , & \downarrow \\ A^{tC_p} & & B^{tC_p}\end{pmatrix}$$

Recollection 2.20 (Tate diagonal)**.** [NS18, Definition III.1.4] The Tate diagonal is a natural transformation $\mathrm{id} \to (-^{\otimes p})^{tC_p}$ of exact functors $\mathrm{Sp} \to \mathrm{Sp}$ where C_p acts on $(A)^{\otimes p}$ via a cyclic permutation.

Recollection 2.21. Given a subgroup inclusion $H \subset G$, the Hill–Hopkins–Ravenel norm [HHR16, Definition A.52] is a (non-exact) functor

$$N_H^G : \mathrm{Sp}^H \to \mathrm{Sp}^G.$$

When $H = \{e\} \subseteq G = C_p$, the norm is uniquely characterized by the existence of natural equivalences $\left(N_e^{C_p}X\right)^{\varphi C_p} \simeq X$ in $\mathrm{Sp}^{\{e\}} \simeq \mathrm{Sp}$ and $\left(N_e^{C_p}X\right)^e \simeq X^{\otimes p}$ in Sp^{BC_p}, where C_p acts on the smash product by permuting the terms. The connecting map $X \to (X^{\otimes p})^{tC_p}$ is given by the Tate diagonal of [NS18, Theorem 1.7]. The functor N_H^G enjoys the properties of being symmetric monoidal and it preserves sifted colimits [HHR16, Proposition A.54], so it lifts to a functor [HHR16, Proposition A.56]

$$N_H^G : \mathrm{Alg}_{\mathbb{E}_\infty}\mathrm{Sp}^H \to \mathrm{Alg}_{\mathbb{E}_\infty}\mathrm{Sp}^G.$$

Lemma 2.22. *The Hill–Hopkins–Ravenel norm $N^{C_p} : \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ preserves all small colimits.*

Proof. By [BH21, Lemma 2.8], it suffices to show that N^{C_p} preserves sifted colimits and finite coproducts. The norm N^{C_p} preserves sifted colimits of algebras because they are computed at the level of underlying spectra, and N^{C_p} preserves finite coproducts of algebras because it is symmetric monoidal with respect to the smash product on Sp and Sp^{C_p} (Recollections 2.2 & 2.21). □

2.3 C_p-$\mathbb{E}_\infty$-rings

In this section we introduce the genuine equivariant algebraic structures of interest via the formalism of parametrized operads of Nardin–Shah [NS22]. We fix notation for the remainder of the paper.

Notation 2.23. Let p be a prime and let $\mathcal{T}$ denote the orbital ∞-category $\mathcal{O}_{C_p}$ of Recollection 2.1.

Definition 2.24. The category $\underline{\mathrm{Fin}}_{\mathcal{T}} = \underline{\mathrm{Fin}}_{C_p}$ of *parametrized $\mathcal{T}$-sets* is the $\mathcal{T}$-∞-category classified by the functor $V \mapsto \mathrm{Fin}_{\mathcal{T}/V}$. Equivalently, it is described by the fiber product $\mathrm{Ar}\left(\mathrm{Fin}_{\mathcal{T}/V}\right) \times_{\mathrm{Fin}_{\mathcal{T}}} \{V\}$. The category $\underline{\mathrm{Fin}}_{\mathcal{T},*}$ of *parametrized pointed $\mathcal{T}$-sets* is the $\mathcal{T}$-∞-category classified by the functor $V \mapsto \left(\mathrm{Fin}_{\mathcal{T}/V}\right)_{\mathrm{id}_V/}$.

$$\underline{\mathrm{Fin}}^{v}_{\mathcal{T}} := \mathrm{Ar}\left(\mathrm{Fin}_{\mathcal{T}/V}\right) \underset{\mathrm{Fin}_{\mathcal{T}}}{\times} \mathcal{T}$$

Example 2.25. We unpack the definition in the case $\mathcal{T} = \mathcal{O}_{C_p}$. For each orbit, the fiber is given by

$$(\underline{\mathrm{Fin}}^{v}_{\mathcal{T}})_{C_p/C_p} \simeq \mathrm{Fin}_{C_p} \qquad (\underline{\mathrm{Fin}}^{v}_{\mathcal{T}})_{C_p} \simeq \mathrm{Fin}^{Free}_{C_p}$$

and the morphism $C_p/C_p \leftarrow C_p$ classifies the functor $\mathrm{Fin}_{C_p} \to \mathrm{Fin}^{Free}_{C_p}$, $V \mapsto V \times_{C_p/C_p} C_p$.

Definition 2.26. [NS22, Definition 2.1.2] Let $\mathcal{T}$ be an atomic orbital ∞-category. The ($\mathcal{T}$-parametrized) ∞-category of *finite pointed $\mathcal{T}$-sets* is

$$\underline{\mathrm{Fin}}_{\mathcal{T},*} = \mathrm{Span}\left(\underline{\mathrm{Fin}}^{v}_{\mathcal{T}}, (\underline{\mathrm{Fin}}^{v}_{\mathcal{T}})^{si}, (\underline{\mathrm{Fin}}^{v}_{\mathcal{T}})^{tdeg}\right).$$

where a morphism $[\phi : f \to g]$ of $\underline{\mathrm{Fin}}^{v}_{\mathcal{T}}$

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ f\downarrow & & \downarrow g \\ V & \xrightarrow{k} & Y \end{array}$$

- belongs to $(\underline{\mathrm{Fin}}^{v}_{\mathcal{T}})^{tdeg}$ if k is degenerate, and
- belongs to $(\underline{\mathrm{Fin}}^{v}_{\mathcal{T}})^{si}$ if $U \to V \times_Y X$ is a summand inclusion.

Definition 2.27. [NS22, Definition 2.1.7] A *$\mathcal{T}$-∞-operad* is a pair $(\mathcal{C}^{\otimes}, p)$ consisting of a $\mathcal{T}$-∞-category $\mathcal{C}^{\otimes}$ and a $\mathcal{T}$-functor $p : \mathcal{C}^{\otimes} \to \underline{\mathrm{Fin}}_{\mathcal{T},*}$ which is a categorical fibration and satisfies the following additional conditions

(1) For every inert morphism $\psi : f_+ \to g_+$ of $\underline{\mathrm{Fin}}_{\mathcal{T},*}$ and every object $x \in \mathcal{C}^{\otimes}_{f_+}$, there is a p-cocartesian edge $x \to y$ covering ψ.

(2) For any object $f_+ = [U_+ \to V]$ of $\underline{\mathrm{Fin}}_{\mathcal{T},*}$, the p-cocartesian edges lying over the characteristic morphisms

$$\left\{\chi_{[W\subseteq U]} : f_+ \to I(W)_+ \mid W \in \mathrm{Orbit}(U)\right\}$$

together induce an equivalence

$$\prod_{W\in\mathrm{Orbit}(U)} \left(\chi_{[W\subseteq U]}\right)_! : \mathcal{C}^{\otimes}_{f_+} \xrightarrow{\sim} \prod_{W\in\mathrm{Orbit}(U)} \mathcal{C}^{\otimes}_{I(W)_+}.$$

(3) For any morphism
$$\psi\colon f_+ = [U_+ \to V] \to g_+ = [U'_+ \to V']$$
of $\underline{\mathrm{Fin}}_{\mathcal{T},*}$, objects $x \in \mathcal{C}^{\otimes}_{f_+}$ and $y \in \mathcal{C}^{\otimes}_{g_+}$, and any choice of p-cocartesian edges
$$\{y \to y_W \mid W \in \mathrm{Orbit}(U')\}$$
lying over the characteristic morphisms
$$\left\{\chi_{[W \subseteq U]} : g_+ \to I(W)_+ \mid W \in \mathrm{Orbit}(U')\right\},$$
the induced map
$$\mathrm{Map}^{\psi}_{\mathcal{C}^{\otimes}}(x, y) \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U')} \mathrm{Map}^{\chi_{[W \subseteq U']} \circ \psi}_{\mathcal{C}^{\otimes}}(x, y_W)$$
is an equivalence.

Given a $\mathcal{T}$-∞-operad $(\mathcal{C}^{\otimes}, p)$, its *underlying* $\mathcal{T}$*-∞-category* is the fiber product
$$\mathcal{C} := \mathcal{T}^{\mathrm{op}} \underset{\underline{\mathrm{Fin}}_{\mathcal{T},*}}{\times} \mathcal{C}^{\otimes}.$$

[NS22, Definition 2.1.8] Given a $\mathcal{T}$-∞-operad $(\mathcal{C}^{\otimes}, p)$, an edge of $\mathcal{C}^{\otimes}$ is *inert* if it is p-cocartesian over an inert edge of $\underline{\mathrm{Fin}}_{\mathcal{T},*}$, and it is *active* if it factors as a p-cocartesian edge followed by an edge lying over a fiberwise active edge in $\underline{\mathrm{Fin}}_{\mathcal{T},*}$.

Example 2.28 (Indexing systems)**.** Let us recall that the C_p-$\mathbb{E}_\infty$-operad is given by $\mathrm{Com}^{\otimes}_{C_p} = \underline{\mathrm{Fin}}_{C_p,*}$ the $\mathcal{O}_{C_p}$-operad corresponding to the maximal indexing system [NS22, Example 2.4.7]. The minimal indexing system $\mathrm{Com}^{\otimes}_{\mathcal{O}^{\simeq}_{C_p}}$ is a C_p-∞ operad with underlying category the wide subcategory of $\underline{\mathrm{Fin}}_{\mathcal{O}_{C_p},*}$ containing those morphisms
$$\begin{array}{ccccc} U & \longleftarrow & Z & \xrightarrow{m} & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Y & = & Y \end{array}$$
where m is a coproduct of (possibly empty) fold maps. The structure map is the natural inclusion $\mathrm{Com}^{\otimes}_{\mathcal{O}^{\simeq}_{C_p}} \subseteq \mathrm{Com}^{\otimes}_{C_p}$.

Definition 2.29. [NS22, Definition 2.2.3] Let $p : \mathcal{C}^{\otimes} \to \underline{\mathrm{Fin}}_{\mathcal{T},*}$ be a fibration of $\mathcal{T}$-∞-operads in which p is moreover a cocartesian fibration. Then we will call $\mathcal{C}^{\otimes}$ a $\mathcal{T}$*-symmetric monoidal* $\mathcal{T}$*-∞-category*.

Recollection 2.30. [NS22, Example 2.4.2; BH21, §9] The C_p-∞-category of C_p-spectra is endowed with a C_p-symmetric monoidal structure via the Hill–Hopkins–Ravenel norm functors as follows: Example 2.4.2 [NS22] and §9 of [BH21] define a functor
$$\begin{aligned} \zeta := \mathbf{SH}^{\otimes} \circ \omega_{C_p} : \mathrm{Span}(\mathrm{Fin}_{C_p}) &\to \mathrm{Alg}_{\mathbb{E}_1}(\mathfrak{C}\mathrm{at}) \\ T &\mapsto \mathbf{SH}^{\otimes} \circ \omega_{C_p}(T) \end{aligned}.$$

Unravelling definitions, this functor takes

$$
\begin{aligned}
\zeta\colon (C_p \twoheadrightarrow C_p/C_p) &\mapsto \mathrm{Sp}^{BC_p} \\
: (C_p = C_p) &\mapsto \mathrm{Sp}^{BC_p} \\
: (C_p/C_p = C_p/C_p) &\mapsto \mathrm{Sp}^{C_p} \\
: \begin{array}{ccc} C_p/C_p & \longleftarrow & C_p \\ \| & & \downarrow \\ C_p/C_p & = & C_p/C_p \end{array} &\mapsto \begin{array}{c} \mathrm{Sp}^{C_p} \\ \downarrow {\scriptstyle (-)^e} \\ \mathrm{Sp}^{BC_p} \end{array} \\
: \begin{array}{ccc} C_p & \longrightarrow & C_p/C_p \\ \downarrow & & \| \\ C_p/C_p & = & C_p/C_p \end{array} &\mapsto \begin{array}{c} \mathrm{Sp} \\ \downarrow {\scriptstyle N^{C_p}} \\ \mathrm{Sp}^{C_p} \end{array} \\
: \begin{array}{ccc} C_p & = & C_p \\ \downarrow & & \| \\ C_p/C_p & \longleftarrow & C_p \end{array} &\mapsto \begin{array}{c} \mathrm{Sp}^{BC_p} \\ \downarrow {\scriptstyle \mathrm{id}} \\ \mathrm{Sp}^{BC_p} \end{array}
\end{aligned}
\quad . \tag{2.31}
$$

Under Theorem 2.3.9 of [NS22], this corresponds to a cocartesian fibration $p : \int \zeta := \left(\underline{\mathrm{Sp}}^{C_p}\right)^{\otimes} \to \underline{\mathrm{Fin}}_{C_p,*}$.

In this paper we use the notion of a C_p-$\mathbb{E}_\infty$-*ring* in the sense of Nardin–Shah [NS22, Definition 2.2.1].

Definition 2.32. Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be fibrations of C_p-∞-operads. A $\mathcal{T}$-functor $p : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is *a morphism of $\mathcal{T}$-∞-operads over $\mathcal{O}$* if p takes inert morphisms in $\mathcal{C}^{\otimes}$ to inert morphisms in $\mathcal{D}^{\otimes}$. Then the category of *$\mathcal{C}^{\otimes}$-algebras in $\mathcal{D}$*

$$\underline{\mathrm{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

is the full $\mathcal{T}$-subcategory of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ on the morphisms of $\mathcal{T}$-∞-operads over $\mathcal{O}$. We write $\mathrm{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$ for the (ordinary) ∞-category of $\mathcal{T}$-objects in $\underline{\mathrm{Alg}}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D})$.

When $\mathcal{O}$ and/or $\mathcal{C}$ are equivalent to $\underline{\mathrm{Fin}}_{\mathcal{T},*}$, we drop them from notation.

We write $\mathrm{Alg}_{\underline{\mathrm{Fin}}_{C_p,*}}\left(\underline{\mathrm{Fin}}_{C_p,*}, \left(\underline{\mathrm{Sp}}^{C_p}\right)^{\otimes}\right) =: C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$.

Example 2.33. The category of C_p-$\mathbb{E}_\infty$-rings in C_p-spectra is $C_p\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})$ the space of sections of $p\colon \left(\underline{\mathrm{Sp}}^{C_p}\right)^{\otimes} \to \underline{\mathrm{Fin}}_{C_p,*}$ (Recollection 2.30) which take inert morphisms to inert morphisms.

The inclusion $\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}} \subseteq \mathrm{Com}^{\otimes}_{\mathcal{T}}$ of Example 2.28 induces a forgetful map

$$G\colon \mathrm{Alg}_{\mathcal{T}}\left(\mathrm{Com}_{C_p}, \mathcal{D}\right) \to \mathrm{Alg}_{\mathcal{T}}\left(\mathrm{Com}_{\mathcal{O}^{\simeq}_{C_p}}, \mathcal{D}\right). \tag{2.34}$$

The discussion immediately following [NS22, Theorem 4.3.4] is summarized by the following result.

Theorem 2.35. *Suppose $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a fibration of $\mathcal{T}$-∞-operads, and let $\mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad. Then the restriction functor*

$$p^* \colon \mathrm{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{E}) \to \mathrm{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C},\mathcal{E})$$

admits a left adjoint $p_!$.

Definition 2.36. Suppose $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a fibration of $\mathcal{T}$-∞-operads, and let $\mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{T}$-∞-operad. Let $A : \mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ be an $\mathcal{O}$-algebra map. Then the $\mathcal{O}$-algebra map $p_! A$ of Theorem 2.35 will be referred to as the *$\mathcal{T}$-operadic left Kan extension* of A.

Remark 2.37. [NS22, Remark 4.0.1] Definition 2.36 specializes to the theory of operadic left Kan extensions of [Lur17, §3.1.2] when $\mathcal{T} = \Delta^0$.

3 Normed rings

In defining the category of C_p-normed rings, §1.1 guides how we axiomatize the information contained in a C_p-$\mathbb{E}_\infty$ ring. We will see that this information is most naturally captured as the limit of a diagram of ∞-categories (Definition 3.11). We then exhibit a formula for mapping spaces in normed rings which will be used in the proof of our main theorem (in particular see Proposition 4.25). Finally, we close out this section by showing in Proposition 3.21 that the category of normed $\mathbb{E}_\infty$-rings is monadic over the category of ordinary $\mathbb{E}_\infty$-algebras.

3.1 Preliminaries

Construction 3.1. Consider the functor

$$\begin{aligned} |-| \colon \mathcal{O}_{C_p} &\to \mathrm{Fin}_* \\ C_p &\mapsto \langle p \rangle \\ C_p/C_p &\mapsto \langle 1 \rangle \end{aligned}$$

which takes the underlying set of a set-with-C_p-action. Since Span(Fin) is 0-semiadditive, the composite $\mathcal{O}_{C_p} \to \mathrm{Fin}_* \subset \mathrm{Span}(\mathrm{Fin})$ induces $\mathrm{Span}\left(\mathcal{O}_{C_p}^{\sqcup}, fold, all\right) \to \mathrm{Span}(\mathrm{Fin})$ which restricts to

$$m := - \times |-| \colon \mathrm{Fin}_* \times \mathcal{O}_{C_p} \to \mathrm{Fin}_*.$$

Denote the adjoint of m by $m^\dagger \colon \mathrm{Fin}_* \to \mathrm{Fun}(\mathcal{O}_{C_p}, \mathrm{Fin}_*)$. Given a symmetric monoidal ∞-category $q \colon \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$, the induced map

$$\mathrm{Fun}_{\mathrm{Fin}_*}\left(\mathcal{O}_{C_p}, \mathcal{C}^{\otimes}\right)^{\otimes} := \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathcal{C}^{\otimes}\right) \underset{\mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathrm{Fin}_*\right), m^\dagger}{\times} \mathrm{Fin}_* \to \mathrm{Fin}_*$$

is a cocartesian fibration of ∞-operads (cf. [Lur17, Remark 2.1.3.4]). Since $\mathcal{C}^{\otimes}$ is symmetric monoidal, given any morphism $h \colon X \to Y$ in $\mathcal{O}_{C_p}$ and any lift $\tilde{X}$ of $|X|$, there is a q-cocartesian morphism $\tilde{h}$ lifting $|h|$, so by [Lur09, Proposition 2.4.4.2] there is a functor

$$\mathrm{Fun}_{\mathrm{Fin}_*}\left(\mathcal{O}_{C_p}, \mathcal{C}^{\otimes}\right) \to \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathcal{C}\right).$$

Restriction along m induces a functor which we also denote by

$$m_{(-)} \colon \mathbb{E}_\infty \mathrm{Alg}\left(\mathcal{C}^{\otimes}\right) \to \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty \mathrm{Alg}(\mathcal{C}^{\otimes})\right). \tag{3.2}$$

Informally, m takes an $\mathbb{E}_\infty$-algebra A to the $\mathcal{O}_{C_p}$-diagram $m_A \colon A^{\otimes p} \to A$.

Notation 3.3. The prime p is left implicit in the notation m_A of Construction 3.1, and when A is understood it may also be dropped from notation.

Remark 3.4. The parametrized norm map $n_A \colon N^{C_p}(A^e) \to A$ is *invariant* with respect to the C_p-action coming from A^e. On the other hand, $(A^e)^{\otimes p}$ has a C_p-action via cyclic permutations and n_A^e may also be regarded as a C_p-equivariant map. The reader is warned to remember the distinction between these two C_p-actions; the following observations clarify how these actions interact differently with the structure maps inherent to a C_p-$\mathbb{E}_\infty$-algebra.

Notation 3.5. Let $A \in \mathrm{Sp}^{BC_p}$ and let $\sigma \in C_p$ be a generator. Write $A^{\otimes^\Delta p}$ for the object in $\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})$ with the *diagonal* action, i.e. the composite

$$\mathbb{E}_\infty \mathrm{Alg}\left(\mathrm{Sp}\right)^{BC_p} \xrightarrow{R \circ m} \mathbb{E}_\infty \mathrm{Alg}\left(\mathrm{Sp}\right)^{BC_p \times BC_p} \xrightarrow{\Delta^*} \mathbb{E}_\infty \mathrm{Alg}\left(\mathrm{Sp}\right)^{BC_p} \tag{3.6}$$

where m is (3.2), R is restriction along the inclusion $BC_p \subseteq \mathcal{O}_{C_p}$, and Δ^* is restriction along the diagonal $\Delta : BC_p \to BC_p \times BC_p$. Informally, we regard $A^{\otimes^\Delta p}$ as being equipped with the C_p-action where σ acts by $\sigma(a_1 \otimes \cdots \otimes a_p) = \sigma(a_p) \otimes \sigma(a_1) \otimes \cdots \otimes \sigma(a_{p-1})$. Write $A^{\otimes^\tau p}$ for the object in $\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})$ with the *transposition* action, i.e. the same definition as in (3.6) but with the map $\{e\} \times \mathrm{id} \colon BC_p \to BC_p \times BC_p$ instead of Δ. Informally, we regard $A^{\otimes^\tau p}$ as being equipped with the C_p-action where σ acts by $\sigma(a_1 \otimes \cdots \otimes a_p) = a_p \otimes a_1 \otimes \cdots \otimes a_{p-1}$.

Observation 3.7. Let $A \in \mathrm{Sp}^{BC_p}$.

(1) The shear endomorphism $sh := \mathrm{id}_A \otimes \sigma \otimes \cdots \otimes \sigma^{p-1}$ of $A^{\otimes p} \in \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ promotes to an equivalence $A \otimes^\Delta A \to A \otimes^\tau A$ in $\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})^{BC_p}$–in particular it is C_p-equivariant.

(2) Moreover, the Tate diagonal $A \to (A^{\otimes^\tau p})^{tC_p}$ is equivariant with respect to the given C_p-action on the source and the diagonal C_p-action on the target.

Definition 3.8. Let $A \in \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{BC_p}\right)$. The *Tate-valued norm* is the $\mathbb{E}_\infty$-ring map defined by the composite

$$\nu_A : A \xrightarrow{\Delta} \left(A^{\otimes^\tau p}\right)^{tC_p} \xrightarrow{sh} \left(A^{\otimes^\Delta p}\right)^{tC_p} \xrightarrow{m^{tC_p}} A^{tC_p}$$

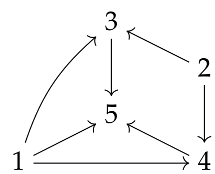
where Δ is the Tate diagonal of Recollection 2.20 and sh is the shear equivalence of Observation 3.7. In particular, it is C_p-equivariant with respect to the given action on A, the diagonal C_p-action on $(A^{\otimes^\tau_S p})$, and the *trivial* action on A^{tC_p}. We regard ν_A as a morphism $A_{hC_p} \to A^{tC_p}$, or equivalently as an object of $\mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})\right)$.

Remark 3.9. Informally, we think of the Tate-valued norm as being $a \mapsto a\gamma(a) \cdots \gamma^{p-1}(a)$, which is a ring homomorphism modulo transfers. Note that when A is equipped with the trivial C_p-action, this is simply the ordinary Tate-valued Frobenius (compare Definition IV.1.1 of [NS18]).

3.2 Definition and properties

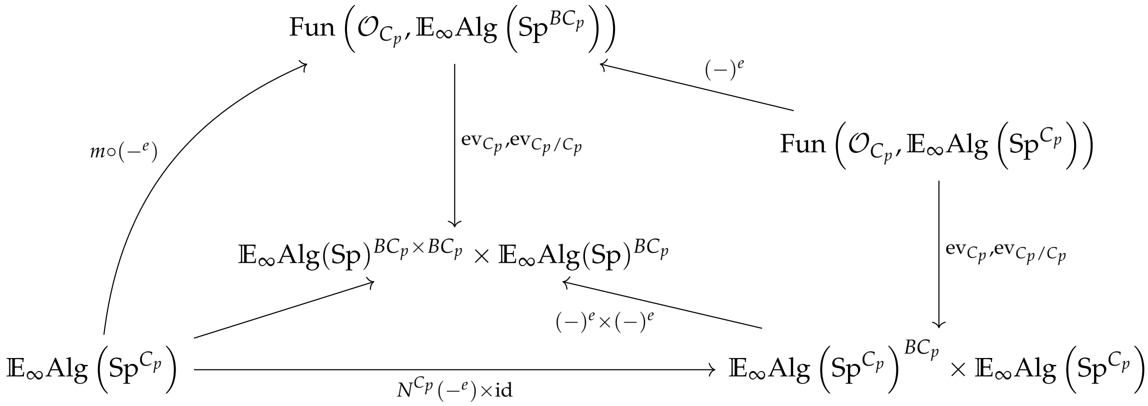
We introduce some notation for the indexing category.

Notation 3.10. Let K denote the ∞-categorical nerve of the 1-category



in which all triangles and squares commute.

Definition 3.11. Consider the diagram $\mathcal{N}\colon K \to \mathcal{C}\mathrm{at}_\infty$ where K is as in Notation 3.10:



(3.12)

where m is the functor of (3.2). Observe that the right-hand trapezoid of (3.12) commutes essentially by definition, and the leftmost triangle commutes because $\left(N^{C_p}A\right)^e \simeq (A^e)^{\otimes p}$. We define the category of *normed* C_p-$\mathbb{E}_\infty$-rings to be the limit of the diagram

$$N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) := \lim_K \mathcal{N}. \tag{3.13}$$

There is a canonical forgetful functor $G'\colon N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ given by the canonical projection to the lower left corner of the diagram (3.12).

Notation 3.14. Write $p_i\colon N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathcal{N}(i)$ for the canonical projection functors.

We will often abuse notation and abbreviate an object of $N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ as a pair $(A, n_A : N^{C_2}A \to A)$ (suppressing the data of the equivalence $n_A^e \simeq m_{A^e}$).

Remark 3.15. Note that all categories in (3.12) are presentable and all functors are left adjoints (Lemma 2.22), so by [Lur09, Proposition 5.5.3.13] we may take the limit in either Pr^L or $\mathcal{C}\mathrm{at}_\infty$.

Proposition 3.16. *The category $N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ can be equivalently described as the limit of the diagram*

$$\begin{array}{ccccc}
 & & \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})\right) & \xleftarrow{\mathrm{ev}_1} & \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathrm{Ar}(\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}))\right) \\
 & {\scriptstyle m^{tC_p}:(-^e)^{\otimes p}\to(-^e)} \nearrow & \downarrow{\scriptstyle \mathrm{ev}_{C_p},\mathrm{ev}_*} & & \downarrow{\scriptstyle \mathrm{ev}_{C_p},\mathrm{ev}_*} \\
 & & \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})^{BC_p}\times\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}) & \xleftarrow{\mathrm{ev}_1} & \\
\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) & \nearrow & \xrightarrow[((-)^{\varphi C_p}\to(-)^{tC_p})\circ(N^{C_p}\times id)]{} & & \mathrm{Ar}\left(\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{BC_p}\right)\right)\times\mathrm{Ar}\left(\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}\right)\right)
\end{array}$$

Proof. Follows from Corollary 2.18. □

Remark 3.17. Recall the description of a limit of ∞-categories given by Corollary 3.3.3.2 of [Lur09]. Combining this with the description of mapping spaces in $\mathrm{Alg}_{\mathbb{E}_\infty}\left(\mathrm{Sp}^{C_p}\right)$ which is a consequence of Corollary 2.18, we may equivalently characterize a normed $\mathbb{E}_\infty$-ring as the data of a $\mathbb{E}_\infty$-algebra A in Sp^{C_p} plus the data of a factorization $n_A^{\varphi C_p}$ in $\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ and a 2-cell making the diagram

$$\begin{array}{ccc}
A & \xrightarrow{n_A} & A^{\varphi C_2} \\
{\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\
\left(A^{\otimes^\tau p}\right)^{tC_p} & \xrightarrow{m^{tC_p}\circ(sh)} & A^{tC_p}
\end{array} \tag{3.18}$$

commute such that, considered as a morphism $g : \Delta \to \alpha$, g is equivariant with respect to the given C_p-action on the source and the trivial C_p-action on the target. Note that the composite of the left arrow followed by the lower arrow in (3.18) is the Tate-valued norm of Definition 3.8.

When C_p acts trivially on A^e, it suffices to produce the 2-cell (3.18). More formally, given a choice of multiplication map $m : A^{\otimes^\Delta p} \to A$, by Corollary 2.18 we have an equivalence of fibers

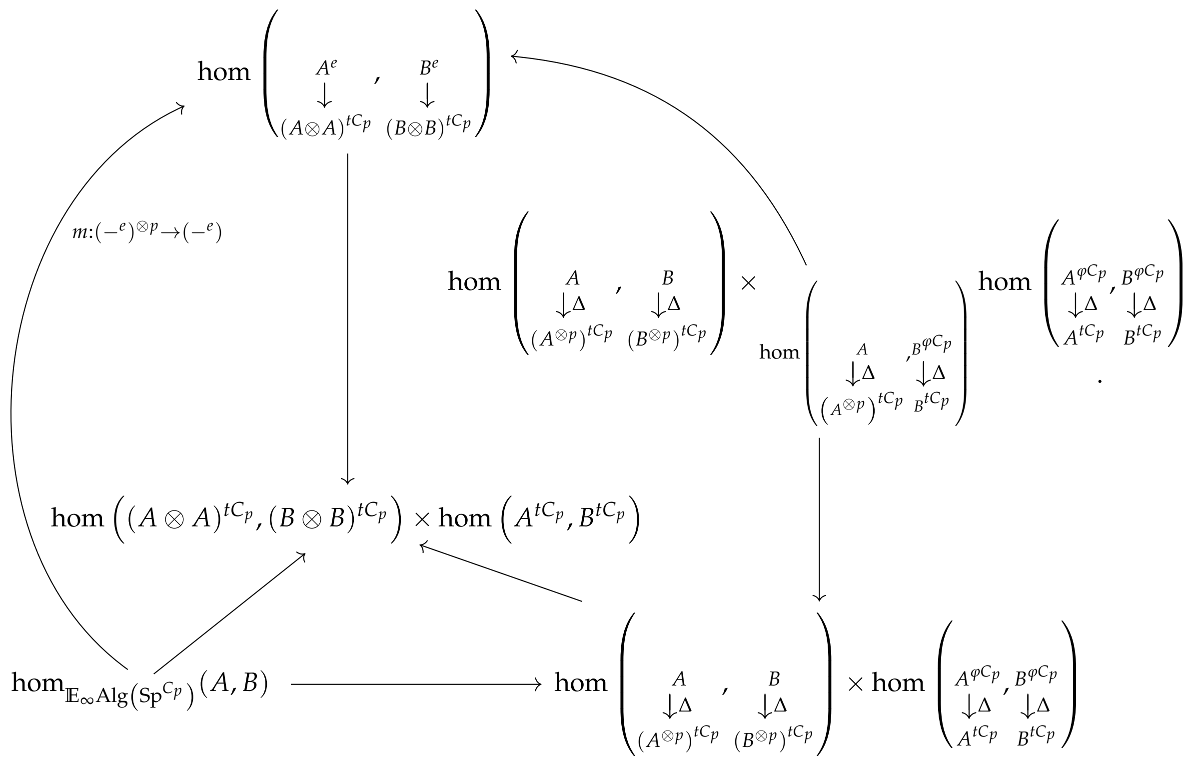
$$\mathrm{fib}_{\{m\}}\left(\hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})^{BC_p}}\left(N^{C_p}(A^e), A\right) \xrightarrow{(-)^e} \hom_{\mathbb{E}_\infty\mathrm{AlgSp}^{BC_p\times BC_p}}\left((A^e)^{\otimes^\Delta p}, A^e\right)\right)$$

$$\simeq \mathrm{fib}_{\{m^{tC_p}\}}\left\{\hom_{\mathbb{E}_\infty\mathrm{Alg}^{\Delta^1\times BC_p}}\left(\begin{array}{cc} A^e & A^{\varphi C_p} \\ \downarrow{\scriptstyle\Delta} & \downarrow{\scriptstyle\alpha} \\ \left(A^{\otimes^\tau_{\mathrm{S}} p}\right)^{tC_p}, & A^{tC_p}\end{array}\right) \xrightarrow{\mathrm{ev}_1} \hom_{\mathbb{E}_\infty\mathrm{Alg}^{BC_p}}\left(\left(A^{\otimes^\tau_{\mathrm{S}} p}\right)^{tC_p}, A^{tC_p}\right)\right\}.$$

Observation 3.19 (Morphism spaces in normed algebras)**.** Let $s, t\colon K \to \int\mathcal{N}$ be objects in the limit $N\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})$, which we identify as spaces of coCartesian sections of $\int\mathcal{N} \to K$ where

$\mathcal{N}\colon K \to \mathcal{C}\mathrm{at}_\infty$ is the diagram defining (3.12). Now by definition of a limit of ∞-categories, we may write the space of morphisms from s to t in $N\mathbb{E}_\infty$ as $\lim_k \hom_{F(k)}(s(k), t(k))$.

Unravelling definitions, given a pair $(A, n_A\colon N^{C_p}A \to A), (B, n_B\colon N^{C_p}B \to B)$ in the limit of (3.12), the morphism space $\mathrm{Hom}_{N\mathbb{E}_\infty}((A, n_A\colon N^{C_p}A \to A), (B, n_B\colon N^{C_p}B \to B))$ is computed as the limit of the diagram



(3.20)

Proposition 3.21. *The forgetful functor $G'\colon N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ of Definition 3.11 is monadic.*

Proof. The functor G' is conservative by inspection.

Recall our notation $p_i\colon N\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p}) \to \mathcal{N}(i)$ for the canonical projection functors. Now suppose given a simplicial object $A\colon \Delta^{\mathrm{op}} \to N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ which is G'-split. Then in particular $p_0 \circ A$ is a colimit diagram of $\mathbb{E}_\infty$-algebras in Sp^{C_p}. Since the norm preserves all colimits of algebras by Lemma 2.22, $p_4 \circ A \simeq (N^{C_p} \times \mathrm{id}) \circ p_0 \circ A$ is a colimit diagram. By [Lur17, Corollary 5.1.2.3(2)] applied to $S = \mathcal{O}_{C_p}$, $p_2 \circ A$ is a colimit diagram. Now by Remark 3.15 and Proposition 5.1.2.2(2) of *loc. cit.* applied to $S = K$, A is a colimit diagram in $N\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})$, and said colimit is preserved by G'. Thus G' is monadic by the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5]. $\square$

4 Comparing C_p-$\mathbb{E}_\infty$ and normed rings

In §4.1 we write down a functor from C_p-$\mathbb{E}_\infty$-algebras to normed C_p-algebras. Our proof strategy will be to show that the comparison functor of Corollary 4.7 exhibits both C_p-$\mathbb{E}_\infty$-algebras and normed $\mathbb{E}_\infty$-algebras as categories of algebras over the *same* monad on $\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$, then appeal to a variant of the Barr–Beck–Lurie theorem. In §4.2 we exhibit a formula for the free C_p-$\mathbb{E}_\infty$-algebra on an $\mathbb{E}_\infty$-algebra, then we show in §4.3 that it induces an equivalence.

4.1 A comparison functor

Since a normed $\mathbb{E}_\infty$-ring is a priori less data than a C_p-$\mathbb{E}_\infty$-algebra, it is most natural to define a 'forgetful' functor from the latter to the former. In order to write down the functor, we need to unpack the definition of a C_2-$\mathbb{E}_\infty$-algebra.

Notation 4.1. Observe that $\mathrm{Span}\left(\mathrm{Fin}_{C_p}^{/V}\right)$ is 0-semiadditive [BH21, Lemma C.3] and define $\underline{\mathrm{Span}}(\mathrm{Fin}_{C_p,*})$ to be the colimit of the functor $V \mapsto \mathrm{Span}\left(\mathrm{Fin}_{C_p}^{/V}\right)$ [Lur09, Corollary 3.3.4.3]. Since 0-semiadditive ∞-categories are closed under all colimits [Har20, Corollary 5.4], $\underline{\mathrm{Span}}(\mathrm{Fin}_{C_p})$ is 0-semiadditive. Moreover, notice that there is an inclusion $\underline{\mathrm{Fin}}_{C_p,*} \subset \underline{\mathrm{Span}}(\mathrm{Fin}_{C_p})$.

Let $\delta : J \to \underline{\mathrm{Fin}}_{C_p,*}$ be a diagram. Under the equivalence of [Har20, Theorem 4.1; BH21, Lemma C.4] the diagram δ classifies a functor $\mathrm{Span}(J^{\sqcup}, \mathrm{fold}, \mathrm{all}) \to \underline{\mathrm{Span}}(\mathrm{Fin}_{C_p})$ which evidently restricts to

$$\iota_J \colon \mathrm{Fin}_* \times J \to \underline{\mathrm{Fin}}_{C_p,*}. \tag{4.2}$$

When $J = \Delta^0$ and δ is the inclusion of a single object $T \in \underline{\mathrm{Fin}}_{C_p,*}$, we write ι_T.

Consider the diagrams $\alpha_2, \alpha_3 \colon \mathcal{O}_{C_p} \to \underline{\mathrm{Fin}}_{C_p,*}$

$$\begin{array}{ccccc} C_p & = & C_p & \longrightarrow & C_p/C_p \\ \downarrow & & \downarrow & & \Vert \\ C_p/C_p & = & C_p/C_p & = & C_p/C_p \end{array} \qquad \begin{array}{ccccc} C_p^{\sqcup p} & = & C_p^{\sqcup p} & \xrightarrow{\nabla} & C_p \\ \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} & & \Vert \\ C_p & = & C_p & = & C_p \end{array}$$

resp., where C_p acts on $C_p^{\sqcup p}$ by permuting the terms of the disjoint union. The preceding discussion shows that there are functors

$$\iota_{\alpha_i} := - \times \alpha_i \colon \mathrm{Fin}_* \times \mathcal{O}_{C_p} \to \underline{\mathrm{Fin}}_{C_p,*}. \tag{4.3}$$

By a similar discussion to that of Construction 3.1, the $\iota_{(-)}$ induce functors

$$\iota_J \colon C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathrm{Fun}\left(J, \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)\right). \tag{4.4}$$

Construction 4.5. Recall that the category of C_p-$\mathbb{E}_\infty$-algebras $C_p\mathbb{E}_\infty\mathrm{Alg}$ is given by sections of the fibration $\left(\underline{\mathrm{Sp}}^{C_p}\right)^{\otimes}$ (Definition 2.32). By (4.4), restricting to certain subcategories of $\underline{\mathrm{Fin}}_{C_p,*}$ gives functors:

(a) $\gamma_1 \colon C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ given by restriction along ι_T of (4.2) for $T = [C_p/C_p = C_p/C_p]$.

(b) $\gamma_4 \colon C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \times \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ given by restriction along $\iota_T \times \iota_S$ of (4.2) for $T = [C_p \twoheadrightarrow C_p/C_p]$ and $S = [C_p/C_p = C_p/C_p]$.

(c) $\gamma_5 \colon C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})^{BC_p \times BC_p} \times \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})^{BC_p}$ given by restriction along $\iota_T \times \iota_S$ of (4.2) for $T = \left[C_p^{\sqcup p} \xrightarrow{\nabla} C_p\right]$ and $S = [C_p = C_p]$, resp. Note that in the former case, C_p acts by permuting the factors of $C_p^{\sqcup p} \simeq C_p \times C_p$ cyclically.

(d) $\gamma_2 \colon C_p\mathbb{E}_\infty\mathrm{Alg} \to \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})\right)$ given by restriction along ι_{α_2} of (4.3).

(e) $\gamma_3 \colon C_p\mathbb{E}_\infty\mathrm{Alg} \to \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})\right)$ given by restriction along ι_{α_3} of (4.3).

Proposition 4.6. *The functors of Construction 4.5 are related in the following way:*

(a) There is an equivalence $m \circ \gamma_1^e \simeq \gamma_3$.

(b) There is an equivalence $\mathrm{ev}_{C_p} \times \mathrm{ev}_{C_p/C_p} \circ \gamma_3 \simeq \gamma_5$.

(c) There is an equivalence $(((-)^e)^{\otimes 2} \times (-)^e) \circ \gamma_1 \simeq \gamma_5$.

(d) There is an equivalence $\left(\mathrm{ev}_{C_p} \times \mathrm{ev}_{C_p/C_p}\right) \circ \gamma_2 \simeq \gamma_4$ *of functors* $C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)^{\times 2}$.

(e) There is an equivalence $\left(N^{C_p} \times id\right) \circ \gamma_1 \simeq \gamma_4$ *of functors* $C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)^{\times 2}$.

(f) There is a commutative diagram

$$\begin{array}{ccc} C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) & \xrightarrow{\gamma_2} & \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)\right) \\ & {\scriptstyle\gamma_3}\searrow \quad \swarrow{\scriptstyle(-)^e} & \\ & \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{BC_p})\right) & \end{array}$$

Corollary 4.7. *There is a canonical functor*

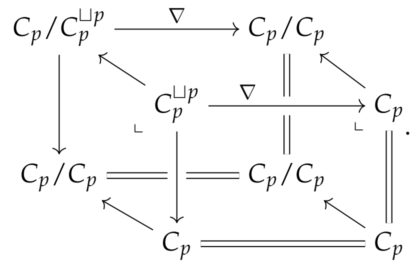
$$\gamma \colon C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right).$$

Proof. The functors of Construction 4.5 may be regarded as

$$\gamma_i : C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathcal{N}(i)$$

where $\mathcal{N} : K \to \mathfrak{Cat}_\infty$ is as in Definition 3.11 and Notation 3.10. Proposition 4.6 shows that the functors γ_i commute with the structure maps in the diagram $\mathcal{N}$. By definition of a homotopy limit, the γ_i assemble to the desired functor γ. □

Proof of Proposition 4.6. (a) Consider the diagram $T := \mathcal{O}_{C_p} \times \Delta^1 \to \underline{\mathrm{Fin}}_{C_p,*}$



(4.8)

Note that $(m \circ \gamma_1)^e \simeq m \circ (\gamma_1^e)$. Now notice that $m \circ \gamma_1$ is given by restriction along $\iota_{\nabla_{C_p/C_p}}$ (i.e. the back face), while restriction along the front face implements γ_3. We may regard ι_T (Notation 4.1) as a natural transformation $\beta : (m \circ \gamma_1)^e \implies \gamma_3$ by (2.31). Since the morphisms from the back face to the front face of (4.8) are inert, β is a natural equivalence.

(b) This is evident.

(c) Follows from (a) and (b).

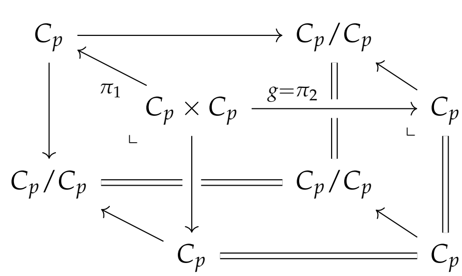
(d) This is evident from the definitions of α_2 and γ_4.

(e) Consider the morphism $w \colon \Delta^1 \to \underline{\mathrm{Fin}}_{C_p,*}$

$$\begin{array}{ccccc} C_p/C_p & \longleftarrow & C_p & = & C_p \\ \| & & \downarrow & & \downarrow \\ C_p/C_p & = & C_p/C_p & = & C_p/C_p \end{array}.$$

Notice that w is inert and recall that morphisms of operads take inert morphisms to inert morphisms. Because a morphism in $\left(\underline{\mathrm{Sp}}^{C_p}\right)^{\otimes}$ factors canonically as a p-cocartesian morphism and a fiberwise morphism, by definition of ζ (2.31) we see that restriction along ι_w gives an equivalence $N^{C_p}(\gamma_1^e) \simeq \pi_1\gamma_4$.

(f) Now consider the diagram $T := \mathcal{O}_{C_p} \times \Delta^1 \to \underline{\mathrm{Fin}}_{C_p,*}$



(4.9)

considered as an inert morphism (in fact, ev_1-cocartesian) from the back face α_2 (Notation 4.1) to the front face.

Identifying the underlying set of C_p with $\{0, 1, \ldots, p-1\}$, notice that the shear equivalence

$$\begin{aligned} sh\colon \{0,1,\ldots,p-1\} \times C_p &\to C_p \times C_p \\ (a,b) &\mapsto (a+b, b) \end{aligned}$$

which is equivariant with respect to the diagonal C_p-action on the target and the action by C_p on the second factor on the source. The shear map identifies g with the fold map $\nabla : C_p^{\sqcup p} \to C_p$, i.e. there is a commutative diagram

$$\begin{array}{ccc} C_p \times C_p & & \\ {\scriptstyle sh}\uparrow\wr & \searrow^{\pi_2 = g} & \\ & & C_p. \\ & \nearrow_{\pi_2 = \nabla} & \\ \{0,1,\ldots,p-1\} \times C_p & & \end{array}$$

Thus we see that the shear map induces an equivalence $\iota_g \simeq \iota_{\alpha_3} \simeq \gamma_3$.

Now restriction along ι_T (Notation 4.1) gives a natural transformation β

$$\begin{array}{ccc} & \nearrow^{\iota_{\alpha_2} \simeq \gamma_2} & \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)\right) \\ C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) & \Leftarrow_{\beta} & \downarrow (-)^e \\ & \searrow_{\iota_g} & \mathrm{Fun}\left(\mathcal{O}_{C_p}, \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{BC_p}\right)\right) \end{array}.$$

Since the back-to-front arrows in (4.9) are inert, β is a natural equivalence. □

4.2 A parametrized monoidal envelope

To apply the Barr–Beck–Lurie theorem [Lur17, Proposition 4.7.3.22], we will need to show that γ of the free C_p-$\mathbb{E}_\infty$-algebra on an $\mathbb{E}_\infty$-algebra A computes the free normed algebra on A. A general strategy for understanding free C_p-$\mathbb{E}_\infty$ algebras is outlined in Remark 4.3.6 of [NS22]; we introduce the ingredients first, then outline the strategy in Recollection 4.13. Then we apply the aforementioned general strategy to exhibit a formula for the free C_p-$\mathbb{E}_\infty$-algebra on an $\mathbb{E}_\infty$-algebra A in Theorem 4.14.

Definition 4.10. [NS22, Definition 2.8.4 & Notation 2.8.3] Let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-operad. let

$$\mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^\otimes) := \mathcal{T}^{\mathrm{op}} \times_{\mathrm{Ar}(\mathcal{T}^{\mathrm{op}})} \mathrm{Ar}^{act}(\mathcal{O}^\otimes)$$

where $\mathrm{Ar}^{act}(\mathcal{O}^\otimes)$ is the $\mathcal{T}$-full subcategory on the active morphisms.

Suppose given a fibration of $\mathcal{T}$-operads $p\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$. The *$\mathcal{O}$-monoidal envelope of $\mathcal{C}^\otimes$* is

$$\pi : \mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^\otimes := \mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathrm{Ar}^{act}_{\mathcal{T}}(\mathcal{O}^\otimes) \to \mathcal{O}^\otimes.$$

When $\mathcal{O} = \underline{\mathrm{Fin}}_{\mathcal{T},*}$ we drop it from notation.

Remark 4.11. More informally, an object of $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}$ is a pair $(c, g\colon p(c) \to o)$ where $c \in \mathcal{C}^{\otimes}$, $o \in \mathcal{O}^{\otimes}$, g is a fiberwise active arrow in $\mathcal{O}^{\otimes}$. The forgetful map π takes this tuple to o. By [NS22, Remark 2.8.5], the underlying $\mathcal{T}$-∞-category of $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}$ is $\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C}) \simeq \mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq},act}$.

Proposition 4.12. *[NS22, Proposition 2.8.7] Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of $\mathcal{T}$-∞-operads, and let $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of $\mathcal{T}$-∞-operads. Let $i : \mathcal{C}^{\otimes} \subseteq \mathrm{Env}_{\mathrm{Com}_{\mathcal{T}}}(\mathcal{C})^{\otimes}$ denote the inclusion of $\mathcal{C}^{\otimes}$ into its monoidal envelope. Then there is an adjunction*

$$i_! : \mathrm{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Alg}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}, \mathcal{D}) : i^*$$

and $i_!$ has essential image the full subcategory of the right-hand side given by $\mathrm{Fun}^{\otimes}_{\mathcal{O},\mathcal{T}}(\mathrm{Env}_{\mathcal{O},\mathcal{T}}(\mathcal{C})^{\otimes}, \mathcal{D})$.

Recollection 4.13. Consider $\mathcal{O} = \underline{\mathrm{Fin}}_{\mathcal{T},*}$, $\mathcal{E} = \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}$ (Notation 3.10), and $\mathcal{C} = \mathrm{Com}_{\mathcal{T}^{\simeq}}$ (Example 2.28) in Theorem 2.35. Then there is an adjunction

$$F : \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_2}) \leftrightarrows C_2\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_2}) : G.$$

where G is from (2.34). By Remarks 2.8.5 & 4.3.6 of [NS22], the free C_2-$\mathbb{E}_\infty$-algebra $F(A)$ on an $\mathbb{E}_\infty$-algebra in Sp^{C_2} is computed by the C_2-left Kan extension of $i_!A^{\otimes}\colon \mathrm{Env}_{\mathcal{T}}(\mathrm{Com}_{\mathcal{T}^{\simeq}})^{\otimes} \to \underline{\mathrm{Sp}}^{C_2}$ along the structure map $\pi : \mathrm{Env}_{\mathcal{T}}(\mathrm{Com}_{\mathcal{T}^{\simeq}})^{\otimes} \to \mathrm{Com}^{\otimes}_{\mathcal{T}}$, where $i_!$ is from Proposition 4.12.

Theorem 4.14. *Let $A \in \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ (also see Lemma 4.30) and consider the adjunction $F \dashv G$ of Recollection 4.13.*

(1) The underlying C_p-spectrum of the free C_p-$\mathbb{E}_\infty$ algebra $F(A)$ on A is given by (via the recollement of Proposition 2.15)

$$F(A) \simeq \begin{array}{ccc} & & A^{\varphi C_p} \otimes A^e_{hC_p} \\ & & \big\downarrow{\scriptstyle s_A \otimes \nu_A} \\ A^e & \longmapsto & A^{tC_p} \end{array} \tag{4.15}$$

where u is the unit, $s_A : A^{\varphi C_p} \to A^{tC_p}$ is the structure map, and ν_A is the twisted Tate-valued Frobenius (Definition 3.8).

(2) There is a canonical $\mathbb{E}_\infty$ ring map $\eta_A\colon A \to GF(A)$ given by $id_{A^{\varphi C_p}} \otimes (\eta_{A^e_{hC_p}} : \mathbb{S}^0 \to A^e_{hC_p})$ on geometric fixed points and the identity on underlying.

Proof. By Recollection 4.13, the C_p-$\mathbb{E}_\infty$-algebra $F(A)$ may be computed as the C_p-left Kan extension of $i_!A^{\otimes}\colon \mathrm{Env}_{\mathcal{T}}(\mathrm{Com}_{\mathcal{T}^{\simeq}})^{\otimes} \to \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}$ along the structure map $\pi : \mathrm{Env}_{\mathcal{T}}(\mathrm{Com}_{\mathcal{T}^{\simeq}})^{\otimes} \to \mathrm{Com}^{\otimes}_{\mathcal{T}}$.

Denote $x = [C_p/C_p = C_p/C_p] \in \mathrm{Com}^{\otimes}_{\mathcal{T}}$. In particular, the C_p-spectrum underlying $F(A)$ may be computed as the C_p-left Kan extension of $i_!A^{\otimes}_{\underline{x}}\colon \left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq},act}\right)_{\underline{x}} \to \underline{\mathrm{Sp}}^{C_p}$ along the structure map $\pi_{\underline{x}} : \left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq},act}\right)_{\underline{x}} \simeq \mathrm{Env}_{\mathcal{T}}(\mathrm{Com}_{\mathcal{T}^{\simeq}})^{\otimes}_{\underline{x}} \to \left(\mathrm{Com}^{\otimes}_{\mathcal{T}}\right)_{\underline{x}} \simeq \mathcal{T}^{\mathrm{op}}$.

Let $I = \left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq},act}\right)_{\underline{x}} \xrightarrow{q} \mathcal{T}^{\mathrm{op}}$ be shorthand for our indexing diagram and write I_{C_p} and I_{C_p/C_p} for the respective fibers (not parametrized fibers). We will write $F|_{C_p}$ for the restriction of a diagram F defined on I to I_{C_p}.

By definition of a C_2-left Kan extension and Definition 5.2 of [Sha18], we seek a $\mathcal{T}^{\mathrm{op}}$-initial lift

$$\begin{array}{ccc} & & \underline{\mathrm{Fun}}_{\mathcal{T}^{\mathrm{op}}}\left(I \star_{\mathcal{T}^{\mathrm{op}}} \mathcal{T}^{\mathrm{op}}, \underline{\mathrm{Sp}}^{C_p}\right) \\ & \overset{\widetilde{F(A)}}{\nearrow} & \downarrow \\ \mathcal{T}^{\mathrm{op}} & \xrightarrow[i_! A^{\otimes}_{\underline{x}}]{} & \underline{\mathrm{Fun}}_{\mathcal{T}^{\mathrm{op}}}\left(I, \underline{\mathrm{Sp}}^{C_p}\right) \end{array} \quad . \tag{4.16}$$

Informally, a lift $\widetilde{F(A)}$ of (4.16) is the data of

- a cocartesian section $F(A)\colon \mathcal{T}^{\mathrm{op}} \to \underline{\mathrm{Sp}}^{C_p}$
- a morphism β from $i_! A^{\otimes}_{\underline{x}}|_{C_p}$ to the constant I_{C_p}-indexed diagram at $F(A)(C_p)$ in Sp^{BC_p}
- a morphism α from $i_! A^{\otimes}_{\underline{x}}|_{C_p/C_p}$ to the constant I_{C_p/C_p}-indexed diagram at $F(A)(C_p/C_p)$ in Sp^{C_p}
- Choose a functor $R\colon I_{C_p/C_p} \to I_{C_p}$ classified by the map $C_p \twoheadrightarrow C_p/C_p$. Then we require the data of an equivalence $(\alpha)^e \simeq \beta \circ R$ of natural transformations.

Now notice that the diagram $i_! A^{\otimes}_{\underline{x}}|_{C_p}$ is defined on $\left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq},act}\right)_{\underline{x},C_p}$, which has a final object $[C_p = C_p]$. Thus for $\widetilde{F(A)}(C_p)$ to be an initial object of the C_p-fiber of (4.16), we must have $F(A)^e \simeq A^e$. An initial object of the C_p/C_p-fiber of (4.16) is equivalently an object $F(A) : \mathcal{T}^{\mathrm{op}} \to \underline{\mathrm{Sp}}^{C_p}$ representing the functor

$$\begin{aligned} \underline{\mathrm{Sp}}^{C_p} &\to \mathrm{Spc} \\ B &\mapsto \hom_{\underline{\mathrm{Fun}}_{\mathcal{T}^{\mathrm{op}}}(I, \underline{\mathrm{Sp}}^{C_p})}\left(i_! A^{\otimes}_{\underline{x}}, q^* B\right). \end{aligned}$$

By a similar argument to our earlier discussion of morphisms in categories of cocartesian sections (Observation 3.19) and Proposition 2.15, the space of morphisms from the diagram $i_! A^{\otimes}_{\underline{x}}$ to $q^* B$ sits in a fiber sequence

$$
\begin{array}{c}
\mathrm{fib}\left(\hom_{\mathrm{Fun}\left(I|_{C_p/C_p}, Ar(\mathrm{Sp})\right)}\left(\begin{array}{cc} (i_! A^{\otimes}_{\underline{x}}|_{C_p/C_p})^{\varphi C_p} & (q^* B|_{C_p/C_p})^{\varphi C_p} \\ \downarrow & \downarrow \\ (i_! A^{\otimes}_{\underline{x}}|_{C_p/C_p})^{tC_p} & q^* B|^{tC_p}_{C_p/C_p} \end{array}\right) \to \hom_{\mathrm{Fun}\left(I|_{C_p}, \mathrm{Sp}\right)}\left((i_! A^{\otimes}_{\underline{x}}|_{C_p})^{tC_p}, q^* B|^{tC_p}_{C_p}\right)\right) \\
\downarrow \\
\hom_{\mathrm{Fun}_{\mathcal{T}^{op}}\left(I, \underline{\mathrm{Sp}}^{C_p}\right)}\left(i_! A^{\otimes}_{\underline{x}}, q^* B\right) \simeq \hom_{\mathrm{Fun}\left(I|_{C_p/C_p}, \underline{\mathrm{Sp}}^{C_p}\right)}\left(i_! A^{\otimes}_{\underline{x}}|_{C_p/C_p}, q^* B|_{C_p/C_p}\right) \\
\big\downarrow {\scriptstyle (-)^e} \\
\hom_{\mathrm{Fun}_{BC_p}\left(I|_{C_p}, \underline{\mathrm{Sp}}^{BC_p}\right)}\left(i_! A^{\otimes}_{\underline{x}}|_{C_p}, q^* B|_{C_p}\right)
\end{array}
.
$$

By the previous discussion, we have $\hom_{\mathrm{Fun}_{BC_p}\left(I|_{C_p}, \underline{\mathrm{Sp}}^{BC_p}\right)}(i_! A^{\otimes}_{\underline{x}}|_{C_p}, q^* B^e) \simeq \hom_{\mathrm{Sp}^{BC_p}}(A^e, B^e)$. Thus we see that for $\widetilde{F(A)}(C_p/C_p)$ to be an initial object of the C_p/C_p-fiber of (4.16), it suffices to take $F(A)^{\varphi C_p}$ to be the colimit of the diagram

$$(i_! A^{\otimes}_{\underline{x}})^{\varphi C_p} : \left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}, act}\right)_{\underline{x}, C_p/C_p} \to \mathrm{Sp}.$$

By Lemma 4.18, the C_p-left Kan extension of $i_! A^{\otimes}_{\underline{x}}$ along $\pi_{\underline{x}} : \mathrm{Env}(\mathrm{Com}_{\mathcal{T}^{\simeq}})_{\underline{x}} \to (\mathrm{Com}_{\mathcal{T}})_{\underline{x}}$ is computed on C_p geometric fixed points by (4.15).

The existence of the unit η_A follows from monadicity (Proposition 4.29), and its exact form follows from tracing through the definition of C_p-left Kan extension. □

Warning 4.17. The G-∞-category of G-spectra $\underline{\mathrm{Sp}}^G$ is *not* an example of the G-category of objects of Construction 2.6. Thus many of the techniques to compute G-left Kan extensions of [Sha18] do not apply to our proof of Theorem 4.14.

Lemma 4.18. *Consider the fiber* $\left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}}\right)_{act, C_p/C_p}$ *(not parametrized fiber) over* C_p/C_p *of* $\left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}}\right)_{act}$*. The inclusion*

$$\iota \colon BC_p \sqcup * \hookrightarrow \left(\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}}\right)_{act, C_p/C_p}$$

of the full subcategory spanned by the object

$$
\begin{array}{ccc}
C_p \sqcup C_p/C_p & \longrightarrow & C_p/C_p \\
\downarrow & & \| \\
C_p/C_p & = & C_p/C_p
\end{array}
$$

is cofinal.

Proof. Since we consider the fiber over C_p/C_p, the target is always C_p/C_p and we omit it throughout the following proof. Observe that there is a pullback diagram of simplicial sets

$$\begin{array}{ccc} BC_p \sqcup * & \xrightarrow{\iota} & (\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}})_{act,C_p/C_p} \\ \downarrow & & \downarrow Q \\ \{\langle 2\rangle, \langle 1\rangle, \{*,2\}\} & \longrightarrow & \mathrm{Sub} \end{array} .$$

By Lemma 4.21 and [Lur09, Proposition 4.1.2.15], Q is smooth. By Remark 4.1.2.10 of *loc. cit.* and Lemma 4.20, ι is cofinal. □

Recollection 4.19. Recall the category Sub of [Lur17, Definition 2.2.3.2] which was defined to have

(a) objects of Sub are triples $(\langle n\rangle, S, T)$ where $S, T \subseteq \langle n\rangle$ such that $S \cup T = \langle n\rangle$ and $S \cap T = \langle n\rangle$.

(b) a morphism of Sub from $(\langle n\rangle, S, T)$ to $(\langle n'\rangle, S', T')$ is a pointed map $f : \langle n\rangle \to \langle n'\rangle$ such that $f(S) \subseteq S'$ and $f(T) \subseteq T'$.

The ∞-category Sub is an ∞-operad by Proposition 2.2.3.5 of *loc. cit.* applied to $\mathcal{C}^{\otimes} = \mathcal{D}^{\otimes} = \mathrm{Fin}_*$. Write $\pi : \mathrm{Sub} \to \mathrm{Fin}_*$ for the structure map.

Lemma 4.20. *Let $\mathcal{A} \subseteq \mathrm{Env}_{\mathrm{Fin}_*}(\mathrm{Sub})$ denote the full subcategory on the object $(a, \pi(a) \to \langle 1\rangle)$ for $a = (\langle 2\rangle, \langle 1\rangle, \{2,*\})$. Then the inclusion*

$$\iota\colon \mathcal{A} \longrightarrow \mathrm{Env}_{\mathrm{Fin}_*}(\mathrm{Sub})$$

is cofinal.

Proof. We verify criterion (2) of [Lur09, Theorem 4.1.3.1]. Observe that for every object $(\langle n\rangle, S, T)$ of Sub, there is a unique morphism $(\langle n\rangle, S, T) \to (\langle 2\rangle, \langle 1\rangle, \{2,*\})$ which sends all $s \in S \setminus \{*\}$ to 1 and all $t \in T \setminus \{*\}$ to 2. □

Lemma 4.21. *There is a functor $Q\colon \mathrm{Com}_{\mathcal{O}^{\simeq}_{C_p}} \to \mathrm{Sub} \simeq \mathrm{Fin}_* \boxplus \mathrm{Fin}_*$ which takes a C_p-set T to its set of C_p-orbits grouped by orbit type, i.e. $Q : C_p \mapsto (\langle 1\rangle, \langle 1\rangle, \{*\})$ and $Q : C_p/C_p \mapsto (\langle 1\rangle, \{*\}, \langle 1\rangle)$. The functor Q is a coCartesian fibration classified by the functor*

$$\mathrm{Sub} \to \mathfrak{Cat}$$
$$(\langle n\rangle, S, T) \mapsto \bigsqcup_S BC_p \sqcup \bigsqcup_T * .$$

Construction 4.22. Let $A \in \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})$. Given any $(B, n_B\colon N^{C_p}B \to B) \in N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$, there is a canonical map

$$f : \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(A, B) \to \hom_{N\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(\gamma F(A), (B, n_B\colon N^{C_p}B \to B)).$$

By Observation 3.19 and Corollary 2.18, we may define f 'componentwise.' Denote $\hom_{\mathcal{N}(i)}(p_iF(A), p_i(B))$ by M_i (Definition 3.11). We have

$$M_0 = \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(F(A), B) \simeq \hom_{\mathbb{E}_\infty\mathrm{Alg}}(A, B) \underset{\hom\left(A^{tC_p}, B^{tC_p}\right)}{\times} \hom\begin{pmatrix} A^e_{hC_p} & B^{\varphi C_p} \\ \downarrow & \downarrow \\ A^{tC_p}, & B^{tC_p} \end{pmatrix}.$$
$$=: M_0' \times_{M^t} M_0''$$

Take the identity on M_0' and define $\hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(A,B) \to M_0''$ to be the composite

$$f_0'' \colon \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(A,B) \xrightarrow{N^{C_p}} \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}\left(N^{C_p}A, N^{C_p}B\right) \xrightarrow{g} M'$$

where g takes $h : N^{C_p}A \to N^{C_p}B$ to the outermost trapezoid in the commutative diagram

$$\begin{array}{ccccc}
A^e & \xrightarrow{\quad h \quad} & B^e & \xrightarrow{n_B^{\varphi C_p}} & B^{\varphi C_p} \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \\
((A^e)^{\otimes p})^{tC_p} & \xrightarrow{(h^{\otimes p})^{tC_p}} & ((B^e)^{\otimes p})^{tC_p} & \xrightarrow{n_B^{tC_p}} & \downarrow \\
\downarrow{\scriptstyle m^{tC_p}} & & & & \\
A^{tC_p} & & \xrightarrow{\quad h^{tC_p} \quad} & & B^{tC_p}
\end{array}$$

where by definition $n_B^{tC_p} \simeq m_{B^e}^{tC_p}$ and the lower trapezoid is the Tate construction $(-)^{tC_p}$ on m of (3.2) applied to h. Clearly f_0' and f_0'' lift canonically to a functor $f_0 : \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(A,B) \to M_0$.

Take $f_4 : \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(A,B) \to M_4$ to be the product $N^{C_p}(-^e) \times \mathrm{id}$. The map f_4 clearly lifts to $f_2 : \hom_{\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_p})}(A,B) \to M_2$ and is identified canonically with $(N^{C_p}(-^e) \times \mathrm{id}) \circ f_0$. Since $F(A)^e \simeq A^e$, we may define f_3 and f_5 as $m_{(-)^e}$ and $(-^e)^{\otimes 2} \times (-)^e$, respectively.

The f_i assemble to give the desired map.

4.3 Proof of main theorem

Equipped with an explicit description of the free C_p-$\mathbb{E}_\infty$-algebra on an $\mathbb{E}_\infty$-algebra A in C_p-spectra from the previous section, here we show that γ of the free C_p-$\mathbb{E}_\infty$-algebra on an $\mathbb{E}_\infty$-algebra A computes the free normed algebra on A using our description of mapping spaces in the category of normed $\mathbb{E}_\infty$-algebras. The main result then follows from an application of the Barr–Beck–Lurie theorem.

Theorem 4.23. *The functor $\gamma \colon C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ of Corollary 4.7 is an equivalence.*

Proof. Consider the diagram of forgetful functors

$$\begin{array}{ccc}
C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) & \xrightarrow{\qquad \gamma \qquad} & N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \\
& {\scriptstyle G}\searrow \qquad \swarrow{\scriptstyle G'} & \\
& \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) &
\end{array} \tag{4.24}$$

where G' is from Definition 3.11 and G is (2.34). The diagram (4.24) evidently commutes.

The functor G is monadic by Proposition 4.29. The functor G' is monadic by Proposition 3.21. Now for any $A \in \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)$, the unit $A \to \gamma F(A)$ of Theorem 4.14 induces an equivalence $F'(A) \simeq \gamma F(A)$ by Corollary 4.27. The result follows from [Lur17, Proposition 4.7.3.16]. □

Proposition 4.25. *Let A be an $\mathbb{E}_\infty$-ring in Sp^{C_p} and let $(B, n_B\colon N^{C_p}B \to B)$ be a normed $\mathbb{E}_\infty$-ring in Sp^{C_p}. Then precomposition with the $\mathbb{E}_\infty$-map $\eta_A : A \to GF(A)$ of Theorem 4.14 induces an equivalence of morphism spaces*

$$\begin{array}{c} Hom_{N\mathbb{E}_\infty}\Big(\big(\gamma F(A), n_{F(A)}\colon N^{C_p}\gamma F(A) \to \gamma F(A)\big), \big(B, n_B\colon N^{C_p}B \to B\big)\Big) \\ \big\downarrow G' \\ Hom_{\mathbb{E}_\infty}(G'\gamma F(A), G'(B)) \\ \big\downarrow \eta^* \\ Hom_{\mathbb{E}_\infty}(A, G'(B)). \end{array} \qquad \text{(the composite is marked } \wr\text{)} \tag{4.26}$$

where G' is the forgetful functor of Definition 3.11 and γ is the functor of Corollary 4.7. That is, $\eta_{(-)}$ is a unit for the functors $(\gamma \circ F, G')$ in the sense of [Lur09, Definition 5.2.2.7].

Corollary 4.27. *The natural transformation $\eta_{(-)}$ exhibits $\gamma \circ F$ as a left adjoint to G'.*

Proof. Follows from [Lur09, Proposition 5.2.2.8] and Proposition 4.25. □

Proof of Proposition 4.25. By Observation 3.19, the space of morphisms $\mathrm{Hom}_{N\mathbb{E}_\infty}((F(A), n_{F(A)}\colon N^{C_p}F(A) \to F(A)), (B, n_B\colon N^{C_p}B \to B))$ is computed by the limit of the diagram

$$\begin{array}{ccc}
 & \mathrm{hom}\left(\begin{array}{c} A^e \\ \downarrow \\ (A^{\otimes p})^{tC_p}\end{array}, \begin{array}{c} B^e \\ \downarrow \\ (B^{\otimes p})^{tC_p}\end{array}\right) & \longleftarrow \;\mathrm{hom}\left(\begin{array}{c} A \\ \downarrow\Delta \\ (A^{\otimes p})^{tC_p}\end{array}, \begin{array}{c} B \\ \downarrow\Delta \\ (B\otimes p)^{tC_p}\end{array}\right) \times \mathrm{hom}\left(\begin{array}{c} A \\ \downarrow\Delta \\ (A^{\otimes p})^{tC_p}\end{array}, \begin{array}{c} B^{\varphi C_p} \\ \downarrow\Delta \\ B^{tC_p}\end{array}\right) \mathrm{hom}\left(\begin{array}{c} A^{\varphi C_p}\otimes A_{hC_p} \\ \downarrow s_A\otimes \nu_A \\ A^{tC_p}\end{array}, \begin{array}{c} B^{\varphi C_p} \\ \downarrow\Delta \\ B^{tC_p}\end{array}\right). \\
 {\scriptstyle m\colon (-^e)^{\otimes p} \to (-^e)} \nearrow & \downarrow & \downarrow \\
 & \mathrm{hom}\left((A^{\otimes p})^{tC_p}, (B^{\otimes p})^{tC_p}\right) \times \mathrm{hom}\left(A^{tC_p}, B^{tC_p}\right) & \nwarrow \\
 \nearrow & & \\
\mathrm{hom}_{\mathbb{E}_\infty \mathrm{Alg}(\mathrm{Sp}^{C_p})}(F(A), B) & \longrightarrow & \mathrm{hom}\left(\begin{array}{c} A \\ \downarrow\Delta \\ (A^{\otimes p})^{tC_p}\end{array}, \begin{array}{c} B \\ \downarrow\Delta \\ (B^{\otimes p})^{tC_p}\end{array}\right) \times \mathrm{hom}\left(\begin{array}{c} A^{\varphi C_p}\otimes A_{hC_p} \\ \downarrow\Delta \\ A^{tC_p}\end{array}, \begin{array}{c} B^{\varphi C_p} \\ \downarrow\Delta \\ B^{tC_p}\end{array}\right)
\end{array} \tag{4.28}$$

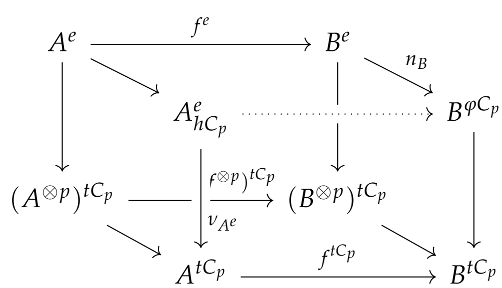
where all morphisms are computed in either $\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{BC_p}\right)$ or $\mathrm{Ar}\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{BC_p}\right)$. Notice that

$$\hom_{\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)}(F(A),B) \simeq \hom_{\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)}(A,B) \underset{\hom\left(A^{tC_p},B^{tC_p}\right)}{\times} \hom\begin{pmatrix} A_{hC_p}, & B^{\varphi C_p} \\ \downarrow \nu_A & \downarrow \Delta \\ A^{tC_p} & B^{tC_p}\end{pmatrix}$$

and moreover the composite

$$\hom_{N\mathbb{E}_\infty\mathrm{Alg}}(F(A),B) \xrightarrow{G'} \hom_{\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)}(F(A),B) \xrightarrow{\pi_1} \hom_{\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)}(A,B)$$

is equivalent to $\eta^* \circ G'$. Unravelling definitions, we see that given a point $f \in \hom_{\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)}(A,B)$, the fiber of $\eta^* \circ G'$ over f is given by the space of fillings of the below diagram to a commutative diagram $\mathcal{O}_{C_p} \times (\Delta^1)^{\times 2} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$:



wherein all but the top and front face are given. This space is contractible by the adjunction

$$(-)^{\mathrm{triv}} \colon \mathrm{Sp} \leftrightarrows \mathrm{Sp}^{C_p} \colon (-)_{hC_p}.$$

Now by Construction 4.22, $\eta^* \circ G'$ admits a right inverse f, hence it is surjective on connected components. Thus the result follows. □

Proposition 4.29. *The forgetful functor $G : C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right)$ of (2.34) is monadic.*

Proof. We consider the commuting triangle of forgetful functors

$$\begin{array}{ccc} C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) & \xrightarrow{G} & \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \\ & \searrow \quad \swarrow & \\ & \mathrm{Sp}^{C_p} & \end{array}.$$

The upper horizontal arrow is given by restricting along the C_p operadic inclusion $\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}} \hookrightarrow \mathrm{Com}^{\otimes}_{\mathcal{T}}$ and applying the equivalence of Lemma 4.30. By [NS22, Corollary 5.1.5], the diagonal arrows are monadic; in particular by the Barr–Beck–Lurie theorem the right diagonal arrow is conservative. By [NS22, Theorem 4.3.4] applied to the morphism $\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}} \hookrightarrow \mathrm{Com}^{\otimes}_{\mathcal{T}}$, the upper horizontal arrow admits a left adjoint. Thus the result follows from [Lur17, Proposition 4.7.3.22.]. □

Lemma 4.30. *For the minimal indexing system $\mathrm{Com}_{\mathcal{T}^{\simeq}}$ (Example 2.28), we have a canonical identification of $\mathrm{Com}_{\mathcal{T}^{\simeq}}$-algebras in $\left(\underline{\mathrm{Sp}}^{C_p}\right)^{\otimes}$ with $\mathcal{O}_{C_p}^{op}$-families of $\mathbb{E}_\infty$-algebras in spectra, or equivalently $\mathbb{E}_\infty$-algebras in Sp^{C_p}.*

Proof. Compare [NS22, Corollary 2.4.15] and [Lur17, Example 2.1.3.5].

Let $A : \mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}} \to \mathcal{C}^{\otimes}$ be a section of $p : \mathcal{C}^{\otimes} \to \underline{\mathrm{Fin}}_{\mathcal{T},*}$. Notice that A is a $\mathrm{Com}_{\mathcal{T}^{\simeq}}$-algebra if and only if A is $\mathcal{T}$-right Kan extended from the full subcategory of $\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq}}$ spanned by coproducts of $[C_p/C_p = C_p/C_p]$. The result follows from [Lur17, Proposition 4.3.2.15]. □

5 Applications & examples

5.1 Examples

The example which will be used in the author's upcoming work on real (C_2-equivariant) trace theories is

Theorem 5.1. *Let k be a discrete commutative ring. The constant C_p-Mackey functor $\underline{k}$ on k uniquely acquires the structure of a C_p-$\mathbb{E}_\infty$-ring.*

Proof. In view of Theorem 4.23, it suffices to show that $\underline{k}$ can be lifted to an object of Definition 3.11. Note that the isotropy separation sequence for $\underline{k}$ is

$$\begin{array}{ccc} k^{C_p} & \longrightarrow & \underline{k}^{\varphi C_p} \\ {\scriptstyle \tau_{\geq 0}}\downarrow & & \downarrow \\ k^{hC_p} & \longrightarrow & k^{tC_p} \end{array} .$$

The left vertical arrow is a connective cover; hence so is the right vertical arrow and $\underline{k}^{\varphi C_p} = \tau_{\geq 0} k^{tC_p}$. Note that $\tau_{\geq 0} k^{tC_p}$ is an $\mathbb{E}_\infty$-ring in spectra because the Tate construction and connective cover are lax symmetric monoidal functors. By Theorem 4.23 and Remark 3.17, it suffices to exhibit a commutative diagram

$$\begin{array}{ccc} k & \overset{n_{\underline{k}}}{\dashrightarrow} & \underline{k}^{\varphi C_p} \\ {\scriptstyle \text{Tate diagonal}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ (k^{\otimes p})^{tC_p} & \xrightarrow{m^{tC_p}} & k^{tC_p} \end{array} .$$

The dotted arrow and 2-cell making the diagram commute exist up to contractible choice because the inclusion of connective $\mathbb{E}_\infty$-algebra spectra into all $\mathbb{E}_\infty$-algebra spectra admits a right adjoint [Lur17, Proposition 7.1.3.13], and our assumption that k is connective. □

There is a natural class of equivariant C_p-spectra for which the data of (3.18) is no extra data at all.

Recollection 5.2. The ∞-category $\mathrm{Sp}^{C_p,\mathrm{Borel}}$ of *Borel C_p*-spectra is the image of Sp^{BC_p} under the fully faithful right adjoint to the 'underlying' spectrum functor of Proposition 2.15. Write $C_p\mathbb{E}_\infty\mathrm{Alg}^{\mathrm{Borel}}\left(\mathrm{Sp}^{C_p}\right)$

for the pullback

$$C_p\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \underset{\mathrm{Sp}^{C_p}}{\times} \mathrm{Sp}^{C_p,\mathrm{Borel}}.$$

In words, this is the category of C_p-$\mathbb{E}_\infty$-algebras whose underlying C_p-spectrum is Borel. A C_p-spectrum is Borel if and only if the structure map $A^{\varphi C_p} \to A^{tC_p}$ is an equivalence.

In view of the expected correspondence between the theories of N_∞-algebras of Blumberg–Hill and the C_p-$\mathbb{E}_\infty$-algebras of Nardin–Shah, we have the following analogue of [BH15, Theorem 6.26].

Proposition 5.3. *Every Borel $\mathbb{E}_\infty$-algebra in C_p-spectra admits an essentially unique structure of a $C_p\mathbb{E}_\infty$-algebra. More precisely, there is an equivalence of categories*

$$\mathbb{E}_\infty\mathrm{Alg}^{\mathrm{Borel}}\left(\mathrm{Sp}^{C_p}\right) \xrightarrow{\exists} C_p\mathbb{E}_\infty\mathrm{Alg}^{\mathrm{Borel}}\left(\mathrm{Sp}^{C_p}\right)$$

with inverse the forgetful functor G (2.34).

This result may also be regarded as a special case of [Hil22, Proposition 3.3.6].

Proof. Let $A \in \mathbb{E}_\infty\mathrm{Alg}^{\mathrm{Borel}}\left(\mathrm{Sp}^{C_p}\right)$. Then by Theorem 4.23, it suffices to produce a lift

$$\begin{array}{ccc} A^e & \overset{\exists ?}{\dashrightarrow} & A^{\varphi C_p} \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle s_A} \\ (A^{\otimes p})^{tC_p} & \xrightarrow{m^{tC_p}} & A^{tC_p} \end{array}$$

which is functorial in A. By definition of Borel spectra, s_A is an equivalence, so the space of choices of $\dashrightarrow$ and a 2-cell making the diagram commute is contractible. ☐

Corollary 5.4. *The real bordism spectrum $MU_\mathbb{R}$ admits a unique refinement to C_2-$\mathbb{E}_\infty$-algebra.*

Proof. Follows from [HK01, Theorem 4.1(1)] and Proposition 5.3. ☐

Proposition 5.5. *Let $B \in \mathbb{E}_\infty\mathrm{Alg}\,(\mathrm{Sp})$ be an $\mathbb{E}_\infty$-algebra. Then $N^{C_p}B$ admits a canonical structure of a C_p-$\mathbb{E}_\infty$-algebra. That is, there is a factorization*

$$\begin{array}{ccc} & \overset{\exists}{\nearrow} & N\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \\ & & \downarrow{\scriptstyle G'} \\ \mathbb{E}_\infty\mathrm{Alg}\,(\mathrm{Sp}) & \xrightarrow{N^{C_p}} & \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_p}\right) \end{array}.$$

Proof. Then by Theorem 4.23, it suffices to produce a dotted arrow and a commutative diagram $\mathcal{O}_{C_p} \times \Delta^1 \to \mathbb{E}_\infty\mathrm{Alg}\,(\mathrm{Sp})$

$$\begin{array}{ccc} B^{\otimes p} & \overset{\exists ?}{\dashrightarrow} & B \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \left(B^{\otimes p^2}\right)^{tC_p} & \xrightarrow{m^{tC_2}} & (B^{\otimes p})^{tC_p} \end{array}$$

which is functorial in B. We can choose the dotted arrow to be m_B and a commutative diagram to be functorial in B because the Tate diagonal is functorial. ☐

5.2 Real motivic invariants

Here, we briefly recall that algebras with C_2-actions naturally give rise to motivic invariants valued in genuine C_2-spectra. These real motivic invariants and their associated real trace theories provided the impetus for this work.

We only provide brief sketches of the required constructions and definitions; readers who are unfamiliar with the following notions should refer to sources cited below for details.

Recollection 5.6 (Real topological K-theory)**.** [Ati66; Dug05] The space $\mathbb{Z} \times BU$ has a C_2-action coming from complex conjugation on the unitary group U, with C_2-fixed points $\mathbb{Z} \times BO$. Furthermore, there is a C_2-equivariant form of Bott periodicity $\mathbb{Z} \times BU \simeq \Omega^{\rho}(\mathbb{Z} \times BU)$. Real K-theory $KU_{\mathbb{R}}$ is the associated C_2-spectrum (under the equivalence in [GM17, §3]).

By Appendix A (see discussion after Theorem A.5) of [LN14], $KU_{\mathbb{R}}$ is an $\mathbb{E}_\infty$ algebra in C_2-spectra.

Recollection 5.7 (Poincaré ∞-categories)**.** There is an ∞-category $\mathcal{C}\mathrm{at}_\infty^p$ of *Poincaré ∞-categories* ([Cal+20a, Definitions 1.2.7-8]) whose objects are pairs $(\mathcal{C}, \Omega)$ consisting of a small stable ∞-category and a quadratic functor $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$, and morphisms are given by duality-preserving exact functors. Moreover, the ∞-category $\mathcal{C}\mathrm{at}_\infty^p$ has a symmetric monoidal structure [Cal+20a, Theorem 5.2.7(iii)] lifting the Lurie tensor product on small stable ∞-categories.

Definition 5.8 (Real algebraic K-theory)**.** Let A be a C_2-$\mathbb{E}_\infty$-algebra. We may associate to A the module with genuine involution $(M = A^e, N = A^{\varphi C_2}, s_A : A^{\varphi C_2} \to A^{tC_2})$ (Definition 3.2.2 of *loc. cit*. To such a module with genuine involution there is an associated Poincaré ∞-category $\left(\mathcal{P}\mathrm{erf}_{A^e}, \Omega^{s_A}_{A^e}\right)$ ([Cal+20a, Construction 3.2.5]). The *real algebraic K-theory* $K_{\mathbb{R}}(A)$ of A is the real algebraic K-theory

$$K_{\mathbb{R}}(A) \simeq \mathrm{GW}^{\mathrm{ghyp}}\left(\mathcal{P}\mathrm{erf}_{A^e}, \Omega^{s_A}_{A^e}\right) \in \mathrm{Sp}^{C_2}$$

in the sense of [Cal+20b, Definition 4.5.1].

Proposition 5.9. *(1) The assignment $A \mapsto \left(\mathcal{P}\mathrm{erf}_{A^e}, \Omega^{s_A}_{A^e}\right)$ of Definition 5.8 (compare examples from §3.2, in particular Example 3.2.11 of [Cal+20a]) promotes to a symmetric monoidal functor*

$$C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \to \mathcal{C}\mathrm{at}_\infty^p.$$

(2) The real algebraic K-theory of a C_2-$\mathbb{E}_\infty$-algebra R canonically refines to an $\mathbb{E}_\infty$-algebra in C_2-spectra.

Proof. To prove (1), it suffices to observe that a morphism of C_2-$\mathbb{E}_\infty$-algebras $\phi\colon A \to B$ induces a canonical triple (δ, γ, σ) in the sense of Corollary 3.3.2 of [Cal+20a] (corresponding to a hermitian functor $\left(\mathcal{P}\mathrm{erf}_{A^e}, \Omega^{s_A}_{A^e}\right) \to \left(\mathcal{P}\mathrm{erf}_{B^e}, \Omega^{s_B}_{B^e}\right)$covering the induction $\phi_*\colon \mathrm{Mod}_{A^e} \to \mathrm{Mod}_{B^e}$). Furthermore, the triple (δ, γ, σ) automatically satisfies the criterion of Lemma 3.3.3 & Definition 3.3.4 of *loc.cit.*, hence the associated hermitian functor is in fact Poincaré.

To prove (2), it suffices to exhibit a composite functor $C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \to \mathcal{C}\mathrm{at}_\infty^p \to \mathrm{Sp}^{C_2}$ which is lax symmetric monoidal. The former functor is lax symmetric monoidal by (1). The latter functor is that of [Cal+20b, Definition 4.5.1]. That it is lax symmetric monoidal will appear in [Cal+]. □

A ring R is said to satisfy the *homotopy limit problem* if its genuine symmetric real K-theory is a Borel C_2-spectrum [Tho83; Cal+21, §3].

Corollary 5.10.
- *Real topological K-theory $KU_{\mathbb{R}}$ admits a unique refinement to a C_2-$\mathbb{E}_\infty$ ring spectrum.*
- *If A satisfies the* homotopy limit problem*, then $K_{\mathbb{R}}(A)$ admits a unique refinement to a C_2-$\mathbb{E}_\infty$ ring spectrum.*

Proof. By [Ati66] (also see [Rog08, proof of Proposition 5.3.1; Dug05, Corollary 7.6]), $KU_{\mathbb{R}}$ is Borel. Both results follow from Proposition 5.3. □

5.3 A relative enhancement

In this section we state a version of our main theorem *relative* to an arbitrary base C_p-$\mathbb{E}_\infty$-algebra A (Example 2.33). In order to make sense of a C_p-$\mathbb{E}_\infty$-algebra over A, we require a C_p-symmetric monoidal structure on the category of A-modules. That this is possible is suggested by the following

Definition 5.11. Let A be a C_p-$\mathbb{E}_\infty$-ring in Sp^{C_p}. The *(A-linear or relative) norm* is the functor

$$\underline{N}_e^{C_p}\colon \mathrm{Mod}_{A^e} \to \mathrm{Mod}_A\left(\mathrm{Sp}^{C_p}\right)$$
$$M \mapsto A \otimes_{N_e^{C_p}(A^e)} N_e^{C_p} M$$

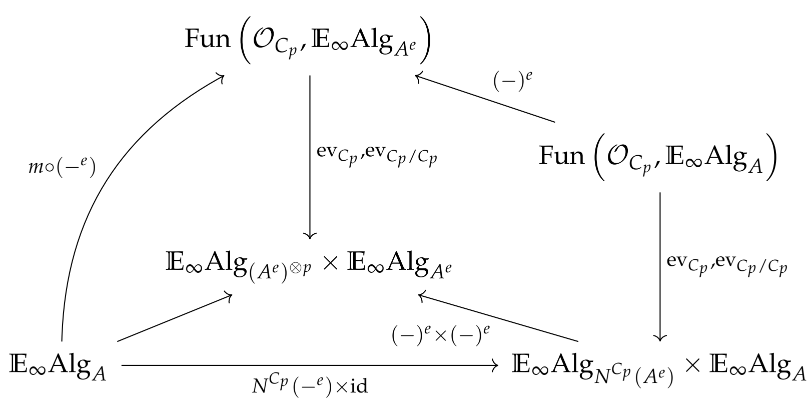
Note that the reasoning of Lemma 2.22 applies to show that $\underline{N}_e^{C_p}$ lifts to a colimit-preserving functor $\mathbb{E}_\infty\mathrm{Alg}_{A^e} \to \mathbb{E}_\infty\mathrm{Alg}_A$.

By Proposition A.9 (communicated by Jay Shah), we may regard the category of A-modules as a C_p-symmetric monoidal ∞-category.

Definition 5.12. Let A be a C_p-$\mathbb{E}_\infty$-algebra (Example 2.33). The ∞-category of C_p-$\mathbb{E}_\infty$-A-algebras is $\mathrm{Alg}_{\underline{\mathrm{Fin}}_{C_p,*}}\left(\underline{\mathrm{Fin}}_{C_p,*}, (\underline{\mathrm{Mod}}_A)^{\otimes}\right) =: C_p\mathbb{E}_\infty\mathrm{Alg}_A$ (Definition 2.32). In other words, it is the category of sections of $\underline{\mathrm{Mod}}_A^{\otimes} \to \underline{\mathrm{Fin}}_{C_p,*}$ which take inert morphisms to inert morphisms.

There is moreover a relative notion of normed algebras over A.

Definition 5.13. Let A be a C_p-$\mathbb{E}_\infty$-ring. We define the category $N\mathbb{E}_\infty\mathrm{Alg}_A$ of *normed $\mathbb{E}_\infty$-A-algebras* to be the limit of the diagram



where we have abbreviated $\mathbb{E}_\infty\mathrm{Alg}_A = \mathbb{E}_\infty\mathrm{Alg}_A\left(\mathrm{Sp}^{C_p}\right)$, $\mathbb{E}_\infty\mathrm{Alg}_{(A^e)^{\otimes p}} = \mathbb{E}_\infty\mathrm{Alg}_{(A^e)^{\otimes p}}\left(\mathrm{Sp}^{BC_p\times BC_p}\right)$, etc.

There is a relative version of the main results of this paper.

Proposition 5.14. *Let A be a C_p-$\mathbb{E}_\infty$-ring. There is a canonical forgetful functor*

$$\gamma_A\colon C_p\mathbb{E}_\infty\mathrm{Alg}_A \to N\mathbb{E}_\infty\mathrm{Alg}_A$$

Proof. Proceeds as in proof of Corollary 4.7. □

Theorem 5.15. *Let A be a C_p-$\mathbb{E}_\infty$-ring. Then the canonical comparison functor γ_A of Proposition 5.14 is an equivalence.*

Proof. Proceeds as in proof of Theorem 4.23. □

A Modules over normed equivariant algebras

In this appendix, we show that the category of modules over a C_p-$\mathbb{E}_\infty$-ring naturally acquires a structure of a C_p-symmetric monoidal ∞-category in the sense of Nardin–Shah via a relative norm (cf. Definition 5.11). The author would like to thank Jay Shah who communicated details of this construction.

Fix κ a regular cardinal and let $\mathcal{K}$ denote the collection of κ-small simplicial sets.

Recollection A.1. [Lur17, Notation 4.8.3.5.] Write $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}_\infty)$. It has objects given by monoidal ∞-categories which are compatible with κ-indexed colimits and whose morphisms are monoidal functors $F:\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ whose preserve κ-indexed colimits. Write $U\colon \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{C}\mathrm{at}_\infty$ for the forgetful functor which forgets the monoidal structure.

[Lur17, Definition 4.8.3.7.] There is an ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}}(\mathcal{K})$. Informally its objects are given by pairs $(\mathcal{C}^\otimes, A)$ where $\mathcal{C}$ is a monoidal ∞-category and A is an algebra object of $\mathcal{C}$. Morphisms from $(\mathcal{C}^\otimes, A)$ to $(\mathcal{D}^\otimes, B)$ are given by monoidal functors $F:\mathcal{C}\to\mathcal{D}$ such that $F(A)\simeq B$.

Similarly, there is an ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Mod}}(\mathcal{K})$ whose objects are pairs $(\mathcal{C}^\otimes,\mathcal{M})$ where $\mathcal{C}$ is a monoidal ∞-category and $\mathcal{M}$ is an ∞-category left tensored over $\mathcal{C}$. In particular, there is a forgetful functor

$$\begin{aligned}\Upsilon\colon \mathcal{C}\mathrm{at}_\infty^{\mathrm{Mod}}(\mathcal{K}) &\to \mathcal{C}\mathrm{at}_\infty \\ (\mathcal{C}^\otimes,\mathcal{M}) &\mapsto \mathcal{M}\end{aligned}. \tag{A.2}$$

[Lur17, Construction 4.8.3.24] There is a functor Θ making the following diagram commute:

$$\begin{array}{ccc}\mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}}(\mathcal{K}) & \xrightarrow{\Theta:(\mathcal{C}^\otimes,A)\mapsto(\mathcal{C}^\otimes,\mathrm{Mod}_A(\mathcal{C}))} & \mathcal{C}\mathrm{at}_\infty^{\mathrm{Mod}}(\mathcal{K}) \\ & {\scriptstyle u_a}\searrow \quad \swarrow{\scriptstyle u_m} & \\ & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}_\infty) & \end{array} \tag{A.3}$$

where the vertical arrows are the universal fibrations classifying families of $\mathbb{E}_1$-algebras in $\mathcal{C}$ and modules over said algebras in $\mathcal{C}$, respectively [Lur17, Remarks 4.8.3.8. & 4.8.3.20.].

Notation A.4. Hereafter, we drop $\kappa, \mathcal{K}$ from notation.

Lemma A.5. *Let A be a C_p-$\mathbb{E}_\infty$-algebra object of $\left(\underline{\mathrm{Sp}}^{C_p}\right)^{\otimes}$, and recall the functor $\zeta \colon \mathrm{Span}(\mathrm{Fin}_{C_p}) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at})$ of Recollection 2.30. Then A lifts to a cocartesian section $\tilde{A}$ of $\int \zeta$.*

Proof. Let $T \to C_p/H$ be an object of $\underline{\mathrm{Fin}}_{C_p,*}$. There is a natural functor $\iota_T := - \times T \colon \mathrm{Fin}_* \to \underline{\mathrm{Fin}}_{C_p,*}$. Moreover, $\iota_{(-)}$ assembles to give the functor

$$\begin{aligned} \iota \colon \mathrm{Fin}_* \times \underline{\mathrm{Fin}}_{C_p,*} &\to \underline{\mathrm{Fin}}_{C_p,*} \\ (S, T \to C_p/H) &\mapsto S \times T \to C_p/H \end{aligned}.$$

Consider the restriction $\iota_T^* A$ of A along ι_T. We may further precompose $\iota_T^* A$ with the structure morphism $\mathrm{Assoc}^{\otimes} \to \mathrm{Fin}_*$ where $\mathrm{Assoc}^{\otimes}$ is the $\mathbb{E}_1$ operad of [Lur17, Definition 4.1.1.3]. Then by the characterization of inert morphisms of Theorem 2.3.3 of [NS22], $\iota_T^* A$ and by definition of morphisms of operads (both parametrized and non-parametrized), $\iota_T^* A$ defines an associative algebra object in $\zeta(T)$. Likewise $\iota^* A$ defines a $\underline{\mathrm{Fin}}_{C_p,*}$-family of associative algebra objects, hence the existence of $\tilde{A}$ follows by the universal property characterizing $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}}$. ◻

Variant A.6. Let $\mathcal{K}_0$ denote the full subcategory of the arrow category $\mathrm{Ar}(\mathrm{Fin}_*)$ on those arrows given by the inclusion of the basepoint $\{*\} \hookrightarrow S$. There is a variant functor

$$\begin{aligned} \iota_\eta \colon \mathcal{K}_0 \times \underline{\mathrm{Fin}}_{C_p,*} &\to \underline{\mathrm{Fin}}_{C_p,*} \\ (\{*\} \hookrightarrow S, T) &\mapsto (\{*\} \hookrightarrow S) \times T \end{aligned}$$

which defines a $\Delta^1 \times \underline{\mathrm{Fin}}_{C_p,*}$-family of associative algebra objects.

Now consider the commutative diagram

$$\begin{array}{ccccccc} \int \zeta & \xrightarrow{u_a^*(\zeta)} & \mathcal{C}\mathrm{at}_\infty^{\mathrm{Alg}} & \xrightarrow{\Theta} & \mathcal{C}\mathrm{at}_\infty^{\mathrm{Mod}} & \xrightarrow{\Upsilon} & \mathcal{C}\mathrm{at}_\infty \\ {\scriptstyle \tilde{A}}\uparrow\downarrow & & \downarrow{\scriptstyle u_a} & & \downarrow{\scriptstyle u_m} & & \\ \underline{\mathrm{Fin}}_{C_p,*} & \xrightarrow{\zeta} & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}) & = & \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}\mathrm{at}) & & \end{array}$$

where $\tilde{A}$ exists by Lemma A.5 and the center square is (A.3).

Definition A.7. Let A be a C_p-$\mathbb{E}_\infty$-algebra in C_p-spectra. Recall the Grothendieck construction [Lur09, Theorem 2.2.1.2]. Let $\underline{\mathrm{Mod}}_A^{\otimes}$ be the C_p-symmetric monoidal ∞-category classified by the morphism $\Upsilon \circ \Theta \circ u_a^*(\zeta) \circ \tilde{A}$. Let $\underline{\mathrm{Mod}}_A$ denote the corresponding underlying C_p-∞-category of $\underline{\mathrm{Mod}}_A^{\otimes}$.

Examples A.8.

(1) When $A = \mathbb{S}^0$, we recover $\underline{\mathrm{Mod}}_A \simeq \underline{\mathrm{Sp}}^{C_p}$.

(2) Suppose A is a C_p-$\mathbb{E}_\infty$-ring. Then $\underline{\mathrm{Mod}}_A$ may be regarded as the $\mathcal{O}_{C_p}$-diagram *of stable ∞-categories*

$$\mathrm{Mod}_A\left(\mathrm{Sp}^{C_p}\right) \xrightarrow{(-)^e} \mathrm{Mod}_{A^e}(\mathrm{Sp}) \circlearrowleft_{\sigma_*}.$$

The morphism $[C_p \to C_p/C_p] \to [C_p/C_p = C_p/C_p]$ in $\underline{\mathrm{Fin}}_{C_p,*}$ classifies the relative norm $\underline{N}^{C_p}$ of Definition 5.11.

The previous discussion shows that

Proposition A.9. *Let A be a C_p-$\mathbb{E}_\infty$-algebra in C_p-spectra. Then the C_p-∞-category of Definition A.7 naturally acquires a C_p-symmetric monoidal structure in the sense of [NS22, Definition 2.2.1].*

Remark A.10. By the proof of Lemma A.5, each A_T for $T \to C_p/H$ an object of $\underline{\mathrm{Fin}}_{C_p,*}$ is in fact an $\mathbb{E}_\infty$-algebra in $\zeta(T)$. Thus we write modules instead of left modules [Lur17, Corollary 4.5.1.6].

Construction A.11 (Parametrized base change)**.** Since $\underline{\mathrm{Fin}}_{C_p,*}$ is unital, the category of C_p-$\mathbb{E}_\infty$-algebras in C_p-spectra has an initial object $\underline{\mathbb{1}}$ [NS22, Definition 5.2.1 & Theorem 5.2.11, resp.] given fiberwise by the sphere spectrum. As in Lemma A.5, $\underline{\mathbb{1}}$ lifts to a coCartesian section $\tilde{\underline{\mathbb{1}}}$ of $\int \zeta$. Suppose A is a C_p-$\mathbb{E}_\infty$-ring spectrum. Variant A.6 shows that the unit map $\eta\colon \underline{\mathbb{1}} \to A$ induces a natural transformation $\tilde{\eta}\colon \tilde{\underline{\mathbb{1}}} \to \tilde{A}$. Under the Grothendieck construction, the unstraightening of $\mathrm{Y} \circ \Theta \circ u_a^*(\zeta)(\eta)$ corresponds to a C_p-functor of C_p-∞-categories which we denote by

$$- \otimes_{\mathbb{S}^0} A\colon \underline{\mathrm{Sp}}^{C_p} \to \underline{\mathrm{Mod}}_A.$$

Categories of modules behave in the expected way.

Proposition A.12. *Let A be a C_p-$\mathbb{E}_\infty$-algebra in C_p-spectra. Then the C_p-functor $- \otimes_{\mathbb{S}^0} A\colon \underline{\mathrm{Sp}}^{C_p} \to \underline{\mathrm{Mod}}_A$ participates in a C_p-adjunction which is fiberwise monadic.*

Proof. Notice that essentially by definition, the C_p-functor $- \otimes_{\mathbb{S}^0} A$ preserves coCartesian arrows. By (the dual to) [Lur17, Proposition 7.3.2.6], it suffices to check that $- \otimes_{\mathbb{S}^0} A$ admits a fiberwise right adjoint, which is classical. □

Remark A.13. The strategy outlined here generalizes straightforwardly to endow the G-∞-category of modules over a normed G-$\mathbb{E}_\infty$-ring spectrum with the structure of a G-symmetric monoidal structure for any finite group G.

Remark A.14. One expects a equivariant form of the Tannaka reconstruction theorem [Lur17, Propositions 7.1.2.6-7] by which a G-$\mathbb{E}_\infty$-ring A can be recovered from its category of modules endowed with its G-symmetric monoidal structure.

References

[Ati66] M. F. Atiyah. "K-theory and reality". In: *Quart. J. Math. Oxford Ser. (2)* 17 (1966), pp. 367–386.

[Bar+16a] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. "Parametrized higher category theory and higher algebra: A general introduction". 2016. arXiv: `1608.03657 [math.AT]`.

[Bar+16b] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. "Parametrized higher category theory and higher algebra: Exposé I – Elements of parametrized higher category theory". 2016. arXiv: `1608.03657 [math.AT]`.

[Bar+17] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. "Parametrized higher category theory and higher algebra: Exposé IX". in preparation. 2017.

[Bar17] Clark Barwick. "Spectral Mackey functors and equivariant algebraic K-theory (I)". In: *Adv. Math.* 304.2 (2017), pp. 646–727.

[BGS16] Clark Barwick, Saul Glasman, and Jay Shah. "Spectral Mackey functors and equivariant algebraic K-theory (II)". 2016. arXiv: 1505.03098v2 [math.AT].

[BH15] Andrew J. Blumberg and Michael A. Hill. "Operadic multiplications in equivariant spectra, norms, and transfers". In: *Adv. Math.* 285 (2015), pp. 658–708.

[BH21] Tom Bachmann and Marc Hoyois. "Norms in motivic homotopy theory". In: *Astérisque* 425 (2021), pp. ix+207.

[Cal+] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories IV*. to appear.

[Cal+20a] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories I: Foundations*. 2020.

[Cal+20b] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity*. 2020.

[Cal+21] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories III: Grothendieck-Witt groups of rings*. 2021. arXiv: 2009.07225 [math.KT].

[Dre71] Andreas W. M. Dress. *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*. With the aid of lecture notes, taken by Manfred Küchler. Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971, iv+158+A28+B31 pp. (loose errata).

[Dug05] Daniel Dugger. "An Atiyah-Hirzebruch spectral sequence for KR-theory". In: *K-Theory* 35.3-4 (2005), 213–256 (2006).

[Gla16] Saul Glasman. "Day convolution for ∞-categories". In: *Math. Res. Lett.* 23.5 (2016), pp. 1369–1385.

[GM17] Bertrand J. Guillou and J. Peter May. "Equivariant iterated loop space theory and permutative G-categories". In: *Algebr. Geom. Topol.* 17.6 (2017), pp. 3259–3339.

[Har20] Yonatan Harpaz. "Ambidexterity and the universality of finite spans". In: *Proc. Lond. Math. Soc. (3)* 121.5 (2020), pp. 1121–1170.

[HH14] M. A. Hill and M. J. Hopkins. "Equivariant multiplicative closure". In: *Algebraic topology: applications and new directions*. Vol. 620. Contemp. Math. Amer. Math. Soc., Providence, RI, 2014, pp. 183–199.

[HH16] Michael A Hill and Michael J Hopkins. "Equivariant symmetric monoidal structures". 2016. arXiv: 1610.03114 [math.AT].

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. "On the nonexistence of elements of Kervaire invariant one". In: *Ann. of Math. (2)* 184.1 (2016), pp. 1–262.

[Hil22] Kaif Hilman. "Norms and periodicities in genuine equivariant hermitian K-theory". PhD thesis. University of Copenhagen, 2022.

[HK01] Po Hu and Igor Kriz. "Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence". In: *Topology* 40.2 (2001), pp. 317–399.

[Joy08] André Joyal. "Notes on quasi-categories". Preprint available at http://www.math.uchicago.edu/~may/IMA/Joyal.pdf. 2008.

[LN14] Tyler Lawson and Niko Naumann. "Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2". In: *Int. Math. Res. Not. IMRN* 10 (2014), pp. 2773–2813.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Vol. 170. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2009.

[Lur17] Jacob Lurie. *Higher algebra*. Manuscript available at https://www.math.ias.edu/~lurie/papers/HA.pdf. 2017.

[May96] J. P. May. *Equivariant homotopy and cohomology theory*. Vol. 91. CBMS Regional Conference Series in Mathematics. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996, pp. xiv+366.

[MNN17] Akhil Mathew, Niko Naumann, and Justin Noel. "Nilpotence and descent in equivariant stable homotopy theory". In: *Adv. Math.* 305 (2017), pp. 994–1084.

[Nar16] Denis Nardin. "Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category". 2016. arXiv: 1608.07704v4 [math.AT].

[Nar17] Denis Nardin. *Stability and distributivity over orbital ∞-categories*. Thesis (Ph.D.)–Massachusetts Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2017, (no paging).

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". In: *Acta Math.* 221.2 (2018), pp. 203–409.

[NS22] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022.

[QS22] J. D. Quigley and Jay Shah. *On the equivalence of two theories of real cyclotomic spectra*. 2022. arXiv: 2112.07462 [math.AT].

[Rog08] John Rognes. "Galois extensions of structured ring spectra. Stably dualizable groups". In: *Mem. Amer. Math. Soc.* 192.898 (2008), pp. viii+137.

[Seg74] Graeme Segal. "Categories and cohomology theories". In: *Topology* 13.3 (1974), pp. 293–312.

[Sha18] Jay Shah. *Parametrized higher category theory*. To appear in *Algebraic and Geometric Topology*. 2018.

[Sha21] Jay Shah. *Recollements and stratification*. Preprint available at https://arxiv.org/abs/2110.06567. 2021.

[Tho83] R. W. Thomason. "The homotopy limit problem". In: *Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*. Vol. 19. Contemp. Math. Amer. Math. Soc., Providence, R.I., 1983, pp. 407–419.

Normed equivariant ring spectra and higher Tambara functors

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Abstract

In this paper we extend equivariant infinite loop space theory to take into account multiplicative norms: For every finite group G, we construct a multiplicative refinement of the comparison between the ∞-categories of connective genuine G-spectra and space-valued Mackey functors, first proven by Guillou–May, and use this to give a description of connective normed equivariant ring spectra as space-valued Tambara functors.

In more detail, we first introduce and study a general notion of homotopy-coherent normed (semi)rings, and identify these with product-preserving functors out of a corresponding ∞-category of bispans. In the equivariant setting, this identifies space-valued Tambara functors with normed algebras with respect to a certain normed monoidal structure on grouplike G-commutative monoids in spaces. We then show that the latter is canonically equivalent to the normed monoidal structure on connective G-spectra given by the Hill–Hopkins–Ravenel norms. Combining our comparison with results of Elmanto–Haugseng and Barwick–Glasman–Mathew–Nikolaus, we produce normed ring structures on equivariant algebraic K-theory spectra.

Contents

1 Introduction

Infinite loop space theory has played an important role in algebraic topology since the 1970's, giving a way to construct interesting examples of spectra from space-level data. At its heart lies the *Recognition Theorem* [May72, BV73, Seg74], which in modern language describes connective spectra as commutative group objects in the ∞-category of spaces, i.e.

$$\mathsf{Sp}^{\geq 0} \simeq \mathsf{CGrp}(\mathsf{Spc}). \tag{1}$$

Such commutative groups arise in nature, for instance, by group-completing the classifying spaces of symmetric monoidal (∞-)categories. As an important example, applying this to the groupoid of finitely generated projective modules over a ring R with symmetric monoidal structure via direct sum yields the algebraic K-theory spectrum of the ring R [May74, Seg74].

In order to obtain spectra with algebraic structures, we need to upgrade (1) to take into account *multiplicative* structures. For 1-categorical inputs, such *multiplicative infinite loop spaces machines* have been considered for example by May [May77, May09] and Elmendorf–Mandell [EM06]. Working in the ∞-categorical framework, Gepner–Groth–Nikolaus [GGN15] both generalized and elucidated these constructions: they show that there is a canonical symmetric monoidal structure on the ∞-category $\mathsf{CMon}(\mathsf{Spc})$ of commutative monoids in spaces, which localizes to commutative groups, and that (1) uniquely upgrades to an equivalence of symmetric monoidal ∞-categories where we equip $\mathsf{Sp}^{\geq 0}$ with the smash product. The tensor product on $\mathsf{CGrp}(\mathsf{Spc})$ is an ∞-categorical analogue of the tensor product of abelian groups, so it is natural to think of a commutative algebra object of $\mathsf{CGrp}(\mathsf{Spc})$ as a commutative ring in Spc; as a direct consequence of the multiplicative comparison, we then have an equivalence

$$\mathsf{CAlg}(\mathsf{Sp}^{\geq 0}) \simeq \mathsf{CAlg}(\mathsf{CGrp}(\mathsf{Spc})) =: \mathsf{CRing}(\mathsf{Spc}) \tag{2}$$

between connective commutative ring spectra and commutative ring spaces.

A multiplicative equivariant recognition theorem

Our first goal is to extend the above story to equivariant spectra over a finite group G. While the study of such *equivariant infinite loop space machines* began in the late 70's (in unpublished work of Segal and Hauschild–May–Waner), the subject has experienced a renaissance in recent years. As part of this, its point-set level foundations have been rewritten and extended by May and his collaborators [GM11, GM17, MMO17], and new ∞-categorical approaches to the subject have been introduced by Barwick and collaborators [Bar17, BDG+16].

In the present paper, we will adopt the latter perspective. For this, recall from [CMNN20, A.1] that the ∞-category Sp_G of G-spectra, defined classically as the Dwyer–Kan localization of a suitable model category of orthogonal or symmetric spectra with G-action, admits a purely ∞-categorical description as

$$\mathsf{Sp}_G \simeq \mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}_G), \mathsf{Sp}),$$

that is, as product-preserving functors from a $(2,1)$-category of spans of finite G-sets to spectra; see also [GM11] for a model-categorical version.

From this equivalence and (1), one immediately obtains an *Equivariant Recognition Theorem*, in the form of a space-level description of connective G-spectra as

$$\mathsf{Sp}_G^{\geq 0} \simeq \mathsf{Fun}^{\times}_{\mathrm{grp}}(\mathsf{Span}(\mathbb{F}_G), \mathsf{Spc}); \tag{3}$$

here the right-hand side consists of functors $F\colon \mathsf{Span}(\mathbb{F}_G) \to \mathsf{Spc}$ that preserve products and that are *grouplike*, in the sense that for every G-set X the commutative multiplication given by

$$F(X) \times F(X) \simeq F(X \amalg X) \longrightarrow F(X)$$

makes $F(X)$ a commutative group. Analogously to the non-equivariant situation, such objects arise naturally from ∞-categorical data, and this provides one possible approach to equivariant algebraic K-theory [BGS20].

To shed some light on this equivalence, recall [Dre71] that a *Mackey functor* M for G consists of abelian groups $M(X)$ for every finite G-set X together with restriction and additive norm (or "transfer") maps

$$f^*\colon M(X) \longrightarrow M(Y), \quad f_{\oplus}\colon M(Y) \longrightarrow M(X)$$

for every morphism $f\colon Y \to X$ of G-sets, such that M takes disjoint unions to products, both restrictions and norms are functorial, and they compose according to a double coset formula. The zeroth homotopy groups of any G-spectrum form a Mackey functor, and Mackey functors are the most general coefficients for ordinary equivariant (co)homology [LMS86, §V.9]. The data of a Mackey functor can be neatly organized into a product-preserving functor

$$\mathsf{Span}(\mathbb{F}_G) \longrightarrow \mathsf{Ab},$$

or equivalently a functor $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{Set}$ that preserves products and such that the induced (commutative) multiplication on the value at every G-set has inverses. Thus we may think of the equivalence (3) as saying that connective G-spectra are *space-valued Mackey functors*.

Just like a space-valued Mackey functor contains more information than a commutative group in the ∞-category of G-spaces (namely, in the form of additive norms), a multiplicative refinement of the equivalence (3) should not just take the ordinary symmetric monoidal structures on both sides into account (arising via the smash product and Day convolution, respectively), but additionally respect suitable *symmetric monoidal norms*. To make this precise, note that if $\mathscr{C}$ is any ∞-category with finite products, we can more generally define a *normed G-monoid* in $\mathscr{C}$ to be a functor

$$M\colon \mathsf{Span}(\mathbb{F}_G) \longrightarrow \mathscr{C}$$

that preserves finite products; this amounts to specifying a commutative monoid $M(G/H)$ with an action of the Weyl group $W_GH := N_GH/H$ for every subgroup H of G, together with restrictions $\mathrm{Res}^K_H\colon M(G/K) \to M(G/H)$ and norm maps $\mathrm{Nm}^K_H\colon M(G/H) \to M(G/K)$ for all subgroups $H \leqslant K \leqslant G$ as well as various coherences. The equivalence (2) can then be restated as saying that genuine G-spectra are normed G-monoids in spectra, while the equivalence (3) says that connective G-spectra are equivalently "grouplike" normed G-monoids in spaces, or *normed G-groups* for short.

On the other hand, taking $\mathscr{C}$ to be Cat_∞, we obtain the notion of a *normed G-∞-category* as the equivariant version of a symmetric monoidal ∞-category. Many important symmetric monoidal ∞-categories studied in equivariant homotopy theory turn out to admit natural refinements to normed G-∞-categories; in particular, there is a normed G-∞-category

$$\underline{\mathsf{Sp}}_G\colon G/H \mapsto \mathsf{Sp}_H$$

whose contravariant functoriality is given by the evident restrictions, and whose covariant functoriality encodes the smash product of equivariant spectra together with the *Hill–Hopkins–Ravenel norms* [HHR16]. We then prove:

Theorem A. (See Theorem 5.6.1) *The G-∞-category*

$$\underline{\mathsf{NMon}}_G(\mathsf{Spc})\colon G/H \mapsto \mathsf{Fun}^\times(\mathsf{Span}(\mathbb{F}_H), \mathsf{Spc})$$

of G-normed monoids in spaces has a canonical normed structure that localizes to $\underline{\mathsf{NGrp}}_G(\mathsf{Spc})$. Furthermore, the equivalence (3) upgrades to an equivalence

$$\underline{\mathsf{Sp}}_G^{\geq 0} \simeq \underline{\mathsf{NGrp}}_G(\mathsf{Spc})$$

of normed G-∞-categories, where the left-hand side carries the restriction of the normed structure described above.

Normed G-ring spectra

As a direct consequence of Theorem A, we obtain equivalences between connective G-spectra equipped with extra "parametrized algebraic structure" and G-commutative groups equipped with the same structure. To make this precise, note that by straightening–unstraightening we can equivalently regard a normed G-∞-category $\mathscr{C}$ as a cocartesian fibration

$$\mathscr{C}^{\otimes} \longrightarrow \mathsf{Span}(\mathbb{F}_G).$$

Following Bachmann–Hoyois [BH21] we define a *G-normed algebra* in $\mathscr{C}$ as a section $\mathsf{Span}(\mathbb{F}_G) \to \mathscr{C}^{\otimes}$ that takes the backward maps to cocartesian morphisms in $\mathscr{C}^{\otimes}$; this amounts to specifying a commutative algebra A in the underlying ∞-category $\mathscr{C}(G/G)$ together with suitably coherent *normed multiplications* $\mathrm{Nm}_H^G \mathrm{Res}_H^G A \to A$ for every $H \leqslant G$.

In particular, we get a notion of G-normed algebras in $\underline{\mathsf{Sp}}_G$, or *normed G-spectra* for short, which are generally expected to be equivalent to the objects obtained as strict commutative algebras in the 1-categories of symmetric or orthogonal G-spectra. Theorem A then shows that *connective* normed G-spectra can equivalently be described as normed algebras in $\underline{\mathsf{NGrp}}_G(\mathsf{Spc})$, i.e. as "normed G-rings." As our second main result, we then build on this comparison to give a space-level description of connective normed G-spectra, generalizing the result for $G = 1$ proven in [CHLL24]:

Theorem B. (See Theorem 5.6.3) *There is an equivalence of ∞-categories*

$$\mathsf{NAlg}_G(\mathsf{Sp}_G^{\geq 0}) \simeq \mathsf{Fun}^{\times}_{\mathrm{grp}}(\mathsf{Bispan}(\mathbb{F}_G), \mathsf{Spc}). \tag{4}$$

Here $\mathsf{Bispan}(\mathbb{F}_G)$ is the $(2,1)$-category of *bispans* of finite G-sets in the sense of [EH23]: its objects are finite G-sets, and morphisms are given by diagrams

$$A \xleftarrow{R} B \xrightarrow{N} C \xrightarrow{T} D; \tag{5}$$

the composition law in $\mathsf{Bispan}(\mathbb{F}_G)$ is somewhat involved and encodes both the Mackey double coset formulas for commuting restrictions past norms and transfers, as well as a distributivity relation between norms and transfers. Moreover, $\mathsf{Fun}^{\times}_{\mathrm{grp}}$ again denotes the full subcategory of those product-preserving functors that are *grouplike* in a suitable sense (see Definition 4.3.5 for details).

Recall [Tam93] that a *Tambara functor* X for a finite group G is an assignment of an abelian group $X(G/H)$ for every subgroup of G together with compatible restriction, transfer, and norm maps for every subgroup inclusion. A Tambara functor is thus a multiplicative enhancement of a Mackey functor, and this is precisely the structure existing on the zeroth equivariant homotopy groups of a strictly commutative G-ring spectrum, see [Bru07, §7.2] and [Ull13]. Tambara functors can equivalently be described [Str12] as grouplike product-preserving functors $\mathsf{Bispan}(\mathbb{F}_G) \to \mathsf{Set}$ (with restrictions, transfers, and norms corresponding to the functoriality in the components R, T, and N of the bispan (5), respectively), and we can therefore think of the equivalence (4) as identifying connective normed G-spectra with *space-valued Tambara functors*.

In fact, we deduce Theorem B from a much more general result: following Bachmann [Bac22], we consider *normed ∞-categories* as functors from suitable span ∞-categories into Cat_∞, and we give a general description of *normed (semi)rings* in this context in terms of product-preserving functors out of an ∞-category of bispans, see Theorems 4.2.4 and 4.3.6. This in particular allows us to deduce a version of Theorem B with fewer normed multiplications, in which case we can describe the corresponding connective normed algebras as a space-valued version of the *incomplete Tambara functors* considered by Blumberg–Hill [BH18].

Multiplicative equivariant K-theory

As a concrete application of Theorem B, we can construct normed multiplicative structures on equivariant algebraic K-theory spectra: Recall that Elmanto and Haugseng [EH23, §4.3] show that if E is a normed G-spectrum, then the assignment

$$H \mapsto \mathsf{Mod}_{E^H}(\mathsf{Sp}_H)$$

extends naturally to a functor

$$\mathsf{Bispan}(\mathbb{F}_G) \longrightarrow \mathsf{Cat}_\infty$$

that preserves products and takes values in the subcategory of stable ∞-categories and polynomial functors. Combining this with the polynomial functoriality of (connective) algebraic K-theory of Barwick, Glasman, Mathew, and Nikolaus [BGMN21], we obtain a space-valued Tambara functor given by

$$H \mapsto \Omega^\infty K(\mathsf{Mod}_{E^H}(\mathsf{Sp}_H)).$$

Now Theorem B identifies this with a normed G-spectrum; as the constructions involved are functorial, we obtain:

Corollary C. *Connective equivariant algebraic K-theory can be enhanced to a functor*

$$K\colon \mathsf{NAlg}_G(\mathsf{Sp}_G) \longrightarrow \mathsf{NAlg}_G(\mathsf{Sp}_G^{\geq 0}).$$

More generally, we obtain normed G-spectra from suitable normed stable ∞-categories [EH23, 4.3.2]. Specializing this as in [EH23, 4.3.9] we in particular obtain a refinement of connective equivariant algebraic K-theory of stable ∞-categories to a functor from symmetric monoidal stable ∞-categories to normed G-spectra. In the case where G is a finite 2-group, an entirely different approach to such a refinement has previously been worked out by Hilman [Hil22b]; to the best of our knowledge, ours is the first construction in this generality.

Related work

During the long history of equivariant infinite loop space theory, a wide range of notions of "G-commutative monoids" have been introduced and studied, for example Shimakawa's *special* Γ-*G-spaces* [Shi89], the operadic models of Guillou–May [GM17], various "ultra-commutative" models [Len20, LS23], and the ∞-categorical model [GM11, Bar17] used in this paper. All of these notions are known to be equivalent to each other [MMO17, Len23, Mar24], and in particular each of them comes with an equivariant recognition theorem relating the corresponding grouplike objects to connective G-spectra.

Since the early days of the subject, much effort went into the search for multiplicative refinements of these comparisons, with several breakthroughs in the last couple of years. In particular, Guillou–May–Merling–Osorno studied multiplicative properties of the operadic machine, culminating in the article [GMMO23] where they refine equivariant infinite loop space theory to an enriched multifunctor, allowing the construction of *non-commutative* G-ring spectra from space-level or categorical data. Yau [Yau24] recently improved this to a *symmetric* enriched multifunctor, which then in particular can also be used to produce *commutative* G-ring spectra.

In contrast to our approach, the aforementioned authors work with *strict* (non-parametrized) algebraic structures on the level of 1-categorical models. While commutative structures on symmetric/orthogonal G-spectra are expected to model ∞-categorical G-normed spectra, a symmetric monoidal or symmetric multifunctor structure on the functor from G-commutative monoids to connective G-spectra does *not* induce any such structure on the inverse functor, and accordingly there is no analogue of our Theorem B known in these settings. In fact, there are serious obstructions to achieving a complete space-level description of connective commutative G-ring spectra along these lines: for example, Lawson [Law09] proved that even for $G = 1$ not all connective commutative ring spectra arise from strictly commutative algebras in Γ-spaces.

Outline

In Section 2 we recall some necessary background about ∞-categories of spans and bispans. We then introduce the framework of *normed monoids*, *normed ∞-categories*, and *normed algebras* in Section 3 as a very mild generalization of work of Bachmann and Hoyois. As the main new result of that section, we construct a *Day convolution* normed structure on certain ∞-categories of product-preserving functors and give a description of normed algebras in it, see Proposition 3.3.1.

Specializing this, we then construct normed ∞-categories of normed monoids in Section 4, which in particular allows us to define *normed (semi)rings*. Combining the description of normed algebras with respect to the Day convolution structure with our results in [CHLL24], we then show that normed rings can be equivalently described as certain higher Tambara functors (Theorem 4.3.6).

In Section 5 we introduce and compare various normed ∞-categories related to equivariant homotopy theory, in particular proving Theorem A. Combining this with the results of the previous section, we then finally deduce Theorem B.

The paper ends with a short appendix on a *Borel construction* due to Hilman [Hil22a] that builds normed G-∞-categories from ordinary symmetric monoidal ∞-categories with G-action, which is used in various constructions in Section 5.

Notations and conventions

- We write $\mathbb{F}$ for the category of finite sets, and $\mathbf{n} := \{1, \ldots, n\}$ for the standard set with n elements. For an ∞-category $\mathscr{C}$, we write $\mathbb{F}[\mathscr{C}]$ for the finite coproduct completion of $\mathscr{C}$. When $\mathscr{C}$ is the orbit category $\mathbf{O}_G$ of a finite group G, we denote the category of finite G-sets by $\mathbb{F}_G = \mathbb{F}[\mathbf{O}_G]$.

- Functors that preserve *finite* products will be referred to as *product-preserving* for short, and similarly for coproducts. We will never speak about arbitrary products and coproducts.

- We write Cat_∞ for the ∞-category of ∞-categories and Spc for the ∞-category of spaces (a.k.a. ∞-groupoids).

- If $\mathscr{C}$ is an ∞-category, then we denote its underlying ∞-groupoid by $\mathscr{C}^{\simeq}$ or $\mathscr{C}_{\mathrm{eq}}$, depending on context.

- We write $\mathsf{Ar}(\mathscr{C}) := \mathsf{Fun}([1], \mathscr{C})$ for the *arrow ∞-category* of $\mathscr{C}$.

- Throughout, we use the word *subcategory* to refer to what is sometimes called a *replete subcategory*: that is, for us a subcategory $\mathscr{C}_0 \subseteq \mathscr{C}$ is always required to contain all equivalences between its objects. A subcategory is called *wide* if it in addition contains all objects.

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2 Spans and bispans

The goal of this section is to recall some basic properties of ∞-categories of spans and bispans, and to formulate conditions guaranteeing that a (bi)span ∞-category admits (co)products.

2.1 Spans

Let us begin by recalling some basic definitions and results concerning ∞-categories of spans. Our main references for this are Barwick's original article [Bar17] and the more recent treatment in [HHLN23].

Definition 2.1.1. A *span pair* $(\mathscr{C}, \mathscr{C}_F)$ consists of an ∞-category $\mathscr{C}$ together with a wide subcategory $\mathscr{C}_F$ of "forward" maps, such that base changes of morphisms in $\mathscr{C}_F$ exist in $\mathscr{C}$ and are again contained in $\mathscr{C}_F$. We write $\mathsf{SpanPair}$ for the ∞-category of span pairs; a morphism $(\mathscr{C}, \mathscr{C}_F) \to (\mathscr{D}, \mathscr{D}_F)$ here is a functor $\mathscr{C} \to \mathscr{D}$ that preserves the forward maps as well as pullbacks along forward maps.

Remark 2.1.2. Both [Bar17] and [HHLN23] work more generally with so-called *adequate triples* $(\mathscr{C}, \mathscr{C}_F, \mathscr{C}_B)$. Span pairs correspond to the special case $\mathscr{C}_B = \mathscr{C}$.

Example 2.1.3. For any ∞-category $\mathscr{C}$, we always have the *minimal* span pair $(\mathscr{C}, \mathscr{C}^{\simeq})$. If $\mathscr{C}$ has all pullbacks, then we also have the *maximal* span pair $(\mathscr{C}, \mathscr{C})$.

[HHLN23, 2.12] constructs a functor

$$\mathsf{Span}\colon \mathsf{SpanPair} \longrightarrow \mathsf{Cat}_\infty,$$

sending a span pair $(\mathscr{C}, \mathscr{C}_F)$ to its *span ∞-category*

$$\mathsf{Span}_F(\mathscr{C}) := \mathsf{Span}(\mathscr{C}, \mathscr{C}_F).$$

This ∞-category has the same objects as $\mathscr{C}$, and a map in $\mathsf{Span}_F(\mathscr{C})$ from x to y is given by a *span*

$$x \xleftarrow{\;b\;} z \xrightarrow{\;f\;} y,$$

where f is in $\mathscr{C}_F$ and b is arbitrary; composition is given by taking pullbacks in $\mathscr{C}$. If $\mathscr{C}$ has all pullbacks, we abbreviate $\mathsf{Span}(\mathscr{C})$ for the span category associated to the span pair $(\mathscr{C}, \mathscr{C})$.

Example 2.1.4 ([HHLN23, 2.15])**.** We have $\mathsf{Span}_{\mathrm{eq}}(\mathscr{C}) = \mathsf{Span}(\mathscr{C}, \mathscr{C}^{\simeq}) \simeq \mathscr{C}^{\mathrm{op}}$.

Remark 2.1.5. The ∞-category $\mathsf{SpanPair}$ has limits, which are computed in Cat_∞ [HHLN23, 2.4], and the functor Span preserves these [HHLN23, 2.18].

We will need the following special case of Barwick's "unfurling" theorem:

Proposition 2.1.6. *Suppose $(\mathcal{B}, \mathcal{B}_F)$ is a span pair and $\Phi\colon \mathcal{B} \to \mathsf{Cat}_\infty$ is a functor such that*

- *for every morphism $f\colon b \to b'$ in $\mathcal{B}_F$, the functor $f_! = \Phi(f)$ has a right adjoint f^*,*
- *and for every pullback square*

$$\begin{array}{ccc} a' & \xrightarrow{f'} & b' \\ {\scriptstyle\alpha}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\beta} \\ a & \xrightarrow[f]{} & b \end{array}$$

in $\mathcal{B}$ with f in $\mathcal{B}_F$, the induced Beck–Chevalley transformation

$$\alpha_! f'^* \longrightarrow f^* \beta_!$$

is an equivalence.

Let $p\colon \mathcal{E} \to \mathcal{B}$ be the cocartesian fibration for Φ, and write $\mathcal{E}_{F\text{-cart}}$ for the subcategory containing the morphisms that are p-cartesian over morphisms in $\mathcal{B}_F$. Then $(\mathcal{E}, \mathcal{E}_{F\text{-cart}})$ is a span pair, p is a morphism of span pairs, and

$$\mathsf{Span}(p)^{\mathrm{op}}\colon \mathsf{Span}_{F\text{-cart}}(\mathcal{E})^{\mathrm{op}} \longrightarrow \mathsf{Span}_F(\mathcal{B})^{\mathrm{op}}$$

is the cocartesian fibration for a functor $\mathsf{Span}_F(\mathcal{B})^{\mathrm{op}} \to \mathsf{Cat}_\infty$ that restricts to Φ on $\mathcal{B}$ and to the functor obtained by passing to right adjoints from Φ on $\mathcal{B}_F^{\mathrm{op}}$.

Proof. It follows from [Bar17, 11.6] that $(\mathcal{E}, \mathcal{E}_{F\text{-cart}})$ is a span pair, that p is a morphism of span pairs, and that $\mathsf{Span}(p)^{\mathrm{op}}$ is a cocartesian fibration. For the convenience of the reader we recall the proof, giving some additional details. It follows from [Bar17, 11.2] that $(\mathcal{E}, \mathcal{E}_{F\text{-cart}})$ is a span pair, and the pullback

$$\begin{array}{ccc} w & \xrightarrow{\bar{f}'} & z \\ {\scriptstyle\bar{g}'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\bar{g}} \\ x & \xrightarrow[\bar{f}]{} & y \end{array}$$

of $\bar{f}\colon x \to y$ in $\mathcal{E}_{F\text{-cart}}$ over $f\colon a \to b$ along a morphism $\bar{g}\colon z \to y$ over $g\colon c \to b$ is obtained by taking the pullback

$$\begin{array}{ccc} d & \xrightarrow{f'} & c \\ {\scriptstyle g'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\ a & \xrightarrow[f]{} & b \end{array}$$

in $\mathcal{B}$, picking a p-cartesian morphism $\bar{f}'\colon w \to z$ over f', and letting $\bar{g}'$ be the unique factorization of $\bar{g}\bar{f}'$ through the p-cartesian morphism $\bar{f}$.

To show that $\mathsf{Span}(p)^{\mathrm{op}}$ is a cocartesian fibration, it suffices to show that any span

$$x \xleftarrow{f} y \xrightarrow{g} z$$

in $\mathsf{Span}_{F\text{-cart}}(\mathcal{E})^{\mathrm{op}}$, where f is p-cartesian over $\mathcal{B}_F$ and g is p-cocartesian, is a cocartesian morphism, since then $\mathsf{Span}_{F\text{-cart}}(\mathcal{E})^{\mathrm{op}}$ has all cocartesian lifts of morphisms in $\mathsf{Span}_F(\mathcal{B})^{\mathrm{op}}$. To see this we apply [Bar17, 12.2], in the guise of [HHLN23, 3.1]:

- Condition (1) is immediate since p is a cocartesian fibration.
- Unwinding the definitions, condition (2) says that given a pullback square

$$\begin{array}{ccc} a & \xrightarrow{g'} & b' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ b & \xrightarrow[g]{} & c \end{array}$$

  in $\mathcal{B}$ with f in $\mathcal{B}_F$ and a commutative square

$$\begin{array}{ccc} f'^{*}x & \xrightarrow{\gamma} & y \\ {\scriptstyle \bar{f}'}\downarrow & & \downarrow{\scriptstyle \phi} \\ x & \xrightarrow[\bar{g}]{} & g_! x \end{array}$$

  where $\bar{g}$ is p-cocartesian over g and $\bar{f}'$ is p-cartesian over f', then γ is p-cocartesian if and only if ϕ lies in $\mathcal{E}_{F\text{-cart}}$ and the square is a pullback. Indeed, in the former case ϕ factors as the canonical map $g'_! f'^{*}x \to f^{*}g_! x$ followed by a cartesian morphism over f, while in the latter case γ factors as a cocartesian morphism over g' followed by the same map. Since this Beck–Chevalley map is by assumption invertible, the two conditions are equivalent.

It remains to identify the fibrations we get over $\mathcal{B}$ and $\mathcal{B}_F^{\mathrm{op}}$. Since the functor $\mathsf{Span}(-)$ is compatible with pullbacks, we see that over $\mathcal{B}$ we recover $p\colon \mathcal{E} \to \mathcal{B}$, while over $\mathcal{B}_F$ we get $\mathsf{Span}(\mathcal{E}_F, \mathcal{E}_{F,\mathrm{fw}}, \mathcal{E}_{F\text{-cart}}) \to \mathcal{B}_F^{\mathrm{op}}$, where $\mathcal{E}_F := \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_F$ and $\mathcal{E}_{F,\mathrm{fw}}$ denotes the subcategory of morphisms that map to equivalences in $\mathcal{B}_F^{\mathrm{op}}$. This is the cocartesian fibration that describes the same functor as the cartesian fibration $\mathcal{E}_F \to \mathcal{B}_F$, by [BGN18, 1.4] or [HHLN23, 3.18]. □

2.2 Products in span ∞-categories

In this subsection we provide criteria for ∞-categories of spans to have products and coproducts.

Definition 2.2.1. Recall that an ∞-category $\mathscr{C}$ is called *extensive* if $\mathscr{C}$ has finite coproducts and the coproduct functor

$$\amalg\colon \prod_{i=1}^{n} \mathscr{C}_{/x_i} \longrightarrow \mathscr{C}_{/\coprod_i x_i}$$

is an equivalence for all objects $x_1, \ldots, x_n \in \mathscr{C}$. A span pair $(\mathscr{C}, \mathscr{C}_F)$ is called *extensive* if the following conditions are satisfied:

- $\mathscr{C}$ is extensive,
- the morphisms in $\mathscr{C}_F$ are closed under finite coproducts,
- and for every $x \in \mathscr{C}$, the maps $\emptyset \to x$ and $x \amalg x \to x$ are in $\mathscr{C}_F$.

More generally, we say that $(\mathscr{C}, \mathscr{C}_F)$ is *weakly extensive* if

- $\mathscr{C}$ has finite coproducts,
- the morphisms in $\mathscr{C}_F$ are closed under finite coproducts,
- and the coproduct functor

  $$\amalg\colon \prod_{i=1}^{n} \mathscr{C}^F_{/x_i} \longrightarrow \mathscr{C}^F_{/\amalg_i x_i}$$

  is an equivalence for all $n \geq 0$ and $x_1, \ldots, x_n \in \mathscr{C}$. Here $\mathscr{C}^F_{/y}$ denotes the full subcategory of $\mathscr{C}_{/y}$ spanned by those maps $z \to y$ that belong to F.

If $\mathscr{C}$ is an extensive ∞-category, then a wide subcategory $\mathscr{C}_F \subseteq \mathscr{C}$ is called a *(weakly) extensive subcategory* if the pair $(\mathscr{C}, \mathscr{C}_F)$ is (weakly) extensive span pair.

Remark 2.2.2. Note that a span pair $(\mathscr{C}, \mathscr{C}_F)$ is extensive if and only if $\mathscr{C}$ and $\mathscr{C}_F$ are both extensive ∞-categories and the inclusion $\mathscr{C}_F \hookrightarrow \mathscr{C}$ preserves finite coproducts. Also note that every extensive span pair is weakly extensive.

Remark 2.2.3. Let $(\mathscr{C}, \mathscr{C}_F)$ be a span pair such that the morphisms in $\mathscr{C}_F$ are closed under finite coproducts. Then the coproduct functor $\prod_{i=1}^{n} \mathscr{C}^F_{/x_i} \to \mathscr{C}^F_{/\amalg_i x_i}$ admits a right adjoint $\mathscr{C}^F_{/\amalg_i x_i} \to \prod_{i=1}^{n} \mathscr{C}^F_{/x_i}$ given by pullback along the maps $x_i \to \amalg_i x_i$, see [Lur09, 5.2.5.1], and it follows that $(\mathscr{C}, \mathscr{C}_F)$ is weakly extensive if and only if this functor is an equivalence. In this case, the canonical squares

$$\begin{array}{ccc} y_i & \longrightarrow & \amalg_i y_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle \amalg_i f_i} \\ x_i & \longrightarrow & \amalg_i x_i \end{array}$$

are pullback squares for all morphisms $f_i\colon y_i \to x_i$ in $\mathscr{C}_F$.

One may similarly characterize extensiveness of $\mathscr{C}$ by means of the right adjoint $\mathscr{C}_{/\amalg_i x_i} \to \prod_{i=1}^n \mathscr{C}_{/x_i}$ to the coproduct functor; in this case we need to assume that the morphism $\emptyset \to x$ is in $\mathscr{C}_F$ for each object x to guarantee that the relevant pullbacks exist.

Proposition 2.2.4. *A span pair $(\mathscr{C}, \mathscr{C}_F)$ is weakly extensive if and only if the following conditions hold:*

- *the ∞-category $\mathscr{C}$ has finite coproducts,*
- *the coproduct functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a morphism of span pairs (i.e. morphisms in $\mathscr{C}_F$ are closed under coproducts and coproducts of pullbacks of morphisms in $\mathscr{C}_F$ are again pullbacks),*
- *and the commutative squares*

$$\begin{array}{ccc} x \amalg x & \longrightarrow & y \amalg y \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array} \qquad \begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

are pullbacks for all morphisms $x \to y$ in $\mathscr{C}_F$.

The pair $(\mathscr{C}, \mathscr{C}_F)$ is extensive if and only if in addition we have:

- *the above two squares are pullbacks for* ***any*** *morphism $x \to y$ in $\mathscr{C}$,*
- *the maps $\emptyset \to x$ and $x \amalg x \to x$ are in $\mathscr{C}_F$ for all $x \in \mathscr{C}$.*

Proof. First assume that $(\mathscr{C}, \mathscr{C}_F)$ is weakly extensive. By assumption, $\mathscr{C}$ has finite coproducts and morphisms in $\mathscr{C}_F$ are closed under finite coproducts. For the second condition, consider pullback squares

$$\begin{array}{ccc} x'_i & \xrightarrow{h_i} & x_i \\ {\scriptstyle f'_i}\downarrow & \lrcorner & \downarrow{\scriptstyle f_i} \\ y'_i & \xrightarrow{g_i} & y_i \end{array}$$

for $i = 1, \ldots, n$, with $f_i \in \mathscr{C}_F$. We need to show that their coproduct is again a pullback square, or, equivalently, that the map

$$\amalg_i x'_i \longrightarrow (\amalg_i x_i) \times_{\amalg_i y_i} (\amalg_i y'_i)$$

is an equivalence in $\mathscr{C}^F_{/\amalg_i y'_i}$. But since $(\mathscr{C}, \mathscr{C}_F)$ is weakly extensive, this may be checked after pulling back along each of the maps $y'_i \to \amalg_i y'_i$, where it becomes clear. For the third condition, we must show that the maps $x \amalg x \to (y \amalg y) \times_y x$ in $\mathscr{C}^F_{/y \amalg y}$ and $\emptyset \to \emptyset \times_y x$ in $\mathscr{C}^F_{/\emptyset}$ are equivalences. The latter is clear since

$\mathscr{C}^F_{/\emptyset} \xrightarrow{\sim} *$. For the former, it again suffices to check this after pulling back along the two inclusions $y \to y \amalg y$, where it is also clear.

For the converse, assume that the first three conditions in the proposition are satisfied. We show that $(\mathscr{C}, \mathscr{C}_F)$ is a weakly extensive span pair. It suffices to prove that the pullback functor

$$\mathscr{C}^F_{/\coprod_{i=1}^n x_i} \longrightarrow \prod_{i=1}^n \mathscr{C}^F_{/x_i}$$

is an equivalence when $n = 0$ and $n = 2$. For $n = 0$ we want to show that $\mathscr{C}^F_{/\emptyset} \simeq *$, which follows because for $x \to \emptyset$ in $\mathscr{C}_F$ we have a pullback square

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & \lrcorner & \downarrow \\ x & \longrightarrow & \emptyset, \end{array}$$

so that $\emptyset \to x$ is an equivalence. For $n = 2$, the coproduct functor determines a left adjoint $\amalg \colon \mathscr{C}^F_{/x_1} \times \mathscr{C}^F_{/x_2} \to \mathscr{C}^F_{/x_1 \amalg x_2}$ of the pullback functor by [Lur09, 5.2.5.1], and it will suffice that both the unit and the counit are equivalences. For the counit, consider a map $y \to x_1 \amalg x_2$ in $\mathscr{C}_F$, and let $y_i \to x_i$ be the pullback of y along the inclusion of x_i in the coproduct, which is again in $\mathscr{C}_F$; we must show that the canonical map $y_1 \amalg y_2 \to y$ is an equivalence. To see this, consider the commutative diagram

$$\begin{array}{ccccc} y_1 \amalg y_2 & \longrightarrow & y \amalg y & \longrightarrow & y \\ \downarrow & & \downarrow & & \downarrow \\ x_1 \amalg x_2 & \longrightarrow & (x_1 \amalg x_2) \amalg (x_1 \amalg x_2) & \longrightarrow & x_1 \amalg x_2. \end{array}$$

Here the left square is cartesian since it's a coproduct of two pullback squares along morphisms in $\mathscr{C}_F$, and the right square is cartesian since it's a square of fold maps for a morphism in $\mathscr{C}_F$. The composite square is then cartesian, and the bottom horizontal composite is the identity, which implies that the top horizontal composite is indeed an equivalence.

We now show that the unit of the adjunction is an equivalence. Given morphisms $y_1 \to x_1$ and $y_2 \to x_2$ in $\mathscr{C}_F$, this amounts to showing that the canonical squares

$$\begin{array}{ccc} y_i & \longrightarrow & y_1 \amalg y_2 \\ \downarrow & & \downarrow \\ x_i & \longrightarrow & x_1 \amalg x_2 \end{array}$$

are pullback squares. Writing $x_1 = x_1 \amalg \emptyset$ and similarly for x_2, y_1 and y_2, these squares can be expressed as a coproduct of squares we know are pullbacks along morphisms in $\mathscr{C}_F$, hence are pullback squares by assumption. This finishes the proof of the characterization of being weakly extensive.

The proof for extensive span pairs is identical; the additional assumption that $\mathscr{C}_F$ contains the maps $\emptyset \to x$ and $x \amalg x \to x$ is to ensure that all the relevant pullbacks that appear in the proof exist in $\mathscr{C}$. □

We will now show that the extensiveness properties on a span pair imply good behavior of products and coproducts in the associated span ∞-category.

Proposition 2.2.5 (cf. [BH21, C.3])**.** *Suppose $(\mathscr{C}, \mathscr{C}_F)$ is a span pair.*

(1) If $(\mathscr{C}, \mathscr{C}_F)$ is weakly extensive, then the coproduct in $\mathscr{C}$ gives a product in $\mathsf{Span}_F(\mathscr{C})$.

(2) If $(\mathscr{C}, \mathscr{C}_F)$ is extensive, then the coproduct in $\mathscr{C}$ is also a coproduct in $\mathsf{Span}_F(\mathscr{C})$. Moreover, the ∞-category $\mathsf{Span}_F(\mathscr{C})$ is semiadditive.

For the last statement, recall that an ∞-category $\mathscr{D}$ is called *semiadditive* if it admits finite products and coproducts, the unique morphism $\emptyset \to *$ is an equivalence, and for all $x_1, x_2 \in \mathscr{D}$, the morphism

$$\begin{pmatrix} \mathrm{id}_{x_1} & 0 \\ 0 & \mathrm{id}_{x_2} \end{pmatrix} : x_1 \amalg x_2 \longrightarrow x_1 \times x_2$$

is an equivalence, where 0 denotes the unique map that factors through $*$.

Proof. For the first part apply [BH21, C.21(2)] together with the characterization from Proposition 2.2.4 to the adjunctions

$$\amalg : \mathscr{C} \times \mathscr{C} \rightleftarrows \mathscr{C} : \Delta \qquad \text{and} \qquad \{\emptyset\} : * \rightleftarrows \mathscr{C} : p.$$

For the second part apply part (1) of the same corollary, to see that $\emptyset$ is also initial and $\mathsf{Span}(\amalg)$ is also *left* adjoint to the restriction, so that $\mathsf{Span}_F(\mathscr{C})$ has finite coproducts. It is then clear that $\mathsf{Span}_F(\mathscr{C})$ is pointed. To see that it is semiadditive, we now observe that in any pointed ∞-category with finite (co)products the canonical comparison map $x \amalg y \to x \times y$ factors as

$$x \amalg y \simeq (x \times 0) \amalg (0 \times y) \longrightarrow (x \amalg 0) \times (0 \amalg y) \simeq x \times y,$$

so it is an equivalence in the case of $\mathsf{Span}_F(\mathscr{C})$ as the coproduct functor is a right adjoint by the above, and hence preserves products. □

Remark 2.2.6. Our definition of "extensive span pairs" is closely related to Barwick's *disjunctive triples* [Bar17, 5.2]. Thus, Proposition 2.2.5 is essentially a variant of the proof of semiadditivity in [Bar17, 4.3 and 5.8].

2.3 Bispans

Finally, let us recall ∞-categories of *bispans* from [EH23].

Definition 2.3.1. A *bispan triple* $(\mathscr{C}, \mathscr{C}_F, \mathscr{C}_L)$ consists of an ∞-category $\mathscr{C}$ together with two wide subcategories $\mathscr{C}_F, \mathscr{C}_L \subseteq \mathscr{C}$ such that both $(\mathscr{C}, \mathscr{C}_F)$ and $(\mathsf{Span}_F(\mathscr{C})^{\mathrm{op}}, \mathscr{C}_L)$ are span pairs. In this case, we define

$$\mathsf{Bispan}_{F,L}(\mathscr{C}) := \mathsf{Span}_L(\mathsf{Span}_F(\mathscr{C})^{\mathrm{op}}).$$

For $\mathscr{C}_L = \mathscr{C}$ we abbreviate this to $\mathsf{Bispan}_F(\mathscr{C})$, and if moreover also $\mathscr{C}_F = \mathscr{C}$, we will simply write $\mathsf{Bispan}(\mathscr{C})$.

Remark 2.3.2. By [EH23, 2.5.2(1)], a triple $(\mathscr{C}, \mathscr{C}_F, \mathscr{C}_L)$ is a bispan triple if and only if it satisfies the following more explicit conditions:

(1) Both $(\mathscr{C}, \mathscr{C}_F)$ and $(\mathscr{C}, \mathscr{C}_L)$ are span pairs.

(2) Let $\mathscr{C}^L_{/x} \subseteq \mathscr{C}_{/x}$ again denote the full subcategory spanned by the maps to x that lie in $\mathscr{C}_L$. Then the functor $f^*\colon \mathscr{C}^L_{/y} \to \mathscr{C}^L_{/x}$ given by pullback along f has a right adjoint f_* for every map f in $\mathscr{C}_F$.

(3) For any pullback square

$$\begin{array}{ccc} x' & \xrightarrow{f'} & y' \\ {\scriptstyle\xi}\downarrow & \lrcorner & \downarrow{\scriptstyle\eta} \\ x & \xrightarrow[f]{} & y \end{array}$$

with f a map in $\mathscr{C}_F$, the commutative square

$$\begin{array}{ccc} \mathscr{C}^L_{/y} & \xrightarrow{f^*} & \mathscr{C}^L_{/x} \\ {\scriptstyle\eta^*}\downarrow & & \downarrow{\scriptstyle\xi^*} \\ \mathscr{C}^L_{/y'} & \xrightarrow[f'^*]{} & \mathscr{C}^L_{/x'} \end{array}$$

is *right adjointable*, i.e. the Beck–Chevalley transformation $\eta^* f_* \to f'_* \xi^*$ is invertible.

Note that if $\mathscr{C}_L = \mathscr{C}$, then condition (2) precisely says that $\mathscr{C}$ is locally cartesian closed. In this case, condition (3) is actually automatic as it can be checked after passing to left adjoints.

Definition 2.3.3. Let $(\mathscr{C}, \mathscr{C}_F, \mathscr{C}_L)$ and $(\mathscr{D}, \mathscr{D}_F, \mathscr{D}_L)$ be bispan triples. A *morphism of bispan triples* is a functor $\Phi\colon \mathscr{C} \to \mathscr{D}$ that induces morphisms of span pairs $(\mathscr{C}, \mathscr{C}_F) \to (\mathscr{D}, \mathscr{D}_F)$, $(\mathscr{C}, \mathscr{C}_L) \to (\mathscr{D}, \mathscr{D}_L)$, and

$$(\mathsf{Span}_F(\mathscr{C})^{\mathrm{op}}, \mathscr{C}_L) \longrightarrow (\mathsf{Span}_F(\mathscr{D})^{\mathrm{op}}, \mathscr{D}_L).$$

Remark 2.3.4. In order to unpack the final condition, let us describe pullbacks in $\mathsf{Span}_F(\mathscr{C})^{\mathrm{op}}$ along morphisms in $\mathscr{C}_L$ more concretely, for which it will be enough to describe pullbacks of backwards and forwards maps individually:

- Given a forward map $x \xleftarrow{=} x \xrightarrow{g} y$, its pullback along a map $l\colon z \to y$ in $\mathscr{C}_L$ is given by

  $$\begin{array}{ccccc} d & = & d & \longrightarrow & z \\ \| & & \| & \lrcorner & \| \\ d & = & d & \longrightarrow & z \\ \downarrow & \llcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle l} \\ x & = & x & \xrightarrow{g} & y, \end{array}$$

  see [EH23, 2.5.10].

- Given a backwards map $x \xleftarrow{f} y \xrightarrow{=} y$ with f in $\mathscr{C}_F$, [EH23, 2.5.12] shows that its pullback along $l\colon z \to y$ is of the form

  $$\begin{array}{ccccc} e & \longleftarrow & d & \xrightarrow{\epsilon} & z \\ \| & & \| & \lrcorner & \| \\ e & \longleftarrow & d & \xrightarrow{\epsilon} & z \\ {\scriptstyle f_*l}\downarrow & \llcorner & \downarrow{\scriptstyle f^*f_*l} & & \downarrow{\scriptstyle l} \\ x & \xleftarrow[f]{} & y & = & y, \end{array}$$

  where ϵ is the counit map $f^*f_*l \to l$.

In particular, we see that if $\Phi\colon \mathscr{C} \to \mathscr{D}$ is such that $(\mathscr{C}, \mathscr{C}_L) \to (\mathscr{D}, \mathscr{D}_L)$ and $(\mathscr{C}, \mathscr{C}_F) \to (\mathscr{D}, \mathscr{D}_F)$ are maps of span pairs, then Φ is a a map of bispan triples if and only if the Beck–Chevalley map

$$\Phi \circ f_* \longrightarrow \Phi(f)_* \circ \Phi$$

induced by the commutative square

$$\begin{array}{ccc} \mathscr{C}^L_{/y} & \xrightarrow{f^*} & \mathscr{C}^L_{/x} \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathscr{D}^L_{/\Phi(y)} & \xrightarrow[\Phi(f)^*]{} & \mathscr{D}^L_{/\Phi(x)} \end{array}$$

is an equivalence.

Proposition 2.3.5. *Suppose $(\mathscr{C}, \mathscr{C}_F, \mathscr{C}_L)$ is a bispan triple such that both of the span pairs $(\mathscr{C}, \mathscr{C}_F)$ and $(\mathscr{C}, \mathscr{C}_L)$ are weakly extensive. Then the span pair $(\mathsf{Span}_F(\mathscr{C})^{\mathrm{op}}, \mathscr{C}_L)$ is also weakly extensive. In particular, $\mathsf{Bispan}_{F,L}(\mathscr{C})$ has finite products, and these are given by coproducts in $\mathscr{C}$.*

Proof. By Proposition 2.2.5(1), the coproduct in $\mathscr{C}$ gives a product in $\mathsf{Span}_F(\mathscr{C})$ and hence a coproduct in $\mathsf{Span}_F(\mathscr{C})^{\mathrm{op}}$.

We now claim that pullback squares along $\mathscr{C}_L$ are closed under finite coproducts. Using the explicit description of pullbacks from the previous remark, the only non-obvious part of this is that, given $f_i \colon y_i \to x_i$ in $\mathscr{C}_F$ and $l_i \colon z_i \to y_i$ in $\mathscr{C}_L$ for $i = 1, 2$, we have

$$(f_1 \amalg f_2)_*(l_1 \amalg l_2) \simeq f_{1,*}(l_1) \amalg f_{2,*}(l_2).$$

To see this we use Proposition 2.2.4 with the argument from [EH23, 2.6.12]: Given $g \colon a \to x_1 \amalg x_2$ in $\mathscr{C}_L$, if $g_i \colon a_i \to x_i$ for $i = 1, 2$ are the pullbacks along the summand inclusions, we get

$$\begin{aligned}
\mathsf{Map}_{\mathscr{C}^L_{/x_1 \amalg x_2}}(g, f_{1,*}(l_1) \amalg f_{2,*}(l_2)) &\simeq \mathsf{Map}_{\mathscr{C}^L_{/x_1}}(g_1, f_{1,*}(l_1)) \times \mathsf{Map}_{\mathscr{C}^L_{/x_2}}(g_2, f_{2,*}(l_2)) \\
&\simeq \mathsf{Map}_{\mathscr{C}^L_{/y_1}}(f_1^* g_1, l_1) \times \mathsf{Map}_{\mathscr{C}^L_{/y_2}}(f_2^* g_2, l_2) \\
&\simeq \mathsf{Map}_{\mathscr{C}^L_{/y_1 \amalg y_2}}(f_1^* g_1 \amalg f_2^* g_2, l_1 \amalg l_2) \\
&\simeq \mathsf{Map}_{\mathscr{C}^L_{/y_1 \amalg y_2}}((f_1 \amalg f_2)^*(g_1 \amalg g_2), l_1 \amalg l_2) \\
&\simeq \mathsf{Map}_{\mathscr{C}^L_{/x_1 \amalg x_2}}(g, (f_1 \amalg f_2)_*(l_1 \amalg l_2)).
\end{aligned}$$

The remaining part of the conditions for a weakly extensive span pair hold because they by assumption hold for $(\mathscr{C}, \mathscr{C}_L)$. With this established, the final statement is another instance of Proposition 2.2.5(1). □

3 Normed ∞-categories

We recall the definition of normed ∞-categories and normed algebras from [BH21, Bac22] and give various examples of normed ∞-categories.

3.1 Normed monoids

Our starting point is the following generalization of the notion of a commutative monoid:

Definition 3.1.1. Let $F = (\mathscr{F}, \mathscr{F}_N)$ be a weakly extensive span pair and let $\mathscr{C}$ be an ∞-category with finite products. An *F-normed monoid in $\mathscr{C}$* is a product-preserving functor

$$M \colon \mathsf{Span}_N(\mathscr{F}) \longrightarrow \mathscr{C}.$$

We denote its contravariant functoriality by $f^* \colon M(Y) \to M(X)$ for morphisms $f \colon X \to Y$ in $\mathscr{F}$, and refer to these maps as *restriction maps*. We denote its covariant functoriality by either $n_\oplus \colon M(X) \to M(Y)$ or $n_\otimes \colon M(X) \to M(Y)$ for morphisms $n\colon X \to Y$ in $\mathscr{F}_N$, and refer to these maps as *(additive/multiplicative) norm maps*. We write

$$\mathsf{NMon}_F(\mathscr{C}) := \mathsf{Fun}^{\times}(\mathsf{Span}_N(\mathscr{F}), \mathscr{C})$$

for the full subcategory of $\mathsf{Fun}(\mathsf{Span}_N(\mathscr{F}), \mathscr{C})$ spanned by the F-normed monoids.

Observation 3.1.2. If F is actually extensive (and not only weakly so), then semiadditivity of $\mathsf{Span}_N(\mathscr{F})$ implies that all its objects carry unique commutative monoid structures, and so the values $M(X)$ of an F-normed monoid at $X \in \mathscr{F}$ inherit commutative monoid structures in $\mathscr{C}$. In fact, by [GGN15, 2.5] we get an equivalence

$$\mathsf{NMon}_F(\mathscr{C}) \simeq \mathsf{NMon}_F(\mathsf{CMon}(\mathscr{C})) \tag{6}$$

inverse to the forgetful functor.

Observation 3.1.3. If $\mathscr{C}$ is presentable and $\mathscr{F}$ is small, then $\mathsf{NMon}_F(\mathscr{C})$ is an accessible localization of $\mathsf{Fun}(\mathsf{Span}_N(\mathscr{F}), \mathscr{C})$, and so is a presentable ∞-category.

Let us discuss various examples of normed monoids:

Example 3.1.4. Our main example of an extensive span pair is the pair $F = (\mathbb{F}_G, \mathbb{F}_G)$ where $\mathbb{F}_G$ is the category of finite G-sets for a finite group G. In this case, F-normed monoids in $\mathscr{C}$ are also known as $\mathscr{C}$-valued *G-Mackey functors*:

$$\mathsf{Mack}_G(\mathscr{C}) := \mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}_G), \mathscr{C}).$$

More generally, we obtain a notion of *incomplete G-Mackey functors* by taking $F = (\mathbb{F}_G, I)$ for some weakly extensive subcategory $I \subseteq \mathbb{F}_G$:

$$\mathsf{Mack}^I_G(\mathscr{C}) := \mathsf{NMon}_{(\mathbb{F}_G, I)}(\mathscr{C}) = \mathsf{Fun}^{\times}(\mathsf{Span}_I(\mathbb{F}_G), \mathscr{C}).$$

These are most typically considered when $I \subseteq \mathbb{F}_G$ is in fact an extensive subcategory of $\mathbb{F}_G$ (and not only weakly extensive), in which case I is usually called an *indexing system for G* [BH18, 1.2 and 1.4].

Remark 3.1.5. To see how our approach relates to classical equivariant infinite loop space theory, consider an indexing system $I \subseteq \mathbb{F}_G$. By the discussion after [Rub21, 3.9], we can associate to I a so-called *N_∞-operad* $\mathscr{O}$ in G-spaces, and all N_∞-operads arise this way; see also [GW18, BP21]. The main result of [Mar24] connects space-valued Mackey-functors to N_∞-algebras by showing that $\mathsf{Mack}^I_G(\mathsf{Spc})$ is equivalent to the Dwyer–Kan localization of the 1-category of $\mathscr{O}$-algebras in G-spaces at a certain class of equivariant weak equivalences.

Example 3.1.6. Specializing example Example 3.1.4 to the trivial group, we obtain the extensive span pair $(\mathbb{F}, \mathbb{F})$, where $\mathbb{F}$ is the category of finite sets. By [BH21, C.1] there is an equivalence

$$\mathsf{NMon}_{\mathbb{F}}(\mathscr{C}) \simeq \mathsf{CMon}(\mathscr{C})$$

between $\mathbb{F}$-normed monoids in $\mathscr{C}$ and commutative monoids in $\mathscr{C}$, defined as functors $\mathbb{F}_* \to \mathscr{C}$ satisfying the Segal condition.

Example 3.1.7. Let $\mathscr{F}$ be an extensive ∞-category and let $\mathscr{F}_{\text{fold}}$ be the wide subcategory whose morphisms are finite coproducts of fold maps $\coprod_n x \to x$ for $x \in \mathscr{F}$ and $n \geq 0$. Then the pair $(\mathscr{F}, \mathscr{F}_{\text{fold}})$ is an extensive span pair, and [BH21, C.5] provides an equivalence

$$\mathsf{NMon}_{(\mathscr{F},\mathscr{F}_{\text{fold}})}(\mathscr{C}) \simeq \mathsf{Fun}^{\times}(\mathscr{F}^{\text{op}}, \mathsf{CMon}(\mathscr{C})).$$

Example 3.1.8. Given a span pair $F = (\mathscr{F}, \mathscr{F}_N)$ and an object $x \in \mathscr{C}$, we may consider the wide subcategory $\mathscr{F}_{/x,N} := \mathscr{F}_{/x} \times_{\mathscr{F}} \mathscr{F}_N$ of the slice $\mathscr{F}_{/x}$ consisting of those morphisms over x that are contained in $\mathscr{F}_N$. We note that $F_{/x} := (\mathscr{F}_{/x}, \mathscr{F}_{/x,N})$ is again a span pair, which is (weakly) extensive if $(\mathscr{F}, \mathscr{F}_N)$ is so. In particular we may speak of $F_{/x}$-normed monoids in $\mathscr{C}$.

Example 3.1.9. Let T be any small ∞-category, and let $\mathbb{F}[T]$ be the ∞-category obtained by freely adjoining finite coproducts to T, i.e. $\mathbb{F}[T]$ is the full subcategory of the ∞-category of presheaves spanned by finite coproducts of representables. An *orbital subcategory of* T [CLL23a, 4.2.2] is a wide subcategory $P \subseteq T$ such that $(\mathbb{F}[T], \mathbb{F}[P])$ is a span pair. In this case, $(\mathbb{F}[T], \mathbb{F}[P])$ is always extensive: indeed, pullbacks in $\mathbb{F}[T]$ are also pullbacks in $\mathsf{Fun}(T^{\text{op}}, \mathsf{Spc})$ as $\mathbb{F}[T]$ contains all representables, whence it suffices to check the compatibility axioms between coproducts and pullbacks in Spc, which is straightforward.

In particular, if T is any *orbital category* in the sense of [Nar16, 4.1] (i.e. T is orbital as a subcategory of itself), then $(\mathbb{F}[T], \mathbb{F}[T])$ is extensive. Note that for $T = \mathbf{O}_G$ the *orbit category* of G (i.e. the 1-category of finite transitive G-sets), we precisely recover Example 3.1.4.

Remark 3.1.10. If $F_T := (\mathbb{F}[T], \mathbb{F}[T])$ is the extensive span pair arising from an orbital ∞-category T, then our definition of F_T-normed monoids fits into the framework for algebraic structures defined by Segal conditions from [CH21]: We can endow $\mathsf{Span}(\mathbb{F}[T])$ with the structure of an *algebraic pattern* where the inert–active factorization system is that given by the backwards and forwards maps, and the objects of T are the elementary objects. Then a Segal $\mathsf{Span}(\mathbb{F}[T])$-object in $\mathscr{C}$ is a functor $M\colon \mathsf{Span}(\mathbb{F}[T]) \to \mathscr{C}$ such that

$$M(\textstyle\coprod_i t_i) \xrightarrow{\sim} \prod_{i=1}^n M(t_i)$$

for all $t_i \in T$, which is equivalent to M preserving finite products.

3.2 Normed ∞-categories and normed algebras

In this subsection we fix a weakly extensive span pair $F = (\mathscr{F}, \mathscr{F}_N)$. Specializing Definition 3.1.1 to Cat_∞ leads to the following definition:

Definition 3.2.1 (Bachmann)**.** An *F-normed ∞-category* is an F-normed monoid in Cat_∞, i.e. a product-preserving functor $\mathscr{C}\colon \mathsf{Span}_N(\mathscr{F}) \to \mathsf{Cat}_\infty$. We denote its contravariant functoriality by $f^*\colon \mathscr{C}(y) \to \mathscr{C}(x)$ for morphisms $f\colon x \to y$ in $\mathscr{F}$, and denote its covariant functoriality by $f_\otimes\colon \mathscr{C}(x) \to \mathscr{C}(y)$ whenever f is in $\mathscr{F}_N$.

Remark 3.2.2. When F is actually an extensive span pair, our definition of a normed ∞-category is identical to that of Bachmann [Bac22, 3.1].

Remark 3.2.3. Since the product in $\mathsf{Span}_N(\mathcal{F})$ is the coproduct in $\mathcal{F}$, a functor $\mathsf{Span}_N(\mathcal{F}) \to \mathsf{Cat}_\infty$ is an F-normed ∞-category if and only if its restriction to $\mathcal{F}^{\mathrm{op}}$ preserves finite products. We will sometimes refer to product-preserving functors $\mathcal{F}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ as *$\mathcal{F}$-∞-categories*, and refer to the restriction of an F-normed ∞-category $\mathcal{C}$ to $\mathcal{F}^{\mathrm{op}}$ as the *underlying $\mathcal{F}$-∞-category* of $\mathcal{C}$. Similarly, we may sometimes refer to F-normed ∞-categories as *N-normed $\mathcal{F}$-∞-categories* whenever we wish to emphasize the collection of morphisms $\mathcal{F}_N$ along which we have norms.

Note that for $F_T = (\mathbb{F}[T], \mathbb{F}[T])$, T some orbital ∞-category, an $\mathbb{F}[T]$-∞-category is equivalently a functor $T^{\mathrm{op}} \to \mathsf{Cat}_\infty$ by the universal property of finite coproduct completion. This is the definition of a *T-∞-category* used e.g. in [BDG+16, Nar16, CLL23a].

Notation 3.2.4. Given an F-normed structure on an ∞-category $\mathcal{C}$, we will denote the corresponding cocartesian and cartesian fibrations by

$$\mathcal{C}^{\otimes} \longrightarrow \mathsf{Span}_N(\mathcal{F}), \quad \mathcal{C}_{\otimes} \longrightarrow \mathsf{Span}_N(\mathcal{F})^{\mathrm{op}}.$$

We say that a morphism in $\mathcal{C}^{\otimes}$ is *inert* if it is cocartesian over a backwards morphism in $\mathsf{Span}_N(\mathcal{F})$; similarly, a morphism in $\mathcal{C}_{\otimes}$ is *inert* if it is cartesian over a (reversed) backwards morphism in $\mathsf{Span}_N(\mathcal{F})^{\mathrm{op}}$.

Definition 3.2.5. Suppose $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes} \to \mathsf{Span}_N(\mathcal{F})$ are F-normed ∞-categories. An *F-normed functor* from $\mathcal{C}$ to $\mathcal{D}$ is a commutative triangle

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{\quad\Phi\quad} & \mathcal{D}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathsf{Span}_N(\mathcal{F}) & \end{array}$$

where Φ preserves cocartesian morphisms. We say that Φ is *lax F-normed* if it instead only preserves inert morphisms. We write

$$\mathsf{Fun}^{\mathrm{lax}}_{/\mathsf{Span}_N(\mathcal{F})}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \subseteq \mathsf{Fun}_{/\mathsf{Span}_N(\mathcal{F})}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$$

for the full subcategory spanned by the lax F-normed functors.

Remark 3.2.6. In the non-parametrized case, i.e. the case $F = (\mathbb{F}, \mathbb{F})$, it follows from [BHS22, 5.1.15] that this definition of lax symmetric monoidal functors agrees with the more standard one, with $\mathbb{F}_*$ in place of $\mathsf{Span}(\mathbb{F})$.

Definition 3.2.7. An *F-normed algebra* in an F-normed ∞-category $\mathcal{C}$ is a lax F-normed functor from $*^{\otimes} = \mathsf{Span}_N(\mathcal{F})$ to $\mathcal{C}$; in other words, it is a section of the cocartesian fibration

$$\mathcal{C}^{\otimes} \longrightarrow \mathsf{Span}_N(\mathcal{F})$$

that takes backward maps in $\mathsf{Span}_N(\mathcal{F})$ to cocartesian morphisms. We write

$$\mathsf{NAlg}_F(\mathcal{C}) := \mathsf{Fun}^{\mathrm{lax}}_{/\mathsf{Span}_N(\mathcal{F})}(\mathsf{Span}_N(\mathcal{F}), \mathcal{C}^{\otimes})$$

for the ∞-category of F-normed algebras in $\mathcal{C}$.

Remark 3.2.8. By an easy extension of [BHS22, 5.2.14], our definitions of F-normed ∞-categories and lax F-normed functors are equivalent to those of Nardin and Shah [NS22] in the case where $F = F_T$ for a so-called "atomic" orbital ∞-category T. In particular, the ∞-categories of F-normed algebras are equivalent, cf. [BHS22, 5.3.17]. For extensive F, our F-normed algebras are also studied in [Bac22], as a generalization of the normed spectra introduced in [BH21, 7.1].

We end this section by considering a construction of normed structures on spans:

Construction 3.2.9. Since the functor $\mathsf{Span}\colon \mathsf{SpanPair} \to \mathsf{Cat}_\infty$ preserves limits, hence in particular finite products, any F-normed monoid in $\mathsf{SpanPair}$ gives rise to an F-normed ∞-category by applying Span pointwise. Observe that an F-normed monoid in $\mathsf{SpanPair}$ is an F-normed ∞-category

$$\mathcal{C}\colon \mathsf{Span}_N(\mathcal{F}) \longrightarrow \mathsf{Cat}_\infty$$

equipped with a subfunctor $\mathcal{C}_Q \subseteq \mathcal{C}$ such that $(\mathcal{C}(X), \mathcal{C}_Q(X))$ is a span pair for every $X \in \mathcal{F}$ and the induced functor $m_\otimes f^*\colon \mathcal{C}(X) \to \mathcal{C}(Y)$ is a map of span pairs for every morphism $X \xleftarrow{f} Z \xrightarrow{m} Y$ in $\mathsf{Span}_N(\mathcal{F})$. In this case, the composite

$$\mathsf{Span}_Q(\mathcal{C}) := \mathsf{Span} \circ (\mathcal{C}, \mathcal{C}_Q)\colon \mathsf{Span}_N(\mathcal{F}) \longrightarrow \mathsf{Cat}_\infty$$

endows $\mathsf{Span}_Q(\mathcal{C})$ with an F-normed structure inherited from that of $\mathcal{C}$.

The following result provides an explicit description of the cocartesian fibrations associated to such normed structures:

Proposition 3.2.10. *Let $p\colon \mathcal{C}_\otimes \to \mathsf{Span}_N(\mathcal{F})^{\mathrm{op}}$ be a cartesian fibration corresponding to an F-normed monoid in $\mathsf{SpanPair}$. Then the cocartesian fibration $\mathsf{Span}_Q(\mathcal{C})^{\otimes} \to \mathsf{Span}_N(\mathcal{F})$ for the induced F-normed structure on spans from Construction 3.2.9 is given by*

$$\mathsf{Span}_Q(\mathcal{C})^{\otimes} \simeq \mathsf{Span}_{Q-\mathrm{fw}}(\mathcal{C}_\otimes),$$

where $(\mathcal{C}_\otimes)_{Q-\mathrm{fw}}$ denotes the subcategory of maps that go to equivalences under p and fiberwise lie in $\mathcal{C}^{\otimes}(-)_Q$.

Proof. This is a special case of [HHLN23, 3.9]. □

3.3 Norms on product-preserving functors

In this subsection we will construct a (low-tech) version of "parametrized Day convolution" for ∞-categories of product-preserving functors. More precisely, we will show the following:

Proposition 3.3.1. *Let $\mathcal{X}$ be a cocomplete ∞-category with finite products, where the product functor preserves colimits in each variable. Suppose $F = (\mathcal{F}, \mathcal{F}_N)$ is a weakly extensive span pair, and consider an F-normed ∞-category $\mathcal{C}\colon \mathsf{Span}_N(\mathcal{F}) \to \mathsf{Cat}_\infty$ such that $\mathcal{C}(X)$ has finite products for every $X \in \mathcal{F}$ (but the morphisms in the diagram need not preserve these).*

(i) There is a functor

$$\mathbb{Q} = \mathsf{Fun}^{\times}(\mathcal{C}(-), \mathcal{X})\colon \mathsf{Span}_N(\mathcal{F}) \longrightarrow \widehat{\mathsf{Cat}}_\infty$$

obtained by left Kan extension from $\mathcal{C}$. This preserves finite products, and so defines another F-normed ∞-category.

(ii) If $\mathcal{C}^{\otimes} \to \mathsf{Span}_N(\mathcal{F})$ is the cocartesian fibration for $\mathcal{C}$, then F-normed algebras in $\mathbb{Q} = \mathsf{Fun}^{\times}(\mathcal{C}(-), \mathcal{X})$ are equivalent to functors

$$A\colon \mathcal{C}^{\otimes} \longrightarrow \mathcal{X}$$

such that

- *for every $X \in \mathcal{F}$ the restriction*

$$A_X\colon \mathcal{C}(X) \longrightarrow \mathcal{X}$$

of A to the fiber over X is a product-preserving functor,

- *and for every morphism $f\colon X \to Y$ in $\mathcal{F}$, viewed as a backward morphism in $\mathsf{Span}_N(\mathcal{F})$, the natural transformation*

$$\begin{array}{ccc} \mathcal{C}(Y) & \xrightarrow{\quad f^* \quad} & \mathcal{C}(X) \\ & \Longrightarrow & \\ {\scriptstyle A_Y} \searrow & & \swarrow {\scriptstyle A_X} \\ & \mathcal{X} & \end{array}$$

exhibits A_X as a left Kan extension of A_Y along f^*.

More precisely, $\mathsf{NAlg}_F(\mathbb{Q})$ is equivalent to the full subcategory $\mathcal{A} \subseteq \mathsf{Fun}(\mathcal{C}^{\otimes}, \mathcal{X})$ of functors that satisfy these conditions, and for every $A \in \mathcal{F}$ this equivalence fits into a commutative diagram

$$\begin{array}{ccc} \mathsf{NAlg}_F(\mathbb{Q}) & \xrightarrow{\quad \simeq \quad} & \mathcal{A}. \\ {\scriptstyle \mathrm{ev}_A} \searrow & & \swarrow {\scriptstyle \mathrm{res}} \\ & \mathsf{Fun}^{\times}(\mathcal{C}(A), \mathcal{X}) & \end{array}$$

The key input to the construction is the following observation about left Kan extensions of product-preserving functors:

Proposition 3.3.2. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are small ∞-categories with finite products, and $\mathscr{C}$ is an ∞-category with small colimits and finite products such that the cartesian product preserves colimits in each variable. If $F\colon \mathscr{A} \to \mathscr{C}$ is a product-preserving functor and $g\colon \mathscr{A} \to \mathscr{B}$ is an arbitrary functor, then the left Kan extension $g_!F$ also preserves finite products. In other words, left Kan extension restricts to a functor*

$$g_!\colon \mathsf{Fun}^{\times}(\mathscr{A}, \mathscr{C}) \longrightarrow \mathsf{Fun}^{\times}(\mathscr{B}, \mathscr{C}).$$

Remark 3.3.3. For 1-categories, a version of this result apparently goes back to Lawvere's thesis [Law04]. See also for instance [Day70, Appendix 2] or [BD77] for generalizations to enriched categories and [Str14] for another variant and a historical discussion.

Proof of Proposition 3.3.2. Our assumptions guarantee that $g_!F$ is computed by the pointwise formula,

$$g_!F(b) \simeq \operatorname{colim}_{\mathscr{A}_{/b}} F.$$

In particular, $g_!F(*)$ is a colimit over $\mathscr{A} \times_{\mathscr{B}} \mathscr{B}_{/*} \simeq \mathscr{A}$; since this has a terminal object $* \to *$, we see

$$g_!F(*) \simeq F(*) \simeq *.$$

For objects $b_1, b_2 \in \mathscr{B}$, consider the functor

$$\pi_{b_1,b_2}\colon \mathscr{A}_{/b_1 \times b_2} \longrightarrow \mathscr{A}_{/b_1} \times \mathscr{A}_{/b_2}$$

given by composition with the projections $b_1 \times b_2 \to b_i$. We claim that this functor has a right adjoint $R = R_{b_1,b_2}$, given on a pair (Φ_1, Φ_2) of objects $\Phi_i := (a_i, \phi_i\colon g(a_i) \to b_i)$ in $\mathscr{A}_{/b_i}$ by $R(\Phi_1, \Phi_2) = (a_1 \times a_2, r(\phi_1, \phi_2))$, where $r(\phi_1, \phi_2)$ is defined as the composite

$$g(a_1 \times a_2) \to g(a_1) \times g(a_2) \xrightarrow{\phi_1 \times \phi_2} b_1 \times b_2.$$

To see this, observe that for an object $\Psi = (x, \psi\colon g(x) \to b_1 \times b_2)$ of $\mathscr{A}_{/b_1 \times b_2}$ the mapping space $\mathsf{Map}_{\mathscr{A}_{/b_1 \times b_2}}(\Psi, R(\Phi_1, \Phi_2))$ sits in a pullback diagram as follows:

$$\begin{array}{ccccc}
\mathsf{Map}_{\mathscr{A}_{/b_1 \times b_2}}(\Psi, R(\Phi_1, \Phi_2)) & \to & \mathsf{Map}_{\mathscr{B}_{/b_1 \times b_2}}(\psi, r(\phi_1, \phi_2)) & \longrightarrow & \{\psi\} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathsf{Map}_{\mathscr{A}}(x, a_1 \times a_2) & \xrightarrow{g} & \mathsf{Map}_{\mathscr{B}}(g(x), g(a_1 \times a_2)) & \xrightarrow{r(\phi_1,\phi_2)\circ -} & \mathsf{Map}_{\mathscr{B}}(g(x), b_1 \times b_2).
\end{array}$$

Under the identification of $\mathsf{Map}_{\mathscr{A}}(x, a_1 \times a_2)$ with $\mathsf{Map}_{\mathscr{A}}(x, a_1) \times \mathsf{Map}_{\mathscr{A}}(x, a_2)$ and of $\mathsf{Map}_{\mathscr{B}}(g(x), b_1 \times b_2)$ with $\mathsf{Map}_{\mathscr{B}}(g(x), b_1) \times \mathsf{Map}_{\mathscr{B}}(g(x), b_2)$, the bottom map turns into a product of the two maps

$$\mathsf{Map}_{\mathscr{A}}(x, a_i) \xrightarrow{g} \mathsf{Map}_{\mathscr{B}}(g(x), g(a_i)) \xrightarrow{\phi_i \circ -} \mathsf{Map}_{\mathscr{B}}(g(x), b_i),$$

and so by passing to fibers we obtain a natural equivalence

$$\mathsf{Map}_{\mathcal{A}_{/b_1\times b_2}}(\Psi, R(\Phi_1,\Phi_2)) \xrightarrow{\sim} \mathsf{Map}_{\mathcal{A}_{/b_1}}((x,\mathrm{pr}_1\psi),\Phi_1)\times \mathsf{Map}_{\mathcal{A}_{/b_2}}((x,\mathrm{pr}_2\psi),\Phi_2).$$

Since the target is canonically identified with $\mathsf{Map}_{\mathcal{A}_{/b_1}\times\mathcal{A}_{/b_2}}(\pi_{b_1,b_2}\Psi,(\Phi_1,\Phi_2))$, this shows that R_{b_1,b_2} is the desired right adjoint.

Since right adjoints are cofinal, composition with R_{b_1,b_2} therefore induces an equivalence

$$g_!F(b_1)\times g_!F(b_2) \simeq \mathrm{colim}_{(a,a')\in\mathcal{A}_{/b_1}\times\mathcal{A}_{/b_2}} F(a\times a') \longrightarrow \mathrm{colim}_{x\in\mathcal{A}_{/b_1\times b_2}} F(x)\simeq g_!F(b_1\times b_2).$$

Moreover, these right adjoints are compatible with composition in $\mathcal{B}$, so for maps $b_1\to c_1, b_2\to c_2$ we get a commutative square

$$\begin{array}{ccc} g_!F(b_1)\times g_!F(b_2) & \xrightarrow{\sim} & g_!F(b_1\times b_2) \\ \downarrow & & \downarrow \\ g_!F(c_1)\times g_!F(c_2) & \xrightarrow{\sim} & g_!F(c_1\times c_2). \end{array}$$

Taking $c_1 = b_1$ and $c_2 = *$, we see in particular that projection to $g_!F(b_1)$ on the left corresponds to composition with $b_1\times b_2\to b_1$ on the right, so that the canonical map $g_!F(b_1\times b_2)\to g_!F(b_1)\times g_!F(b_2)$ is an equivalence. In other words, the functor $g_!F$ is product-preserving, as required. □

Lemma 3.3.4. *Suppose $\mathcal{A}_1,\ldots,\mathcal{A}_n$ and $\mathcal{B}$ are ∞-categories with finite products. If $\mathcal{A} := \prod_i \mathcal{A}_i$, then left Kan extension along the projections $\pi_i\colon \mathcal{A}\to\mathcal{A}_i$ gives an equivalence*

$$\mathsf{Fun}^{\times}(\mathcal{A},\mathcal{B}) \xrightarrow{\sim} \prod_i \mathsf{Fun}^{\times}(\mathcal{A}_i,\mathcal{B}),$$

with inverse given by

$$(F_i\colon \mathcal{A}_i\longrightarrow\mathcal{B}) \;\mapsto\; \left(\prod_i F_i\circ\pi_i\colon \mathcal{A}\longrightarrow\mathcal{B}\right)$$

Proof. For $i=0$ we indeed have $\mathsf{Fun}^{\times}(*,\mathcal{B})\simeq *$ as the only product-preserving functor is the one constant at the terminal object. Suppose therefore that $i>1$. The pointwise left Kan extension of $F\colon\mathcal{A}\to\mathcal{B}$ along π_i, if it exists, is given at $x\in\mathcal{A}_i$ by taking a colimit over

$$\mathcal{A}_{/x}\simeq \mathcal{A}_{i/x}\times\prod_{j\neq i}\mathcal{A}_j$$

This has a terminal object, so the colimit (and hence the pointwise Kan extension) always exists, and is given by

$$(\pi_{i,!}F)(x)\simeq F(*,\ldots,*,x,*,\ldots,*).$$

The functor we claim is an equivalence is the composite

$$\mathsf{Fun}^{\times}(\mathscr{A},\mathscr{B}) \longrightarrow \prod_i \mathsf{Fun}^{\times}(\mathscr{A},\mathscr{B}) \xrightarrow{\prod_i \pi_{i,!}} \prod_i \mathsf{Fun}^{\times}(\mathscr{A}_i,\mathscr{B}).$$

Since $\mathsf{Fun}^{\times}(\mathscr{A},\mathscr{B})$ has finite products (computed pointwise), this functor has a right adjoint, given by

$$\prod_i \mathsf{Fun}^{\times}(\mathscr{A}_i,\mathscr{B}) \xrightarrow{\prod \pi_i^*} \prod_i \mathsf{Fun}^{\times}(\mathscr{A},\mathscr{B}) \xrightarrow{\times} \mathsf{Fun}^{\times}(\mathscr{A},\mathscr{B}).$$

To see that this adjunction is in fact an equivalence, it suffices to observe that for $F_i \in \mathsf{Fun}^{\times}(\mathscr{A}_i,\mathscr{B})$ we have

$$\Big(\pi_{j,!}\Big(\prod_i F_i \circ \pi_i\Big)\Big)(x) \simeq F_j(x) \times \prod_{i\neq j} F_i(*) \simeq F_j(x)$$

and that for $F \in \mathsf{Fun}^{\times}(\mathscr{A},\mathscr{B})$ we have

$$F(x_1,\ldots,x_n) \xrightarrow{\sim} \prod_i (\pi_{i,!}F)(x_i),$$

since $(x_1,\ldots,x_n)$ is the finite product

$$(x_1,*,\cdots *) \times (*,x_2,*,\ldots,*) \times \cdots (*,\ldots,*,x_n)$$

in $\mathscr{A}$. ◻

Remark 3.3.5. Let $\mathscr{R}$ be any collection of diagram shapes containing both the empty set and the two-point set. Then the same argument shows that the categories of $\mathscr{R}$-shaped limit preserving functors satisfy $\mathsf{Fun}^{\mathscr{R}\text{-lim}}(\prod_{i=1}^n \mathscr{A}_i,\mathscr{B}) \simeq \prod_{i=1}^n \mathsf{Fun}^{\mathscr{R}\text{-lim}}(\mathscr{A}_i,\mathscr{B})$.

We now come to our main construction:

Construction 3.3.6. Let $\mathfrak{X}$ be a cocomplete ∞-category with finite products, such that the cartesian product preserves colimits in each variable, and let $F\colon \mathscr{I} \to \mathsf{Cat}_\infty$ be a functor such that $F(i)$ has finite products for all $i \in \mathscr{I}$ (but these are not necessarily preserved by the morphisms in the diagram).

Let $p\colon \mathscr{E} \to \mathscr{I}$ be the cartesian fibration for the functor

$$\mathsf{Fun}(F(-),\mathfrak{X})\colon \mathscr{I}^{\mathrm{op}} \longrightarrow \widehat{\mathsf{Cat}}_\infty,$$

and note that by [GHN17, 7.3] there is a natural equivalence

$$\mathsf{Fun}_{/\mathscr{I}}(\mathscr{K},\mathscr{E}) \simeq \mathsf{Fun}(\mathscr{K} \times_{\mathscr{I}} \mathscr{F},\mathfrak{X}), \tag{7}$$

where $\mathscr{F} \to \mathscr{I}$ is the cocartesian fibration for F. Here p is also a cocartesian fibration, since we can left Kan extend functors to $\mathfrak{X}$. Moreover, if we define $\mathscr{E}'$

as the full subcategory containing the functors $F(i) \to \mathfrak{X}$ that preserve products for all i, then Proposition 3.3.2 implies that $\mathscr{E}' \to \mathscr{I}$ is again a cocartesian fibration. Note that for $f: i \to j$ in $\mathscr{I}$, a morphism ϕ in $\mathscr{E}$ over f corresponds under the equivalence (7) to a functor $[1] \times_{\mathscr{I}} \mathscr{F} \to \mathfrak{X}$. Here the source is the cocartesian fibration over $[1]$ encoding the functor $F(f): F(i) \to F(j)$ and so can be described as the pushout $F(i) \times [1] \amalg_{F(i)\times\{1\}} F(j)$, see [GHN17, 3.1]. We can thus identify the morphism ϕ with a natural transformation

$$\begin{array}{ccc} F(i) & \xrightarrow{F(f)} & F(j) \\ & \searrow \;\Longrightarrow\; \swarrow & \\ & \mathfrak{X}, & \end{array}$$

and ϕ is a cocartesian morphism if and only if this diagram exhibits $F(j) \to \mathfrak{X}$ as a left Kan extension of $F(i) \to \mathfrak{X}$ along $F(f)$.

Proof of Proposition 3.3.1. To prove that $\mathbb{Q}$ is F-normed we must show that given a finite coproduct $X \simeq \coprod_i X_i$ in $\mathscr{F}$, with $\iota_j: X_j \to X$ the summand inclusions, the functor

$$(\pi_{j,!})_j : \mathsf{Fun}^\times(\mathscr{C}(X), \mathfrak{X}) \longrightarrow \prod_j \mathsf{Fun}^\times(\mathscr{C}(X_j), \mathfrak{X}),$$

where $\pi_j := \mathscr{C}(\iota_j)$, is an equivalence. This is the content of Lemma 3.3.4.

Part (ii) follows immediately from Construction 3.3.6 specialized to $\mathscr{I} = \mathsf{Span}_N(\mathscr{F})$: note that the straightening of the cocartesian fibration $\mathscr{E} \to \mathscr{I}$ dicussed there agrees by definition with the functor $X \mapsto \mathsf{Fun}(\mathscr{C}(X), \mathfrak{X})$ with functoriality via left Kan extension, so that the cocartesian subfibration $\mathscr{E}' \to \mathscr{I}$ classifies the functor $\mathsf{Fun}^\times(\mathscr{C}(-), \mathfrak{X})$ in question. □

Observation 3.3.7. In the situation above, suppose the functor $f^*: \mathscr{C}(Y) \to \mathscr{C}(X)$ has a right adjoint f_* for every backwards map f. Then the condition for $A: \mathscr{C}^\otimes \to \mathfrak{X}$ to define an F-normed algebra in $\mathbb{Q}$ can be rephrased as requiring an equivalence

$$A_X \simeq A_Y \circ f_*.$$

In this case $\mathscr{C}^\otimes$ also has *cartesian* morphisms over backwards maps, and we can phrase this condition more precisely as: If $\bar{X}$ is in $\mathscr{C}^\otimes_X$ and $\phi: \bar{Y} \to \bar{X}$ is cartesian over a backwards map in $\mathsf{Span}_N(\mathscr{F})$, then $A(\phi)$ is an equivalence.

In the special case $F = (\mathbb{F}, \mathbb{F})$, the resulting normed structure on $\mathsf{Fun}^\times(\mathscr{C}, \mathfrak{X})$ corresponds by Example 3.1.6 to a symmetric monoidal structure. We will now compare it to the Day convolution monoidal structure:

Proposition 3.3.8. *Let $\mathfrak{X}$ be a cocomplete ∞-category with finite products, where the product functor preserves colimits in each variable. Suppose $\mathscr{C}: \mathsf{Span}(\mathbb{F}) \to \mathsf{Cat}_\infty$ is a symmetric monoidal ∞-category whose underlying ∞-category has finite products.*

- *The symmetric monoidal structure on* $\mathsf{Fun}^{\times}(\mathcal{C},\mathcal{X})$ *from Proposition 3.3.1 is a full symmetric monoidal subcategory of the Day convolution on* $\mathsf{Fun}(\mathcal{C},\mathcal{X})$.
- *If* $\mathcal{X}$ *is presentable and the tensor product on* $\mathcal{C}$ *preserves finite products in each variable, it is moreover a symmetric monoidal localization.*

The proof will require some preparations.

Lemma 3.3.9. *Let* $\mathcal{C},\mathcal{D}\to\mathcal{I}$ *be cocartesian fibrations, and let* $F\colon\mathcal{C}\to\mathcal{D}$ *be a functor over* $\mathcal{I}$. *Then the following are equivalent:*

(i) F *preserves cocartesian edges.*

(ii) For every cocartesian edge $[1]\to\mathcal{C}$ *the composite* $[1]\to\mathcal{D}$ *is the relative left Kan extension (over* $\mathcal{I}$*) of its restriction to* 0.

(iii) For every $i\in\mathcal{I}$ *and every* $\mathcal{I}_{i/}\to\mathcal{C}$ *over* $\mathcal{I}$ *landing in the subcategory of cocartesian edges, the composite* $\mathcal{I}_{i/}\to\mathcal{D}$ *is relatively left Kan extended from* $\mathrm{id}_i\in\mathcal{I}_{i/}$.

Proof. Recall that if $\mathcal{J}\to\mathcal{I}$ is arbitrary and $\mathcal{J}$ has an initial object $\emptyset$, then the relative left Kan extension along $\{\emptyset\}\hookrightarrow\mathcal{J}$ exists for every cocartesian fibration $\mathcal{E}\to\mathcal{I}$, and $\mathcal{J}\to\mathcal{E}$ is relatively left Kan extended if and only if it factors through cocartesian edges [Lur24, Tag 043G]. The equivalence between (1) and (2) follows immediately, while for the equivalence between (1) and (3) it suffices to observe in addition that every cocartesian edge $x\to y$ of $\mathcal{C}$ is contained in the image of some cocartesian $\mathcal{I}_{i/}\to\mathcal{C}$: namely, if i is the image of x, then the relative left Kan extension of x along $\{\emptyset\}\hookrightarrow\mathcal{I}_{i/}$ has the required properties. □

Proposition 3.3.10. *Let* $\mathcal{C}\colon\mathsf{Span}(\mathbb{F})\to\mathsf{Cat}_\infty$ *be a symmetric monoidal* ∞*-category, let* $\mathcal{X}$ *be a cocomplete category with finite products such that the product preserves colimits in each variable, and let* $\mathcal{E}'\to\mathsf{Span}(\mathbb{F})$ *denote the cocartesian fibration classifying the functor* $\mathsf{Fun}^{\times}(\mathcal{C}(-),\mathcal{X})$ *with functoriality via left Kan extension.*

If $\mathcal{O}^{\otimes}\to\mathsf{Span}(\mathbb{F})$ *is any symmetric monoidal* ∞*-category, then* $\mathcal{C}^{\otimes}\times_{\mathsf{Span}(\mathbb{F})}\mathcal{O}^{\otimes}$ *has finite products, and a functor* $\mathcal{O}^{\otimes}\to\mathcal{E}'$ *over* $\mathsf{Span}(\mathbb{F})$ *is lax symmetric monoidal if and only if the functor* $\tilde{F}\colon\mathcal{C}^{\otimes}\times_{\mathsf{Span}(\mathbb{F})}\mathcal{O}^{\otimes}\to\mathcal{X}$ *associated to* F *via (7) preserves finite products.*

Proof. We first recall from [Hau23, 2.2.6] that $\mathcal{C}^{\otimes}$ and $\mathcal{O}^{\otimes}$ have finite products and that the maps $\mathcal{C}^{\otimes}\to\mathsf{Span}(\mathbb{F})$ and $\mathcal{O}^{\otimes}\to\mathsf{Span}(\mathbb{F})$ preserve them; thus, $\mathcal{C}^{\otimes}\times_{\mathsf{Span}(\mathbb{F})}\mathcal{O}^{\otimes}$ again has finite products, which are computed componentwise. The cited reference moreover shows that any $X\in\mathcal{O}_{\mathbf{n}}$ is the product of its cocartesian pushforwards along the backwards maps $\mathbf{n}\leftarrow\mathbf{1}=\mathbf{1}$; thus, we see that a functor $\mathcal{C}^{\otimes}\times_{\mathsf{Span}(\mathbb{F})}\mathcal{O}^{\otimes}\to\mathcal{X}$ preserves products if and only if its restriction to $\mathcal{C}^{\otimes}\times_{\mathsf{Span}(\mathbb{F})}(\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}$ does so for every map $(\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}\to\mathcal{O}^{\otimes}$ over $\mathsf{Span}(\mathbb{F})$ landing in cocartesian edges.

Write now $\mathcal{E}\to\mathsf{Span}(\mathbb{F})$ for the cocartesian fibration from Construction 3.3.6 classifying $\mathsf{Fun}(\mathcal{C}(-),\mathcal{X})$, of which $\mathcal{E}'\to\mathsf{Span}(\mathbb{F})$ is a subfibration.

We then have for every $\mathbf{n} \in \mathbb{F}$ and $(\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}} \to \mathcal{O}^{\otimes}$ over $\mathsf{Span}(\mathbb{F})$ a commutative diagram

$$\begin{array}{ccc} \mathsf{Fun}_{/\mathsf{Span}(\mathbb{F})}(\mathcal{O}^{\otimes}, \mathcal{E}) & \xrightarrow{\sim} & \mathsf{Fun}(\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \mathcal{O}^{\otimes}, \mathcal{X}) \\ \downarrow & & \downarrow \\ \mathsf{Fun}_{/\mathsf{Span}(\mathbb{F})}((\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}, \mathcal{E}) & \xrightarrow{\sim} & \mathsf{Fun}(\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}, \mathcal{X}) \\ \downarrow & & \downarrow \\ \mathsf{Fun}(\{O\}, \mathsf{Fun}(\mathcal{C}_{\mathbf{n}}, \mathcal{X})) & \xrightarrow{\sim} & \mathsf{Fun}(\underbrace{\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \{O\}}_{=\mathcal{C}_{\mathbf{n}}}, \mathcal{X}) \end{array} \tag{8}$$

where the horizontal equivalences are as in Construction 3.3.6 and the vertical maps are the restrictions. By the previous lemma applied to $\mathcal{I} = \mathbb{F}^{\mathrm{op}}$, $F\colon \mathcal{O}^{\otimes} \to \mathcal{E}$ preserves inert edges if and only if restriction to $(\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}$ is contained in the image of the left adjoint of the lower left vertical map for every $(\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}} \to \mathcal{O}^{\otimes}$ factoring through cocartesian edges. It follows formally from commutativity of (8) that this is equivalent to the restriction of the corresponding functor $\tilde{F}$ to $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}$ being left Kan extended from $\mathcal{C}_{\mathbf{n}}$; it remains to show that if F factors through $\mathcal{E}' \subseteq \mathcal{E}$ (i.e. if the restriction of $\tilde{F}$ to $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \{P\}$ preserves products for every $P \in \mathcal{O}^{\otimes}$), then the latter condition is in turn equivalent to the restriction of $\tilde{F}$ to $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}} \to \mathcal{X}$ preserving products.

Proposition 3.3.2 shows that the left Kan extension of any product-preserving functor $\mathcal{C}_{\mathbf{n}} \to \mathcal{X}$ to $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}$ is again product-preserving, so the above condition is indeed sufficient for $\tilde{F}$ to preserve products. To see that it is also necessary, it will suffice to show that any product-preserving functor $G\colon \mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}} \to \mathcal{X}$ whose restriction to $\mathcal{C}_{\mathbf{n}}$ again preserves products is left Kan extended from its restriction to $\mathcal{C}_{\mathbf{n}}$.

If we let $j\colon \mathcal{C}_{\mathbf{n}} \hookrightarrow \mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}$ denote the inclusion, then the counit $j_! j^* G \to G$ is an equivalence for any $(X, \mathrm{id}_n) \in \mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}$ by full faithfulness of j. We claim that it is in fact also an equivalence for every $(X, i\colon \mathbf{1} \to \mathbf{n})$; with this established, the claim will follow as both G (by assumption) and $j_! j^* G$ (by the above) preserve products and every object of $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} (\mathbb{F}_{/\mathbf{n}})^{\mathrm{op}}$ decomposes as a finite product of objects $(X, \mathbf{1} \to \mathbf{n})$.

To prove the claim, note that for any $X_1, \ldots, X_n \in \mathcal{C}_{\mathbf{1}}$

$$\mathcal{C}_{\mathbf{n}} \ni (X_1, \ldots, X_n; \mathrm{id}_{\mathbf{n}}) \simeq \prod_{k=1}^{n} (X_k, k\colon \mathbf{1} \longrightarrow \mathbf{n});$$

thus, the product of the counits $j_! j^* G(X_k, k\colon \mathbf{1} \to \mathbf{n}) \to G(X_k, k\colon \mathbf{1} \to \mathbf{n})$ is an equivalence as G and $j_! j^* G$ preserve products. Specializing to $X_k = *$ for $k \neq i$, it will therefore suffice that $G(*, k\colon \mathbf{1} \to \mathbf{n}) \simeq * \simeq j_! j^* G(*, k\colon \mathbf{1} \to \mathbf{n})$ for every $1 \leq k \leq n$. For this, we further set $X_i = *$ to see that

$$\prod_{k=1}^{n} G(*, k\colon \mathbf{1} \longrightarrow \mathbf{n}) \simeq G(*, \mathrm{id}_{\mathbf{n}}) \simeq j_! j^* G(*, \mathrm{id}_{\mathbf{n}}) \simeq \prod_{k=1}^{n} j_! j^* G(*, k\colon \mathbf{1} \longrightarrow \mathbf{n}).$$

As the restriction of G to $\mathcal{C}_{\mathbf{n}}$ preserves finite products, $G(*, \mathrm{id}_{\mathbf{n}})$ is terminal; since a product is terminal if and only if all of its factors are, this completes the proof of the claim and hence of the proposition. □

Proof of Proposition 3.3.8. Recall first that the Day convolution of $F, G\colon \mathcal{C} \to \mathcal{X}$ is given by the left Kan extension of

$$\mathcal{C} \times \mathcal{C} \xrightarrow{F\times G} \mathcal{X} \times \mathcal{X} \xrightarrow{\prod} \mathcal{X} \tag{9}$$

along $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ [Lur17, 2.2.6.17]. If F and G preserve products, so does (9), whence so does the Day convolution by Proposition 3.3.2. On the other hand, the unit is given by the left Kan extension of $*$ along $\{\mathbb{1}\} \hookrightarrow \mathcal{C}$, and the same argument shows that this is again product preserving, i.e. the Day convolution structure indeed restricts to $\mathsf{Fun}^{\times}(\mathcal{C}, \mathcal{X})$. Moreover, we saw in [CHLL24, 3.3.4] that this is also a symmetric monoidal localization when $\mathcal{X}$ is presentable and the tensor product on $\mathcal{C}$ preserves products in each variable.

It remains to compare this symmetric monoidal structure to the one above, for which we will show that both represent the same functor in the ∞-category of symmetric monoidal ∞-categories and *lax* symmetric monoidal functors.

For this we first recall that Day convolution is defined in such a way that lax symmetric monoidal functors $\mathcal{O}^{\otimes} \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})^{\otimes}_{\mathrm{Day}}$ correspond bijectively to lax symmetric monoidal functors $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \mathcal{O}^{\otimes} \to \mathcal{X}^{\times}$, which are in turn identified with product-preserving functors $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \mathcal{O}^{\otimes} \to \mathcal{X}$ [CHLL24, 2.4.5 and 2.4.6]. Thus, functors into the restricted Day convolution on $\mathsf{Fun}^{\times}(\mathcal{C}, \mathcal{X})$ correspond bijectively to product-preserving functors $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \mathcal{O}^{\otimes} \to \mathcal{X}$ such that in addition the restriction $\mathcal{C}_{\mathbf{n}} \simeq \mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \{O\} \to \mathcal{X}$ preserves products for all $n \geq 0$, $O \in \mathcal{O}_{\mathbf{n}}$.

On the other hand, we have seen in Construction 3.3.6 that functors $F\colon \mathcal{O}^{\otimes} \to \mathsf{Fun}^{\times}(\mathcal{C}, \mathcal{X})^{\otimes}$ over $\mathsf{Span}(\mathbb{F})$ correspond to functors

$$\tilde{F}\colon \mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \mathcal{O}^{\otimes} \longrightarrow \mathcal{X} \tag{10}$$

that preserve products when restricted to each $\mathcal{C}^{\otimes} \times_{\mathsf{Span}(\mathbb{F})} \{O\}$, and Proposition 3.3.10 shows that such a functor F is indeed lax symmetric monoidal if and only if $\tilde{F}$ preserves finite products. □

3.4 The cartesian normed structure on $\mathcal{F}$

Consider an extensive ∞-category $\mathcal{F}$ with pullbacks, taken to be fixed throughout this subsection. We will see that we may equip $\mathcal{F}$ with a "cartesian" normed structure whenever $\mathcal{F}$ is suitably locally cartesian closed.

Notation 3.4.1. We define a parametrized version $\underline{\mathcal{F}}$ of $\mathcal{F}$ as the functor

$$\underline{\mathcal{F}}\colon \mathcal{F}^{\mathrm{op}} \to \mathsf{Cat}_{\infty}, \quad X \mapsto \mathcal{F}_{/X},$$

with functoriality coming from pullbacks. Since $\mathcal{F}$ is extensive, this functor preserves products, and hence defines an $\mathcal{F}$-∞-category.

Definition 3.4.2. Given a weakly extensive subcategory $\mathcal{F}_N \subseteq \mathcal{F}$, we say that $\mathcal{F}$ is *N-locally cartesian closed* if the functor $f^*\colon \mathcal{F}_{/Y} \to \mathcal{F}_{/X}$ given by pullback along $f\colon X \to Y$ in $\mathcal{F}_N$ has a right adjoint f_*.

Proposition 3.4.3. *Let $\mathcal{F}_N \subseteq \mathcal{F}$ be a weakly extensive subcategory and suppose $\mathcal{F}$ is N-locally cartesian closed. Then the following hold:*

- *The pair $(\mathsf{Ar}(\mathcal{F}), \mathsf{Ar}(\mathcal{F})_{N-\mathrm{pb}})$ is a span pair, where $\mathsf{Ar}(\mathcal{F})_{N-\mathrm{pb}}$ consists of the pullback squares along morphisms in $\mathcal{F}_N$.*
- *The functor $\mathrm{ev}_1\colon \mathsf{Ar}(\mathcal{F}) \to \mathcal{F}$ is a morphism of span pairs.*
- *The functor*

$$\mathsf{Span}(\mathrm{ev}_1)^{\mathrm{op}}\colon \mathsf{Span}_{N-\mathrm{pb}}(\mathsf{Ar}(\mathcal{F}))^{\mathrm{op}} \longrightarrow \mathsf{Span}_N(\mathcal{F})^{\mathrm{op}}$$

is the cartesian fibration for an N-normed structure on $\underline{\mathcal{F}}$.

Proof. Consider the cocartesian fibration

$$\mathrm{ev}_1\colon \mathsf{Ar}(\mathcal{F}) \longrightarrow \mathcal{F}$$

classified by the functor $\mathcal{F}_{/(-)}\colon \mathcal{F} \to \mathsf{Cat}_\infty$, with functoriality given by composition. By assumption we have right adjoints (given by pullback) for morphisms in $\mathcal{F}_N$, and by unpacking the definitions and applying the pasting lemma for pullbacks we see that these satisfy base change. Applying Proposition 2.1.6 to this situation, we obtain the first two bullet points, and we get that $\mathsf{Span}(\mathrm{ev}_1)^{\mathrm{op}}$ is a cocartesian fibration. To see that it is also a cartesian fibration, it suffices by [Lur09, 5.2.2.4$^{\mathrm{op}}$] to show that it is a locally cartesian fibration, which we can check separately over $\mathcal{F}$ and $\mathcal{F}_N^{\mathrm{op}}$. Over $\mathcal{F}_N$ we get the cocartesian fibration for the functor $\mathcal{F}_{/(-)}\colon \mathcal{F}_N^{\mathrm{op}} \to \mathsf{Cat}_\infty$, with functoriality given by pullback; since these pullback functors have right adjoints due to N-locally cartesian closedness of $\mathcal{F}$, it is also a cartesian fibration over $\mathcal{F}_N^{\mathrm{op}}$. On the other hand, over $\mathcal{F}$ we get the functor $\mathrm{ev}_1\colon \mathsf{Ar}(\mathcal{F}) \to \mathcal{F}$. Since this is the cartesian fibration for $\underline{\mathcal{F}}$, we indeed get the cartesian fibration for an F-normed structure on $\underline{\mathcal{F}}$. □

Notation 3.4.4. In the context of Proposition 3.4.3, we write

$$\underline{\mathcal{F}}_\times \coloneqq \mathsf{Span}_{N-\mathrm{pb}}(\mathsf{Ar}(\mathcal{F}))^{\mathrm{op}}$$

and refer to it as the *cartesian* normed structure on $\underline{\mathcal{F}}$. In the non-parametrized case, this construction indeed gives the cartesian fibration for the cartesian symmetric monoidal structure on $\mathbb{F}$ by [CHLL24, 3.1.4]. We expect that our construction more generally agrees with [NS22, 2.4.1] whenever the two frameworks overlap; however, as this won't be relevant for the purposes of this paper, we will not prove this here.

4 Normed rings

In this section, we introduce the notion of a *normed ring* and show it may equivalently be encoded as a *space-valued Tambara functor*.

4.1 Normed semirings

We want to consider notions of normed semirings where we have two potentially different families of ("additive" and "multiplicative") norms, generalizing the addition and multiplication operations that exist in an ordinary semiring. To capture such structures, we introduce the following definition:

Definition 4.1.1. A *semiring context* $F = (\mathcal{F}, \mathcal{F}_M, \mathcal{F}_A)$ consists of an extensive ∞-category $\mathcal{F}$ together with two weakly extensive subcategories $\mathcal{F}_M$ and $\mathcal{F}_A$ such that:

(1) $\mathcal{F}$ has pullbacks.

(2) For $m\colon X \to Y$ in $\mathcal{F}_M$, the pullback functor $m^*\colon \mathcal{F}_{/Y} \to \mathcal{F}_{/X}$ has a right adjoint $m_*\colon \mathcal{F}_{/X} \to \mathcal{F}_{/Y}$ which preserves morphisms whose image in $\mathcal{F}$ lies in $\mathcal{F}_A$.

We write $F_M := (\mathcal{F}, \mathcal{F}_M)$ and $F_A := (\mathcal{F}, \mathcal{F}_A)$ for the resulting two weakly extensive span pairs.

Observation 4.1.2. Every semiring context is a bispan triple:

- The Beck–Chevalley condition for the functors m_* is automatically satisfied, since it may be checked after passing to left adjoints.
- For $m\colon X \to Y$ in $\mathcal{F}_M$, the functor $m_*\colon \mathcal{F}_{/X} \to \mathcal{F}_{/Y}$ preserves terminal objects, hence it sends $\mathcal{F}^A_{/X}$ into $\mathcal{F}^A_{/Y}$.

Example 4.1.3. When $\mathcal{F}_A = \mathcal{F}^{\simeq}$, the triple $(\mathcal{F}, \mathcal{F}_M, \mathcal{F}^{\simeq})$ is a semiring context if and only if $\mathcal{F}$ is extensive, admits pullbacks, and is M-locally cartesian closed, in the sense of Definition 3.4.2.

Example 4.1.4. Let $\mathcal{F}$ be an extensive ∞-category that is locally cartesian closed. Then the triple $(\mathcal{F}, \mathcal{F}, \mathcal{F})$ is a semiring context.

Example 4.1.5. For a finite group G, the category $\mathbb{F}_G$ of finite G-sets is extensive and locally cartesian closed, so that the triple $(\mathbb{F}_G, \mathbb{F}_G, \mathbb{F}_G)$ is a semiring context. More generally we obtain a semiring context $(\mathbb{F}_G, I, \mathbb{F}_G)$ for every weakly extensive subcategory $I \subseteq \mathbb{F}_G$.

We fix a semiring context $F = (\mathcal{F}, \mathcal{F}_M, \mathcal{F}_A)$. Our goal in the rest of this subsection is to construct the ∞-category $\mathsf{NRig}_F(\mathcal{X})$ of *F-normed semirings in $\mathcal{X}$* for suitable choices of ∞-categories $\mathcal{X}$. We start by constructing a certain F_M-normed ∞-category $\mathsf{Span}_A(\underline{\mathcal{F}})$ of spans in $\mathcal{F}$.

Construction 4.1.6. As $\mathcal{F}$ admits pullbacks, the evaluation map $\mathrm{ev}_1\colon \mathsf{Ar}(\mathcal{F}) \to \mathcal{F}$ is a cartesian fibration, classifying the functor $\underline{\mathcal{F}}\colon \mathcal{F}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ from Notation 3.4.1. Since morphisms in $\mathcal{F}_A$ are closed under base change, we obtain a functor

$$\mathcal{F}^{\mathrm{op}} \longrightarrow \mathsf{SpanPair}, \qquad X \mapsto (\mathcal{F}_{/X}, \mathcal{F}_{/X,A})$$

where $(\mathcal{F}_{/X}, \mathcal{F}_{/X,A})$ is the span pair from Example 3.1.8; we define the $\mathcal{F}$-∞-category $\mathsf{Span}_A(\underline{\mathcal{F}})$ by composing with the functor $\mathsf{Span}\colon \mathsf{SpanPair} \to \mathsf{Cat}_\infty$. Since limits in $\mathsf{SpanPair}$ are computed in Cat_∞ and $\mathsf{Span}(-)$ preserves limits, this is indeed an $\mathcal{F}$-∞-category.

Lemma 4.1.7. *The F_M-normed ∞-category $\underline{\mathcal{F}}$ from Proposition 3.4.3 induces, via Construction 3.2.9, an F_M-normed structure on $\mathsf{Span}_A(\underline{\mathcal{F}})$.*

Proof. Equipping each $\mathcal{F}_{/X}$ with the span pair structure $(\mathcal{F}_{/X}, \mathcal{F}_{/X,A})$ from Construction 4.1.6, the functors $m_* f^*\colon \mathcal{F}_{/X} \to \mathcal{F}_{/Y}$ are morphisms of span pairs by our assumptions on F, hence we obtain an F_M-normed ∞-category $\mathsf{Span}_A(\underline{\mathcal{F}})^{\otimes}$ using Construction 3.2.9. □

Definition 4.1.8. Let $\mathcal{X}$ be a cocomplete ∞-category with finite products, such that the cartesian product preserves colimits in each variable. We define $\underline{\mathsf{NMon}}_{F_A}(\mathcal{X})$ to be the F_M-normed ∞-category $\underline{\mathsf{Fun}}^{\times}(\mathsf{Span}_A(\underline{\mathcal{F}}), \mathcal{X})$ obtained by applying Proposition 3.3.1 to the F_M-normed structure on $\mathsf{Span}_A(\underline{\mathcal{F}})$ from Construction 4.1.6.

Note that the value of $\underline{\mathsf{NMon}}_{F_A}(\mathcal{X})$ at $X \in \mathcal{F}$ is the ∞-category of $(\mathcal{F}_{/X}, \mathcal{F}_{/X,A})$-normed monoids in $\mathcal{X}$.

Example 4.1.9. Combining Proposition 3.3.8 with [CHLL24, 3.3.5], we see that when $\mathcal{X}$ is a cartesian closed presentable ∞-category, then the symmetric monoidal structure on $\mathsf{NMon}_{\mathbb{F}}(\mathcal{X}) \simeq \mathsf{CMon}(\mathcal{X})$ from Definition 4.1.8 agrees with the "standard" one constructed in [GGN15].

Remark 4.1.10. Let $f\colon X \to Y$ be any map in $\mathcal{F}$. By [BH21, C.21(2)], the adjunction $f_!\colon \mathcal{F}_{/X} \rightleftarrows \mathcal{F}_{/Y} : f^*$ induces a "wrong way" adjunction

$$f^*\colon \mathsf{Span}_A(\mathcal{F}_{/Y}) \rightleftarrows \mathsf{Span}_A(\mathcal{F}_{/X}) : f_!.$$

The underlying $\mathcal{F}$-∞-category of $\underline{\mathsf{NMon}}_{F_A}(\mathcal{X})$ therefore admits the following alternative description: it is given by the composite

$$\mathcal{F}^{\mathrm{op}} \xrightarrow{\mathsf{Span}_A(\mathcal{F}_{/-})} (\mathsf{Cat}_\infty^{\times})^{\mathrm{op}} \xrightarrow{\mathsf{Fun}^{\times}(-,\mathcal{X})} \mathsf{Cat}_\infty,$$

i.e. its functoriality is given via restriction along pushforwards.

Remark 4.1.11. If $\mathcal{X}$ is presentable and $(\mathcal{F}, \mathcal{F}_A) = (\mathbb{F}[T], \mathbb{F}[P])$ for a small ∞-category T and a left-cancellable orbital subcategory $P \subseteq T$ consisting of truncated maps, then the $\mathcal{F}$-∞-category $\underline{\mathsf{NMon}}_{F_A}(\mathcal{X})$ is studied in [CLL24, §9.2] under the name $\underline{\mathsf{Mack}}^P_T(\mathcal{X})$. Corollary 9.9 of said article establishes a universal property for this $\mathcal{F}$-∞-category, and shows that whenever P is a so-called *atomic* orbital subcategory it agrees with the $\mathcal{F}$-∞-category $\underline{\mathsf{CMon}}^P_T(\underline{\mathcal{X}}_T)$ of [CLL23a, §4.8]. In particular, if in addition $P = T$, this further agrees with Nardin's $\underline{\mathsf{CMon}}_T(\underline{\mathcal{X}}_T)$ [Nar16, 4.9].

Definition 4.1.12. Let $\mathcal{X}$ be as in Definition 4.1.8. An *F-normed semiring* in $\mathcal{X}$ is an F_M-normed algebra in $\underline{\mathsf{NMon}}_{F_A}(\mathcal{X})$; we write

$$\mathsf{NRig}_F(\mathcal{X}) := \mathsf{NAlg}_{F_M}\big(\underline{\mathsf{NMon}}_{F_A}(\mathcal{X})\big).$$

Example 4.1.13. Let $F = (\mathbb{F}, \mathbb{F}, \mathbb{F})$. Combining Examples 3.1.6 and 4.1.9, we see that $\mathsf{NRig}_F(\mathcal{X})$ agrees with the ∞-category $\mathcal{R}ig_{\mathbb{E}_\infty}(\mathcal{X})$ of $\mathbb{E}_\infty$-semirings considered by Gepner, Groth, and Nikolaus [GGN15, 7.1].

4.2 The Lawvere theory of normed semirings

Let $F = (\mathcal{F}, \mathcal{F}_M, \mathcal{F}_A)$ be a semiring context. Since F is in particular a bispan triple by Observation 4.1.2, we may form its bispan category $\mathsf{Bispan}_{M,A}(\mathcal{F})$. Our goal in this subsection is to show that the ∞-category $\mathsf{Bispan}_{M,A}(\mathcal{F})$ is the *Lawvere theory for F-normed semirings*: for any ∞-category $\mathcal{X}$ satisfying the conditions from Definition 4.1.8, the ∞-category of F-normed semirings in $\mathcal{X}$ is equivalent to the ∞-category of product-preserving functors $\mathsf{Bispan}_{M,A}(\mathcal{F}) \to \mathcal{X}$. This in particular allows us to think of an F-normed semiring R in $\mathcal{X}$ as a family of objects $R(X)$ for all $X \in \mathcal{F}$ equipped with restrictions $f^*\colon R(Y) \to R(X)$, additive norms $a_\oplus\colon R(X) \to R(Y)$, and multiplicative norms $m_\otimes\colon R(X) \to R(Y)$, which satisfy various compatibility relations exhibited by the respective composition laws in $\mathsf{Bispan}_{M,A}(\mathcal{F})$.

We start with some preliminary statements.

Proposition 4.2.1. *The F_M-normed structure on $\mathsf{Span}_A(\underline{\mathcal{F}})$ induced by the cartesian F_M-normed structure on $\underline{\mathcal{F}}$ is given by*

$$\mathsf{Span}_A(\underline{\mathcal{F}})^\otimes \simeq \mathsf{Bispan}_{M-\mathrm{pb},A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F})) = \mathsf{Span}_{A-\mathrm{fw}}(\mathsf{Span}_{M-\mathrm{pb}}(\mathsf{Ar}(\mathcal{F}))^{\mathrm{op}})$$

where $\mathsf{Ar}(\mathcal{F})_{M-\mathrm{pb}}$ denotes the wide subcategory of $\mathsf{Ar}(\mathcal{F})$ whose morphisms are pullback squares over $\mathcal{F}_M$, and $\mathsf{Span}_{M-\mathrm{pb}}(\mathsf{Ar}(\mathcal{F}))_{A-\mathrm{fw}}$ denotes the subcategory of morphisms whose image under ev_1 is an equivalence in $\mathsf{Span}_M(\mathcal{F})$ and whose forward part lives over $\mathcal{F}_A$.

Proof. This follows by combining Proposition 3.2.10 with the description of $\underline{\mathcal{F}}_\times$ from Proposition 3.4.3. □

Explicitly this means a morphism in $\mathsf{Span}_A(\underline{\mathcal{F}})^{\otimes}$ is represented by a diagram

$$\begin{array}{ccccccc} X & \longleftarrow & Y & \longrightarrow & X' & \xrightarrow{a} & X'' \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ S & \longleftarrow & T & \xrightarrow[m]{} & S' & = & S' \end{array}$$

with m in $\mathcal{F}_M$ and a in $\mathcal{F}_A$; the cocartesian fibration to $\mathsf{Span}_M(\mathcal{F})$ is given by restricting to the bottom row.

Example 4.2.2. For the semiring context $F = (\mathcal{F}, \mathcal{F}_M, \mathcal{F}^{\simeq})$ from Example 4.1.3, we obtain the F_M-normed structure on $\underline{\mathcal{F}}^{\mathrm{op}}$ given by

$$\mathsf{Span}_{M-\mathrm{pb}}(\mathsf{Ar}(\mathcal{F}))^{\mathrm{op}} \simeq (\underline{\mathcal{F}}_{\times})^{\mathrm{op}}.$$

Moreover, the underlying $\mathcal{F}$-∞-category of $\underline{\mathsf{NMon}}_{T_A}(\mathfrak{X})$ is the $\mathcal{F}$-∞-category

$$\underline{\mathfrak{X}}_{\mathcal{F}} := \mathsf{Fun}^{\times}(\underline{\mathcal{F}}^{\mathrm{op}}, \mathfrak{X}) = \mathsf{Fun}^{\times}(-, \mathfrak{X}) \circ \underline{\mathcal{F}}^{\mathrm{op}}$$

of $\mathcal{F}$-objects in $\mathfrak{X}$.

Corollary 4.2.3. *The ∞-category $\mathsf{NRig}_F(\mathfrak{X})$ of F-normed semirings in $\mathfrak{X}$ is naturally equivalent to the full subcategory $\mathcal{R} \subseteq \mathsf{Fun}(\mathsf{Bispan}_{M-\mathrm{pb},A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F})), \mathfrak{X})$ spanned by functors*

$$\Phi\colon \mathsf{Bispan}_{M-\mathrm{pb},A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F})) \longrightarrow \mathfrak{X}$$

such that

(1) *For every object $E \to X$ in $\mathsf{Bispan}_{M-\mathrm{pb},A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F}))$, where E decomposes as a coproduct $\coprod_{i=1}^n E_i$ in $\mathcal{F}$, evaluating Φ at the morphisms*

$$\begin{array}{ccccccc} E & \longleftarrow & E_i & = & E_i & = & E_i \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ X & = & X & = & X & = & X \end{array} \qquad (11)$$

gives an equivalence

$$\Phi(E \longrightarrow X) \xrightarrow{\sim} \prod_{i=1}^{n} \Phi(E_i \longrightarrow X).$$

(2) *Φ takes morphisms of the form*

$$\begin{array}{ccccccc} E & = & E & = & E & = & E \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ Y & \longleftarrow & X & = & X & = & X \end{array} \qquad (12)$$

in $\mathsf{Bispan}_{M-\mathrm{pb},A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F}))$ to equivalences in $\mathfrak{X}$.

Moreover, for every $X \in \mathcal{F}$ this equivalence fits into a commutative diagram

$$\begin{array}{ccc} \mathsf{NRig}_F(\mathcal{X}) & \xrightarrow{\simeq} & \mathcal{R} \\ & {\scriptstyle \mathrm{ev}_A} \searrow \quad \swarrow {\scriptstyle \iota_X^*} & \\ & \mathsf{Fun}^\times(\mathsf{Span}_A(\mathcal{F}_{/X}), \mathcal{X}) & \end{array} \tag{13}$$

where ι_X is induced by $(\mathcal{F}_{/X}, (\mathcal{F}_{/X})^{\simeq}, \mathcal{F}_{/X,A}) \hookrightarrow (\mathsf{Ar}(\mathcal{F}), \mathsf{Ar}(\mathcal{F})_{M-\mathrm{pb}}, \mathsf{Ar}(\mathcal{F})_{A-\mathrm{fw}})$.

Proof. In light of Proposition 4.2.1, we only have to show that the description of the full subcategory $\mathcal{R}$ is equivalent to the description given in Proposition 3.3.1.

Since products in $\mathsf{Span}_A(\mathcal{F}_{/X})$ are given by coproducts in $\mathcal{F}$, the first condition amounts to asking for the restriction

$$\Phi_X \colon \mathsf{Span}_A(\mathcal{F}_{/X}) \longrightarrow \mathcal{X}$$

of Φ to preserve products for every $X \in \mathcal{F}$, which is the first condition formulated in Proposition 3.3.1.

We now claim that (12) defines a cartesian edge over $Y \leftarrow X = X$; Observation 3.3.7 will then immediately show that our second condition is indeed equivalent to the second condition of Proposition 3.3.1. For this we observe that restricting in the target to $\mathcal{F}^{\mathrm{op}} = \mathsf{Span}_{\mathrm{eq}}(\mathcal{F})$ recovers the map $\mathsf{Span}_{A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F})) \to \mathcal{F}^{\mathrm{op}}$ classifying $\mathsf{Span}_A(\underline{\mathcal{F}})$. The subfibration $\mathsf{Ar}(\mathcal{F})^{\mathrm{op}} \to \mathcal{F}^{\mathrm{op}}$ is both cartesian and cocartesian, with cartesian edges given by the squares in question (the *co*cartesian edges of $\mathsf{Ar}(\mathcal{F}) \to \mathcal{F}$). We therefore want to show that this is still cartesian in $\mathsf{Span}_{A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F}))$. However, this simply means that the adjunction $f^* \colon (\mathcal{F}_{/X})^{\mathrm{op}} \rightleftarrows (\mathcal{F}_{/Y})^{\mathrm{op}} : f_!$ ought to extend to $\mathsf{Span}_A(\mathcal{F}_{/X}) \rightleftarrows \mathsf{Span}_A(\mathcal{F}_{/Y})$, which was observed in Remark 4.1.10 above. ◻

Theorem 4.2.4. *Composition with* $\mathrm{ev}_0 \colon \mathsf{Bispan}_{M-\mathrm{pb},A-\mathrm{fw}}(\mathsf{Ar}(\mathcal{F})) \to \mathsf{Bispan}_{M,A}(\mathcal{F})$ *induces an equivalence*

$$\mathsf{NRig}_F(\mathcal{X}) \xrightarrow{\sim} \mathsf{Fun}^\times(\mathsf{Bispan}_{M,A}(\mathcal{F}), \mathcal{X})$$

fitting into commutative diagrams

$$\begin{array}{ccc} \mathsf{NRig}_F(\mathcal{X}) & \xrightarrow{\simeq} & \mathsf{Fun}^\times(\mathsf{Bispan}_{M,A}(\mathcal{F}), \mathcal{X}) \\ & {\scriptstyle \mathrm{ev}_X} \searrow \quad \swarrow {\scriptstyle p_X^*} & \\ & \mathsf{Fun}^\times(\mathsf{Span}_A(\mathcal{F}_{/X}), \mathcal{X}) & \end{array} \tag{14}$$

where p_X is induced by the forgetful map $(\mathcal{F}_{/X}, (\mathcal{F}_{/X})^{\simeq}, \mathcal{F}_{/X,A}) \to (\mathcal{F}, \mathcal{F}_M, \mathcal{F}_A)$.

Proof. By [CHLL24, 4.2.2], the functor ev_0 on bispans is a localization. Let W be the class of morphisms it takes to equivalences, which we can immediately simplify to those of the form

$$\begin{array}{ccccccc} X & = & X & = & X & = & X \\ \downarrow & & \downarrow \lrcorner & & \downarrow & & \downarrow \\ S & \longleftarrow & T & \longrightarrow & S' & = & S', \end{array}$$

and let $S \subseteq W$ be the morphisms of the form (12). Arguing as in the proof of [CHLL24, 4.3.1], we see that a functor that inverts S must invert all of W. From Corollary 4.2.3 we know that an F-normed semiring inverts the morphisms in S, and it therefore factors through the localization. All that remains for the equivalence $\mathsf{NRig}_F(\mathfrak{X}) \simeq \mathsf{Fun}^{\times}(\mathsf{Bispan}_{M,A}(\mathcal{F}), \mathfrak{X})$ is to show that a functor $\Phi\colon \mathsf{Bispan}_{M,A}(\mathcal{F}) \to \mathfrak{X}$ preserves products if and only if the composite $\Psi := \Phi \circ \mathrm{ev}_0$ satisfies

$$\Psi(E \longrightarrow X) \xrightarrow{\sim} \prod_i \Psi(E_i \longrightarrow X)$$

for $E \simeq \coprod_i E_i$. Since the product in $\mathsf{Bispan}_{M,A}(\mathcal{F})$ is given by the coproduct in $\mathcal{F}$ by Proposition 2.3.5, the condition for Ψ is immediate if Φ preserves products. Conversely, for any coproduct decomposition $E \simeq \coprod E_i$, we can apply the condition for Ψ with $X = E$ to conclude that Φ preserves this product.

Finally, the commutativity of (14) follows at once from the commutativity of (13) and the observation $p_X = \mathrm{ev}_0 \circ \iota_X$. □

Remark 4.2.5. By Theorem 4.2.4, an F-normed semiring in $\mathfrak{X}$ may be identified with a product-preserving functor $R\colon \mathsf{Bispan}_{M,A}(\mathcal{F}) \to \mathfrak{X}$, and thus gives rise to maps $f^*\colon R(Y) \to R(X)$, $a_\oplus\colon R(X) \to R(Y)$ and $m_\otimes\colon R(X) \to R(Y)$ for morphisms $f, a, m\colon X \to Y$ in $\mathcal{F}$ such that $a \in \mathcal{F}_A$ and $m \in \mathcal{F}_M$. Each of these classes of maps are compatible with composition in $\mathcal{F}$. Furthermore, given pullback squares

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle a'}\downarrow & \lrcorner & \downarrow{\scriptstyle a} \\ Y' & \xrightarrow{f} & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle m'}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ Y' & \xrightarrow{f} & Y \end{array}$$

with $a \in \mathcal{F}_A$ and $m \in \mathcal{F}_M$, the composition relations in $\mathsf{Bispan}_{M,A}(\mathcal{F})$ give rise to relations $f^* a_\oplus \simeq a'_\oplus g^*$ and $f^* m_\otimes \simeq m'_\otimes g^*$ of maps $R(X) \to R(Y')$. Finally, given morphisms $a\colon X \to Y$ in $\mathcal{F}_A$ and $m\colon Y \to Z$ in $\mathcal{F}_M$, we may consider their associated "distributivity diagram" [EH23, 2.4.1]

$$\begin{array}{ccccc} X & \xleftarrow{e} & m^* m_*(X) & \xrightarrow{m'} & m_*(X) \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle b'} & \lrcorner & \downarrow{\scriptstyle b = m_*(a)} \\ & & Y & \xrightarrow{m} & Z, \end{array}$$

where e is the counit of the adjunction $m^* \dashv m_*$; we then obtain a distributivity relation $m_\otimes a_\oplus \simeq b_\oplus m'_\otimes e^*$ of maps $R(X) \to R(Z)$.

Example 4.2.6. Specializing Theorem 4.2.4 to $F = (\mathbb{F}, \mathbb{F}, \mathbb{F})$ as in Example 4.1.13, we obtain equivalences $\mathcal{R}ig_{\mathbb{E}_\infty}(\mathcal{X}) \simeq \mathsf{NRig}_F(\mathcal{X}) \simeq \mathsf{Fun}^\times(\mathsf{Bispan}(\mathbb{F}), \mathcal{X})$, recovering the main result of [CHLL24].

Example 4.2.7. Our main interest is the case $F = (\mathbb{F}_G, \mathbb{F}_G, \mathbb{F}_G)$ for a finite group G, which will be discussed extensively in Section 5.

Example 4.2.8. Applying Theorem 4.2.4 to the case for trivial additive norms from Example 4.2.2, we deduce that F-normed monoids in $\mathcal{X}$ admit an interpretation as F-normed algebras:

$$\mathsf{NAlg}_F(\mathsf{Fun}^\times(\underline{\mathcal{F}}^{\mathrm{op}}, \mathcal{X})) \simeq \mathsf{Fun}^\times(\mathsf{Span}_N(\mathcal{F}), \mathcal{X}) = \mathsf{NMon}_F(\mathcal{X}).$$

We may think of this as the normed analogue of the statement that commutative monoids in $\mathcal{X}$ are the commutative algebras with respect to the cartesian symmetric monoidal structure on $\mathcal{X}$.

4.3 Normed rings and Tambara functors

Fix a cocomplete ∞-category $\mathcal{X}$ with finite products such that the cartesian product preserves colimits in each variable. Among the normed semirings in $\mathcal{X}$, we are especially interested in those that behave like *rings*, in the sense that their underlying additive monoid is in fact a group:

Definition 4.3.1. Suppose $F_A = (\mathcal{F}, \mathcal{F}_A)$ is an extensive span pair (and not only a *weakly* extensive one). We then say an F_A-normed monoid $M\colon \mathsf{Span}_A(\mathcal{F}) \to \mathcal{X}$ is *grouplike* if the induced commutative monoid structure on $M(X)$ from Observation 3.1.2 is grouplike in the usual sense for every $X \in \mathcal{F}$. We also refer to grouplike F_A-normed monoids as *F_A-normed groups* and write $\mathsf{NGrp}_{F_A}(\mathcal{X}) \subseteq \mathsf{NMon}_{F_A}(\mathcal{X})$ for the full subcategory of these. Note that under the equivalence (6), the full subcategory $\mathsf{NGrp}_{F_A}(\mathcal{X})$ corresponds to the subcategory

$$\mathsf{NMon}_{F_A}(\mathsf{CGrp}(\mathcal{X})) \subseteq \mathsf{NMon}_{F_A}(\mathsf{CMon}(\mathcal{X})) \simeq \mathsf{NMon}_{F_A}(\mathcal{X}).$$

If $\mathcal{X}$ is presentable, then $\mathsf{NGrp}_{F_A}(\mathcal{X})$ is an accessible localization of $\mathsf{NMon}_{F_A}(\mathcal{X})$, and so is again presentable.

Definition 4.3.2. A *ring context* is a semiring context $F = (\mathcal{F}, \mathcal{F}_M, \mathcal{F}_A)$ such that the span pair $F_A = (\mathcal{F}, \mathcal{F}_A)$ is extensive. An *F-normed ring in $\mathcal{X}$* is an F-normed semiring $R\colon \mathsf{Span}_M(\mathcal{F}) \to \underline{\mathsf{NMon}_{F_A}}(\mathcal{X})^\otimes$ such that for all $X \in \mathcal{F}$ the resulting $F_{/X,A}$-normed monoid $R_X\colon \mathsf{Span}_A(\mathcal{F}_{/X}) \to \mathcal{X}$ is an $F_{/X,A}$-normed group; here $F_{/X,A} = (\mathcal{F}_{/X}, \mathcal{F}_{/X,A})$ denotes the span pair from Example 3.1.8.

Example 4.3.3. The semiring contexts arising in equivariant mathematics, discussed in Example 4.1.5, are always ring contexts.

Warning 4.3.4. In the generality of our setup, the F_M-normed structure on $\underline{\mathsf{NMon}}_{F_A}(\mathcal{X})$ need not descend to grouplike objects: the latter may only form what should be called a F_M-∞-*operad*, and the previous definition could then be more succinctly phrased as saying that an F-normed ring is an F_M-normed algebra in this parametrized ∞-operad.

However, we will show in the next section that such a normed structure *does* exist in the setting of equivariant homotopy theory, which is the main case of interest to us.

Definition 4.3.5. A product-preserving functor $\mathsf{Bispan}_{M,A}(\mathcal{F}) \to \mathcal{X}$ is called an ($\mathcal{X}$-valued) F-*Tambara functor* if its restriction to $\mathsf{Span}_A(\mathcal{F}) \simeq \mathsf{Bispan}_{\mathrm{eq},A}(\mathcal{F})$ is grouplike in the sense of Definition 4.3.1. We write

$$\mathsf{Tamb}_F(\mathcal{X}) \subseteq \mathsf{Fun}^\times(\mathsf{Bispan}_{M,A}(\mathcal{F}), \mathcal{X})$$

for the full subcategory spanned by the Tambara functors.

Theorem 4.3.6. *Let $F = (\mathcal{F}, \mathcal{F}_M, \mathcal{F}_A)$ be a ring context. Then the equivalence $\mathsf{NRig}_F(\mathcal{X}) \simeq \mathsf{Fun}^\times(\mathsf{Bispan}_{M,A}(\mathcal{F}), \mathcal{X})$ constructed in Theorem 4.2.4 restricts to*

$$\mathsf{NRing}_F(\mathcal{X}) \simeq \mathsf{Tamb}_F(\mathcal{X}).$$

Proof. Write $\Phi\colon \mathsf{NRig}_F(\mathcal{X}) \to \mathsf{Fun}^\times(\mathsf{Bispan}_{M,A}(\mathcal{F}), \mathcal{X})$ for the equivalence from Theorem 4.2.4; we have to show that an F-normed semiring $R \in \mathsf{NRig}_F(\mathcal{X})$ is an F-normed ring if and only if $\Phi(R)$ is grouplike.

By commutativity of (14), R is an F-normed ring if and only if $\Phi(R) \circ p_X$ is grouplike for every $X \in \mathcal{F}$; we have to show that this is in turn equivalent to the composite $\phi\colon \mathsf{Span}_A(\mathcal{F}) \to \mathsf{Bispan}_{M,A}(\mathcal{F}) \to \mathcal{X}$ being grouplike. But indeed, if $\hat{\phi}\colon \mathsf{Span}_A(\mathcal{F}) \to \mathsf{CMon}(\mathcal{X})$ is the unique lift of ϕ, its restriction along $\mathsf{Span}_A(\mathcal{F}_{/X}) \to \mathsf{Span}_A(\mathcal{F})$ is a lift of $\Phi(R) \circ p_X$; the claim follows as the functors $\mathsf{Span}_A(\mathcal{F}_{/X}) \to \mathsf{Span}_A(\mathcal{F})$ for varying $X \in \mathcal{F}$ are jointly surjective. □

5 Normed equivariant spectra

In this section, we will prove the main results of our paper: In particular, we will define the G-normed ∞-category of G-spectra, compare it to G-commutative groups, and then finally specialize the results of the previous sections to describe connective normed G-spectra in terms of Tambara functors.

Convention 5.0.1. We will fix the ring context $F = (\mathbb{F}_G, \mathbb{F}_G, \mathbb{F}_G)$ throughout the whole section, write "G-∞-category" instead of "$\mathbb{F}_G$-∞-category", and write "normed" instead of "F-normed." It will be convenient to think of normed G-∞-categories as functors $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{CMon}(\mathsf{Cat}_\infty)$, via Observation 3.1.2. We will furthermore repurpose the notation $\mathcal{C}^\otimes$ to refer to a normed G-∞-category $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{CMon}(\mathsf{Cat}_\infty)$ with underlying G-∞-category $\mathcal{C}\colon \mathbb{F}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

5.1 Presentable G-∞-categories

Unlike in the rest of this paper, we will need a fair bit of parametrized higher category theory [BDG⁺16, MW21] in this section, and we begin by recalling some of the basic terminology. For simplicity, we will restrict to the case of G-∞-categories here, although our references work in much greater generality.

Construction 5.1.1. The ∞-category $\mathsf{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \simeq \mathsf{Fun}(\mathbf{O}_G^{\mathrm{op}}, \mathsf{Cat}_\infty)$ of G-∞-categories is cartesian closed. We write $\underline{\mathsf{Fun}}_G$ for the internal hom, and $\mathsf{Fun}_G = \mathrm{ev}_{G/G} \circ \underline{\mathsf{Fun}}_G$ for its underlying ordinary ∞-category.

Using Fun_G, we can view the ∞-category of G-∞-categories as an $(\infty, 2)$-category; all that we will need below is that this enhances the homotopy 1-category to a 2-category. In particular, we obtain a natural notion of *adjunctions* between G-∞-categories. The following recognition principle will be useful:

Lemma 5.1.2 ([MW21, 3.2.9 and 3.2.11])**.** *A functor $F\colon \mathscr{C} \to \mathscr{D}$ of G-∞-categories admits a right adjoint if and only if the following hold:*

(1) For each $X \in \mathbf{O}_G$ (or equivalently for each $X \in \mathbb{F}_G$), the functor $F_X\colon \mathscr{C}(X) \to \mathscr{D}(X)$ admits a right adjoint G_X in the usual sense.

(2) For each $f\colon X \to Y$ in $\mathbf{O}_G$ (or equivalently for f in $\mathbb{F}_G$) the Beck–Chevalley transformation $f^*G_Y \to G_X f^*$ is invertible. □

Definition 5.1.3. A G-∞-category $\mathscr{C}\colon \mathbb{F}_G \to \mathsf{Cat}_\infty$ is called *presentable* if it satisfies all of the following conditions:

(1) It factors through Pr^{L}.

(2) For each $g\colon C \to D$ in $\mathbb{F}_G$ the functor $g^*\colon \mathscr{C}(D) \to \mathscr{C}(C)$ admits a left adjoint $g_!$, and for any pullback square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ C & \xrightarrow[g]{} & D \end{array}$$

the Beck–Chevalley map $f_!p^* \to q^*g_!$ is invertible.

Corollary 5.1.4 ([MW22, 6.3.1])**.** *A functor $F\colon \mathscr{C} \to \mathscr{D}$ of presentable G-∞-categories is a left adjoint if and only if it* ***preserves G-colimits*** *in the following sense: F is left adjointable, and for every $X \in \mathbf{O}_G$ (or equivalently $\mathbb{F}_G$) the functor $\mathscr{C}(X) \to \mathscr{D}(X)$ preserves ordinary colimits.*

Proof. By the Adjoint Functor Theorem the latter condition is equivalent to each $\mathscr{C}(X) \to \mathscr{D}(X)$ admitting a right adjoint G_X. By passing to total mates, the former condition is then equivalent to the Beck–Chevalley maps $f^*G_Y \to G_X f^*$ being invertible. Thus, the claim follows from Lemma 5.1.2. □

Definition 5.1.5. We denote the ∞-category of presentable G-∞-categories and left adjoint (equivalently: G-cocontinuous) functors by $\mathsf{Pr}^{\mathrm{L}}_{G}$.

Remark 5.1.6. For every $\mathscr{C}, \mathscr{D} \in \mathsf{Pr}^{\mathrm{L}}_{G}$, there is a G-subcategory $\underline{\mathsf{Fun}}^{\mathrm{L}}_{G}(\mathscr{C}, \mathscr{D}) \subseteq \underline{\mathsf{Fun}}_{G}(\mathscr{C}, \mathscr{D})$ of the internal hom, given in degree G/G by the full subcategory of left adjoint functors $\mathscr{C} \to \mathscr{D}$; we refer the reader to [CLL23a, 2.3.22] or [MW21, discussion before 3.3.6] for details. As we will recall in the proof of Theorem 5.4.10 below, this is the internal hom for a parametrized analogue of the Lurie tensor product.

5.2 The G-∞-category of G-spaces

As a warm-up and an ingredient for the construction of the normed G-∞-category of G-spectra, we will recall two equivalent constructions of the G-∞-category of G-*spaces* in this subsection, and show that it admits a unique normed structure interacting suitably with (pointwise) colimits.

We begin with a construction via classical equivariant homotopy theory:

Construction 5.2.1. Let SSet be the 1-category of simplicial sets. Applying Construction A.1.1, we obtain a *Borel G-category* $\mathsf{SSet}^\flat$, given slightly informally as follows: $\mathsf{SSet}^\flat$ sends G/H to the category of simplicial sets with (strict) H-action, with contravariant functoriality via restricting the action.

We now equip each $\mathsf{SSet}^\flat(G/H) = \mathsf{Fun}(BH, \mathsf{SSet})$ with the H-equivariant weak equivalences, i.e. those maps f such that f^K is a weak homotopy equivalence for every $K \leqslant H$. As these are clearly stable under restriction, this defines a lift of $\mathsf{SSet}^\flat$ to a functor from $\mathbf{O}_G^{\mathrm{op}}$ to relative categories. Postcomposing with Dwyer–Kan localization, we therefore obtain a G-∞-category $\underline{\mathfrak{S}}_G$.

We moreover write $\underline{\mathfrak{F}}_G$ for the Borel G-category $\mathbb{F}^\flat$; equivalently, this is the full subcategory of $\underline{\mathfrak{S}}_G$ spanned in degree G/H by the finite H-sets.

Next, let us compare this to a purely ∞-categorical construction.

Construction 5.2.2. We write $\underline{\mathbb{F}}_G$ for the G-∞-category $X \mapsto (\mathbb{F}_G)_{/X}$ and $\underline{\mathsf{Spc}}_G$ for the G-∞-category $\mathscr{P}_\Sigma(\underline{\mathbb{F}}_G) \coloneqq \mathsf{Fun}^\times(\underline{\mathbb{F}}_G^{\mathrm{op}}, \mathsf{Spc})$, with functoriality via left Kan extension (cf. Proposition 3.3.2). By [HHLN23, 8.1] the Yoneda embeddings assemble into a G-functor $\underline{\mathbb{F}}_G \hookrightarrow \underline{\mathsf{Spc}}_G$, exhibiting the target as the pointwise sifted cocompletion.

Remark 5.2.3. Equivalently, $\underline{\mathsf{Spc}}_G$ is obtained from the "co-G-∞-category" $\mathbb{F}_G \to \mathsf{Cat}_\infty, X \mapsto (\mathbb{F}_G)_{/X}$ (functoriality via pushforward) by applying the *contravariant* functor $\mathsf{Fun}^\times(-, \mathsf{Spc})$. As the inclusion $(\mathbf{O}_G)_{/X} \hookrightarrow (\mathbb{F}_G)_{/X}$ is the finite coproduct completion for every $X \in \mathbf{O}_G$, we can also describe this as the functor $\mathbf{O}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty, X \mapsto \mathsf{Fun}((\mathbf{O}_G)_{/X}^{\mathrm{op}}, \mathsf{Spc})$. The latter description serves as the definition of $\underline{\mathsf{Spc}}_G$ in [CLL23b].

Theorem 5.2.4. *There are unique equivalences $\underline{\mathfrak{S}}_G \simeq \underline{\mathsf{Spc}}_G$ and $\underline{\mathfrak{F}}_G \simeq \underline{\mathbb{F}}_G$ of G-∞-categories. Moreover, these equivalences fit into a commutative diagram*

$$\begin{array}{ccc} \underline{\mathfrak{F}}_G & \hookrightarrow & \underline{\mathfrak{S}}_G \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \underline{\mathbb{F}}_G & \underset{\text{Yoneda}}{\hookrightarrow} & \underline{\mathsf{Spc}}_G. \end{array}$$

Proof. By [CLL23b, 5.12] there exists a unique equivalence $\Phi\colon \underline{\mathfrak{S}}_G \xrightarrow{\sim} \underline{\mathsf{Spc}}_G$. By virtue of being an equivalence, this preserves (levelwise) terminal objects and G-colimits.

Let now $K \leqslant H \leqslant G$. Then both the restriction $i^*\colon \mathsf{Fun}(BH, \mathsf{SSet}) \to \mathsf{Fun}(BK, \mathsf{SSet})$ as well as its 1-categorical left adjoint $i_!$ are homotopical; thus, the ∞-categorical left adjoint $i_!\colon \underline{\mathfrak{S}}_G(G/K) \to \underline{\mathfrak{S}}_G(G/H)$ can simply be computed by the 1-categorical left adjoint. In the same way, we see that terminal objects and coproducts in $\underline{\mathfrak{S}}_G$ can be computed in the 1-category SSet. In particular, we have $H/K = i_!i^*(1_K)$ in $\underline{\mathfrak{S}}_G$, whence

$$\Phi(H/K) \simeq i_!i^*(\mathrm{id}_{G/H}) \simeq (G/K \twoheadrightarrow G/H) \in (\mathbb{F}_G)_{/(G/H)}.$$

It follows by direct inspection that Φ maps $\mathbf{O}_H \subseteq \underline{\mathfrak{F}}_G(G/H)$ essentially surjectively into $(\mathbf{O}_G)_{/(G/H)} \subseteq \underline{\mathbb{F}}_G(G/H)$, and closing up under finite coproducts we see that the equivalence Φ restricts to a functor $\phi\colon \underline{\mathbb{F}}_G \to \underline{\mathfrak{F}}_G$ that is essentially surjective, and hence itself an equivalence.

Finally, $\underline{\mathbb{F}}_G$ has no non-trivial automorphisms by [CLL23a, 4.2.17], which completes the proof of the proposition. □

Remark 5.2.5. As recalled in [CLL23a, 2.4.11], $\underline{\mathsf{Spc}}_G$ is the *free presentable G-∞-category on a point* in the following sense: for any $\mathscr{C} \in \mathsf{Pr}^{\mathrm{L}}_G$ evaluation at the terminal object of $\mathsf{Spc}_G(G/G)$ defines an equivalence $\underline{\mathsf{Fun}}^{\mathrm{L}}_G(\underline{\mathsf{Spc}}_G, \mathscr{C}) \xrightarrow{\sim} \mathscr{C}$.

We can also give a pointed version of the above comparison:

Construction 5.2.6. Consider the category SSet_* of pointed simplicial sets. As before, we can associate to this a G-1-category $\mathsf{SSet}^\flat_*$, $G/H \mapsto \mathsf{Fun}(BH, \mathsf{SSet}_*)$, which we then localize at the (underlying) equivariant weak equivalences to obtain a G-∞-category $\underline{\mathfrak{S}}_{G,*}$. As the equivariant weak equivalences are part of a left proper model structure, we get a natural equivalence $\underline{\mathfrak{S}}_{G,*} \simeq (\underline{\mathfrak{S}}_G)_*$ compatible with the forgetful functors.

We further write $\underline{\mathfrak{F}}_{G,*}$ for the full subcategory spanned in degree G/H by the finite pointed H-sets, so that $\underline{\mathfrak{F}}_{G,*} \simeq (\underline{\mathfrak{F}}_G)_*$.

Corollary 5.2.7. *There is a commutative diagram*

$$\begin{array}{ccc} \underline{\mathfrak{F}}_{G,*} & \hookrightarrow & \underline{\mathfrak{S}}_{G,*} \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ (\underline{\mathbb{F}}_G)_* & \hookrightarrow & (\underline{\mathsf{Spc}}_G)_*. \end{array}$$

in which the vertical maps are equivalences and the top and bottom vertical arrow exhibit their targets as sifted cocompletion of the respective sources.

Proof. In light of Theorem 5.2.4, the only non-trivial statement is that the horizontal maps define sifted cocompletions. For this it will be enough to consider the bottom arrow, where this is an immediate consequence of [BH21, 4.1] as every $(Y \to X \to Y) \in \left((\mathbb{F}_G)_{/Y}\right)_*$ is disjointly based. □

Next, we turn our attention to normed structures on these G-∞-categories; we restrict to the pointed case here (as this is the only one we will need below), although the unbased case is analogous.

Proposition 5.2.8. *There exists a unique normed structure on $\underline{\mathfrak{F}}_{G,*}$ with unit S^0 such that the symmetric monoidal product on $\underline{\mathfrak{F}}_{G,*}(G/e) = \mathbb{F}_*$ preserves finite coproducts in each variable.*

Proof. By Corollary A.2.4, it will suffice that $\mathbb{F}_*$ (with trivial G-action) has a unique lift to $\mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}_\infty))$ with unit S^0 and for which the tensor product preserves finite coproducts in each variable. Consider for this the version of the Lurie tensor product on $\mathsf{Cat}^{\amalg}$ representing functors that preserve finite coproducts in each variable. Then $(\mathbb{F}_*, S^0)$ is an idempotent object for this tensor product by e.g. [CLL23a, 4.7.6], whence it is also idempotent in $\mathsf{Fun}(BG, \mathsf{Cat}^{\amalg})$ with the levelwise symmetric monoidal structure. The claim now follows from [Lur17, 4.8.2.9]. □

Combining this with Corollary 5.2.7 and the universal property of sifted cocompletion, we get:

Corollary 5.2.9. *There exists a unique G-normed structure on $\underline{\mathfrak{S}}_{G,*}$ together with a lift of $\underline{\mathfrak{F}}_{G,*} \hookrightarrow \underline{\mathfrak{S}}_{G,*}$ to a normed G-functor such that the following two conditions are satisfied:*

(1) For each $X \in \mathbb{F}_G$, $\underline{\mathfrak{S}}_{G,*}(X)$ is presentably symmetric monoidal.

(2) For each $f: X \to Y$ in $\mathbb{F}_G$, the functor $f_\otimes: \underline{\mathfrak{S}}_{G,*}(X) \to \underline{\mathfrak{S}}_{G,*}(Y)$ preserves sifted colimits.

The analogous statement for $\underline{\mathbb{F}}_{G,*} \hookrightarrow \underline{\mathsf{Spc}}_{G,*}$ holds, and for these normed structures there is a unique normed equivalence $\underline{\mathfrak{S}}^{\otimes}_{G,*} \simeq \underline{\mathsf{Spc}}^{\otimes}_{G,*}$. □

Let us make the normed structure from Corollary 5.2.9 explicit for our favorite model:

Construction 5.2.10. We equip SSet_* with the symmetric monoidal structure coming from the smash product. This then yields a normed structure on the Borel G-category $\mathsf{SSet}^\flat_*$ via Proposition A.2.1. The symmetric monoidal structure on the individual categories $\mathsf{SSet}^\flat_*(G/H)$ is then given by the usual smash product (Observation A.2.5), while Corollary A.3.6 shows that the map

$i_\otimes\colon \mathsf{Fun}(BK, \mathsf{SSet}_*) \to \mathsf{Fun}(BH, \mathsf{SSet}_*)$ for subgroups $K \leqslant H \leqslant G$ is given by the classical *symmetric monoidal norm*, i.e. it sends a K-simplicial set X to $X^{\wedge n}$ where $n = |H/K|$ and H acts on $X^{\wedge n}$ by restricting the natural $\Sigma_n \wr K$-action along a certain homomorphism $H \to \Sigma_n \wr K$; see Construction A.3.4 for details.

Proposition 5.2.11. *The previous construction localizes to a normed structure on $\underline{\mathfrak{S}}_{G,*}$, and this is the normed structure from Corollary 5.2.9.*

Proof. Since $\mathsf{SSet}^\flat_*$ comes with a normed G-functor $\mathbb{F}^\flat_* \to \mathsf{SSet}^\flat_*$ by construction, the only non-trivial statement is that this localizes to a normed structure satisfying the assumptions (1) and (2) of Corollary 5.2.9.

It is clear that the smash product of pointed H-simplicial sets preserves weak equivalences in each variable and is a left Quillen bifunctor (with respect to the model structures where cofibrations are levelwise injections). Thus, it descends to make each $\underline{\mathfrak{S}}_{G,*}(G/H)$ into a presentably symmetric monoidal ∞-category. It remains to show that for every $K \leqslant H$ the symmetric monoidal norm functor $i_\otimes\colon \mathsf{Fun}(BK, \mathsf{SSet}_*) \to \mathsf{Fun}(BH, \mathsf{SSet}_*)$ preserves weak equivalences, and that the resulting functor on localizations preserves sifted colimits.

For the first statement, let f be a K-equivariant weak equivalence; we have to show that for any $j\colon L \hookrightarrow H$ the map $(j^*i_\otimes f)^L$ is a weak equivalence. Rewriting the cospan $G/H \to G/K \leftarrow G/L$ as a span, we see that this splits as a smash product of maps $(i'_\otimes j'^* f)^L$, i.e. after renaming we are reduced to showing that $(i_\otimes f)^H$ is a weak equivalence. But this map agrees with f^K by direct inspection.

For the second statement, we claim that the functor of 1-categories $X \mapsto X^{\wedge n}$ preserves filtered colimits and geometric realization up to *isomorphism*: the first statement is clear, while the second one follows from the fact that geometric realization is given by taking the diagonal of the associated bisimplicial set. As both of these operations are homotopical by [Len20, 1.1.2 and 1.2.57], it follows that $i_\otimes\colon \underline{\mathfrak{S}}_{G,*}(G/K) \to \underline{\mathfrak{S}}_{G,*}(G/H)$ commutes with filtered colimits and Δ^{op}-shaped colimits, hence with all sifted colimits as claimed. □

5.3 Norms on G-Mackey functors

As a next step, we will show that also the G-∞-category of normed G-monoids/G-Mackey functors from Example 3.1.4 admits a unique normed structure. As a special case of Definition 4.1.8, we obtain one such normed structure

$$\underline{\mathsf{NMon}}_G \coloneqq \underline{\mathsf{NMon}}_G(\mathsf{Spc}) = \mathsf{Fun}^\times(\mathsf{Span}(\underline{\mathbb{F}}_G), \mathsf{Spc}).$$

We begin by relating it to the unstable world:

Proposition 5.3.1. *There exists a unique G-left adjoint $\mathbb{P}\colon \underline{\mathfrak{S}}_{G,*} \to \underline{\mathsf{NMon}}_G$ sending S^0 to $\mathsf{Map}(1, -)$. Moreover, this functor upgrades (canonically) to a normed G-functor.*

Proof. For the first statement, we may equivalently consider $\underline{\mathsf{Spc}}_{G,*} = \underline{\mathsf{Spc}}_G \otimes \mathsf{Spc}_*$ in lieu of $\underline{\mathfrak{S}}_{G,*}$. In this case, the existence and uniqueness of the G-left adjoint $\mathbb{P}$

follows via [CLL24, 7.39] from the universal property of $\underline{\mathsf{Spc}}_G$ (Remark 5.2.5) and the fact that the non-parametrized presentable ∞-category Spc_* is the mode for pointed presentable ∞-categories [Lur17, 4.8.2.11].

To complete the proof, we will now construct a G-left adjoint normed G-functor $\underline{\mathfrak{S}}^{\otimes}_{G,*} \to \underline{\mathsf{NMon}}^{\otimes}_G$. For this, note first that by [CLL23a, 4.7.6] the inclusion $\mathbb{F} \hookrightarrow \mathsf{Span}(\mathbb{F})$ extends (uniquely) to a coproduct-preserving functor $j\colon \mathbb{F}_* \to \mathsf{Span}(\mathbb{F})$, and as both sides are idempotents in $\mathsf{Cat}^{\amalg}$ (see [Har20, 5.3] for the target) this uniquely upgrades to a symmetric monoidal functor. Passing to Borel G-∞-categories we obtain a normed G-functor

$$\underline{\mathfrak{F}}_{G,*} = \underline{\mathbb{F}}_G^{\flat} \longrightarrow \mathsf{Span}(\mathbb{F})^{\flat} \simeq \mathsf{Span}(\underline{\mathbb{F}}_G)$$

sending S^0 to $\mathbf{1}$; here the final equivalence uses that $\mathsf{Span}(\underline{\mathbb{F}}_G) \simeq \mathsf{Span} \circ \underline{\mathfrak{F}}_{G,*}$ is a Borel G-∞-category as postcomposing with the limit preserving functor Span preserves right Kan extensions. Passing to sifted cocompletions and using that

$$\mathsf{Fun}^{\times}(\mathsf{Span}(\underline{\mathbb{F}}_G), \mathsf{Spc}) \simeq \mathsf{Fun}^{\times}(\mathsf{Span}(\underline{\mathbb{F}}_G)^{\mathrm{op}}, \mathsf{Spc}) = \mathcal{P}_{\Sigma}(\mathsf{Span}(\mathbb{F}_G)),$$

we then get a normed G-functor $\underline{\mathfrak{S}}^{\otimes}_{G,*} \to \underline{\mathsf{NMon}}^{\otimes}_G$, whose underlying G-functor agrees up to equivalence with $\mathcal{P}_{\Sigma}(j^{\flat})\colon \mathcal{P}_{\Sigma}(\mathbb{F}_*^{\flat}) \to \mathcal{P}_{\Sigma}(\mathsf{Span}(\mathbb{F})^{\flat})$, i.e. it is the restriction of the left Kan extension along $(j^{\flat})^{\mathrm{op}}$ to product-preserving functors.

To see that this is a G-left adjoint, we first note that each $\mathcal{P}_{\Sigma}(j^{\flat})(G/H)$ admits a right adjoint (given by restriction); it therefore only remains to check the Beck–Chevalley condition of Corollary 5.1.4. For this we observe that for any inclusion $i\colon K \hookrightarrow H$ of subgroups of G, the functors $i^*\colon \mathsf{Fun}(BH, \mathbb{F}_*) \to \mathsf{Fun}(BK, \mathbb{F}_*)$ and $\mathsf{Fun}(BH, \mathsf{Span}(\mathbb{F})) \to \mathsf{Fun}(BK, \mathsf{Span}(\mathbb{F}))$ admit left adjoints $i_!$, given non-equivariantly by an $|H/K|$-fold coproduct. Thus, we may check the Beck–Chevalley condition before passing to sifted cocompletions, i.e. we want to show that $i_! \circ \mathsf{Fun}(BK, j) \to \mathsf{Fun}(BH, j) \circ i_!$ is an equivalence of functors $\mathsf{Fun}(BK, \mathbb{F}_*) \to \mathsf{Fun}(BH, \mathsf{Span}(\mathbb{F}))$. But this may be checked after forgetting to $\mathsf{Span}(\mathbb{F})$, where this follows from the fact that j preserves finite coproducts by construction. □

Restricting, we in particular get a normed structure on the G-functor $\underline{\mathfrak{F}}_{G,*} \to \underline{\mathsf{NMon}}_G$. In fact, this once again uniquely characterizes the normed structure if we in addition impose compatibility with colimits:

Proposition 5.3.2. *There exists a unique pair of a normed structure on $\underline{\mathsf{NMon}}_G$ and a normed structure on the G-functor $\underline{\mathfrak{F}}_{G,*} \to \underline{\mathsf{NMon}}_G$ such that the following conditions are satisfied:*

(1) For each $H \leqslant G$, the symmetric monoidal ∞-category $\underline{\mathsf{NMon}}_G(G/H)$ is presentably symmetric monoidal.

(2) For each $K \leqslant H \leqslant G$ the norm $\underline{\mathsf{NMon}}_G(G/H) \to \underline{\mathsf{NMon}}_G(G/K)$ preserves sifted colimits.

Proof. We will first prove this statement with $\underline{\mathfrak{F}}_{G,*}$ replaced by $\mathsf{Span}(\underline{\mathfrak{F}}_G)$. As $\underline{\mathsf{NMon}}_G$ is defined as the sifted cocompletion of the latter, the same argument as in Corollary 5.2.9 reduces this to showing that $(\mathsf{Span}(\mathbb{F}), 1)$ is idempotent in $\mathsf{Cat}^{\amalg}$, which was already recalled above.

To complete the proof, we now observe that the data in question is equivalent to a normed structure on $\underline{\mathsf{NMon}}_G$ (satisfying the above two axioms) that preserves the full subcategory $\mathsf{Span}(\underline{\mathfrak{F}}_G)$, together with a lift of $\underline{\mathfrak{F}}_{G,*} \to \mathsf{Span}(\underline{\mathfrak{F}}_G)$ to a normed G-functor. The former is no data by the above, while Corollary A.2.2 together with the idempotency of $\mathbb{F}_*$ and $\mathsf{Span}(\mathbb{F})$ shows that also the latter is unique. □

5.4 G-spectra and their symmetric monoidal structure

Let us begin by giving two equivalent descriptions of the G-∞-category of G-spectra:

Construction 5.4.1. We define the G-∞-category $\underline{\mathsf{Sp}}_G$ of G-spectra as the pointwise stabilization of $\underline{\mathsf{NMon}}_G$, i.e. it is the G-∞-category

$$X \mapsto \mathsf{Fun}^{\times}(\mathsf{Span}((\mathbb{F}_G)_{/X}), \mathsf{Sp})$$

with functoriality via restriction along pushforwards. This comes with a natural *stabilization map* $\ell\colon \underline{\mathsf{NMon}}_G \to \underline{\mathsf{Sp}}_G$, induced by the usual stabilization/delooping map $\mathsf{CMon}(\mathsf{Spc}) \to \mathsf{Sp}$. We write $\mathbb{S}_G$ for the image of $\mathsf{Map}(1, -) \in \underline{\mathsf{NMon}}_G(G/G)$ under ℓ.

Construction 5.4.2. Write Sp^{Σ} for the 1-category of symmetric spectra in simplicial sets. For each finite group H, the category Sp^{Σ} carries an *equivariant flat model structure* [Haus17, 4.7] whose weak equivalences are the so-called H-equivariant weak equivalences and whose cofibrations are the so-called *flat cofibrations*; the latter are independent of the group H. We write $\mathsf{Sp}^{\Sigma}_{\text{flat}}$ for the full subcategory of flat spectra (i.e. those X for which $\emptyset \to X$ is a flat cofibration).

We now consider the Borel G-category $(\mathsf{Sp}^{\Sigma}_{\text{flat}})^{\flat}$, and we equip each category $(\mathsf{Sp}^{\Sigma}_{\text{flat}})^{\flat}(G/H) = \mathsf{Fun}(BH, \mathsf{Sp}^{\Sigma}_{\text{flat}})$ with the H-equivariant weak equivalences. By [Haus17, §5.2] these are preserved under restriction, so we can Dwyer–Kan localize this to obtain a G-∞-category $\underline{\mathfrak{Sp}}_G$.

Remark 5.4.3. The inclusion $(\mathsf{Sp}^{\Sigma}_{\text{flat}})^{\flat} \hookrightarrow (\mathsf{Sp}^{\Sigma})^{\flat}$ induces an equivalence of Dwyer–Kan localizations (being pointwise the inclusion of the cofibrant objects of a model category), so we could equivalently have worked without restricting to flat spectra. However, flatness will come in handy below to define the symmetric monoidal and normed structures on $\underline{\mathfrak{Sp}}_G$.

Theorem 5.4.4. *There is a unique equivalence $\underline{\mathfrak{Sp}}_G \simeq \underline{\mathsf{Sp}}_G$ sending $\mathbb{S}_G$ to $\mathbb{S}_G$.*

Proof. Combine [CLL23b, 9.13] with [CLL24, 9.9]. □

Our goal is to make both sides into normed ∞-categories and then upgrade the above equivalence to a normed equivalence. As a stepping stone for this, we will first prove a comparison that does not take norms into account. We therefore introduce:

Definition 5.4.5. A (naïve) *symmetric monoidal G-∞-category* is a functor $\mathbb{F}_G^{\mathrm{op}} \to \mathsf{CMon}(\mathsf{Cat}_\infty)$.

Equivalently, we can view a symmetric monoidal G-∞-category as a commutative monoid in the (ordinary) ∞-category of G-∞-categories with respect to the cartesian product. Restricting along $\mathbb{F}_G^{\mathrm{op}} \hookrightarrow \mathsf{Span}(\mathbb{F}_G)$, every normed G-∞-category has an underlying symmetric monoidal G-∞-category.

We will be particularly interested in the case where the underlying G-∞-category $\mathscr{C}$ is presentable and the tensor product $- \otimes -\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ preserves G-colimits in each variable, i.e. for each G/H the symmetric monoidal structure on $\mathscr{C}(G/H)$ is closed, and for all $i\colon H \hookrightarrow K$ the *projection map*

$$i_!(X \otimes i^*Y) \longrightarrow i_!X \otimes Y$$

(the Beck–Chevalley map associated to $i^*X \otimes i^*Y \simeq i^*(X \otimes Y)$) is invertible. We call a symmetric monoidal G-∞-category *G-presentably symmetric monoidal* in this case. Similarly we say that a normed G-∞-category is *G-presentably normed* if the underlying symmetric monoidal G-∞-category is G-presentably symmetric monoidal.

Example 5.4.6. We have already seen that the normed structure on $\underline{\mathfrak{S}}_{G,*}$ coming from the smash product is presentably symmetric monoidal in each degree. As all functors in sight are homotopical, the projection formula can be checked on the pointset level, where it is a trivial computation.[1]

Example 5.4.7. The usual smash product of (H-)symmetric spectra is homotopical when restricted to flat spectra [Haus17, 6.1], making $\underline{\mathfrak{Sp}}_G$ into a symmetric monoidal G-∞-category. This is again G-presentably symmetric monoidal: the statement for the levelwise tensor product is again clear, while for the projection map we observe that the corresponding non-derived map is again an isomorphism by direct inspection, and that all functors in sight are homotopical on flat spectra.

Example 5.4.8. Also the normed structure on $\underline{\mathsf{NMon}}_G$ is G-presentably symmetric monoidal. For this note first that Proposition 3.3.8 shows that each $\underline{\mathsf{NMon}}_G(X)$ is presentably symmetric monoidal. For the projection formula $i_!(X \otimes i^*Y) \simeq i_!X \otimes Y$ we observe that both sides preserve colimits in X and Y, so we may assume that X and Y are both in the image of the free functor $\mathbb{P}\colon \underline{\mathfrak{S}}_{G,*} \to \underline{\mathsf{NMon}}_G$. In this case, the claim follows by Proposition 5.3.1 together with Example 5.4.6.

[1] This isomorphism of pointed G-(simplicial) sets is sometimes referred to as the *shearing isomorphism*.

Example 5.4.9. As Sp is idempotent, $\underline{\mathsf{Sp}}_G = \mathsf{Sp} \otimes \underline{\mathsf{NMon}}_G$ inherits a symmetric monoidal structure from $\underline{\mathsf{NMon}}_G$ such that each $\underline{\mathsf{Sp}}_G(X)$ is presentably symmetric monoidal, see [GGN15, 5.1]. This is again G-presentably symmetric monoidal: by the universal property of stabilization, the projection formula can be checked after restricting along $\underline{\mathsf{NMon}}_G \to \underline{\mathsf{Sp}}_G$, where this was verified in the previous example.

In fact, the G-presentably symmetric monoidal structures considered in the above examples are unique:

Theorem 5.4.10.

(1) The G-∞-categories $\underline{\mathsf{Spc}}_{G,*}$ and $\underline{\mathfrak{S}}_{G,*}$ admit unique G-presentably symmetric monoidal structures with unit S^0.

(2) The G-∞-category $\underline{\mathsf{NMon}}_G$ admits a unique G-presentably symmetric monoidal structure with unit $\mathsf{Map}(1,-)$.

(3) The G-∞-categories $\underline{\mathsf{Sp}}_G$ and $\underline{\mathfrak{Sp}}_G$ admit unique G-presentably symmetric monoidal structures with unit $\mathbb{S}_G$.

Moreover, the G-functors $\underline{\mathfrak{S}}_{G,*} \to \underline{\mathsf{NMon}}_G \to \underline{\mathsf{Sp}}_G$ considered above enhance uniquely to maps of symmetric monoidal G-∞-categories, as do the equivalences $\underline{\mathsf{Spc}}_{G,*} \simeq \underline{\mathfrak{S}}_{G,*}$ and $\underline{\mathsf{Sp}}_G \simeq \underline{\mathfrak{Sp}}_G$.

Proof. By [MW22, §8.2], the ∞-category of presentable G-∞-categories comes with a *parametrized Lurie tensor product*, corepresenting bifunctors that preserve G-colimits in each variable. The unit is the G-∞-category $\underline{\mathsf{Spc}}_G$, and the tensor product can be computed by the formula

$$\mathscr{C} \otimes \mathscr{D} = \underline{\mathsf{Fun}}^{\mathrm{R}}_G(\mathscr{C}^{\mathrm{op}}, \mathscr{D}) \simeq \underline{\mathsf{Fun}}^{\mathrm{L}}_G(\mathscr{C}, \mathscr{D}^{\mathrm{op}})^{\mathrm{op}}$$

with functoriality in $\mathscr{C}$ given via precomposition, see [MW22, 8.2.11].

It now suffices to show that $(\underline{\mathsf{Spc}}_{G,*}, S^0)$, $(\underline{\mathsf{NMon}}_G, \mathsf{Map}(1,-))$, and $(\underline{\mathfrak{Sp}}_G, \mathbb{S}_G)$ are all idempotent with respect to this tensor product. By the above explicit formula for the tensor product, the statement for $\underline{\mathfrak{Sp}}_G$ is an instance of [CLL23b, 9.13(2)], while the statement for $\underline{\mathsf{NMon}}_G$ follows by combining [CLL23a, 4.8.11] with [CLL24, 9.9].

Finally, for $\underline{\mathsf{Spc}}_{G,*} = \underline{\mathsf{Spc}}_G \otimes \mathsf{Spc}_*$, we recall that $\underline{\mathsf{Spc}}_G \otimes -\colon \mathsf{Pr}^{\mathrm{L}} \to \mathsf{Pr}^{\mathrm{L}}_G$ admits a strong symmetric monoidal structure [MW22, end of 8.3.8]. In particular, it sends the idempotent Spc_* to an idempotent, finishing the proof. □

5.5 Normed structures on G-spectra

In this subsection we will finally construct the G-normed structure on $\underline{\mathsf{Sp}}_G$; in particular, we will show:

Theorem 5.5.1. *There exists a unique pair of a normed structure on $\underline{\mathsf{Sp}}_G$ together with a lift of $\ell\colon \underline{\mathsf{NMon}}_G \to \underline{\mathsf{Sp}}_G$ to a normed G-functor $\ell^{\otimes}\colon \underline{\mathsf{NMon}}_G^{\otimes} \to \underline{\mathsf{Sp}}_G^{\otimes}$ that satisfies the following two properties:*

(1) For each $X \in \mathbb{F}_G$, $\underline{\mathsf{Sp}}_G^{\otimes}(X)$ is presentably symmetric monoidal.

(2) For each $f\colon X \to Y$ in $\mathbb{F}_G$, the norm functor $f_{\otimes}\colon \underline{\mathsf{Sp}}_G^{\otimes}(X) \to \underline{\mathsf{Sp}}_G^{\otimes}(Y)$ preserves sifted colimits.

This will require some further preparations.

Lemma 5.5.2. *Let $\mathcal{J}$ be a small ∞-category with finite coproducts equipped with a symmetric monoidal structure that preserves coproducts in each variable. Then the Day convolution on $\mathsf{Fun}(\mathcal{J}^{\mathrm{op}}, \mathsf{Spc})$ restricts to a symmetric monoidal structure on $\mathcal{P}_{\Sigma}(\mathcal{J}) = \mathsf{Fun}^{\times}(\mathcal{J}^{\mathrm{op}}, \mathsf{Spc})$. Moreover, this is a presentably symmetric monoidal structure, and the tensor product of compact objects is compact again.*

Proof. The fact that this restricts is the content of Proposition 3.3.8, where it is also shown that this is equivalently the localization of the Day convolution structure, whence in particular presentably symmetric monoidal.

For the second statement, we now claim that any compact object in $\mathcal{P}_{\Sigma}(\mathcal{J})$ is a retract of a finite colimit of representables. To prove the claim, consider any $X \in \mathcal{P}_{\Sigma}(\mathcal{J})$, and write it as a colimit $\mathrm{colim}_{i\in I}\, x_i$ in $\mathcal{P}(\mathcal{J}) = \mathsf{Fun}(\mathcal{J}^{\mathrm{op}}, \mathsf{Spc})$ of representables. Applying the localization functor $\mathcal{P}(\mathcal{J}) \to \mathcal{P}_{\Sigma}(\mathcal{J})$ we then also get such a colimit decomposition in $\mathcal{P}_{\Sigma}(\mathcal{J})$. Restricting along a cofinal functor, we may assume that I is a poset [Lur09, 4.2.3.15], and filtering it by its finite subsets, we can express X as a filtered colimit in $\mathcal{P}_{\Sigma}(\mathcal{J})$ of finite colimits of representables, i.e. we have an equivalence

$$\phi\colon X \xrightarrow{\simeq} \mathrm{colim}_{J\subseteq I \text{ finite}}\, \mathrm{colim}_{j\in J}\, x_j.$$

By compactness of X, ϕ has to factor through a map $\psi\colon X \to \mathrm{colim}_{j\in J}\, x_j$ for some finite $J \subseteq I$. The composite $\mathrm{colim}_{j\in J}\, x_j \to \mathrm{colim}_{J\subseteq I \text{ finite}}\, \mathrm{colim}_{j\in J}\, x_j \simeq X$ is then a retraction of ψ, finishing the proof of the claim.

With this, we can now easily prove the second statement: if X and Y are compact, then the above shows that $X \otimes Y$ is again a retract of a finite colimit of representables (using that $\otimes$ preserves colimits in each variable). As representables are compact in $\mathcal{P}$, and hence also in $\mathcal{P}_{\Sigma}$ (using that the latter is closed under filtered colimits), this immediately implies that $X \otimes Y$ is compact, as desired. □

Lemma 5.5.3. *Let $\mathcal{D}$ be presentably symmetric monoidal and pointed. Then the stabilization map $\Sigma^{\infty}\colon \mathcal{D} \to \mathsf{Sp}(\mathcal{D})$ lifts uniquely to a map in $\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}})$, and this lift is symmetric monoidal inversion of $\Sigma\mathbb{1}$.*

Proof. The existence and uniqueness of the symmetric monoidal structure follows from idempotency of $(\mathsf{Sp}, \mathbb{S})$ in Pr^{L}, see [GGN15, 5.1]. It therefore only remains to prove that this map is symmetric monoidal inversion.

As the symmetric monoidal product $\otimes$ preserves colimits in each variable, $\Sigma\mathbb{1} \otimes -$ is equivalent to the suspension functor Σ. Expressing $\Sigma^\infty\colon \mathscr{D} \to \mathsf{Sp}(\mathscr{D})$ as the sequential colimit in Pr^{L} along Σ, the claim is therefore an instance of [GM23, C.6] once we show that $\Sigma\mathbb{1}$ is a *symmetric object*, in the sense that for some $n \geq 2$ the automorphism of $(\Sigma\mathbb{1})^{\otimes n}$ induced by the permutation $\sigma :=$ $(1\ 2\ \ldots\ n) \in \Sigma_n$ is trivial. This is in fact true for any odd n as in this case σ has sign $+1$, so that the induced automorphism of S^n also has degree 1. □

Proposition 5.5.4. *For any $X \in \mathbb{F}_G$, the functor $\ell\colon \underline{\mathsf{NMon}}_G(X) \to \underline{\mathsf{Sp}}_G(X)$ is symmetric monoidal inversion of the object $\Sigma\mathbb{1}$ in both $\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}})$ and $\mathsf{CAlg}(\mathsf{Cat}^{\mathrm{sifted}})$.*

Proof. Observe first that $\mathbb{1} = \mathsf{Map}(1, -)$ is compact, whence so is $\Sigma\mathbb{1}$. Moreover, we have seen in the proof of Lemma 5.5.3 that $\Sigma\mathbb{1}$ is a symmetric object, while Lemma 5.5.2 shows that the the symmetric monoidal product preserves compact objects and colimits in each variable. Thus, [BH21, 4.1] shows that the universal map in $\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}})$ inverting $\Sigma\mathbb{1}$ agrees with the universal map in $\mathsf{CAlg}(\mathsf{Cat}^{\mathrm{sifted}})$. The claim therefore follows from the previous lemma. □

Lemma 5.5.5. *Let $f\colon X \to Y$ be any map in $\mathbb{F}_G$. Then the symmetric monoidal functor $\ell\colon \underline{\mathsf{NMon}}_G(Y) \to \underline{\mathsf{Sp}}_G(Y)$ sends $f_\otimes(\Sigma\mathbb{1})$ to an invertible object.*

Proof. Consider first the special case that f is the projection $G/H \to G/K$ for some $H \leqslant K$. By Theorem 5.4.10, it will be enough to show that the composite

$$\mathfrak{L}\colon \underline{\mathsf{NMon}}_G \xrightarrow{\ell} \underline{\mathsf{Sp}}_G \xrightarrow{\sim} \underline{\mathfrak{Sp}}_G$$

sends $f_\otimes \Sigma\mathsf{Map}(1, -)$ to an invertible object (with respect to the derived smash product of K-equivariant symmetric spectra). For this we compute

$$\mathfrak{L}(f_\otimes \Sigma\mathsf{Map}(1, -)) = \mathfrak{L} f_\otimes \Sigma\mathbb{P}(S^0) \simeq (\mathfrak{L}\mathbb{P})(f_\otimes S^1) \simeq (\mathfrak{L}\mathbb{P})(S^{K/H})$$

where $\mathbb{P}\colon \underline{\mathfrak{S}}_{G,*} \to \underline{\mathsf{NMon}}_G$ is the normed G-left adjoint from Proposition 5.3.1, and the last equation uses the explicit description of the normed structure on $\underline{\mathfrak{S}}_{G,*}$. Now $\mathfrak{L} \circ \mathbb{P}\colon \underline{\mathfrak{S}}_{G,*} \to \underline{\mathfrak{Sp}}_G$ is a G-left adjoint sending S^0 to $\mathbb{S}_G$, so it is necessarily the suspension spectrum functor. But the representation sphere $\Sigma^\infty S^{K/H}$ is invertible with respect to the smash product of K-spectra by [Haus17, 4.9(i)], finishing the proof of the special case.

In the case of a general map $f\colon X \to Y$ in $\mathbb{F}_G$, we first note that an object is invertible in $\underline{\mathsf{Sp}}_G(Y)$ if and only if it is so after restricting to each orbit. By the double coset formula, we may therefore assume that $Y = G/H$. Decomposing X into its orbits then provides a factorization of $X \to G/H$ as

$$X = \coprod_{i=1}^r X_i \xrightarrow{\amalg f_i} \coprod_{i=1}^r G/H \xrightarrow{\nabla} G/H,$$

whence $\ell(f_\otimes \Sigma\mathsf{Map}(1, -)) \simeq \ell\big(\bigotimes_{i=1}^r f_{i\otimes} \Sigma(\mathsf{Map}(1, -))\big) \simeq \bigotimes_{i=1}^r \ell f_{i\otimes}(\Sigma\mathsf{Map}(1, -))$. As invertible elements are closed under tensor product, this completes the proof of the lemma. □

Proof of Theorem 5.5.1. By Theorem 5.4.10, ℓ lifts uniquely to a natural transformation of functors into the ∞-category of presentably symmetric monoidal ∞-categories and sifted-colimit-preserving functors, and by Proposition 5.5.4 this map is pointwise given by symmetric monoidal inversion of $\Sigma\mathbb{1}$. This then uniquely extends to the desired map of normed G-∞-categories (viewed as functors $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{CMon}(\mathsf{Cat}_\infty)$ as per our standing convention) by the universal property of symmetric monoidal inversion combined with the previous lemma. □

Let us now give alternative interpretations of this normed G-∞-category:

Proposition 5.5.6 (cf. [BH21, 9.11 and 9.13])**.** *For every $H \leqslant G$, the symmetric monoidal functor $(\ell \circ \mathbb{P})(G/H)\colon \underline{\mathsf{Spc}}^{\otimes}_{G,*}(G/H) \to \underline{\mathsf{Sp}}^{\otimes}_G(G/H)$ is given by universally inverting the objects of the form $f_\otimes\Sigma\mathbb{1}$ in $\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}})$, or equivalently in $\mathsf{CAlg}(\mathsf{Cat}^{\mathrm{sifted}})$.*

Note that Nardin and Shah use this as the *definition* of the normed structure on $\underline{\mathsf{Sp}}_G$ [NS22, 2.4.2], following [BH21, §9.2]. Thus, this result in particular shows that our approach agrees with their construction.

Proof. First note that this holds for $\Sigma^\infty\colon \underline{\mathfrak{S}}_{G,*}(G/H) \to \underline{\mathfrak{Sp}}_G(G/H)$ by [GM23, C.7] together with [Haus17, 7.5].[2] Thus, it will suffice to lift the commutative triangle

$$\begin{array}{ccc} & \underline{\mathfrak{S}}_{G,*} & \\ {}^{\Sigma^\infty}\swarrow & & \searrow^{\ell\circ\mathbb{P}} \\ \underline{\mathfrak{Sp}}_G & \xrightarrow[\simeq]{} & \underline{\mathsf{Sp}}_G \end{array} \tag{15}$$

observed in the proof of Theorem 5.5.1 to a commutative diagram of symmetric monoidal G-∞-categories with respect to the symmetric monoidal structures inherited from the normed structures considered above. This is however clear from Theorem 5.4.10 (using that all of these structures are indeed G-presentably symmetric monoidal by the above). □

Finally, we can also describe the normed structure in terms of models:

Proposition 5.5.7. *The normed structure on $(\mathsf{Sp}^{\Sigma}_{\mathrm{flat}})^\flat$ given by the smash product localizes to a G-presentably normed structure on $\underline{\mathfrak{Sp}}_G$ such that the norm $f_\otimes\colon \underline{\mathfrak{Sp}}^{\otimes}(X) \to \underline{\mathfrak{Sp}}^{\otimes}(Y)$ preserves sifted colimits for every $f\colon X \to Y$ in $\mathbb{F}_G$.*

Proof. In view of Example 5.4.7 it only remains to show that the symmetric monoidal norms $N_K^H\colon \mathsf{Fun}(BK, \mathsf{Sp}^{\Sigma}_{\mathrm{flat}}) \to \mathsf{Fun}(BH, \mathsf{Sp}^{\Sigma}_{\mathrm{flat}})$ (known as the *Hill–Hopkins–Ravenel norms*) are homotopical and that the resulting functors on localizations preserve sifted colimits.

[2] Hausmann a priori just provides a Quillen equivalence without referring to the symmetric monoidal structures on both sides, but the left Quillen functor from symmetric to orthogonal spectra is strong symmetric monoidal with respect to Day convolution, so this automatically gives an equivalence of symmetric monoidal ∞-categories.

The first statement is [Haus17, 6.8]. For the second statement we first observe that both filtered colimits and geometric realization in Sp^Σ are homotopical: namely, both are left Quillen (with respect to the projective model structures on the respective source functor categories) and moreover preserve levelwise weak equivalences (maps $f\colon X \to Y$ such that each $f(A)$ is a $(G \times \Sigma_A)$-weak equivalences) as observed in the proof of Proposition 5.2.11. We moreover claim that both of these constructions preserve flat spectra. To see this, we recall that a symmetric spectrum X is flat if and only if for each finite set A a certain natural *latching map* $L_A(X) \to X(A)$ is levelwise injective, see [Haus17, 2.18]; all that we will need to know is that L_A is defined as a certain colimit, and in particular commutes with geometric realization and all colimits. The two claims now immediately follow as injections of simplicial sets are preserved by filtered colimits and geometric realization.

With this established, it will be enough to show that N_H^G commutes with geometric realization and filtered colimits on the point-set level, up to *isomorphism*. In particular, we can forget about all the actions and simply consider the endofunctor $X \mapsto X^{\wedge n}$ of $\mathsf{Sp}^\Sigma \supseteq \mathsf{Sp}^\Sigma_{\text{flat}}$. The statement about filtered colimits is then clear as the smash product preserves colimits in each variable. Similarly, the statement about geometric realizations reduces to showing that for any $X_\bullet\colon \Delta^{\text{op}} \to \mathsf{Sp}^\Sigma_{\text{flat}}$ and $n \geq 1$ the map

$$\int^{[k]\in\Delta^{\text{op}}} X_k \wedge \Delta^k_+ \longrightarrow \int^{[k_1],\dots,[k_n]\in(\Delta^{\text{op}})^n} X_{k_1} \wedge \cdots \wedge X_{k_n} \wedge \Delta^{k_1}_+ \wedge \cdots \wedge \Delta^{k_n}_+$$

induced by the diagonal embedding is an isomorphism. By induction, we reduce to proving this for the map

$$\int^{[k]} Y_{k,k} \wedge \Delta^k_+ \longrightarrow \int^{[k_1]}\int^{[k_2]} Y_{k_1,k_2} \wedge \Delta^{k_1}_+ \wedge \Delta^{k_2}_+$$

for any bisimplicial object Y in Sp^Σ. Arguing levelwise, this follows at once from the fact that the geometric realization of a simplicial object in (pointed) simplicial sets is just given by its diagonal. □

Theorem 5.5.8. *The equivalence $\underline{\mathfrak{Sp}}_G \simeq \underline{\mathsf{Sp}}_G$ of G-∞-categories upgrades canonically to an equivalence $\underline{\mathfrak{Sp}}^\otimes_G \simeq \underline{\mathsf{Sp}}^\otimes_G$ of normed G-∞-categories.*

Proof. Of the maps of G-∞-categories comprising the diagram (15), all except for the lower one have been lifted to maps in $\mathsf{NAlg}_G(\mathsf{Cat}_\infty)$ above. As Σ^∞ is given by universally inverting representation spheres, and we have shown that they become invertible in $\underline{\mathsf{Sp}}_G$, there is then a unique normed pointwise left adjoint $\underline{\mathfrak{Sp}}^\otimes_G \to \underline{\mathsf{Sp}}^\otimes_G$ making the diagram commute. It only remains to show that this map forgets to our equivalence $\underline{\mathfrak{Sp}}_G \simeq \underline{\mathsf{Sp}}_G$.

For this we simply note that after forgetting to symmetric monoidal G-∞-categories there is still a unique map making the diagram commute, and we have lifted the equivalence $\underline{\mathfrak{Sp}}_G \simeq \underline{\mathsf{Sp}}_G$ to such a map in the proof of Proposition 5.5.6. □

5.6 The multiplicative equivariant recognition theorem

As an upshot of all the hard work done in the previous subsections, we can now easily prove Theorem A from the introduction:

Theorem 5.6.1.

(1) *The normed structure on* $\underline{\mathsf{NMon}}_G$ *localizes to a normed structure on* $\underline{\mathsf{NGrp}}_G$.

(2) *The normed structure on* $\underline{\mathfrak{Sp}}_G$ *restricts to a normed structure on the full G-subcategory* $\underline{\mathfrak{Sp}}_G^{\geq 0}$ *spanned by the connective equivariant spectra.*

(3) *The delooping functor* $\underline{\mathsf{NMon}}_G \to \underline{\mathfrak{Sp}}_G$ *acquires a canonical normed structure, and this restricts to a normed equivalence* $\underline{\mathsf{NGrp}}_G^{\otimes} \simeq (\underline{\mathfrak{Sp}}_G^{\geq 0})^{\otimes}$.

Proof. By Theorem 5.5.8, we may replace $\underline{\mathfrak{Sp}}_G$ by $\underline{\mathsf{Sp}}_G$; under this identification, the G-subcategory $\underline{\mathfrak{Sp}}_G^{\geq 0}$ corresponds to $\underline{\mathsf{Sp}}_G^{\geq 0} := \mathsf{Fun}(\mathsf{Span}((\mathbb{F}_G)_{/-}), \mathsf{Sp}^{\geq 0})$.

We now observe that the essential image of $\ell\colon \underline{\mathsf{NGrp}}_G \to \underline{\mathsf{Sp}}_G$ is precisely $\underline{\mathsf{Sp}}_G^{\geq 0}$. As we have lifted ℓ to a normed functor in Theorem 5.5.1, this shows that $\underline{\mathsf{Sp}}_G^{\geq 0}$ is indeed a normed subcategory. Since ℓ factors as the localization functor $\underline{\mathsf{NMon}}_G \to \underline{\mathsf{NGrp}}_G$ followed by an equivalence $\underline{\mathsf{NGrp}}_G \simeq \underline{\mathsf{Sp}}_G^{\geq 0}$, this then immediately implies the remaining statements. □

Corollary 5.6.2. *The cocartesian fibration* $\underline{\mathsf{NMon}}_G^{\otimes} \to \mathsf{Span}(\mathbb{F}_G)$ *restricts to a cocartesian fibration* $\underline{\mathsf{NGrp}}_G^{\otimes} \to \mathsf{Span}(\mathbb{F}_G)$*, and the inclusion* $\iota\colon \underline{\mathsf{NGrp}}_G^{\otimes} \hookrightarrow \underline{\mathsf{NMon}}_G^{\otimes}$ *is lax normed.*

Proof. For each $X \in \mathbb{F}_G$, the functor $\ell\colon \underline{\mathsf{NMon}}_G(X) \to \underline{\mathsf{Sp}}_G^{\geq 0}(X)$ has a fully faithful right adjoint, induced by the right adjoint $\mathsf{Sp}^{\geq 0} \to \mathsf{CMon}$ of the delooping functor, and this induces a relative adjunction of cocartesian fibrations over $\mathbb{F}_G^{\mathrm{op}}$. As a consequence of [Lur17, 7.3.2.6] (parallel to [Lur17, 7.3.2.8]), the right adjoint ι then canonically lifts to a lax normed functor $\iota^{\otimes}\colon (\underline{\mathsf{Sp}}_G^{\geq 0})^{\otimes} \hookrightarrow \underline{\mathsf{NMon}}_G^{\otimes}$, which is fully faithful with essential image the subcategory $\underline{\mathsf{NGrp}}_G$. □

We now immediately obtain the following generalization of Theorem B from the introduction:

Theorem 5.6.3. *For a weakly extensive subcategory* $I \subseteq \mathbb{F}_G$*, there is an equivalence between*

- *the* ∞*-category* $\mathsf{NAlg}_I((\underline{\mathfrak{Sp}}_G^{\geq 0})^{\otimes}) := \mathsf{NAlg}_{(\mathbb{F}_G, I)}((\underline{\mathfrak{Sp}}_G^{\geq 0})^{\otimes})$ *of connective* ***I-normed G-spectra****, and*
- *the* ∞*-category* $\mathsf{Tamb}_{(\mathbb{F}_G, I)}(\mathsf{Spc}) \subseteq \mathsf{Fun}^{\times}(\mathsf{Bispan}_I(\mathbb{F}_G), \mathsf{Spc})$ *of space-valued* $(\mathbb{F}_G, I)$*-Tambara functors.*

Proof. By Theorem 5.6.1, $\mathsf{NAlg}_{(\mathbb{F}_G,I)}((\underline{\mathfrak{Sp}}_G^{\geq 0})^{\otimes})$ is equivalent to $\mathsf{NAlg}_{(\mathbb{F}_G,I)}(\underline{\mathsf{NGrp}}_G^{\otimes})$. The inclusion $\mathsf{NAlg}_{(\mathbb{F}_G,I)}(\underline{\mathsf{NGrp}}_G^{\otimes}) \hookrightarrow \mathsf{NAlg}_{(\mathbb{F}_G,I)}(\underline{\mathsf{NMon}}_G^{\otimes}) = \mathsf{NRig}_{(\mathbb{F}_G,I)}(\mathsf{Spc})$ identifies its source with the subcategory $\mathsf{NRing}_{(\mathbb{F}_G,I)}(\mathsf{Spc}) \subseteq \mathsf{NRig}_{(\mathbb{F}_G,I)}(\mathsf{Spc})$ of normed G-rings in the ∞-category of spaces. By Theorem 4.3.6, the equivalence between $\mathsf{NRig}_{(\mathbb{F}_G,I)}(\mathsf{Spc}) \simeq \mathsf{Fun}^{\times}(\mathsf{Bispan}_I(\mathbb{F}_G), \mathsf{Spc})$ of Theorem 4.2.4 restricts to an equivalence between $\mathsf{NRing}_{(\mathbb{F}_G,I)}(\mathsf{Spc})$ and $\mathsf{Tamb}_{(\mathbb{F}_G,I)}(\mathsf{Spc})$. Combining these three equivalences gives the result. □

If $I \subseteq \mathbb{F}_G$ is even an extensive subcategory (i.e. an indexing system), then set-valued $(\mathbb{F}_G, I)$-Tambara functors are known under the name *incomplete Tambara functors* [BH18, 4.1]; thus, we may think of the right-hand side of the theorem as "higher" incomplete Tambara functors.

Remark 5.6.4. Let $I \subseteq \mathbb{F}_G$ be an indexing system and $\mathscr{C}^{\otimes}$ an I-normed G-∞-category. As we will now explain, the ∞-category $\mathsf{NAlg}_I(\mathscr{C})$ can be identified with that of algebras for the G-∞-operad $\mathrm{Com}_I^{\otimes}$ as defined by Nardin and Shah [NS22, 2.4.10]: By definition, the ∞-category $\mathsf{NAlg}_I(\mathscr{C}^{\otimes})$ is that of sections $\mathsf{Span}_I(\mathbb{F}_G) \to \mathscr{C}^{\otimes}$ that are cocartesian over $\mathbb{F}_G^{\mathrm{op}}$. Here the inclusion $\mathsf{Span}_I(\mathbb{F}_G) \to \mathsf{Span}(\mathbb{F}_G)$ exhibits $\mathsf{Span}_I(\mathbb{F}_G)$ as an equivariant ∞-operad when these are defined over the base $\mathsf{Span}(\mathbb{F}_G)$ (see [BHS22, §5.2]), and I-normed algebras in $\mathscr{C}^{\otimes}$ are precisely algebras for this ∞-operad. In [NS22] the theory of equivariant ∞-operads is instead developed over a different base $\underline{\mathbb{F}}_{G,*}$ (a specific model of the cocartesian unstraightening of the functor $\underline{\mathbb{F}}_{G,*}\colon \mathbb{F}_G \to \mathsf{Cat}_{\infty}$ considered above), but these two versions of G-∞-operads were shown to be equivalent under pullback along a certain functor $\underline{\mathbb{F}}_{G,*} \to \mathsf{Span}(\mathbb{F}_G)$ in [BHS22, 5.2.14]. It is clear from the definitions that $\mathrm{Com}_I^{\otimes}$ is precisely the pullback $\underline{\mathbb{F}}_{G,*} \times_{\mathsf{Span}(\mathbb{F}_G)} \mathsf{Span}_I(\mathbb{F}_G)$, so by [BHS22, 5.3.17] we get an equivalence between the ∞-category of $\mathsf{Span}_I(\mathbb{F}_G)$- and $\mathrm{Com}_I^{\otimes}$-algebras in $\mathscr{C}^{\otimes}$. In particular, our I-normed G-spectra are equivalently $\mathrm{Com}_I^{\otimes}$-algebras in $\underline{\mathsf{Sp}}_G^{\otimes}$ in the sense of [NS22].

Remark 5.6.5. Recall from Remark 3.1.5 that any indexing system $I \subseteq \mathbb{F}_G$ has an associated N_{∞}-operad $\mathcal{O}$ in G-spaces. It is generally expected that the ∞-category of $\mathrm{Com}_I^{\otimes}$-algebras in $\underline{\mathsf{Sp}}_G^{\otimes}$ is modelled by $\mathcal{O}$-algebras in a good model category of G-spectra, like G-symmetric spectra; however, to our knowledge no rigorous proof of this comparison has appeared in the literature.

A The Borel construction

In this appendix, we recall from [Hil22a] how any ∞-category with G-action gives rise to a G-∞-category and how similarly any symmetric monoidal ∞-category with G-action yields a normed G-∞-category.

A.1 Borel G-∞-categories

We start by constructing the functor

$$(-)^\flat \colon \mathsf{Fun}(BG, \mathsf{Cat}_\infty) \longrightarrow \mathsf{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty)$$

from ∞-categories with G-action to G-∞-categories, which is used, for instance, to define the G-∞-categories $\underline{\mathfrak{S}}_G$ and $\underline{\mathfrak{Sp}}_G$.

Construction A.1.1. Write $k\colon (BG)^{\mathrm{op}} \hookrightarrow \mathbb{F}_G$ for the inclusion of the full subcategory on the free G-set G. Then k is fully faithful, so $k^*\colon \mathsf{Fun}(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \to \mathsf{Fun}(BG, \mathsf{Cat}_\infty)$ has a fully faithful right adjoint $(-)^\flat$, which is uniquely characterized by demanding that we have a counit equivalence $\epsilon\colon k^*(-)^\flat \to \mathrm{id}$ and that each individual $\mathscr{C}^\flat$ be right Kan extended.

We will now give an explicit construction of $(-)^\flat$. For this we note that the inclusion $\mathsf{Fun}(BG, \mathsf{Spc}) \hookrightarrow \mathsf{Fun}(BG, \mathsf{Cat}_\infty)$ is cocontinuous, hence (by the universal property of presheaves) left Kan extended from the functor $(BG)^{\mathrm{op}} \to \mathsf{Fun}(BG, \mathsf{Cat}_\infty)$ classifying the right G-set G. Restricting to a full subcategory, we see that also the inclusion $i\colon \mathbb{F}_G \hookrightarrow \mathsf{Fun}(BG, \mathsf{Cat}_\infty)$ is left Kan extended from the same functor. Thus, $\mathsf{Fun}_G(i(-), \mathscr{C})$ is right Kan extended, and we see that $(-)^\flat$ is given by the assignment $\mathscr{C} \mapsto \mathsf{Fun}_G(i(-), \mathscr{C})$, where the right-hand side denotes the internal hom in $\mathsf{Fun}(BG, \mathsf{Cat}_\infty)$; the counit is the evident equivalence $\mathsf{Fun}_G(G, \mathscr{C}) \simeq \mathscr{C}$. Note that $(-)^\flat$ lands in the ∞-category $\mathsf{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty)$ of G-∞-categories, so that we obtain an adjunction

$$k^*\colon \mathsf{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \rightleftarrows \mathsf{Fun}(BG, \mathsf{Cat}_\infty) : (-)^\flat.$$

Definition A.1.2. We will refer to G-∞-categories in the essential image of $(-)^\flat$ as *Borel G-∞-categories*.

Remark A.1.3. In all of our examples, we apply the above right adjoint $(-)^\flat$ to an ordinary ∞-category, which is then to be understood as coming equipped with the trivial G-action. By adjointness, the resulting functor is given by

$$\begin{aligned} \mathsf{Cat}_\infty &\longrightarrow \mathsf{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \simeq \mathsf{Fun}(\mathbf{O}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \\ \mathscr{C} &\longmapsto \mathsf{Fun}(i(-)_{hG}, \mathscr{C}). \end{aligned}$$

In particular, the value of $\mathscr{C}^\flat$ on an orbit G/H is the ∞-category $\mathscr{C}^{BH} := \mathsf{Fun}(BH, \mathscr{C})$ of objects of $\mathscr{C}$ with an H-action, with the evident restriction functoriality. This suggests the following alternative description of the Borel G-∞-category $\mathscr{C}^\flat$ that connects it to the constructions of [CLL23a, CLL23b]:

Write Orb for the ∞-category of finite connected groupoids and faithful functors: in other words, the objects are groupoids of the form BH for a finite group H, and the morphisms $BK \to BH$ are those induced by *injective* group homomorphisms $K \to H$. By [CLL23b, 5.10] there is an equivalence $\mathbf{O}_G \simeq \mathsf{Orb}_{/BG}$ sending G to the homomorphism $1 \to BG$; postcomposing with the forgetful functor and the inclusion yields a functor $v\colon \mathbf{O}_G \to \mathsf{Cat}_\infty$. We claim that v agrees with $i(-)_{hG}$. For this we observe that both agree on the full subcategory spanned by the object G (where they are constant with value the terminal object), so it suffices that both are left Kan extended from this subcategory.

For $i(-)_{hG}$ this is clear since it is the restriction of a cocontinuous functor $\mathsf{Fun}(BG, \mathsf{Spc}) \to \mathsf{Cat}_\infty$. For v, it suffices that $\mathsf{Orb}_{/BG} \to \mathsf{Cat}_\infty$ is left Kan extended. But $\mathsf{Orb}_{/BG}$ is a full subcategory of $(\mathsf{Spc})_{/BG}$, so it will be enough that the forgetful functor $\mathsf{Spc}_{/BG} \to \mathsf{Cat}_\infty$ is left Kan extended from the full subcategory spanned by $1 \to BG$. However, straightening–unstraightening provides an equivalence $\mathsf{Spc}_{/BG} \simeq \mathsf{Fun}(BG, \mathsf{Spc})$ sending $1 \to BG$ to the corepresented functor $G = \mathsf{Map}_{BG}(*, -)$ so this follows again from cocontinuity.

A.2 Normed structures on Borel G-∞-categories

Recall from Definition 3.2.1 that a *normed structure* on a G-∞-category $\mathbb{F}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty$ is an extension to a product-preserving functor $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{Cat}_\infty$. Given a symmetric monoidal ∞-category $\mathscr{C}$ with G-action, the Borel G-∞-category $\mathscr{C}^\flat$ comes equipped with a canonical normed structure that we will refer to as the *Borel normed structure*:

Proposition A.2.1 ([Pü24, 3.4 and 3.6], [Hil22a, 3.3.3])**.** *The adjunction from Construction A.1.1 lifts to an adjunction*

$$\mathsf{Fun}^\times(\mathsf{Span}(\mathbb{F}_G), \mathsf{Cat}_\infty) \rightleftarrows \mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}_\infty)) : (-)^\flat,$$

i.e. the forgetful functor $\mathsf{Fun}^\times(\mathsf{Span}(\mathbb{F}_G), \mathsf{Cat}_\infty) \to \mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}_\infty))$ *has a right adjoint* $(-)^\flat$*, and the Beck–Chevalley transformation*

$$\begin{array}{ccc} \mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}_\infty)) & \xrightarrow{(-)^\flat} & \mathsf{Fun}^\times(\mathsf{Span}(\mathbb{F}_G), \mathsf{Cat}_\infty) \\ {\scriptstyle \mathbb{U}}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle \mathbb{U}} \\ \mathsf{Fun}(BG, \mathsf{Cat}_\infty) & \xrightarrow[(-)^\flat]{} & \mathsf{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \end{array}$$

is invertible. □

Corollary A.2.2. *The right adjoint*

$$(-)^\flat\colon \mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}_\infty)) \longrightarrow \mathsf{Fun}^\times(\mathsf{Span}(\mathbb{F}_G), \mathsf{Cat}_\infty)$$

is fully faithful, with essential image those normed G-∞-categories whose underlying G-∞-category is Borel.

Proof. As $\mathbb{U}$ is conservative, the Beck–Chevalley condition readily implies that the counit is invertible, as it is so for the original adjunction $\mathsf{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \rightleftarrows \mathsf{Fun}(BG, \mathsf{Cat}_\infty)$, proving full faithfulness. Arguing in the same way about the units yields the characterization of the essential image. □

Corollary A.2.3. *Let* $F\colon \mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}_\infty)) \to \mathsf{Fun}^\times(\mathsf{Span}(\mathbb{F}_G), \mathsf{Cat}_\infty)$ *be any functor equipped with a natural equivalence* $\epsilon\colon \mathrm{ev}_G \circ F \to \mathrm{id}$*, and assume that* F *takes values in Borel* G*-*∞*-categories. Then* F *is right adjoint to the evaluation functor, with counit given by* ϵ*.*

Proof. By adjointness, there is a unique natural transformation $F \to (-)^\flat$ that upon evaluation at $G \in \mathbb{F}_G$ recovers ϵ. As this evaluation functor is conservative on Borel G-∞-categories, this map is then an equivalence as desired. ◻

In the same way one shows the following pointwise version:

Corollary A.2.4. *If the* G*-*∞*-category* $\mathscr{C}\colon \mathbb{F}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *is Borel, then any* G*-equivariant symmetric monoidal structure on* $\mathscr{C}(G) \in \mathsf{Fun}(BG, \mathsf{Cat}_\infty)$ *lifts uniquely to a normed structure on* $\mathscr{C}$*.* ◻

Observation A.2.5. If $\mathscr{C}$ is a symmetric monoidal ∞-category with G-action, then we get two natural symmetric monoidal structures on $\mathscr{C}^{hH} \simeq \mathscr{C}^\flat(G/H)$ for any $H \leqslant G$: on the one hand, we can equip $\mathscr{C}^{hH}$ with the symmetric monoidal structure obtained from the one on $\mathscr{C}$ by taking homotopy fixed points; on the other hand, we can restrict $\mathscr{C}^\flat$ along the functor $\mathsf{Span}(\mathbb{F}) \to \mathsf{Span}(\mathbb{F}_G)$ induced by $G/H \times -$. The Eckmann–Hilton argument then shows that these two structures agree, i.e. the covariant functoriality of $\mathscr{C}^\flat$ in fold maps is induced by the given symmetric monoidal structure by passing to homotopy fixed points.

A.3 The classical case

Let $\mathscr{C}$ be a symmetric monoidal ∞-category with G-action. So far, we have completely described the contravariant functoriality of $\mathscr{C}^\flat$, as well as the covariant functoriality with respect to fold maps. The goal of this final subsection is to provide the only missing piece of information, namely the covariant functoriality with respect to maps $G/K \to G/H$ for $K \leqslant H \leqslant G$, when $\mathscr{C}$ is a 1-category.

It turns out that the hardest part of this is actually not understanding the ∞-categorical side, but rather translating this back through the equivalence between (classical, biased) symmetric monoidal 1-categories and the ∞-category of commutative monoids in Cat [SS79, Sha20]. We therefore take a somewhat different route here: namely, we will give a general result characterizing the structure maps $\mathscr{C}^{hK} \to \mathscr{C}^{hH}$ uniquely, and then give a (classical) construction satisfying these assumptions. As an upshot, we will never need to recall *how* the functor Φ from symmetric monoidal 1-categories to symmetric monoidal ∞-categories actually works, except for the following basic facts:

- Φ is fully faithful with essential image those symmetric monoidal ∞-categories whose underlying category is a 1-category.
- Φ preserves underlying categories, and for every $\mathscr{C}$ the maps $* \to \mathscr{C}$ and $\mathscr{C}^{\times 2} \to \mathscr{C}$ obtained via the functoriality of $\Phi(\mathscr{C})$ in $\mathbf{0} \to \mathbf{1}$ and $\mathbf{2} \to \mathbf{1}$ are given by the inclusion of the unit and the tensor product, respectively.

For definiteness, we fix Φ to be the equivalence from [Sha20, 6.19] between $\mathsf{CMon}(\mathsf{Cat})$ and the ∞-category obtained from the 1-category $\mathsf{PermCat}_1^{\text{strict}}$ of *permutative categories* (i.e. symmetric monoidal categories in which associativity and unitality hold on the nose) and *strict* symmetric monoidal functors by Dwyer–Kan localizing at the underlying equivalences of categories. Below, we will frequently and implicitly extend Φ to the analogous localizations of the 1-categories $\mathsf{SymMonCat}_1^{\text{strict}}$ of symmetric monoidal categories and *strict* symmetric monoidal functors as well as $\mathsf{SymMonCat}_1^{\text{strong}}$ using the following reformulation of Mac Lane's coherence theorem, as refined symmetrically in [May74, 4.2]:

Lemma A.3.1. *The inclusions*

$$\mathsf{PermCat}_1^{\text{strict}} \hookrightarrow \mathsf{SymMonCat}_1^{\text{strict}} \hookrightarrow \mathsf{SymMonCat}_1^{\text{strong}}$$

induce equivalences on Dwyer–Kan localizations.

Proof. For the composite $\mathsf{PermCat}_1^{\text{strict}} \hookrightarrow \mathsf{SymMonCat}_1^{\text{strong}}$ this appears for example as [Len21, 1.19]. As part of the proof, the reference constructs a functor $\Pi\colon \mathsf{SymMonCat}_1^{\text{strong}} \to \mathsf{PermCat}_1^{\text{strict}}$ together with a natural *strong* symmetric monoidal equivalence $\nu\colon \mathscr{C} \to \Pi\mathscr{C}$ for any symmetric monoidal category $\mathscr{C}$. It will therefore suffice to show that there exists a natural zig-zag of *strict* symmetric monoidal equivalences between $\mathscr{C}$ and $\Pi\mathscr{C}$.

This is actually an instance of a general construction: Define $\Xi\mathscr{C}$ to be the category with objects triples of an object $X \in \mathscr{C}$, an object $Y \in \Pi\mathscr{C}$, and an isomorphism $\sigma\colon \nu(X) \xrightarrow{\sim} Y$. A morphism in $(X, Y, \sigma) \to (X', Y', \sigma')$ is given by a pair of a map $X \to X'$ and a map $Y \to Y'$ making the obvious diagram commute. This becomes a functor in $\mathscr{C}$ in the obvious way, and the forgetful maps provide natural equivalences $\mathscr{C} \xleftarrow{\sim} \Xi\mathscr{C} \xrightarrow{\sim} \Pi\mathscr{C}$.

We now make $\Xi\mathscr{C}$ into a symmetric monoidal category as follows: the tensor product of objects is given by

$$(X, Y, \sigma) \otimes (X', Y', \sigma') = \big(X \otimes X', Y \otimes Y', (\sigma \otimes \sigma') \circ \psi^{-1}\big)$$

where $\psi\colon \nu(X) \otimes \nu(X') \xrightarrow{\sim} \nu(X \otimes X')$ denotes the structure isomorphism of the symmetric monoidal functor ν. The unit is given by the inverse structure isomorphism $\nu(\mathbb{1}) \xrightarrow{\sim} \mathbb{1}$ of ν, while the tensor product of morphisms as well as the associativity, unitality, and symmetry isomorphisms for $\Xi\mathscr{C}$ are simply given pointwise. We omit the straightforward verification that this is well-defined and a symmetric monoidal category. It is then clear from the definitions that the projections $\mathscr{C} \leftarrow \Xi\mathscr{C} \to \Pi\mathscr{C}$ are strict symmetric monoidal. By direct inspection, they are still natural when considered as maps in $\mathsf{SymMonCat}_1^{\text{strong}}$, which then completes the proof of the lemma. ◻

We can now state our key technical lemma, whose proof will be given below after some preparations.

Proposition A.3.2. *Let $K \leqslant H \leqslant G$, and let $h_1, \ldots, h_r$ be orbit representatives for H/K. Then there exists a **unique** natural transformation*

$$\nu \colon \mathscr{C}^{hK} \longrightarrow \mathscr{C}^{hH}$$

of functors $\mathsf{Fun}(BG, \mathsf{SymMonCat}_1^{\mathrm{strict}}) \to \mathsf{Cat}$ *lifting* $\mathscr{C} \to \mathscr{C}, X \mapsto \bigotimes_{i=1}^r h_i.X$.

Example A.3.3. If $\mathscr{C}$ is any symmetric monoidal ∞-category with G-action, then the structure map $\mathscr{C}^\flat(G/K = G/K \to G/H) \colon \mathscr{C}^{hK} \to \mathscr{C}^{hH}$ lifts the twisted r-fold tensor product $X \mapsto \bigotimes_{i=1}^r h_i.X$: this follows at once by computing the composition

$$\begin{array}{ccccccc} & G/K & & & & G & \\ /\!\!/ & & \searrow & & \swarrow & & \backslash\!\!\backslash \\ G/K & & & G/H & & & G \end{array}$$

in $\mathsf{Span}(\mathbb{F}_G)$, cf. [Hil22a, 3.2.1].

Construction A.3.4. Let $\mathscr{C}$ be a symmetric monoidal 1-category with G-action. Recall that the *symmetric monoidal norm* $\mathrm{Nm}_K^H \colon \mathscr{C}^{hK} \to \mathscr{C}^{hH}$ is given as follows: we send a K-homotopy fixed point X (with structure isomorphisms $\phi_k \colon X \to k.X$) of $\mathscr{C}$ to $\bigotimes_{i=1}^r h_i.X$ with structure isomorphisms

$$\psi_h \colon \bigotimes_{i=1}^r h_i.X \longrightarrow \bigotimes_{i=1}^r hh_i.x$$

given as follows: if $\sigma \in \Sigma_n$ and $\ell_1, \ldots, \ell_r \in K$ satisfy $hh_i = h_{\sigma(i)}\ell_i$ for $i = 1, \ldots, r$, then ψ_h is given as the composite

$$\bigotimes_{i=1}^r h_i.X \xrightarrow{\sim} \bigotimes_{i=1}^r h_{\sigma(i)}.X \xrightarrow{\bigotimes h_{\sigma(i)}.\phi_{\ell_i}} \bigotimes_{i=1}^r h_{\sigma(i)}\ell_i.X = \bigotimes_{i=1}^r hh_i.x$$

where the unlabelled isomorphism is given by permuting the tensor factors according to σ; on morphisms, Nm_H^K is simply given by $f \mapsto \bigotimes_{i=1}^r h_i.f$.

We omit the straightforward but rather lengthy verification that this is well-defined. Note that this is clearly natural in *strict* symmetric monoidal functors, and hence also satisfies the assumptions of the proposition.

Observation A.3.5. If $\mathscr{C}$ carries the trivial G-action, the above construction simplifies as follows: the assignment $h \mapsto (\sigma; \ell_1, \ldots, \ell_r)$ defines a homomorphism $\iota \colon H \to \Sigma_r \wr K$, and the H-object $\mathrm{Nm}_K^H X$ is given by equipping $\bigotimes_{i=1}^r X$ with the restriction of the natural $\Sigma_r \wr K$-action on X (by permuting the factors and via the individual K-actions) along ι.

The uniqueness part of Proposition A.3.2 now immediately implies:

Corollary A.3.6. *Let $\mathscr{C}$ be a symmetric monoidal 1-category with G-action. Then the structure map $\mathscr{C}^{hK} = \mathscr{C}^\flat(G/K) \to \mathscr{C}^\flat(G/H) = \mathscr{C}^{hH}$ of the associated Borel G-∞-category is given by the classical symmetric monoidal norm of Construction A.3.4.* $\square$

It remains to prove the proposition.

Lemma A.3.7. *Let* $\mathbb{P}\colon \mathsf{Fun}(BG, \mathsf{Cat}_\infty) \to \mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}_\infty))$ *denote the left adjoint of the forgetful functor. Then the restriction of* $\mathbb{P}$ *to* $\mathbb{F}_G$ *is given by* $X \mapsto (\mathbb{F}_{/X})^{\simeq}$*, with functoriality and* G*-action via postcomposition. The unit is given by* $X \to (\mathbb{F}_{/X})^{\simeq}, x \mapsto (x\colon \{*\} \to X)$.

Proof. Note that the claim is clear for $G = 1$ and $X = \{*\}$. For trivial G and general X, we now observe that since $\mathbb{P}$ is a left adjoint, it in particular preserves finite coproducts. As $\mathsf{CMon}(\mathsf{Cat}_\infty)$ is semiadditive, it therefore suffices that $\prod_{x\in X} \mathbb{F}_{/\{x\}} \simeq \mathbb{F}_{/X}$ via the coproduct functor, which is simply the statement that $\mathbb{F}$ is extensive.

This finishes the proof for trivial G. The lemma follows as the left adjoint for general G is simply given by taking the non-equivariant left adjoint and equipping it with the induced G-action. □

Proof of Proposition A.3.2. The existence of such a map was observed in Example A.3.3, so it only remains to prove uniqueness.

As both $(-)^{hK}$ and $(-)^{hH}$ preserve underlying equivalences of categories, we may replace the source by its Dwyer–Kan localization. Combining the above with [Len20, 4.1.36], we see that this localization is given by $\mathsf{Fun}(BG, \mathsf{CMon}(\mathsf{Cat}))$ (with the evident localization functor).

On the other hand, we observe that since the forgetful functor $\mathscr{C}^{hH} \to \mathscr{C}$ is faithful (here it is crucial that $\mathscr{C}$ is a 1-category!), a functor $f\colon \mathscr{C}^{hK} \to \mathscr{C}^{hH}$ is uniquely described by its effect on cores together with the composition $\mathscr{C}^{hK} \to \mathscr{C}^{hH} \to \mathscr{C}$. Thus, we are altogether reduced to showing that there is at most one natural transformation $((-)^{hK})^{\simeq} \to ((-)^{hH})^{\simeq}$ of functors $\mathsf{CMon}(\mathsf{Cat}) \to \mathsf{Spc}$ lifting the twisted r-fold tensor product.

Combining the Yoneda lemma with the representability result proven in Lemma A.3.7, this translates to saying that the object

$$\bigotimes_{i=1}^{r} h_i.(1 \longrightarrow G/K) \cong (H/K \hookrightarrow G/K)$$

of $\mathbb{F}^{\simeq}_{/(G/K)}$ admits at most one lift to an H-homotopy fixed point. But this is immediate since it admits no non-trivial automorphisms. □

References

[Bac22] Tom Bachmann, *Motivic spectral Mackey functors* (2022), available at arXiv:2205.13926.

[Bar17] Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Adv. Math. **304** (2017), 646–727, available at arXiv:1404.0108.

[BD77] Francis Borceux and Brian J. Day, *On product-preserving Kan extensions*, Bull. Austral. Math. Soc. **17** (1977), no. 2, 247–255.

[BDG+16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, *Parametrized higher category theory and higher algebra: Exposé I – Elements of parametrized higher category theory* (2016), available at arXiv:1608.03657.

[BGN18] Clark Barwick, Saul Glasman, and Denis Nardin, *Dualizing cartesian and cocartesian fibrations*, Theory Appl. Categ. **33** (2018), 67–94, available at arXiv:1409.2165.

[BGS20] Clark Barwick, Saul Glasman, and Jay Shah, *Spectral Mackey functors and equivariant algebraic K-theory, II*, Tunis. J. Math. **2** (2020), no. 1, 97–146, available at arXiv:1505.03098.

[BGMN21] Clark Barwick, Saul Glasman, Akhil Mathew, and Thomas Nikolaus, *K-theory and polynomial functors* (2021), available at arXiv:2102.00936.

[BH18] Andrew J. Blumberg and Michael A. Hill, *Incomplete Tambara functors*, Algebr. Geom. Topol. **18** (2018), no. 2, 723–766, available at arXiv:1603.03292.

[BH21] Tom Bachmann and Marc Hoyois, *Norms in motivic homotopy theory*, Astérisque **425** (2021), available at arXiv:1711.03061.

[BHS22] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner, *Envelopes for algebraic patterns* (2022), available at arXiv:2208.07183.

[BP21] Peter Bonventre and Luís A. Pereira, *Genuine equivariant operads*, Adv. Math. **381** (2021), Paper No. 107502, available at arXiv:1707.02226.

[Bru07] Morten Brun, *Witt vectors and equivariant ring spectra applied to cobordism*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 2, 351–385, available at arXiv:math/0411567.

[BV73] J. Michael Boardman and Rainer M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973.

[CH21] Hongyi Chu and Rune Haugseng, *Homotopy-coherent algebra via Segal conditions*, Adv. Math. **385** (2021), Paper No. 107733, available at arXiv:1907.03977.

[CHLL24] Bastiaan Cnossen, Rune Haugseng, Tobias Lenz, and Sil Linskens, *Homotopical commutative rings and bispans* (2024), available at arXiv:2403.06911.

[CLL23a] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens, *Parametrized stability and the universal property of global spectra* (2023), available at arXiv:2301.08240.

[CLL23b] ———, *Partial parametrized presentability and the universal property of equivariant spectra* (2023), available at arXiv:2307.11001.

[CLL24] ———, *Parametrized higher semiadditivity and the universality of spans* (2024), available at 2403.07676.

[CMNN20] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel, *Descent and vanishing in chromatic algebraic K-theory via group actions* (2020), available at 2011.08233. To appear in *Ann. Sci. Éc. Norm. Supér.*

[Day70] Brian J. Day, *Construction of Biclosed Categories* (1970). PhD thesis, University of New South Wales; available at `http://www.math.mq.edu.au/~street/DayPhD.pdf`.

[Dre71] Andreas W. M. Dress, *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*, Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971. With the aid of lecture notes, taken by Manfred Küchler.

[EH23] Elden Elmanto and Rune Haugseng, *On distributivity in higher algebra I: the universal property of bispans*, Compos. Math. **159** (2023), no. 11, 2326–2415, available at arXiv:2010.15722.

[EM06] Anthony D. Elmendorf and Michael A. Mandell, *Rings, modules, and algebras in infinite loop space theory*, Adv. Math. **205** (2006), no. 1, 163–228, available at arXiv:0403403.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153, available at arXiv:1305.4550.

[GHN17] David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Doc. Math. **22** (2017), 1225–1266, available at arXiv:1501.02161.

[GM11] Bertrand J. Guillou and J. Peter May, *Models of G-spectra as presheaves of spectra* (2011), available at arXiv:1110.3571. To appear in *Algebr. Geom. Topol.*

[GM17] ———, *Equivariant iterated loop space theory and permutative G-categories*, Algebr. Geom. Topol. **17** (2017), no. 6, 3259–3339.

[GM23] David Gepner and Lennart Meier, *On equivariant topological modular forms*, Compos. Math. **159** (2023), no. 12, 2638–2693, available at arXiv:2004.10254.

[GMMO23] Bertrand J. Guillou, J. Peter May, Mona Merling, and Angélica M. Osorno, *Multiplicative equivariant K-theory and the Barratt-Priddy-Quillen theorem*, Adv. Math. **414** (2023), Paper No. 108865.

[GW18] Javier J. Gutiérrez and David White, *Encoding equivariant commutativity via operads*, Algebr. Geom. Topol. **18** (2018), no. 5, 2919–2962, available at arXiv:1707.02130.

[Har20] Yonatan Harpaz, *Ambidexterity and the universality of finite spans*, Proc. Lond. Math. Soc. (3) **121** (2020), no. 5, 1121–1170, available at arXiv:1703.09764.

[Hau23] Rune Haugseng, *From analytic monads to ∞-operads through Lawvere theories* (2023), available at arXiv:2301.01199.

[Haus17] Markus Hausmann, *G-symmetric spectra, semistability and the multiplicative norm*, J. Pure Appl. Algebra **221** (2017), no. 10, 2582–2632, available at arXiv:1411.2290.

[HHLN23] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten, *Two-variable fibrations, factorisation systems and ∞-categories of spans*, Forum Math. Sigma **11** (2023), Paper No. e111, available at arXiv:2011.11042.

[HHR16] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262, available at arXiv:0908.3724.

[Hil22a] Kaif Hilman, *Norms and periodicities in genuine equivariant hermitian K-theory* (2022). PhD thesis, University of Copenhagen; available at `https://drive.google.com/file/d/1HONOkztsG4Zqpq9RJxPzSQrEQBE5oSTg/view`.

[Hil22b] ———, *Parametrised noncommutative motives and equivariant cubical descent in algebraic K-theory* (2022), available at arXiv:2202.02591.

[Law04] F. William Lawvere, *Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories*, Repr. Theory Appl. Categ. **5** (2004), 1–121. Reprinted from *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 869–872 and *Reports of the Midwest Category Seminar. II*, 41–61, Springer, Berlin, 1968.

[Law09] Tyler Lawson, *Commutative Γ-rings do not model all commutative ring spectra*, Homology Homotopy Appl. **11** (2009), no. 2, 189–194.

[Len20] Tobias Lenz, *G-global homotopy theory and algebraic K-theory* (2020), available at arXiv:2012.12676. To appear in *Mem. Amer. Math. Soc.*

[Len21] ———, *Parsummable categories as a strictification of symmetric monoidal categories*, Theory Appl. Categ. **37** (2021), Paper No. 17, 482–529.

[Len23] ———, *Genuine versus naïve symmetric monoidal G-categories*, Doc. Math. **28** (2023), no. 5, 1079–1161, available at arXiv:2203.02277.

[LMS86] L. Gaunce Lewis Jr., J. Peter May, and Mark Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[LS23] Tobias Lenz and Anna Marie Schröter, *Simplicial $*$-modules and mild actions* (2023), available at arXiv:2307.11002. To appear in *Homology Homotopy Appl.*

[Lur09] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available from http://math.ias.edu/~lurie/.

[Lur17] ______, *Higher Algebra*, 2017. Available at http://math.ias.edu/~lurie/.

[Lur24] ______, *Kerodon*, 2024. https://kerodon.net.

[Mar24] Gregoire Marc, *A higher Mackey functor description of algebras over an N_∞-operad* (2024), available at arXiv:2402.12447.

[May72] J. Peter May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York, 1972. Lecture Notes in Mathematics, Vol. 271.

[May74] ______, *E_∞ spaces, group completions, and permutative categories*, New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), London Math. Soc. Lecture Note Ser., vol. No. 11, Cambridge Univ. Press, London-New York, 1974, pp. 61–93.

[May77] ______, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics, vol. Vol. 577, Springer-Verlag, Berlin–New York, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave.

[May09] ______, *The construction of E_∞ ring spaces from bipermutative categories*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, available at arXiv:0903.2818.

[MMO17] J. Peter May, Mona Merling, and Angélica M. Osorno, *Equivariant infinite loop space theory, the space level story* (2017), available at arXiv:1704.03413. To appear in *Mem. Amer. Math. Soc.*

[MW21] Louis Martini and Sebastian Wolf, *Colimits and cocompletions in internal higher category theory* (2021), available at arXiv:2111.14495.

[MW22] ______, *Presentable categories internal to an ∞-topos* (2022), available at arXiv:2209.05103.

[Nar16] Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category* (2016), available at arXiv:1608.07704.

[NS22] Denis Nardin and Jay Shah, *Parametrized and equivariant higher algebra* (2022), available at arXiv:2203.00072.

[Pü24] Phil Pützstück, *Parametrized higher algebra and global Picard spectra* (2024). MSc thesis, University of Bonn.

[Rub21] Jonathan Rubin, *Detecting Steiner and linear isometries operads*, Glasg. Math. J. **63** (2021), no. 2, 307–342, available at arXiv:1903.08723.

[Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Sha20] Amit Sharma, *Symmetric monoidal categories and Γ-categories*, Theory Appl. Categ. **35** (2020), Paper No. 14, 417–512.

[Shi89] Kazuhisa Shimakawa, *Infinite loop G-spaces associated to monoidal G-graded categories*, Publ. Res. Inst. Math. Sci. **25** (1989), no. 2, 239–262.

[SS79] Nobuo Shimada and Kazuhisa Shimakawa, *Delooping symmetric monoidal categories*, Hiroshima Math. J. **9** (1979), no. 3, 627–645.

[Str14] Ross Street, *Kan extensions and cartesian monoidal categories* (2014), available at arXiv:1409.6405v1.

[Str12] Neil Strickland, *Tambara functors* (2012), available at arXiv:1205.2516.

[Tam93] Daisuke Tambara, *On multiplicative transfer*, Comm. Algebra **21** (1993), no. 4, 1393–1420.

[Ull13] John Ullman, *Tambara functors and commutative ring spectra* (2013), available at arXiv:1304.4912.

[Yau24] Donald Yau, *Multifunctorial equivariant algebraic K-theory* (2024), available at arXiv:2404.02794.

Part IV

The global equivariant setting

PARAMETRIZED STABILITY AND THE UNIVERSAL PROPERTY OF GLOBAL SPECTRA

BASTIAAN CNOSSEN, TOBIAS LENZ, AND SIL LINSKENS

Abstract. We develop a framework of parametrized semiadditivity and stability with respect to so-called atomic orbital subcategories of an indexing ∞-category T, extending work of Nardin. Specializing this framework, we introduce *global ∞-categories* and the notions of equivariant semiadditivity and stability, yielding a higher categorical version of the notion of a Mackey 2-functor studied by Balmer-Dell'Ambrogio. As our main result, we identify the free presentable equivariantly stable global ∞-category with a natural global ∞-category of global spectra for finite groups, in the sense of Schwede and Hausmann.

Contents

1. Introduction

Equivariant homotopy theory combines classical homotopy theory with ideas from representation theory to study geometric objects with symmetries. Many constructions from homotopy theory carry over to the equivariant setting, leading for example to equivariant analogues of important cohomology theories like topological K-theory and (stable) bordism. The resulting tools and methods have been successfully applied to various other branches of mathematics, for example in the proof of the *Atiyah-Segal Completion Theorem* [AS69], Carlson's proofs of the *Segal* [Car84] and *Sullivan Conjecture* [Car91], or in the resolution of the Kervaire invariant one problem by Hill, Hopkins, and Ravenel [HHR16].

While one can study equivariant homotopy theory for a single group G at a time, there are many equivariant phenomena which occur uniformly and compatibly in large families of groups, such as the families of all finite groups or all compact Lie groups. The study of such phenomena is known as *global homotopy theory* [GH07, Boh14, Sch18, Hau19, Len20, LNP22]. This framework has led to improved understanding of a variety of equivariant phenomena, where previously a direct description for each individual group was either much more opaque or not available, for example for equivariant stable bordism and equivariant formal group laws [Hau22]. The study of global homotopy theory moreover admits connections to the geometry of stacks [GH07, Jur20, Par20, SS20].

Just like non-equivariant and G-equivariant homotopy theory, global homotopy theory comes in various different flavours: *unstable global homotopy theory* studies *global spaces* [GH07] while *stable global homotopy theory* is concerned with so-called *global spectra* [Sch18]; in-between, one can also consider a variety of algebraic structures on global spaces [Bar21], with the most prominent example being *ultra-commutative monoids* or the equivalent notion of *special global* Γ*-spaces* [Len20]. The goal of this article is to understand the relationship between these different variants.

Stability and equivariant semiadditivity. Classically the passage from the homotopy theory of spaces to the homotopy theory of spectra is known as *stabilization.* More generally, a homotopy theory $\mathcal{C}$ (e.g. given in the form of a model category or an ∞-category) is said to be *stable* if it is pointed and the suspension-loop adjunction in $\mathcal{C}$ is an equivalence. Stability of a homotopy theory leads to a lot of algebraic structure: for example, its homotopy category $\mathrm{Ho}(\mathcal{C})$ is additive, and it canonically admits the structure of a triangulated category. If $\mathcal{C}$ is not yet (known to be) stable, there is a universal way to stabilize it by forming a homotopy theory $\mathrm{Sp}(\mathcal{C})$ of suitable *spectrum objects* in $\mathcal{C}$.

Although one may apply this stabilization procedure to the homotopy theory of global spaces, the resulting theory is in many ways not sufficient, and in particular does not yield the homotopy theory of global spectra. This issue in fact already arises in the case of equivariant homotopy theory for a fixed group G: applying the general stabilization procedure to the homotopy theory of G-spaces for some finite group G only results in the homotopy theory of *naive G-spectra*, which for example does not support a good theory of duality. Instead, one defines the homotopy theory of *genuine G-spectra* by stabilizing more generally with respect to the *representation spheres* S^V for each finite-dimensional G-representation V. This genuine stabilization leads to a much richer algebraic structure on the associated homotopy category than naive stabilization: for example, the homotopy category of genuine G-spectra admits a canonical enrichment in *Mackey functors*, refining the enrichment in abelian groups.

Non-equivariantly, the algebraic structure on hom sets in a stable homotopy theory comes from semiadditivity: finite coproducts agree with finite products. In a similar way, the Mackey enrichment of the homotopy theory of genuine G-spectra comes from a form of *equivariant semiadditivity.* To explain what this means, consider a subgroup H of the finite group G; the restriction functor from genuine G-spectra to genuine H-spectra then admits both a left adjoint ind_H^G and a right adjoint coind_H^G, called *induction* and *coinduction*, respectively. From the perspective of this article, the main feature of genuine equivariant spectra is that there is a natural equivalence $\mathrm{ind}_H^G \simeq \mathrm{coind}_H^G$ between these two functors, called the *Wirthmüller isomorphism* [Wir74]. If we think of ind_H^G as a 'G-coproduct over G/H' and coind_H^G as a 'G-product over G/H,' this may be seen as an equivariant analogue of the usual notion of semiadditivity. These Wirthmüller isomorphisms are then precisely what gives rise to the transfer maps in the aforementioned Mackey enrichment.

Parametrized higher category theory. In light of the above, it is natural to ask whether one can modify the stabilization procedure for G-spaces in a way that additionally enforces equivariant semiadditivity, and, if so, whether this will result in the homotopy theory of genuine G-spectra. One subtlety with this question is that the Wirthmüller isomorphisms described above do not only depend on the homotopy theory of genuine G-spectra but also on the homotopy theories of genuine

H-spectra for every subgroup H of G, together with all the restriction functors relating them. Based on suggestions by Mike Hill in 2012, Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah [BDG+16] began developing the theory of *G-∞-categories* for a finite group G, in which these ideas could be made precise. More generally, given an ∞-category T, they introduced the notion of a *T-∞-category*, thought of as an family of ∞-categories parametrized by T, and showed that many concepts and foundational results from the theory of ∞-categories have analogues in this parametrized setting. Using this framework, Nardin [Nar16] worked out a notion of *parametrized semiadditivity* which neatly recovers the equivariant Wirthmüller isomorphisms described earlier. He further sketched a proof that the G-∞-category of genuine G-spectra is obtained from the G-∞-category of G-spaces by enforcing both stability and parametrized semiadditivity.

1.1. **Global ∞-categories.** The goal of this article is to develop an analogue of the above story, and in particular of Nardin's result, in the global setting. A distinguishing feature that was not present in the equivariant setting is the appearance of *inflation functors*: restriction functors along *surjective* group homomorphisms $\alpha\colon H \twoheadrightarrow G$. This extra structure leads to the notion of a *global ∞-category.* Roughly speaking, such an object consists of

(i) an ∞-category $\mathcal{C}(G)$ for every finite group G;
(ii) a restriction functor $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$ for every homomorphism $\alpha\colon H \to G$;
(iii) higher structure which in particular witnesses that conjugations act as the identity.

Examples of global ∞-categories abound in representation theory, and more generally equivariant mathematics; here we only mention categories of representations, genuine equivariant spectra, and equivariant Kasparov categories, referring the reader to [BD20] for a detailed discussion of these examples. In this paper, on the other hand, we will be mainly interested in examples coming from *G-global homotopy theory* in the sense of [Len20]; namely, we consider:

- the global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ of *global spaces*, given at a group G by G-global spaces (see Section 3.2 for a precise definition);
- the global ∞-category $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}$ of *special global Γ-spaces*, given at a group G by special G-global Γ-spaces (see Section 5.1 for a precise definition);
- the global ∞-category $\underline{\mathscr{S}p}^{\mathrm{gl}}$ of *global spectra*, given at a group G by G-global spectra (see Section 7.1 for a precise definition).

As the main results of this paper we establish universal properties for these three global ∞-categories:

Presentability. A global ∞-category $\mathcal{C}$ is said to be *presentable* if $\mathcal{C}(G)$ is presentable for all G and the restriction functors $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$ admit left and right adjoints $a_!$ and α_* for all $\alpha\colon H \to G$ satisfying a base change condition, which may be thought of as a categorified version of the Mackey double coset formula. We refer to Section 2.4 for a precise definition. The universal example is provided by G-global homotopy theory:

Theorem A (Universal property of global spaces, 3.3.2)**.** The global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ is presentable. For every presentable global ∞-category $\mathcal{D}$, evaluation at the point $* \in \underline{\mathscr{S}}^{\mathrm{gl}}(1)$ induces an equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}}^{\mathrm{gl}}, \mathcal{D}) \to \mathcal{D}$$

of global ∞-categories. Put differently, $\underline{\mathscr{S}}^{\mathrm{gl}}$ is the free presentable global ∞-category on one generator $*$.

We will in fact provide a stronger version of Theorem A based on a notion of *global cocompleteness*, see Section 2.3. Our proof of this result can be regarded as a highly coherent version of Schwede's global Elmendorf theorem [Sch20].

Equivariant semiadditivity. Following ideas of [Nar16], we introduce a notion of *equivariant semiadditivity* in our context; namely, a global ∞-category $\mathcal{C}$ is equivariantly semiadditive if the following conditions are satisfied:

- *Fiberwise semiadditivity:* The ∞-category $\mathcal{C}(G)$ is semiadditive for every G and the functor $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$ preserves finite biproducts for every $\alpha\colon H \to G$;
- *Ambidexterity:* For every *injective* homomorphism $i\colon H \to G$, the restriction functor $i^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$ admits a both left adjoint $i_!$ and a right adjoint i_* satisfying a base change condition as before, and a certain *norm map* $\mathrm{Nm}_i\colon i_! \to i_*$ is a natural equivalence between these two adjoints.

A 2-categorical analogue of this definition was studied under the name *Mackey 2-functor* by Balmer-Dell'Ambrogio [BD20]. The examples of representations, equivariant spectra, and Kasparov categories referred to above are all equivariantly semiadditive – for example, in the case of equivariant spectra, ambidexterity precisely comes from the Wirthmüller isomorphism. Once again, G-global homotopy theory provides the universal example in this setting:

Theorem B (Universal property of global Γ-spaces, 5.3.5)**.** The global ∞-category $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}$ is presentable and equivariantly semiadditive. For every presentable equivariantly semiadditive global ∞-category $\mathcal{D}$, evaluation at the free special global Γ-space $\mathbb{P}(*)$ induces an equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

of global ∞-categories. Put differently, $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}$ is the free presentable equivariantly semiadditive global ∞-category on one generator $\mathbb{P}(*)$.

Equivariant stability. A global ∞-category $\mathcal{C}$ is called *equivariantly stable* if it is equivariantly semiadditive and *fiberwise stable*, meaning that the ∞-category $\mathcal{C}(G)$ is stable for every finite group G and the restriction functors $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$ are exact for all $\alpha\colon H \to G$.

Theorem C (Universal property of global spectra, 7.3.2)**.** The global ∞-category $\underline{\mathscr{S}p}^{\mathrm{gl}}$ is presentable and equivariantly stable. For every presentable equivariantly stable global ∞-category $\mathcal{D}$, evaluation at the global sphere spectrum $\mathbb{S}$ defines an equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}p}^{\mathrm{gl}}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

of global ∞-categories. Put differently, $\underline{\mathscr{S}p}^{\mathrm{gl}}$ is the free presentable equivariantly stable global ∞-category on one generator $\mathbb{S}$.

Combining this with Theorem A, this makes precise that $\underline{\mathscr{S}p}^{\mathrm{gl}}$ is obtained from $\underline{\mathscr{S}}^{\mathrm{gl}}$ by universally stabilizing and enforcing Wirthmüller isomorphisms, answering the question from the beginning. In particular, global ∞-categories provide a natural and convenient home for studying global homotopy theory. Conversely, once one is interested in global ∞-categories, global (and more generally G-global) homotopy theory appears naturally in the form of the universal examples. For example one can show using the above that for every equivariantly stable global ∞-category $\mathcal{C}$, the ∞-category $\mathcal{C}(G)$ is canonically enriched over G-global spectra, with strong compatibilities as the group G varies.

1.2. **Parametrized higher category theory.** In setting up the formalism of equivariant semiadditivity and stability, we work in the more general context of T-∞-categories for an arbitrary ∞-category T, in the sense of [BDG$^+$16]. Global ∞-categories arise as the special case where T is the $(2,1)$-category Glo of finite connected groupoids, see Example 2.1.3. We introduce the notion of an *atomic orbital* subcategory $P \subseteq T$, generalizing a notion due to [Nar16]; in this setting, we can then more generally define *P-semiadditivity* and *P-stability*, which for the subcategory Orb $\subseteq$ Glo of *faithful morphisms* specializes to the notions of equivariant semiadditivity/stability discussed before.

Given a T-∞-category $\mathcal{C}$ with sufficiently many parametrized limits, we provide a universal way to turn it into a P-semiadditive T-∞-category by passing to the T-∞-category $\underline{\mathrm{CMon}}^P(\mathcal{C})$ of *P-commutative monoids*, a parametrized version of commutative monoid objects in higher algebra. In a similar way, we construct a universal P-stabilization $\underline{\mathrm{Sp}}^P(\mathcal{C})$ of $\mathcal{C}$. Combining this with Theorem A, the key step in the proof of Theorem B and Theorem C is then to produce equivalences of global ∞-categories

$$\underline{\Gamma\mathscr{S}}^{\mathrm{gl,spc}} \simeq \underline{\mathrm{CMon}}^{\mathrm{Orb}}(\underline{\mathscr{S}}^{\mathrm{gl}}), \qquad \underline{\mathscr{S}p}^{\mathrm{gl}} \simeq \underline{\mathrm{Sp}}^{\mathrm{Orb}}(\underline{\mathscr{S}}^{\mathrm{gl}}).$$

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2. Parametrized higher category theory

In this section, we will recall some of the basic notions of parametrized higher category theory. A first development of such theory was given by Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin and Jay Shah, cf. [BDG$^+$16, Sha21, Nar16]. From the perspective of categories internal to ∞-topoi, an alternative development was given by Louis Martini and Sebastian Wolf [Mar21, MW21, MW22].

2.1. T-∞-categories.

We introduce the notion of a T-∞-category for a small ∞-category T and discuss various constructions and examples.

Definition 2.1.1. Let T be a small ∞-category. A *T-∞-category* is a functor $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$. If $\mathcal{C}$ and $\mathcal{D}$ are T-∞-categories, then a *T-functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a natural transformation from $\mathcal{C}$ to $\mathcal{D}$. The ∞-category Cat_T of T-∞-categories is defined as the functor category $\mathrm{Cat}_T := \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty)$.

If $\mathcal{C}$ is a T-∞-category and $f\colon A \to B$ is a morphism in T, we will write f^* for the functor $\mathcal{C}(f)\colon \mathcal{C}(B) \to \mathcal{C}(A)$ and refer to this as *restriction along f*.

Example 2.1.2. Let G be a finite group and let Orb_G denote the *orbit category of G*, defined as the full subcategory of the category of G-sets spanned by the orbits G/H for subgroups $H \leqslant G$. When $T = \mathrm{Orb}_G$, T-∞-categories are referred to as *G-∞-categories*, c.f. [BDG$^+$16].

We will be mainly interested in the following example.

Example 2.1.3. Define **Glo** as the strict $(2,1)$-category of finite groups, group homomorphisms, and conjugations. In particular, **Glo** comes with a fully faithful functor $B\colon \mathbf{Glo} \hookrightarrow \mathbf{Grpd}$ into the $(2,1)$-category of groupoids given by sending a finite group G to the corresponding 1-object groupoid BG, a homomorphism $f\colon G \to H$ to the functor $Bf\colon BG \to BH$ given on homomorphisms by f, and a conjugation $h\colon f \Rightarrow f'$ (i.e. an $h \in H$ such that $f'(g) = hf(g)h^{-1}$ for all $g \in G$) to the natural transformation $Bf \Rightarrow Bf'$ whose value at the unique object of BG is the edge h.

We define the ∞-category Glo as the Duskin nerve of the $(2,1)$-category **Glo**. We will use the term *global ∞-category* for a Glo-∞-category, *global functor* for a Glo-functor, etc.

Remark 2.1.4. The straightening-unstraightening equivalence (see [Lur09, Theorem 3.2.0.1]) provides an equivalence of ∞-categories $\mathrm{Cat}_T \simeq (\mathrm{Cat}_\infty)^{\mathrm{cocart}}_{/T^{\mathrm{op}}}$, where $(\mathrm{Cat}_\infty)^{\mathrm{cocart}}_{/T^{\mathrm{op}}}$ denotes the (non-full) subcategory of the slice $(\mathrm{Cat}_\infty)_{/T^{\mathrm{op}}}$ spanned by the cocartesian fibrations over T^{op} and the functors over T^{op} that preserve cocartesian edges. The cocartesian fibration over T^{op} corresponding to a T-∞-category $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is denoted by $\int \mathcal{C} \to T^{\mathrm{op}}$ and is referred to as the cocartesian unstraightening of $\mathcal{C}$. A T-functor $F\colon \mathcal{C} \to \mathcal{D}$ corresponds to a functor $\int F\colon \int \mathcal{C} \to \int \mathcal{D}$ over T^{op} which preserves cocartesian edges. In fact, in the articles [BDG$^+$16], [Sha21] and [Nar16], a T-∞-category is *defined* as a cocartesian fibration over T^{op}.

Definition 2.1.5. Let $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a T-∞-category. We define the *underlying ∞-category* $\Gamma(\mathcal{C})$ of $\mathcal{C}$ as the limit of $\mathcal{C}$:

$$\Gamma(\mathcal{C}) := \lim_{B \in T^{\mathrm{op}}} \mathcal{C}(B).$$

This defines a functor $\Gamma\colon \mathrm{Cat}_T \to \mathrm{Cat}_\infty$. Note that when T has a final object, $\Gamma(\mathcal{C})$ is obtained by evaluating $\mathcal{C}$ at the final object.

Remark 2.1.6. By [Lur09, Corollary 3.3.3.2], the ∞-category $\Gamma(\mathcal{C})$ is equivalent to the ∞-category of cocartesian sections of $\int \mathcal{C} \to T^{\mathrm{op}}$.

We discuss some important examples of T-∞-categories.

Example 2.1.7. Every ∞-category $\mathcal{E}$ gives rise to a T-∞-category $\mathrm{const}_{\mathcal{E}}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ given by $\mathrm{const}_{\mathcal{E}}(t) = \mathcal{E}$ for all $t \in T$. This provides a functor $\mathrm{const}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_T$. We will refer to T-∞-categories in the essential image of this functor as *constant T-∞-categories.*

Remark 2.1.8. Note that the functor $\mathrm{const}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_T$ is left adjoint to the underlying ∞-category functor $\Gamma\colon \mathrm{Cat}_T \to \mathrm{Cat}_\infty$: for every T-∞-category $\mathcal{C}$ and every ∞-category $\mathcal{E}$ there is an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_T}(\mathrm{const}_{\mathcal{E}}, \mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}, \Gamma(\mathcal{C})).$$

Example 2.1.9. Every presheaf $B\colon T^{\mathrm{op}} \to \mathrm{Spc}$ on T gives rise to a T-∞-category $\underline{B}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ by composing it with the inclusion $\mathrm{Spc} \subseteq \mathrm{Cat}_\infty$ of ∞-groupoids into all ∞-categories, and we obtain a fully faithful inclusion

$$\mathrm{PSh}(T) = \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Spc}) \hookrightarrow \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty) = \mathrm{Cat}_T\,.$$

The T-∞-categories in the essential image of this functor will be referred to as *T-∞-groupoids.*

In particular, every object $B \in T$ gives rise to a T-∞-category $\underline{B}$ via the Yoneda embedding $T \hookrightarrow \mathrm{PSh}(T)$.

Remark 2.1.10. The inclusion $\mathrm{PSh}(T) \subseteq \mathrm{Cat}_T$ admits a right adjoint $\iota\colon \mathrm{Cat}_T \to \mathrm{PSh}(T)$. It is given on $\mathcal{C}$ by $\iota \circ \mathcal{C}$, where $\iota\colon \mathrm{Cat}_\infty \to \mathrm{Spc}$ is the functor which assigns to an ∞-category its core, the largest ∞-groupoid contained in it.

Example 2.1.11. Let $\mathcal{E}$ be an ∞-category. A *T-object in $\mathcal{E}$* is a functor $T^{\mathrm{op}} \to \mathcal{E}$. We obtain a T-∞-category $\underline{\mathcal{E}}_T$ of *T-objects in $\mathcal{E}$* by assigning to an object $B \in T$ the ∞-category $\mathrm{Fun}((T_{/B})^{\mathrm{op}}, \mathcal{E})$ of $T_{/B}$-objects in $\mathcal{E}$. More precisely, the T-∞-category $\underline{\mathcal{E}}_T$ is defined as the following composite

$$T^{\mathrm{op}} \xrightarrow{B \mapsto (T_{/B})^{\mathrm{op}}} (\mathrm{Cat}_\infty)^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathcal{E})} \mathrm{Cat}_\infty,$$

where the functoriality of the first functor is via post-composition in T, i.e. the straightening of the cocartesian fibration $\mathrm{ev}_1\colon T^{[1]} \to T$. It is clear that sending $\mathcal{E}$ to $\underline{\mathcal{E}}_T$ gives rise to a functor $\mathrm{Cat}_\infty \to \mathrm{Cat}_T$.

As a special case, we obtain the following T-∞-categories:

(1) taking $\mathcal{E} = \mathrm{Spc}$ gives a T-∞-category $\underline{\mathrm{Spc}}_T$ of *T-spaces* or *T-∞-groupoids.*
(2) taking $\mathcal{E} = \mathrm{Spc}_*$ gives a T-∞-category $\underline{\mathrm{Spc}}_{T,*} \coloneqq \underline{\mathrm{Spc}_*}_T$ of *pointed T-spaces.*
(3) taking $\mathcal{E} = \mathrm{Sp}$ gives a T-∞-category $\underline{\mathrm{Sp}}_T$ of *naive T-spectra.*[1]
(4) taking $\mathcal{E} = \mathrm{cat}_\infty$, the ∞-category of small ∞-categories, gives a T-∞-category $\underline{\mathrm{cat}}_T \coloneqq \underline{\mathrm{cat}_\infty}_T$ of *small T-∞-categories.*

Remark 2.1.12. For every T-∞-category $\mathcal{C}$ and every ∞-category $\mathcal{E}$, there is an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_T}(\mathcal{C}, \underline{\mathcal{E}}_T) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\textstyle\int \mathcal{C}, \mathcal{E})$$

which is natural in $\mathcal{C}$ and $\mathcal{E}$. In other words, the construction of Example 2.1.11 provides a right adjoint to the cocartesian unstraightening $\int\colon \mathrm{Cat}_T \to \mathrm{Cat}_\infty$ which

[1]The term 'naive spectra' is used in equivariant homotopy theory to contrast it with 'genuine spectra'.

assigns to a T-∞-category $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ the total category $\int \mathcal{C}$ of its unstraightening $\int \mathcal{C} \to T^{\mathrm{op}}$. We will prove this in Lemma 2.2.13 below.

Remark 2.1.13. One may alternatively describe T-∞-categories as Cat_∞-valued sheaves on the presheaf ∞-topos $\mathrm{PSh}(T) = \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Spc})$, i.e., as limit-preserving functors $\mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Indeed, the functor

$$\mathrm{Fun}(\mathrm{PSh}(T)^{\mathrm{op}}, \mathrm{Cat}_\infty) \to \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty)$$

given by precomposition with the Yoneda embedding $T \hookrightarrow \mathrm{PSh}(T)$ becomes an equivalence when restricting the domain to the full subcategory of limit-preserving functors, see [Lur09, Theorem 5.1.5.6].

Remark 2.1.14. For an ∞-topos $\mathcal{B}$, the ∞-category $\mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ of sheaves of ∞-categories on $\mathcal{B}$ is equivalent to the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{B})$ spanned by the *internal ∞-categories* (or *complete Segal objects*). We refer to [Mar21, Definition 3.2.4] for a precise definition of an internal ∞-category, and to [Mar21, Section 3.5] for a proof of this equivalence. By Remark 2.1.13, the study of T-∞-categories is thus equivalent to the study of ∞-categories internal to the presheaf topos $\mathrm{PSh}(T)$. Although we will never use this perspective in this article, we will not hesitate to cite results from [Mar21, MW21, MW22] which are formulated in the language of internal ∞-categories.

Convention 2.1.15. Henceforth, we will abuse notation and denote the extension of a T-∞-category $\mathcal{C}$ to a limit preserving functor $\mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ again by $\mathcal{C}$. At various points in this article, we will write expressions such as $A \times A$ or $A \times_B A$ for objects $A, B \in T$, meaning implicitly that this pullback is taken in the presheaf ∞-category $\mathrm{PSh}(T)$. In particular, when we write $\mathcal{C}(A \times B)$ or $\mathcal{C}(A \times_B A)$, we are referring to the values of the limit-extension $\mathcal{C}\colon \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ at the relevant objects. This abuse of notation is justified by the fact that the Yoneda embedding $T \hookrightarrow \mathrm{PSh}(T)$ preserves all limits that exist in T, cf. [Lur09, Proposition 5.1.3.2]. In a similar way, all colimits of objects of T are understood to be taken in the presheaf ∞-category $\mathrm{PSh}(T)$: expressions such as $\coprod_{i=1}^n A_i$ will always mean formal disjoint union.

Remark 2.1.16. It will be useful to observe that the limit-extension of the T-∞-category $\underline{\mathrm{Spc}}_T$ of T-spaces is equivalent to the slice functor

$$\begin{aligned} \mathrm{PSh}(T)_{/-}\colon \mathrm{PSh}(T)^{\mathrm{op}} &\to \mathrm{Cat}_\infty, \\ A &\mapsto \mathrm{PSh}(T)_{/A}, \\ [f\colon A \to B] &\mapsto f^*\colon \mathrm{PSh}(T)_{/B} \to \mathrm{PSh}(T)_{/A}, \end{aligned}$$

which is defined as the functor which classifies the cartesian fibration

$$t\colon \mathrm{Ar}(\mathrm{PSh}(T)) \to \mathrm{PSh}(T)\colon (A \to B) \mapsto B.$$

Indeed, since this slice functor preserves limits by [Lur09, Theorem 6.1.3.9, Proposition 6.1.3.10], it suffices to show that its restriction to T^{op} is equivalent to $\underline{\mathrm{Spc}}_T$. Consider the Yoneda embedding $T \hookrightarrow \mathrm{PSh}(T)$. By considering the functoriality in over-categories on both sides we obtain a natural transformation

$$T_{/-} \to \mathrm{PSh}(T)_{/-}$$

of functors in T. The universal property of presheaves implies that this extends to a natural equivalence

$$\mathrm{PSh}(T_{/-}) \xrightarrow{\sim} \mathrm{PSh}(T)_{/-}.$$

By the naturality of the Yoneda embedding (see [HHLN22b, Theorem 8.1] or [Ram22, Theorem 2.4]) we get that upon passing to right adjoints the diagram $\mathrm{PSh}(T_{/-})$ agrees with $\underline{\mathrm{Spc}}_T$, completing the proof.

Example 2.1.17. For an object $B \in T$ there is an adjunction

$$\pi_B \colon \mathrm{PSh}(T)_{/B} \rightleftarrows \mathrm{PSh}(T) : - \times B,$$

where π_B is the forgetful functor. Since both functors preserve colimits we obtain by precomposition an adjunction

$$\pi_B^* \colon \mathrm{Cat}_T \rightleftarrows \mathrm{Cat}_{T/B} : (\pi_B)_* = (- \times B)^*.$$

Lemma 2.1.18. *Consider an object $B \in T$. Then there is for every ∞-category $\mathcal{E}$ an equivalence of $T_{/B}$-∞-categories*

$$\pi_B^* \underline{\mathcal{E}}_T \simeq \underline{\mathcal{E}}_{T_{/B}},$$

natural in $\mathcal{E}$.

Proof. It will suffice to prove that the composite

$$T_{/B} \xrightarrow{\pi_B} T \xrightarrow{A \mapsto (T_{/A})^{\mathrm{op}}} \mathrm{Cat}_\infty$$

is equivalent to the slice functor of $T_{/B}$. This is immediate from the observation that the target map $\mathrm{ev}_1 \colon (T_{/B})^{[1]} \to T_{/B}$ of $T_{/B}$ is the pullback along π_B of the target map $\mathrm{ev}_1 \colon T^{[1]} \to T$. □

2.2. **Parametrized functor categories.** In this subsection, we establish a variety of basic results on parametrized functor categories.

Definition 2.2.1. Since T is small and Cat_∞ is cartesian closed, the ∞-category $\mathrm{Cat}_T = \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty)$ is again cartesian closed. Given two T-∞-categories $\mathcal{C}$ and $\mathcal{D}$, we define the *T-∞-category of T-functors* $\mathcal{C} \to \mathcal{D}$, denoted $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$, as the internal hom-object between $\mathcal{C}$ and $\mathcal{D}$ in the ∞-category Cat_T. In particular, for any triple of T-∞-categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ there is a natural equivalence

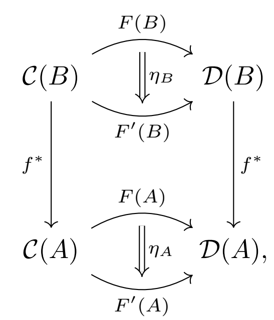
$$\underline{\mathrm{Fun}}_T(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \underline{\mathrm{Fun}}_T(\mathcal{C}, \underline{\mathrm{Fun}}_T(\mathcal{D}, \mathcal{E})).$$

Definition 2.2.2. Given two T-∞-categories $\mathcal{C}$ and $\mathcal{D}$, we define the ∞-category $\mathrm{Fun}_T(\mathcal{C}, \mathcal{D})$ of T-functors $\mathcal{C} \to \mathcal{D}$ as the underlying ∞-category of the T-∞-category $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$:

$$\mathrm{Fun}_T(\mathcal{C}, \mathcal{D}) := \Gamma(\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})).$$

Remark 2.2.3. The objects of $\mathrm{Fun}_T(\mathcal{C}, \mathcal{D})$ may be identified with T-functors $\mathcal{C} \to \mathcal{D}$. If F and F' are two such T-functors, we refer to a morphism $\alpha \colon F \to F'$ in $\mathrm{Fun}_T(\mathcal{C}, \mathcal{D})$ as a *natural transformation of T-functors*. A natural transformation of

T-functors is given by a collection of natural transformations $\eta_A\colon F(A) \to F'(A)$ together with a coherent collection of 3-cells which fill the cylinders



for every morphism $f\colon A \to B$ in T.

Example 2.2.4. The T-functors of the form $\mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Spc}}_T$ are called *T-presheaves on $\mathcal{C}$*. There is an analogue of the Yoneda embedding,

$$y\colon\ \mathcal{C} \to \underline{\mathrm{Fun}}_T(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Spc}}_T),$$

see [BDG$^+$16, Section 10] or [Mar21, Section 4.7], which is fully faithful by [Mar21, Theorem 4.7.8].

Natural transformations between ordinary categories induce natural transformations between their associated T-∞-categories of T-objects.

Construction 2.2.5. Given ∞-categories $\mathcal{E}$ and $\mathcal{E}'$, we will construct a functor

$$\mathrm{Fun}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}_T(\underline{\mathcal{E}}_T, \underline{\mathcal{E}'}_T)$$

which on groupoid cores reduces to the functoriality of the construction $\mathcal{E} \mapsto \underline{\mathcal{E}}_T$ of Example 2.1.11. By adjunction we may equivalently specify a T-functor of the form

$$\mathrm{const}_{\mathrm{Fun}(\mathcal{E},\mathcal{E}')} \times \underline{\mathcal{E}}_T \to \underline{\mathcal{E}'}_T.$$

At level $B \in T$, we define this as the composition functor

$$\mathrm{Fun}(\mathcal{E}, \mathcal{E}') \times \mathrm{Fun}((T_{/B})^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Fun}((T_{/B})^{\mathrm{op}}, \mathcal{E}').$$

By precomposing with the functors $T^{\mathrm{op}}_{/A} \to T^{\mathrm{op}}_{/B}$ this specifies a T-functor.

The following result of [MW21] relates the ∞-category of T-functors from Definition 2.2.2 to the identically named ∞-category of T-functors from [BDG$^+$16, p.3].

Proposition 2.2.6 ([MW21, Proposition 3.2.1])**.** *For any two T-∞-categories $\mathcal{C}$ and $\mathcal{D}$ there is a natural equivalence*

$$\mathrm{Fun}_T(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/T^{\mathrm{op}}}(\textstyle\int \mathcal{C}, \int \mathcal{D}),$$

where the right-hand side denotes the full subcategory of $\mathrm{Fun}_{/T^{\mathrm{op}}}(\int \mathcal{C}, \int \mathcal{D})$ *spanned by those functors $\int \mathcal{C} \to \int \mathcal{D}$ over T^{op} that preserve cocartesian edges.* □

To give a pointwise description of the parametrized functor category $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$, we need the following enhanced version of the Yoneda lemma.

Lemma 2.2.7 (Categorical Yoneda lemma)**.** *For every $B \in T$ and $\mathcal{C} \in \mathrm{Cat}_T$, evaluation at the identity $\mathrm{id}_B \in \mathrm{Hom}_T(B, B) = \underline{B}(B)$ induces a natural equivalence of ∞-categories*

$$\mathrm{Fun}_T(\underline{B}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}(B).$$

Proof. By the Yoneda lemma and Remark 2.1.10 there is a natural equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_T}(\underline{B}, \mathcal{C}) \simeq \iota(\mathcal{C}(B))$$

between the ∞-groupoid of T-functors $\underline{B} \to \mathcal{C}$ and the groupoid core of the ∞-category $\mathcal{C}(B)$, so the statement holds on groupoid cores. To obtain the statement on the level of categories, we use that the ∞-category Cat_T is cotensored over Cat_∞: for every T-∞-category $\mathcal{C}$ and every ∞-category $\mathcal{E}$, the cotensor $\mathcal{C}^{\mathcal{E}}$ is given at $B \in T$ by $\mathcal{C}^{\mathcal{E}}(B) \simeq \mathrm{Fun}(\mathcal{E}, \mathcal{C}(B))$. It follows that for any ∞-category $\mathcal{E}$ we have natural equivalences

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}, \mathrm{Fun}_T(\underline{B}, \mathcal{C})) &\simeq \mathrm{Hom}_{\mathrm{Cat}_T}(\underline{B}, \mathcal{C}^{\mathcal{E}}) \simeq \iota(\mathcal{C}^{\mathcal{E}}(B)) \\ &\simeq \iota(\mathrm{Fun}(\mathcal{E}, \mathcal{C}(B)) = \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}, \mathcal{C}(B)),\end{aligned}$$

and thus the claim follows from the Yoneda lemma. □

By limit-extending the previous equivalence to presheaves, we immediately obtain:

Corollary 2.2.8. *There is a unique natural equivalence $\mathrm{Fun}_T(\underline{X}, \mathcal{C}) \simeq \mathcal{C}(X)$ of functors $\mathrm{PSh}(T) \times \mathrm{Cat}_T \to \mathrm{Cat}$ that for representable presheaves recovers the equivalence from the previous lemma.* □

Corollary 2.2.9. *Let $X \in \mathrm{PSh}(T)$ and let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories. Then there are natural equivalences of ∞-categories*

$$\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(X) \simeq \mathrm{Fun}_T(\underline{X}, \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}_T(\underline{X} \times \mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}_T(\mathcal{C}, \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})).$$

Proof. The first equivalence is Corollary 2.2.8, while the others are immediate. □

Our next goal is to give an alternative description of the functor T-∞-category $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$.

Construction 2.2.10. Let $B \in T$ and let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories. We define a $T_{/B}$-functor

$$\pi_B^* \colon \pi_B^* \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D}) \to \underline{\mathrm{Fun}}_{T_{/B}}(\pi_B^* \mathcal{C}, \pi_B^* \mathcal{D})$$

as adjoint to the composite

$$\pi_B^* \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D}) \times \pi_B^* \mathcal{C} \simeq \pi_B^*(\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D}) \times \mathcal{C}) \xrightarrow{\pi_B^*(\mathrm{ev})} \pi_B^* \mathcal{D}\,.$$

We obtain T-functors

$$\underline{\mathrm{Fun}}_T(\underline{B}, \mathcal{D}) \to (\pi_B)_* \pi_B^* \mathcal{D} \qquad \text{and} \qquad (\pi_B)_! \pi_B^* \mathcal{C} \to \underline{B} \times \mathcal{C}\,.$$

The first one is adjoint to the composite $T_{/B}$-functor

$$\pi_B^* \underline{\mathrm{Fun}}_T(\underline{B}, \mathcal{D}) \xrightarrow{\pi_B^*} \underline{\mathrm{Fun}}_{T_{/B}}(\pi_B^* \underline{B}, \pi_B^* \mathcal{D}) \xrightarrow{\mathrm{ev}_{\mathrm{id}_B}} \pi_B^* \mathcal{D},$$

while the second one is adjoint to the composite

$$\pi_B^* \mathcal{C} \xrightarrow{(\iota, \mathrm{id})} \pi_B^* \underline{B} \times \pi_B^* \mathcal{C} \simeq \pi_B^*(\underline{B} \times \mathcal{C}).$$

Here $\mathrm{ev}_{\mathrm{id}_B}$ denotes evaluation at the object $\mathrm{id}_B \in \Gamma(\pi_B^*\underline{B}) = \mathrm{Hom}_T(B,B)$, and ι picks out $\mathrm{id}_B \in \Gamma(\pi_B^*\underline{B}) = \mathrm{Hom}_T(B,B)$. Observe that the resulting map $(\pi_B)_!\pi_B^*\mathcal{C} \to \underline{B}\times\mathcal{C}$ is the total mate of the map $\underline{\mathrm{Fun}}_T(\underline{B},\mathcal{D}) \to (\pi_B)_*\pi_B^*\mathcal{D}$.

Corollary 2.2.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories and let $B \in T$. The following hold:*

(1) The T-functors

$$\underline{\mathrm{Fun}}_T(\underline{B},\mathcal{D}) \xrightarrow{\sim} (\pi_B)_*\pi_B^*\mathcal{D} \qquad \text{and} \qquad (\pi_B)_!\pi_B^*\mathcal{C} \xrightarrow{\sim} \underline{B}\times\mathcal{C}$$

from Construction 2.2.10 are equivalences of T-∞-categories.

(2) The $T_{/B}$-functor

$$\pi_B^*\colon \pi_B^*\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_{T_{/B}}(\pi_B^*\mathcal{C},\pi_B^*\mathcal{D})$$

from Construction 2.2.10 is an equivalence of $T_{/B}$-∞-categories.

(3) In particular, passing to global sections gives an equivalence of ∞-categories

$$\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})(B) \xrightarrow{\sim} \mathrm{Fun}_{T_{/B}}(\pi_B^*\mathcal{C},\pi_B^*\mathcal{D}).$$

Proof. For part (1), it suffices to show the first equivalence, since the second equivalence follows by passing to total mates. For this, we have to show that for every object $A \in T$ the induced map $\underline{\mathrm{Fun}}_T(\underline{B},\mathcal{D})(A) \to (\pi_B)_*\pi_B^*\mathcal{D}(A) = \mathcal{D}(A\times B)$ is an equivalence. Given Lemma 2.2.7, it will suffice to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Fun}_T(\underline{A},\underline{\mathrm{Fun}}_T(\underline{B},\mathcal{D})) & \xrightarrow{\sim} & \mathrm{Fun}_T(\underline{A\times B},\mathcal{D}) \\
{\scriptstyle 2.2.7}\downarrow{\scriptstyle\sim} & & {\scriptstyle 2.2.7}\downarrow{\scriptstyle\sim} \\
\underline{\mathrm{Fun}}_T(\underline{B},\mathcal{D})(A) & \longrightarrow & \mathcal{D}(A\times B).
\end{array}$$

This follows from unwinding the definitions, using the observation that the equivalence $\mathrm{Fun}_T(\underline{B},\mathcal{D}) \to \mathcal{D}(B)$ from Lemma 2.2.7 is the map induced on global sections by the map $\underline{\mathrm{Fun}}_T(\underline{B},\mathcal{D}) \to (\pi_B)_*\pi_B^*\mathcal{D}$.

We will next prove part (3). It suffices to show that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Fun}_T(\underline{B},\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})) & \xrightarrow{\sim} & \mathrm{Fun}_T(\mathcal{C},\underline{\mathrm{Fun}}_T(\underline{B},\mathcal{D})) & \xrightarrow[(1)]{\sim} & \mathrm{Fun}_T(\mathcal{C},(\pi_B)_*\pi_B^*\mathcal{D}) \\
{\scriptstyle 2.2.7}\downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\sim} \\
\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})(B) & & \xrightarrow{\pi_B^*(B)} & & \mathrm{Fun}_{T_{/B}}(\pi_B^*\mathcal{C},\pi_B^*\mathcal{D}).
\end{array}$$

This is again a matter of unwinding definitions, using that the equivalence of (1) is defined in terms of the map π_B^* and evaluation at id_B.

Finally, for part (2) it remains to show that the map π_B^* induces an equivalence when evaluated at every object $A \in T_{/B}$. To see this, consider the following two $T_{/A}$-functors:

$$\pi_A^*\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D}) \to \pi_A^*\underline{\mathrm{Fun}}_{T_{/B}}(\pi_B^*\mathcal{C},\pi_B^*\mathcal{D}) \to \underline{\mathrm{Fun}}_{T_{/A}}(\pi_A^*\mathcal{C},\pi_A^*\mathcal{D}).$$

Here we abuse notation by writing A both for an object in $T_{/B}$ and for its underlying object in T. By part (3), both the second map and the composite map induce equivalences on global sections, and therefore so does the first. This finishes the proof. $\square$

For later use let us describe the functoriality of $\underline{\mathrm{Fun}}_T(\pi_B^*\mathcal{C}, \pi_B^*\mathcal{D})$ in B. While this can be done in a fully coherent fashion using the results and techniques of [HHLN22a], for the purposes of the present paper the following more elementary lemma will be sufficient:

Lemma 2.2.12. *Let $f\colon A \to B$ be a map in T, and let $\sigma\colon \pi_B \circ T_{/f} \simeq \pi_A$ be the usual equivalence. Then the diagram*

$$\begin{array}{ccccc}
\mathrm{Hom}_{T_{/B}}(\pi_B^*\mathcal{C}, \pi_B^*\mathcal{D}) & \xrightarrow{T_{/f}^*} & \mathrm{Hom}_{T_{/A}}(T_{/f}^*\pi_B^*\mathcal{C}, T_{/f}^*\pi_B^*\mathcal{D}) & \xrightarrow{\sigma^*} & \mathrm{Hom}_{T_{/A}}(\pi_A^*\mathcal{C}, \pi_A^*\mathcal{D}) \\
\big\downarrow & & & & \big\downarrow \\
\mathrm{Fun}_{T_{/B}}(\pi_B^*\mathcal{C}, \pi_B^*\mathcal{D}) & & & & \mathrm{Fun}_{T_{/A}}(\pi_A^*\mathcal{C}, \pi_A^*\mathcal{D}) \\
\sim\big\downarrow 2.2.11 & & & & \sim\big\downarrow 2.2.11 \\
\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(B) & & \xrightarrow[f^*]{\hspace{6em}} & & \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(A)
\end{array}$$

of natural transformations of functors $\mathrm{Cat}_T^{\mathrm{op}} \times \mathrm{Cat}_T \to \mathrm{Cat}_\infty$ commutes up to homotopy.

Proof. Unravelling the definitions and using the naturality of the equivalences from the previous corollary, it suffices to construct a homotopy filling

$$\begin{array}{ccccc}
\mathrm{Hom}_{T_{/B}}(\pi_B^*\mathcal{C}, \pi_B^*\mathcal{D}) & \xrightarrow{T_{/f}^*} & \mathrm{Hom}_{T_{/A}}(T_{/f}^*\pi_B^*\mathcal{C}, T_{/f}^*\pi_B^*\mathcal{D}) & \xrightarrow{\sigma^*} & \mathrm{Hom}_{T_{/A}}(\pi_A^*\mathcal{C}, \pi_A^*\mathcal{D}) \\
\sim\big\downarrow & & & & \big\downarrow\sim \\
\mathrm{Hom}_T(\pi_{B!}\pi_B^*\mathcal{C}, \mathcal{D}) & & & & \mathrm{Hom}_T(\pi_{A!}\pi_A^*\mathcal{C}, \mathcal{D}) \\
j_B^*\big\downarrow & & & & \big\downarrow j_A^* \\
\mathrm{Hom}_T(\underline{B} \times \mathcal{C}, \mathcal{D}) & & \xrightarrow[(f\times\mathcal{C})^*]{\hspace{6em}} & & \mathrm{Hom}_T(\underline{A} \times \mathcal{C}, \mathcal{D})
\end{array}$$

where the unlabelled equivalences come from adjunction and $j_A\colon \pi_{A!}\pi_A^*\mathcal{C} \to \underline{A} \times \mathcal{C}$, $j_B\colon \pi_{B!}\pi_B^*\mathcal{C} \to \underline{B} \times \mathcal{C}$ are as in Corollary 2.2.11 again.

Obviously, we can make the lower half commute by adding the restriction along the composite $\pi_{A!}\pi_A^*\mathcal{C} \simeq \underline{A} \times \mathcal{C} \to \underline{B} \times \mathcal{C} \simeq \pi_{B!}\pi_B^*$ as the middle arrow. Similarly, we can make the upper portion commute by taking the restriction along

$$f_\Diamond\colon (\pi_A)_!\pi_A^*\mathcal{C} \xrightarrow[\sim]{\sigma^*} (\pi_A)_!T_{/f}^*\pi_B^*\mathcal{C} \longrightarrow (\pi_B)_!\pi_B^*\mathcal{C}$$

instead, where the second map is the mate of σ^*. It will therefore suffice to show that these two natural transformations $\pi_{A!}\pi_A^* \Rightarrow \pi_{B!}\pi_B^*$ are in fact homotopic. Plugging in the definitions of the equivalences j_A and j_B, this amounts to saying that $j_B f_\Diamond$ is adjunct to the map $\pi_A^*\mathcal{C} \to \pi_A^*\underline{B} \times \pi_A^*\mathcal{C}$ picking out $f \in (\pi_A^*\underline{B})(\mathrm{id}_A) = \mathrm{Hom}(A, B)$.

Further plugging in definitions, the adjunct of $jf_\Diamond$ is the top right composite in

$$\begin{array}{ccccccc}
\pi_A^*\mathcal{C} & \xrightarrow{\sigma^*} & T_{/f}^*\pi_B^*\mathcal{C} & \xrightarrow{\eta} & T_{/f}^*\pi_B^*(\pi_B)_!\pi_B^*\mathcal{C} & \xrightarrow{(\sigma^{-1})^*} & \pi_A^*(\pi_B)_!\pi_B^*\mathcal{C} \\
& & & \searrow & j_B\big\downarrow & & \big\downarrow j_B \\
& & & & T_{/f}^*\pi_B^*(\underline{B} \times \mathcal{C}) & \xrightarrow[(\sigma^{-1})^*]{} & \pi_A^*(\underline{B} \times \mathcal{C}).
\end{array}$$

The square on the right commutes by naturality and the dashed composite is by definition the adjunct of j_B, i.e. it is induced by the map $\pi_B^*\mathcal{C} \to \pi_B^*(\underline{B} \times \mathcal{C})$ classifying $\mathrm{id}_B \in \underline{B}(B)$. It follows that after postcomposing with the projection $\pi_A^*(\underline{B} \times \mathcal{C}) \to \pi_A^*\mathcal{C}$ the above is simply the identity, and it remains to show that the natural map $\pi_A^*\mathcal{C} \to \pi_A^*\underline{B}$ obtained by postcomposing with the other projection classifies $f \in \underline{B}(A)$. But indeed, as a functor of Cat_T the right hand side is constant, so any natural transformation into it is determined by its value on the terminal object. Together with application of the Yoneda lemma it therefore suffices that for $\mathcal{C} = *$ the map $(\pi_A^* *)(\mathrm{id}_A) \to (\pi_A^*\underline{B})(\mathrm{id}_A)$ hits f. However, by the above commutative diagram this can be identified with $(\pi_B^* *)(f) \to (\pi_B^*\underline{B})(f)$, which hits $f = f^*(\mathrm{id}_B)$ for formal reasons. □

We will now prove the adjunction between $\mathcal{E} \mapsto \underline{\mathcal{E}}_T$ and $\mathcal{C} \mapsto \int\mathcal{C}$ promised in Remark 2.1.12.

Lemma 2.2.13. *The functor $\int\colon \mathrm{Cat}_T \to \mathrm{Cat}_\infty$, sending a T-∞-category $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ to the total space $\int\mathcal{C}$ of the cocartesian fibration $\int\mathcal{C} \to T^{\mathrm{op}}$ it classifies, admits a right adjoint given by the construction $\mathcal{E} \mapsto \underline{\mathcal{E}}_T$ of Example 2.1.11.*

Proof. The functor $\int\colon \mathrm{Cat}_T \to \mathrm{Cat}_\infty$ can be expanded into the following composite functor:

$$\mathrm{Cat}_T \overset{2.1.4}{\simeq} (\mathrm{Cat}_\infty)^{\mathrm{cocart}}_{/T^{\mathrm{op}}} \hookrightarrow (\mathrm{Cat}_\infty)_{/T^{\mathrm{op}}} \xrightarrow{\mathrm{fgt}} \mathrm{Cat}_\infty$$

By [Lur17, Example B.2.10, Remark B.0.28], the functor in the middle is the underlying functor of a left Quillen functor between model categories, so that it admits a right adjoint by [Hin16, Proposition 1.5.1]. The second functor clearly admits a right adjoint. It follows that $\int\colon \mathrm{Cat}_T \to \mathrm{Cat}_\infty$ admits a right adjoint $R\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_T$.

As a formal consequence we obtain for each T-∞-category $\mathcal{C}$ and for each ∞-category $\mathcal{E}$ a natural equivalence

$$\mathrm{Fun}_T(\mathcal{C}, R(\mathcal{E})) \simeq \mathrm{Fun}(\textstyle\int\mathcal{C}, \mathcal{E})$$

between the ∞-category T-functors $\mathcal{C} \to R(\mathcal{E})$ and the ∞-category of functors $\int\mathcal{C} \to \mathcal{E}$: for every other ∞-category $\mathcal{E}'$ there is a natural equivalence

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}_T}(\mathcal{E}', \mathrm{Fun}_T(\mathcal{C}, R(\mathcal{E}))) &\simeq \mathrm{Hom}_{\mathrm{Cat}_T}(\mathcal{C} \times \mathrm{const}_{\mathcal{E}'}, R(\mathcal{E})) \\ &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\textstyle\int(\mathcal{C} \times \mathrm{const}_{\mathcal{E}'}), \mathcal{E}) \\ &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\textstyle\int\mathcal{C} \times \mathcal{E}', \mathcal{E}) \\ &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}', \mathrm{Fun}(\textstyle\int\mathcal{C}, \mathcal{E})),\end{aligned}$$

where we use that the cocartesian unstraightening of $\mathrm{const}_{\mathcal{E}'}$ is $T^{\mathrm{op}} \times \mathcal{E}'$ and that the inclusion $(\mathrm{Cat}_\infty)^{\mathrm{cocart}}_{/T^{\mathrm{op}}} \hookrightarrow (\mathrm{Cat}_\infty)_{/T^{\mathrm{op}}}$ preserves finite products. The claim now follows from the Yoneda lemma.

The description of R as the functor $\mathcal{E} \mapsto \underline{\mathcal{E}}_T$ from Example 2.1.11 now follows immediately by recalling that the cocartesian unstraightening of the functor $\underline{B} = \mathrm{Hom}_T(-, B)\colon T^{\mathrm{op}} \to \mathrm{Spc}$ is by definition given by the target functor $(T_{/B})^{\mathrm{op}} \to T^{\mathrm{op}}$. Namely for any $\mathcal{E} \in \mathrm{Cat}_\infty$ and $B \in T$ we have a natural equivalence

$$R(\mathcal{E})(B) \overset{2.2.7}{\simeq} \mathrm{Fun}_T(\underline{B}, R(\mathcal{E})) \simeq \mathrm{Fun}(\textstyle\int\underline{B}, \mathcal{E}) \simeq \mathrm{Fun}((T_{/B})^{\mathrm{op}}, \mathcal{E}) = \underline{\mathcal{E}}_T(B). \quad (1)$$

This finishes the proof. □

Remark 2.2.14. Combining the previous lemma and Corollary 2.2.9 we obtain a natural equivalence

$$\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{E}_T) \simeq \mathrm{Fun}(\textstyle\int \mathcal{C} \times \underline{(-)}, \mathcal{E}).$$

Remark 2.2.15. Let $B \in T$ arbitrary. Unravelling the chain of equivalences (1) we see that the diagram

$$\begin{array}{ccc} \mathrm{Fun}_T(\underline{B}, \underline{\mathcal{E}}_T) & \xrightarrow{\text{adjunction}} & \mathrm{Fun}(\int \underline{B}, \mathcal{E}) \\ {\scriptstyle\text{Yoneda}}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \underline{\mathcal{E}}_T(B) & = & \mathrm{Fun}(T_{/B}, \mathcal{E}) \end{array}$$

of equivalences commutes up to natural equivalence where f is the chosen identification of $\int \underline{B}$ with $T_{/B}$ over T.

Now assume T has a final object 1. Specializing the above to $B = 1$ (and identifying $T_{/1}$ with T as usual), we see that

$$\begin{array}{ccc} \mathrm{Fun}_T(\underline{1}, \underline{\mathcal{E}}_T) & \underset{\simeq}{\xrightarrow{\text{adjunction}}} & \mathrm{Fun}(\int \underline{1}, \mathcal{E}) \\ {\scriptstyle\text{Yoneda}}\big\downarrow{\scriptstyle\simeq} & & \big\uparrow{\scriptstyle \pi^*} \\ \mathcal{E}_T(1) & = & \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{E}) \end{array}$$

commutes up to natural equivalence, where $\pi\colon \int \underline{1} \to T^{\mathrm{op}}$ is the cocartesian projection. Combining this with the naturality of the adjunction equivalence, we conclude that we have for every T-∞-category $\mathcal{C}$ and $c \in \mathcal{C}(1)$ a natural equivalence filling

$$\begin{array}{ccc} \mathrm{Fun}_T(\mathcal{C}, \underline{\mathcal{E}}_T) & \underset{\simeq}{\xrightarrow{\text{adjunction}}} & \mathrm{Fun}(\int \mathcal{C}, \mathcal{E}) \\ {\scriptstyle\mathrm{ev}_c}\big\downarrow & & \big\downarrow{\scriptstyle \hat{c}^*} \\ \underline{\mathcal{E}}_T(1) & = & \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{E}) \end{array}$$

where $\hat{c}\colon T^{\mathrm{op}} \to \int \mathcal{C}$ is the essentially unique map over T^{op} sending the fiber over $1 \in T$ to c (i.e. the unstraightening of c viewed as a T-functor $\underline{1} \to \mathcal{C}$).

Remark 2.2.16. We can make the equivalence $\mathrm{Fun}_T(\mathcal{C}, \underline{\mathcal{E}}_T) \simeq \mathrm{Fun}(\int \mathcal{C}, \mathcal{E})$ of Lemma 2.2.13 more explicit. Consider a functor $\tilde{F}\colon \int \mathcal{C} \to \mathcal{E}$. The associated T-functor $F\colon \mathcal{C} \to \underline{\mathcal{E}}_T$ is given at $B \in T$ by the functor

$$F_B\colon \mathcal{C}(B) \to \mathrm{Fun}(T^{\mathrm{op}}_{/B}, \mathcal{E}),$$

where $F_B(X)(h\colon C \to B) = \tilde{F}(h^*(X))$, the value of $\tilde{F}$ on the cocartesian pushforward of $X \in \mathcal{C}(B)$ along h to $\mathcal{C}(C)$. The value of $F_B(X)$ on a triangle

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & B & \end{array}$$

is given by applying $\tilde{F}$ to the cocartesian edge over f from $g^*(X)$ to $h^*(X)$. More generally, for another object $B' \in T$ and a functor $\tilde{F}\colon \int(\mathcal{C} \times \underline{B'}) \to \mathcal{E}$, the associated

T-functor $F\colon \mathcal{C} \to \underline{\mathrm{Fun}}(\underline{B}, \mathcal{E})$ is given at $B \in T$ by the functor $F_B\colon \mathcal{C}(B) \to \mathrm{Fun}(T^{\mathrm{op}}_{/B\times B'}, \mathcal{E})$ given by

$$F_B(X)(A \xrightarrow{(f_B, f_{B'})} B \times B') = \tilde{F}(A, f_B^* X, f_{B'}).$$

2.3. **Parametrized adjunctions, limits and colimits.** We will briefly recall the parametrized versions of adjunctions, limits and colimits, following Sections 3 and 4 of [MW21] (see Remark 2.1.14). An alternative treatment in the language of cocartesian fibrations over T^{op} is given by [Sha21, Sections 8 and 9].

Definition 2.3.1 ([MW21, Definition 3.1.1])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ is a tuple $(L, R, \eta, \varepsilon)$, where $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ are T-functors and where $\eta\colon \mathrm{id}_{\mathcal{D}} \to RL$ and $\varepsilon\colon LR \to \mathrm{id}_{\mathcal{C}}$ are natural transformations of T-functors fitting in commutative triangles

$$\begin{array}{ccc} L & \xrightarrow{L\eta} & LRL \\ & \searrow & \downarrow{\scriptstyle \varepsilon L} \\ & & L \end{array} \qquad \text{and} \qquad \begin{array}{ccc} RLR & \xleftarrow{\eta R} & R \\ {\scriptstyle R\varepsilon}\downarrow & \swarrow & \\ R. & & \end{array}$$

Note that the notion of an adjunction between two T-∞-categories only depends on the (homotopy) 2-category associated to Cat_T and in particular many of the standard 2-categorical results about adjunctions hold in this setting.

Example 2.3.2. Every adjunction $\mathcal{E} \rightleftarrows \mathcal{E}'$ of ∞-categories gives rise to an adjunction $\mathrm{const}_{\mathcal{E}} \rightleftarrows \mathrm{const}_{\mathcal{E}'}$ on associated constant T-∞-categories.

Example 2.3.3. By Construction 2.2.5, every adjunction $\mathcal{E} \rightleftarrows \mathcal{E}'$ of ∞-categories gives rise to an adjunction $\underline{\mathcal{E}}_T \rightleftarrows \underline{\mathcal{E}'}_T$ on associated T-∞-categories of T-objects.

Important will be the following 'pointwise' criterion for checking that a T-functor has a parametrized adjoint.

Proposition 2.3.4 ([MW21, Proposition 3.2.8 and Corollary 3.2.10])**.** *A T-functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a (parametrized) right adjoint if and only if the following two conditions hold:*

(1) For every object $B \in T$, the induced functor $F(B)\colon \mathcal{C}(B) \to \mathcal{D}(B)$ admits a right adjoint $G(B)\colon \mathcal{D}(B) \to \mathcal{C}(B)$;
(2) For every morphism $f\colon A \to B$ in T, the Beck-Chevalley transformation

$$f^* \circ G(B) \implies G(A) \circ f^*$$

given as the mate of the naturality square

$$\begin{array}{ccc} \mathcal{C}(B) & \xrightarrow{F(B)} & \mathcal{D}(A) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{C}(A) & \xrightarrow{F(A)} & \mathcal{D}(A) \end{array}$$

is an equivalence.

If this is the case, the right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ of F is given on an object $B \in T$ by the functor $G(B)\colon \mathcal{D}(B) \to \mathcal{C}(B)$. Moreover, if $Y \in \mathrm{PSh}(T)$ is any presheaf, then also the functor $F(Y)\colon \mathcal{C}(Y) \to \mathcal{D}(Y)$ admits a right adjoint $G(Y)$ in this case, and *for any map* $f\colon X \to Y$ *in* $\mathrm{PSh}(T)$ *the Beck-Chevalley map* $f^* \circ G(Y) \Rightarrow G(X) \circ f^*$ *is an equivalence.*

The dual statement for parametrized left adjoints also holds. □

We will now move to parametrized limits and colimits, of which we will only give a brief treatment sufficient for the purposes of the present article.

Definition 2.3.5. Let K and $\mathcal{C}$ be T-∞-categories. We say that $\mathcal{C}$ *admits* K-*indexed colimits* if the diagonal functor $\mathrm{diag}\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(K, \mathcal{C})$ given by precomposing with $K \to \underline{1}$ admits a left adjoint $\mathrm{colim}_K \colon \underline{\mathrm{Fun}}_T(K, \mathcal{C}) \to \mathcal{C}$. Similarly we say that $\mathcal{C}$ *admits* K*-indexed limits* if diag admits a right adjoint $\lim_K \colon \underline{\mathrm{Fun}}_T(K, \mathcal{C}) \to \mathcal{C}$.

Definition 2.3.6. Let K, $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories and assume that $\mathcal{C}$ and $\mathcal{D}$ admit K-indexed colimits. We will say that a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ *preserves* K-*indexed colimits* if the Beck-Chevalley transformation $\mathrm{colim}_K \circ \underline{\mathrm{Fun}}_T(K, F) \implies F \circ \mathrm{colim}_K$ of the naturality square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathrm{diag}} & \underline{\mathrm{Fun}}_T(K, \mathcal{C}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \underline{\mathrm{Fun}}_T(K,F)} \\ \mathcal{D} & \xrightarrow{\mathrm{diag}} & \underline{\mathrm{Fun}}_T(K, \mathcal{D}) \end{array}$$

is an equivalence.

In the non-parametrized context, one often asks an ∞-category to admit (co)limits for a certain class of indexing diagrams. In the parametrized setting, one should work with the following parametrized notion of 'class of indexing diagrams'.

Definition 2.3.7. Let T be an ∞-category. A *class of* T-∞-*categories* is a full parametrized subcategory $\mathbf{U} \subseteq \underline{\mathrm{cat}}_T$ of the T-∞-category of small T-∞-categories.

Definition 2.3.8 ([MW21, Definition 5.2.1 and Remark 5.2.4])**.** Let $\mathbf{U}$ be a class of T-∞-categories and let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories.

(1) We will say that $\mathcal{C}$ *admits* $\mathbf{U}$-*colimits* if the $T_{/B}$-∞-category $\pi_B^* \mathcal{C}$ of Example 2.1.17 admits K-indexed $T_{/B}$-colimits for every $B \in T$ and $K \in \mathbf{U}(B) \subseteq \mathrm{Cat}(T_{/B})$.
(2) If $\mathcal{C}$ and $\mathcal{D}$ admit $\mathbf{U}$-colimits, a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to *preserve* $\mathbf{U}$-*colimits* if $\pi_B^* F$ preserves K-indexed $T_{/B}$-colimits for every $B \in T$ and $K \in \mathbf{U}(B)$.

Dually, $\mathcal{C}$ is said to *admit* $\mathbf{U}$-*limits* if for every $B \in T$ and $K \in \mathbf{U}(B)$, the $T_{/B}$-∞-category $\pi_B^* \mathcal{C}$ admits K-indexed $T_{/B}$-limits. A T-functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to *preserve* $\mathbf{U}$-*limits* if $\pi_B^* F$ preserved K-indexed $T_{/B}$-limits for every $B \in T$ and $K \in \mathbf{U}(B)$.

If $\mathbf{U} = \underline{\mathrm{cat}}_T$ consists of *all* T-∞-categories, we will say that $\mathcal{C}$ is T-*cocomplete* or T-*complete* respectively.

From the pointwise criterion Proposition 2.3.4 of parametrized adjunctions, we immediately obtain characterizations of T-(co)limits indexed by constant T-∞-categories and T-∞-groupoids, respectively. We start with the case of constant T-∞-categories.

Lemma 2.3.9 (cf. [MW21, Example 4.1.10 and Remark 4.1.11])**.** *Let $\mathcal{C}$ be a T-∞-category, let K be an ∞-category, and let const_K be the associated constant T-∞-category. Then the following conditions are equivalent:*

(1) The T-∞-category $\mathcal{C}$ admits const_K-indexed colimits;
(2) For every object $B \in T$ the ∞-category $\mathcal{C}(B)$ admits K-indexed colimits, and for every morphism $\beta\colon B' \to B$ in T the restriction functor $\beta^*\colon \mathcal{C}(B) \to \mathcal{C}(B')$ preserves K-indexed colimits.
(3) For every presheaf $Y \in \mathrm{PSh}(T)$, the ∞-category $\mathcal{C}(Y)$ admits K-indexed colimits, and for every morphism $\beta\colon Y' \to Y$ in $\mathrm{PSh}(T)$ the restriction $\beta^*\colon \mathcal{C}(Y) \to \mathcal{C}(Y')$ preserves K-indexed colimits.

The dual statement for limits also holds.

Proof. We apply the natural identification

$$\begin{aligned}\underline{\mathrm{Fun}}_T(\mathrm{const}_K, \mathcal{C})(B) &\overset{2.2.9}{\simeq} \mathrm{Fun}_T(\mathrm{const}_K, \underline{\mathrm{Fun}}_T(\underline{B}, \mathcal{C}))\\ &\overset{2.1.8}{\simeq} \mathrm{Fun}(K, \mathrm{Fun}_T(\underline{B}, \mathcal{C}))\\ &\overset{2.2.7}{\simeq} \mathrm{Fun}(K, \mathcal{C}(B)).\end{aligned}$$

Because each equivalence above is natural in K, we find that under this identification the T-functor $\mathrm{diag}\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(K, \mathcal{C})$ corresponds at $B \in T$ to the standard diagonal functor. Furthermore the Beck-Chevalley transformation associated to the naturality square

$$\begin{array}{ccc}\mathcal{C}(B) & \xrightarrow{\mathrm{diag}} & \mathrm{Fun}(K, \mathcal{C}(B))\\ {\scriptstyle \beta^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Fun}(K,\beta^*)}\\ \mathcal{C}(B') & \xrightarrow{\mathrm{diag}} & \mathrm{Fun}(K, \mathcal{C}(B'))\end{array}$$

is the standard comparison $\mathrm{colim} \circ \mathrm{Fun}(K, \beta^*) \Rightarrow F \circ \mathrm{colim}_K$. Therefore the equivalence of the first two statements is an instance of Proposition 2.3.4.

The equivalence between the first and the third statement is proven in exactly the same way. □

The following result is proved similarly and will be left to the reader.

Lemma 2.3.10. *Let K be an ∞-category and let $\mathcal{C}$ and $\mathcal{D}$ be two T-∞-categories that admit const_K-indexed T-colimits. Then a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves const_K-indexed T-colimits if and only if for each $B \in T$ the functor $F(B)\colon \mathcal{C}(B) \to \mathcal{D}(B)$ preserves K-indexed colimits. Moreover, in this case $F(Y)\colon \mathcal{C}(Y) \to \mathcal{D}(Y)$ preserves K-indexed colimits for all $Y \in \mathrm{PSh}(T)$.*

The dual statement for limits also holds. □

Definition 2.3.11. If the equivalent conditions of Lemma 2.3.9 are satisfied, we say that $\mathcal{C}$ *admits fiberwise K-indexed colimits.* If S is a collection of small ∞-categories such that $\mathcal{C}$ admits fiberwise K-indexed colimits for every $K \in S$, we say that $\mathcal{C}$ *admits fiberwise S-indexed colimits*. We say that $\mathcal{C}$ is *fiberwise cocomplete* if $\mathcal{C}$ admits fiberwise K-indexed colimits for every small ∞-category K.

Dually one defines when $\mathcal{C}$ *admits fiberwise K-indexed limits* or is *fiberwise complete.*

We next describe parametrized colimits indexed by T-∞-groupoids.

Definition 2.3.12. A *class of T-∞-groupoids*[2] is a full parametrized subcategory $\mathbf{U} \subseteq \underline{\mathrm{Spc}}_T$ of the T-∞-category of T-∞-groupoids. A morphism $f\colon X \to Y$ in $\mathrm{PSh}(T)$ is said to be in $\mathbf{U}$ if it is an object in the full subcategory $\mathbf{U}(Y) \subseteq \mathrm{PSh}(T)_{/Y}$.

Remark 2.3.13. In the above definition, we have again viewed $\mathbf{U}$ as a sheaf on $\mathrm{PSh}(T)$ via limit extension. For later use, let us make explicit what this means in terms of the original functor $T^{\mathrm{op}} \to \mathrm{Cat}_\infty$: a map $f\colon X \to Y$ in $\mathrm{PSh}(T)$ belongs to $\mathbf{U}$ if and only if for every map $\beta\colon B \to Y$ from a representable presheaf $B \in T$ the pulled back map $\beta^* f\colon \beta^* X \to B$ is an object of $\mathbf{U}(B)$.

Lemma 2.3.14 (cf. [MW21, Example 4.1.9], [Sha21, Proposition 5.12])**.** *Let $\mathbf{U}$ be a class of T-∞-groupoids. Then a T-∞-category $\mathcal{C}$ admits $\mathbf{U}$-colimits if and only if for every morphism $p\colon A \to B$ in $\mathbf{U}$, with $B \in T$, the restriction functor $p^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $p_!\colon \mathcal{C}(A) \to \mathcal{C}(B)$, and for every pullback square*

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\beta} & B \end{array} \tag{2}$$

in $\mathrm{PSh}(T)$ with $\beta\colon B' \to B$ in T and $p\colon A \to B$ in $\mathbf{U}$, the Beck-Chevalley transformation $p'_! \circ \alpha^* \Rightarrow \beta^* \circ p_!$ associated to the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(B) & \xrightarrow{\beta^*} & \mathcal{C}(B') \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p'^*} \\ \mathcal{C}(A) & \xrightarrow{\alpha^*} & \mathcal{C}(A') \end{array}$$

is a natural equivalence.

Dually, $\mathcal{C}$ admits $\mathbf{U}$-limits if and only if $p^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a right adjoint $p_*\colon \mathcal{C}(A) \to \mathcal{C}(B)$ for every morphism $p\colon A \to B$ in $\mathbf{U}$ and for every pullback square (2), the Beck-Chevalley transformation $\beta^* \circ p_* \Rightarrow p'_* \circ \alpha^*$ is a natural equivalence.

Proof. Let $(p\colon A \to B) \in \mathbf{U}(B) \subseteq \mathrm{PSh}(T)_{/B}$ be a morphism in $\mathbf{U}$. It suffices to show that the $T_{/B}$-∞-category $\pi_B^* \mathcal{C}$ admits $\underline{A}$-indexed colimits if and only if for every pullback diagram

$$\begin{array}{ccccc} A'' & \xrightarrow{\alpha'} & A' & \xrightarrow{\alpha} & A \\ {\scriptstyle p''}\downarrow & \lrcorner & {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B'' & \xrightarrow{\beta'} & B' & \xrightarrow{\beta} & B \end{array}$$

the functors p'^* and p''^* admit left adjoints $p'_!$ and $p''_!$, and the Beck-Chevalley transformation $p''_! \circ \alpha'^* \Rightarrow \beta'^* \circ p'_!$ is a natural equivalence. By replacing T by $T_{/B}$, we may assume $B = 1$ is a terminal object of T. Using the natural identifications

$$\underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{C})(B') \overset{2.2.9}{\simeq} \mathrm{Fun}_T(\underline{A} \times \underline{B'}, \mathcal{C})) \simeq \mathrm{Fun}_T(\underline{A \times B'}, \mathcal{C})) \overset{2.2.7}{\simeq} \mathcal{C}(A \times B'),$$

this is an instance of Proposition 2.3.4 applied to the T-∞-category $\underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{C})$. □

[2] This is called a 'subuniverse' in [Mar21, Definition 3.9.13]

Remark 2.3.15. If $\mathcal{C}$ is $\mathbf{U}$-cocomplete, then [MW21, Remark 5.2.4] shows that the left adjoint $p_!\colon \mathcal{C}(A) \to \mathcal{C}(B)$ exists more generally for any presheaf $B \in \mathrm{PSh}(T)$ and any $p \in \mathbf{U}(B)$; similarly, the Beck-Chevalley condition holds for any pullback square (2) in which $p \in \mathbf{U}(B)$, $B \in \mathrm{PSh}(T)$. Put differently, we can drop all representability conditions in the above lemma.

The following lemma is proved in a similar way and is left to the reader.

Lemma 2.3.16. *Let $\mathbf{U}$ be a class of T-∞-groupoids and let $\mathcal{C}$ and $\mathcal{D}$ be two T-∞-categories which admit $\mathbf{U}$-colimits. Then a T-functor F preserves $\mathbf{U}$-colimits if and only if for every $B \in T$ morphism $f\colon A \to B$ in $\mathbf{U}$, the Beck-Chevalley transformation $f_! \circ F(A) \Rightarrow F(B) \circ f_!$ is an equivalence. Moreover, in this case the Beck-Chevalley map is an equivalence more generally for any presheaf $B \in \mathrm{PSh}(T)$ and any $f \in \mathbf{U}(B)$.*

The dual statement for preserving $\mathbf{U}$-limits also holds. □

Using this we now give an easy criterion ensuring that cocontinuity is preserved under changing the indexing category:

Lemma 2.3.17. *Let $f\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ be a cocontinuous functor preserving pullbacks, let $\mathbf{U}$ be a class of S-∞-groupoids, and let $\mathbf{V}$ be a class of T-∞-groupoids such that $f(u) \in \mathbf{V}(f(B))$ for any $B \in S$ and $(u\colon A \to B) \in \mathbf{U}(B)$.*

Then $f^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ sends $\mathbf{V}$-cocomplete T-∞-categories to $\mathbf{U}$-cocomplete S-∞-categories and $\mathbf{V}$-cocontinuous T-functors to $\mathbf{U}$-cocontinuous S-functors.

Proof. Let $\mathcal{C}$ be a $\mathbf{V}$-cocomplete T-∞-category. If $B \in S$, $(u\colon A \to B) \in \mathbf{U}(S)$, then $u^*\colon (f^*\mathcal{C})(B) \to (f^*\mathcal{C})(A)$ agrees with $(f(u))^*\colon \mathcal{C}(f(B)) \to \mathcal{C}(f(A))$, so it admits a left adjoint by Remark 2.3.15 and $\mathbf{V}$-cocompleteness of $\mathcal{C}$. Similarly, given any pullback in $\mathrm{PSh}(S)$ as on the left

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[\beta]{} & B \end{array} \qquad\qquad \begin{array}{ccc} f(A') & \xrightarrow{f(\alpha)} & f(A) \\ {\scriptstyle f(p')}\downarrow & \lrcorner & \downarrow{\scriptstyle f(p)} \\ f(B') & \xrightarrow[f(\beta)]{} & f(B) \end{array}$$

with B, B' representable and $p \in \mathbf{U}(B)$, also the diagram on the right is a pullback by assumption, and the Beck-Chevalley map $p'_!\alpha^* \Rightarrow \beta^* p_!$ for $f^*\mathcal{C}$ agrees with the Beck-Chevalley map $f(p')_! f(\alpha)^* \Rightarrow f(\beta)^* f(p)_!$ for $\mathcal{C}$. In particular, it is an equivalence again, so Lemma 2.3.14 shows that $f^*\mathcal{C}$ is $\mathbf{U}$-cocomplete.

The statement about cocontinuity follows similarly from the previous lemma. □

It turns out that the parametrized colimits indexed by the constant T-∞-categories and the T-∞-groupoids already determine all parametrized colimits.

Proposition 2.3.18 ([MW21, Proposition 4.7.1])**.** *A T-∞-category is T-cocomplete if and only if it admits fiberwise colimits and $\underline{\mathrm{Spc}}_T$-colimits. A T-functor between T-cocomplete T-∞-categories preserves T-colimits if and only if it preserves fiberwise colimits and $\underline{\mathrm{Spc}}_T$-colimits.* □

Corollary 2.3.19. *Let $f\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ as in Lemma 2.3.17. Then the restriction $f^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ sends T-cocomplete T-∞-categories to S-cocomplete S-∞-categories and T-cocontinuous T-functors to S-cocontinuous S-functors.*

Proof. Clearly, f^* preserves fiberwise cocompleteness and cocontinuity. The claim therefore follows from the previous proposition together with Lemma 2.3.17. $\square$

An important example of a T-(co)complete T-∞-category is the T-∞-category of T-spaces.

Example 2.3.20. The T-∞-category $\underline{\mathrm{Spc}}_T$ is both T-cocomplete and T-complete. Recall from Remark 2.1.16 that $\underline{\mathrm{Spc}}_T(B) \simeq \mathrm{PSh}(T)_{/B}$ for every $B \in T$, with functoriality given via pullback in $\mathrm{PSh}(T)$. The functor $f^*\colon \mathrm{PSh}(T)_{/B} \to \mathrm{PSh}(T)_{/A}$ admits a left adjoint given by postcomposition with f, and since $\mathrm{PSh}(T)$ is locally cartesian closed it also admits a right adjoint. It follows that $\underline{\mathrm{Spc}}_T$ admits all fiberwise limits and colimits. The left Beck-Chevalley condition is a consequence of the pasting law of pullback squares. The right Beck-Chevalley condition follows from this by passing to total mates.

Example 2.3.21. It follows directly from Example 2.3.20 that also the T-∞-categories $\underline{\mathrm{Spc}}_{T,*}$ and $\underline{\mathrm{Sp}}_T$ of pointed T-spaces and naive T-spectra are both T-cocomplete and T-complete, since they may be obtained from $\underline{\mathrm{Spc}}_T$ by pointwise tensoring with Spc_* and Sp inside Pr^{L}, respectively. For later use, we will make the left adjoint functors $p_!$ of $\underline{\mathrm{Spc}}_{T,*}$ explicit. First note that giving a basepoint to an object $(X, f : X \to A) \in \underline{\mathrm{Spc}}_T(A) \simeq \mathrm{PSh}(T)_{/A}$ amounts to providing a section $s\colon A \to X$ of the map f, so that we can identify objects of $\underline{\mathrm{Spc}}_{T,*}(A)$ with triples (X, f, s). Given a morphism $p\colon A \to B$ in $\mathrm{PSh}(T)$, we get $p_!(X, f, s) \simeq (X', f', s')$ defined via the following pushout diagram:

$$\begin{array}{ccccc} A & \xrightarrow{s} & X & \xrightarrow{f} & A \\ {\scriptstyle p}\downarrow & \ulcorner & \downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{s'} & X' & \xrightarrow{f'} & B. \end{array}$$

We end this subsection with a discussion of categories of T-cocontinuous functors.

Definition 2.3.22. Let $\mathcal{C}, \mathcal{D}$ be T-cocomplete T-∞-categories, and let $A \in T$. We write $\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\mathcal{C}, \mathcal{D})(A) \subset \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ for the full subcategory spanned by the $T_{/A}$-cocontinuous functors $\pi_A^* \mathcal{C} \to \pi_A^* \mathcal{D}$.

Lemma 2.3.23. *This defines a T-subcategory $\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\mathcal{C}, \mathcal{D}) \subset \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$.*

Proof. By Lemma 2.2.12 it suffices to show that for any $f\colon A \to B$ in T and $T_{/B}$-cocontinuous $F\colon \mathcal{C}' \to \mathcal{D}'$, the restriction $T_{/f}^* F$ is $T_{/A}$-cocontinuous. This follows at once from Corollary 2.3.19 as $(T_{/f})_!\colon \mathrm{PSh}(T_{/A}) \to \mathrm{PSh}(T_{/B})$ agrees up to equivalence with the pullback preserving functor $\mathrm{PSh}(T)_{/f}\colon \mathrm{PSh}(T)_{/A} \to \mathrm{PSh}(T)_{/B}$. $\square$

We will now give an alternative description in terms of the adjunct functors $F\colon \mathcal{C} \to \pi_{A*}\pi_A^* \mathcal{D} \simeq \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})$, which will in particular allow us to describe the value of $\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\mathcal{C}, \mathcal{D})$ at non-representable presheaves. For this we will need:

Proposition 2.3.24 ([MW21, Proposition 4.3.1])**.** *Let K and $\mathcal{D}$ be T-∞-categories such that $\mathcal{D}$ admits all K-indexed parametrized limits. Then $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ admits all K-indexed limits for any T-∞-category $\mathcal{C}$. Furthermore, the precomposition functor $i^*\colon \underline{\mathrm{Fun}}_T(\mathcal{C}', \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ preserves K-indexed limits for every T-functor $i\colon \mathcal{C} \to \mathcal{C}'$. The dual statement for colimits is true as well.* $\square$

Combining this with Corollary 2.2.11 we get:

Corollary 2.3.25. *Let $\mathcal{C}, \mathcal{D}$ be T-∞-categories.*

1. *If $\mathcal{D}$ is $\mathbf{U}$-(co)complete for some $\mathbf{U} \subset \underline{\mathrm{Spc}}_T$, then so is $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$. Moreover, if $F\colon \mathcal{C} \to \mathcal{C}'$ is any functor, then $F^*\colon \underline{\mathrm{Fun}}_T(\mathcal{C}', \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ is $\mathbf{U}$-(co)continuous.*
2. *If $\mathcal{D}$ is fiberwise (co)complete, then so is $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$. For any $\mathcal{C} \to \mathcal{C}'$, the restriction $\underline{\mathrm{Fun}}_T(\mathcal{C}', \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ is fiberwise (co)continuous.* □

Proposition 2.3.26. *Let $\mathcal{C}$ be a T-cocomplete T-∞-category and let $\mathcal{D}$ be a $T_{/A}$-cocomplete T-∞-category. Then $\pi_{A*}\mathcal{D}$ is T-cocomplete, and a functor $F\colon \pi_A^* \to \mathcal{D}$ is $T_{/A}$-cocontinuous if and only if its adjunct $\tilde{F}\colon \mathcal{C} \to \pi_{A*}\mathcal{D}$ is T-cocontinuous.*

Proof. Assume first that F is $T_{/A}$-cocontinuous. Its adjunct $\tilde{F}$ is then given by

$$\mathcal{C} \xrightarrow{\eta} \pi_{A*}\pi_A^*\mathcal{C} \xrightarrow{\pi_{A*}F} \pi_{A*}\mathcal{D}\,.$$

Applying Corollary 2.3.19 to $A \times -\colon \mathrm{PSh}(T) \to \mathrm{PSh}(T)_{/A} \simeq \mathrm{PSh}(T_{/A})$, we see that $\pi_{A*}\mathcal{D}$ is T-cocomplete and $\pi_{A*}F$ is T-cocontinuous, so it suffices to show that η is T-cocontinuous. Unravelling definitions, this simply amounts to the functor $\mathcal{C} \to \mathcal{C}(A \times -)$ given by restriction along the projections $A \times B \to B$, which is clearly fiberwise cocontinuous. On the other hand, if $p\colon B' \to B$ is arbitrary, then the Beck-Chevalley map $p_!\eta \Rightarrow \eta p_!$ is precisely the Beck-Chevalley map $(A \times p)_! \circ \mathrm{pr}^* \Rightarrow \mathrm{pr}^* \circ p_!$ associated to the pullback

$$\begin{array}{ccc} A \times B' & \xrightarrow{\mathrm{pr}} & B' \\ {\scriptstyle A\times p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ A \times B & \xrightarrow[\mathrm{pr}]{} & B \end{array}$$

and hence an equivalence by T-cocompleteness of $\mathcal{C}$.

Conversely, assume $\tilde{F}$ is T-cocontinuous. Then F factors as

$$\pi_A^*\mathcal{C} \xrightarrow{\pi_A^*\tilde{F}} \pi_A^*\pi_{A*}\mathcal{D} \xrightarrow{\varepsilon} \mathcal{D},$$

where the first functor is $T_{/A}$-cocontinuous by Corollary 2.3.19 (or in fact, simply by definition). Similarly to the above, the counit is given by restriction along the unit maps $B \to A \times \pi_A(B)$, and the claim follows by observing that we also have pullbacks

$$\begin{array}{ccc} B' & \xrightarrow{\eta} & A \times \pi_A(B') \\ {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle A\times\pi_A(p)} \\ B & \xrightarrow[\eta]{} & A \times \pi_A(B) \end{array}$$

in $\mathrm{PSh}(T_{/A})$ for any $p\colon B' \to B$ in $T_{/A}$. □

Remark 2.3.27. Analogously one sees that for a class $\mathbf{U}$ of T-∞-groupoids a functor $F\colon \pi_A^*\mathcal{C} \to \mathcal{D}$ is $\pi_A^*\mathbf{U}$-cocontinuous if and only if its adjunct is $\mathbf{U}$-cocontinuous.

Proposition 2.3.28. *Let $\mathcal{C}, \mathcal{D}$ be T-cocomplete T-∞-categories, let $X \in \mathrm{PSh}(T)$, and let $(F\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})) \in \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(X)$. Then F belongs to $\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\mathcal{C}, \mathcal{D})(X)$ if and only if F is T-cocontinuous.*

Proof. If X is representable, this is immediate from Proposition 2.3.26. It therefore suffices to show that for general X a functor $F\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})$ is T-cocontinuous if and only if for every map $A \to X$ from a representable the composite $\mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})$ is T-cocontinuous.

The 'only if' part is immediate from Proposition 2.3.24. For the converse we observe that the functors $\underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})$ exhibit the left hand side as a limit, and so are jointly conservative. It follows immediately that F is fiberwise cocontinuous. For the Beck-Chevalley condition we now let $p\colon C \to D$ be any map in T and consider

$$\begin{array}{ccccc} \mathcal{C}(D) & \xrightarrow{F} & \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})(D) & \longrightarrow & \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})(D) \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ \mathcal{C}(C) & \xrightarrow[F]{} & \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})(C) & \longrightarrow & \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})(C). \end{array}$$

By cocontinuity of the restriction functors, the mate of the right hand square is an equivalence, and so is the mate of the total rectangle by assumption. By the compatibility of mates with pasting we see that the mate of the left hand square is an equivalence after postcomposing with $\underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})(C) \to \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})(C)$; the claim follows from joint conservativity again. $\square$

2.4. **Presentable T-∞-categories.** For the statement of various universal properties we need to restrict to *presentable* T-∞-categories. The notion of parametrized presentability was introduced by Nardin [Nar17] and was subsequently further developed by Hilman [Hil22] in the case where the ∞-category T is *orbital* (in the sense of Definition 4.2.2 below). A more general theory of parametrized presentability which works for arbitrary T was developed by Martini and Wolf [MW22] in terms of internal higher category theory. In this subsection, we will recall the main results on parametrized presentability.

Definition 2.4.1. A T-∞-category $\mathcal{C}$ is called *presentable* if the following two conditions hold:

(1) $\mathcal{C}$ is fiberwise presentable, meaning that the functor $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors (necessarily uniquely) through Pr^{L};
(2) $\mathcal{C}$ is T-cocomplete.

Observe that fiberwise presentability guarantees that $\mathcal{C}$ has fiberwise colimits, so that condition (2) holds if and only if $\mathcal{C}$ admits $\underline{\mathrm{Spc}}_T$-indexed colimits.

By [MW22, Theorem A], this definition agrees with the definition of [MW22, Section 6] applied to the ∞-topos $\mathrm{PSh}(T)$. When T is orbital, this definition agrees with that of [Hil22, Section 4].

Remark 2.4.2. Any presentable T-∞-category $\mathcal{C}$ is automatically T-complete: fiberwise completeness and the existence of right adjoints $f_*\colon \mathcal{C}(A) \to \mathcal{C}(B)$ follow from fiberwise presentability, and for every pullback square of the form (2), the Beck-Chevalley map $\beta^* \circ p_* \Rightarrow p'_* \circ \alpha^*$ is the total mate of the Beck-Chevalley map $\alpha_! \circ p'^* \Rightarrow p^* \circ \beta_!$ and thus an equivalence.

Definition 2.4.3. We define $\mathrm{Pr}^{\mathrm{L}}_T$ to be the (non-full) subcategory of Cat_T spanned by the presentable T-∞-categories and left adjoint T-functors between them. Similarly we define $\mathrm{Pr}^{\mathrm{R}}_T$ to be the (non-full) subcategory of Cat_T spanned by the presentable T-∞-categories and right adjoint T-functors between them. There is a canonical equivalence $\mathrm{Pr}^{\mathrm{L}}_T \simeq (\mathrm{Pr}^{\mathrm{R}}_T)^{\mathrm{op}}$, see [MW22, Proposition 6.4.7].

Example 2.4.4. The T-∞-category $\underline{\mathrm{Spc}}_T$ of T-spaces is presentable: fiberwise presentability follows from presentability of $\mathrm{PSh}(T)$ while T-cocompleteness was argued for in Example 2.3.20.

Example 2.4.5. Let K be a small T-∞-category and let $\mathcal{C}$ be a presentable T-∞-category. Then the functor T-∞-category $\underline{\mathrm{Fun}}_T(K, \mathcal{C})$ is again presentable [MW22, Corollary 6.2.6], [Hil22, Lemma 4.6.1].

Example 2.4.6. Accessible Bousfield localizations of presentable T-∞-category are again presentable.

In more detail, let $\mathcal{C}$ be a presentable T-∞-category and let S be a parametrized family of morphisms in $\mathcal{C}$, i.e. a specification of a set $S(B)$ of morphisms of $\mathcal{C}(B)$ for every $B \in T$ such that $f^*(u) \in S(A)$ for every $u \in S(B)$ and every morphism $f\colon A \to B$ in T. An object $X \in \mathcal{C}(B)$ is said to be *S-local* if for every morphism $f\colon A \to B$ in T the object $f^*X \in \mathcal{C}(A)$ is $S(A)$-local, meaning that for every morphism $u\colon Y \to Z$ in $S(A)$ the induced map of spaces $\mathrm{Hom}_{\mathcal{C}(A)}(Z, f^*X) \to \mathrm{Hom}_{\mathcal{C}(A)}(Y, f^*X)$ is an equivalence. We let $\mathrm{Loc}_S(\mathcal{C}) \subseteq \mathcal{C}$ denote the full subcategory spanned by the S-local objects.

By [MW22, Lemma 6.1.3, Corollary 6.2.8] the T-∞-category $\mathrm{Loc}_S(\mathcal{C})$ is again presentable and the inclusion $\mathrm{Loc}_S(\mathcal{C}) \subset \mathcal{C}$ admits a left adjoint.

Remark 2.4.7. It follows from the previous three examples that the subcategory of S-local objects of a T-∞-category of T-presheaves $\underline{\mathrm{PSh}}_T(K) := \underline{\mathrm{Fun}}_T(K^{\mathrm{op}}, \underline{\mathrm{Spc}}_T)$ is presentable whenever S is a parametrized family of morphisms in $\underline{\mathrm{PSh}}_T(K)$. Conversely, any presentable T-∞-category is of this form, see [MW22, Theorem B], [Hil22, Theorem 4.1.2]

Proposition 2.4.8 (Adjoint functor theorem, [MW22, Proposition 6.3.1])**.** *If $\mathcal{C}$ and $\mathcal{D}$ are large T-∞-categories such that $\mathcal{C}$ is presentable and $\mathcal{D}$ is locally small, a T-functor $\mathcal{C} \to \mathcal{D}$ preserves T-colimits if and only if it admits a right adjoint.* □

Given a small T-∞-category K, the T-∞-category $\underline{\mathrm{PSh}}_T(K)$ is freely generated under parametrized colimits by K:

Theorem 2.4.9 ([MW21, Theorem 6.1.1])**.** *Let K be a small T-∞-category and let $\mathcal{D}$ be a T-cocomplete T-∞-category. Then restriction along the Yoneda embedding $y\colon K \hookrightarrow \underline{\mathrm{PSh}}_T(K)$ induces an equivalence of T-∞-categories*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{PSh}}_T(K), \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_T(K, \mathcal{D}).$$ □

Remark 2.4.10. Let $A \in T$ and let $f\colon \pi_A^*K \to \pi_A^*\mathcal{D}$ define an element of $\underline{\mathrm{Fun}}_T(K, \mathcal{D})(A)$, which by the proposition then extends to a left adjoint T-functor $F\colon \pi_A^*\underline{\mathrm{PSh}}_T(K) \to \pi_A^*\mathcal{D}$. As in the classical non-parametrized situation, the *right adjoint G* of F is actually easy to describe [MW21, Remark 7.1.4]: it is given by the composition

$$\pi_A^*\mathcal{D} \xrightarrow{y} \underline{\mathrm{Fun}}_{T_{/A}}(\pi_A^*\mathcal{D}^{\mathrm{op}}, \underline{\mathrm{Spc}}_{T_{/A}}) \xrightarrow{f^*} \underline{\mathrm{Fun}}_{T_{/A}}(\pi_A^*K, \underline{\mathrm{Spc}}_{T_{/A}}) \simeq \pi_A^*\underline{\mathrm{PSh}}_T(K).$$

Applying the theorem to the case where K is the terminal T-∞-category $\underline{1}$, we see that the T-∞-category $\underline{\mathrm{Spc}}_T$ is the free T-cocomplete T-∞-category on a single generator:

Corollary 2.4.11. *Let $\mathcal{D}$ be a T-cocomplete T-∞-category. Then evaluation at the terminal object $1 \in \mathrm{PSh}(T) = \Gamma(\underline{\mathrm{Spc}}_T)$ induces an equivalence of T-∞-categories*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{Spc}}_T, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}\,. \qquad \square$$

3. The universal property of global spaces

In this section we will give a parametrized interpretation of unstable global homotopy theory in the sense of [Sch18, Chapter 1] with respect to *finite* groups. For this, the key idea will be to more generally consider unstable *G-global homotopy theory* in the sense of [Len20, Chapter 1] for finite groups G, which we recall in Subsection 3.1 below. In 3.2 we will then explain how these models for varying G assemble into a global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ (in the sense of Example 2.1.3), and in Subsection 3.3 we will finally provide a universal description of $\underline{\mathscr{S}}^{\mathrm{gl}}$ as the free cocomplete global ∞-category generated by the terminal object.

3.1. A reminder on global and G-global homotopy theory.

Let G be a finite group; [Len20, Chapter 1] studies various models of *unstable G-global homotopy theory*. We will recall two of these models that will be particularly convenient for us:

Definition 3.1.1. We write $\mathcal{M}$ for the monoid (under composition) of injective self-maps of the countably infinite set $\omega := \{0, 1, \dots\}$.

The functor $\mathbf{SSet} \to \mathbf{Set}, X \mapsto X_0$ sending a simplicial set to its set of vertices admits a right adjoint E, given explicitly by $(EX)_n = X^{1+n}$ with functoriality induced by the identification $X^{1+n} \cong \mathrm{Hom}(\{0, \dots, n\}, X)$; equivalently, this is the nerve of the groupoid with objects X and a unique map between any two objects. As a right adjoint, E in particular preserves products, so $E\mathcal{M}$ inherits a natural monoid structure from $\mathcal{M}$.

We occasionally call the resulting simplicial monoid $E\mathcal{M}$ the 'universal finite group.' While $E\mathcal{M}$ is of course neither finite nor a group, this terminology is motivated by the fact that we can embed any finite group into $E\mathcal{M}$ in a particularly nice way:

Definition 3.1.2. Let H be a finite group. A countable H-set $\mathcal{U}$ is called a *complete H-set universe* if every other countable H-set embeds equivariantly into $\mathcal{U}$.

Definition 3.1.3. A finite subgroup $H \subset \mathcal{M}$ is called *universal* if the tautological H-action on ω makes the latter into a complete H-set universe.

Lemma 3.1.4 (See [Len20, Lemma 1.2.8])**.** *Let H be a finite group. Then there exists an injective homomorphism $i\colon H \to \mathcal{M}$ with universal image. If $j\colon H \to \mathcal{M}$ is another such map, then there exists an invertible $\varphi \in \mathcal{M}$ such that $i(h) = \varphi j(h)\varphi^{-1}$ for all $h \in H$.* $\square$

Remark 3.1.5. Somewhat loosely speaking, the reason to pass from the discrete monoid $\mathcal{M}$ to the simplicial monoid $E\mathcal{M}$ is to eliminate the indeterminacy of the invertible element φ in the above lemma, see [Len20, Subsections 1.2.2–1.2.3] for more details.

Definition 3.1.6. Let G be any group. We write $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** for the 1-category (or simplicially enriched category) of simplicial sets with a strict action of the simplicial monoid $E\mathcal{M}\times G$, together with the strictly $(E\mathcal{M}\times G)$-equivariant maps.

The category $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** will be our first model for G-global homotopy theory. In order to define the weak equivalences of this model structure we recall the following notation:

Notation 3.1.7. Let G_1, G_2 be groups, let $H \subset G_1$, and let $\varphi\colon H \to G_2$ be a homomorphism. The *graph subgroup* $\Gamma_{H,\varphi} \subset G_1 \times G_2$ is the subgroup $\{(h,\varphi(h)) : h \in H\}$. If X is a $(G_1\times G_2)$-simplicial set, then we abbreviate $X^\varphi := X^{\Gamma_{H,\varphi}}$, and similarly for $(G_1 \times G_2)$-equivariant maps.

Proposition 3.1.8. *The category* $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** *carries a (unique) combinatorial model structure in which a map is a weak equivalence or fibration if and only if f^φ is a weak homotopy equivalence or Kan fibration, respectively, for every universal subgroup $H \subset \mathcal{M}$ and homomorphism $\varphi\colon H \to G$. We call this the* G-global model structure *and its weak equivalences the* G-global weak equivalences*.*

Moreover, there is also a unique model structure on $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** *whose weak equivalences are the G-global weak equivalences and whose cofibrations are the* injective cofibrations*, i.e. the levelwise injections. We call this the* injective G-global model structure*.*

Proof. These are special cases of [Len20, Propositions 1.1.2 and 1.1.15], respectively; also see Corollary 1.2.34 of *op. cit.* for the former model structure. □

For $G = 1$ the above recovers a version of Schwede's global homotopy theory where one only considers equivariant information for finite groups ('$\mathcal{F}in$-global homotopy theory'), see Remark 3.1.14 below. On the other hand, for general finite G one can exhibit ordinary G-equivariant homotopy theory explicitly as a Bousfield localization of G-global homotopy theory, see [Len20, Subsection 1.2.6]. In this sense, G-global homotopy theory can be thought of as a 'synthesis' of the usual equivariant and global approaches.

Lemma 3.1.9 (See [Len20, Corollaries 1.2.76–1.2.79])**.** *Let $\alpha\colon G \to G'$ be any group homomorphism. Then the restriction functor $\alpha^*\colon$* $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G'}$-**SSet** $\to$ $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** *is homotopical and it takes part in Quillen adjunctions*

$$\alpha_!\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_{G\text{-gl}} \rightleftarrows \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}_{G'\text{-gl}} : \alpha^*$$
$$\alpha^*\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}_{\text{inj. }G'\text{-gl}} \rightleftarrows \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_{\text{inj. }G\text{-gl}} : \alpha_*.$$

Moreover, if α is injective*, then we also have Quillen adjunctions*

$$\alpha_!\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_{\text{inj. }G\text{-gl}} \rightleftarrows \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}_{\text{inj. }G'\text{-gl}} : \alpha^*$$
$$\alpha^*\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}_{G'\text{-gl}} \rightleftarrows \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_{G\text{-gl}} : \alpha_*.$$ □

Next, we come to another model in terms of suitable 'diagram spaces' that will become useful later to relate the unstable and stable theory to each other:

Definition 3.1.10. We write I for the category of finite sets and injections. Moreover, we write $\mathcal{I}$ for the simplicially enriched category obtained by applying $E\colon \mathbf{Set} \to \mathbf{SSet}$ to all hom-sets.

We write $\boldsymbol{\mathcal{I}}$**-SSet** for the category $\mathbf{Fun}(\mathcal{I}, \mathbf{SSet})$ of simplicially enriched functors $\mathcal{I} \to \mathbf{SSet}$. Moreover, if G is any group, then we write $\boldsymbol{G}$-$\boldsymbol{\mathcal{I}}$**-SSet** for the category of G-objects in $\boldsymbol{\mathcal{I}}$**-SSet**.

Construction 3.1.11. Let X be any $\mathcal{I}$-simplicial set. Then we define

$$X(\omega) := \operatorname*{colim}_{\substack{A \subset \omega \\ \text{finite}}} X(A).$$

This admits an $E\mathcal{M}$-action via the original functoriality of X in $\mathcal{I}$, see [Len20, Construction 1.4.14] for details, giving rise to a functor $\mathrm{ev}_\omega \colon \boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} \to \boldsymbol{E\mathcal{M}}\text{-}\mathbf{SSet}$. If G is any group, then we obtain a functor $\mathrm{ev}_\omega \colon \boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} \to \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}$ by pulling through the G-actions.

Theorem 3.1.12 (See [Len20, Proposition 1.4.3 and Theorem 1.4.30])**.** *There is a unique model structure on* $\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}$ *with*

- *weak equivalences those maps f for which $\mathrm{ev}_\omega f =: f(\omega)$ is a G-global weak equivalence,* and
- *acyclic fibrations those maps f for which $f(A)^\varphi$ is an acyclic Kan fibration for every finite set A, $H \subset \Sigma_A$, and $\varphi \colon H \to G$.*

We call this the G-global model structure *and its weak equivalences the* G-global weak equivalences *again.*

Moreover, the functor ev_ω *is the left half of a Quillen equivalence* $\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} \rightleftarrows \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_{\text{inj. } G\text{-gl}}$. □

Remark 3.1.13. One can also define a G-global model structure on the category $\boldsymbol{G}\text{-}\boldsymbol{I}\text{-}\mathbf{SSet}$ (whose weak equivalences are somewhat intricate). The forgetful functor $\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} \to \boldsymbol{G}\text{-}\boldsymbol{I}\text{-}\mathbf{SSet}$ is then the right half of a Quillen equivalence, see [Len20, Theorem 1.4.31].

Remark 3.1.14. Schwede [Sch18, Theorem 1.2.21] originally studied unstable global homotopy theory in terms of so-called *orthogonal spaces*, which are topologically enriched functors from the topological category L of finite dimensional inner product spaces and linear isometric embeddings into **Top**. While Schwede's *global equivalences* on $\boldsymbol{L}$**-Top** see equivariant information for all compact Lie groups, there is a natural notion of '$\mathcal{F}in$-global weak equivalences' [Len20, Definition 1.5.13], and with respect to these the evident forgetful functor $\boldsymbol{L}\text{-}\mathbf{Top} \to \boldsymbol{I}\text{-}\mathbf{SSet}$ becomes an equivalence of homotopy theories, see [Len20, Corollary 1.5.29]. In this sense, the above two models generalize global homotopy theory with respect to *finite* groups.

Finally, we again have suitable restriction functoriality analogous to Lemma 3.1.9. We will only recall one aspect that we will need later:

Lemma 3.1.15 (See [Len20, Lemma 1.4.40])**.** *Let $\alpha \colon G \to G'$ be any group homomorphism. Then the adjunction*

$$\alpha_! \colon \boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} \rightleftarrows \boldsymbol{G'}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} : \alpha^*$$

is a Quillen adjunction with homotopical right adjoint. □

3.2. **The global ∞-category of global spaces.** We will now bundle the ∞-categories associated to the above model categories into a *global ∞-category*, i.e. an ∞-category parametrized over the ∞-category Glo from Example 2.1.3:

Construction 3.2.1. We define the strict 2-functor $\boldsymbol{E\mathcal{M}}$-$\bullet$-**SSet** as the composition

$$\mathbf{Glo}^{\mathrm{op}} \xhookrightarrow{B} \mathbf{Grpd}^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\boldsymbol{E\mathcal{M}}\text{-}\mathbf{SSet})} \mathbf{Cat}; \tag{3}$$

put differently, this sends a finite group G to the 1-category $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet**, a homomorphism $\alpha\colon G \to G'$ to the restriction map $\alpha^*\colon \boldsymbol{E\mathcal{M}}$-$\boldsymbol{G'}$-**SSet** $\to \boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet**, and a 2-cell $g'\colon \alpha \Rightarrow \beta$ in Glo to the transformation $\alpha^* \Rightarrow \beta^*$ given by acting with g'.

We now want to obtain a *global ∞-category of global spaces* by pointwise localizing at the G-global weak equivalences. To this end we recall:

Definition 3.2.2. A *relative category* is a 1-category $\mathcal{C}$ together with a wide subcategory $W \subseteq \mathcal{C}$, whose morphisms we call *weak equivalences*. We let **RelCat** denote the $(2,1)$-category of relative categories, weak equivalence preserving functors, and natural isomorphisms, and we write RelCat for its Duskin nerve.

By Lemma 3.1.9, the restriction functor $\alpha^*\colon \boldsymbol{E\mathcal{M}}$-$\boldsymbol{G'}$-**SSet** $\to \boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** sends G'-global weak equivalences to G-global weak equivalences for any homomorphism $\alpha\colon G \to G'$. In particular, (3) lifts to a 2-functor into **RelCat** this way.

Construction 3.2.3. To every relative category $(\mathcal{C}, W)$, one can associate an ∞-category $\mathcal{C}[W^{-1}]$ together with a functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ that exhibits it as a Dwyer-Kan localization of $\mathcal{C}$ at W in the sense of [Lur17, Definition 1.3.4.1]. We will now recall the argument of [GM20, Section C.1] that the ∞-category $\mathcal{C}[W^{-1}]$ is in fact functorial in the pair $(\mathcal{C}, W)$.

Let $\iota\colon \mathrm{Cat}_\infty \to \mathrm{Spc}$ denote the left adjoint to the inclusion $\mathrm{Spc} \subseteq \mathrm{Cat}_\infty$ of ∞-groupoids into ∞-categories. Sending an ∞-category $\mathcal{C}$ to the adjunction counit $\iota\mathcal{C} \hookrightarrow \mathcal{C}$ refines to a functor

$$R\colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty).$$

We let $L_\infty\colon \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$ denote a left adjoint to this functor. By associating to a relative category $(\mathcal{C}, W)$ the inclusion $W \hookrightarrow \mathcal{C}$ and regarding both W and $\mathcal{C}$ as ∞-categories via their nerve, we obtain a functor $\mathrm{RelCat} \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$. In particular we obtain a localization functor

$$L\colon \mathrm{RelCat} \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty) \xrightarrow{L_\infty} \mathrm{Cat}_\infty\,.$$

It follows directly from the definition of L_∞ that L is on objects given by sending a relative category $(\mathcal{C}, W)$ to the Dwyer-Kan localization $L(\mathcal{C}, W) \simeq \mathcal{C}[W^{-1}]$.

Postcomposing with this, we get a global ∞-category $L\mathscr{C}$ from any global relative category $\mathscr{C}$, and this comes with a global functor $\mathscr{C} \to L\mathscr{C}$ that is pointwise a Dwyer-Kan localization. By uniqueness of adjoints, this actually pins down $L\mathscr{C}$ up to essentially unique equivalence; in particular, we can (and will at times) freely choose a specific construction of the above localization for a given $\mathscr{C}$.

Definition 3.2.4. We define the global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ of *global spaces* as the composite

$$\mathrm{Glo}^{\mathrm{op}} = \mathrm{N}_\Delta(\mathbf{Glo})^{\mathrm{op}} \xrightarrow{\mathrm{N}_\Delta(\boldsymbol{E\mathcal{M}}\text{-}\bullet\text{-}\mathbf{SSet})} \mathrm{N}_\Delta(\mathbf{RelCat}) = \mathrm{RelCat} \xrightarrow{L} \mathrm{Cat}_\infty\,.$$

In particular, for a finite group G the ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}(G) =: \mathscr{S}^{\mathrm{gl}}_G$ is the ∞-category of G-global spaces and for a group homomorphism $\alpha\colon G \to G'$, the functor $\underline{\mathscr{S}}^{\mathrm{gl}}(\alpha)$ is induced by the restriction functor $\alpha^*\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet} \to \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}$.

Analogously, we get a global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$ sending G to the Dwyer-Kan localization of $\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}$, with functoriality via restrictions.

By design, the maps ev_ω are homotopical and strictly compatible with restrictions, and so they assemble into a strictly 2-natural transformation between functors $\mathbf{Glo}^{\mathrm{op}} \to \mathbf{RelCat}$. Upon localization, we therefore get a global functor $\underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ that we again call ev_ω. Theorem 3.1.12 then implies:

Corollary 3.2.5. *The global functor* $\mathrm{ev}_\omega\colon \underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ *is an equivalence of global ∞-categories.* □

3.3. Proof of Theorem A.

As a basis for the universal properties of special global Γ-spaces and global spectra, we will now relate the global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ (defined above in terms of a purely model categorical construction) to the global ∞-category $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ (constructed using parametrized higher category theory alone). Namely we will prove:

Theorem 3.3.1. *The global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ is presentable. Moreover, the essentially unique globally cocontinuous functor $\underline{\mathrm{Spc}}_{\mathrm{Glo}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ that sends the terminal object of $\mathrm{Spc}_{\mathrm{Glo}}(1)$ to the terminal object of $\mathscr{S}^{\mathrm{gl}} = \mathscr{S}^{\mathrm{gl}}_1$ is an equivalence.*

Together with Corollary 2.4.11 this will then immediately imply Theorem A from the introduction:

Theorem 3.3.2. *The presentable global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ is freely generated under global colimits by $* \in \underline{\mathscr{S}}^{\mathrm{gl}}$, i.e. for any globally cocomplete global ∞-category $\mathcal{D}$ evaluating at $*$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}}^{\mathrm{gl}}, \mathcal{D}) \to \mathcal{D}$$

of global ∞-categories. □

Corollary 3.2.5 then shows:

Corollary 3.3.3. *The global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$ is presentable, and it is freely generated under global colimits by $* \in \underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$, i.e. for any globally cocomplete global ∞-category $\mathcal{D}$ evaluating at $*$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}, \mathcal{D}) \to \mathcal{D}$$

of global ∞-categories. □

The way Theorem 3.3.1 is phrased naturally suggests a proof strategy: show that the (fiberwise presentable) global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$ is globally cocomplete, use the universal property to construct the map, and then check that it is an equivalence. In fact, one can use the functoriality properties of Lemma 3.1.9 together with [Len20, Proposition 1.1.22] to verify global cocompleteness, and it is not hard to show using some adjunction yoga that the resulting functor sends corepresented objects to the standard 'generators' of G-global homotopy theory (see Proposition 3.3.5 below) while a concrete computation reveals that the mapping spaces on both sides are *abstractly* equivalent. However, proving that actually the universal functor induces equivalences between these mapping spaces is a totally different story, and in fact the authors do not know a direct argument for this.

Instead, our proof of the theorem will proceed backwards: we will construct an equivalence between $\underline{\mathscr{S}}^{\mathrm{gl}}$ and $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ by hand, and deduce the remaining statements from this. Since this comparison is somewhat lengthy, let us outline the general strategy first: by definition, $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ is levelwise given by ∞-categories of presheaves, and the first step will be to likewise express the levels of $\underline{\mathscr{S}}^{\mathrm{gl}}$ in terms of *model categories* of presheaves. To complete the proof, we will then give a comparison between the indexing categories on both sides, as well as a comparison between presheaves in the model categorical and ∞-categorical setting.

3.3.1. *The G-global Elmendorf Theorem.* Recall that the classical *Elmendorf Theorem* [Elm83] expresses the homotopy theory of G-CW-complexes in terms of *fixed point systems*, yielding a presheaf model of unstable G-equivariant homotopy theory. We will now recall a G-global version of this, which is most easily formulated using the model of $E\mathcal{M}$-G-simplicial sets:

Construction 3.3.4. Let G be finite. We write $\mathbf{O}^{\mathrm{gl}}_G$ for the full simplicial subcategory of $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** spanned by the objects $E\mathcal{M}\times_\varphi G := (E\mathcal{M}\times G)/H$ for all universal subgroups $H\subset\mathcal{M}$ and homomorphisms $\varphi\colon H\to G$, where H acts on $E\mathcal{M}$ from the right in the evident way and on G from the right via φ.

We now define a functor

$$\Phi\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}\to\mathbf{Fun}((\mathbf{O}^{\mathrm{gl}}_G)^{\mathrm{op}},\mathbf{SSet}),$$

where **Fun** denotes the 1-category of simplicially enriched functors, via the formula $\Phi(X)(E\mathcal{M}\times_\varphi G)=\mathrm{maps}(E\mathcal{M}\times_\varphi G,X)$ with the evident (enriched) functoriality in each variable, i.e. Φ is the composition

$$\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}\xrightarrow{\text{Yoneda}}\mathbf{Fun}(\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}^{\mathrm{op}},\mathbf{SSet})\xrightarrow{\text{restriction}}\mathbf{Fun}((\mathbf{O}^{\mathrm{gl}}_G)^{\mathrm{op}},\mathbf{SSet}).$$

Proposition 3.3.5. *For any finite group G the above functor Φ is homotopical and the right half of a Quillen equivalence for the projective model structure on the target. In particular, it descends to an equivalence between the ∞-categorical localization at the G-global weak equivalences and the ∞-categorical localization at the levelwise weak homotopy equivalences.*

Proof. This is a special case of [Len20, Corollary 1.1.13]. □

Remark 3.3.6. We can describe the simplicial category $\mathbf{O}^{\mathrm{gl}}_G$ combinatorially as follows, see also [Len20, Remark 1.2.40]: n-simplices of $\mathrm{maps}(E\mathcal{M}\times_\varphi G,E\mathcal{M}\times_\psi G)$ correspond bijectively to n-simplices $[u_0,\dots,u_n;g]\in(E\mathcal{M}\times_\psi G)^\varphi$ via evaluation at $[1;1]\in E\mathcal{M}\times_\varphi G$; note that the right hand side is the nerve of a groupoid (as H acts freely on $E\mathcal{M}$), so $\mathbf{O}^{\mathrm{gl}}_G$ can be equivalently viewed a $(2,1)$-category. Under this correspondence, composition is given by $[u_0,\dots,u_n;g][u'_0,\dots,u'_n;g']=[u'_0u_0,\dots,u'_nu_n;g'g]$ (note the flipped order of multiplication).

More generally, if X is any $E\mathcal{M}$-G-simplicial set, then evaluation at $[1;1]$ induces a natural isomorphism $\varepsilon\colon\Phi(X)(E\mathcal{M}\times_\varphi G)=\mathrm{maps}(E\mathcal{M}\times_\varphi G,X)\to X^\varphi$. A direct computation shows that under this isomorphism restriction along an $(n+1)$-cell

$[u_0, \ldots, u_n; g] : E\mathcal{M} \times_\varphi G \to E\mathcal{M} \times_\psi G$ in $\mathbf{O}_G^{\mathrm{gl}}$ corresponds to action by the same element, i.e. the following diagram commutes:

$$\begin{array}{ccc} \Phi(X)(E\mathcal{M} \times_\psi G) & \xrightarrow{\Phi[u_0,\ldots,u_n;g]} & \Phi(X)(E\mathcal{M} \times_\varphi G) \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ X^\psi & \xrightarrow[(u_0,\ldots,u_n;g).-]{} & X^\varphi. \end{array} \tag{4}$$

3.3.2. *Comparisons of enriched presheaves.* While one can extend the assignment $G \mapsto \mathbf{O}_G^{\mathrm{gl}}$ to a strict 2-functor in Glo, and so assemble the localizations of the categories $\mathbf{Fun}((\mathbf{O}_G^{\mathrm{gl}})^{\mathrm{op}}, \mathbf{SSet})$ into a global ∞-category, the maps Φ will not be strictly natural with respect to this structure, but only pseudonatural. In order to avoid talking about all the coherences required to make this precise, we will now give a more 'combinatorial' version of the simplicial categories $\mathbf{O}_G^{\mathrm{gl}}$ and the functors Φ that will also become relevant in Section 5.

Construction 3.3.7. Let G be a finite group. We define a strict $(2,1)$-category $\mathfrak{O}_G^{\mathrm{gl}}$ as follows: an object of $\mathfrak{O}_G^{\mathrm{gl}}$ is a pair (H, φ) of a universal subgroup $H \subset \mathcal{M}$ and a homomorphism $\varphi\colon H \to G$. For any two such objects $(H, \varphi), (K, \psi)$ the hom-category $\mathbf{Hom}((H, \varphi), (K, \psi))$ has objects the triples (u, g, σ) with $u \in \mathcal{M}$, $g \in G$ and $\sigma\colon H \to K$ a homomorphism such that $hu = u\sigma(h)$ for all $h \in H$ and moreover $\varphi = c_g\psi\sigma$, where c_g denotes conjugation by g. If (u', g', σ') is another object of the hom-category, then a morphism $(u, g, \sigma) \to (u', g', \sigma')$ is a $k \in K$ such that $\sigma' = c_k\sigma$ and $g'\psi(k) = g$. Composition in $\mathbf{Hom}((H, \varphi), (K, \psi))$ is induced by multiplication in K; we omit the easy verification that this is a well-defined groupoid.

If (L, ζ) is another object and $(u_1, g_1, \sigma_1)\colon (H, \varphi) \to (K, \psi)$, $(u_2, g_2, \sigma_2)\colon (K, \psi) \to (L, \zeta)$ are composable maps, then we define their composition as $(u_1u_2, g_1g_2, \sigma_2\sigma_1)$ (note the flipped order of composition in the first two components!); this is indeed a map $(H, \varphi) \to (L, \zeta)$ as $hu_1u_2 = u_1\sigma_1(h)u_2 = u_1u_2\sigma_2\sigma_1(h)$ for all $h \in H$ and moreover $\varphi = c_{g_1}\psi\sigma_1 = c_{g_1g_2}\zeta\sigma_2\sigma_1$.

Finally, if $(u_1', g_1', \sigma_1')\colon (H, \varphi) \to (K, \psi)$ and $(u_2', g_2', \sigma_2')\colon (K, \psi) \to (L, \zeta)$ are further morphisms and $k_1\colon (u_1, g_1, \sigma_1) \to (u_1', g_1', \sigma_1')$, $k_2\colon (u_2, g_2, \sigma_2) \to (u_2', g_2', \sigma_2')$ are 2-cells, then the composite of k_1 and k_2 is $k_2\sigma_2(k_1)$; note that this is indeed well-defined as $\sigma_2'\sigma_1' = c_{k_2}\sigma_2c_{k_1}\sigma_1 = c_{k_1\sigma_2(k_2)}\sigma_2\sigma_1$ while $g_1g_2 = g_1'\psi(k_1)g_2'\zeta(k_2) = g_1'g_2'\zeta\sigma_2'(k_1)\zeta(k_2) = g_1'g_2'\zeta(\sigma_2'(k_1)k_2) = g_1'g_2'\zeta(k_2\sigma_2(k_1))$ where the second equality uses that (u_2', g_2', σ_2') is a morphism and the final equality uses that k_2 is a 2-cell.

We omit the straight-forward verification that this is suitably associative and unital with units the maps of the form $(1, 1, \mathrm{id})$, making $\mathfrak{O}_G^{\mathrm{gl}}$ into a strict $(2,1)$-category.

Construction 3.3.8. We define $\mu\colon \mathfrak{O}_G^{\mathrm{gl}} \to \mathbf{O}_G^{\mathrm{gl}}$ as follows: an object (H, φ) is sent to $E\mathcal{M} \times_\varphi G$, a morphism $(u, g, \sigma)\colon (H, \varphi) \to (K, \psi)$ is sent to the map $E\mathcal{M} \times_\varphi G \to E\mathcal{M} \times_\psi G$ represented by $[u; g]$ while a 2-cell $k\colon (u, g, \sigma) \to (u', g', \sigma')$ is sent to $[u'k, u; g]$.

Lemma 3.3.9. *The above μ is well-defined (i.e. these are indeed morphisms and 2-cells in $\mathbf{O}_G^{\mathrm{gl}}$) and an equivalence of $(2,1)$-categories.*

Proof. First observe that $[u; g]$ is indeed φ-fixed as $[hu; \varphi(h)g] = [u\sigma(h); g\psi\sigma(h)] = [u; g]$ by definition of the morphisms of $\mathfrak{O}_G^{\mathrm{gl}}$; moreover, any 1-cell in the target is of this form by [Len20, Lemma 1.2.38]. On the other hand, Lemma 1.2.74 of *op. cit.* shows that $[u'k, u; g]$ is indeed a 2-cell $[u; g] \Rightarrow [u'; g']$ and that this assignment is bijective. Thus, it only remains to show that μ is a strict 2-functor.

To prove that $\mu\colon \mathbf{Hom}((H,\varphi),(K,\psi)) \to (E\mathcal{M} \times_\psi G)^\varphi$ is a functor, it suffices to prove compatibility with composition (as both sides are groupoids), for which we note that for all $k\colon (u, g, \sigma) \to (u', g', \sigma')$ and $k'\colon (u', g', \sigma') \to (u'', g'', \sigma'')$

$$\begin{aligned}\mu(k')\mu(k) &= [u''k', u'; g'][u'k, u; g] = [u''k'k, u'k; \underbrace{g'\psi(k)}_{=g}][u'k, u; g] = [u''k'k, u; g]\\ &= \mu(k'k).\end{aligned}$$

Next, we have to show that μ is compatible with horizontal composition of 2-cells, hence in particular with composition of 1-cells. For this we note that if $k\colon (u_1, g_1, \sigma_1) \Rightarrow (u_1', g_1', \sigma_1')$ is a 2-cell between morphisms $(H, \varphi) \to (K, \psi)$ and $\ell\colon (u_2, g_2, \sigma_2) \Rightarrow (u_2', g_2', \sigma_2')$ is a 2-cell between morphisms $(K, \psi) \to (L, \zeta)$, then the horizontal composition $\mu(\ell) \odot \mu(k)$ is given by

$$\begin{aligned}[u_2'\ell, u_2, g_2] \odot [u_1'k, u_1; g_1] &= [u_1'ku_2'\ell, u_1u_2; g_1g_2] = [u_1'u_2'\sigma_2'(k)\ell, u_1u_2; g_1g_2]\\ &= [u_1'u_2'\ell\sigma_2(k), u_1u_2; g_1g_2]\end{aligned}$$

where the final equality uses that $\sigma_2'(k)\ell = \ell\sigma_2(k)$ as ℓ is a 2-cell. On the other hand, by definition $\ell \odot k = \ell\sigma_2(k)\colon (u_1u_2, g_1g_2, \sigma_2\sigma_1) \to (u_1'u_2', g_1'g_2', \sigma_2'\sigma_1')$, so $\mu(\ell \odot k) = \mu(\ell) \odot \mu(k)$ as desired.

Finally, $\mu(1, 1, \mathrm{id}) = [1; 1]$ by construction, i.e. μ also preserves identity 1-cells. □

Construction 3.3.10. Let G be a finite group. We define $\Psi\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet} \to \mathbf{PSh}(\mathfrak{O}_G^{\mathrm{gl}}) := \mathbf{Fun}((\mathfrak{O}_G^{\mathrm{gl}})^{\mathrm{op}}, \mathbf{SSet})$ as follows: for any $E\mathcal{M}$-G-simplicial set X, the enriched functor $\Psi(X)\colon (\mathfrak{O}_G^{\mathrm{gl}})^{\mathrm{op}} \to \mathbf{SSet}$ is given on objects by $\Psi(X)(H, \varphi) = X^\varphi \subset X$; if (K, ψ) is another object, then we send an n-simplex

$$(u_0, g_0, \sigma_0) \xRightarrow{k_1} (u_1, g_1, \sigma_1) \xRightarrow{k_2} \cdots \xRightarrow{k_n} (u_n, g_n, \sigma_n) \in \mathrm{maps}((H,\varphi),(K,\psi))_n \quad (5)$$

to the action of $(u_nk_n \cdots k_1, u_{n-1}k_{n-1} \cdots k_1, \dots, u_1k_1, u_0; g_0)$ on X, cf. Remark 3.3.6. If $f\colon X \to Y$ is any map of $E\mathcal{M}$-G-simplicial sets, then we define $\Psi(f)$ via $\Psi(f)(H, \varphi) = f^\varphi$.

Proposition 3.3.11. *The assignment $\Psi\colon \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet} \to \mathbf{PSh}(\mathfrak{O}_G^{\mathrm{gl}})$ is well-defined (i.e. $\Psi(X)$ is a simplicially enriched functor and $\Psi(f)$ is an enriched natural transformation) and constitutes a functor. Furthermore, it descends to an equivalence on ∞-categorical localizations.*

Proof. We will simultaneously prove that Ψ is well-defined and that it is isomorphic to the composite

$$\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet} \xrightarrow{\Phi} \mathbf{PSh}(\mathbf{O}_G^{\mathrm{gl}}) \xrightarrow{\mu^*} \mathbf{PSh}(\mathfrak{O}_G^{\mathrm{gl}});$$

the claim then follows from Proposition 3.3.5 together with Lemma 3.3.9.

To prove this, we first fix an $E\mathcal{M}$-G-simplicial set X, and we will show that $\Psi(X)$ is a well-defined simplicial functor isomorphic to $\Phi(X) \circ \mu$. To this end, we recall that we have for every $(H, \varphi) \in \mathfrak{O}_G^{\mathrm{gl}}$ an isomorphism

$$\Phi(X)(\mu(H,\varphi)) = \mathrm{maps}(E\mathcal{M} \times_\varphi G, X) \xrightarrow{\varepsilon} X^\varphi = \Psi(X)(H,\varphi)$$

given by evaluation at $[1;1]$. It follows formally that there is a unique way to extend the assignment $(H,\varphi)\mapsto X^\varphi$ to a simplicially enriched functor $(\mathfrak{O}^{\mathrm{gl}}_G)^{\mathrm{op}}\to \mathbf{SSet}$ in such a way that the ε's assemble into an enriched natural isomorphism from $\Phi(X)\circ\mu$, namely in terms of the composites

$$\begin{aligned}\mathrm{maps}_{\mathfrak{O}^{\mathrm{gl}}}\big((H,\varphi),(K,\psi)\big) &\xrightarrow{\mu} \mathrm{maps}_{\mathbf{O}^{\mathrm{gl}}}(E\mathcal{M}\times_\varphi G, E\mathcal{M}\times_\psi G)\\ &\xrightarrow{\Phi} \mathrm{maps}_{\mathbf{SSet}}\big(\mathrm{maps}_{\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}}(E\mathcal{M}\times_\psi G, X),\\ &\qquad\qquad \mathrm{maps}_{\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}}(E\mathcal{M}\times_\varphi G, X)\big)\\ &\xrightarrow{\varepsilon_*(\varepsilon^{-1})^*} \mathrm{maps}_{\mathbf{SSet}}(X^\psi, X^\varphi)\end{aligned}$$

and we only have to show that this recovers the above definition of Ψ. By commutativity of (4) this then amounts to saying that

$$\mathrm{maps}\big((H,\varphi),(K,\psi)\big)\xrightarrow{\mu}\mathrm{maps}(E\mathcal{M}\times_\varphi G, E\mathcal{M}\times_\psi G)\xrightarrow{\varepsilon}(E\mathcal{M}\times_\psi G)^\varphi\subset E\mathcal{M}\times_\psi G$$

sends (5) to $(u_nk_n\cdots k_1,\dots,u_1k_1,u_0;g_0)$. As $E\mathcal{M}\times_\psi G$ is the nerve of a groupoid, it will be enough to show this after restricting to each edge $0\to m$ ($0\le m\le n$), i.e. that $\mu(k_m\cdots k_0)=(u_mk_m\cdots k_1,u_0;g_0)$. However, this is precisely the definition.

Thus, we have altogether shown that $\Psi(X)$ is indeed a well-defined simplicial functor and that the maps ε assemble into an isomorphism $\Psi(X)\cong\Phi(X)\circ\mu$. We can now show that Ψ is a well-defined functor: indeed, if $f\colon X\to Y$ is $(E\mathcal{M}\times G)$-equivariant, then $\Psi(f)$ is enriched natural as the enriched functor structure on both sides is given by acting with simplices of $E\mathcal{M}\times G$. It is then clear that Ψ preserves composition and identities as this can be checked after evaluating at each (H,φ).

Finally, we have to establish that the isomorphisms ε are natural in X. However, we can again check this after evaluating at each (H,φ), where this is obvious. $\square$

Construction 3.3.12. We extend the assignment $G\mapsto\mathfrak{O}^{\mathrm{gl}}_G$ to a strict $(2,1)$-functor $\mathfrak{O}^{\mathrm{gl}}_\bullet\colon\mathbf{Glo}\to\mathbf{Cat}_\Delta$ into the 2-category of simplicial categories as follows: if $\alpha\colon G\to G'$ is a homomorphism, then $\alpha_!\colon\mathfrak{O}^{\mathrm{gl}}_G\to\mathfrak{O}^{\mathrm{gl}}_{G'}$ is given on objects by $\alpha_!(H,\varphi)=(H,\alpha\varphi)$, on 1-cells by $\alpha_!(u,g,\sigma)=(u,\alpha(g),\sigma)$, and on 2-cells by the identity; we omit the easy verification that this is well-defined and strictly functorial in α. Moreover, if $g\in G'$ defines a natural transformation $\alpha_1\Rightarrow\alpha_2$ (i.e. $\alpha_2=c_g\alpha_1$), then we define the natural transformation $g_!\colon\alpha_{1!}\Rightarrow\alpha_{2!}$ on (H,φ) as $(1,g^{-1},\mathrm{id}_H)\colon(H,\alpha_1\varphi)\to(H,\alpha_2\varphi)$. We again omit the easy verification that this is well-defined and yields a strict 2-functor.

This 2-functor structure then induces a 2-functor structure on the assignment $G\mapsto(\mathfrak{O}^{\mathrm{gl}}_G)^{\mathrm{op}}$; note that in this the 2-cells get inverted, i.e. $g\colon\alpha_1\Rightarrow\alpha_2$ is now sent to the natural transformation $g_!^{\mathrm{op}}$ given pointwise by $(1,g,\mathrm{id})$.

Proposition 3.3.13. *The maps* Ψ *are strictly* 2*-natural in* $\mathbf{Glo}$.

Proof. Let us first check 1-naturality, i.e. that for every $\alpha\colon G\to G'$ the diagram

$$\begin{array}{ccc}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet} & \xrightarrow{\alpha^*} & \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}\\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi}\\ \mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_{G'}) & \xrightarrow[(\alpha_!)^*]{} & \mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_G)\end{array}$$

of ordinary categories commutes.

The above diagram commutes on the level of objects: Let X be an $E\mathcal{M}$-G-simplicial set; we have to show that $\Psi(\alpha^*X) = \Psi(X) \circ \alpha_!$. On the level of objects, this just amounts to the relation $(\alpha^*X)^\varphi = X^{\alpha\circ\varphi}$ for all $(H, \varphi\colon H \to G) \in \mathfrak{O}^{\mathrm{gl}}_G$. To prove commutativity on the level of morphism spaces, we let (K, ψ) be any other object and we consider an n-simplex

$$(u_\bullet, g_\bullet, \sigma_\bullet) := \big((u_0, g_0, \sigma_0) \xRightarrow{k_1} (u_1, g_1, \sigma_1) \xRightarrow{k_2} \cdots \xRightarrow{k_n} (u_n, g_n, \sigma_n)\big)$$

of maps$((H, \varphi), (K, \psi))$. Then $\Psi(\alpha^*X)(u_\bullet, g_\bullet, \sigma_\bullet)$ is by definition given by acting with $(u_nk_n \cdots k_1, \dots, u_1k_1, u_0; g_0) \in E\mathcal{M}_n \times G$ on α^*X, or equivalently by acting with $(u_nk_n \cdots k_1, \dots, u_1k_1, u_0; \alpha(g_0)) \in E\mathcal{M}_n \times G'$ on X. As $\alpha_!\colon \mathfrak{O}^{\mathrm{gl}}_G \to \mathfrak{O}^{\mathrm{gl}}_{G'}$ sends $(u_\bullet, g_\bullet, \sigma_\bullet)$ to

$$(u_0, \alpha(g_0), \sigma_0) \xRightarrow{k_1} (u_1, \alpha(g_1), \sigma_1) \xRightarrow{k_2} \cdots \xRightarrow{k_n} (u_n, \alpha(g_n), \sigma_n)$$

by definition, we see that $\Psi(X)(\alpha_!(u_\bullet, g_\bullet, \sigma_\bullet))$ is given by acting with the same element. Since in addition both $\Psi(X)(\alpha_!(u_\bullet, g_\bullet, \sigma_\bullet))$ and $\Psi(\alpha^*X)(u_\bullet, g_\bullet, \sigma_\bullet)$ are (higher) maps between the same two objects, this completes the proof that they agree, so that $\Psi(\alpha^*X) = \Psi(X) \circ \alpha_!$ as desired.

The above diagram commutes on the level of morphisms: As we already know that the diagram commutes on the level of objects, it is enough to check the claim after evaluating at each (H, φ). However, in this case both paths through the diagram send a morphism $f\colon X \to Y$ to the restriction $X^{\alpha\varphi} \to Y^{\alpha\varphi}$ of f.

Finally, we can now very easily prove 2-*naturality* by the same argument: namely, it only remains to show that for every 2-cell $g\colon \alpha_1 \Rightarrow \alpha_2$ in Glo, every $E\mathcal{M}$-G-simplicial set X, and every $(H, \varphi) \in \mathfrak{O}^{\mathrm{gl}}_G$ the maps $\Psi(X)(g_!^{\mathrm{op}}\colon (H, \alpha_2\varphi) \to (H, \alpha_1\varphi))$ and $\Psi(g.{-}\colon \alpha_1^*X \to \alpha_2^*X)(H, \varphi)$ agree. However, plugging in the definitions, both are simply given by acting with g on X. ☐

Construction 3.3.14. Let G be a finite group. We define a strict 2-functor $\gamma\colon \mathfrak{O}^{\mathrm{gl}}_G \to \mathbf{Glo}_{/G}$ into the 2-categorical slice as follows: an object (H, φ) is sent to $\varphi\colon H \to G$ and a morphism $(u, g, \sigma)\colon (H, \varphi) \to (K, \psi)$ is sent to the morphism

$$\begin{array}{ccc} H & \xrightarrow{\quad\sigma\quad} & K; \\ & {\scriptstyle g^{-1}}\,\Nearrow & \\ {\scriptstyle\varphi}\searrow & & \swarrow{\scriptstyle\psi} \\ & G & \end{array} \tag{6}$$

note that g^{-1} indeed defines such a 2-cell in **Glo** since $\varphi = c_g\sigma\psi$ by assumption, whence $\sigma\psi = c_{g^{-1}}\varphi$. Finally, a 2-cell $k\colon (u, g, \sigma) \Rightarrow (u, g, \sigma)$ is sent to the 2-cell $k\colon \sigma \Rightarrow \sigma'$.

Lemma 3.3.15. *For any finite G, γ defines an equivalence $\mathfrak{O}^{\mathrm{gl}}_G \simeq \mathbf{Glo}_{/G}$ of strict $(2,1)$-categories.*

Proof. One easily checks by plugging in the definitions that γ is indeed a strict 2-functor. Essential surjectivity of γ follows from the fact that any finite group is isomorphic to a universal subgroup (Lemma 3.1.4). Moreover, given a general 1-cell as depicted in (6), there exists by [Len20, Corollary 1.2.39] a $u \in \mathcal{M}$ with $hu = u\sigma(h)$ for all $h \in H$; (u, g, σ) then clearly defines a 1-cell $(H, \varphi) \to (K, \psi)$ in

$\mathfrak{O}^{\mathrm{gl}}_G$, and this is a preimage of (6). This shows that γ is essentially surjective on hom groupoids. Finally, γ is bijective on 2-cells by direct inspection. ☐

Lemma 3.3.16. *The maps γ define a strictly 2-natural transformation $\mathfrak{O}^{\mathrm{gl}}_\bullet \Rightarrow$ $\mathbf{Glo}_{/\bullet}$.*

Proof. Let us first show that γ is 1-natural, i.e. for every homomorphism $\alpha\colon G \to G'$ the diagram

$$\begin{array}{ccc} \mathfrak{O}^{\mathrm{gl}}_G & \xrightarrow{\gamma} & \mathbf{Glo}_{/G} \\ {\scriptstyle \alpha_!}\downarrow & & \downarrow{\scriptstyle \mathbf{Glo}_{/\alpha}} \\ \mathfrak{O}^{\mathrm{gl}}_{G'} & \xrightarrow[\gamma]{} & \mathbf{Glo}_{/G'} \end{array}$$

of strict 2-functors commutes strictly. This just amounts to plugging in the definitions: both paths through the diagram send an object (H,φ) to $\alpha\varphi\colon H \to G'$, a 1-cell as in (6) to

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & K, \\ & \overset{\Rightarrow}{\alpha(g)^{-1}} & \\ {\scriptstyle \alpha\varphi}\searrow & & \swarrow{\scriptstyle \alpha\psi} \\ & G' & \end{array}$$

and a 2-cell $\sigma \Rightarrow \sigma'$ represented by k to a 2-cell represented by the same k.

For 2-functoriality it then only remains to show that for any 2-cell $g\colon \alpha \Rightarrow \alpha'$ of maps $G \to G'$ in **Glo** the two pastings

$$\mathfrak{O}^{\mathrm{gl}}_G \underset{\alpha'_!}{\overset{\alpha_!}{\rightrightarrows}}{\scriptstyle \Downarrow g_!}\ \mathfrak{O}^{\mathrm{gl}}_{G'} \xrightarrow{\gamma} \mathbf{Glo}_{/G'} \qquad \text{and} \qquad \mathfrak{O}^{\mathrm{gl}}_G \xrightarrow{\gamma} \mathbf{Glo}_{/G} \underset{\alpha'_!}{\overset{\alpha_!}{\rightrightarrows}}{\scriptstyle \Downarrow g_!}\ \mathbf{Glo}_{/G'}$$

agree pointwise. However, by direct inspection both are given on an object (H,φ) of $\mathfrak{O}^{\mathrm{gl}}_G$ simply as the 1-cell

$$\begin{array}{ccc} H & \xrightarrow{=} & H \\ & \overset{\Rightarrow}{g} & \\ {\scriptstyle \alpha\varphi}\searrow & & \swarrow{\scriptstyle \alpha'\varphi} \\ & G & \end{array}$$

which completes the proof of the lemma. ☐

3.3.3. *Putting the pieces together.* Now we are finally ready to deduce our comparison result:

Proof of Theorem 3.3.2. As mentioned in the beginning of this subsection, we will first construct an equivalence $\underline{\mathscr{S}}^{\mathrm{gl}} \simeq \underline{\mathrm{Spc}}_{\mathrm{Glo}}$ by hand:

Proposition 3.3.13 says that the maps Ψ define a 2-natural transformation between the global categories $\boldsymbol{E\mathcal{M}}$**-•-SSet** and $\mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_\bullet)\colon G \mapsto \mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_G)$. If we equip $\boldsymbol{E\mathcal{M}}$**-*G*-SSet** with the G-global weak equivalences for varying G and each $\mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_G)$ with the levelwise weak homotopy equivalences, this lifts to a map of global relative categories, which in turn decends to an equivalence between the global ∞-categories obtained by pointwise localization according to Proposition 3.3.11. Note that the localization of $\boldsymbol{E\mathcal{M}}$**-•-SSet** is the global ∞-category $\underline{\mathscr{S}}^{\mathrm{gl}}$

by definition; it will now be useful to pick a very specific localization of $\mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_\bullet)$ for the purposes of this proof:

Namely, we pick a *simplicially enriched* fibrant replacement functor for the Kan-Quillen model structure (for example via the enriched small object argument [Rie14, Theorem 13.5.2] or simply by using the geometric realization-singular set adjunction), which provides us with an enriched functor $r\colon \mathbf{SSet} \to \mathbf{Kan}$ together with enriched natural transformations $\mathrm{id} \Rightarrow ir$ and $\mathrm{id} \Rightarrow ri$ that are levelwise weak homotopy equivalences, where $i\colon \mathbf{Kan} \hookrightarrow \mathbf{SSet}$ is the inclusion. As an upshot, if A is any simplicially enriched category, then $r \circ -\colon \mathbf{PSh}(A) \to \mathbf{PSh}^{\mathbf{Kan}}(A) := \mathbf{Fun}(A^{\mathrm{op}}, \mathbf{Kan})$ is a homotopy equivalence with respect to the levelwise weak homotopy equivalences, so it induces an equivalence of ∞-categorical localizations. Specializing this to our situation, the maps r assemble into a map of global relative categories from $\mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_\bullet)$ to $\mathbf{PSh}^{\mathbf{Kan}}(\mathfrak{O}^{\mathrm{gl}}_\bullet)$. Finally, for any simplicial category A the enriched-natural comparison map

$$\mathrm{N}\big(\mathbf{PSh}^{\mathbf{Kan}}(A)\big) = \mathrm{N}\mathbf{Fun}(A^{\mathrm{op}}, \mathbf{Kan}) \to \mathrm{Fun}(\mathrm{N}_\Delta(A^{\mathrm{op}}), \mathrm{N}_\Delta(\mathbf{Kan})) = \mathrm{PSh}(\mathrm{N}_\Delta A)$$

is a localization at the levelwise weak homotopy equivalences as a consequence of [Lur09, Proposition 4.2.4.4], see also [Len20, Proposition A.1.18], where this argument is spelled out in detail. Thus, we altogether get a map of global ∞-categories

$$\mathrm{N}\big(\mathbf{PSh}(\mathfrak{O}^{\mathrm{gl}}_\bullet)\big) \xrightarrow{r\circ -} \mathrm{N}\big(\mathbf{PSh}^{\mathbf{Kan}}(\mathfrak{O}^{\mathrm{gl}}_\bullet)\big) \xrightarrow{\text{canonical}} \mathrm{PSh}(\mathrm{N}_\Delta(\mathfrak{O}^{\mathrm{gl}}_\bullet))$$

that is pointwise a localization, whence induces an equivalence $\underline{\mathscr{S}}^{\mathrm{gl}} \simeq \mathrm{PSh}(\mathrm{N}_\Delta\mathfrak{O}^{\mathrm{gl}}_\bullet)$ of global ∞-categories.

Restricting along the strictly 2-natural equivalence $\gamma\colon \mathfrak{O}^{\mathrm{gl}}_\bullet \Rightarrow \mathbf{Glo}_{/\bullet}$ of 2-functors $\mathbf{Glo} \to \mathbf{Cat}_\Delta$ (see Lemmas 3.3.15 and 3.3.16) yields an equivalence of global ∞-categories $\mathrm{PSh}(\mathrm{N}_\Delta(\mathbf{Glo}_{/\bullet})) \simeq \mathrm{PSh}(\mathrm{N}_\Delta\mathfrak{O}^{\mathrm{gl}}_\bullet)$. By Proposition A.1 the left hand side is then further equivalent to $\mathrm{PSh}(\mathrm{Glo}_{/\bullet}) = \underline{\mathrm{Spc}}_{\mathrm{Glo}}$. This completes the construction of an equivalence $\underline{\mathrm{Spc}}_{\mathrm{Glo}} \simeq \underline{\mathscr{S}}^{\mathrm{gl}}$ of global ∞-categories.

As $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ is presentable (Example 2.4.4), so is $\underline{\mathscr{S}}^{\mathrm{gl}}$. Moreover, the universal property of $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ shows that the equivalence $F\colon \underline{\mathrm{Spc}}_{\mathrm{Glo}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ constructed above is characterized essentially uniquely by the image of the terminal object $* \in \underline{\mathrm{Spc}}_{\mathrm{Glo}}(1)$, so it only remains to verify that F sends this to the terminal object of $\mathscr{S}^{\mathrm{gl}}$. However, this follows simply from the fact that $F(1)\colon \underline{\mathrm{Spc}}_{\mathrm{Glo}}(1) \to \mathscr{S}^{\mathrm{gl}}$ is an equivalence of ordinary ∞-categories. □

4. Parametrized semiadditivity

The goal of this section is to introduce the parametrized analogue of the familiar notion of semiadditivity of an ∞-category, following the ideas introduced by Nardin [Nar16]. In the parametrized setting, the notion of semiadditivity comes in various flavors, parametrized by suitable subcategories $P \subseteq T$: roughly speaking, a T-∞-category $\mathcal{C}$ is P-semiadditive if it is pointwise semiadditive, admits left adjoints $p_!$ and right adjoints p_* for the morphisms $p\colon A \to B$ in P satisfying base change, and a canonical *norm map* $\mathrm{Nm}_p\colon p_! \to p_*$ between these two adjoints is an equivalence.

4.1. **Pointed T-∞-categories.** As a first step towards defining parametrized semiadditivity, we introduce the notion of pointedness for T-∞-categories. Recall that a *zero object* of an ∞-category is an object which is both initial and terminal. An ∞-category is called *pointed* if it admits a zero object. This has the following parametrized analogue.

Definition 4.1.1. Let $\mathcal{C}$ be a T-∞-category and let $c\colon \underline{1} \to \mathcal{C}$ be a T-∞-functor. We say that c is a *T-zero object* of $\mathcal{C}$ if $c(B) \in \mathcal{C}(B)$ is a zero object for every $B \in T$. We say that $\mathcal{C}$ is *pointed* if it admits a T-zero object; equivalently, $\mathcal{C}(B)$ is a pointed ∞-category for every $B \in T$ and $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ preserves the zero object for every $f\colon A \to B$ in T.

Similarly, we say that $c\colon \underline{1} \to \mathcal{C}$ is a *T-initial object* (resp. a *T-final object*) if $c(B) \in \mathcal{C}(B)$ is an initial object (resp. a final object) for all $B \in T$.

Denote by $\mathrm{Cat}_T^* \subseteq \mathrm{Cat}_T$ the (non-full) subcategory spanned by the T-∞-categories admitting a T-final object and the T-functors that preserve the T-final object. We let $\mathrm{Cat}_T^{\mathrm{pt}} \subseteq \mathrm{Cat}_T^*$ denote the full subcategory spanned by the pointed T-∞-categories.

Definition 4.1.2. For T-∞-categories $\mathcal{C}$ and $\mathcal{D}$ which admit a T-final object, we let

$$\underline{\mathrm{Fun}}_T^*(\mathcal{C}, \mathcal{D}) \subseteq \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$$

be the full parametrized subcategory spanned at $B \in T$ by the *pointed $T_{/B}$-functors*, i.e. those $F\colon \pi_B^* \mathcal{C} \to \pi_B^* \mathcal{D}$ which preserve the $T_{/B}$-final object.

In the non-parametrized setting, an ∞-category is pointed if and only if it admits an initial object $\emptyset$ and a terminal object 1, and the canonical map $\emptyset \to 1$ is an equivalence. In other words: the limit and colimit of the empty diagram in $\mathcal{C}$ exist and are canonically equivalent. For our discussion of parametrized semiadditivity, we will need an enhancement of this statement to the parametrized setting which identifies more generally the (parametrized) limit and colimit corresponding to a *disjoint summand inclusion.*

Definition 4.1.3. A map $f\colon A \to B$ in an ∞-category $\mathcal{E}$ is called a *disjoint summand inclusion* if there exists another morphism $g\colon C \to B$ in $\mathcal{E}$ such that the maps f and g exhibit B as a coproduct of A and C in $\mathcal{E}$.

Lemma 4.1.4. *Let $\mathcal{C}$ be a T-∞-category and let $f\colon A \to B$ be a disjoint summand inclusion in* $\mathrm{PSh}(T)$.

(1) If $\mathcal{C}$ admits a T-initial object, then the restriction functor $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a fully faithful left adjoint $f_!\colon \mathcal{C}(A) \to \mathcal{C}(B)$;
(2) If $\mathcal{C}$ admits a T-final object, then the restriction functor $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a fully faithful right adjoint $f_*\colon \mathcal{C}(A) \to \mathcal{C}(B)$;
(3) If $\mathcal{C}$ admits both a T-initial object and a T-final object, then there is a unique map

$$\mathrm{Nm}_f\colon f_! \implies f_*$$

whose restriction $f^* \mathrm{Nm}_f\colon f^*f_! \Rightarrow f^*f_*$ is the equivalence inverse to the composite

$$f^*f_* \xrightarrow[\sim]{c_f^*} \mathrm{id} \xrightarrow[\sim]{u_f^!} f^*f_!;$$

(4) If $\mathcal{C}$ is pointed, this map $\mathrm{Nm}_f\colon f_! \Rightarrow f_*$ is an equivalence.

Proof. Let $g\colon C \to B$ denote the disjoint complement of f. As $\mathcal{C}\colon \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sends colimits in $\mathrm{PSh}(T)$ to limits of ∞-categories, the maps f and g induce an equivalence

$$(f^*, g^*)\colon \mathcal{C}(B) \xrightarrow{\simeq} \mathcal{C}(A) \times \mathcal{C}(C),$$

and under this equivalence the restriction functor $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ corresponds to the first projection map $\mathcal{C}(A) \times \mathcal{C}(C) \to \mathcal{C}(A)$. If $\mathcal{C}$ admits a T-initial object, then this projection has a fully faithful left adjoint given by $X \mapsto (X, \emptyset)$, where $\emptyset \in \mathcal{C}(C)$ denotes the initial object. It follows that f^* admits a fully faithful left adjoint $f_!$. Similarly if $\mathcal{C}$ admits a T-final object, the projection has a fully faithful right adjoint given by $X \mapsto (X, 1)$, where $1 \in \mathcal{C}(C)$ is a final object, and thus f^* admits a right adjoint f_*. If $\mathcal{C}$ satisfies both, then inserting the unique map $\emptyset \to 1$ in the second variable gives rise to a natural transformation $\mathrm{Nm}_f\colon f_! \Rightarrow f_*$, which is uniquely determined by requiring that its restriction along f is the canonical identification $f^*f_! \simeq f^*f_*$ in (3). It is clear that Nm_f is an equivalence whenever $\mathcal{C}(C)$ is pointed. □

Before moving on, we record a useful characterization of the disjoint summand inclusions in a presheaf category:

Lemma 4.1.5. *Let $f\colon X \to Y$ be a map in $\mathrm{PSh}(T)$. Then the following are equivalent:*

(1) The map f is a disjoint summand inclusion;
(2) For every map $t\colon A \to Y$ from a representable object $A \in T$, the base change $t^*f\colon A \times_Y X \to A$ of f is a disjoint summand inclusion (i.e. either t^*f is an equivalence or $A \times_Y X = \emptyset$).

Proof. It is clear that (1) implies (2) as disjoint summand inclusions in $\mathrm{PSh}(T)$ are closed under pullback. We thus assume that (2) is satisfied and prove that it implies (1).

By the co-Yoneda Lemma, there are equivalences $X \simeq \mathrm{colim}_{A \in T_{/X}} A$ and $Y \simeq \mathrm{colim}_{B \in T_{/Y}} B$. Under these equivalences, the map f corresponds to restriction of indexing diagrams along the functor $T_{/f}\colon T_{/X} \to T_{/Y}$. It will therefore suffice to show that this functor is a disjoint summand inclusion of ∞-categories, or equivalently that it is fully faithful and any object of $T_{/X}$ admitting a map to or from the image of $T_{/f}$ already belongs to the essential image.

For this, we will first show that f is a monomorphism. This will immediately imply that $\mathrm{PSh}(T)_{/f}$ is fully faithful, hence so is $T_{/f}$. To this end, we observe that the natural map $\tau\colon \coprod_{A \in T_{/X}} A \to \mathrm{colim}_{A \in T_{/X}} A \simeq X$ is an effective epimorphism [ABFJ22, Example 2.3.6], so pullbacks along it detect monomorphisms by [Lur09, Proposition 6.2.3.17]. However, by universality of colimits, τ^*f is simply given as the coproduct of all the pullbacks of f along all the maps $A \to X$, and each of these is in particular a monomorphism by assumption.

For the closure of the image, consider objects $t\colon A \to X$ in $T_{/X}$ and $u\colon B \to Y$ in $T_{/Y}$. If there is a map $\alpha\colon u \to T_{/f}(t)$ in $T_{/Y}$, then $ft\alpha \sim u$, so $t\alpha$ is a preimage of

u. Conversely, a map $\beta\colon T_{/f}(t) \to u$ amounts to a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle u} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{7}$$

in $\mathrm{PSh}(T)$. By assumption, the pullback $B \times_Y X$ is either empty or the projection to B is an equivalence. However, the first case is impossible as $B \times_Y X$ receives a map from A induced by (7), so we see there exists a (pullback) square of the form

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle v}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

The map v is the desired preimage of u, finishing the proof of the lemma. □

Given a T-∞-category $\mathcal{C}$ admitting a T-final object, one may form the T-∞-category $\mathcal{C}_*$ of pointed objects of $\mathcal{C}$. We will need several basic properties of this construction.

Construction 4.1.6. Let $\mathcal{C}$ be a T-∞-category which admits a T-final object. We define the T-∞-category $\mathcal{C}_*$ of *pointed objects of* $\mathcal{C}$ as the composite

$$T^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Cat}^*_\infty \xrightarrow{(-)_*} \mathrm{Cat}^{\mathrm{pt}}_\infty,$$

where the second functor sends an ∞-category $\mathcal{E}$ with terminal object $*$ to the slice $\mathcal{E}_* := \mathcal{E}_{*/}$. This construction is functorial in $\mathcal{C}$ and assembles into a functor $(-)_*\colon \mathrm{Cat}^*_T \to \mathrm{Cat}^{\mathrm{pt}}_T$.

Corollary 4.1.7. *The functor* $(-)_*\colon \mathrm{Cat}^*_T \to \mathrm{Cat}^{\mathrm{pt}}_T$ *is right adjoint to the fully faithful inclusion* $\mathrm{Cat}^{\mathrm{pt}}_T \hookrightarrow \mathrm{Cat}^*_T$.

Proof. This is immediate from the fact that the functor $(-)_*\colon \mathrm{Cat}^*_\infty \to \mathrm{Cat}^{\mathrm{pt}}_\infty$ is right adjoint to the fully faithful inclusion $\mathrm{Cat}^{\mathrm{pt}}_\infty \subseteq \mathrm{Cat}^*_\infty$. □

Corollary 4.1.8. *For* $\mathcal{C} \in \mathrm{Cat}^{\mathrm{pt}}_T$ *and* $\mathcal{D} \in \mathrm{Cat}^*_T$, *composition with the adjunction counit* $\mathcal{D}_* \to \mathcal{D}$ *induces an equivalence of T-∞-categories* $\underline{\mathrm{Fun}}^*_T(\mathcal{C}, \mathcal{D}_*) \xrightarrow{\sim} \underline{\mathrm{Fun}}^*_T(\mathcal{C}, \mathcal{D})$.

Proof. We will prove that the induced functor $\mathrm{Fun}^*_T(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}^*_T(\mathcal{C}, \mathcal{D})$ on underlying ∞-categories is an equivalence. For every $B \in T$ this thus gives an equivalence $\mathrm{Fun}^*_{T/B}(\pi^*_B\,\mathcal{C}, \pi^*_B\,\mathcal{D}_*) \to \mathrm{Fun}^*_{T/B}(\pi^*_B\,\mathcal{C}, \pi^*_B\,\mathcal{D})$ which proves the claim. By the Yoneda lemma it suffices to prove that for any ∞-category $\mathcal{E}$ the above map induces an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}, \mathrm{Fun}^*_T(\mathcal{C}, \mathcal{D}_*)) \to \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}, \mathrm{Fun}^*_T(\mathcal{C}, \mathcal{D})).$$

Observe that the cotensor $\mathcal{D}^{\mathcal{E}}$ of $\mathcal{D}$ by $\mathcal{E}$ again has fiberwise final objects, and that there is a canonical equivalence $(\mathcal{D}^{\mathcal{E}})_* \simeq (\mathcal{D}_*)^{\mathcal{E}}$. The cotensoring adjunction gives rise to an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}, \mathrm{Fun}^*_T(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Hom}_{\mathrm{Cat}^*_T}(\mathcal{C}, \mathcal{D}^{\mathcal{E}})$$

and similarly for $\mathcal{D}_*$. It thus suffices to show that for every ∞-category $\mathcal{E}$ the map $(\mathcal{D}^{\mathcal{E}})_* \to \mathcal{D}^{\mathcal{E}}$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}^*_T}(\mathcal{C}, (\mathcal{D}^{\mathcal{E}})_*) \to \mathrm{Hom}_{\mathrm{Cat}^*_T}(\mathcal{C}, \mathcal{D}^{\mathcal{E}}),$$

which is true by the adjunction of Corollary 4.1.7. □

It follows that the condition of being pointed is closed under passing to parametrized functor categories.

Corollary 4.1.9. *Consider T-∞-categories $\mathcal{C}$ and $\mathcal{D}$ admitting a T-final object. If either $\mathcal{C}$ or $\mathcal{D}$ is pointed, the T-∞-category $\underline{\mathrm{Fun}}^*_T(\mathcal{C}, \mathcal{D})$ is pointed as well.*

Proof. The case where $\mathcal{D}$ is pointed is clear from Proposition 2.3.24. When $\mathcal{C}$ is pointed, we have by Corollary 4.1.8 an equivalence $\underline{\mathrm{Fun}}^*_T(\mathcal{C}, \mathcal{D}_*) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^*_T(\mathcal{C}, \mathcal{D})$, which reduces to the previous case since $\mathcal{D}_*$ is pointed. □

Lemma 4.1.10. *Let $\mathbf{U}$ be a class of T-∞-groupoids and let $\mathcal{D}$ be a $\mathbf{U}$-complete T-∞-category admitting a T-final object. Then $\mathcal{D}_*$ is also $\mathbf{U}$-complete and the forgetful functor $\mathcal{D}_* \to \mathcal{D}$ preserves $\mathbf{U}$-limits.*

Proof. Let $B \in T$ and let $(f\colon A \to B) \in \mathbf{U}(B)$. Consider objects $X \in \mathcal{D}(B)$ and $Y \in \mathcal{D}(A)_*$, and assume we are given a map $\varphi\colon f^*X \to Y$ in $\mathcal{D}(A)$ which exhibits X as a right adjoint object to Y under $f^*\colon \mathcal{D}(B) \to \mathcal{D}(A)$, in the sense that for all $Z \in \mathcal{C}(B)$ the composite

$$\mathrm{Hom}_{\mathcal{C}(B)}(Z, X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}(A)}(f^*Z, f^*X) \xrightarrow{\varphi\circ -} \mathrm{Hom}_{\mathcal{C}(A)}(f^*Z, Y)$$

is an equivalence. Taking $Z = *$ gives X a canonical basepoint which turns the map $f^*X \to Y$ into a map in $\mathcal{D}(A)_*$. One now observes that this map exhibits $X \in \mathcal{D}(A)_*$ as a right adjoint object to $Y \in \mathcal{D}(B)_*$ under $f^*\colon \mathcal{D}(B)_* \to \mathcal{D}(A)_*$. This proves the claim. □

4.2. **Orbital subcategories.** In order to obtain a parametrized analogue of semi-additivity, we first need a parametrized analogue of the notion of *finite (co)products*. In the non-parametrized setting, an ∞-category $\mathcal{E}$ admits finite (co)products if and only if it admits (co)limits indexed by finite sets (regarded as discrete ∞-categories). To generalize this to the parametrized setting, we would thus need a parametrized analogue of the notion of finite set.

In general, there might be various natural choices for such a generalization. A large family of examples comes from certain subcategories P of T that we call *orbital*, extending the terminology of [BDG+16]. To every orbital subcategory P, we assign a class of T-∞-groupoids called the *finite P-sets*, and a T-∞-category $\mathcal{C}$ is said to admit *finite P-coproducts* if it admits parametrized colimits indexed by finite P-sets.

Definition 4.2.1. Let $\mathbb{F}_T$ be the finite coproduct completion of T, defined as the full subcategory of $\mathrm{PSh}(T)$ spanned by the finite disjoint unions $\coprod_{i=1}^n A_i$ of representable presheaves $A_i \in T$. We refer to $\mathbb{F}_T$ as the *∞-category of finite T-sets*.

For a wide subcategory $P \subseteq T$, we let $\mathbb{F}_T^P \subseteq \mathbb{F}_T$ denote the wide subcategory spanned by all the morphisms which are a disjoint union of morphisms of the form

$(p_i)\colon \bigsqcup_{i=1}^n A_i \to B$ where each morphism $p_i\colon A_i \to B$ is in P. We refer to $\mathbb{F}_T^P$ as the *∞-category of finite P-sets.*

Note that $\mathbb{F}_T^P$ is equivalent to the finite coproduct completion of the ∞-category P.

Definition 4.2.2. A wide subcategory $P \subseteq T$ is called *orbital* if the base change of a morphism in $\mathbb{F}_T^P$ along an arbitrary morphism in $\mathbb{F}_T$ exists and is again in $\mathbb{F}_T^P$. Equivalently, for every pullback diagram

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\beta} & B \end{array}$$

in $\mathrm{PSh}(T)$, with $A, B, B' \in T$ and $p\colon A \to B$ in P, the morphism $p'\colon A' \to B'$ can be decomposed as a disjoint union $(p_i)_{i=1}^n\colon \bigsqcup_{i=1}^n A_i \to B'$ for morphisms $p_i\colon A_i \to B'$ in P.

The ∞-category T is called *orbital* if it is orbital when regarded as a subcategory of itself.

Remark 4.2.3. An ∞-category T is orbital in our sense if and only if it is orbital in the sense of [BDG+16], [Sha21], [Nar16, Definition 4.1].

Example 4.2.4. Every ∞-category T has a minimal orbital subcategory given by ιT, the core of T.

The following is the main example of an orbital subcategory in this article.

Example 4.2.5. We define $\mathrm{Orb} \subset \mathrm{Glo}$ to be the subcategory spanned by all objects and the injective group homomorphisms. We claim that Orb is an orbital subcategory of Glo. Observe that the ∞-category of finite Glo-sets is equivalent to the $(2,1)$-category of finite groupoids, which admits all homotopy-pullbacks. The subcategory of finite Orb-sets is the wide subcategory on the faithful maps of groupoids, and thus the orbitality of Orb is equivalent to the observation that pullbacks of faithful maps of groupoids are again faithful.

The following two examples are variations of Example 4.2.5.

Example 4.2.6. The orbit category Orb_G of a finite group G is orbital. More generally, for a Lie group G, let $\mathrm{Orb}_G^{f.i.}$ be the wide subcategory of the orbit ∞-category Orb_G spanned by the morphisms equivalent to projections $G/K \to G/H$ for subgroups $K \subseteq H \subseteq G$ where K has finite index in H. Then $\mathrm{Orb}_G^{f.i.}$ is an orbital subcategory of Orb_G. Indeed, the pullback of $G/K \to G/H$ along a morphism $G/H' \to G/H$ is computed via a double coset formula, namely the finite disjoint union $\bigsqcup_{[g]\in H'\backslash H/K} G/(H' \cap {}^gK)$.

Example 4.2.7. Mixing Example 4.2.5 with Example 4.2.6, one can define an ∞-category $\mathrm{Glo}_{\mathrm{Lie}}$ whose objects are compact Lie groups G and whose morphism space $\mathrm{Hom}_{\mathrm{Glo}}(G, H)$ is given by the homotopy orbit space $\mathrm{Hom}_{\mathrm{TopGrp}}(G, H)_{hH}$, where H acts on the space of continuous homomorphisms $G \to H$ via conjugation. See [GH07, Section 4.1] or [Rez14, Section 2.2]. Let $\mathrm{Orb}_{\mathrm{Lie}}^{f.i.} \subseteq \mathrm{Glo}_{\mathrm{Lie}}$ be the wide subcategory whose morphisms are given by the injective homomorphisms $G \hookrightarrow H$ of finite index. Then $\mathrm{Orb}_{\mathrm{Lie}}^{f.i.}$ is an orbital subcategory. The relevant pullbacks are again computed by a double coset formula.

Orbital subcategories are closed under various constructions:

Example 4.2.8. (1) (Slice) Let $P \subseteq T$ be an orbital subcategory and let $B \in T$ be an object. Then the wide subcategory of $T_{/B}$ spanned by those morphisms over B contained in P is again an orbital subcategory. We will often abuse notation and denote this subcategory again by P.

(2) (Preimage) More generally, if $f\colon S \to T$ is a right fibration, then the preimage $Q := f^{-1}(P) \subseteq S$ of an orbital subcategory $P \subseteq T$ is again orbital. Indeed, note that $\mathbb{F}_Q = f^{-1}(\mathbb{F}_P)$, and that the extension $\mathbb{F}_Q \to \mathbb{F}_P$ of f is still a right fibration. The claim is then an instance of [HHLN22b, Proposition 2.6].

(3) (Intersection) The intersection $\bigcap_{i\in I} P_i$ of any non-empty collection of orbital subcategories $P_i \subseteq T$ is again orbital.

Example 4.2.9. Let G be a finite group. Combining part (2) from Example 4.2.8 with Example 4.2.6, we find that for a G-space $X\colon \mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Spc}$, the ∞-category $\int X$ of points of X (that is, the total category of the right fibration classified by X) is orbital.

So far, all the given examples of orbital subcategories are equivariant in nature, being a variation of the orbit category of a group; these are the examples we are most interested in in this article. In the following example we mention some orbital subcategories that appear in completely different contexts.

Example 4.2.10. Let T be an ∞-category, and assume $P \subseteq T$ is a wide subcategory such that base changes of morphisms in P exist in T and are again in P. Then P is orbital.

In particular, many geometric examples give rise to orbital subcategories. For example:

(1) If $T = \mathrm{Diff}$ is the ordinary category of smooth manifolds, the wide subcategory on the local diffeomorphisms is orbital.
(2) If $T = \mathrm{Sm}_S$ is the ordinary category of smooth schemes over some base scheme S, the wide subcategory on the étale morphisms is orbital.

For the remainder of this subsection, we will fix an orbital subcategory $P \subseteq T$.

Definition 4.2.11. We define the *T-∞-category of finite P-sets* $\underline{\mathbb{F}}_T^P$. Given $B \in T$, we let

$$\underline{\mathbb{F}}_T^P(B) \subseteq \mathrm{PSh}(T)_{/B}$$

be the full subcategory spanned by those morphisms $p\colon A \to B$ in $\mathrm{PSh}(B)$ which can be decomposed as a coproduct $(p_i)\colon \bigsqcup_{i=1}^n A_i \to B$ such that each morphism $p_i\colon A_i \to B$ is in P. By orbitality of P, $\underline{\mathbb{F}}_T^P$ forms a parametrized subcategory of $\underline{\mathrm{Spc}}_T$. When $P = T$, we simply write $\underline{\mathbb{F}}_T$ for $\underline{\mathbb{F}}_T^T$.

Since $\underline{\mathbb{F}}_T^P$ forms a class of T-∞-groupoids (see Definition 2.3.12) it makes sense to speak of parametrized colimits indexed by $\underline{\mathbb{F}}_T^P$.

Definition 4.2.12. Let $P \subseteq T$ be an orbital subcategory of T. We say that a T-∞-category $\mathcal{C}$ *admits finite P-coproducts* if it admits $\underline{\mathbb{F}}_T^P$-colimits, in the sense of Definition 2.3.8. Dually, we say $\mathcal{C}$ *admits finite P-products* if it admits $\underline{\mathbb{F}}_T^P$-limits.

Definition 4.2.13. Let $\mathcal{C}$ and $\mathcal{D}$ be two T-∞-categories which admit $\underline{\mathbb{F}}_T^P$-limits. We define $\underline{\mathrm{Fun}}^{P\text{-}\times}(\mathcal{C},\mathcal{D})$ to be the full parametrized subcategory of $\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$ spanned in level B by the $T_{/B}$-functors $F\colon \pi_B^*\,\mathcal{C} \to \pi_B^*\,\mathcal{D}$ which preserve P-products (i.e. preserves $\pi_B^{-1}(P)$-products, c.f. Example 4.2.8). Dually we define $\underline{\mathrm{Fun}}^{P\text{-}\sqcup}(\mathcal{C},\mathcal{D})$.

When $P = T$, a T-∞-category $\mathcal{C}$ admits finite T-coproducts in the sense of Definition 4.2.12 if and only if it has finite T-coproducts in the sense of Shah [Sha21, Definition 5.10].

The following result gives a more explicit characterization of the condition for a T-∞-category to admit finite P-(co)products.

Proposition 4.2.14 (cf. [Sha21, Proposition 5.12], [Nar16, Proposition 2.11])**.** *Let $P \subseteq T$ be an orbital subcategory and let $\mathcal{C}$ be a T-∞-category. Then $\mathcal{C}$ admits finite P-coproducts if and only if the following two conditions hold:*

(1) $\mathcal{C}$ admits fiberwise finite coproducts, see Definition 2.3.11;
(2) for every morphism $p\colon A \to B$ in P, the restriction functor $p^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $p_!\colon \mathcal{C}(A) \to \mathcal{C}(B)$ and for every pullback square as in Lemma 2.3.14(2) with $A, B, B' \in T$ and $f\colon A \to B$ in P, the Beck-Chevalley transformation $p'_! \circ \alpha^* \Rightarrow \beta^* \circ p_!$ is an equivalence.

Dually, $\mathcal{C}$ admits finite P-products if and only the dual conditions hold.

Proof. By definition, every morphism in $\mathbb{F}_T^P$ with target $B \in T$ can be written as a composite

$$\bigsqcup_{i=1}^n B_i \xrightarrow{\bigsqcup_{i=1}^n p_i} \bigsqcup_{i=1}^n B \xrightarrow{\nabla} B \tag{8}$$

for morphisms $p_i\colon B_i \to B$ in P, where $\nabla\colon \bigsqcup_{i=1}^n B \to B$ denotes the fold map in $\mathrm{PSh}(T)$. As the functor $\mathcal{C}\colon\ \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sends colimits in $\mathrm{PSh}(T)$ to limits of ∞-categories, the condition of left $\mathbb{F}_T^P$-adjointability splits up as left adjointability for the maps $\nabla\colon \bigsqcup_{i=1}^n B \xrightarrow{\nabla} B$ and left adjointability for the maps in P. Spelling out the definitions, one observes that the former is equivalent to condition (1) while the latter is equivalent to condition (2). □

A similar argument gives the following analogous result for preservation of finite P-coproducts:

Proposition 4.2.15. *Let $P \subseteq T$ be an orbital subcategory and let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories that admits finite P-coproducts. Then a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves finite P-coproducts if and only if it preserves fiberwise finite coproducts and for every morphism $p\colon A \to B$ in P, the Beck-Chevalley transformation $p_! \circ F_A \Rightarrow F_B \circ p_!$ is an equivalence.*

The dual statement for preservation of finite P-products also holds. □

We end this subsection by showing that the T-∞-category $\underline{\mathbb{F}}_T^P$ can be characterized by a universal property: it is the free T-∞-category admitting finite P-coproducts.

Corollary 4.2.16. *The T-∞-category $\underline{\mathbb{F}}_T^P$ admits finite P-coproducts and the inclusion $\underline{\mathbb{F}}_T^P \hookrightarrow \underline{\mathrm{Spc}}_T$ preserves finite P-coproducts.*

Proof. By Example 2.3.20 it suffices to show that the subcategory $\underline{\mathbb{F}}_T^P \hookrightarrow \underline{\mathrm{Spc}}_T$ is closed under finite P-coproducts. But this is clear from Proposition 4.2.14 since it is closed under fiberwise coproducts and under composition with morphisms in P by construction. □

Corollary 4.2.17. *Let $\mathcal{D}$ be a T-∞-category admitting finite P-coproducts. Let $*\colon \underline{1} \to \underline{\mathbb{F}}_T^P$ denote the T-final object, given at $B \in T$ by the identity $\mathrm{id}_B \in \underline{\mathbb{F}}_T^P(B)$. Then composition with $*\colon \underline{1} \to \underline{\mathbb{F}}_T^P$ induces an equivalence of T-∞-categories*

$$\underline{\mathrm{Fun}}_T^{P\text{-}\sqcup}(\underline{\mathbb{F}}_T^P, \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\underline{1}, \mathcal{D}) \simeq \mathcal{D}\,.$$

Proof. It follows directly from Corollary 4.2.16 that the subcategory $\underline{\mathbb{F}}_T^P \subseteq \underline{\mathrm{Spc}}_T$ is the smallest subcategory which contains the T-final object and is closed under finite P-coproducts, meaning it is equivalent to $\underline{\mathrm{PSh}}_T^{\mathbb{F}_T^P}(\underline{1})$ in the notation of [MW21, Definition 6.1.6]. The claim is then an instance of [MW21, Theorem 6.1.10]. □

4.3. **Atomic orbital subcategories and norm maps.** Let P be an orbital subcategory of T. In this subsection, we will define what it means for P to be an *atomic orbital* subcategory of T, generalizing a definition of [Nar16]. The atomicity condition on P will allow us to define *norm maps* $\mathrm{Nm}_p\colon p_! \to p_*$ in a pointed T-∞-category $\mathcal{C}$, making it possible to compare finite P-coproducts in $\mathcal{C}$ to finite P-products in $\mathcal{C}$. We may therefore think of the atomic orbital subcategories as classifying the various potential 'levels of semiadditivity' that a T-∞-category might have.

Definition 4.3.1. Suppose T is an ∞-category and let $P \subseteq T$ be an orbital subcategory. We say that P is *atomic orbital* if for every morphism $p\colon A \to B$ in P the diagonal $\Delta\colon A \to A \times_B A$ in $\mathrm{PSh}(T)$ is a disjoint summand inclusion in the sense of Definition 4.1.3. An ∞-category T is called *atomic orbital* if it is atomic orbital as a subcategory of itself.

For a subcategory $P \subset T$, being an atomic orbital subcategory is a very restrictive condition: since every disjoint summand inclusion in $\mathrm{PSh}(T)$ is in particular a monomorphism, it implies that all the morphisms in P have to be 0-truncated.

The following lemma provides an alternative characterization of atomic subcategories in terms of the triviality of certain retracts. The case $P = T$ of this lemma immediately implies that our definition of atomic orbital ∞-categories is equivalent to that of [Nar16, Definition 4.1].

Lemma 4.3.2. *Let $P \subseteq T$ be an orbital subcategory. Then P is atomic orbital if and only if any morphism $p\colon A \to B$ in P which admits a section in T is an equivalence.*

Proof. Assume first that P is atomic orbital. Let $p\colon A \to B$ be a morphism in P and assume that p admits a section $s\colon B \to A$ in T. We will show that p is an equivalence with inverse s. Since we are given an equivalence $ps \simeq \mathrm{id}_B$, it remains to show that $sp \simeq \mathrm{id}_A$. Equivalently, we may show that the map $(\mathrm{id}_A, sp)\colon A \to A \times_B A$ factors through the diagonal $\Delta_p\colon A \to A \times_B A$. By assumption this diagonal is equivalent to an inclusion $A \hookrightarrow A \sqcup C$ for some presheaf $C \in \mathrm{PSh}(T)$, and since A is a representable presheaf it follows that the map $(\mathrm{id}_A, sp)\colon A \to A \times_B A \simeq A \sqcup C$ must either factor through $\Delta_p\colon A \hookrightarrow A \sqcup C$ or through $C \hookrightarrow A \sqcup C$. Assume for contradiction that we are in the latter situation. Then the pullback of $\Delta_p \colon A \to A \times_B A$ and $(\mathrm{id}_A, sp) \colon A \to A \times_B A$ is the empty presheaf. But this pullback is also equivalent to B, by the following pullback diagram:

$$\begin{array}{ccccc}
B & \xrightarrow{s} & A & \xrightarrow{p} & B \\
{\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{id}_A, sp)} & \lrcorner & \downarrow{\scriptstyle s} \\
A & \xrightarrow{\Delta_p} & A \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\
 & & {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
 & & A & \xrightarrow{p} & B.
\end{array}$$

Since B is not the empty presheaf, this leads to a contradiction, showing that $(\mathrm{id}_A, sp) \colon A \to A \times_B A$ factors through Δ_p as desired.

Conversely, assume that any map in P which admits a section in T is an equivalence. Let $p \colon A \to B$ be a morphism in P. Since P is orbital, the projection map $\mathrm{pr}_1 \colon A \times_B A \to A$ in $\mathrm{PSh}(T)$ can be decomposed as a disjoint union $(p_i)_{i=1}^n \colon \bigsqcup_{i=1}^n A_i \to A$ of morphisms $p_i \colon A_i \to B$ in P. Since A is representable, the diagonal $\Delta_p \colon A \to A \times_B A \simeq \bigsqcup_{i=1}^n A_i$ has to factor through one of the inclusions $A_i \hookrightarrow A \times_B A$, so that the morphism $p_i \colon A_i \to A$ admits a section $A \to A_i$ in T. By the assumption on P, this means that p_i is an equivalence. It follows that the diagonal Δ_p of p is the inclusion of a disjoint summand $A \simeq A_i \hookrightarrow \bigsqcup_{i=1}^n A_i$, as desired. □

Example 4.3.3. Recall the subcategory $\mathrm{Orb} \subset \mathrm{Glo}$ spanned by the injective homomorphisms. Clearly, any injective homomorphism that admits a section is also surjective, hence an isomorphism. Together with Example 4.2.5, we conclude that Orb is an atomic orbital subcategory of Glo. By direct computation one sees that the diagonal in PSh(Glo) of a non-injective group homomorphism is never a disjoint summand inclusion, and thus it follows that Orb is in fact the *maximal* atomic orbital subcategory of Glo.

Remark 4.3.4. There is a classification of the atomic orbital subcategories of Glo in terms of *global transfer systems* in the sense of Barrero [Bar23]. Recall from *op. cit.* that a global transfer system (for the family of finite groups) is a partial order $\leqslant_T$ on the collection of finite groups which refines the subgroup relation and which is closed under preimages, meaning that for a group homomorphism $\alpha \colon G' \to G$, if $H \leqslant_T G$ then $\alpha^{-1}(H) \leqslant G'$. We may assign to $\leqslant_T$ a wide subcategory $\mathrm{Orb}_T \subseteq \mathrm{Orb}$ which contains those injections $i \colon H \hookrightarrow G$ for which $i(H) \leqslant_T G$. It is not difficult to show that Orb_T is an atomic orbital subcategory of Glo, and that conversely every atomic orbital subcategory of Glo is of the form Orb_T for some global transfer system $\leq_T$.

A convenient feature of atomic orbital subcategories is that they are *left cancellable*, in the sense that for morphisms $f \colon A \to B$ and $g \colon B \to C$ in T, if both g and gf are in P then also f is in P.

Lemma 4.3.5. *Every atomic orbital subcategory $P \subseteq T$ is left cancellable.*

Proof. Let $f \colon A \to B$ and $g \colon B \to C$ be morphisms in T, and assume that both g and gf are in P. We will show that then also f is in P. This is a classical argument [Gro60, Remarque 5.5.12]: we may factor f as a composite

$$A \xrightarrow{(1,f)} A \times_C B \xrightarrow{\mathrm{pr}_B} B$$

in $\mathbb{F}_T$, and it will suffice to show that both of these morphisms are morphisms in $\mathbb{F}_T^P$. The projection $\mathrm{pr}_B\colon A\times_C B\to B$ is the base change of $gf\colon A\to C$ along $B\to C$, so it is in $\mathbb{F}_T^P$ by orbitality of P and the assumption on gf. In turn, the morphism $(1,f)\colon A\to A\times_C B$ is a base change of the diagonal map $\Delta_g\colon B\to B\times_C B$, which is by assumption a disjoint summand inclusion and thus in particular in $\mathbb{F}_T^P$. This finishes the proof. ☐

Corollary 4.3.6. *Let $P\subseteq T$ be an atomic orbital subcategory. Then for every $B\in T$, the inclusion $P_{/B}\hookrightarrow T_{/B}$ is fully faithful. In particular, there is an equivalence $\underline{\mathbb{F}}_T^P(B)\simeq(\mathbb{F}_T^P)_{/B}$.* ☐

While atomicity a priori only requires the diagonals of maps in P be disjoint summand inclusions, the next proposition shows that this in fact holds for a more general class of maps in $\mathrm{PSh}(T)$. Recall from Remark 2.3.15 that, given a presheaf B on T, we write $\underline{\mathbb{F}}_T^P(B)\subseteq\mathrm{PSh}(T)_{/B}$ for the full subcategory containing those morphisms $p\colon A\to B$ of presheaves whose base change to any representable $B'\in T$ lives in $\mathbb{F}_T^P$.

Proposition 4.3.7. *Let $P\subseteq T$ atomic orbital, let $Y\in\mathrm{PSh}(T)$ and let $(p\colon X\to Y)\in\underline{\mathbb{F}}_T^P(Y)$. Then the diagonal $\Delta_p\colon X\to X\times_Y X$ in $\mathrm{PSh}(T)$ is a disjoint summand inclusion.*

Proof. By Lemma 4.1.5, it will suffice to show that the base change of $\Delta_p\colon X\to X\times_Y X$ along any map $\alpha=(\alpha_1,\alpha_2)\colon A\to X\times_Y X$ from a representable $A\in T$ is a disjoint summand inclusion. Observe that the map α factors as the following composite:

$$A\xrightarrow{(\mathrm{id},\alpha_2)}A\times_Y X\xrightarrow{\alpha_1\times\mathrm{id}}X\times_Y X.$$

As base changes of disjoint summand inclusions are again disjoint summand inclusions, it will thus suffice to show that the base change of Δ_f along the map $\alpha_1\times\mathrm{id}$ is a disjoint summand inclusion. To this end, consider the following commutative diagram:

$$\begin{array}{ccccc}
A & \xrightarrow{\alpha_1} & X & & \\
{\scriptstyle(\mathrm{id},\alpha_1)}\downarrow & \lrcorner & \downarrow{\scriptstyle\Delta_f} & & \\
A\times_Y X & \xrightarrow{\alpha_1\times\mathrm{id}} & X\times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\
{\scriptstyle\mathrm{pr}_1}\downarrow & \lrcorner & {\scriptstyle\mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
A & \xrightarrow{\alpha_1} & X & \xrightarrow{f} & Y.
\end{array}$$

It follows readily from the pasting law of pullback squares that each square is a pullback square. Observe that the projection map $\mathrm{pr}_1\colon A\times_Y X\to A$ is the base change of f along $f\alpha_1$, hence it lies in $\mathbb{F}_T^P$ by assumption. It follows that the map (id,α_1) is a section of a morphism in $\mathbb{F}_T^P$, hence is a disjoint summand inclusion by the same argument as in the proof of Lemma 4.3.2. This finishes the proof. ☐

For the remainder of this subsection, we will fix an atomic orbital subcategory $P\subseteq T$. We are now ready to define the norm map $\mathrm{Nm}_p\colon p_!\to p_*$ for p as in Proposition 4.3.7.

Construction 4.3.8 (Norm map, cf. [Lur17, Construction 6.1.6.8], [NS18, Construction I.1.7], [HL13, Construction 4.1.8])**.** Let $\mathcal{C}$ be a pointed T-∞-category, let $B \in \mathrm{PSh}(T)$ and let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$. Consider the following pullback diagram

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ A & \xrightarrow{p} & B \end{array} \tag{9}$$

in $\mathrm{PSh}(T)$, and let $\Delta\colon A \to A \times_B A$ denote the diagonal of p. By Proposition 4.3.7, Δ is a disjoint summand inclusion, so that Lemma 4.1.4 provides adjunctions $\Delta_! \dashv \Delta^* \dashv \Delta_*$ and an equivalence $\mathrm{Nm}_\Delta\colon \Delta_! \simeq \Delta_*$.

(1) Define a natural transformation $\alpha\colon \mathrm{pr}_1^* \Rightarrow \mathrm{pr}_2^*$ as the following composite:

$$\mathrm{pr}_2^* \overset{u_\Delta^*}{\Longrightarrow} \Delta_*\Delta^*\mathrm{pr}_2{}^* \simeq \Delta_* \overset{\mathrm{Nm}_\Delta^{-1}}{\simeq} \Delta_! \simeq \Delta_!\Delta^*\,\mathrm{pr}_1^* \overset{c_\Delta^!}{\Longrightarrow} \mathrm{pr}_1^*\,.$$

(2) Assume that $\mathcal{C}$ admits finite P-coproducts, so that the pullback square (9) gives a left base change equivalence $p^*p_! \simeq \mathrm{pr}_{1!}\,\mathrm{pr}_2^*$. We define the *adjoint norm transformation* $\widetilde{\mathrm{Nm}}_p\colon p^*p_! \Rightarrow \mathrm{id}$ *of* p *in* $\mathcal{C}$ as the composite

$$\widetilde{\mathrm{Nm}}_p\colon p^*p_! \overset{l.b.c.}{\simeq} \mathrm{pr}_{1!}\,\mathrm{pr}_2^* \xRightarrow{\mathrm{pr}_{1!}\alpha} \mathrm{pr}_{1!}\,\mathrm{pr}_1^* \overset{c_{\mathrm{pr}_1}^!}{\Longrightarrow} \mathrm{id}\,.$$

(3) Assume that $\mathcal{C}$ admits finite P-products, so that the pullback square (9) gives a right base change equivalence $p^*p_* \simeq \mathrm{pr}_{2*}\,\mathrm{pr}_1^*$. We define the *dual adjoint norm transformation* $\overline{\mathrm{Nm}}_p\colon \mathrm{id} \Rightarrow p^*p_*$ *of* p *in* $\mathcal{C}$ as the composite

$$\overline{\mathrm{Nm}}_p\colon \mathrm{id} \overset{u_{\mathrm{pr}_2}^*}{\Longrightarrow} \mathrm{pr}_{2*}\,\mathrm{pr}_2^* \xRightarrow{\mathrm{pr}_{2*}\alpha} \mathrm{pr}_{2*}\,\mathrm{pr}_1^* \overset{r.b.c.}{\simeq} p^*p_*.$$

(4) Assume that $\mathcal{C}$ admits both finite P-products and finite P-coproducts. We define the *norm transformation of* p *in* $\mathcal{C}$

$$\mathrm{Nm}_p\colon p_! \implies p_*$$

as the map adjoint to the adjoint norm transformation $\widetilde{\mathrm{Nm}}_p\colon p^*p_! \Rightarrow \mathrm{id}$.

We will sometimes write $\widetilde{\mathrm{Nm}}_p^{\mathcal{C}}$, $\overline{\mathrm{Nm}}_p^{\mathcal{C}}$ or $\mathrm{Nm}_p^{\mathcal{C}}$ to emphasize the dependence on $\mathcal{C}$.

Remark 4.3.9. Unwinding the definitions, the map $\widetilde{\mathrm{Nm}}_p\colon p^*p_! \Rightarrow \mathrm{id}$ may be given more directly as the composite

$$p^*p_! \overset{l.b.c.}{\simeq} \mathrm{pr}_{1!}\,\mathrm{pr}_2^* \overset{u_\Delta^*}{\Longrightarrow} \mathrm{pr}_{1!}\Delta_*\Delta^*\,\mathrm{pr}_2^* \overset{\mathrm{Nm}_\Delta^{-1}}{\simeq} \mathrm{pr}_{1!}\Delta_!\Delta^*\,\mathrm{pr}_2^* \simeq \mathrm{id}_{\mathcal{C}(A)}\,.$$

Similarly, the map $\overline{\mathrm{Nm}}_p\colon \mathrm{id} \Rightarrow p^*p_*$ unwinds to the following composite:

$$\mathrm{id}_{\mathcal{C}(A)} \simeq \mathrm{pr}_{2*}\Delta_*\Delta^*\,\mathrm{pr}_1^* \overset{\mathrm{Nm}_\Delta^{-1}}{\simeq} \mathrm{pr}_{2*}\Delta_!\Delta^*\,\mathrm{pr}_1^* \overset{c_\Delta^!}{\Longrightarrow} \mathrm{pr}_{2*}\,\mathrm{pr}_1^* \overset{r.b.c.}{\simeq} p^*p_*.$$

The description of the adjoint norm map $\widetilde{\mathrm{Nm}}$ given above is precisely the definition of the map $\nu_p^{(0)}\colon p^*p_! \Rightarrow \mathrm{id}$ of [HL13, Construction 4.1.8], applied to the Beck-Chevalley fibration $\int \mathcal{C} \to \mathrm{PSh}(T)$ classified by the functor $\mathcal{C}\colon \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}_\infty$. In particular, the norm map $\mathrm{Nm}_p\colon p_! \to p_*$ defined above agrees with the norm map Nm_p of [HL13, Construction 4.1.12].

Remark 4.3.10. Let $(f\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$ be a morphism in $\mathrm{PSh}(T)$ which happens to be a disjoint summand inclusion. Then the norm map $\mathrm{Nm}_f\colon f_! \Rightarrow f_*$ of Construction 4.3.8 agrees with the map $\mathrm{Nm}_f\colon f_! \Rightarrow f_*$ constructed in Lemma 4.1.4.

The map $\alpha\colon \mathrm{pr}_2^* \Rightarrow \mathrm{pr}_1^*$ defined in Construction 4.3.8(1) may be thought of as some kind of 'diagonal matrix': as the next lemma shows, it restricts to the identity when restricted along the diagonal $\Delta\colon A \hookrightarrow A \times_B A$, and restricts to the zero map on the complement of the diagonal.

Lemma 4.3.11. *Let $\mathcal{C}$ be a pointed T-∞-category and let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$. Let $j\colon C \hookrightarrow A \times_B A$ denote the disjoint complement of the diagonal inclusion $\Delta\colon A \hookrightarrow A \times_B A$. Then the following hold:*

(1) The composite $\mathrm{id}_{\mathcal{C}(A)} \simeq \Delta^* \mathrm{pr}_2^* \xrightarrow{\Delta^*\alpha} \Delta^* \mathrm{pr}_1^* \simeq \mathrm{id}_{\mathcal{C}(A)}$ is homotopic to the identity transformation.
(2) The map $j^*\alpha\colon j^* \mathrm{pr}_2^* \Rightarrow j^* \mathrm{pr}_1^*$ is the zero transformation, in the sense that it factors through the zero functor $0\colon \mathcal{C}(A) \to \mathcal{C}(C)$.

Proof. The proof of (1) follows from the following commutative diagram:

$$\begin{array}{ccccccccc}
\Delta^* \mathrm{pr}_2^* & = & \Delta^* \mathrm{pr}_2^* & \xrightarrow{\simeq} & \mathrm{id} & \xrightarrow{\simeq} & \Delta^* \mathrm{pr}_1^* & = & \Delta^* \mathrm{pr}_1^* \\
{\scriptstyle u_\Delta^*}\downarrow & \nearrow{\scriptstyle c_\Delta^*} & & \nearrow{\scriptstyle c_\Delta^*} & & \searrow{\scriptstyle u_\Delta^!} & & \searrow{\scriptstyle u_\Delta^!} & \uparrow{\scriptstyle c_\Delta^!} \\
\Delta^*\Delta_*\Delta^* \mathrm{pr}_2^* & \xrightarrow[\simeq]{} & \Delta^*\Delta_* & & \xrightarrow[\mathrm{Nm}_\Delta^{-1}]{} & & \Delta^*\Delta_! & \xrightarrow[\simeq]{} & \Delta^*\Delta_!\Delta^* \mathrm{pr}_2^*.
\end{array}$$

The triangles on the two sides commute by the triangle identity, the rhombi commute by naturality and the triangle in the middle commutes by the defining property of the norm map Nm_Δ of Lemma 4.1.4.

For (2), note that by definition of α the map $j^*\alpha$ factors through the functor $j^*\Delta_*$. Since coproducts are disjoint in $\mathrm{PSh}(T)$, the fiber product $C \times_{A \times_B A} A$ is the empty presheaf. It then follows from base change that the functor $j^*\Delta_*$ factors through the ∞-category $\mathcal{C}(\emptyset) \simeq *$, which forces it to be the zero functor. $\square$

Remark 4.3.12. In the setting of Mackey 2-functors, Balmer and Dell'Ambrogio [BD20, Theorem 3.3.4] have produced a similar transformation $\Theta_i\colon i_! \Rightarrow i_*$ for i a faithful map of groupoids, i.e. a morphism in $\mathbb{F}_{\mathrm{Glo}}^{\mathrm{Orb}}$. It follows from Lemma 4.3.11 and [BD20, Proposition 3.2.1] that the transformation $\mathrm{Nm}_i\colon i_! \Rightarrow i_*$ of Construction 4.3.8 specializes to the transformation Θ_i of Balmer and Dell'Ambrogio in the case $T = \mathrm{Glo}$ and $P = \mathrm{Orb}$. In particular, if $\mathcal{C}$ is a pointed global ∞-category admitting finite Orb-(co)products, it follows from [BD20, Theorem 3.4.2] that the norm maps Nm_i are equivalences for every faithful map of groupoids $i\colon H \to G$ if and only if there exist abstract equivalences $i_! \simeq i_*$ for every such i.

4.4. **Properties of norm maps.** We will next establish a variety of results about the calculus of norm maps.

To start with, we address the obvious asymmetry in the construction of the norm map: we could just as well have considered the map $p_! \Rightarrow p_*$ adjoint to the *dual* adjoint norm map $\overline{\mathrm{Nm}}_p\colon \mathrm{id} \Rightarrow p^*p_*$. The following lemma shows that these two maps agree.

Lemma 4.4.1. *Assume that $\mathcal{C}$ is a pointed T-∞-category which admits both finite P-products and finite P-coproducts. For every $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$, the maps $\widetilde{\mathrm{Nm}}_p\colon p^*p_! \Rightarrow \mathrm{id}$ and $\overline{\mathrm{Nm}}_p\colon \mathrm{id} \Rightarrow p^*p_*$ adjoin to the same map $\mathrm{Nm}_p\colon p_! \Rightarrow p_*$.*

Proof. We have to show that dual adjoint norm map $\overline{\mathrm{Nm}}_p$ is the total mate of the adjoint norm map $\widetilde{\mathrm{Nm}}_p$. A mundane exercise in 2-category theory shows that the total mate of the Beck-Chevalley equivalence $p^*p_! \simeq \mathrm{pr}_{1!}\,\mathrm{pr}_2^*$ is the Beck-Chevalley equivalence $\mathrm{pr}_{2*}\,\mathrm{pr}_1^* \simeq p^*p_*$. Furthermore, it follows directly from the triangle identity that the total mate of the composite

$$\mathrm{pr}_{1!}\,\mathrm{pr}_2^* \xRightarrow{\mathrm{pr}_{1!}\alpha} \mathrm{pr}_{1!}\,\mathrm{pr}_1^* \xRightarrow{c^!_{\mathrm{pr}_1}} \mathrm{id}$$

is given by the composite

$$\mathrm{id} \xRightarrow{u^*_{\mathrm{pr}_2}} \mathrm{pr}_{2*}\,\mathrm{pr}_2^* \xRightarrow{\mathrm{pr}_{2*}\alpha} \mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,.$$

Since the total mate of a composite of transformations is given by composing in opposite order the individual total mates of these transformations, this finishes the proof. □

The norm map Nm_p can be written in terms of the double Beck-Chevalley map $p_!\mathrm{pr}_{2*} \Rightarrow p_*\mathrm{pr}_{1!}$ associated to the pullback square (9):

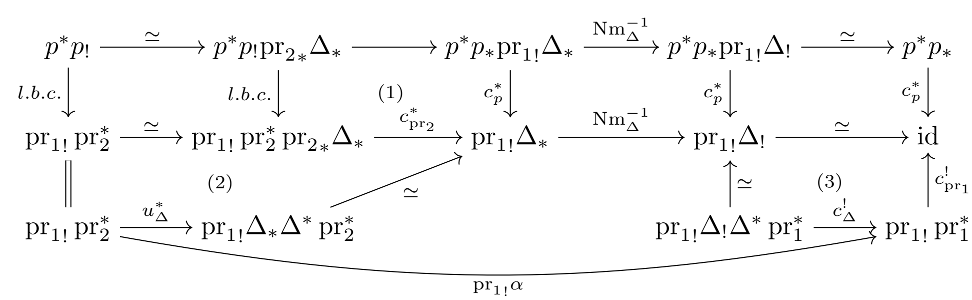
Lemma 4.4.2. *Assume that $\mathcal{C}$ is a pointed T-∞-category which admits both finite P-products and finite P-coproducts, and let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$. Then the norm map Nm_p is homotopic to the composite*

$$p_! \simeq p_!\mathrm{pr}_{2*}\Delta_* \to p_*\mathrm{pr}_{1!}\Delta_* \xrightarrow{\mathrm{Nm}_\Delta^{-1}} p_*\mathrm{pr}_{1!}\Delta_! \simeq p_*.$$

Proof. By adjunction, it suffices to show that the adjoint norm map $\widetilde{\mathrm{Nm}}_p\colon p^*p_! \to \mathrm{id}$ is given by the composite

$$p^*p_! \simeq p^*p_!\mathrm{pr}_{2*}\Delta_* \to p^*p_*\mathrm{pr}_{1!}\Delta_* \xrightarrow{\mathrm{Nm}_\Delta^{-1}} p^*p_*\mathrm{pr}_{1!}\Delta_! \simeq p^*p_* \xrightarrow{c_p^*} \mathrm{id}\,.$$

This follows from the following commutative diagram:



The unlabeled squares commute by naturality. Commutativity of (1) is by the triangle identity, while commutativity of (2) and (3) follows from the equivalence $\mathrm{pr}_1 \circ \Delta \simeq \mathrm{id} \simeq \mathrm{pr}_2 \circ \Delta$ and the fact that the (co)unit of a composite of adjunctions is the composite of the individual (co)units. □

As was shown by Hopkins and Lurie [HL13], the norm maps behave well under composition and base change of morphisms in $\underline{\mathbb{F}}_T^P$.

Proposition 4.4.3 ([HL13, Proposition 4.2.1]). *Assume that $\mathcal{C}$ is a pointed T-∞-category which admits finite P-coproducts. Consider a pullback square*

$$\begin{array}{ccc} A' & \xrightarrow{g_A} & A \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{g_B} & B \end{array}$$

in $\mathrm{PSh}(T)$ *such that $p \in \underline{\mathbb{F}}_T^P(B)$ and (hence) $p' \in \underline{\mathbb{F}}_T^P(B')$. Then there is a commutative diagram*

$$\begin{array}{ccccc} p'^* p'_! g_A^* & \xrightarrow[\sim]{l.b.c.} & p'^* g_B^* p_! & \xrightarrow{\sim} & g_A^* p^* p_! \\ {\scriptstyle \widetilde{\mathrm{Nm}}_{p'}}\downarrow & & & & \downarrow{\scriptstyle \widetilde{\mathrm{Nm}}_p} \\ g_A^* & & \xlongequal{\qquad\qquad} & & g_A^*. \end{array}$$
□

Corollary 4.4.4 ([HL13, Remark 4.2.3]). *In the situation of Proposition 4.4.3, assume that $\mathcal{C}$ furthermore admits finite P-products. Then the composite*

$$p'_! g_A^* \overset{l.b.c.}{\simeq} g_B^* p_! \xrightarrow{g_B^* \mathrm{Nm}_p} g_B^* p_* \overset{r.b.c.}{\simeq} p'_* g_A^*$$

is homotopic to the map $\mathrm{Nm}_{p'} g_A^*$. □

Proposition 4.4.5 ([HL13, Proposition 4.2.2]). *Assume that $\mathcal{C}$ is a pointed T-∞-category which admits finite P-coproducts. Let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$ and $(q\colon B \to C) \in \underline{\mathbb{F}}_T^P(C)$, so that also $(qp\colon A \to C) \in \underline{\mathbb{F}}_T^P(C)$. Then the adjoint norm map $\widetilde{\mathrm{Nm}}_{qp}$ is homotopic to the composite*

$$(qp)^*(qp)_! \simeq p^* q^* q_! p_! \xrightarrow{\widetilde{\mathrm{Nm}}_q} p^* p_! \xrightarrow{\widetilde{\mathrm{Nm}}_p} \mathrm{id}\,.$$
□

Corollary 4.4.6 ([HL13, Remark 4.2.4]). *In the situation of Proposition 4.4.5, assume that $\mathcal{C}$ furthermore admits finite P-products. Then the composite transformation*

$$(qp)_! \simeq q_! p_! \xrightarrow{\mathrm{Nm}_q} q_* p_! \xrightarrow{\mathrm{Nm}_p} q_* p_* \simeq (qp)_*$$

is homotopic to the norm map Nm_{qp}. □

The norm maps are also suitably functorial in the T-∞-category $\mathcal{C}$: as we will now show, any pointed T-functor $G\colon \mathcal{C} \to \mathcal{D}$ transforms norm maps in $\mathcal{C}$ into norm maps in $\mathcal{D}$.

Lemma 4.4.7. *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a pointed T-functor of pointed T-categories and let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$. Then the diagram*

$$\begin{array}{ccc} \mathrm{pr}_2^* G & \xrightarrow{\alpha G} & \mathrm{pr}_1^* G \\ \simeq\downarrow & & \downarrow\simeq \\ G\,\mathrm{pr}_2^* & \xrightarrow[G\alpha]{} & G\,\mathrm{pr}_1^* \end{array}$$

of transformations between functors $\mathcal{C}(A) \to \mathcal{D}(A \times_B A)$ commutes.

Proof. Spelling out the definition of α, this is a direct consequence of the following three commutative diagrams:

$$
\begin{array}{ccc}
G & \xrightarrow{u^*_\Delta G} & \Delta_*\Delta^*G \\
{\scriptstyle Gu^*_\Delta}\downarrow & & \downarrow{\scriptstyle \simeq} \\
G\Delta_*\Delta^* & \xrightarrow[\mathrm{BC}_*]{} & \Delta_*G\Delta^*,
\end{array}
\qquad
\begin{array}{ccc}
\Delta_!G & \xrightarrow{\mathrm{Nm}_\Delta G} & \Delta_*G \\
{\scriptstyle \mathrm{BC}_!}\downarrow & & \uparrow{\scriptstyle \mathrm{BC}_*} \\
G\Delta_! & \xrightarrow[G\,\mathrm{Nm}_\Delta]{} & G\Delta_*,
\end{array}
\qquad
\begin{array}{ccc}
\Delta_!\Delta^*G & \xrightarrow{c^!_\Delta G} & G \\
{\scriptstyle \simeq}\uparrow & & \uparrow{\scriptstyle Gc^!_\Delta} \\
\Delta_!G\Delta^* & \xrightarrow[\mathrm{BC}_!]{} & G\Delta_!\Delta^*.
\end{array}
$$

The left and right squares commute by definition of the Beck-Chevalley maps, using the triangle identities. The fact that the middle square commutes follows directly from pointedness of G and the construction of Nm_Δ in Lemma 4.1.4. ☐

Lemma 4.4.8. *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a pointed T-functor between two pointed T-∞-categories which admit finite P-coproducts. Then for every $(p\colon A \to B) \in \underline{\mathbb{F}}^P_T(B)$, the diagram*

$$
\begin{array}{ccccc}
p^*p_!G_A & \xrightarrow{\mathrm{BC}_!} & p^*G_Bp_! & \xrightarrow{\simeq} & G_Ap^*p_! \\
{\scriptstyle \widetilde{\mathrm{Nm}}^{\mathcal{D}}_p G_A}\downarrow & & & & \downarrow{\scriptstyle G_A\widetilde{\mathrm{Nm}}^{\mathcal{C}}_p} \\
G_A & & = & & G_A
\end{array}
$$

commutes.

Proof. Consider the diagram

$$
\begin{array}{ccccc}
p^*p_!G_A & \xrightarrow{\mathrm{BC}_!} & p^*G_Bp_! & \xrightarrow{\simeq} & G_Ap^*p_! \\
\downarrow{\scriptstyle l.b.c.} & & & & {\scriptstyle l.b.c.}\downarrow \\
\mathrm{pr}_{1!}\,\mathrm{pr}_2^*\,G_A & \xrightarrow{\simeq} & \mathrm{pr}_{1!}G_{A\times_BA}\,\mathrm{pr}_2^* & \xrightarrow{\mathrm{BC}_!} & G_A\mathrm{pr}_{1!}\,\mathrm{pr}_2^* \\
\downarrow{\scriptstyle \mathrm{pr}_{1!}\alpha G_A} & & \downarrow{\scriptstyle \mathrm{pr}_{1!}G_{A\times_BA}\alpha} & & {\scriptstyle G_A\mathrm{pr}_{1!}\alpha}\downarrow \\
\mathrm{pr}_{1!}\,\mathrm{pr}_1^*\,G_A & \xrightarrow{\simeq} & \mathrm{pr}_{1!}G_{A\times_BA}\,\mathrm{pr}_1^* & \xrightarrow{\mathrm{BC}_!} & G_A\mathrm{pr}_{1!}\,\mathrm{pr}_1^* \\
\downarrow{\scriptstyle c^!_{\mathrm{pr}_1}G_A} & & & & {\scriptstyle G_Ac^!_{\mathrm{pr}_1}}\downarrow \\
G_A & & = & & G_A.
\end{array}
$$

We are interested in the outer square. The right middle square commute by naturality. The left middle square commutes by Lemma 4.4.7. The bottom rectangle commutes by definition of the Beck-Chevalley map, using the triangle identity. Finally, the upper rectangle commutes as the two composites are the Beck-Chevalley transformations associated to the following two equivalent composite squares:

$$
\begin{array}{ccccc}
\mathcal{C}(B) & \xrightarrow{G_B} & \mathcal{D}(B) & \xrightarrow{p^*} & \mathcal{D}(A) \\
{\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle \mathrm{pr}_1^*} \\
\mathcal{C}(A) & \xrightarrow{G_A} & \mathcal{D}(A) & \xrightarrow{\mathrm{pr}_2^*} & \mathcal{D}(A\times_BA)
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
\mathcal{C}(B) & \xrightarrow{p^*} & \mathcal{C}(A) & \xrightarrow{G_A} & \mathcal{D}(A) \\
{\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1^*} & & \downarrow{\scriptstyle \mathrm{pr}_1^*} \\
\mathcal{C}(A) & \xrightarrow{\mathrm{pr}_2^*} & \mathcal{C}(A\times_BA) & \xrightarrow{G_{A\times_BA}} & \mathcal{D}(A\times_BA).
\end{array}
$$

This finishes the proof. ☐

We end the subsection with the the following technical lemma, needed for the proof of Proposition 4.5.8 below. We recommend the reader skip this lemma on first reading.

Lemma 4.4.9. *Let $\mathcal{C}$ be a pointed T-∞-category which admits finite P-products. Let $(p\colon A \to B) \in \underline{\mathbb{F}}^P_T(B)$, and assume that p admits a section $s\colon B \to A$ which is a disjoint summand inclusion. Then the composite*

$$s^* \xrightarrow{\simeq} p_*s_*s^* \xrightarrow{p_*\operatorname{Nm}_s^{-1} s^*} p_*s_!s^* \xrightarrow{p_*c^!_s} p_*$$

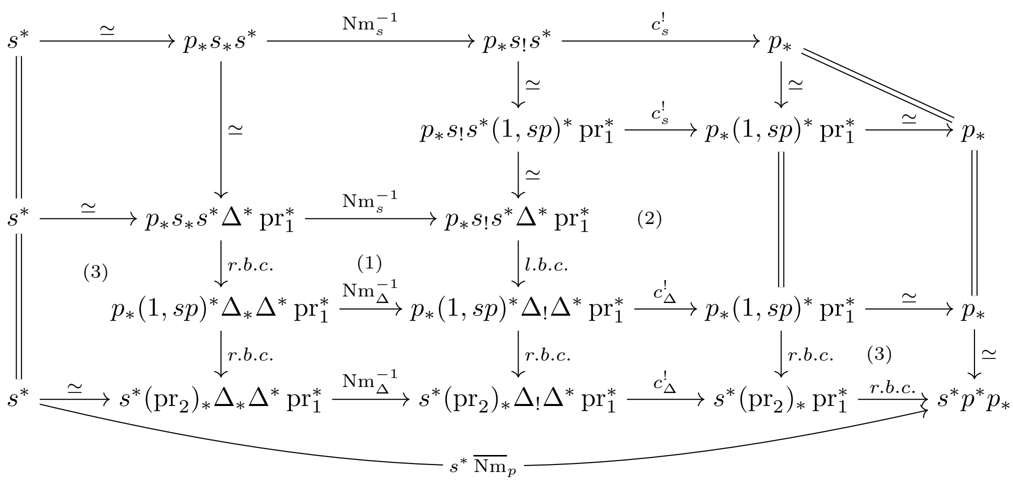
is homotopic to the composite

$$s^* \xrightarrow{s^*\overline{\operatorname{Nm}}_p} s^*p^*p_* \simeq p_*.$$

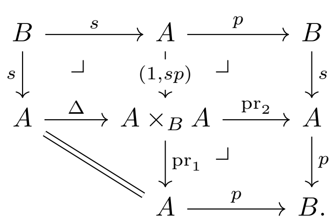
Proof. Recall from Remark 4.3.9 that the map $\overline{\operatorname{Nm}}_p\colon \mathrm{id} \to p^*p_*$ is given by the following composite:

$$\mathrm{id} \simeq \mathrm{pr}_{2*}\Delta_*\Delta^*\,\mathrm{pr}_1^* \xrightarrow{\operatorname{Nm}_\Delta^{-1}} \mathrm{pr}_{2*}\Delta_!\Delta^*\,\mathrm{pr}_1^* \xrightarrow{c^!_\Delta} \mathrm{pr}_{2*}\,\mathrm{pr}_1^* \overset{r.b.c.}{\simeq} p^*p_*.$$

We thus see that the composite $s^* \xrightarrow{s^*\overline{\operatorname{Nm}}_p} s^*p^*p_* \simeq p_*$ is given by the composite along the left, bottom and right in the following large diagram:



The composite along the top of this diagram is the other map appearing in the statement of the lemma, so it will suffice to prove that the diagram commutes. All unlabeled equivalences in this diagram come from identifications on the level of maps in $\mathrm{PSh}(T)$, e.g. we have $p_*s_* \simeq (ps)_* \simeq \mathrm{id}_* \simeq \mathrm{id}$, etcetera. The maps labeled *l.b.c.* and *r.b.c.* are the left/right base change equivalences associated with one of the following three pullback squares in $\mathbb{F}^P_T$:



Except for the squares labelled (1), (2) and (3), all squares in the above diagram commute by naturality. The commutativity of (1) is an instance of Corollary 4.4.4 applied to the previous pullback square exhibiting s as a base change of Δ along $(1, sp)\colon A \to A \times_B A$. The commutativity of (2) follows directly from the definition of the left base change equivalence $s_!s^* \xrightarrow{\sim} (1, sp)^*\Delta_!$, using the triangle identity. Finally, the two squares labeled (3) use that the composite of two right base change equivalences is the right base change equivalence for the composite, which in both cases is just equivalent to the identity. □

4.5. ***P*-semiadditive *T*-∞-categories.** In this section, we will introduce and discuss the notion of a P-semiadditive T-∞-category for a fixed atomic orbital subcategory $P \subseteq T$.

Definition 4.5.1 (cf. [Nar16, Definition 5.3])**.** Let $\mathcal{C}$ be a pointed T-∞-category which admits both finite P-products and finite P-coproducts. We say that $\mathcal{C}$ is P-*semiadditive* if for every morphism $p\colon A \to B$ in $\mathbb{F}_T^P$ the norm map $\mathrm{Nm}_p\colon p_! \Rightarrow p_*$ is an equivalence.

We let $\mathrm{Cat}_T^{P\text{-}\times} \subseteq \mathrm{Cat}_T$ denote the (non-full) subcategory spanned by the T-∞-categories which admit finite P-products and the T-functors which preserve finite P-products. We let $\mathrm{Cat}_T^{P\text{-}\oplus} \subseteq \mathrm{Cat}_T^{P\text{-}\times}$ denote the full subcategory spanned by the P-semiadditive T-∞-categories.

Example 4.5.2. The previous definition applied to the pair $\mathrm{Orb} \subset \mathrm{Glo}$ gives a notion of Orb-semiadditivity for global ∞-categories. We will refer to this as *equivariant semiadditivity.*

It follows directly that also the norm maps for more general morphisms in $\underline{\mathbb{F}}_T^P$ are equivalences:

Corollary 4.5.3. *Let $\mathcal{C}$ be a P-semiadditive T-∞-category, let $B \in \mathrm{PSh}(T)$ and let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$. Then the norm map $\mathrm{Nm}_p\colon p_! \Rightarrow p_*$ is an equivalence.*

Proof. We may write the presheaf B as a colimit $\mathrm{colim}_i\, B_i$ of representables $B_i \in T$, which gives rise to an equivalence of ∞-categories $\mathcal{C}(B) \simeq \lim_i \mathcal{C}(B_i)$. It will thus suffice to show that for every representable $B' \in T$ and any morphism $g\colon B' \to B$ of presheaves, the transformation $g^*\,\mathrm{Nm}_p\colon g^*p_! \Rightarrow g^*p_*$ is an equivalence. By Corollary 4.4.4, it will suffice to show that the transformation $\mathrm{Nm}_{p'}\colon p'_! \Rightarrow p'_*$ is an equivalence, where $p'\colon A \times_B B' \to B'$ is the base change of p along g. Since this base change is a morphism in $\mathbb{F}_T^P$, this holds by assumption on $\mathcal{C}$. □

We will next discuss various alternative characterizations of P-semiadditivity. We start by observing that this condition is self-dual.

Lemma 4.5.4. *Let $\mathcal{C}$ be a pointed T-∞-category. Then the following conditions are equivalent:*

(1) The T-∞-category $\mathcal{C}$ is P-semiadditive;
(2) The opposite T-∞-category $\mathcal{C}^{\mathrm{op}}$ is P-semiadditive;
(3) The T-∞-category $\mathcal{C}$ admits finite P-coproducts and for every morphism $p\colon A \to B$ in $\mathbb{F}_T^P$ the adjoint norm map $\widetilde{\mathrm{Nm}}_p\colon p^*p_! \Rightarrow \mathrm{id}$ is the counit of an adjunction $p^* \dashv p_!$;
(4) The T-∞-category $\mathcal{C}$ admits finite P-products and for every morphism $p\colon A \to B$ in $\mathbb{F}_T^P$ the dual adjoint norm map $\overline{\mathrm{Nm}}_p\colon \mathrm{id} \Rightarrow p^*p_*$ is the unit of an adjunction $p_* \dashv p^*$.

Proof. Observe that the dual adjoint norm map $\overline{\mathrm{Nm}}_p\colon \mathrm{id} \Rightarrow p^*p_*$ may be obtained by applying the construction of the adjoint norm map $\widetilde{\mathrm{Nm}}\colon p^*p_! \Rightarrow \mathrm{id}$ to the T-∞-category $\mathcal{C}^{\mathrm{op}}$. The equivalence between (1) and (2) is then immediate from Lemma 4.4.1. The equivalence between (1) and (3) is clear since the norm map $\mathrm{Nm}_p\colon p_! \Rightarrow p_*$ is adjoint to $\widetilde{\mathrm{Nm}}\colon p^*p_! \Rightarrow \mathrm{id}$. The equivalence between (2) and (4) is obtained dually by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$. ☐

Every choice of an atomic orbital subcategory $P \subseteq T$ gives a different notion of parametrized semiadditivity for a T-∞-category $\mathcal{C}$. The weakest form of parametrized semiadditivity is fiberwise semiadditivity:

Definition 4.5.5. A T-∞-category $\mathcal{C}$ is called *fiberwise semiadditive* if for every $B \in T$ the ∞-category $\mathcal{C}(B)$ is semiadditive and for every morphism $f\colon A \to B$ in T the restriction functor $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ preserves finite biproducts.

Lemma 4.5.6. *Let $\mathcal{C}$ be a pointed T-∞-category which admits fiberwise finite products and coproducts. Then the following three conditions are equivalent:*

(1) The T-∞-category $\mathcal{C}$ is fiberwise semiadditive;
(2) The norm map $\mathrm{Nm}_\nabla\colon \nabla_! \to \nabla_*$ associated to the fold map $\nabla\colon \bigsqcup_{i=1}^n B \to B$ is an equivalence for every $n \geq 0$ and every $B \in T$;
(3) The T-∞-category $\mathcal{C}$ is P-semiadditive for $P = \iota T$, the core of T.

Proof. When $P = \iota T$ is the core of T, any map in $\mathbb{F}_T^P$ is equivalent to a fold map $\nabla\colon \bigsqcup_{i=1}^n B \to B$ for some $B \in T$, and thus the equivalence between (2) and (3) is clear. It remains to show that (1) and (2) are equivalent. The ∞-category $\mathcal{C}(\bigsqcup_{i=1}^n B)$ is equivalent to the n-fold product $\prod_{i=1}^n \mathcal{C}(B)$ of $\mathcal{C}(B)$. Given an object $X = (X_i) \in \prod_{i=1}^n \mathcal{C}(B)$, there are equivalences $\nabla_!(X) \simeq \bigoplus_{i=1}^n X_i$ and $\nabla_*(X) \simeq \prod_{i=1}^n X_i$. By Lemma 4.3.11, the map $\alpha(X)$ is a morphism in $\prod_{i=1}^n \prod_{j=1}^n \mathcal{C}(B)$ which we may visually display as

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \colon \begin{pmatrix} X_1 & X_2 & \dots & X_n \\ X_1 & X_2 & \dots & X_n \\ \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & \dots & X_n \end{pmatrix} \to \begin{pmatrix} X_1 & X_1 & \dots & X_1 \\ X_2 & X_2 & \dots & X_2 \\ \vdots & \vdots & \ddots & \vdots \\ X_n & X_n & \dots & X_n \end{pmatrix},$$

where 1 denotes an identity map while 0 denotes the zero map. In particular, the induced norm map $\mathrm{Nm}_p\colon \bigoplus_{i=1}^n X_i \to \prod_{j=1}^n X_j$ is induced by the family of maps $\{X_i \to X_j\}_{i,j}$ given by the identity when $i = j$ and the zero-map when $i \neq j$. This is precisely the norm map defining ordinary semiadditivity for ∞-categories, finishing the proof. ☐

As the next result shows, the condition of P-semiadditivity for general P is a combination of fiberwise semiadditivity and norm equivalences $\mathrm{Nm}_p\colon p_! \simeq p_*$ for morphisms p in P.

Corollary 4.5.7. *Let $\mathcal{C}$ be a T-∞-category. Then $\mathcal{C}$ is P-semiadditive if and only if it is fiberwise semiadditive and for every morphism $p\colon A \to B$ in P the norm map $\mathrm{Nm}_p\colon p_! \Rightarrow p_*$ is an equivalence.*

Proof. As in the proof of Proposition 4.2.14, every morphism in $\mathbb{F}_T^P$ with representable domain $B \in T$ can be written as a composite $\bigsqcup_{i=1}^n A_i \xrightarrow{\bigsqcup_{i=1}^n p_i} \bigsqcup_{i=1}^n A_i \xrightarrow{\nabla}$

B for morphisms $p_i\colon A_i \to B$ in P, where ∇ denotes the fold map. The norm map of $\bigsqcup_{i=1}^n p_i\colon \bigsqcup_{i=1}^n A_i \to \bigsqcup_{i=1}^n B$ is equivalent to the product of the norm maps for each individual $p_i\colon A_i \to B$. By Corollary 4.4.6, the norm map of a composite morphism can be written as a composite of norm maps, and it follows that $\mathcal{C}$ is P-semiadditive if and only if the norm maps of all the fold maps $\nabla\colon \bigsqcup_{i=1}^n B \to B$ and of all morphisms $p\colon A \to B$ in P are equivalences. But by Lemma 4.5.6 the norm maps for the fold maps are equivalences if and only if $\mathcal{C}$ is fiberwise semiadditive. □

We finish this subsection with a recognition criterion for P-semiadditivity along the lines of [Lur17, Proposition 2.4.3.19].

Proposition 4.5.8. *Let $\mathcal{C}$ be a pointed T-∞-category admitting finite P-products. Assume that for every morphism $p\colon A \to B$ in $\mathbb{F}_T^P$, there is a natural transformation $\mu_p\colon p_*p^* \Rightarrow \mathrm{id}_{\mathcal{C}(B)}$ of functors $\mathcal{C}(B) \to \mathcal{C}(B)$ satisfying the following two conditions:*

(a) for every $X \in \mathcal{C}(B)$, the composite

$$p^*X \xrightarrow{\overline{\mathrm{Nm}}_p\, p^*X} p^*p_*p^*X \xrightarrow{p^*\mu_p X} p^*X$$

is homotopic to the identity;

(b) for every $Y \in \mathcal{C}(A)$, the following diagram commutes

$$\begin{tikzcd} p_*p^*p_*Y \arrow[r, "\simeq", "r.b.c."'] \arrow[dr, "\mu_p p_* Y"'] & p_*(\mathrm{pr}_2)_*\,\mathrm{pr}_1^*\,Y \arrow[r, "\simeq"] & p_*(\mathrm{pr}_1)_*\,\mathrm{pr}_1^*\,Y \arrow[dl, "p_*\mu_{\mathrm{pr}_1} Y"] \\ & p_*Y. & \end{tikzcd}$$

Then the T-∞-category $\mathcal{C}$ is P-semiadditive.

Proof. To show that $\mathcal{C}$ is P-semiadditive, we may by Lemma 4.5.4 equivalently show that for every map $p\colon A \to B$ in $\mathbb{F}_T^P$ and every object $Y \in \mathcal{C}(A)$, the dual adjoint norm map $\overline{\mathrm{Nm}}_p\, Y\colon Y \Rightarrow p^*p_*Y$ exhibits p_*Y as a left adjoint object to Y under the functor $p^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$, i.e. that for every $X \in \mathcal{C}(B)$ the composite

$$\mathrm{Hom}_{\mathcal{C}(B)}(p_*Y, X) \xrightarrow{p^*} \mathrm{Hom}_{\mathcal{C}(A)}(p^*p_*Y, p^*X) \xrightarrow{-\circ\overline{\mathrm{Nm}}_p\, Y} \mathrm{Hom}_{\mathcal{C}(A)}(Y, p^*X)$$

is an equivalence. We claim that an inverse is given by

$$\mathrm{Hom}_{\mathcal{C}(A)}(Y, p^*X) \xrightarrow{p_*} \mathrm{Hom}_{\mathcal{C}(B)}(p_*Y, p_*p^*X) \xrightarrow{\mu_p X\circ -} \mathrm{Hom}_{\mathcal{C}(B)}(p_*Y, X).$$

By naturality of μ_p and Nm_p, it suffices to prove that the following two composites are homotopic to the identity for every *fixed* X:[3]

$$p^*X \xrightarrow{\overline{\mathrm{Nm}}_p\, p^*X} p^*p_*p^*X \xrightarrow{p^*\mu_p X} p^*X,$$

$$p_*Y \xrightarrow{p_*\,\overline{\mathrm{Nm}}_p\, Y} p_*p^*p_*Y \xrightarrow{\mu_p p_* Y} p_*Y.$$

The first composite is homotopic to the identity by condition (a), so we focus on the second composite. Plugging in the description of $\overline{\mathrm{Nm}}_p$ given in Remark 4.3.9,

[3]While this suffices to show that Nm_p is a unit of an adjunction, it does not show that μ_p is the corresponding counit, as we do not provide homotopies that are functorial in X and Y.

this composite expands to

$$\begin{array}{l} p_*Y \xrightarrow{\sim} p_*\mathrm{pr}_{2*}\Delta_*\Delta^*\,\mathrm{pr}_1^*\,Y \xrightarrow[\sim]{\mathrm{Nm}_\Delta^{-1}} p_*\mathrm{pr}_{2*}\Delta_!\Delta^*\,\mathrm{pr}_1^*\,Y \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow c^!_\Delta \\ \qquad\qquad\qquad p_*\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,Y \xrightarrow[\sim]{r.b.c.} p_*p^*p_*Y \xrightarrow{\mu_p p_* Y} p_*Y, \end{array}$$

which, using condition (b) and the equivalence $p \circ \mathrm{pr}_1 \simeq p \circ \mathrm{pr}_2$, is homotopic to the composite

$$p_*Y \simeq p_*\mathrm{pr}_{1*}\Delta_*\Delta^*\,\mathrm{pr}_1^*\,Y \xrightarrow{\mathrm{Nm}_\Delta^{-1}} p_*\mathrm{pr}_{1*}\Delta_!\Delta^*\,\mathrm{pr}_1^*\,Y \xrightarrow{c^!_\Delta} p_*\mathrm{pr}_{1*}\,\mathrm{pr}_1^*\,Y \xrightarrow{p_*\mu_{\mathrm{pr}_1}Y} p_*Y.$$

Applying Lemma 4.4.9 to the map $\mathrm{pr}_1 \colon A \times_B A \to A$ with section $\Delta\colon A \to A \times_B A$, we see that this map is homotopic to the following composite:

$$p_*Y \simeq p_*\Delta^*\,\mathrm{pr}_1^*\,Y \xrightarrow{p_*\Delta^*\,\overline{\mathrm{Nm}}_{\mathrm{pr}_1}\,\mathrm{pr}_1^*\,Y} p_*\Delta^*\,\mathrm{pr}_1^*\,\mathrm{pr}_{1*}\,\mathrm{pr}_1^*\,Y \xrightarrow{p_*\Delta^*\,\mathrm{pr}_1^*\,\mu_{\mathrm{pr}_1}Y} p_*\Delta^*\,\mathrm{pr}_1^*\,Y \simeq p_*Y.$$

This map is homotopic to the identity by assumption (a) applied to the map $\mathrm{pr}_1 \colon A \times_B A \to A$, finishing the proof. □

4.6. ***P*-semiadditive *T*-functors.** We continue to fix an atomic orbital subcategory $P \subseteq T$. In this subsection we will define what it means for a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ to be P-*semiadditive*: roughly speaking, it means that F turns finite P-coproducts in $\mathcal{C}$ into finite P-products in $\mathcal{D}$. The main result of this subsection is Proposition 4.6.13, which states that the T-subcategory $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C},\mathcal{D})$ of $\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})$ spanned by the P-semiadditive T-functors is P-semiadditive.

We start by constructing a 'relative' variant of the norm map.

Construction 4.6.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a T-functor such that $\mathcal{C}$ is pointed and admits finite P-coproducts and $\mathcal{D}$ admits finite P-products, let $B \in \mathrm{PSh}(T)$ and let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$. We define the *norm transformation of* p *relative to* F

$$\mathrm{Nm}_p^F \colon F_B \circ p_! \implies p_* \circ F_A$$

as the transformation adjoint to the composite $p^*F_Bp_! \simeq F_Ap^*p_! \xRightarrow{F_A\,\widetilde{\mathrm{Nm}}_p^{\mathcal{C}}} F_A$, where the first equivalence uses that the parametrized functor $F\colon \mathcal{C} \to \mathcal{D}$ commutes with the restriction functors.

Note that when $\mathcal{D}$ is equal to $\mathcal{C}$ and F is the identity on $\mathcal{C}$, the transformation Nm_p^F reduces to the norm transformation $\mathrm{Nm}_p^{\mathcal{C}}\colon p_! \Rightarrow p_*$ of Construction 4.3.8.

Definition 4.6.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a T-functor such that $\mathcal{C}$ is pointed and admits finite P-coproducts and $\mathcal{D}$ admits finite P-products. We will say that F is P-*semiadditive* if it satisfies the following condition:

(∗) For each morphism $p\colon A \to B$ in $\mathbb{F}_T^P$, the transformation $\mathrm{Nm}_p^F\colon F_B \circ p_! \Rightarrow p_* \circ F_A$ defined in Construction 4.6.1 is a natural equivalence.

By Example 4.2.8(1) we also obtain a notion of P-semiadditive $T_{/B}$-functors for all $B \in T$. Note that $\mathcal{C}$ is P-semiadditive if and only if the identity $\mathrm{id}\colon \mathcal{C} \to \mathcal{C}$ is P-semiadditive. Also note that condition (∗) specializes for $A = \emptyset$ to the condition that the functor $F_B\colon \mathcal{C}(B) \to \mathcal{D}(B)$ sends the zero object of $\mathcal{C}(B)$ to the final object of $\mathcal{D}(B)$.

Just like in Corollary 4.5.3, one immediately deduces that the relative norm maps are equivalnces for arbitrary morphisms in $\underline{\mathbb{F}}_T^P$:

Corollary 4.6.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ as in Construction 4.6.1 and assume that F is P-semiadditive. Then for every $B \in \mathrm{PSh}(T)$ and every $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$, the transformation $\mathrm{Nm}_p^F\colon F_B \circ p_! \Rightarrow p_* \circ F_A$ is an equivalence.* □

While not necessary for our work, we show for completeness that our norm map generalizes the analogous construction in [Nar16].

Proposition 4.6.4. *Let T be an atomic orbital ∞-category, let $B \in T$ and let $p\colon A \to B$ be a morphism in $\mathbb{F}_T$. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a T-functor with $\mathcal{C}$ and $\mathcal{D}$ satisfying the assumptions of Construction 4.6.1. Then the norm transformation $\mathrm{Nm}_p^F\colon F_B \circ p_! \Rightarrow p_* \circ F_A$ of Construction 4.6.1 is homotopic to the transformation defined in* [Nar16, Construction 5.2].

Proof. We will first give an alternative description of the norm map in this special case, and then argue why it agrees with the construction of Nardin. By definition of $\mathbb{F}_T$, we may assume $p\colon A \to B$ to be of the form $p = (p_i)\colon \bigsqcup_{i=1}^n A_i \to B$, where each $A_i \in T$ is representable. Let $\iota_i\colon A_i \hookrightarrow \bigsqcup_{i=1}^n A_i = A$ denote the canonical inclusion, so that $p_i = p \circ \iota_i\colon A_i \to B$. The functor $p_*\colon \mathcal{D}(A) \to \mathcal{D}(B)$ may be decomposed as

$$\mathcal{D}(A) = \mathcal{D}(\bigsqcup_{i=1}^n A_i) \xrightarrow[\simeq]{(\iota_i^*)_i} \prod_{i=1}^n \mathcal{D}(A_i) \xrightarrow{\prod_{i=1}^n p_{i*}} \prod_{i=1}^n \mathcal{D}(B) \xrightarrow{\prod} \mathcal{D}(B),$$

where the last map denotes the multiplication in $\mathcal{D}(B)$. For an object $X = (X_i) \in \mathcal{C}(A) \simeq \prod_{i=1}^n \mathcal{C}(A_i)$ the norm map $\mathrm{Nm}_p^F(X)\colon F_B(p_!(X)) \to p_*F_A(X) \simeq \prod_{i=1}^n p_{i*}(F_{A_i}(X_i))$ is the product of n maps $F_B(p_!(X)) \to p_{i*}(F_{A_i}(X_i))$, where the i-th one is obtained by adjunction from the composite

$$p_i^*F_B(p_!(X)) \simeq F_{A_i}p_i^*p_!X \simeq F_{A_i}\iota_i^*p^*p_!X \xrightarrow{F_{A_i}\iota_i^*\widetilde{\mathrm{Nm}}_p} F_{A_i}\iota_i^*X = F_{A_i}X_i.$$

We will now expand the definition of the map $\iota_i^*\widetilde{\mathrm{Nm}}_p\colon p_i^*p_!X \to X_i$. First notice that the map $\widetilde{\mathrm{Nm}}_p\colon p^*p_!X \to X$ is given by the following composite:

$$p^*p_!X \overset{l.b.c.}{\simeq} \mathrm{pr}_{1!}\,\mathrm{pr}_2^*\,X \xRightarrow{u_\Delta^*} \mathrm{pr}_{1!}\Delta_*\Delta^*\,\mathrm{pr}_2^*\,X \xRightarrow[\simeq]{\mathrm{Nm}_\Delta^{-1}} \mathrm{pr}_{1!}\Delta_!\Delta^*\,\mathrm{pr}_2^*\,X \simeq X.$$

Applying left base change to the pullback diagram

$$\begin{array}{ccccc} A_i \times_B A & \xrightarrow{\iota_i \times_B A} & A_i \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ A_i & \xrightarrow{\iota_i} & A_i & \xrightarrow{p_i} & B \end{array}$$

gives an equivalence $p_i^*p_!X \simeq \mathrm{pr}_{1!}\,\mathrm{pr}_2^*\,X$. Since T is atomic, the diagonal $\Delta_{p_i}\colon A_i \to A_i \times_B A_i \hookrightarrow \bigsqcup_{i=1}^n A_i \times_B A_i = A_i \times_B A$ is a disjoint summand inclusion. Writing $g\colon C \to A_i \times_B A$ for the complement summand, we observe that $\mathcal{C}(A_i \times_B A) = \mathcal{C}(A_i \sqcup C) \simeq \mathcal{C}(A_i) \times \mathcal{C}(C)$ and that the object $\mathrm{pr}_2^*\,X \in \mathcal{C}(A_i \times_B A)$ corresponds to the pair (X_i, X_C) for some $X_C \in \mathcal{C}(C)$. Plugging in the map $X_C \to *$ to the zero-object $*$ of $\mathcal{C}(C)$ thus gives a map $\mathrm{pr}_{1!}(X_i, X_C) \to \mathrm{pr}_{1!}(X_i, *) \simeq X_i$. Looking at the construction of Nm_Δ in Lemma 4.1.4, one sees that the resulting composite $p_i^* p_! X \to X_i$ is precisely $\iota_i^* \widetilde{\mathrm{Nm}}_p$.

One may now observe that this second description of the norm map is precisely the construction of [Nar16], after making the following translations in notation:

$$B \leftrightarrow V, \qquad A \leftrightarrow U, \qquad p \leftrightarrow I, \qquad \bigsqcup_{i=1}^n A_i \leftrightarrow \bigsqcup_{W \in \mathrm{Orbit(U)}} W,$$
$$p_! \leftrightarrow \bigsqcup_I, \qquad p_i^* \leftrightarrow \delta_{W/V}, \qquad \iota_i \leftrightarrow (W \subseteq U), \qquad \iota_i^* \widetilde{\mathrm{Nm}}_p \leftrightarrow (\chi_{[W\subseteq U]})_*.$$

This finishes the proof. □

Next, we will show that the P-semiadditive T-functors from $\mathcal{C}$ to $\mathcal{D}$ form a parametrized subcategory of $\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})$. This will rely on the following general criterion in the spirit of Lemma 2.3.17:

Lemma 4.6.5. *Let $f\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ be a cocontinuous functor that preserves pullbacks. Let $P \subset S$ and $Q \subset T$ be atomic orbital subcategories and assume that for every $p\colon A \to B$ in P we have $(f(p)\colon f(A) \to f(B)) \in \underline{\mathbb{F}}_T^Q(f(B))$. Then:*

1. *(1) The functor $f^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ sends (pointed) T-∞-categories with finite Q-coproducts to (pointed) S-∞-categories with finite P-coproducts, and dually for finite Q-products and finite P-products.*
2. *(2) If $F\colon \mathcal{C} \to \mathcal{D}$ is a T-functor such that $\mathcal{C}$ is pointed with finite Q-coproducts and $\mathcal{D}$ has finite Q-products, then the relative norm map $\mathrm{Nm}_p^{f^*F}$ for any $B \in \mathrm{PSh}(T), p \in \underline{\mathbb{F}}_S^P(p)$ agrees with the relative norm map $\mathrm{Nm}_{f(p)}^F$.*
3. *(3) The functor $f^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ sends Q-semiadditive T-categories to P-semiadditive S-categories and Q-semiadditive T-functors to P-semiadditive S-functors.*

Proof. It is clear that f^* preserves pointedness. Moreover, as f preserves coproducts, it more generally maps $\underline{\mathbb{F}}_S^P(B)$ into $\underline{\mathbb{F}}_T^Q(f(B))$, so part (1) is an instance of Lemma 2.3.17 and its dual. Part (2) follows similarly by direct inspection of the construction of the norm maps, and (3) is an immediate consequence of (2). □

Definition 4.6.6. Let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories such that $\mathcal{C}$ is pointed and admits finite P-coproducts and $\mathcal{D}$ admits finite P-products. We define $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C},\mathcal{D})$ as the full subcategory $\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})$ spanned at level $B \in T$ by the P-semiadditive $T_{/B}$-functors $F\colon \pi_B^*\mathcal{C} \to \pi_B^*\mathcal{D}$ for $B \in T$.

This does indeed form a T-subcategory by the previous lemma applied to the maps $T_{/f}\colon T_{/A} \to T_{/B}$ for all $f\colon A \to B$ in T, cf. the proof of Lemma 2.3.23.

We think of a P-semiadditive T-functor as a functor which sends finite P-coproducts to finite P-products. Hence we expect that this condition should be preserved when precomposing (resp. postcomposing) with a T-functor which preserves finite P-coproducts (resp. finite P-products). The following result shows that this is indeed the case.

Proposition 4.6.7. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a P-semiadditive T-functor, where $\mathcal{C}$ and $\mathcal{D}$ are as in Definition 4.6.2, and let $(p\colon A \to B) \in \underline{\mathbb{F}}_T^P(B)$.*

(1) Let $\mathcal{C}'$ be another pointed T-category admitting finite P-coproducts and let $G\colon \mathcal{C}' \to \mathcal{C}$ be a pointed T-functor which preserves finite P-coproducts. Then the norm map $\mathrm{Nm}_p^{FG}\colon F_B G_B p_! \Rightarrow p_* F_B G_B$ of FG with respect to p is given by the composite

$$\begin{tikzcd} \mathcal{C}'(A) \arrow[r, "G_A"] \arrow[d, "p_!"'] & \mathcal{C}(A) \arrow[r, "F_A"] \arrow[d, "p_!"'] & \mathcal{D}(A) \arrow[d, "p_*"] \\ \mathcal{C}'(B) \arrow[r, "G_B"'] \arrow[ur, Rightarrow, "\mathrm{BC}_!^{-1}"] & \mathcal{C}(B) \arrow[r, "F_B"'] \arrow[ur, Rightarrow, "\mathrm{Nm}_p^F"] & \mathcal{D}(B), \end{tikzcd}$$

where $\mathrm{BC}_!\colon p_! G(A) \xrightarrow{\simeq} G(B) p_!$ denotes the Beck-Chevalley equivalence of G. In particular the composite $F \circ G\colon \mathcal{C}' \to \mathcal{D}$ is again P-semiadditive.

(2) Let $\mathcal{D}'$ be another T-∞-category which admits finite P-products and let $H\colon \mathcal{D} \to \mathcal{D}'$ be a T-functor which preserves finite P-products. Then the norm map $\mathrm{Nm}_p^{HF}\colon H_B F_B p_! \Rightarrow p_* H_B F_B$ of HF at p is given by the composite

$$\begin{tikzcd} \mathcal{C}(A) \arrow[r, "F_A"] \arrow[d, "p_!"'] & \mathcal{D}(A) \arrow[r, "H_A"] \arrow[d, "p_*"] & \mathcal{D}'(A) \arrow[d, "p_*"] \\ \mathcal{C}(B) \arrow[r, "F_B"'] \arrow[ur, Rightarrow, "\mathrm{Nm}_p^F"] & \mathcal{D}(B) \arrow[r, "H_B"'] \arrow[ur, Rightarrow, "\mathrm{BC}_*"] & \mathcal{D}'(B), \end{tikzcd}$$

where $\mathrm{BC}_*\colon H(A) p_* \xrightarrow{\simeq} p_* H(A)$ denotes the Beck-Chevalley equivalence of H. In particular the composite $H \circ F\colon \mathcal{C} \to \mathcal{D}$ is again P-semiadditive.

Proof. The description of Nm_p^{FG} follows from the commutative diagram

$$\begin{tikzcd} F_B G_B p_! \arrow[r, "u_p^*"] \arrow[d, "\mathrm{BC}_!^{-1}"'] & p_* p^* F_B G_B p_! \arrow[r, "\simeq"] \arrow[d, "\mathrm{BC}_!^{-1}"] & p_* F_A p^* G_B p_! \arrow[r, "\simeq"] \arrow[d, "\mathrm{BC}_!^{-1}"] & p_* F_A G_A p^* p_! \arrow[d, "p_* F_A G_A \widetilde{\mathrm{Nm}}_p"] \\ F_B p_! G_A \arrow[r, "u_p^*"] \arrow[rrr, bend right, "\mathrm{Nm}_p^F\, G_A"'] & p_* p^* F_B p_! G_A \arrow[r, "\simeq"] & p_* F_A p^* p_! G_A \arrow[r, "p_* F_A \widetilde{\mathrm{Nm}}_p\, G_A"] & p_* F_A G_A. \end{tikzcd}$$

The middle and left square commute by naturality, and the right square by Lemma 4.4.8. The description of Nm_p^{HF} follows from the commutative diagram

$$\begin{tikzcd} H_B F_B p_! \arrow[r, "H_B u_p^*"'] \arrow[rrr, bend left, "H_B \mathrm{Nm}_p^F"] \arrow[d, "u_p^*"'] & H_B p_* p^* F_B p_! \arrow[r, "\simeq"] \arrow[d, "\mathrm{BC}_*"', "\simeq"] & H_B p_* F_A p^* p_! \arrow[r, "H_B p_* F_A \widetilde{\mathrm{Nm}}_p"'] \arrow[d, "\mathrm{BC}_*"', "\simeq"] & H_B p_* F_A \arrow[d, "\simeq"', "\mathrm{BC}_*"] \\ p_* p^* H_B F_B p_! \arrow[r, "\simeq"'] & p_* H_A p^* F_B p_! \arrow[r, "\simeq"'] & p_* H_A F_A p^* p_! \arrow[r, "p_* H_A F_A \widetilde{\mathrm{Nm}}_p"'] & p_* H_A F_A, \end{tikzcd}$$

where the middle and right square commute by naturality while the left-most square commutes by definition of the Beck-Chevalley equivalence BC_* and the triangle identity. ☐

Corollary 4.6.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories such that $\mathcal{C}$ is pointed and admits finite P-coproducts, and $\mathcal{D}$ admits finite P-products. Then post-composition with the forgetful functor $\mathcal{D}_* \to \mathcal{D}$ induces an equivalence of T-∞-categories*

$$\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D}_*) \xrightarrow{\sim} \underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D}).$$

Proof. By Corollary 4.1.8 it remains to show that a pointed T-functor $\mathcal{C} \to \mathcal{D}_*$ is P-semiadditive if and only if its composition with $\mathcal{D}_* \to \mathcal{D}$ is P-semiadditive. This follows from Proposition 4.6.7 since the T-functor $\mathcal{D}_* \to \mathcal{D}$ is conservative and preserves T-limits by Lemma 4.1.10. □

Corollary 4.6.9. *Let $\mathcal{C}$ be a pointed T-∞-category which admits finite P-coproducts and let $\mathcal{D}$ be a T-∞-category which admits finite P-products. Let $B \in T$ and consider a $T_{/B}$-functor $F\colon \pi_B^* \mathcal{C} \to \mathcal{D}$. Then $\pi_B^* \mathcal{C}$ is pointed with finite P-coproducts, $\pi_{B*} \mathcal{D}$ has finite P-products, and F is P-semiadditive if and only if the corresponding functor $\widetilde{F}\colon \mathcal{C} \to \pi_{B*} \mathcal{D}$ is P-semiadditive.*

Proof. This follows from Lemma 4.6.5 and Proposition 4.6.7 by the same arguments as in the proof of Proposition 2.3.28. □

Corollary 4.6.10. *Let $\mathcal{C}$ be a pointed T-∞-category with finite coproducts, let $\mathcal{D}$ be a T-∞-category with finite products, and let $X \in \mathrm{PSh}(T)$ arbitrary. Then $(F\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})) \in \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(X)$ defines an object of $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})(X)$ if and only if it is P-semiadditive.*

Proof. If X is representable, this is an instance of the previous proposition. In the general case, we then simply observe analogously to the proof of Proposition 2.3.28 that the functors $\underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})$ for maps $\underline{A} \to \underline{X}$ with $A \in T$ are jointly conservative and preserve finite P-products, so that the claim follows from Proposition 4.6.7. □

Lemma 4.6.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories such that $\mathcal{C}$ is pointed and admits finite P-coproducts and $\mathcal{D}$ admits finite P-products. Let $\mathbf{U}$ be a class of T-∞-categories, and assume that $\mathcal{D}$ admits $\mathbf{U}$-limits. Then the T-∞-category $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})$ also admits $\mathbf{U}$-limits and the inclusion $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D}) \hookrightarrow \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ preserves $\mathbf{U}$-limits.*

Proof. First note that the T-∞-category $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ admits $\mathbf{U}$-limits by Proposition 2.3.24. Let $K \in \mathbf{U}(B)$ be a $T_{/B}$-∞-category in $\mathbf{U}$, and let $F\colon \pi_B^* \mathcal{C} \to \underline{\mathrm{Fun}}_T(K, \pi_B^* \mathcal{D})$ be a P-semiadditive $T_{/A}$-functor. We need to show that the $T_{/B}$-functor $\lim_K F\colon \pi_B^* \mathcal{C} \to \pi_B^* \mathcal{D}$ is again P-semiadditive. To simplify the notation, we will assume that B is the final object of T by replacing T by $T_{/B}$, and thus we may identify $\pi_B^* \mathcal{C}$ and $\pi_B^* \mathcal{D}$ with $\mathcal{C}$ and $\mathcal{D}$, respectively. Since parametrized limits in $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ are computed pointwise by Proposition 2.3.24, the functor $\lim_K F\colon \mathcal{C} \to \mathcal{D}$ is given by the composite

$$\mathcal{C} \xrightarrow{F} \underline{\mathrm{Fun}}_T(K, \mathcal{D}) \xrightarrow{\lim_K} \mathcal{D}\,.$$

Note that the T-functor $\lim_K\colon \underline{\mathrm{Fun}}_T(K, \mathcal{D}) \to \mathcal{D}$, being right adjoint to the diagonal $\mathcal{D} \to \underline{\mathrm{Fun}}_T(K, \mathcal{D})$, preserves all parametrized limits and thus in particular all finite P-products. It then follows from Proposition 4.6.7 that $\lim_K F$ is P-semiadditive as desired. □

Corollary 4.6.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be pointed T-∞-categories admitting finite P-coproducts, and let $\mathcal{E}$ be a T-∞-category admitting finite P-products. Then the composite equivalence*

$$\underline{\mathrm{Fun}}_T(\mathcal{C}, \underline{\mathrm{Fun}}_T(\mathcal{D}, \mathcal{E})) \simeq \underline{\mathrm{Fun}}_T(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \underline{\mathrm{Fun}}_T(\mathcal{D}, \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{E}))$$

restricts to an equivalence

$$\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{D}, \mathcal{E})) \simeq \underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{D}, \underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{E})).$$

Proof. It follows immediately from Lemma 4.6.11 and Proposition 2.3.24 that both sides correspond to the full subcategory of $\underline{\mathrm{Fun}}_T(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by those T-functors which are P-semiadditive in both variables. Here we say a T-functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is *P-semiadditive in both variables* if for every $B \in T$ and $X : \underline{B} \to \mathcal{C}$, the T-functor

$$F(X, -)\colon \mathcal{D} \to \underline{\mathrm{Fun}}_T(\underline{B}, \mathcal{E})$$

adjoint to the composite $\underline{B} \times \mathcal{D} \xrightarrow{X \times \mathcal{D}} \mathcal{C} \times \mathcal{D} \xrightarrow{F} \mathcal{E}$ is P-semiadditive and similarly for every $Y : \underline{B} \to \mathcal{D}$ the T-functor

$$F(-, Y)\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{B}, \mathcal{E})$$

adjoint to $\mathcal{C} \times \underline{B} \xrightarrow{\mathcal{C} \times Y} \mathcal{C} \times \mathcal{D} \xrightarrow{F} \mathcal{E}$ is P-semiadditive. □

We now come to the main result of this subsection: the P-semiadditivity of the T-∞-category $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})$.

Proposition 4.6.13 (cf. [Nar16, Proposition 5.8]). *Let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories such that $\mathcal{C}$ is pointed and admits finite P-coproducts and $\mathcal{D}$ admits finite P-products. Then the T-∞-category $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})$ is P-semiadditive.*

Proof. By Corollary 4.6.8, we may assume that $\mathcal{D}$ is pointed. It follows from Corollary 4.1.9 that $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})$ is pointed and from Lemma 4.6.11 that $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})$ admits finite P-products. These are computed pointwise, meaning that for $p\colon A \to B$ in $\mathbb{F}_T^P$ the map

$$p_*\colon \underline{\mathrm{Fun}}^{P\text{-}\oplus}(\mathcal{C}, \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})) \to \underline{\mathrm{Fun}}^{P\text{-}\oplus}(\mathcal{C}, \underline{\mathrm{Fun}}_T(\underline{B}, \mathcal{D}))$$

is given by post-composition with $p_*\colon \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\underline{B}, \mathcal{D})$.

To show that $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})$ is P-semiadditive, we will apply the recognition principle from Proposition 4.5.8. For every morphism $p\colon A \to B$ in $\mathbb{F}_T^P$ and every P-semiadditive $T_{/B}$-functor $G\colon \pi_B^* \mathcal{C} \to \pi_B^* \mathcal{D}$, we define a natural transformation $\mu_p G\colon p_* p^* G \to G$. For notational simplicity, we will construct this in the case where $B = 1$ is a terminal object of T; the general case is obtained by replacing T by $T_{/B}$. In this case, $\mu_p G$ is defined as the following composite:

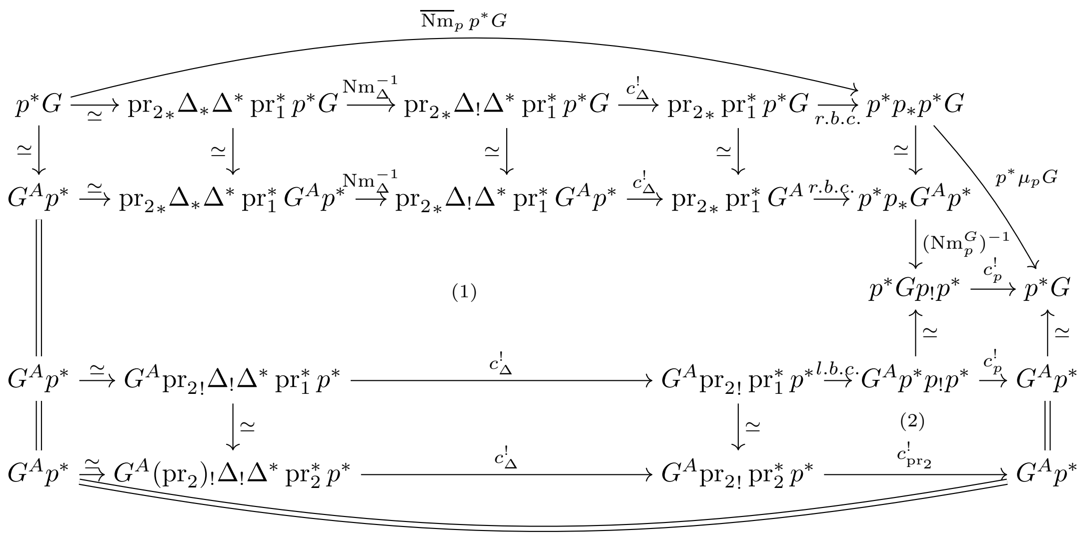
$$p_* p^* G \simeq p_* G^A p^* \xrightarrow{(\mathrm{Nm}_p^G)^{-1}} G p_! p^* \xrightarrow{G c_p^!} G;$$

here we denote by $G^A\colon \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{C}) \to \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})$ the T-functor induced by G. We need to check that conditions (a) and (b) of Proposition 4.5.8 are satisfied. Condition (b) follows directly from the definitions, using Proposition 4.6.7(2) to compute the norm map of $p_* F$ in terms of the norm map of F and the right base change equivalence $p^* p_* \simeq (\mathrm{pr}_2)_* \mathrm{pr}_1^*$. For condition (a), we need to show that for every P-semiadditive T-functor $G\colon \mathcal{C} \to \mathcal{D}$, the composite

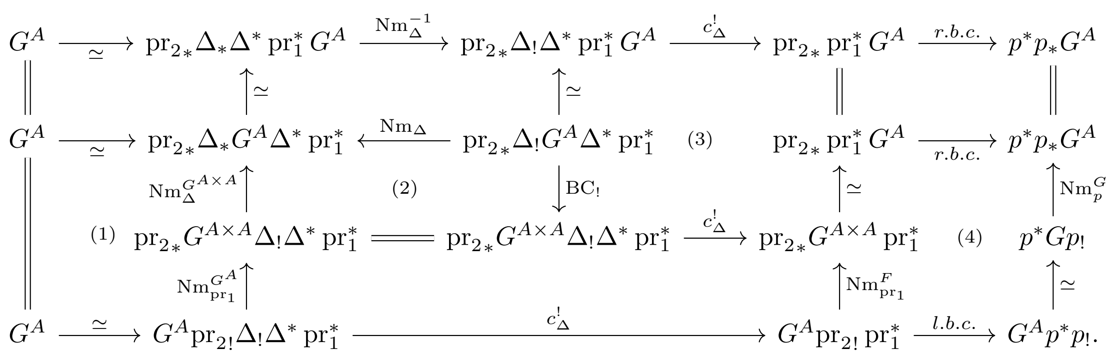
$$p^* G \xrightarrow{\overline{\mathrm{Nm}}_p\, p^* G} p^* p_* p^* G \xrightarrow{p^* \mu_p G} p^* G$$

is homotopic to the identity in $\mathrm{Fun}_{T/A}^{P\text{-}\sqcup}(\pi_A^* \mathcal{C}, \pi_A^* \mathcal{D}) \simeq \mathrm{Fun}_T^{P\text{-}\oplus}(\mathcal{C}, \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D}))$. Observe that pointedness of $\mathcal{D}$ guarantees that the transformation $\overline{\mathrm{Nm}}_p\, p^* G\colon p^* G \to p^* p_* p^* G$ is given by whiskering $p^* G$ with the transformation $\overline{\mathrm{Nm}}_p^{\mathcal{D}}\colon \mathrm{id} \to p^* p_*$.

Spelling out the definitions, we are therefore interested in the composite along the top right in the following diagram:



As the composite along the bottom left is the identity, it remains to show that this diagram commutes. Except for (1) and (2), all squares commute either by definition or by naturality, and the commutativity of square (2) follows from the triangle identity. The commutativity of (1) follows from the following commutative diagram:



The unlabeled squares commute by naturality. The fact that (1) commutes follows from Corollary 4.4.6, while the commutativity of (4) follows from Corollary 4.4.4. The commutativity of (2) and (3) easily follows from the definitions. This finishes the proof. □

Proposition 4.6.14. *Let $\mathcal{C}$ be a pointed T-∞-category which admits finite P-coproducts, and suppose $\mathcal{D}$ is P-semiadditive. Then a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ is P-semiadditive if and only if it preserves finite P-coproducts. In particular we get that $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C},\mathcal{D})$ and $\underline{\mathrm{Fun}}_T^{P\text{-}\sqcup}(\mathcal{C},\mathcal{D})$ are the same subcategory of $\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})$.*

Analogously, suppose $\mathcal{C}$ is a P-semiadditive T-∞-category, and suppose $\mathcal{D}$ admits finite P-products. Then a T-functor $G\colon \mathcal{C} \to \mathcal{D}$ is P-semiadditive if and only if it *preserves finite P-products. In particular $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C},\mathcal{D})$ and $\underline{\mathrm{Fun}}_T^{P\text{-}\times}(\mathcal{C},\mathcal{D})$ are the same subcategory of $\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})$.*

Proof. We start with the first case. Observe that in both cases F is pointed so that Lemma 4.4.8 applies. Adjoining over p^* to the right gives a commutative triangle

$$\begin{array}{ccc} F_B p_! & & \\ {\scriptstyle \mathrm{BC}_!}\uparrow & \searrow^{\mathrm{Nm}_p^F} & \\ p_! F_A & \xrightarrow[\mathrm{Nm}_p^{\mathcal{D}} F_A]{} & p_* F_A. \end{array}$$

Since $\mathcal{D}$ is a P-semiadditive, the bottom map is an equivalence. It thus follows from the two-out-of-three property that $BC_!\colon p_!F_A \Rightarrow F_B p_!$ is an equivalence if and only if $\mathrm{Nm}_p^F : F_B p_! \Rightarrow p_* F_A$ is, proving the result.

Next we consider the second case. Just as before the result follows from the commutativity of the triangle

$$\begin{array}{ccc} & & p_* F_A \\ & \nearrow^{\mathrm{Nm}_p^F} & \uparrow{\scriptstyle \mathrm{BC}_*} \\ F_B p_! & \xrightarrow[F_B\,\mathrm{Nm}_p^{\mathcal{C}}]{} & F_B p_*, \end{array}$$

which in turn follows from the commutative diagram

$$\begin{array}{ccccc} p_* p^* F_B p_! & \xrightarrow{\simeq} & p_* F_A p^* p_! & \xrightarrow{\widetilde{\mathrm{Nm}}_p^{\mathcal{C}}} & p_* F_A \\ \uparrow{\scriptstyle u_p^*} & & \uparrow{\scriptstyle \mathrm{BC}_*} & & \uparrow{\scriptstyle \mathrm{BC}_*} \\ F_B p_! & \xrightarrow[u_p^*]{} & F_B p_* p^* p_! & \xrightarrow[\widetilde{\mathrm{Nm}}_p^{\mathcal{C}}]{} & F_B p_* \end{array}$$

The left square commutes by the triangle identity and the right by naturality. □

Corollary 4.6.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be P-semiadditive T-∞-categories. Then a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves finite P-coproducts if and only if it preserves finite P-products.* □

There exists a characterization of P-semiadditivity which does not make reference to the norm maps: it suffices for finite P-products to commute with finite P-coproducts.

Corollary 4.6.16. *Let $\mathcal{C}$ be a pointed T-∞-category which admits finite P-products and finite P-coproducts. Then the following conditions are equivalent:*

(1) The T-∞-category $\mathcal{C}$ is P-semiadditive
(2) For every morphism $p\colon A \to B$ in $\mathbb{F}_T^P$, the $T_{/B}$-functor

$$p_*\colon \underline{\mathrm{Fun}}_{T_{/B}}(\underline{A}, \pi_B^*\,\mathcal{C}) \to \pi_B^*\,\mathcal{C}$$

preserves finite P-coproducts.

Proof. Suppose $\mathcal{C}$ is P-semiadditive. Then so are the $T_{/B}$-∞-categories $\pi_B^*\,\mathcal{C}$ and $\mathrm{Fun}_{T_{/B}}(\underline{A}, \pi_B^*\,\mathcal{C})$. Given a morphism $p\colon A \to B$, the $T_{/B}$-functor p_* is a right adjoint of p^* so preserves finite P-products. By Corollary 4.6.15, it follows that p_* also preserves finite P-coproducts, proving that (1) implies (2).

Conversely, applying (2) to the finite P-coproduct $p_!$ gives that the double Beck-Chevalley map $p_!\mathrm{pr}_{2*} \Rightarrow p_*\mathrm{pr}_{1!}$ associated to the pullback square (9) is an equivalence. It thus follows from Lemma 4.4.2 that the norm map Nm_p is an equivalence, showing that (2) implies (1). □

We finish this subsection by observing that passing to the T-∞-category of P-semiadditive T-functors out of a small T-∞-category $\mathcal{C}$ preserves presentability.

Proposition 4.6.17. *Let $\mathcal{C}$ be a small pointed T-∞-category which admits finite P-coproducts. Let $\mathcal{D}$ be a presentable T-∞-category, so that $\mathcal{D}$ in particular admits finite P-products by Remark 2.4.2. Then the T-∞-category $\underline{\mathrm{Fun}}^{P\text{-}\oplus}(\mathcal{C},\mathcal{D})$ is again presentable and the inclusion*

$$\underline{\mathrm{Fun}}^{P\text{-}\oplus}(\mathcal{C},\mathcal{D}) \subset \underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$$

admits a left adjoint.

Proof. We will exhibit $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C},\mathcal{D})$ as the T-∞-category of S-local objects for a parametrized family S of morphisms in $\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})$ (i.e. a set $S(B)$ of morphisms of $\mathrm{Fun}_{T_{/B}}(\pi_B^*\mathcal{C},\pi_B^*\mathcal{D})$ for every $B\in T$ which are closed under restriction). Then Example 2.4.6 implies both statements of the proposition. Since we may prove the statement after pulling back to every slice of T, we may assume without loss of generality that T has a final object. We will describe a set $S'(1)$ of morphisms in $\mathrm{Fun}_T(\mathcal{C},\mathcal{D})$ such that F is P-semiadditive if and only if F is S'-local; the set $S'(B)$ at any other object $B\in T$ is given by the analogous procedure applied to the slice $T_{/B}$. We then define $S(B)$ to be the union of the restriction of $S'(A)$ along every map $A\to B$ in T. Note that a functor F is $S(A)$-local if and only if f_*F is $S'(B)$-local for every $f\colon A\to B$ in T. By Lemma 4.6.11 this is equivalent to F being $S'(A)$-local.

By definition, a T-functor $F\colon \mathcal{C}\to\mathcal{D}$ is T-semiadditive if and only if it preserves T-final objects and the norm map $\mathrm{Nm}_p\colon F_B\circ p_! \Rightarrow p_*\circ F_A$ is an equivalence for every $p\colon A\to B$ in $\mathbb{F}_T^P$. By presentability of $\mathcal{D}(B)$, there exists a set $\{d_i\}$ of generating objects of $\mathcal{D}(B)$ for every $B\in\mathbb{F}_T$, which we may assume to be closed under restriction along maps in $\mathbb{F}_T$. It follows that F is semiadditive if and only if for every morphism $p\colon A\to B$ in $\mathbb{F}_T^P$, every generator $d_i\in\mathcal{D}(B)$ and every $x\in\mathcal{C}(A)$ the following two maps of spaces are equivalences:

(1) $\mathrm{Hom}_{\mathcal{D}(B)}(d_i,F_B(*))\to\mathrm{Hom}_{\mathcal{D}(B)}(d_i,*)\simeq *$;
(2) $\mathrm{Hom}_{\mathcal{D}(B)}(d_i,F_B(p_!(x)))\to\mathrm{Hom}_{\mathcal{D}(B)}(d_i,p_*(F_A(x)))\simeq\mathrm{Hom}_{\mathcal{D}(A)}(p^*(d_i),F_A(x))$.

Note that this is a set worth of conditions. We claim that these maps of spaces are obtained by applying $\mathrm{Hom}_{\mathrm{Fun}_T(\mathcal{C},\mathcal{D})}(-,F)$ to certain maps $S'(1)$ in $\mathrm{Fun}_T(\mathcal{C},\mathcal{D})$. Since the maps are natural in the functor F, it suffices to prove that the source and target of each map are corepresented. Note that the functor $F\mapsto *$ is corepresented by the initial object of $\mathrm{Fun}_T(\mathcal{C},\mathcal{D})$. Therefore it will suffice to show that functors in F of the form $\mathrm{Hom}_{\mathcal{D}(B)}(y,F_B(x))$ are corepresented. First recall the standard fact that the assignment $F\mapsto\mathrm{Hom}_{\mathcal{D}(B)}(d_i,F_B(x))$ is corepresented by the functor $y(x)\otimes d_i\colon \mathcal{C}(B)\to\mathcal{D}(B)$ in $\mathrm{Fun}(\mathcal{C}(B),\mathcal{D}(B))$. Here $y(x)=\mathrm{Hom}_{\mathcal{C}(B)}(x,-)\colon \mathcal{C}(B)\to\mathrm{Spc}$ denotes the Yoneda embedding, while the functor $-\otimes d_i\colon \mathrm{Spc}\to\mathcal{D}(B)$ denotes the standard tensoring over spaces in the cocomplete category $\mathcal{D}(B)$. To prove the claim, it thus remains to show that the evaluation functor

$$\mathrm{ev}_B\colon \mathrm{Fun}_T(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}(B),\mathcal{D}(B))$$

admits a left adjoint. Note that by Proposition 2.3.24 it preserves colimits and limits. Since both source and target are presentable the existence of the required left adjoint follows immediately from the adjoint functor theorem [Lur09, Corollary 5.5.2.9]. □

4.7. **Finite pointed P-sets.** We will now introduce the T-∞-category $\underline{\mathbb{F}}^P_{T,*}$ of *finite pointed P-sets* for an orbital subcategory $P \subseteq T$ and prove that it is the free pointed T-∞-category admitting finite P-coproducts.

Definition 4.7.1. Let $P \subseteq T$ be an orbital subcategory. We define the subcategory $\underline{\mathbb{F}}^P_{T,*} \subseteq \underline{\mathrm{Spc}}_{T,*}$ of *finite pointed P-sets* as the inverse image of the subcategory $\underline{\mathbb{F}}^P_T \subseteq \underline{\mathrm{Spc}}_T$ under the forgetful functor $\underline{\mathrm{Spc}}_{T,*} \to \underline{\mathrm{Spc}}_T$: it contains those pointed T-spaces $(X,f,s) \in \underline{\mathrm{Spc}}_{T,*}(B)$ whose underlying T-space $(f\colon X \to B)$ is in $\underline{\mathbb{F}}^P_T$.

Note that $\underline{\mathbb{F}}^P_{T,*}$ is equivalent to $(\underline{\mathbb{F}}^P_T)_*$, the pointed objects in the T-∞-category of finite P-sets.

Notation 4.7.2. By Example 2.3.3, the forgetful functor $\underline{\mathrm{Spc}}_{T,*} \to \underline{\mathrm{Spc}}_T$ admits a left adjoint $(-)_+\colon \underline{\mathrm{Spc}}_T \to \underline{\mathrm{Spc}}_{T,*}$. It is given at $B \in T$ by the functor

$$(-)_+\colon\ \mathrm{PSh}(T)_{/B} \to (\mathrm{PSh}(T)_{/B})_*\colon (X,f) \mapsto (X_+, f_+, s),$$

where $X_+ := X \sqcup B$, where $f_+ := (f,\mathrm{id})\colon X \sqcup B \to B$ and where $s\colon B \hookrightarrow X \sqcup B$ is the canonical inclusion. We will often abuse notation and write X_+ or $(X,f)_+$ instead of (X_+, f_+, s).

Observe that the T-functor $(-)_+\colon \underline{\mathrm{Spc}}_T \to \underline{\mathrm{Spc}}_{T,*}$ of Notation 4.7.2 restricts to a T-functor $(-)_+\colon \underline{\mathbb{F}}^P_T \to \underline{\mathbb{F}}^P_{T,*}$ which is left adjoint to the forgetful functor $\mathrm{fgt}\colon \underline{\mathbb{F}}^P_{T,*} \to \underline{\mathbb{F}}^P_T$.

Lemma 4.7.3. *Let $P \subseteq T$ be an atomic orbital subcategory. Then the T-functor $(-)_+\colon \underline{\mathbb{F}}^P_T \to \underline{\mathbb{F}}^P_{T,*}$ is essentially surjective: any finite pointed P-set $(Y,p,s) \in \underline{\mathbb{F}}^P_{T,*}(B)$ is equivalent to one of the form X_+ for some $(X,q) \in \underline{\mathbb{F}}^P_T(B)$.*

Proof. By definition, we may write $Y = \bigsqcup_{i=1}^n A_i$ as a finite disjoint union such that each map $p_i\colon A_i \to B$ is in P. The section $s\colon B \to \bigsqcup_{i=1}^n A_i$ must factor as $B \to A_i \hookrightarrow \bigsqcup_{i=1}^n A_i$ for some i. But this implies that the map $B \to A_i$ is a section of $p_i\colon A_i \to B$, so by Lemma 4.3.2 it must be an equivalence, exhibiting B as a disjoint summand of Y. Defining X as the disjoint union of the remaining summands gives the desired equivalence $Y \simeq X_+$ over B. □

Notation 4.7.4. When $P \subseteq T$ is atomic orbital, we will assume all pointed P-set over $B \in T$ are given to us in the form $X_+ = X \sqcup B$ for $(X,q) \in \underline{\mathbb{F}}^P_T(B)$. This convention is justified by Lemma 4.7.3. We emphasize that the maps $X_+ \to Y_+$ of finite pointed P-sets over B are not assumed to respect this decomposition, i.e. they might not be induced by maps in $\mathbb{F}^P_T(B)$.

Lemma 4.7.5. *The T-∞-category $\underline{\mathbb{F}}^P_{T,*}$ from Definition 4.7.1 admits finite P-coproducts and the inclusion $\underline{\mathbb{F}}^P_{T,*} \hookrightarrow \underline{\mathrm{Spc}}_{T,*}$ preserves finite P-coproducts. Furthermore, for any other T-∞-category $\mathcal{D}$ which admits finite P-coproducts, a T-functor*

$F\colon \underline{\mathbb{F}}^P_{T,*} \to \mathcal{D}$ preserves finite P-coproducts if and only if the composite $F \circ (-)_+$ does.

Proof. By Example 2.3.21, it suffices to prove that $\underline{\mathbb{F}}^P_{T,*}$ is closed under finite P-coproducts in $\underline{\mathrm{Spc}}_{T,*}$. By Corollary 4.2.16, the T-category $\underline{\mathbb{F}}^P_T$ admits finite P-coproducts and these are preserved by the (left adjoint) T-functor $(-)_+\colon \underline{\mathbb{F}}^P_T \to \underline{\mathbb{F}}^P_{T,*}$. Conversely it follows from Lemma 4.7.3 that every cocone in $\underline{\mathbb{F}}^P_{T,*}$ indexed by a finite P-set comes from $\underline{\mathbb{F}}^P_T$. The claim follows. □

Let $S^0\colon \underline{1} \to \underline{\mathbb{F}}^P_{T,*}$ denote the T-functor given at $B \in T$ by the object $B_+ \in \underline{\mathbb{F}}^P_{T,*}(B)$. The goal of the remainder of this subsection is to show that this map exhibits the T-∞-category $\underline{\mathbb{F}}^P_{T,*}$ as the free pointed T-∞-category admitting finite P-coproducts.

If $\mathcal{E}$ is an ∞-category admitting a final object $*$, we let $\mathcal{E}_+ \subseteq \mathcal{E}_*$ denote the full subcategory of pointed objects $* \to Z$ for which there exists a pointed equivalence $Z \simeq X \sqcup *$ for some $X \in \mathcal{E}$. If $\mathcal{E}$ admits finite coproducts, then $\mathcal{E}_+$ also admits finite coproducts and the functor $(-)_+\colon \mathcal{E} \to \mathcal{E}_+\colon X \mapsto X_+ := X \sqcup *$ preserves finite coproducts. Furthermore $\mathcal{E}_+$ is pointed. We will show that the functor $(-)_+\colon \mathcal{E} \to \mathcal{E}_+$ is universal among coproduct preserving functors from $\mathcal{E}$ into a pointed ∞-category.

Lemma 4.7.6. *Let $\mathcal{E}$ and $\mathcal{D}$ be ∞-categories admitting finite coproducts. Assume that $\mathcal{E}$ admits a final object and that $\mathcal{D}$ is pointed. Then precomposition with the functor $(-)_+\colon \mathcal{E} \to \mathcal{E}_+$ induces an equivalence*

$$\mathrm{Fun}^{\sqcup,*}(\mathcal{E}_+, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\sqcup}(\mathcal{E}, \mathcal{D}).$$

Proof. We claim an inverse is given by sending a finite-coproduct-preserving functor $F\colon \mathcal{E} \to \mathcal{D}$ to the functor $\widetilde{F}\colon \mathcal{E}_+ \to \mathcal{D}$ defined by the formula

$$\widetilde{F}(X_+) := \mathrm{cofib}(F(*) \to F(X_+)).$$

Observe that this colimit exists and is equivalent to $F(X)$ by the following pushout diagram:

$$\begin{array}{ccccc} * & \longrightarrow & F(*) & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow & & \downarrow \\ F(X) & \longrightarrow & F(X_+) & \longrightarrow & F(X). \end{array}$$

Here the left square is a pushout since $\mathcal{D}$ is pointed and F preserves finite coproducts, and it thus follows from the pasting law of pushout diagrams that the right square is a pushout as well. This proves that the composition $\widetilde{F} \circ (-)_+$ is equivalent to F. It is easily observed that $\widetilde{F}$ is pointed and preserves finite coproducts.

Now assume we are given a pointed functor $\widetilde{F}\colon \mathcal{E}_+ \to \mathcal{D}$ which preserves finite coproducts. It remains to show that for every object $Z \in \mathcal{E}_+$ the canonical map

$$\mathrm{cofib}(\widetilde{F}(*_+) \to \widetilde{F}(Z_+)) \to \widetilde{F}(Z)$$

is an equivalence. This follows from the fact that Z_+ is a coproduct in $\mathcal{E}_+$ of Z and $*_+$ and that $\widetilde{F}$ preserves coproducts by assumption. □

Let $\mathrm{Cat}_\infty^{\sqcup} \subseteq \mathrm{Cat}_\infty$ denote the (non-full) subcategory consisting of ∞-categories which admit finite coproducts and functors which preserve finite coproducts. Let $\mathrm{Cat}_\infty^{\sqcup,\mathrm{pt}} \subseteq \mathrm{Cat}_\infty^{\sqcup,*} \subseteq \mathrm{Cat}_\infty^{\sqcup}$ denote the full subcategories spanned by those ∞-categories with finite coproducts which admit a zero object or admit a final object, respectively.

Corollary 4.7.7. *The inclusion* $\mathrm{Cat}_\infty^{\sqcup,\mathrm{pt}} \hookrightarrow \mathrm{Cat}_\infty^{\sqcup,*}$ *admits a left adjoint*

$$(-)_+ \colon \mathrm{Cat}_\infty^{\sqcup,*} \to \mathrm{Cat}_\infty^{\sqcup,\mathrm{pt}}$$

which on objects sends $\mathcal{E}$ *to* $\mathcal{E}_+$.

Proof. We need to show that for any $\mathcal{E} \in \mathrm{Cat}_\infty^{\sqcup,*}$ and any $\mathcal{D} \in \mathrm{Cat}_\infty^{\sqcup,\mathrm{pt}}$, the precomposition with the map $(-)_+ \colon \mathcal{E} \to \mathcal{E}_+$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_\infty^{\sqcup}}(\mathcal{E}_+, \mathcal{D}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}_\infty^{\sqcup}}(\mathcal{E}, \mathcal{D}).$$

This is immediate from Lemma 4.7.6. $\square$

Corollary 4.7.8. *Let* $\mathcal{D}$ *be a pointed* T*-*∞*-category* $\mathcal{D}$ *which admits finite* P*-coproducts. Then composition with* $S^0 \colon \underline{1} \to \underline{\mathbb{F}}_{T,*}^P$ *induces an equivalence of* T*-*∞*-categories*

$$\underline{\mathrm{Fun}}_T^{P\text{-}\sqcup,*}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\underline{1}, \mathcal{D}) \simeq \mathcal{D}\,.$$

Proof. Note that S^0 is the composite $\underline{1} \xrightarrow{*} \underline{\mathbb{F}}_T^P \xrightarrow{(-)_+} \underline{\mathbb{F}}_{T,*}^P$. By Corollary 4.2.17 it thus suffices to show that composition with the T-functor $(-)_+ \colon \underline{\mathbb{F}}_T^P \to \underline{\mathbb{F}}_{T,*}^P$ induces an equivalence $\underline{\mathrm{Fun}}_T^{P\text{-}\sqcup,*}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_T^{P\text{-}\sqcup}(\underline{\mathbb{F}}_T^P, \mathcal{D})$. It in fact suffices to show that it induces an equivalence between T-∞-categories of *fiberwise coproduct* preserving functors. Namely by the last part of Lemma 4.7.5 this equivalence will restrict to the subcategories of P-coproduct preserving functors on either side. Replacing T by $T_{/B}$ for every $B \in T$, it suffices to prove this on underlying ∞-categories. Note that the subcategory $\mathrm{Cat}_T^{\sqcup} \subseteq \mathrm{Cat}_T$ is closed under cotensoring by Cat_∞ and that there is a canonical equivalence $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{E}, \mathrm{Fun}_T^{\sqcup}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Hom}_{\mathrm{Cat}_T^{\sqcup}}(\mathcal{C}, \mathcal{D}^{\mathcal{E}})$ for $\mathcal{E} \in \mathrm{Cat}_\infty$ and $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_T^{\sqcup}$. By the Yoneda lemma it will thus suffice to show that the functor $(-)_+ \colon \underline{\mathbb{F}}_T^P \to \underline{\mathbb{F}}_{T,*}^P$ induces an equivalence $\mathrm{Hom}_{\mathrm{Cat}_T^{\sqcup}}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}_T^{\sqcup}}(\underline{\mathbb{F}}_T^P, \mathcal{D})$. This is immediate from Corollary 4.7.7. $\square$

4.8. *P*-commutative monoids.

In this subsection we will introduce the notion of a *P-commutative monoid* in a T-∞-category $\mathcal{D}$ admitting finite P-products. Furthermore we will show that the T-∞-category $\underline{\mathrm{CMon}}^P(\mathcal{D})$ of P-commutative monoids in $\mathcal{D}$ is the terminal P-semiadditive T-∞-category equipped with a finite P-product preserving T-functor to $\mathcal{D}$.

Definition 4.8.1 (P-commutative monoids, cf. [Nar16, Definition 5.9])**.** Let $\mathcal{D}$ be a T-∞-category which admits finite P-products. A *P-commutative monoid object of* $\mathcal{D}$ is a P-semiadditive T-functor $M \colon \underline{\mathbb{F}}_{T,*}^P \to \mathcal{D}$. We define the T-∞-category $\underline{\mathrm{CMon}}^P(\mathcal{D})$ of *P-commutative monoids* in $\mathcal{D}$ as

$$\underline{\mathrm{CMon}}^P(\mathcal{D}) := \underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}).$$

We define the forgetful functor $\mathbb{U} \colon \underline{\mathrm{CMon}}^P(\mathcal{D}) \to \mathcal{D}$ to be given by precomposition with the T-functor $S^0 \colon \underline{1} \to \underline{\mathbb{F}}_{T,*}^P$.

As a special case, we define the T-∞-category $\underline{\mathrm{CMon}}_T^P$ of *P-commutative monoids* as

$$\underline{\mathrm{CMon}}_T^P := \underline{\mathrm{CMon}}^P(\underline{\mathrm{Spc}}_T).$$

Combining our previous results, we can immediately deduce the universal property of P-commutative monoids. We spell this out in the following series of statements.

Proposition 4.8.2. *For every T-∞-category $\mathcal{D}$ admitting finite P-products, the T-∞-category $\underline{\mathrm{CMon}}^P(\mathcal{D})$ is P-semiadditive. Furthermore, the forgetful functor $\underline{\mathrm{CMon}}^P(\mathcal{D}) \to \mathcal{D}$ preserves finite P-products.*

Proof. The first statement is a special case of Proposition 4.6.13 for $\mathcal{C} = \underline{\mathbb{F}}_{T,*}^P$. The second statement is a special case of Lemma 4.6.11 combined with Proposition 2.3.24. □

Proposition 4.8.3. *Given a T-∞-category $\mathcal{D}$ admitting finite P-products,*

$$\mathbb{U} \colon \underline{\mathrm{CMon}}^P(\mathcal{D}) \to \mathcal{D}$$

is an equivalence if and only if $\mathcal{D}$ is P-semiadditive.

Proof. As $\underline{\mathrm{CMon}}^P(\mathcal{D})$ is P-semiadditive by Proposition 4.8.2, one direction is immediate. Conversely, if $\mathcal{D}$ is P-semiadditive, then Proposition 4.6.14 provides an equivalence

$$\underline{\mathrm{CMon}}^P(\mathcal{D}) = \underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}) \simeq \underline{\mathrm{Fun}}_T^{P\text{-}\sqcup}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}).$$

The result thus follows from Corollary 4.7.8. □

Corollary 4.8.4 (cf. [Nar16, Corollary 5.11.1])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be T-∞-categories such that $\mathcal{C}$ is pointed and admits finite P-coproducts and $\mathcal{D}$ admits finite P-products. Then postcomposition with the forgetful functor $\mathbb{U} \colon \underline{\mathrm{CMon}}^P(\mathcal{D}) \to \mathcal{D}$ induces an equivalence*

$$\mathrm{Fun}_T^{P\text{-}\sqcup}(\mathcal{C}, \underline{\mathrm{CMon}}^P(\mathcal{D})) \to \mathrm{Fun}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D}).$$

Proof. By Proposition 4.6.14, the left-hand side is equal to the T-∞-category of P-semiadditive T-functors $\mathcal{C} \to \underline{\mathrm{CMon}}^P(\mathcal{D})$. By Corollary 4.6.12 this is in turn equivalent to $\underline{\mathrm{CMon}}^P(\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D}))$. The claim thus follows from Proposition 4.8.2 and Proposition 4.8.3. □

Corollary 4.8.5. *The inclusion $\mathrm{Cat}_T^{P\text{-}\oplus} \hookrightarrow \mathrm{Cat}_T^{P\text{-}\times}$ of the T-∞-category of P-semiadditive T-∞-categories and P-semiadditive T-functors into the T-∞-category of T-∞-categories admitting finite P-products and the finite P-product preserving T-functors admits a right adjoint given by*

$$\underline{\mathrm{CMon}}^P(-) \colon \mathrm{Cat}_T^{P\text{-}\times} \to \mathrm{Cat}_T^{P\text{-}\oplus}. \qquad \square$$

We are also interested in a presentable version of Corollary 4.8.5.

Lemma 4.8.6. *Let $\mathcal{C}$ be presentable. Then $\mathbb{U} \colon \underline{\mathrm{CMon}}^P(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\mathbb{P}$.*

Proof. The functor $\mathrm{ev}_{S^0} \colon \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \mathcal{C}) \to \mathcal{C}$ admits a left adjoint by [MW21, Theorem 6.3.5 and Corollary 6.3.7]. The claim follows as also the inclusion $\underline{\mathrm{CMon}}^P(\mathcal{C}) \hookrightarrow \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \mathcal{C})$ admits a left adjoint by Proposition 4.6.17. □

Definition 4.8.7. We define $\mathrm{Pr}_T^{\mathrm{R},P\text{-}\oplus}$ to be the full subcategory of $\mathrm{Pr}_T^{\mathrm{R}}$ spanned by those presentable T-∞-categories which are moreover P-semiadditive. Similarly we define $\mathrm{Pr}_T^{\mathrm{L},P\text{-}\oplus}$.

Proposition 4.8.8. *The functor* $\underline{\mathrm{CMon}}^P$ *restricts to a functor*

$$\underline{\mathrm{CMon}}^P\colon \mathrm{Pr}_T^{\mathrm{R}} \to \mathrm{Pr}_T^{\mathrm{R},P\text{-}\oplus}$$

right adjoint to the inclusion.

Proof. Let $\mathcal{C}$ be a presentable T-∞-category. Note that by Proposition 4.6.17, $\underline{\mathrm{CMon}}^P(\mathcal{C})$ is again presentable. Furthermore suppose $G\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint between presentable T-∞-categories, and denote its left adjoint by F. Note that G preserves finite P-products, and so induces a functor $\underline{\mathrm{CMon}}^P(G)\colon \underline{\mathrm{CMon}}^P(\mathcal{C}) \to \underline{\mathrm{CMon}}^P(\mathcal{D})$. Because G preserves local objects, the composite

$$\underline{\mathrm{CMon}}^P(\mathcal{D}) \hookrightarrow \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \mathcal{D}) \xrightarrow{F} \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \mathcal{C}) \xrightarrow{L^{P\text{-}\oplus}(-)} \underline{\mathrm{CMon}}^P(\mathcal{C})$$

is left adjoint to $\underline{\mathrm{CMon}}^P(R)$, where $L^{P\text{-}\oplus}$ refers to the left adjoint of the inclusion $\underline{\mathrm{CMon}}^P \subset \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \mathcal{C})$ constructed in Proposition 4.6.17.

Finally, the unit $\mathbb{U}$ is a right adjoint by Lemma 4.8.6 while the counit is even an equivalence by Proposition 4.8.3. □

Corollary 4.8.9. *There exists an adjunction*

$$\underline{\mathrm{CMon}}^P(-)\colon \mathrm{Pr}_T^{\mathrm{L}} \rightleftarrows \mathrm{Pr}_T^{\mathrm{L},P\text{-}\oplus} :\mathrm{incl}.$$

Furthermore the unit $\mathbb{P}\colon \mathcal{C} \to \underline{\mathrm{CMon}}^P(\mathcal{C})$ *is left adjoint to the forgetful functor* $\mathbb{U}$.

Proof. Consider the adjunction constructed in Proposition 4.8.8 and apply the equivalence $\mathrm{Pr}_T^{\mathrm{L}} \simeq (\mathrm{Pr}_T^{\mathrm{R}})^{\mathrm{op}}$. □

For ease of reference we record the strongest results obtained above in one omnibus theorem:

Theorem 4.8.10. *Let* $\mathcal{C}$ *be a* T-∞-*category with finite* P-*products. The functor* $\mathbb{U}\colon \underline{\mathrm{CMon}}^P(\mathcal{C}) \to \mathcal{C}$ *exhibits* $\underline{\mathrm{CMon}}^P(\mathcal{C})$ *as the* P-*semiadditive envelope of* $\mathcal{C}$, *i.e. for every* P-*semiadditive* T-∞-*category* $\mathcal{D}$ *postcomposition with* $\mathbb{U}$ *induces an equivalence*

$$\underline{\mathrm{Fun}}^{P\text{-}\times}(\mathcal{D}, \mathbb{U})\colon \underline{\mathrm{Fun}}^{P\text{-}\oplus}(\mathcal{D}, \underline{\mathrm{CMon}}^P(\mathcal{C})) \to \underline{\mathrm{Fun}}^{P\text{-}\times}(\mathcal{D}, \mathcal{C}).$$

Suppose now that $\mathcal{D}$ *is moreover presentable. Then the left adjoint* $\mathbb{P}$ *of* $\mathbb{U}$ *exhibits* $\underline{\mathrm{CMon}}^P(\mathcal{C})$ *as the presentable* P-*semiadditive completion of* $\mathcal{C}$, *i.e. for any presentable* P-*semiadditive* T-∞-*category* $\mathcal{D}$ *precomposition with* $\mathbb{P}$ *yields an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathbb{P}, \mathcal{D})\colon \underline{\mathrm{Fun}}^{\mathrm{L}}(\underline{\mathrm{CMon}}^P(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$ □

Combining the result above with the universal property of $\underline{\mathrm{Spc}}_T$ already shows that we have for any presentable P-semiadditive T-∞-category $\mathcal{D}$ an equivalence $\underline{\mathrm{Fun}}_T^{\mathrm{L}}(\underline{\mathrm{CMon}}_T^P, \mathcal{D}) \simeq \mathcal{D}$ of T-∞-categories. As our final result in this subsection we will generalize this to the case where $\mathcal{D}$ is merely assumed to be T-cocomplete:

Theorem 4.8.11. *Let $\mathcal{D}$ be a locally small T-cocomplete P-semiadditive T-∞-category. Then evaluation at $\mathbb{P}(*)$ defines an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}. \tag{10}$$

Proof. Appealing to the universal property of $\underline{\mathrm{Spc}}_T$ and passing to adjoints, we see that (10) agrees up to equivalence with the map

$$\underline{\mathrm{Fun}}^{\mathrm{R}}_T(\mathcal{D}, \mathbb{U})\colon \underline{\mathrm{Fun}}^{\mathrm{R}}_T(\mathcal{D}, \underline{\mathrm{CMon}}^P_T) \to \underline{\mathrm{Fun}}^{\mathrm{R}}_T(\mathcal{D}, \underline{\mathrm{Spc}}_T)$$

between parametrized categories of *right adjoint* functors. In particular it is fully faithful by the first half of Theorem 4.8.10, so it only remains to prove essential surjectivity.

Replacing $\mathcal{D}$ by $\underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})$ for $A \in T$, it will be enough to construct for every $X \in \Gamma(\mathcal{D})$ a T-left adjoint $F\colon \underline{\mathrm{CMon}}^P \to \mathcal{D}$ with $F(\mathbb{P}(*)) \simeq X$.

For this, we use the universal property of $\underline{\mathbb{F}}^P_{T,*}$ (Lemma 4.7.8) to obtain a P-coproduct preserving functor $\varphi\colon \underline{\mathbb{F}}^P_{T,*} \to \mathcal{D}^{\mathrm{op}}$ sending S^0 to X, which we may then extend to a left adjoint $\Phi\colon \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T) \to \mathcal{D}$ via Proposition 2.4.9. To complete the proof it suffices now to prove that Φ factors through the Bousfield localization $L^{P\text{-}\oplus}\colon \mathrm{Fun}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T) \to \underline{\mathrm{CMon}}^P$, or equivalently that its right adjoint takes values in $\underline{\mathrm{CMon}}^P$. However, by Remark 2.4.10 the value of this right adjoint on $Y \in \mathcal{D}(A)$ is given by the composite

$$\pi_A^*\underline{\mathbb{F}}^P_{T,*} \xrightarrow{\pi_A^*\varphi} \pi_A^*\mathcal{D}^{\mathrm{op}} \xrightarrow{\mathrm{maps}(-,Y)} \mathrm{Spc}_{T/A} \simeq \pi_A^*\underline{\mathrm{Spc}}_T$$

and the first functor sends π_A^*P-coproducts to π_A^*P-products by construction of φ and semiadditivity of $\mathcal{D}$ while the second one even preserves all π_A^*T-limits that exist in $\pi_A^*\mathcal{D}^{\mathrm{op}}$ [MW21, Corollary 4.4.9]. □

For use in future work, we record the following result elaborating on the construction of the inverse to (10):

Proposition 4.8.12. *Write $j\colon (\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}} \to \underline{\mathrm{CMon}}^P_T$ for the unique finite P-product preserving functor sending S^0 to $\mathbb{P}(*)$. Then the restriction $j^*\colon \underline{\mathrm{Fun}}_T(\underline{\mathrm{CMon}}^P_T, \mathcal{D}) \to \underline{\mathrm{Fun}}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}, \mathcal{D})$ admits a left adjoint $j_!$, and j^* and $j_!$ restrict to mutually inverse equivalences $\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T, \mathcal{D}) \simeq \underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}, \mathcal{D})$.*

Proof. Let us write $\bar{\jmath}$ for the composition of the Yoneda embedding $y\colon (\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}} \to \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T)$ with the localization $L^{P\text{-}\oplus}$. We will first prove the proposition with $\bar{\jmath}$ in lieu of j, and then conclude in the end that in fact $j \simeq \bar{\jmath}$.

[MW21, Theorem 7.1.1] shows that for any T-cocomplete (P-semiadditive) $\mathcal{D}$ the restriction $y^*\colon \underline{\mathrm{Fun}}_T(\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T), \mathcal{D}) \to \underline{\mathrm{Fun}}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}, \mathcal{D})$ has a left adjoint $y_!$ inducing an equivalence $\underline{\mathrm{Fun}}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}, \mathcal{D}) \simeq \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T), \mathcal{D})$, while Theorem 6.3.5 and Corollary 6.3.7 of *op. cit.* show that $\bar{\jmath}^*$ admits a left adjoint $\bar{\jmath}_!$.

We claim that $\bar{\jmath}_!$ and $\bar{\jmath}^*$ restrict to functors $\underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}, \mathcal{D}) \rightleftarrows \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T, \mathcal{D})$, which are then automatically adjoint to each other again. As the right adjoint $\bar{\jmath}^*$ in this adjunction is then moreover an equivalence by the previous theorem together with Corollary 4.7.8, they will then be mututally inverse equivalences.

To prove the claim, note that we have seen in the proof of the previous theorem that $y_!\colon \underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}, \underline{\mathrm{Spc}}_T) \to \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T), \mathcal{D})$ factors through the fully faithful functor $(L^{P\text{-}\oplus})^*\colon \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T,\mathcal{D}) \to \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*},\underline{\mathrm{Spc}}_T),\mathcal{D})$. If we write f for the resulting functor $\underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\underline{\mathrm{Spc}}_T) \to \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T,\mathcal{D})$, then for any $A\in T$, $X\in \underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\mathcal{D})(A)$, and $Y\in \underline{\mathrm{Fun}}_T(\underline{\mathrm{CMon}}^P_T,\mathcal{D})(A)$

$$\begin{aligned}\mathrm{maps}(f(X),Y) &\simeq \mathrm{maps}((L^{P\text{-}\oplus})^*f(X),(L^{P\text{-}\oplus})^*Y) \simeq \mathrm{maps}(y_!X,(L^{P\text{-}\oplus})^*Y)\\ &\simeq \mathrm{maps}(X,y^*(L^{P\text{-}\oplus})^*Y) = \mathrm{maps}(X,\bar{\jmath}^*Y) \simeq \mathrm{maps}(\bar{\jmath}_!X,Y)\end{aligned}$$

by full faithfulness, the definition of f, and adjunction, respectively. It follows from the ordinary Yoneda Lemma that $f(X)\simeq \bar{\jmath}_!(X)$, and in particular the latter lives in $\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T,\mathcal{D})(A)$, i.e. $\bar{\jmath}_!$ maps $\underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\mathcal{D})$ into $\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T,\mathcal{D})$.

Conversely, any T-cocontinuous $\underline{\mathrm{CMon}}^P_T\to\mathcal{D}$ has to arise via the construction of the previous theorem, i.e. f is essentially surjective. But $\bar{\jmath}^*f = y^*(L^{P\text{-}\oplus})^*f\simeq y^*y_!\simeq \mathrm{id}$, so $\bar{\jmath}^*$ restricts to

$$\bar{\jmath}^*\colon \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P_T,\mathcal{D}) \to \underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\mathcal{D}). \tag{11}$$

It only remains to show that $\bar{\jmath}$ agrees with j as constructed in the statement of the proposition. As $\bar{\jmath}(S^0)\simeq\mathbb{P}(*)$, we only need to show that $\bar{\jmath}$ preserves finite P-products. But this follows at once by taking $\mathcal{D}=\underline{\mathrm{CMon}}^P_T$ and chasing the identity through (11). □

We can now slightly strengthen the second half of Theorem 4.8.10 in the case of $\underline{\mathrm{Spc}}_T$:

Corollary 4.8.13. *Let $\mathcal{S}$ be a T-∞-category equivalent to $\underline{\mathrm{Spc}}_T$ and let $\mathcal{D}$ be any locally small P-semiadditive T-cocomplete T-∞-category. Then precomposition with the T-functor $\mathbb{P}\colon\mathcal{S}\to\underline{\mathrm{CMon}}^P(\mathcal{S})$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{CMon}}^P(\mathcal{S}),\mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\mathcal{S},\mathcal{D}).$$ □

Remark 4.8.14. We will prove in forthcoming work that Corollary 4.8.13 in fact holds for any presentable T-∞-category $\mathcal{S}$.

4.9. **Commutative monoids in $\underline{\mathcal{E}}_T$.** Let $\mathcal{E}$ be an ∞-category. Recall that a T-functor $F\colon\underline{\mathbb{F}}^P_{T,*}\to\underline{\mathcal{E}}_T$ corresponds to a functor $\widetilde{F}\colon\int\underline{\mathbb{F}}^P_{T,*}\to\mathcal{E}$ of ∞-categories, see Lemma 2.2.13. We will now give a characterization of those functors $\widetilde{F}$ whose associated T-functor F is a P-semiadditive monoid in $\underline{\mathcal{E}}_T$. We start with an explicit description of the adjoint norm map $\widetilde{\mathrm{Nm}}_p\colon p^*p_!\Rightarrow\mathrm{id}$ associated to $\underline{\mathbb{F}}^P_{T,*}$.

Lemma 4.9.1. *Let $P\subseteq T$ be an atomic orbital subcategory. Consider a map $p\colon A\to B$ in $\mathbb{F}^P_T$ and let $f\colon X\to A$ and $g\colon Y\to A$ be a morphisms in $\mathrm{PSh}(T)$. Then the map $1\times_p 1\colon X\times_A Y\to X\times_B Y$ is a disjoint summand inclusion.*

Proof. Using Proposition 4.3.7, this follows directly from the observation that the map $X\times_A Y\to X\times_B Y$ is a base change of the disjoint summand inclusion $\Delta\colon A\to A\times_B A$ along the map $f\times_B g\colon X\times_B Y\to A\times_B A$. □

Construction 4.9.2. Consider a morphism $p\colon A\to B$ in $\mathbb{F}^P_T$. For any finite P-set $(X,q)\in\underline{\mathbb{F}}^P_T(A)$, the unit map $(1,q)\colon X\to X\times_B A = p_*p_!X$ is a disjoint summand inclusion by Lemma 4.9.1, and thus we may choose an identification

$$X\times_B A\simeq X\sqcup J_X$$

for some finite P-set $J_X \in \underline{\mathbb{F}}_T^P(A)$. In particular we obtain a map $p^*p_!(X_+) \to X_+$ in $\underline{\mathbb{F}}_T^P(A)$ defined as the following composite:

$$p^*p_!(X_+) \simeq (X \times_B A)_+ \simeq (X \sqcup J_X)_+ \to X_+,$$

where the last map projects away the disjoint component J_X to the disjoint basepoint.

Lemma 4.9.3. *The map $p^*p_!(X_+) \to X_+$ constructed in Construction 4.9.2 is homotopic to the adjoint norm map $\widetilde{\mathrm{Nm}}_p\colon p^*p_!(X_+) \to X_+$ associated to the T-∞-category $\underline{\mathbb{F}}_{T,*}^P$.*

Proof. Choose a map $J_A \hookrightarrow A \times_B A$ exhibiting J_A as a complement of the disjoint summand inclusion $\Delta\colon A \hookrightarrow A \times_B A$. The resulting equivalence $A \times_B A \simeq A \sqcup J_A$ induces an equivalence $\underline{\mathbb{F}}_{T,*}^P(A \times_B A) \simeq \underline{\mathbb{F}}_{T,*}^P(A \sqcup J_A) \simeq \underline{\mathbb{F}}_{T,*}^P(A) \times \underline{\mathbb{F}}_{T,*}^P(J_A)$. Pulling back the decomposition $A \times_B A \simeq A \sqcup J_A$ along the map $X \times_B A \to A \times_B A$ gives a decomposition $X \times_B A \simeq X \sqcup J_X$, and it follows that the object $\mathrm{pr}_2^*(X_+) \simeq (X \times_B A)_+ \in \underline{\mathbb{F}}_{T,*}^P(A \times_B A)$ corresponds to the pair $(X_+, J_{X+}) \in \underline{\mathbb{F}}_{T,*}^P(A) \times \underline{\mathbb{F}}_{T,*}^P(J_A)$. By Lemma 4.3.11, the transformation $\alpha\colon \mathrm{pr}_2^* \Rightarrow \mathrm{pr}_1^*$ corresponds to a transformation of functors into $\underline{\mathbb{F}}_{T,*}^P(A) \times \underline{\mathbb{F}}_{T,*}^P(J_A)$ which on the first component is the identity and on the second component is the zero-map which projects everything onto the disjoint basepoint. The description from Construction 4.9.2 follows. □

Notation 4.9.4. We will abuse notation and denote objects of the unstraightening $\int \underline{\mathbb{F}}_{T,*}^P$ by pairs (A, X_+), where $A \in T$ and $(X, q\colon X \to A) \in \underline{\mathbb{F}}_T^P(A)$ is a finite P-set. We will specify q explicitly whenever confusion might arise.

Construction 4.9.5 (Parametrized Segal map)**.** Consider a map $p\colon A \to B$ in P, a map $C \to B$ in T and a finite pointed P-set $X_+ \to A$ in $\underline{\mathbb{F}}_{T,*}^P(A)$. Since p is in P, the pullback $A \times_B C$ of p along $C \to B$ may be written as a disjoint union of maps $p_i\colon C_i \to C$ in P:

$$\begin{array}{ccc} \bigsqcup_{i=1}^n C_i & \longrightarrow & A \\ {\scriptstyle (p_i)_{i=1}^n}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \longrightarrow & B. \end{array}$$

We will we construct for each $i \in \{1, \dots, n\}$ a *parametrized Segal map*

$$\rho_i\colon (C, (X \times_B C)_+) \to (C_i, (X \times_A C_i)_+)$$

in $\int \underline{\mathbb{F}}_{T,*}^P$. To give such a map, we need to provide a map $C_i \to C$ in T, which we simply take to be the map $p_i\colon C_i \to C$, and a map $p_i^*(X \times_B C)_+ \simeq (X \times_B C_i)_+ \to (X \times_A C_i)_+$ in $\underline{\mathbb{F}}_{T,*}^P(C_i)$. Recall from Lemma 4.9.1 that the map $X \times_A C_i \to X \times_B C_i$ is a disjoint summand inclusion, so that we may choose an equivalence

$$(X \times_B C_i) \simeq (X \times_A C_i) \sqcup J_i,$$

where $J_i \to C_i$ is some finite P-set. The required map $(X \times_B C_i)_+ \simeq (X \times_A C_i)_+ \vee J_{i+} \to (X \times_A C_i)_+$ is now given by projecting away the second summand.

Proposition 4.9.6. *Let $\mathcal{E}$ be an ∞-category and consider a T-functor $F\colon \underline{\mathbb{F}}_{T,*}^P \to \underline{\mathcal{E}}_T$. Denote by $\widetilde{F}\colon \int \underline{\mathbb{F}}_{T,*}^P \to \mathcal{E}$ the functor associated to F under the equivalence of Lemma 2.2.13. Then F is a P-semiadditive monoid in $\underline{\mathcal{E}}_T$ if and only if F is* *fiberwise semiadditive and for every map $p\colon A \to B$ in P, every map $f\colon C \to B$ in T and every finite pointed P-set $X_+ \in \underline{\mathbb{F}}^P_{T,*}(A)$, the map*

$$(\widetilde{F}(\rho_i))_{i=1}^n \colon \widetilde{F}(C, (X \times_B C)_+) \to \prod_{i=1}^n \widetilde{F}(C_i, (X \times_A C_i)_+)$$

induced by the parametrized Segal maps is an equivalence.

Proof. By Corollary 4.5.7, the T-functor F is P-semiadditive if and only if it is fiberwise semiadditive and for all maps $p\colon A \to B$ in P the transformation $\mathrm{Nm}^F_p \colon F_B \circ p_! \Rightarrow p_* \circ F_A$ of functors $\underline{\mathbb{F}}^P_{T,*}(A) \to \underline{\mathcal{E}}_T(B) = \mathrm{Fun}(T^{\mathrm{op}}_{/B}, \mathcal{E})$ is an equivalence. Since we may check this pointwise, it suffices to show that for every finite P-set $X_+ \in \underline{\mathbb{F}}^P_{T,*}(A)$ and every object $f\colon C \to B$ of $T_{/B}$, the induced map

$$F_B(p_!(A, X_+))(C, f) \to (p_*(F_A(A, X_+))(C, f)$$

is an equivalence. By definition, this map is given by the composite

$$\begin{array}{ccc} F_B((B, X_+))(C, f) \xrightarrow{u_p^*} & p_* p^*(F_B((B, X_+)))(C, f) & \\ & \Big\downarrow \sim & \\ & F_A(p^* p_!(A, X_+))(p^*(C, f)) & \xrightarrow{\widetilde{\mathrm{Nm}}_p} F_A(A, X_+)(p^*(C, f)). \end{array}$$

To make this composite explicit it will be useful to consider the objects of $\underline{\mathcal{E}}_T(B)$ as functors from $(\mathbb{F}_{T/B})^{\mathrm{op}}$ to $\mathcal{E}$ by limit extending. Similarly it will be useful to consider F as a natural transformation of functors from $\mathbb{F}_T$ to Cat_∞ by again limit extending. If we make both of these extensions we may again apply Lemma 2.2.13 to conclude that F is induced by a functor $\bar{F}\colon \int_{\mathbb{F}_T} \underline{\mathbb{F}}^P_{T,*} \to \mathcal{E}$. Namely we recall from Remark 2.2.16 that given a T-set X and a pointed P-set $Y \to X$ over X, $F_X(X, Y_+)(f\colon Z \to X) = \bar{F}(f^*(X, Y_+)) = \bar{F}(Z, (Y \times_X Z)_+)$. Using this identification we find that the composite above is equivalent to

$$\bar{F}(C, X \times_B C) \xrightarrow{\bar{F}(\varphi_p)} \bar{F}(C \times_B A, X \times_B (C \times_B A)) \xrightarrow{\bar{F}(\widetilde{\mathrm{Nm}}_p)} \bar{F}(C \times_B A, X \times_A (C \times_B A)),$$

where φ_p is a cocartesian edge expressing $X \times_B (C \times_B A)$ as a pullback of $X \times_B C$ along $u_p\colon C \times_B A \to C$. Now recall that $\bar{F}$ was defined to be the limit extension of F, and so given a decomposition $C \times_B A \simeq \coprod C_i$, we find that

$$\bar{F}(C \times_B A, X \times_A (C \times_B A)) \xrightarrow{\simeq} \prod \bar{F}(C_i, X \times_A C_i).$$

To conclude we would like to show that projecting the composite above to any factor agrees with the map constructed in Construction 4.9.5. For this observe that by definition applying $\bar{F}$ to a cocartesian edge over $\iota\colon C_j \hookrightarrow C \times_B A$ gives the projection

$$\mathrm{pr}_j \colon \prod_i \bar{F}(C_i, X \times_A C_i) \to \bar{F}(C_j, X \times_A C_j)$$

Therefore we can compute the top-right way around the following commutative diagram

$$\begin{array}{ccccc} \bar{F}(C, X \times_B C) & \xrightarrow{\bar{F}(\varphi_p)} & \bar{F}(C \times_B A, X \times_B (C \times_B A)) & \xrightarrow{\bar{F}(\widetilde{\mathrm{Nm}}_p)} & \bar{F}(C \times_B A, X \times_A (C \times_B A)) \\ & \searrow^{\bar{F}(\varphi_{p_j})} & \Big\downarrow \bar{F}(\varphi_\iota) & & \Big\downarrow \bar{F}(\varphi_\iota) \\ & & \bar{F}(C_j, X \times_B C_j) & \xrightarrow[\bar{F}(\iota^*(\widetilde{\mathrm{Nm}}_p))]{} & \bar{F}(C_j, X \times_A C_j) \end{array}$$

by instead going along the bottom. Once again φ_ι is our notation for a cocartesian edge over ι. Because cocartesian edges compose we see that φ_{p_j} is a cocartesian edge witnessing $X \times_B C_j$ as the pullback of $X \times_B C$ along the map $C_j \to C$. Using the description of $\widetilde{\mathrm{Nm}}_p$ given in Lemma 4.9.3 we find that $\iota^*(\widetilde{\mathrm{Nm}}_p))$ is equivalent to the map $X \times_B C_i \to X \times_A C_i$ given in Construction 4.9.5. Finally note that by definition $\bar{F}$ agrees with $\widetilde{F}$ on the full subcategory over $T \subset \mathbb{F}_T$. Therefore the proposition follows. $\square$

We now show that the P-semiadditivity of a functor $\tilde{F}\colon \int \underline{\mathbb{F}}^P_{T,*} \to \mathcal{E}$ in fact follows from substantially less than the previous proposition suggests.

Observation 4.9.7. Let $X_+ \in \underline{\mathbb{F}}^P_{T,*}(A)$ be a finite pointed P-set, and let $p\colon A \to B$ be a map in P. Furthermore let $C \to B$ be the identity of B. Considering the parametrized Segal maps associated to this data, we note that $A \times_B B = A$, so there is just one. We call this map $\rho_{p,X}$. If $X = A_+$, we simply write ρ_p.

Proposition 4.9.8. *Let $\mathcal{E}$ be an ∞-category and consider a T-functor $F\colon \underline{\mathbb{F}}^P_{T,*} \to \underline{\mathcal{E}}_T$ which corresponds to a functor $\widetilde{F}\colon \int \underline{\mathbb{F}}^P_{T,*} \to \mathcal{E}$ of ∞-categories. Then F is a P-semiadditive monoid in $\underline{\mathcal{E}}_T$ if and only if F is fiberwise semiadditive and for every map $p\colon A \to B$ in P, the map*

$$\widetilde{F}(\rho_p)\colon \widetilde{F}(B, A_+) \to \widetilde{F}(A, A_+)$$

is an equivalence.

Proof. First we observe that F is a P-semiadditive monoid in $\underline{\mathcal{E}}_T$ if and only if F is fiberwise semiadditive and for every map $p\colon A \to B$ in P and every finite pointed P-set $X_+ \in \underline{\mathbb{F}}^P_{T,*}(A)$, the map

$$\widetilde{F}(\rho_{p,X})\colon \widetilde{F}(B, X_+) \to \widetilde{F}(A, X_+)$$

is an equivalence. For this it suffices to observe that the following triangle commutes

$$\begin{array}{ccc} \widetilde{F}(C, (X \times_B C)_+) & \xrightarrow{(\widetilde{F}(\rho_i))_{i=1}^n} & \prod_{i=1}^n \widetilde{F}(C_i, (X \times_A C_i)_+) \\ & \searrow & \uparrow {\scriptstyle (\widetilde{F}(\rho_{p_i, X \times_A C_i}))_{i=1}^n} \\ & & \prod_{i=1}^n \widetilde{F}(C, (X \times_A C_i)_+). \end{array}$$

Next suppose that $X = \coprod C_i$. We note that by fiberwise semi-additivity of F, $\tilde{F}(\rho_{p,X})$ is equal to a product of the $\tilde{F}(\rho_{p,C_i})$, and therefore we can further reduce to the case where $X = C$ is in T. Write $q\colon C \to A$ for the map in P expressing C as a finite P-set over A. Finally we claim that the following diagram

$$\begin{array}{ccc} (B, C_+) & \xrightarrow{\rho_{p,C}} & (A, C_+) \\ & \searrow^{\rho_{pq}} & \downarrow {\scriptstyle \rho_p} \\ & & (C, C_+) \end{array}$$

commutes in $\int \underline{\mathbb{F}}^P_{T,*}$. This can readily be checked from the definitions. Therefore after applying $\tilde{F}$, the 2-out-of-3 property implies that it suffices to assume that $\tilde{F}(\rho_p)$ is an equivalence for all $p \in P$. $\square$

Remark 4.9.9. While Proposition 4.9.8 gives an explicit description of the underlying ∞-category of $\underline{\mathrm{CMon}}^P(\underline{\mathcal{E}_T})$, a similar analysis in fact describes the whole T-∞-category $\underline{\mathrm{CMon}}^P(\underline{\mathcal{E}_T})$. At an object $B' \in T$, it consists of those T-functors $F\colon \underline{\mathbb{F}}^P_{T,*} \times \underline{B'} \to \underline{\mathcal{E}}_T$ whose curried map $F'\colon \underline{\mathbb{F}}^P_{T,*} \to \underline{\mathrm{Fun}}(\underline{B'}, \underline{\mathcal{E}}_T)$ is P-semiadditive, see Corollary 4.6.9. On the other hand, the T-functor F corresponds to a functor $\tilde{F}\colon \int(\underline{\mathbb{F}}^P_{T,*} \times \underline{B'}) \to \mathcal{E}$ by Lemma 2.2.13. Carrying out the same analysis as in the proofs of Proposition 4.9.6 and Proposition 4.9.8 shows that F corresponds to a P-semiadditive functor $F'\colon \underline{\mathbb{F}}^P_{T,*} \to \underline{\mathrm{Fun}}_T(\underline{B'}, \underline{\mathcal{E}}_T)$ if and only if the following conditions are satisfied:

- The T-functor F' is fiberwise semiadditive; put differently, for any $f\colon B \to B'$ the restriction of $\tilde{F}$ to the (non-full) subcategory $\underline{\mathbb{F}}^P_{T,*}(B) \times \{f\} \subset \underline{\mathbb{F}}^P_{T,*}(B) \times \underline{B'}(B) \subset \int(\underline{\mathbb{F}}^P_{T,*} \times \underline{B'})$ is semiadditive in the usual sense.
- For every map $p\colon A \to B$ in P and every map $f\colon B \to B'$ in T, the map
$$\tilde{F}(\rho_p, p)\colon \tilde{F}(B, A_+, f) \to \tilde{F}(A, A_+, p \circ f)$$
is an equivalence.

5. The universal property of special global Γ-spaces

In this section we want to identify the global ∞-category of Orb-commutative monoids in global spaces with the various models of *globally* and *G-globally coherently commutative monoids* studied in [Sch18, Chapter 2] and [Len20, Chapter 2]. In particular, after evaluating at the trivial group, this will yield an equivalence between the underlying ordinary ∞-category of Orb-commutative monoids in global spaces with Schwede's *ultra-commutative monoids* with respect to finite groups.

For this, the model based on so-called *(special) G-global Γ-spaces* will be the most convenient; we recall the relevant theory in 5.1 below and show how G-global Γ-spaces assemble into a global ∞-category $\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}$. In 5.2 we will then identify $\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}$ with a certain parametrized functor category, from which we will deduce the desired comparison between *special* G-global Γ-spaces and $\underline{\mathrm{CMon}}^{\mathrm{Orb}}(\underline{\mathrm{Spc}}_{\mathrm{Glo}})$ in 5.3. This will then immediately imply various universal properties of global Γ-spaces, including Theorem B from the introduction.

5.1. A reminder on G-global Γ-spaces.

Segal [Seg74] introduced *(special) Γ-spaces* as a model of commutative monoids in the ∞-category of spaces, and an equivariant generalization of his theory was later established by Shimakawa [Shi89]. We will be concerned with the following G-global refinement [Len20, Section 2.2] of this story:

Definition 5.1.1. We write Γ for the category of finite pointed sets and pointed maps. For any $n \geq 0$ we let $n^+ := \{0, \dots, n\}$ with basepoint 0.

We moreover write $\mathbf{\Gamma}$-$\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-$\mathbf{SSet}$ for the category of functors $\Gamma \to \boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-$\mathbf{SSet}$. A map $f\colon X \to Y$ in $\mathbf{\Gamma}$-$\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-$\mathbf{SSet}$ (i.e. a natural transformation) is called a *G-global level weak equivalence* if $f(S_+)\colon X(S_+) \to Y(S_+)$ is a $(G \times \Sigma_S)$-global weak equivalence (with respect to the Σ_S-action induced by the tautological action on S) for every finite set S.

Similarly, we write $\mathbf{\Gamma}$-$\boldsymbol{G}$-$\boldsymbol{\mathcal{I}}$-$\mathbf{SSet}$ for the category of functors $X\colon \Gamma \to \boldsymbol{G}$-$\boldsymbol{\mathcal{I}}$-$\mathbf{SSet}$, and we define G-global level weak equivalences in $\mathbf{\Gamma}$-$\boldsymbol{G}$-$\boldsymbol{\mathcal{I}}$-$\mathbf{SSet}$ analogously.

We will refer to objects of either of these categories as *G-global Γ-spaces*. Beware that [Len20] reserves this name for functors X for which $X(0^+)$ is a terminal object, while for us the above definition will be more useful. However, we will later only be interested in so-called *special* G-global Γ-spaces, for which this technicality will turn out to be irrelevant, see Proposition 5.1.7 below.

5.1.1. *Model categorical properties.* Just like in the unstable case we have the following Elmendorf type theorem expressing the homotopy theory of special G-global Γ-spaces in terms of enriched presheaves:

Proposition 5.1.2. *The G-global level weak equivalences are part of a simplicial combinatorial model structure on $\mathbf{\Gamma}$-$\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-$\mathbf{SSet}$.*

Moreover, if we write $\mathbf{O}_\Gamma^{G\text{-gl}} \subset \mathbf{\Gamma}$-$\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-$\mathbf{SSet}$ for the full subcategory spanned by the objects $\Gamma_{H,S,\varphi} := (\Gamma(S_+,-) \times E\mathcal{M} \times G_\varphi)/H$ (where H is a finite group, S a finite H-set, $\varphi\colon H \to G$ a homomorphism, and G_φ denotes G with H acting from the right via φ), then the enriched Yoneda embedding induces a functor

$$\Phi_\Gamma\colon \mathbf{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet} \to \mathbf{PSh}(\mathbf{O}_\Gamma^{G\text{-gl}})$$

which is the right half of a Quillen equivalence when we equip the right hand side with the projective model structure.

Proof. For any finite group H, any finite H-set S, and any homomorphism $\varphi\colon H \to G$, the functor $X \mapsto X(S_+)^\varphi$ preserves filtered colimits, pushouts along injections, and it is corepresented by $\Gamma_{H,S,\varphi}$ (via evaluation at $[\mathrm{id},1,1]$). Thus, the objects of $\mathbf{O}_\Gamma^{G\text{-gl}}$ form a *set of orbits* in the sense of [DK84, 2.1], and the above statements are instances of Theorems 2.2 and 3.1 of *op. cit.* □

Remark 5.1.3. We can make the morphism spaces in $\mathbf{O}_\Gamma^{G\text{-gl}}$ explicit, analogously to Remark 3.3.6: as observed in the above proof, we have for any (H,S,φ) as above and any G-global Γ-space X an isomorphism

$$\varepsilon\colon \mathrm{maps}(\Gamma_{H,S,\varphi}, X) \to X(S_+)^\varphi$$

given by evaluation at $[\mathrm{id},1,1]$. Specializing this to $X = \Gamma_{K,T,\psi}$, we see that $\mathbf{O}_\Gamma^{G\text{-gl}}$ is a $(2,1)$-category (the quotient $\Gamma_{K,T,\psi} = (\Gamma(T_+,-) \times E\mathcal{M} \times G_\psi)/K$ being the nerve of a groupoid as K acts freely on $E\mathcal{M}$) and that n-simplices of $\mathrm{maps}(\Gamma_{H,S,\varphi}, \Gamma_{K,T,\psi})$ correspond to φ-fixed classes $[f; u_0, \dots, u_n; g]$ where $f\colon T_+ \to S_+$, $u_0, \dots, u_n \in \mathcal{M}$, and $g \in G$.

Moreover, a direct computation shows that under the above identification composition is given by

$$[f'; u_0', \dots, u_n'; g'][f; u_0, \dots, u_n; g] = [ff'; u_0u_0', \dots, u_nu_n'; gg']$$

and that the following diagram commutes for any $X \in \mathbf{\Gamma}$-$\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-$\mathbf{SSet}$:

$$\begin{array}{ccc} \Phi(X)(\Gamma_{K,T,\psi}) & \xrightarrow{\Phi(X)[f;u_0,\dots,u_n;g]} & \Phi(X)(\Gamma_{H,S,\varphi}) \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ X(T_+)^\psi & \xrightarrow[X(f)\circ\left([u_0,\dots,u_n;g]\cdot-\right)]{} & X(S_+)^\varphi. \end{array}$$

5.1.2. *The global ∞-category of global Γ-spaces.* Letting G vary, the categories $\boldsymbol{\Gamma}$-$\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** together with the G-global weak equivalences assemble into a global relative category with functoriality given by restrictions (apply Lemma 3.1.9 with α replaced by $\alpha \times \Sigma_S$). Localizing, we then get a global ∞-category $\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}$. Analogously, we obtain a global ∞-category $\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$ whose value at a finite group G is the localization of $\boldsymbol{\Gamma}$-$\boldsymbol{G}$-$\boldsymbol{\mathcal{I}}$-**SSet** at the G-global weak equivalences, with functoriality given via restrictions.

Proposition 5.1.4. *The evaluation functor* ev_ω *induces an equivalence* $\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}} \simeq \underline{\Gamma\mathscr{S}}^{\mathrm{gl}}$.

Proof. Precisely the same argument as in [Len20, Theorem 2.2.33] shows that the functor $\mathrm{ev}_\omega\colon \boldsymbol{\Gamma}\text{-}\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} \to \boldsymbol{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}$ admits a homotopy inverse for any G (given by applying the homotopy inverse of $\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet} \to \boldsymbol{E\mathcal{M}}\text{-}\mathbf{SSet}$ levelwise). □

For every G-global Γ-space X, evaluating at 1^+ (with trivial action) yields an *underlying G-global space* $X(1^+)$, and this obviously yields a global functor $\mathbb{U}\colon \underline{\Gamma\mathscr{S}}^{\mathrm{gl}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$. For later use we record:

Lemma 5.1.5. *The global functor* $\mathbb{U}$ *admits a left adjoint, which is pointwise induced by* $\Gamma(1^+,-) \times -$.

Proof. By the Yoneda Lemma we have an adjunction

$$\Gamma(1^+,-) \times -\colon \boldsymbol{E\mathcal{M}}\text{-}\mathbf{SSet} \rightleftarrows \boldsymbol{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\mathbf{SSet} : \mathrm{ev}_{1^+},$$

and for every finite group G pulling through the G-actions yields an adjunction $\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet} \rightleftarrows \boldsymbol{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}$ of 1-categories such that both functors are homotopical. In particular, $\mathbb{U}$ admits a pointwise adjoint of the above form.

For the Beck-Chevalley condition it suffices now to observe that since all functors are homotopical, the Beck-Chevalley comparison map of ∞-categorical localizations can be modelled by the 1-categorical Beck-Chevalley map, and the latter is even the identity by construction. □

5.1.3. *Specialness.* Just like in the non-equivariant case, in the theory of global coherent commutativity one typically isn't interested in *all* G-global Γ-spaces, but only those satisfying a certain 'specialness' condition (although the fact that there are *non-special* G-global Γ-spaces is what will make this model so convenient for our comparison):

Definition 5.1.6 (cf. [Len20, Definition 2.2.50])**.** A G-global Γ-space $X\colon \Gamma \to \boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}$ is called *special* if for every finite set S the *Segal map*

$$\rho\colon X(S_+) \to \prod_{s\in S} X(1^+)$$

induced by the characteristic maps $\chi_s\colon S_+ \to 1^+$ of the elements $s \in S$ is a $(G\times\Sigma_S)$-global weak equivalence.

We write $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}} \subset \underline{\Gamma\mathscr{S}}^{\mathrm{gl}}$ for the full global subcategory spanned in degree G by the special G-global Γ-spaces, and $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{*} \subset \underline{\Gamma\mathscr{S}}^{\mathrm{gl}}$ for those special Γ-spaces X for which $X(0^+)$ is terminal in the 1-categorical sense (and not just G-globally weakly equivalent to a terminal object).

Analogously, we define specialness for elements of **Γ-*G*-*𝓘*-SSet**, yielding nested full global subcategories $\underline{\Gamma\mathscr{S}}^{\text{gl, spc}}_{\mathcal{I},*} \subset \underline{\Gamma\mathscr{S}}^{\text{gl, spc}}_{\mathcal{I}} \subset \underline{\Gamma\mathscr{S}}^{\text{gl}}_{\mathcal{I}}$.

Proposition 5.1.7. *All maps in the commutative diagram*

$$\begin{array}{ccc} \underline{\Gamma\mathscr{S}}^{\text{gl, spc}}_{\mathcal{I},*} & \hookrightarrow & \underline{\Gamma\mathscr{S}}^{\text{gl, spc}}_{\mathcal{I}} \\ {\scriptstyle \text{ev}_\omega}\downarrow & & \downarrow{\scriptstyle \text{ev}_\omega} \\ \underline{\Gamma\mathscr{S}}^{\text{gl, spc}}_{*} & \hookrightarrow & \underline{\Gamma\mathscr{S}}^{\text{gl, spc}} \end{array}$$

of global ∞-categories are equivalences.

Proof. For the left hand vertical arrow this is part of [Len20, Corollary 2.2.53]. We will now show that the lower horizontal inclusion is an equivalence; the argument for the top inclusion is then similar, and with this established the proposition will follow by 2-out-of-3.

To prove the claim, we now fix a finite group G and observe that the inclusion **Γ-*EM*-*G*-SSet**$_*$ $\hookrightarrow$ **Γ-*EM*-*G*-SSet** of those G-global Γ-spaces X with $X(0^+) = *$ admits a left adjoint given by quotienting out $X(0^+)$, i.e. forming the pushout

$$\begin{array}{ccc} \operatorname{const} X(0^+) & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & X/X(0^+) \end{array}$$

where the top map is induced by the unique pointed maps $0^+ \to S_+$ for varying S. It will therefore be enough that the right hand vertical map is a G-global level weak equivalence if X is special. But indeed, in this case $\operatorname{const} X(0^+) \to *$ is a G-global level weak equivalence (as $X(0^+)$ is G-globally and hence also $(G \times \Sigma_T)$-globally weakly contractible for any T by the special case $S = \varnothing$ of the Segal condition), while for *any* Γ-space the top map is an injective cofibration as $X(0^+) \to X(S_+)$ admits a retraction via functoriality. The claim then follows as pushouts along injective cofibrations preserve G-global level weak equivalences by [Len20, Lemma 1.1.14] applied levelwise. ☐

5.2. **Global Γ-spaces as parametrized functors.** In this section we will prove the key computational ingredient to the universal property of special global Γ-spaces in form of the following description of the global ∞-category $\underline{\Gamma\mathscr{S}}^{\text{gl}}$ of all global Γ-spaces:

Theorem 5.2.1. *There exists an equivalence of global ∞-categories*

$$\Xi\colon \underline{\Gamma\mathscr{S}}^{\text{gl}} \simeq \underline{\text{Fun}}_{\text{Glo}}(\underline{\mathbb{F}}^{\text{Orb}}_{\text{Glo},*}, \underline{\text{Spc}})$$

together with a natural equivalence filling

$$\begin{array}{ccc} \underline{\Gamma\mathscr{S}}^{\text{gl}} & \xrightarrow{\ \Xi\ } & \underline{\text{Fun}}_{\text{Glo}}(\underline{\mathbb{F}}^{\text{Orb}}_{\text{Glo},*}, \underline{\text{Spc}}) \\ {\scriptstyle \mathbb{U}}\downarrow & & \downarrow{\scriptstyle \text{ev}_{(\text{id}_1)_+}} \\ \underline{\mathscr{S}}^{\text{gl}} & \xrightarrow[\simeq]{} & \underline{\text{Spc}}_{\text{Glo}} \end{array}$$

where the unlabelled arrow on the bottom is 'the' essentially unique equivalence (see Theorem 3.3.2).

5.2.1. *A model of finite* Orb*-sets.* The proof of the theorem will occupy this whole subsection. As the first step, we will recognize $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo}}$ and $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}$ as some familiar global 1-categories:

Construction 5.2.2. For any finite group G, we write $\mathcal{F}_G$ for the category of finite G-sets. The assignment $G \mapsto \mathcal{F}_G$ becomes a strict 2-functor in $\mathbf{Glo}^{\mathrm{op}}$ via restrictions, and we denote the resulting global category by $\mathcal{F}_\bullet$.

We moreover write $\mathcal{F}_\bullet^+$ for the corresponding global category of pointed finite G-sets.

Lemma 5.2.3. *There is an essentially unique equivalence of global ∞-categories $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo}} \simeq \mathrm{N}\mathcal{F}_\bullet$. Up to isomorphism, this sends $(H \hookrightarrow G) \in \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo}}(G)$ to $G/H \in \mathcal{F}_G$ for all finite groups $H \subset G$.*

Proof. By Corollary 4.2.17 there is an essentially unique global functor $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo}} \to \mathrm{N}\mathcal{F}_\bullet$ that preserves Orb-coproducts and the terminal object. It remains to construct any such equivalence and prove that it admits the above description.

By construction the left hand side is a subcategory of $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$. On the other hand, we have a fully faithful functor of global ∞-categories $\iota\colon \mathrm{N}\mathcal{F}_\bullet \to \underline{\mathscr{S}}^{\mathrm{gl}}$ that is given by sending a finite G-set X to X considered as a discrete simplicial set with trivial $E\mathcal{M}$-action. It then suffices to show that the unique equivalence $F\colon \underline{\mathrm{Spc}}_{\mathrm{Glo}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ restricts accordingly and admits the above description.

For this we first observe that indeed $F(i\colon H \hookrightarrow G) \simeq G/H$ for every $H \subset G$: namely, i can be identified with $i_!p^*(*)$ where $p\colon H \to 1$ is the unique homomorphism, and since F is an equivalence it follows that $F(i) \simeq i_!p^*F(*) = i_!p^*(*)$, which can in turn be identified with G/H by Lemma 3.1.9.

As a consequence of Corollary 4.2.16, each $\mathbb{F}^{\mathrm{Orb}}_{\mathrm{Glo}}(G)$ is closed under (ordinary) finite coproducts, so F preserves them (as a functor to $\underline{\mathscr{S}}^{\mathrm{gl}}$). Together with the above computation, it immediately follows that F restricts to an essentially surjective functor $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo}} \to \operatorname{ess\,im} \iota$ as claimed. ☐

Corollary 5.2.4. *There is an essentially unique equivalence $\theta\colon \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \simeq \mathrm{N}\mathcal{F}_\bullet^+$. Up to isomorphism, this sends $(H \hookrightarrow G)_+$ to G/H_+ for all finite groups $H \subset G$.*

Proof. The existence of such an equivalence is immediate from the previous lemma. For the uniqueness part, it suffices by Corollary 4.7.8 that any autoequivalence of $\mathcal{F}_1^+$ preserves 1^+ up to isomorphism, which is immediate from the observation that this is the only non-zero object without non-trivial automorphisms. ☐

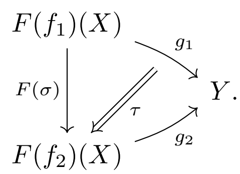
5.2.2. *Grothendieck constructions.* Thanks to Remark 2.2.14, understanding the global functor category $\underline{\mathrm{Fun}}_{\mathrm{Glo}}(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}, \underline{\mathrm{Spc}})$ is equivalent to understanding the unstraightenings $\int \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G}$ of the diagram $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ naturally in $G \in \mathrm{Glo}$. However as an upshot of the previous subsection, the functors $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G}$ are modelled by strict 2-functors of strict $(2,1)$-categories, which will allow us to give a reasonably explicit description in terms of the classical Grothendieck construction:

Construction 5.2.5. Let $\mathscr{C}$ be a strict $(2,1)$-category. We recall (see [Buc14, Construction 2.2.1] or [HNP19, Definition 6.1]) the Grothendieck construction $\iint F$ for a strict 2-functor $F\colon \mathscr{C} \to \mathbf{Cat}_{(2,1)}$ into the $(2,1)$-category of $(2,1)$-categories:

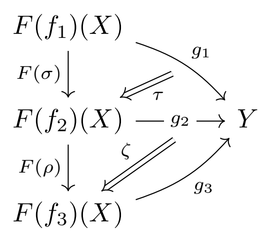
(1) The objects of $\iint F$ are given by pairs (c, X) with $c \in \mathscr{C}$ and $X \in F(c)$
(2) A morphism from (c, X) to (d, Y) is given by a pair of a map $f\colon c \to d$ and a map $g\colon F(f)(X) \to Y$ in $F(d)$; if $(f', g')\colon (d, Y) \to (e, Z)$ is another morphism, then their composite is

$$(f', g')(f, g) = \big(f'f, F(f'f)(X) = F(f')F(f)(X) \xrightarrow{F(f')(g)} F(f')(Y) \xrightarrow{g'} Z\big).$$

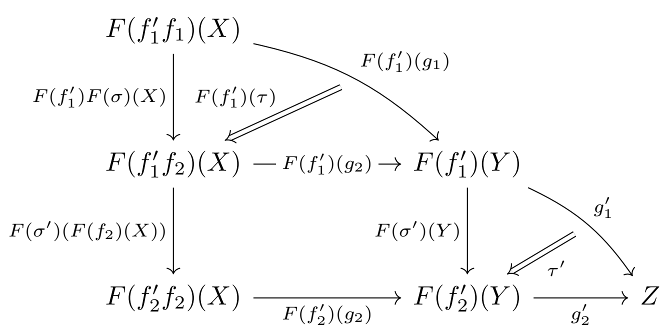
(3) A 2-cell $(f_1, g_1) \Rightarrow (f_2, g_2)$ between maps $(c, X) \to (d, Y)$ is given by a 2-cell $\sigma\colon f_1 \Rightarrow f_2$ in $\mathscr{C}$ together with a 2-cell



in $F(d)$. If $(\rho, \zeta)\colon (f_2, g_2) \Rightarrow (f_3, g_3)$ is another 2-cell, then the composite $(\rho, \zeta) \circ (\sigma, \tau)$ is given by the composition in $\mathscr{C}$ and the pasting



in $F(d)$. Moreover, if $(\sigma', \tau')\colon (f_1', g_1') \Rightarrow (f_2', g_2')$ is a 2-cell between maps $(d, Y) \to (e, Z)$, then the horizontal composite $(\sigma', \tau') \odot (\sigma, \tau)$ is given by the horizontal composite $\sigma' \odot \sigma$ and the pasting



where the square commutes as $F(\sigma')$ is a natural transformation $F(f_1') \Rightarrow F(f_2')$.

This comes with a natural strict 2-functor $\pi\colon \iint F \to \mathscr{C}$ given by projecting onto the first coordinate. By [HNP19, Proposition 2.15] the homotopy coherent nerve of this functor is a cocartesian fibration representing $\mathrm{N}_\Delta \circ F$. Put differently, there is a natural equivalence $\int(\mathrm{N}_\Delta \circ F) \simeq \mathrm{N}_\Delta(\iint F)$ over $\mathrm{N}_\Delta(C)$ from the usual marked unstraightening to the homotopy coherent nerve of the 2-categorical Grothendieck construction which preserves cocartesian edges.

We can also describe the behaviour of this equivalence on fibers as follows: for any $c \in \mathscr{C}$ the composition

$$\mathrm{N}_\Delta F(c) \hookrightarrow \mathrm{N}_\Delta(\textstyle\iint F) \simeq \int(\mathrm{N}_\Delta \circ F)$$

of the natural embedding with the above equivalence agrees with the usual identification of $\mathrm{N}_\Delta F(c)$ with the fiber of the unstraightening $\int(\mathrm{N}_\Delta \circ F)$ over c, see [HNP19, proof of Proposition 6.25]. In particular, for the cocartesian fibration $\mathrm{N}_\Delta(\iint F)$ the notation (c, X) (with $X \in F(c)$) for vertices is compatible with Notation 4.9.4. As the above equivalence moreover preserves cocartesian edges, we also immediately deduce the analogous statement for 1-simplices.

Construction 5.2.6. For every finite group G, we define a strict $(2,1)$-category $\iint \mathfrak{F}_G^{\mathrm{gl},+}$ as follows: Sending a finite group H to the product of the strict $(2,1)$-category $\mathcal{F}_H^+$ of finite pointed H-sets and the groupoid $\mathbf{Glo}(H, G) := \mathbf{Hom}_{\mathbf{Glo}}(H, G)$ of group homomorphisms $H \to G$ and conjugations defines a strict 2-functor

$$\mathcal{F}_\bullet^+ \times \mathbf{Glo}(-, G)\colon \mathbf{Glo}^{\mathrm{op}} \to \mathbf{Cat}_{(2,1)}.$$

Composing this functor with the equivalence of strict $(2,1)$-categories $\gamma\colon \mathfrak{O}^{\mathrm{gl}} \xrightarrow{\sim} \mathbf{Glo}$ from Construction 3.3.14, we obtain a strict 2-functor

$$\mathfrak{F}_G^{\mathrm{gl},+} := \big(\mathcal{F}_\bullet^+ \times \mathbf{Glo}(-, G)\big) \circ \gamma\colon (\mathfrak{O}^{\mathrm{gl}})^{\mathrm{op}} \to \mathbf{Cat}_{(2,1)}.$$

As before, we let $\iint \mathfrak{F}_G^{\mathrm{gl},+}$ denote the 2-categorical Grothendieck construction of $\mathfrak{F}_G^{\mathrm{gl},+}$. The assignment $G \mapsto \iint \mathfrak{F}_G^{\mathrm{gl},+}$ then becomes a strict 2-functor $\iint \mathfrak{F}_\bullet^{\mathrm{gl},+}\colon \mathbf{Glo} \to \mathbf{Cat}_{(2,1)}$ via (post)composition in $\mathbf{Glo}$.

As promised we can now prove:

Proposition 5.2.7. *There exists an equivalence*

$$\Theta_G\colon \mathrm{N}_\Delta\big(\textstyle\iint \mathfrak{F}_G^{\mathrm{gl},+}\big) = \mathrm{N}_\Delta\big(\textstyle\iint (\mathcal{F}_\bullet^+ \times \mathbf{Glo}(-, G)) \circ \gamma\big) \xrightarrow{\simeq} \int \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G}$$

of ∞-categories natural in $G \in$ Glo with the following properties:

(1) For all $H \in \mathfrak{O}^{\mathrm{gl}}$ and $\varphi\colon H \to G$ in $\mathbf{Glo}$, the following diagram commutes up to equivalence:

$$\begin{array}{ccc} \mathrm{N}_\Delta(\mathcal{F}_H^+ \times \{\varphi\}) & \xrightarrow{\theta_H} & \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \{\varphi\} \\ \downarrow & & \downarrow \\ \mathrm{N}_\Delta(\mathcal{F}_H^+ \times \mathbf{Glo}(H, G)) & & \mathbb{F}^{\mathrm{Orb}}_{\mathrm{Glo}} \times \underline{G}(H) \\ \downarrow & & \downarrow \\ \mathrm{N}_\Delta(\iint \mathfrak{F}_G^{\mathrm{gl},+}) & \xrightarrow[\Theta_G]{} & \int \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G} \end{array} \tag{12}$$

where θ is the equivalence from Corollary 5.2.4 and the bottom vertical arrows are the chosen identifications of the fibers over H.

In particular, Θ_G restricts to an equivalence between the non-full subcategory $\mathrm{N}_\Delta(\iint \mathfrak{F}_G^{\mathrm{gl},+})_\varphi \subset \mathrm{N}_\Delta(\iint \mathfrak{F}_G^{\mathrm{gl},+})$ with objects of the form $(H; X, \varphi)$ (for $X \in \mathcal{F}_H^+$) and morphisms only those that are the identities in H and φ (i.e. the image of $\mathcal{F}_H^+ \times \{\varphi\}$ under the chosen identification) and the analogous full subcategory on the right.

(2) *For all maps* $(\alpha, u)\colon K \to H$ *in* $\mathfrak{O}^{\mathrm{gl}}$ *and* $f\colon \alpha^*X \to Y$ *in* $\mathcal{F}_K^+$, *the map* $\Theta_G(\alpha, u; f, \mathrm{id}_{\varphi\alpha})$ *agrees up to equivalence with* $(\alpha; \theta_K(f), \mathrm{id}_{\varphi\alpha})$.

Proof. Specializing the above discussion we have a natural equivalence

$$\mathrm{N}_\Delta\big(\iint \mathfrak{F}_G^{\mathrm{gl},+}\big) \simeq \int \mathrm{N}_\Delta \circ \mathfrak{F}_G^{\mathrm{gl},+} = \int \mathrm{N}_\Delta \circ (\mathcal{F}_\bullet^+ \times \mathbf{Glo}(-, G)) \circ \gamma$$

between the 2-categorical and ∞-categorical Grothendieck construction sending the map $(\alpha, u; f, \mathrm{id}_{\varphi\alpha})$ on the left to the map of the same name on the right and such that for every $H \in \mathfrak{O}^{\mathrm{gl}}$ the induced map on fibers respects the identifications with $\mathcal{F}_H^+ \times \mathbf{Glo}(H, G)$.

On the other hand, as $\gamma\colon \mathfrak{O}^{\mathrm{gl}} \to \mathbf{Glo}$ is an equivalence, the right hand side is in turn naturally equivalent to the unstraightening $\int \mathrm{N}_\Delta \circ (\mathcal{F}_\bullet^+ \times \mathbf{Glo}(-, G))$ over $\mathrm{Glo}^{\mathrm{op}}$ by an equivalence sending $(\alpha, u; f, \mathrm{id}_{\varphi\alpha})$ to $(\alpha; f, \mathrm{id}_{\varphi\alpha})$ up to equivalence; again, under our chosen identifications this is just the identity on fibers.

Finally, by construction of the ∞-categorical Yoneda embedding we have an equivalence $\upsilon\colon \mathrm{N}_\Delta(\mathbf{Glo}(L, G)) \simeq \mathrm{Glo}(L, G) = \underline{G}(L)$ natural in both variables sending $\psi\colon L \to G$ to ψ, which together with the global equivalence θ from Corollary 5.2.4 induces an equivalence $\int \mathrm{N}_\Delta(\mathcal{F}_\bullet^+ \times \mathbf{Glo}(-, G)) \simeq \int \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G}$ sending $(\alpha; f, \mathrm{id}_{\varphi\alpha})$ to $(\alpha; \theta_K(f), \mathrm{id}_{\varphi\alpha})$ and that is given under the chosen identifications of the fibers over H by $\theta_H \times \upsilon$. The commutativity of (12) follows immediately, which completes the proof of the proposition. □

5.2.3. *Global* Γ*-spaces as enriched functors.* Thanks to the above proposition, we can replace the somewhat mysterious ∞-categorical unstraightenings $\int \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G}$ by the homotopy coherent nerves of the much more explicit $(2,1)$-categories $\iint \mathfrak{F}_G^{\mathrm{gl},+}$. These are suitably combinatorial to in turn admit a comparison to the $\mathbf{O}_\Gamma^{G\text{-gl}}$'s:

Construction 5.2.8. Let G be a finite group. We define $\delta\colon (\iint \mathfrak{F}_G^{\mathrm{gl},+})^{\mathrm{op}} \to \mathbf{O}_\Gamma^{G\text{-gl}}$ as follows:

(1) An object $(H; S_+, \varphi)$ consisting of a universal subgroup $H \subset \mathcal{M}$, a finite pointed H-set S_+ and a homomorphism $\varphi\colon H \to G$ is sent to $(\Gamma(S_+, -) \times E\mathcal{M} \times G_\varphi)/H$.
(2) A morphism $(u \in \mathcal{M}, \sigma\colon H \to K; f\colon \sigma^*T_+ \to S_+; g \in G)$ is sent to the map induced by $\Gamma(f, -) \times (-\cdot(u, g))$, i.e. the map corresponding to $[f; u; g]$ under the identification from Remark 5.1.3.
(3) A 2-cell $k\colon (u, \sigma; f, g) \Rightarrow (u', \sigma'; f', g')$ (for $k \in K \subset \mathcal{M}$) is sent to the 2-cell corresponding to $[f; u'k, u; g]$.

Proposition 5.2.9. *The assignment* δ *is well-defined (i.e. the above indeed represent morphisms and* 2*-cells in* $\mathbf{O}_\Gamma^{G\text{-gl}}$*) and is an equivalence of* $(2,1)$*-categories.*

Proof. We break this up into several steps.

It is well-defined on morphisms and a full 1-functor: If $(u, \sigma; f; g)$ is a morphism $(H, S_+, \varphi) \to (K, T_+, \psi)$ in the opposite of the Grothendieck construction, then $hu = u\sigma(h)$ for all $h \in H$ as (u, σ) is a morphism $H \to K$ in $\mathfrak{O}^{\mathrm{gl}}$; moreover, $c_g\psi\sigma = \varphi$ as g is a morphism $\varphi \to \psi\sigma$ in $\mathbf{Glo}(K, G)$, while $(h\cdot-) \circ f = f \circ (\sigma(h)\cdot-)$ for all $h \in H$ as f is a map of (pointed) H-sets. Thus,

$$(h, \varphi(h))\cdot[f; u; g] = [(h\cdot-)\circ f; hu; \varphi(h)g] = [f\circ(\sigma(h)\cdot-); u\sigma(h); g\psi(\sigma(h))] = [f; u; g],$$

i.e. $[f;u;g]$ is indeed φ-fixed. Note that we can also deduce this statement from (the easy direction of) [Len20, Lemma 1.2.38]: namely, if we consider $\Gamma(S_+,T_+)\times G_\psi$ as a $(G\times H)\times K$-biset, where G acts on G from the left, H acts on S_+ from the left, and K acts from the right via its given action on T_+ and its action on G via ψ, then swapping the factors defines an isomorphism

$$(\Gamma(T_+,S_+)\times E\mathcal{M}\times G_\psi)/K)^\varphi \cong \big(E\mathcal{M}\times_K(\Gamma(T_+,S_+)\times G)\big)^{(\mathrm{id}_H,\varphi)}$$

where the right hand side is the usual balanced product; *loc. cit.* then says that $[u;f;g]$ defines a vertex of the right hand side if *and only if* there exists a homomorphism $\sigma\colon H\to K$ (necessarily unique) such that $hu=u\sigma(h)$ for all $h\in H$ and moreover $(h,\varphi(h))\cdot(f,g)=(f,g)\cdot\sigma(h)$, i.e. f is equivariant as a map $\sigma^*T_+\to S_+$ and $\varphi=c_g\psi\sigma$. From the 'only if' part we then immediately deduce that the above is surjective on morphisms: a preimage of $[u;f;g]$ is given by $(u,\sigma;f;g)$.

The equality $\delta(1,1;\mathrm{id}_{S_+},1)=[1;\mathrm{id}_{S_+};1]$ shows that δ preserves identities. To see that it is also compatible with composition of 1-morphisms (whence a 1-functor), we let $(u',\sigma';f',g')$ be a map $(K;T_+;\psi)\to(L;U_+;\zeta)$ in the opposite category (so that $\sigma'\colon K\to L$ is a homomorphism and $f'\colon(\sigma')^*U_+\to T_+$ an equivariant map). Then indeed

$$\begin{aligned}\delta\big((u',\sigma';f',g')(u,\sigma;f,g)\big) &\overset{(*)}{=} \delta(uu',\sigma'\sigma;ff';gg')\\ &= [ff';uu';gg'] = \delta(u';f';g')\delta(u;f;g)\end{aligned}$$

where the somewhat surprising formula $(*)$ for the composition in the Grothendieck construction comes from the fact that σ^* does not change underlying maps of sets nor the group elements representing maps in $\mathbf{Glo}(\text{–},G)$.

It is well-defined on 2*-cells and a locally fully faithful* 2*-functor:* First, let us show that δ defines fully faithful functors

$$\mathrm{maps}\big((H;S_+,\varphi),(K;T_+,\psi)\big)\to\mathrm{maps}\big(\Gamma_{H,S,\varphi},\Gamma_{K,T,\psi}\big) \tag{13}$$

for all objects $(H;S_+,\varphi)$ and $(K;T_+,\psi)$. For this it will be enough to prove this after postcomposing with the isomorphism ε to $(\Gamma_{K,T;\psi})^\varphi$.

If now $(u_1,\sigma_1;f_1;g_1)$ and $(u_2,\sigma_2;f_2;g_2)$ are morphisms $(H;S_+;\varphi)\rightrightarrows(K;T_+;\psi)$, then [Len20, Lemma 1.2.74] shows that we have a bijection between morphisms $[f_1;u_1;g_1]\to[f_2;u_2;g_2]$ in $\Gamma^\varphi_{K,T,\psi}$ and elements $k\in K$ such that $f_1=f_2(k\cdot\text{–})$, $g_1=g_2\psi(k)$, and $\sigma_2=c_k\sigma_1$, which is explicitly given by $k\mapsto[f_1;u_2k,u_1;g_1]$. The last condition precisely says that k is a 2-cell $(u_1,\sigma_1)\Rightarrow(u_2,\sigma_2)$ in $\mathfrak{O}^{\mathrm{gl}}$, while the remaining two conditions say that $(f_2;g_2)\circ\mathfrak{F}^{\mathrm{gl},+}_G(k)=(f_1;g_1)$, which is precisely the compatibility condition for 2-cells in the Grothendieck construction. Thus, (13) is well-defined and bijective on morphisms. To see that it is indeed a functor, we observe that $\delta(1)=\mathrm{id}$ by design, and that for any further 2-cell $k'\colon(u_2,\sigma_2;f_2;g_2)\Rightarrow(u_3,\sigma_3;f_3;g_3)$ we have

$$\begin{aligned}\delta(k')\circ\delta(k) &= [f_2;u_3k',u_2;g_2]\circ[f_1;u_2k;u_1;g_1]\\ &\overset{(*)}{=} [f_1;u_3k'k,u_2k;g_1]\circ[f_1;u_2k;u;g_1]\\ &= [f_1;u_3k'k,u_1;g_1] = \delta(k'k)=\delta(k'\circ k),\end{aligned}$$

where the equality $(*)$ uses $[f_2;u_3k',u_2;g_2]=[f_2(k\cdot\text{–}),u_3k'k,u_2k;g_2\psi(k)]$ together with the above relations.

To complete the current step, it now only remains to show that δ is compatible with horizontal composition of 2-cells, i.e. if $(u_1', \sigma_1'; f_1'; g_1'), (u_2', \sigma_2'; f_2'; g_2') \colon (K; T_+; \psi) \rightrightarrows (L; U_+; \zeta)$ are parallel morphisms and $\ell \colon (u_1', \sigma_1'; f_1'; g_1') \Rightarrow (u_2', \sigma_2'; f_2'; g_2')$, then $\delta(\ell \odot k) = \delta(\ell) \odot \delta(k)$. Plugging in the definitions, the left hand side is given by $\delta(\ell\sigma_2(k)) = [f_1 f_1'; u_2 u_2' \ell \sigma_2(k), u_1 u_1'; g_1 g_1']$ while the right hand side evaluates to $[f_1'; u_2'\ell, u_1'; g_1'] \odot [f_1; u_2 k, u_1; g_1] = [f_1 f_1'; u_2 k u_2' \ell; u_1 u_1'; g_1 g_1']$. But $k u_2' = u_2' \sigma_2'(k)$ as (u_2', σ_2') is a morphism, while $\sigma_2'(k)\ell = \ell\sigma_2(k)$ as ℓ is a 2-cell, whence $u_2 k u_2' \ell = u_2 u_2' \ell \sigma_2(k)$ as desired.

The 2-functor δ is an equivalence: We have shown above that δ is a 2-functor, surjective on 1-cells, and bijective on 2-cells. As it is clearly surjective on objects, the claim follows immediately. □

Together with the Elmendorf Theorem for G-global Γ-spaces, we can now describe the global relative category of global Γ-spaces in terms of suitable simplicially enriched functor categories. The structure of the argument is very similar to the arguments following Construction 3.3.10.

Construction 5.2.10. We define

$$\Psi_\Gamma \colon \boldsymbol{\Gamma\text{-}E\mathcal{M}\text{-}G\text{-}\mathbf{SSet}} \to \mathbf{Fun}(\textstyle\iint \mathfrak{F}_G^{\mathrm{gl},+}, \mathbf{SSet})$$

as follows:

(1) If X is any G-global Γ-space, then $\Psi_\Gamma(X)(H; S_+; \varphi) = X(S_+)^\varphi$. If $(K; T_+; \psi)$ is another object, then an n-simplex

$$(u_0, \sigma_0; f_0; g_0) \xRightarrow{k_1} (u_1, \sigma_1; f_1; g_1) \xRightarrow{k_2} \cdots \xRightarrow{k_n} (u_n, \sigma_n; f_n; g_n) \tag{14}$$

of $\mathrm{maps}((K; T_+; \psi), (H; S_+; \varphi))$ is sent to the composition

$$\big((u_n k_n \cdots k_1, \dots, u_1 k_1, u_0; g_0) \cdot -\big) \circ X(f_0).$$

(2) If $f \colon X \to Y$ is any map of G-global Γ-spaces, then

$$\Psi_\Gamma(f)(H; S_+; \varphi) = f(S_+)^\varphi \colon X(S_+)^\varphi \to Y(S_+)^\varphi.$$

Proposition 5.2.11. *The assignment Ψ_Γ is well-defined and it descends to an equivalence when we localize the source at the G-global level weak equivalences and the target at the levelwise weak homotopy equivalences.*

Proof. One argues precisely as in the proof of Proposition 3.3.11 that Ψ_Γ is well-defined and isomorphic (via corepresentability) to the composite

$$\boldsymbol{\Gamma\text{-}E\mathcal{M}\text{-}G\text{-}\mathbf{SSet}} \xrightarrow{\Phi_\Gamma} \mathbf{Fun}((\mathbf{O}_\Gamma^{G\text{-gl}})^{\mathrm{op}}, \mathbf{SSet}) \xrightarrow{\delta^*} \mathbf{Fun}(\textstyle\iint \mathfrak{F}_G^{\mathrm{gl},+}, \mathbf{SSet}).$$

The claim now follows from Proposition 5.1.2 together with Proposition 5.2.9. □

Proposition 5.2.12. *The maps Ψ_Γ are strictly 2-natural in* **Glo** *(where the right hand side is a 2-functor in G as before).*

Proof. We again break this up into two steps:

The Ψ_Γ's are 1-natural: Let $\alpha \colon G \to G'$ be a group homomorphism. We will first show that we have for every G-global Γ-space X an equality of enriched functors $\Psi_\Gamma(\alpha^* X) = \Psi_\Gamma(X) \circ \big(\iint(\mathcal{F}_\bullet^+ \times \mathbf{Glo}(-, \alpha)) \circ \gamma\big)$. To prove this, we first observe that this holds on objects as $X(S_+)^{\alpha\varphi} = (\alpha^* X)(S_+)^\varphi$ for all universal $H \subset \mathcal{M}, \varphi \colon H \to$

G. Given now an n-simplex (14) of maps$((H;S_+;\varphi),(K;T_+;\psi))$, it is straightforward to check that both $\Psi_\Gamma(\alpha^*X)$ and $\Psi_\Gamma(X)\circ\big(\iint(\mathcal{F}^+_\bullet\times\mathbf{Glo}(-,\alpha))\circ\gamma\big)$ send this to the restriction of the composite

$$\big((u_nk_n\cdots k_1,\dots,u_1k_1,u_0;\alpha(g))\cdot-\big)\circ X(f).$$

With this established, naturality on morphisms can be checked levelwise, i.e. after evaluating at each $(H;S_+;\varphi)$. However, for any map f both $\Psi(\alpha^*f)(H;S_+;\varphi)$ and $\Psi(f)(\iint(\mathcal{F}^+_\bullet\times\mathbf{Glo}(-,\alpha))\circ\gamma)(H;S_+;\varphi)$ are simply given by a restriction of $f(S_+)$.

The Ψ_Γ's are 2-natural: It only remains to show that for each $\alpha,\beta\colon G\to G'$ and $g'\colon\alpha\Rightarrow\beta$ the two pastings

$$\mathbf{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}\ \underset{(\alpha')^*}{\overset{\alpha^*}{\rightrightarrows}}\ \Downarrow g'\ \mathbf{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}\xrightarrow{\Psi_\Gamma}\mathbf{Fun}(\textstyle\iint\mathfrak{F}^{\mathrm{gl},+}_G,\mathbf{SSet})$$

and

$$\mathbf{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}\xrightarrow{\Psi_\Gamma}\mathbf{Fun}(\textstyle\iint\mathfrak{F}^{\mathrm{gl},+}_{G'},\mathbf{SSet})\ \underset{(\alpha')^*}{\overset{\alpha^*}{\rightrightarrows}}\ \Downarrow g'\ \mathbf{Fun}\big(\textstyle\iint\mathfrak{F}^{\mathrm{gl},+}_G,\mathbf{SSet}\big)$$

agree. However, as we have already established 1-naturality, this can be again checked pointwise in $\mathbf{\Gamma}\text{-}\boldsymbol{E\mathcal{M}}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}$ and levelwise in $\iint\mathfrak{F}^{\mathrm{gl},+}_G$, where both are simply given by restriction of the action of g'. $\square$

5.2.4. *The comparison.* Putting everything together we now get:

Proof of Theorem 5.2.1. Arguing precisely as in the proof of Theorem 3.3.2, we deduce from Propositions 5.2.11 and 5.2.12 that we have an equivalence of global ∞-categories

$$\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}\simeq\mathrm{Fun}(\mathrm{N}_\Delta\textstyle\iint\mathfrak{F}^{\mathrm{gl},+}_\bullet,\mathrm{Spc})$$

given on objects in degree G by sending a G-global Γ-space X to $\mathrm{N}_\Delta(P\circ\Psi_\Gamma(X))$ where P is our favourite simplicially enriched Kan fibrant replacement functor.

On the other hand, Proposition 5.2.7 provides an equivalence between the right hand side and $\mathrm{Fun}(\int\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo}}\times\underline{(-)},\mathrm{Spc})$. The desired equivalence now follows as Remark 2.2.14 also gives a natural equivalence

$$\underline{\mathrm{Fun}}_{\mathrm{Glo}}(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*},\underline{\mathrm{Spc}}_{\mathrm{Glo}})\simeq\mathrm{Fun}(\textstyle\int(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}\times\underline{(-)}),\mathrm{Spc}).\tag{15}$$

It remains to construct an equivalence filling the diagram on the left in

$$\begin{array}{ccc}\underline{\Gamma\mathscr{S}}^{\mathrm{gl}} & \xrightarrow{\Xi} & \underline{\mathrm{Fun}}_{\mathrm{Glo}}(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*},\underline{\mathrm{Spc}}_{\mathrm{Glo}})\\ {\scriptstyle\mathbb{U}}\downarrow & & \downarrow{\scriptstyle\mathrm{ev}_{\mathrm{id}_+}}\\ \underline{\mathscr{S}}^{\mathrm{gl}} & \xrightarrow{\simeq} & \underline{\mathrm{Spc}}_{\mathrm{Glo}}\end{array}\qquad\begin{array}{ccc}\underline{\Gamma\mathscr{S}}^{\mathrm{gl}} & \xrightarrow{\Xi} & \underline{\mathrm{Fun}}_{\mathrm{Glo}}(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*},\underline{\mathrm{Spc}}_{\mathrm{Glo}})\\ {\scriptstyle\Gamma(1^+,-)\times-}\uparrow & & \uparrow{\scriptstyle\text{left Kan ext.}}\\ \underline{\mathscr{S}}^{\mathrm{gl}} & \xrightarrow{\simeq} & \underline{\mathrm{Spc}}_{\mathrm{Glo}}\end{array}$$

for which it is enough by passing to vertical left adjoints (as the horizontal maps are equivalences) to construct an equivalence filling the diagram on the right. By the universal property of $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ it is in turn enough for this to chase through the terminal object. Now the forgetful functor $\boldsymbol{E\mathcal{M}}\text{-}\mathbf{SSet}\to\mathbf{SSet}$ sending an

$E\mathcal{M}$-simplicial set to its underlying non-equivariant homotopy type is obviously homotopical right Quillen with left adjoint given by $E\mathcal{M}\times-$; passing to associated ∞-categories, we obtain an adjunction $\underline{\mathscr{S}}^{\text{gl}}(1)\rightleftarrows\text{Spc}$ and as $E\mathcal{M}\simeq *$ by [Len20, Example 1.2.35], we see that the left adjoint preserves the terminal objects. On the other hand, as 1 is a terminal object of Glo, the evaluation functor $\text{ev}_1\colon \underline{\text{Spc}}_{\text{Glo}}(1)\to \text{Spc}$ similarly admits a left adjoint given by const: $\text{Spc}\to\underline{\text{Spc}}_{\text{Glo}}(1)$, which again preserves the terminal object. In particular, we see by another application of the universal property of $\underline{\text{Spc}}_{\text{Glo}}$ that the equivalence $\underline{\mathscr{S}}^{\text{gl}}\simeq\underline{\text{Spc}}_{\text{Glo}}$ is compatible with these adjunctions.

We are therefore reduced to constructing a natural equivalence filling the diagram on the left in

$$\begin{array}{ccc} \underline{\Gamma\mathscr{S}}^{\text{gl}}(1) & \xrightarrow{\Xi} & \text{Fun}_{\text{Glo}}(\underline{\mathbb{F}}^{\text{Orb}}_{\text{Glo},*},\underline{\text{Spc}}_{\text{Glo}}) \\ {\scriptstyle \Gamma(1^+,-)\times E\mathcal{M}\times-}\uparrow & & \uparrow \\ \text{Spc} & = & \text{Spc} \end{array}
\qquad
\begin{array}{ccc} \underline{\Gamma\mathscr{S}}^{\text{gl}}(1) & \xrightarrow{\Xi} & \text{Fun}_{\text{Glo}}(\underline{\mathbb{F}}^{\text{Orb}}_{\text{Glo},*},\underline{\text{Spc}}_{\text{Glo}}) \\ {\scriptstyle \text{forget}\circ\mathbb{U}}\downarrow & & \downarrow{\scriptstyle \text{ev}_1\circ\text{ev}_{\text{id}_+}} \\ \text{Spc} & = & \text{Spc}, \end{array}$$

for which it is then by the same argument as before enough to construct a natural equivalence filling the diagram on the right. By Remark 2.2.15, the composite of the right hand vertical map with the equivalence (15) from the construction of Ξ is given by evaluating at $(1;1,1)$. However, by the description of Θ_1 from Proposition 5.2.7, $\Theta_1(1;1,1)=(1;1,1)$, so it follows by construction of Ξ that the upper path through this diagram is induced by the homotopical functor $P\circ\Psi_\Gamma(-)(1;1,1)\colon \boldsymbol{\Gamma\text{-}E\mathcal{M}\text{-SSet}}\to\textbf{Kan}$. However, by definition $\Psi_\Gamma(-)(1;1,1)$ is precisely the functor sending a global Γ-space X to $X(1^+)$ considered as a non-equivariant space, so the claim follows. $\square$

5.3. **Proof of Theorem B.** Building on the above we will now prove a comparison between *special* G-global Γ-spaces and $\underline{\text{CMon}}^{\text{Orb}}_{\text{Glo}}(\underline{\text{Spc}}_{\text{Glo}})$. Recall from Example 4.5.2 the notion of equivariant semiadditivity.

Theorem 5.3.1. *There exists an essentially unique pair of an equivariantly semiadditive functor* $\Xi\colon\underline{\Gamma\mathscr{S}}^{\text{gl, spc}}\to\underline{\text{CMon}}^{\text{Orb}}(\underline{\text{Spc}}_{\text{Glo}})$ *together with a natural equivalence filling*

$$\begin{array}{ccc} \underline{\Gamma\mathscr{S}}^{\text{gl, spc}} & \xrightarrow{\Xi} & \underline{\text{CMon}}^{\text{Orb}}(\underline{\text{Spc}}_{\text{Glo}}) \\ {\scriptstyle\mathbb{U}}\downarrow & & \downarrow{\scriptstyle\mathbb{U}=\text{ev}_{\text{id}_+}} \\ \underline{\mathscr{S}}^{\text{gl}} & \xrightarrow[\simeq]{} & \underline{\text{Spc}}^{\text{Glo}}. \end{array} \tag{16}$$

Moreover, Ξ *is an equivalence.*

As the notation suggest, we will in fact show that the equivalence Ξ from Theorem 5.2.1 restricts accordingly and is still an equivalence. For this let us first translate our definition of specialness into something that is more akin to the characterization of equivariant semiadditivity given in Subsection 4.9:

Proposition 5.3.2. *A* G*-global* Γ*-space* X *is special if and only if the following conditions are satisfied for every universal subgroup* $H\subset\mathcal{M}$ *and every homomorphism* $\varphi\colon H\to G$*:*

(1) For all finite H*-sets* S,T *the collapse maps* $S_+\leftarrow S_+\vee T_+\to T_+$ *induce a weak homotopy equivalence* $X(S_+\vee T_+)^\varphi\to X(S_+)^\varphi\times X(T_+)^\varphi$*.*

(2) For all $K \subset H$ the composite map

$$X(H/K_+)^\varphi \hookrightarrow X(H/K_+)^{\varphi|_K} \xrightarrow{X(\chi)^{\varphi|_K}} X(1^+)^{\varphi|_K},$$

is a weak homotopy equivalence, where $\chi\colon H/K_+ \to 1^+$ is the characteristic map of $[1] = K \in H/K$.

Proof. Let us first assume that X is special. Then we have a commutative diagram

$$\begin{array}{ccc} X(S_+ \vee T_+) & \longrightarrow & X(S_+) \times X(T_+) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho\times\rho} \\ \prod_{S\sqcup T} X(1^+) & \xrightarrow[\cong]{} & \prod_S X(1^+) \times \prod_T X(1^+) \end{array}$$

where the top horizontal map is again induced by the collapse maps. By assumption, the left hand vertical map is a $(G \times \Sigma_{S\sqcup T})$-global weak equivalence, hence also a $(G\times H)$-global weak equivalence with respect to the H-action on $S \sqcup T$. Similarly, one shows that the right hand vertical map is a $(G \times H)$-global weak equivalence, and hence so is the top horizontal map by 2-out-of-3. Taking fixed points with respect to $(\varphi, \mathrm{id})\colon H \to G \times H$ then establishes Condition (1).

In order to verify Condition (2), we first note that we have for any H-space Y an isomorphism $\left(\prod_{H/K} Y\right)^H \cong Y^K$ via projection to the factor indexed by [1]. Applying this to $Y = (\varphi, \mathrm{id}_H)^* X(1^+)$ we then get a commutative diagram

$$\begin{array}{ccc} & \overset{\rho}{\nearrow} & \left(\prod_{H/K} X(1^+)\right)^\varphi \\ X(H/K_+)^\varphi & & \cong\downarrow \mathrm{pr}_{[1]} \\ & \underset{X(\chi)}{\searrow} & X(1^+)^{\varphi|_K} \end{array} \tag{17}$$

in which the top map is a weak homotopy equivalence by specialness. The claim follows by 2-out-of-3.

Conversely, assume X is a G-global Γ-space satisfying Conditions (1) and (2). We want to show that for every finite set S the Segal map $X(S_+) \to \prod_S X(1^+)$ is a $(G \times \Sigma_S)$-global weak equivalence, i.e. for every universal subgroup $H \subset \mathcal{M}$, every H-action on S (i.e. homomorphism $\rho\colon H \to \Sigma_S$), and every homomorphism $\varphi\colon H \to G$ it induces a weak homotopy equivalence $X(S_+)^\varphi \to \left(\prod_S X(1^+)\right)^\varphi$. Using Condition (1) one readily reduces to the case that S is transitive, i.e. $S = H/K$ for some $K \subset H$; however, in this case the claim again follows by applying 2-out-of-3 to the commutative diagram (17). □

In order to relate this to our characterization of equivariant semiadditive functors into $\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ we note:

Lemma 5.3.3. *Let $p\colon K \hookrightarrow H$ be an inclusion of finite groups (hence a map in* Orb*). Then the essentially unique equivalence $\theta\colon \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \simeq \mathrm{N}\mathcal{F}^+_\bullet$ (see Corollary 5.2.4) sends the map $\rho_p\colon p^*p_!(\mathrm{id}_+) \to \mathrm{id}_+$ in $\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}(K)$ from Observation 4.9.7 up to isomorphism to the map $\chi\colon H/K_+ \to 1^+$ in $\mathcal{F}^+_K$ from Proposition 5.3.2.*

Proof. By construction, ρ_p is characterized by the properties that $\rho_p\eta = \mathrm{id}$ and $\rho_{p,L}j = 0$ for some (hence any) complement $j\colon C \to p^*p_!(\iota)$ of η. Now the inclusion $1^+ \to H/K_+$ of the coset [1] qualifies as a unit $1^+ \to p^*p_!1^+$, and with respect to this choice of η the map $\chi\colon H/K_+ \to 1^+$ obviously admits the analogous description.

If we now assume for ease of notation that $\theta(\mathrm{id}_1)_+ = 1^+$ (instead of them just being isomorphic), then the calculus of mates provides us with an isomorphism $\alpha\colon H/K_+ \cong \theta(p^*p_!(\mathrm{id}_+))$ in $\mathcal{F}_K^+$ fitting into a commutative diagram

$$\begin{array}{ccc} 1^+ & = & \theta(\mathrm{id}_+) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\theta(\eta)} \\ H/K_+ & \xrightarrow[\alpha]{\cong} & \theta(p^*p_!\,\mathrm{id}_+), \end{array} \tag{18}$$

and we claim that χ is actually equal to $\theta(\rho_p)\alpha$. Indeed,

$$\chi\eta = \mathrm{id}_{1^+} = \theta(\mathrm{id}_{\mathrm{id}_+}) = \theta(\rho_p\eta) = \theta(\rho_p)\theta(\eta) = \theta(\rho_p)\alpha\eta,$$

where the last equation uses the commutativity of (18). On the other hand, if $j\colon C \to p^*p_!\,\mathrm{id}_+$ is a complement of η, then $\theta(j)$ is a complement of $\theta(\eta)$ (as θ preserves coproducts), so $\alpha^{-1}\theta(j)$ is a complement of $\eta\colon 1^+ \to H/K_+$ in $\mathcal{F}_K^+$ by commutativity of (18) again. But then $\chi(\alpha^{-1}\theta(j)) = 0 = \theta(0) = \theta(\rho_p j) = \theta(\rho_{p,L})\alpha(\alpha^{-1}\theta(j))$, which finishes the proof. ☐

Proof of Theorem 5.3.1. By the universal property of $\underline{\mathrm{CMon}}^{\mathrm{Orb}}(\underline{\mathrm{Spc}}_{\mathrm{Glo}})$ it will suffice to construct such an equivalence, for which we will show that the equivalence Ξ from Theorem 5.2.1 restricts accordingly, i.e. that a G-global Γ-space X is special if and only if $\Xi(X)\colon \pi_G^*\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \to \pi_G^*\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ is π_G^* Orb-semiadditive.

For this, let us write $\hat{\Xi}(X)$ for the functor $\int \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \times \underline{G} \to \mathrm{Spc}$ corresponding to $\Xi(X)$. Plugging in the construction of Ξ, this is simply given by the restriction of $\mathrm{N}_\Delta(P \circ \Psi_\Gamma(X))\colon \mathrm{N}_\Delta(\iint \mathfrak{F}^{\mathrm{gl},+}) \to \mathrm{N}_\Delta(\mathbf{Kan}) = \mathrm{Spc}$ (where P is a fixed fibrant replacement again) along the inverse of the equivalence $\Theta_G\colon \int \underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*} \simeq \mathrm{N}_\Delta \iint \mathfrak{F}^{\mathrm{gl},+}$ from Proposition 5.2.7. On the other hand, Remark 4.9.9 shows that $\Xi(X)$ is semiadditive if and only if $\hat{\Xi}(X)$ is fiberwise semiadditive and sends the Segal maps (defined there) to equivalences.

Fiberwise semiadditivity. We will first show that X satisfies Condition (1) of Proposition 5.3.2 if and only if $\hat{\Xi}(X)$ is fiberwise semiadditive. Namely, $\hat{\Xi}(X)$ is fiberwise semiadditive if and only if its restriction to the non-full subcategories spanned by the objects $(H; X, \varphi)$ and the maps of the form $(\mathrm{id}; f, \mathrm{id})$ for each *universal* $H \subset \mathcal{M}$ and $\varphi\colon H \to G$ is semiadditive (as the universal subgroups of $\mathcal{M}$ account for all objects of Glo up to isomorphism). As Θ_G identifies this with the corresponding full subcategory $\mathrm{N}_\Delta(\iint \mathfrak{F}_G^{\mathrm{gl},+})_\varphi \subset \mathrm{N}_\Delta(\iint \mathfrak{F}_G^{\mathrm{gl},+})$ via an equivalence by Proposition 5.2.7, we conclude that $\hat{\Xi}(X)$ is fiberwise semiadditive if and only if $\Theta_G^*\hat{\Xi}(X)$ is semiadditive when restricted to each $\mathrm{N}_\Delta(\iint \mathfrak{F}_G^{\mathrm{gl},+})_\varphi$. But by the explicit construction of Ψ_Γ, we immediately see that the latter condition for $\Theta_G^*\hat{\Xi}(X) \simeq \mathrm{N}_\Delta(P \circ \Psi_\Gamma(X))$ is equivalent for every fixed φ to $X(-)^\varphi$ sending coproducts of finite pointed H-sets to products, which is precisely what we wanted to prove.

Segal maps. To complete the proof, it will now suffice to show that X satisfies Condition (2) of Proposition 5.3.2 if and only if $\hat{\Xi}(X)$ sends the parametrized Segal maps $\rho\colon (H;\iota_+,\varphi)\to(K;\mathrm{id}_+,\varphi\iota)$ (where $\iota\colon K\hookrightarrow H$ is an inclusion of universal subgroups and $\varphi\colon H\to G$ is a homomorphism) in $\int\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}\times\underline{G}$ to equivalences. However, by the description of Θ_G given in Proposition 5.2.7 together with the computation in Lemma 5.3.3, we conclude that $\Theta_G^{-1}(\rho)$ is given up to equivalence by $(\iota,1;\chi,\mathrm{id}_{\varphi\iota})\colon (H;H/K_+,\varphi)\to(K;1^+,\varphi\iota)$, and by the explicit construction of Ψ_Γ we see that $P\circ\Psi_\Gamma$ sends this up to weak equivalence to the map $X(H/K_+)^\varphi\to X(1^+)^{\varphi|_K}$ from Proposition 5.3.2 as desired. ☐

We can now leverage the above comparison in order to deduce a universal property of $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}$.

Theorem 5.3.4. *The functor $\mathbb{U}\colon\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}\to\underline{\mathscr{S}}^{\mathrm{gl}}$ exhibits $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}$ as the equivariantly semiadditive envelope of $\underline{\mathscr{S}}^{\mathrm{gl}}$, i.e. for every equivariantly semiadditive global ∞-category $\mathcal{C}$ we have an equivalence*

$$\underline{\mathrm{Fun}}^{P\text{-}\times}_{\mathrm{Glo}}(\mathcal{C},\mathbb{U})\colon\underline{\mathrm{Fun}}^{P\text{-}\oplus}_{\mathrm{Glo}}(\mathcal{C},\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}})\xrightarrow{\simeq}\underline{\mathrm{Fun}}^{P\text{-}\times}_{\mathrm{Glo}}(\mathcal{C},\underline{\mathscr{S}}^{\mathrm{gl}}).$$

Moreover, $\mathbb{U}$ admits a left adjoint $\mathbb{P}$ which exhibits $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}$ as the equivariantly semiadditive completion in the following sense: for every globally cocomplete equivariantly semiadditive global ∞-category $\mathcal{D}$ we have an equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\mathbb{P},\mathcal{D})\colon\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}},\mathcal{D})\xrightarrow{\simeq}\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}}^{\mathrm{gl}},\mathcal{D}).$$

Proof. The existence of the left adjoint follows formally from Theorem 5.3.1 and the fact that $\mathbb{U}\colon\underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Glo}}\to\underline{\mathrm{Spc}}_{\mathrm{Glo}}$ admits a left adjoint (see Corollary 4.8.9).

Now the free-forgetful adjunction $\underline{\mathscr{S}}^{\mathrm{gl}}\rightleftarrows\underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Glo}}(\underline{\mathscr{S}}^{\mathrm{gl}})$ has both of the above universal properties by Theorem 4.8.10 and Corollary 4.8.13), so it suffices to show that the equivalence Ξ from Theorem 5.3.1 is compatible with the free-forgetful adjunctions in the sense that there are natural equivalences filling

$$\begin{array}{ccc} \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}} & \xrightarrow{\Xi} & \underline{\mathrm{CMon}}^{\mathrm{Orb}}(\underline{\mathrm{Spc}}_{\mathrm{Glo}}) \\ {\scriptstyle\mathbb{U}}\downarrow & & \downarrow{\scriptstyle\mathbb{U}=\mathrm{ev}_{\mathrm{id}_+}} \\ \underline{\mathscr{S}}^{\mathrm{gl}} & \xrightarrow[\simeq]{} & \underline{\mathrm{Spc}}^{\mathrm{Glo}}. \end{array} \quad\text{and}\quad \begin{array}{ccc} \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}} & \xrightarrow{\Xi} & \underline{\mathrm{CMon}}^{\mathrm{Orb}}(\underline{\mathrm{Spc}}_{\mathrm{Glo}}) \\ {\scriptstyle\mathbb{P}}\uparrow & & \uparrow{\scriptstyle\mathbb{P}} \\ \underline{\mathscr{S}}^{\mathrm{gl}} & \xrightarrow[\simeq]{} & \underline{\mathrm{Spc}}^{\mathrm{Glo}}. \end{array}$$

However, as Ξ is an equivalence it suffices to prove the first statement, which is simply the defining property of Ξ. ☐

Together with Theorem 4.8.11 we moreover get Theorem B from the introduction:

Theorem 5.3.5. *Let $\mathcal{D}$ be a globally cocomplete and equivariantly semiadditive global ∞-category. Then evaluation at $\mathbb{P}(*)$ provides an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}},\mathcal{D})\xrightarrow{\sim}\mathcal{D}.$$

Put differently, $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}$ is the free globally cocomplete (or presentable) equivariantly semiadditive global ∞-category on one generator (namely, the free global special Γ-space $\mathbb{P}(*)$). ☐

Using Propositions 5.1.4 and 5.1.7 we can deduce several variants of the above theorems. Let us make two of them explicit:

Corollary 5.3.6. *There exist equivalences*

$$\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}} \simeq \underline{\mathrm{Fun}}_{\mathrm{Glo}}(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}, \underline{\mathrm{Spc}}_{\mathrm{Glo}}) \qquad \textit{and} \qquad \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*} \simeq \underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Glo}}$$

fitting into a commutative diagram

$$\begin{array}{ccccc} \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*} & \hookrightarrow & \underline{\Gamma\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}} & \xrightarrow{\mathrm{ev}_{1+}} & \underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}} \\ \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq \\ \underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Glo}} & \hookrightarrow & \underline{\mathrm{Fun}}_{\mathrm{Glo}}(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}, \underline{\mathrm{Spc}}_{\mathrm{Glo}}) & \xrightarrow[\mathrm{ev}_{(\mathrm{id}_1)_+}]{} & \underline{\mathrm{Spc}}_{\mathrm{Glo}} \end{array}$$

where the equivalence on the right is the unique one (see Corollary 3.3.3). □

Corollary 5.3.7. *The forgetful functor* $\mathbb{U}\colon \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*} \to \underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$ *exhibits* $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*}$ *as the universal equivariantly semiadditive envelope of* $\underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$*. Moreover, it admits a left adjoint* $\mathbb{P}$*, exhibiting* $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*}$ *as the equivariantly semiadditive completion of* $\underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$*.* □

Remark 5.3.8. [Len20] also discusses various other models of 'G-globally coherently commutative monoids,' for example *G-ultra-commutative monoids* (Definition 2.1.25 of *op. cit.*) or *G-parsummable simplicial sets* (Definition 2.1.10). Similarly, [Len22, Definition 3.9] introduces a notion of *global E_∞-operads*, and for any global E_∞-operad $\mathcal{O}$, considering $\mathcal{O}$-algebras in $\boldsymbol{E\mathcal{M}}$-$\boldsymbol{G}$-**SSet** (with respect to the trivial G-action on $\mathcal{O}$) yields a concept of *G-global E_∞-algebras.*

All of these models are related via suitable zig-zags of Quillen equivalences by [Len20, Chapter 2] and [Len22, Section 4], and while these can be somewhat complicated (especially on the operadic side of things), in each case they are by design strictly compatible with restrictions along group homomorphisms and moreover at least one of the adjoints is homotopical, so that they lift to equivalences of associated global ∞-categories in the same way as before. As moreover each of them is readily seen to be compatible with the respective forgetful functors, we obtain universal properties in the above spirit for each of these models.

Conversely, while each of these comparisons comes from a concrete (and sometimes ad-hoc) model categorical construction, this tells us that *a posteriori*, once we have passed to parametrized ∞-categories, these comparisons are actually canonical and completely characterized by lying over the forgetful functors.

6. Parametrized stability

In this section, we will introduce the notion of a P-stable T-∞-category: a T-∞-category which is both P-semiadditive and fiberwise stable.

6.1. Fiberwise stable T-∞-categories.

Definition 6.1.1. We say a T-∞-category $\mathcal{C}$ is *fiberwise stable* if the following conditions are satisfied:

(1) For every object $B \in T$, the ∞-category $\mathcal{C}(B)$ is stable;
(2) For every morphism $\beta\colon B' \to B$, the restriction functor $\beta^*\colon \mathcal{C}(B) \to \mathcal{C}(B')$ is exact.

Equivalently, $\mathcal{C}$ is fiberwise stable if the functor $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors through the (non-full) subcategory $\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty$ of stable ∞-categories and exact functors. We let $\mathrm{Cat}_T^{\mathrm{st}}$ denote the ∞-category $\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathrm{st}})$ of fiberwise stable T-∞-categories.

Definition 6.1.2. Denote by $\mathrm{Cat}_\infty^{\mathrm{lex}} \subseteq \mathrm{Cat}_\infty$ the (non-full) subcategory spanned by the ∞-categories admitting finite limits and the finite-limit-preserving functors between them. We let $\mathrm{Cat}_T^{\mathrm{lex}}$ denote the functor ∞-category $\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathrm{lex}})$ of T-∞-categories $\mathcal{C}$ admitting fiberwise finite limits (cf. Definition 2.3.11) and T-functors preserving fiberwise finite limits.

Definition 6.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two T-∞-categories with finite limits. We write $\underline{\mathrm{Fun}}_T^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ spanned on level $B \in T$ by those functors $F\colon \pi_B^* \mathcal{C} \to \pi_B^* \mathcal{D}$ which preserve fiberwise finite limits.

When $\mathcal{C}$ and $\mathcal{D}$ are both fiberwise stable, we will write $\underline{\mathrm{Fun}}_T^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ for $\underline{\mathrm{Fun}}_T^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$.

Construction 6.1.4 (Fiberwise stabilization)**.** Let $\mathcal{C} \in \mathrm{Cat}_T^{\mathrm{lex}}$ be a T-∞-category which has fiberwise finite limits. We define the T-∞-category $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$, called the *fiberwise stabilization of* $\mathcal{C}$, as the composite

$$T^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Cat}_\infty^{\mathrm{lex}} \xrightarrow{\mathrm{Sp}} \mathrm{Cat}_\infty^{\mathrm{st}}.$$

This construction assembles into a functor $\underline{\mathrm{Sp}}^{\mathrm{fib}}\colon \mathrm{Cat}_T^{\mathrm{lex}} \to \mathrm{Cat}_T^{\mathrm{st}}$.

Example 6.1.5. The T-∞-category $\underline{\mathrm{Sp}}_T$ of naive T-spectra is the fiberwise stabilization of the T-∞-category $\underline{\mathrm{Spc}}_T$ of T-spaces.

More generally, if $\mathcal{E}$ is an ∞-category admitting finite limits, then the fiberwise stabilization of the T-∞-category $\underline{\mathcal{E}}_T$ of T-objects in $\mathcal{E}$ is the T-∞-category $\underline{\mathrm{Sp}(\mathcal{E})}_T$ of T-objects in the stabilization $\mathrm{Sp}(\mathcal{E})$. Indeed, this follows easily from the equivalence $\mathrm{Sp}(\mathrm{Fun}(-, \mathcal{E})) \simeq \mathrm{Fun}(-, \mathrm{Sp}(\mathcal{E}))$ from [Lur17, Remark 1.4.2.9].

Remark 6.1.6. As a right adjoint, the stabilization functor $\mathrm{Sp}\colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ preserves limits, which in both the source and target are computed in Cat_∞. It follows that the limit extension of $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ to the presheaf category $\mathrm{PSh}(T)$ is given by postcomposing the limit extension of $\mathcal{C}$ to $\mathrm{PSh}(T)$ with the functor Sp.

Remark 6.1.7. We will use that the functor $\mathrm{Sp}\colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ is in fact functorial in natural transformations of finite limit preserving functors, i.e. that Sp refines to a 2-functor between homotopy 2-categories. Given that taking functor categories forms such a functor, this immediately follows from the definition of $\mathrm{Sp}(\mathcal{C})$ as a full subcategory of $\mathrm{Fun}(\mathrm{Spc}_*^{\mathrm{fin}}, \mathcal{C})$, see [Lur17, Definition 1.4.2.8]. (Using the same argument, one can in fact show that Sp is an $(\infty, 2)$-functor.)

It follows in particular that stabilization preserves adjunctions between left exact functors.

Proposition 6.1.8. *The functor* $\underline{\mathrm{Sp}}^{\mathrm{fib}}\colon \mathrm{Cat}_T^{\mathrm{lex}} \to \mathrm{Cat}_T^{\mathrm{st}}$ *is right adjoint to the fully faithful inclusion* $\mathrm{Cat}_T^{\mathrm{st}} \subseteq \mathrm{Cat}_T^{\mathrm{lex}}$.

Proof. Since $\mathrm{Fun}(T^{\mathrm{op}}, -)\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ preserves adjunctions, this is immediate from the fact that the stabilization functor $\mathrm{Sp}\colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ is right adjoint to the fully faithful inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty^{\mathrm{lex}}$ by [Lur17, Corollary 1.4.2.23]. ∎

Lemma 6.1.9. *Consider $\mathcal{C} \in \mathrm{Cat}_T^{\mathrm{st}}$ and $\mathcal{D} \in \mathrm{Cat}_T^{\mathrm{lex}}$. Composition with the adjunction counit $\Omega^\infty \colon \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence of T-∞-categories*

$$\underline{\mathrm{Fun}}_T^{\mathrm{ex}}(\mathcal{C}, \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{D})) \xrightarrow{\sim} \underline{\mathrm{Fun}}_T^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}).$$

Proof. It immediately follows from Proposition 6.1.8 that the map

$$\iota\,\mathrm{Fun}_T(\mathcal{C}, \Omega^\infty) \colon \iota\,\mathrm{Fun}_T^{\mathrm{ex}}(\mathcal{C}, \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{D})) \to \iota\,\mathrm{Fun}_T^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$$

is an equivalence. We will now show that this already holds before passing to cores. Replacing T by $T_{/B}$ for varying $B \in T$ then yields the proof of the proposition. For this it will be enough to show that for every small ∞-category K the induced map $\iota\big(\mathrm{Fun}_T^{\mathrm{ex}}(\mathcal{C}, \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{D}))^K\big) \to \iota\big(\mathrm{Fun}_T^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})^K\big)$ is an equivalence. But this agrees up to equivalence with the map induced by $(\Omega^\infty)^K \colon \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{D})^K \to \mathcal{D}^K$; the claim follows as this is again the stabilization of $\mathcal{D}^K$. □

The fiberwise stabilization of a T-∞-category $\mathcal{C}$ inherits certain parametrized limits from $\mathcal{C}$. Since this is clear for limits along constant T-∞-categories, we focus on limits along T-∞-groupoids.

Lemma 6.1.10. *Let $\mathbf{U}$ be a class of T-∞-groupoids, and let $\mathcal{C}$ be a $\mathbf{U}$-complete T-∞-category which admits fiberwise finite limits. Then $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is $\mathbf{U}$-complete and the T-functor $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C}) \to \mathcal{C}$ preserves $\mathbf{U}$-limits.*

Proof. We will use the characterization of Lemma 2.3.14. Given a morphism $p \colon A \to B$ in $\mathbf{U}$, applying the functor $\mathrm{Sp} \colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ to the adjunction

$$p^* \colon \mathcal{C}(B) \rightleftarrows \mathcal{C}(A) : p_*$$

shows that the functor $\mathrm{Sp}(p^*) \colon \mathrm{Sp}(\mathcal{C}(B)) \to \mathrm{Sp}(\mathcal{C}(A))$ admits a right adjoint given by $\mathrm{Sp}(p_*) \colon \mathrm{Sp}(\mathcal{C}(A)) \to \mathrm{Sp}(\mathcal{C}(B))$. Furthermore, for a pullback square

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\beta} & B \end{array}$$

in $\mathrm{PSh}(T)$ with $p \colon A \to B$ in $\mathbf{U}$ and $\beta \colon B' \to B$ in T, the resulting Beck-Chevalley transformation $\mathrm{Sp}(p_*) \circ \mathrm{Sp}(\beta^*) \Rightarrow \mathrm{Sp}(\alpha^*) \circ \mathrm{Sp}(p'_*)$ is given by applying Sp to the Beck-Chevalley transformation $p_* \circ \beta^* \Rightarrow \alpha^* \circ p'_*$, and thus is again an equivalence. This shows that $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is again $\mathbf{U}$-complete. It is immediate from this construction that the T-functor $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C}) \to \mathcal{C}$ preserves $\mathbf{U}$-limits, finishing the proof. □

Fiberwise stabilization preserves parametrized presentability.

Definition 6.1.11. We define $\mathrm{Pr}_T^{\mathrm{R,st}}$ to be the full subcategory of $\mathrm{Pr}_T^{\mathrm{R}}$ spanned by those presentable T-∞-categories which are also fiberwise stable. The subcategory $\mathrm{Pr}_T^{\mathrm{L,st}} \subseteq \mathrm{Pr}_T^{\mathrm{L}}$ is defined similarly.

Proposition 6.1.12. *The functor $\underline{\mathrm{Sp}}^{\mathrm{fib}} \colon \mathrm{Cat}_T^{\mathrm{lex}} \to \mathrm{Cat}_T^{\mathrm{st}}$ restricts to a functor*

$$\underline{\mathrm{Sp}}^{\mathrm{fib}} \colon \mathrm{Pr}_T^{\mathrm{R}} \to \mathrm{Pr}_T^{\mathrm{R,st}}$$

which is right adjoint to the inclusion $\mathrm{Pr}_T^{\mathrm{R,st}} \hookrightarrow \mathrm{Pr}_T^{\mathrm{R}}$.

Proof. We first show that the fiberwise stabilization of a presentable T-∞-category $\mathcal{C}$ is again presentable. By [Lur17, Proposition 1.4.4.4, Example 4.8.1.23], $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is given by the composite

$$T^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Pr}^{\mathrm{L}} \xrightarrow{-\otimes \mathrm{Sp}} \mathrm{Pr}^{\mathrm{L}},$$

proving that $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is again fiberwise presentable. Since the functor $-\otimes\mathrm{Sp}\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ preserves adjunctions, one deduces the existence of left adjoints $f_!$ for all morphisms $f\colon A \to B$ in $\mathrm{PSh}(T)$ satisfying the Beck-Chevalley conditions, similar to the proof of Lemma 6.1.10. This shows that $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is again a presentable T-∞-category. One can similarly show that if $L \dashv R$ is an adjunction between presentable T-∞-categories, then $L\otimes\mathrm{Sp} \dashv \underline{\mathrm{Sp}}^{\mathrm{fib}}(R)$ is again an adjunction. This shows that $\underline{\mathrm{Sp}}^{\mathrm{fib}}$ restricts to a functor $\mathrm{Pr}^{\mathrm{R}}_T \to \mathrm{Pr}^{\mathrm{R,st}}_T$. It is right adjoint to the inclusion $\mathrm{Pr}^{\mathrm{R,st}}_T \hookrightarrow \mathrm{Pr}^{\mathrm{R}}_T$ by Proposition 6.1.8. □

Applying the equivalence $(\mathrm{Pr}^{\mathrm{R}}_T)^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{L}}_T$, we obtain:

Corollary 6.1.13. *The construction* $\mathcal{C} \mapsto \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ *defines a functor*

$$\underline{\mathrm{Sp}}^{\mathrm{fib}}\colon \mathrm{Pr}^{\mathrm{L}}_T \to \mathrm{Pr}^{\mathrm{L,st}}_T$$

which is left adjoint to the inclusion functor $\mathrm{incl}\colon \mathrm{Pr}^{\mathrm{L,st}}_T \hookrightarrow \mathrm{Pr}^{\mathrm{L}}_T$. □

6.2. *P*-stable *T*-∞-categories.

Definition 6.2.1. We say a T-∞-category $\mathcal{C}$ has *finite P-limits* if it has fiberwise finite limits and finite P-products. We define $\mathrm{Cat}_T^{P\text{-lex}}$ to be the (non-full) subcategory of Cat_T spanned by the T-∞ categories which admit finite P-limits and those functors which preserve finite P-limits.

Let $\mathcal{C}$ and $\mathcal{D}$ be two T-∞-categories with finite P-limits, we define $\mathrm{Fun}_T^{P\text{-lex}}(\mathcal{C},\mathcal{D})$ to be the full subcategory of $\mathrm{Fun}_T(\mathcal{C},\mathcal{D})$ spanned on level B by those functors $F\colon \pi_B^*\mathcal{C} \to \pi_B^*\mathcal{D}$ which preserve finite P-limits. This is a T-subcategory by the dual of Lemma 2.3.17.

Definition 6.2.2 (cf. [Nar16, Definition 7.1])**.** A T-∞-category $\mathcal{C}$ is *P-stable* if it is fiberwise stable and P-semiadditive. We define $\mathrm{Cat}_T^{P\text{-st}}$ to be the full subcategory of $\mathrm{Cat}_T^{P\text{-lex}}$ spanned by the P-stable T-∞-categories.

When $\mathcal{C}$ and $\mathcal{D}$ are both P-stable T-∞-categories, we will write $\mathrm{Fun}_T^{P\text{-ex}}(\mathcal{C},\mathcal{D})$ for $\mathrm{Fun}_T^{P\text{-lex}}(\mathcal{C},\mathcal{D})$.

Example 6.2.3. Applied to the pair Orb ⊂ Glo we obtain a notion of Orb-stability for global ∞-categories. We will refer to this as *equivariant stability.*

Lemma 6.2.4. *Let $\mathcal{C}$ be a T-∞-category. If $\mathcal{C}$ admits finite P-limits, then so does* $\underline{\mathrm{CMon}}^P(\mathcal{C})$*.*

Proof. This is a special case of Lemma 4.6.11. □

Definition 6.2.5 ([Nar16, Definition 7.3])**.** Let $\mathcal{C}$ be a T-∞-category which admits finite P-limits. Then the *P-stabilization of* $\mathcal{C}$ is the T-∞-category $\underline{\mathrm{Sp}}^P(\mathcal{C})$ defined as

$$\underline{\mathrm{Sp}}^P(\mathcal{C}) := \underline{\mathrm{Sp}}^{\mathrm{fib}}(\underline{\mathrm{CMon}}^P(\mathcal{C})),$$

the fiberwise stabilization of the T-∞-category of P-commutative monoids in $\mathcal{C}$. As a special case, we define the T-∞-category $\underline{\mathrm{Sp}}_T^P$ of *P-genuine T-spectra* as

$$\underline{\mathrm{Sp}}_T^P := \underline{\mathrm{Sp}}^P(\underline{\mathrm{Spc}}_T),$$

the P-stabilization of the T-∞-category of T-spaces.

The next lemma shows that the P-stabilization of a T-∞-category with finite P-limits is indeed P-stable.

Lemma 6.2.6. *Let $\mathcal{C}$ be a P-semiadditive T-∞-category with finite P-limits. Then $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is again P-semiadditive, and thus in particular P-stable.*

Proof. The T-∞-category $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is fiberwise pointed and admits finite P-products by Lemma 6.1.10. By Lemma 4.5.4, it will suffice to show that for every morphism $p\colon A \to B$ in $\mathbb{F}_T^P$ the dual adjoint norm map $\overline{\mathrm{Nm}}_p\colon \mathrm{id} \to \mathrm{Sp}(p^*)\,\mathrm{Sp}(p_*)$ exhibits $\mathrm{Sp}(p^*)$ as a right adjoint of $\mathrm{Sp}(p_*)$. Since the adjunction data for $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is inherited from $\mathcal{C}$ by fiberwise stabilizing, the dual adjoint norm map for $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is obtained by applying the stabilization functor to the map $\overline{\mathrm{Nm}}_p^{\mathcal{C}}\colon \mathrm{id} \to p^*p_*$. As stabilization preserves adjunctions, the claim thus follows from P-semiadditivity of $\mathcal{C}$. □

Corollary 6.2.7. *The functor $\underline{\mathrm{Sp}}^P\colon \mathrm{Cat}_T^{P\text{-lex}} \to \mathrm{Cat}_T^{P\text{-st}}$ is right adjoint to the inclusion $\mathrm{Cat}_T^{P\text{-st}} \hookrightarrow \mathrm{Cat}_T^{P\text{-lex}}$.*

Proof. Lemma 6.2.6 shows that the adjunction of Proposition 6.1.8 restricts to an adjunction

$$\mathrm{incl}\colon \mathrm{Cat}_T^{P-\mathrm{st}} \rightleftarrows \mathrm{Cat}_T^{\mathrm{lex},P\text{-}\oplus} :\underline{\mathrm{Sp}}^{\mathrm{fib}}(-).$$

Composing this with the adjunction of Corollary 4.8.5 gives the statement. □

From the adjunction of ∞-categories from Corollary 6.2.7, we may immediately deduce an equivalence at the level of T-∞-categories of functors.

Definition 6.2.8. We define the T-functor $\Omega^\infty\colon \underline{\mathrm{Sp}}^P(\mathcal{C}) \to \mathcal{C}$ to be the counit of the adjunction from Corollary 6.2.7. Explicitly it is given by the composite

$$\underline{\mathrm{Sp}}^{\mathrm{fib}}(\underline{\mathrm{CMon}}^P(\mathcal{C})) \xrightarrow{\Omega^\infty} \underline{\mathrm{CMon}}^P(\mathcal{C}) \xrightarrow{\mathbb{U}} \mathcal{C},$$

where the first functor is the infinite loop space functor and the second functor is given by evaluation at $S^0\colon \underline{1} \to \underline{\mathbb{F}}_{T,*}^P$.

Proposition 6.2.9. *Let $\mathcal{D}$ be a T-∞-category with finite P-limits. For every P-stable T-∞-category $\mathcal{C}$, composition with $\Omega^\infty\colon \underline{\mathrm{Sp}}^P(\mathcal{C}) \to \mathcal{C}$ induces an equivalence of T-∞-categories*

$$\underline{\mathrm{Fun}}_T(\mathcal{C}, \Omega^\infty)\colon \underline{\mathrm{Fun}}_T^{P\text{-ex}}(\mathcal{C}, \underline{\mathrm{Sp}}^P(\mathcal{D})) \to \underline{\mathrm{Fun}}_T^{P\text{-lex}}(\mathcal{C}, \mathcal{D}).$$

Proof. This follows by combining Corollary 4.8.4 and Lemma 6.1.9. □

Lemma 6.2.10. *Let $\mathbf{U}$ be a family of T-∞-groupoids, and let $\mathcal{C}$ be a $\mathbf{U}$-complete T-∞-category which admits finite P-limits. Then also $\underline{\mathrm{Sp}}^P(\mathcal{C})$ is $\mathbf{U}$-complete and the T-functor $\Omega^\infty\colon \underline{\mathrm{Sp}}^P(\mathcal{C}) \to \mathcal{C}$ preserves $\mathbf{U}$-limits.*

Proof. This follows immediately from Lemma 6.1.10 and Lemma 4.6.11. □

As before, P-stabilization restricts to an adjunction on presentable T-∞-categories.

Lemma 6.2.11. *The construction $\mathcal{C} \mapsto \underline{\mathrm{Sp}}^P(\mathcal{C})$ defines a functor*

$$\underline{\mathrm{Sp}}^P \colon \mathrm{Pr}^{\mathrm{L}}_T \to \mathrm{Pr}^{\mathrm{L},P\text{-st}}_T$$

which is left adjoint to the inclusion $\mathrm{Pr}^{\mathrm{L},P\text{-st}}_T \hookrightarrow \mathrm{Pr}^{\mathrm{L}}_T$.

Proof. Combine Corollary 6.1.13 and Corollary 4.8.9. □

Definition 6.2.12. We write $\Sigma^\infty_+ \colon \mathcal{C} \to \underline{\mathrm{Sp}}^P(\mathcal{C})$ for the left adjoint of the forgetful functor $\Omega^\infty \colon \underline{\mathrm{Sp}}^P(\mathcal{C}) \to \mathcal{C}$. It is the unit of the adjunction in Lemma 6.2.11.

We record the results of this section in the following theorem for easy reference:

Theorem 6.2.13. *Let $\mathcal{C}$ be a T-∞-category with finite P-limits. The functor $\Omega^\infty \colon \underline{\mathrm{Sp}}^P(\mathcal{C}) \to \mathcal{C}$ exhibits $\underline{\mathrm{Sp}}^P(\mathcal{C})$ as the P-stable envelope of $\mathcal{C}$, i.e. for every P-stable T-∞-category $\mathcal{D}$ postcomposition with Ω^∞ induces an equivalence*

$$\underline{\mathrm{Fun}}^{P\text{-lex}}(\mathcal{D}, \Omega^\infty) \colon \underline{\mathrm{Fun}}^{P\text{-ex}}(\mathcal{D}, \underline{\mathrm{Sp}}^P(\mathcal{C})) \to \underline{\mathrm{Fun}}^{P\text{-lex}}(\mathcal{D}, \mathcal{C}).$$

Suppose now that $\mathcal{C}$ is moreover presentable. Then the left adjoint Σ^∞_+ of Ω^∞ exhibits $\underline{\mathrm{Sp}}^P(\mathcal{C})$ as the presentable P-stable completion of $\mathcal{C}$, i.e. for any presentable P-stable T-∞-category $\mathcal{D}$ precomposition with Σ^∞_+ yields an equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\Sigma^\infty_+, \mathcal{D}) \colon \underline{\mathrm{Fun}}^{\mathrm{L}}(\underline{\mathrm{Sp}}^P(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$ □

As a simple consequence, we get that the T-∞-category $\underline{\mathrm{Sp}}^P_T$ of P-genuine T-spectra is the free presentable P-stable T-∞-category on a single generator. As in the P-semiadditive setting of Section 4.9, we can strengthen this to the T-cocomplete setting:

Theorem 6.2.14. *Let $\mathcal{D}$ be a locally small T-cocomplete P-stable T-∞-category. Then evaluating at $\Sigma^\infty_+(*)$ yields an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{Sp}}^P_T, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}.$$

For the proof we will first consider the following non-parametrized version strengthening of [Lur17, Corollary 1.4.4.5]:

Lemma 6.2.15. *Let $\mathcal{C}$ be a presentable ∞-category and let $\mathcal{D}$ be cocomplete and stable. Then we have equivalences*

$$\mathrm{Fun}^{\mathrm{L}}(\Sigma^\infty_+, \mathcal{D}) \colon \mathrm{Fun}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$$
$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \Omega^\infty) \colon \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C})$$

of categories of left adjoint and categories of right adjoint functors, respectively.

Proof. It suffices to prove the second statement. Since full faithfulness follows from the usual universal property of spectrification [Lur17, Corollary 1.4.2.23], it only remains to prove essential surjectivity, i.e. for every right adjoint $G \colon \mathcal{D} \to \mathcal{C}$ we can find a *right adjoint* $G_\infty \colon \mathcal{D} \to \mathrm{Sp}(\mathcal{C})$ such that $\Omega^\infty G_\infty \simeq G$.

For this we first observe that G lifts to a functor $G_* \colon \mathcal{D} \simeq \mathcal{D}_* \to \mathcal{C}_*$ as $\mathcal{D}$ is pointed and G preserves terminal objects; moreover, this is again a right adjoint functor by the dual of [Lur09, Proposition 5.2.5.1]. Replacing $\mathcal{C}$ by $\mathcal{C}_*$ if necessary, we may therefore assume without loss of generality that $\mathcal{C}$ is pointed.

We now define $G_i := G\Sigma^i \colon \mathcal{D} \to \mathcal{C}$ for all $i \geq 0$. Then we have equivalences

$$\Omega G_{i+1} = \Omega G \Sigma^{i+1} \simeq G\Omega\Sigma^{i+1} \simeq G\Sigma^i = G_i,$$

and so we get

$$G_\infty \colon \mathcal{D} \to \mathrm{Sp}(\mathcal{C}) = \lim\big(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}\big)$$

with $\Omega^\infty G_\infty \simeq G_0 = G$ by passing to limits. However, each G_i for $i < \infty$ is a right adjoint (as G is and since Σ^i is even an equivalence by stability), whence so is the limit map G_∞ by [HY17, Theorem B]. □

Corollary 6.2.16. *In the above situation, let $G \colon \mathcal{D} \to \mathrm{Sp}(\mathcal{C})$ be an exact functor. Then G admits a left adjoint if and only if $\Omega^\infty \circ G$ does.* □

Proposition 6.2.17. *Let $\mathcal{C}$ be a presentable T-∞-category and let $\mathcal{D}$ be a T-cocomplete fiberwise stable T-∞-category. Then we have equivalences*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\Sigma^\infty_+, \mathcal{D}) \colon \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\mathcal{C}, \mathcal{D})$$
$$\underline{\mathrm{Fun}}^{\mathrm{R}}_T(\mathcal{D}, \Omega^\infty) \colon \underline{\mathrm{Fun}}^{\mathrm{R}}_T(\mathcal{D}, \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\mathrm{R}}_T(\mathcal{D}, \mathcal{C}).$$

Proof. Arguing as before, it suffices to show that any right adjoint $g \colon \mathcal{D} \to \mathcal{C}$ lifts to a *right adjoint* $G \colon \mathcal{D} \to \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$. However, by Lemma 6.1.9 there exists a fiberwise left exact functor G lifting g, and by the previous corollary this admits a *pointwise* left adjoint F; it only remains to show that for every $t \colon A \to B$ in T the Beck-Chevalley map $Ft^* \Rightarrow t^*F$ is an equivalence.

However, for the diagram

$$\begin{array}{ccccc} \mathcal{D}(A) & \xrightarrow{G} & \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})(A) & \xrightarrow{\Omega^\infty} & \underline{\mathcal{C}}(A) \\ {\scriptstyle t^*}\uparrow & & \uparrow{\scriptstyle t^*} & & \uparrow{\scriptstyle t^*} \\ \mathcal{D}(B) & \xrightarrow[G]{} & \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})(B) & \xrightarrow[\Omega^\infty]{} & \mathcal{C}(B) \end{array}$$

both the mate of the total square as well as the mate of the right hand square are equivalences as g and Ω^∞ are parametrized right adjoints. By the compatiblity of mates with pasting we conclude that $Ft^* \Rightarrow t^*F$ becomes an equivalence after precomposition with $\Sigma^\infty_+ \colon \underline{\mathcal{C}}(B) \to \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})(B)$. Therefore the claim follows by the first half of Lemma 6.2.15. □

Proof of Theorem 6.2.14. By the same reduction as in the semiadditive case (Theorem 4.8.11), we only have to construct for each given $X \in \Gamma(\mathcal{D})$ a left adjoint functor $F \colon \underline{\mathrm{Sp}}^P_T \to \mathcal{D}$ with $F(\Sigma^\infty_+(1)) \simeq X$.

To this end, we simply observe that Theorem 4.8.11 provides us with a left adjoint $f \colon \underline{\mathrm{CMon}}^P_T \to \mathcal{D}$ with $f(\mathbb{P}(1)) \simeq X$, and by the previous proposition f factors as

$$\underline{\mathrm{CMon}}^P_T \xrightarrow{\Sigma^\infty} \underline{\mathrm{Sp}}^{\mathrm{fib}}(\underline{\mathrm{CMon}}^P_T) = \underline{\mathrm{Sp}}^P_T \xrightarrow{F} \mathcal{D}$$

for some left adjoint F, which is then the desired functor. □

Corollary 6.2.18. *Let $\mathcal{S}$ be a T-∞-category equivalent to $\underline{\mathrm{Spc}}_T$ and let $\mathcal{D}$ be a locally small T-cocomplete P-stable T-∞-category. Then we have an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_T(\Sigma^\infty_+, \mathcal{D}) \colon \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\underline{\mathrm{Sp}}^P(\mathcal{S}), \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}_T(\mathcal{S}, \mathcal{D}).$$ □

7. The universal property of global spectra

In this section, we will prove the main result of this article: an interpretation of the global ∞-category of global spectra, defined via certain localizations of symmetric G-spectra generalizing [Sch18, Hau19], in terms of the abstract stabilization procedure we have described in the previous section.

7.1. Stable G-global homotopy theory.

We start by recalling the ∞-category of G-global spectra for a finite group G, and then show how these assemble for varying G into a global ∞-category $\underline{\mathscr{S}p}^{\mathrm{gl}}$.

Definition 7.1.1. We write **Spectra** for the category of *symmetric spectra* in the sense of [HSS00, Definition 1.2.2]. We will as usual evaluate symmetric spectra more generally at all finite sets (and not only at the standard sets $\{1, \ldots, n\}$ for $n \geq 0$), see e.g. [Hau17, 2.4].

We write $\boldsymbol{G}$**-Spectra** for the category of G-objects in **Spectra** and call its objects *(symmetric) G-spectra*.

For a finite group G, we refer the reader to [Hau17, Definition 2.35] for the definition of *G-stable equivalences* of symmetric G-spectra, to which we will refer as *G-weak equivalences* below.

Definition 7.1.2. Let G be a finite group and let $f\colon X \to Y$ be a map of symmetric G-spectra. We call f a *G-global weak equivalence* if $\varphi^* f$ is an H-weak equivalence for every group homomorphism $\varphi\colon H \to G$ (not necessarily injective).

Theorem 7.1.3 (See [Len20, Proposition 3.1.20 and Theorem 3.1.41])**.** *There is a unique (combinatorial) model structure on* $\boldsymbol{G}$**-Spectra** *with*

- *weak equivalences the G-global weak equivalences* and
- *acyclic fibrations those maps f such that $f(A)^\varphi$ is an acyclic Kan fibration for all finite sets A, all $H \subset \Sigma_A$, and all $\varphi\colon H \to G$.*

We call this the projective G-global model structure. □

Remark 7.1.4. For $G = 1$ the above was first studied by Hausmann [Hau19], who also exhibited it as a Bousfield localization of Schwede's *global orthogonal spectra* [Sch18, 4.1] at certain '$\mathcal{F}in$-global weak equivalences,' see [Hau19, Theorem 5.3].

Lemma 7.1.5 (See [Len20, Lemma 3.1.49])**.** *Let $\alpha\colon G \to H$ be a homomorphism. Then the adjunction*

$$\alpha_!\colon \boldsymbol{G}\textbf{-Spectra}_{G\text{-gl proj}} \rightleftarrows \boldsymbol{H}\textbf{-Spectra}_{H\text{-gl proj}} :\alpha^*$$

is a Quillen adjunction with homotopical right adjoint. □

There are also *injective* analogues of the above model structures that will become useful below:

Theorem 7.1.6 (See [Len20, Corollary 3.1.46])**.** *There is a unique (combinatorial) model structure on* $\boldsymbol{G}$**-Spectra** *with*

- *weak equivalences the G-global weak equivalences* and
- *cofibrations the injective cofibrations (i.e. levelwise injections).*

We call this the injective G-global model structure. □

7.1.1. *Relation to unstable G-global homotopy theory.* Passing to pointwise localizations as before, we get a global ∞-category $\underline{\mathscr{S}p}^{\mathrm{gl}}$ such that $\underline{\mathscr{S}p}^{\mathrm{gl}}(G) = \mathscr{S}p^{\mathrm{gl}}_G$ is the ∞-category of G-global spectra, with functoriality given via restriction. Let us now relate this to the unstable models from 3.1.

Construction 7.1.7. Let X be an $\mathcal{I}$-space (or, more generally, an I-space). Then we define its *unbased suspension spectrum* $\Sigma^\bullet_+ X$, cf. [SS12, discussion before Proposition 3.19], via

$$(\Sigma^\bullet_+ X)(A) := S^A \wedge X(A)_+ = \Sigma^A_+ X(A)$$

with the diagonal Σ_A-action and with structure maps given by

$$\begin{aligned} S^A \wedge (\Sigma^\bullet_+ X)(B) = S^A \wedge \big(S^B \wedge X(B)_+\big) &\cong S^{A\amalg B} \wedge X(B)_+ \\ &\xrightarrow{S^{A\amalg B}\wedge X(\mathrm{incl})} S^{A\amalg B} \wedge X(A\amalg B)_+ = (\Sigma^\bullet_+ X)(A\amalg B) \end{aligned}$$

where the unlabelled isomorphism is the canonical one.

This has a right adjoint $\Omega^\bullet$ (e.g. by the Special Adjoint Functor Theorem); for any finite group G, we get an induced adjunction $\boldsymbol{G}$-$\boldsymbol{\mathcal{I}}$-**SSet** $\rightleftarrows$ $\boldsymbol{G}$-**Spectra** by pulling through the G-actions, which we again denote by $\Sigma^\bullet_+ \dashv \Omega^\bullet$.

Warning 7.1.8. Beware that [Len20] uses different (more elaborate) notation for the right adjoint, reserving the above for the right adjoint of $\Sigma^\bullet_+\colon \boldsymbol{G}$-$\boldsymbol{I}$-**SSet** $\to$ $\boldsymbol{G}$-**Spectra**. However, as the latter adjoint will play no role here, we have decided to use the above, simpler notation.

Lemma 7.1.9 (See [Len20, Proposition 3.2.2, Corollary 3.2.6, and Remark 3.2.7])**.** *The above functor $\Sigma^\bullet_+$ preserves G-global weak equivalences and it is part of a Quillen adjunction*

$$\Sigma^\bullet_+ \colon \boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}_{\text{G-gl}} \rightleftarrows \boldsymbol{G}\text{-}\mathbf{Spectra}_{G\text{-gl proj}} : \Omega^\bullet. \qquad \square$$

In particular, we get a global functor $\Sigma^\bullet_+ \colon \underline{\mathscr{S}}^{\mathrm{gl}} \to \underline{\mathscr{S}p}^{\mathrm{gl}}$, and this admits a pointwise adjoint $\mathbf{R}\Omega^\bullet$ as Quillen adjunctions induce adjunctions of ∞-categories. In fact we have:

Proposition 7.1.10. *The global functor $\Sigma^\bullet_+ \colon \underline{\mathscr{S}}^{\mathrm{gl}} \to \underline{\mathscr{S}p}^{\mathrm{gl}}$ admits a parametrized right adjoint, given pointwise by the right derived functors $\mathbf{R}\Omega^\bullet$.*

We will denote this right adjoint simply by $\mathbf{R}\Omega^\bullet$ again.

Proof. As we already know that these form pointwise right adjoints, it only remains to verify the Beck-Chevalley condition, i.e. that for every $\alpha\colon H \to G$ the canonical mate $\alpha^*\mathbf{R}\Omega^\bullet \Rightarrow \mathbf{R}\Omega^\bullet\alpha^*$ is an equivalence. This can be checked on the level of homotopy categories, for which we pick a fibrant replacement functor for the projective H-global model structure on $\boldsymbol{H}$-**Spectra**, i.e. an endofunctor P taking values in projectively fibrant objects together with a natural transformation $\iota\colon \mathrm{id} \Rightarrow P$ that is levelwise an H-global weak equivalence. As $\Sigma^\bullet_+$ and α^* are homotopical (Lemma 7.1.9 and Lemma 7.1.5, respectively) and $\Omega^\bullet$ is right Quillen (Lemma 7.1.9 again), the mate is then represented for any fibrant G-spectrum X

by the lower composite $\alpha^*\Omega^\bullet X \to \Omega^\bullet P\alpha^* X$ in the diagram

$$\begin{array}{ccccccc}
\alpha^*\Omega^\bullet X & \xrightarrow{\eta} & \Omega^\bullet\Sigma^\bullet_+\alpha^*\Omega^\bullet X & = & \Omega^\bullet\alpha^*\Sigma^\bullet_+\Omega^\bullet X & \xrightarrow{\varepsilon} & \Omega^\bullet\alpha^* X \\
 & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\
 & & \Omega^\bullet P\Sigma^\bullet_+\alpha^*\Omega^\bullet X & = & \Omega^\bullet P\alpha^*\Sigma^\bullet_+\Omega^\bullet X & \xrightarrow[\varepsilon]{} & \Omega^\bullet P\alpha^* X
\end{array}$$

in which the two squares commute simply by naturality. However, the top composite is simply the identity (as the adjunction was defined by pulling through the actions); on the other hand, $\iota\colon \alpha^*X \to P\alpha^*X$ is an H-global weak equivalence of fibrant objects (α^* being right Quillen), hence $\Omega^\bullet\iota\colon \Omega^\bullet\alpha^*X \to \Omega^\bullet P\alpha^*X$ is an H-global weak equivalence by Ken Brown's Lemma ($\Omega^\bullet$ being right Quillen). The claim now follows by 2-out-of-3. □

7.1.2. *A t-structure.* The model structures from Theorems 7.1.3 and 7.1.6 are stable [Len20, Proposition 3.1.48], and so $\mathscr{S}\!p^{\mathrm{gl}}_G$ is a stable ∞-category. We will close this discussion by establishing a t-structure on it which generalizes Schwede's t-structure on the global stable homotopy category from [Sch18, Theorem 4.4.9]. For this we first introduce:

Construction 7.1.11. Let H be a finite group, let $\varphi\colon H \to G$ be a homomorphism, and let $k \in \mathbb{Z}$. If X is any G-global spectrum, then the *k-th φ-equivariant homotopy group* $\pi^\varphi_k(X)$ is the usual equivariant homotopy group $\pi^H_k(\varphi^*X)$, i.e. the hom set $[\Sigma^k\mathbb{S}, \varphi^*X]$ in the H-equivariant stable homotopy category, with the group structure coming from semiadditivity.

Equivalently (but more intrinsically), we can also describe $\pi^\varphi_k(X)$ as the hom set $[\Sigma^{\bullet+k}_+ I(H,–)\times_\varphi G, X]$ in the homotopy category of $\mathscr{S}\!p^{\mathrm{gl}}_G$, see [Len20, Corollary 3.3.4].

Theorem 7.1.12. *The stable ∞-category $\mathscr{S}\!p^{\mathrm{gl}}_G$ is compactly generated by the objects $\Sigma^\bullet_+ I(H,–)\times_\varphi G$ for homomorphisms $\varphi\colon H \to G$ from finite groups to G. Moreover, it admits a right complete t-structure with*

(1) connective part $(\mathscr{S}\!p^{\mathrm{gl}}_G)_{\ge 0}$ those G-global spectra that are G-globally connective*, i.e. $\pi^\varphi_k X = 0$ for all $k < 0$,*
(2) coconnective part $(\mathscr{S}\!p^{\mathrm{gl}}_G)_{\le 0}$ those G-global spectra that are G-globally coconnective*, i.e. $\varphi^\varphi_k X = 0$ for all $k > 0$.*

Here we recall [Lur17, p. 44] that a t-structure on a stable ∞-category $\mathscr{C}$ is called *right complete* if the truncations exhibit $\mathscr{C}$ as the inverse limit

$$\cdots \xrightarrow{\tau_{\ge -2}} \mathscr{C}_{\ge -2} \xrightarrow{\tau_{\ge -1}} \mathscr{C}_{\ge -1} \xrightarrow{\tau_{\ge 0}} \mathscr{C}_{\ge 0}.$$

By [Lur17, Proposition 1.2.1.19] this is equivalent to demanding that $\bigcap_n \mathscr{C}_{\le -n}$ consist only of zero objects.

Proof. We first observe that the G-global spectra $\Sigma^\bullet_+ I(H,–)\times_\varphi G$ for finite groups H (up to isomorphism) and homomorphisms $\varphi\colon H \to G$ form a set of compact generators. Indeed, the φ-equivariant homotopy groups for varying φ detect zero objects as the H-equivariant homotopy groups for every H do [Hau17, Proposition 4.9-(iii)], and they moreover commute with coproducts as the classical equivariant homotopy groups do (by the same argument) and since $\varphi^*\colon \mathscr{S}\!p^{\mathrm{gl}}_G \to \mathscr{S}\!p_H$ is a left adjoint by [Len20, Corollary 3.3.4].

With this established, [Lur17, Proposition 1.4.4.11] yields a t-structure on $\mathscr{S}p^{\mathrm{gl}}_G$ with connective part $(\mathscr{S}p^{\mathrm{gl}}_G)_{\geq 0}$ the smallest subcategory closed under small colimits and extensions containing all the $\Sigma^\bullet_+ I(H,-) \times_\varphi G$. We claim that this has the desired properties.

To see this, we let Y be a G-global spectrum. Then the non-negative homotopy groups of Y vanish if and only if $\mathrm{maps}(\Sigma^\bullet_+ I(H,-) \times_\varphi G, Y) \simeq 0$ for all $\varphi\colon H \to G$. On the other hand, the class of objects X for which $\mathrm{maps}(X, Y) \simeq 0$ is closed under colimits and extensions, so it has to contain all of $(\mathscr{S}p^{\mathrm{gl}}_G)_{\geq 0}$ in this case, i.e. $(\mathscr{S}p^{\mathrm{gl}}_G)_{\leq -1}$ consists precisely of those objects with trivial non-negative homotopy groups. As suspension shifts (H-equivariant, hence G-global) homotopy groups, this proves the characterization of the coconnective objects.

On the other hand, the connective G-global spectra contain all the $\Sigma^\bullet_+ I(H,-) \times_\varphi G$'s and they are closed under small coproducts (see above) as well as cofibers and extensions (by the long exact sequence), i.e. every object in $(\mathscr{S}p^{\mathrm{gl}}_G)_{\geq 0}$ is G-globally connective. Conversely, if X is G-globally connective, then there is a cofiber sequence $X_{\geq 0} \to X \to X_{\leq -1}$ with $X_{\geq 0} \in (\mathscr{S}p^{\mathrm{gl}}_G)_{\geq 0}$ and $X_{\leq -1} \in (\mathscr{S}p^{\mathrm{gl}}_G)_{\leq -1}$ by what it means to be a t-structure. But then $X_{\geq 0}$ is G-globally connective by the above, whence so is the cofiber $X_{\leq -1}$. But on the other hand $X_{\leq -1}$ has trivial non-negative homotopy groups, so $X_{\leq -1} \simeq 0$ and hence $X \simeq X_{\geq 0} \in (\mathscr{S}p^{\mathrm{gl}}_G)_{\geq 0}$ as claimed.

This finishes the construction of the desired t-structure. Right completeness is immediate as any object in $\bigcap_{n\geq 0} (\mathscr{S}p^{\mathrm{gl}}_G)_{\leq -n}$ has trivial homotopy groups. □

7.2. **From global Γ-spaces to global spectra.** Segal's classical Delooping Theorem [Seg74] relates (very special) Γ-spaces to connective spectra. We will now recall a G-global refinement of this from [Len20] and interpret it in the parametrized context.

Construction 7.2.1. We define a functor $\mathcal{E}^\otimes\colon \mathbf{\Gamma\text{-}\mathcal{I}\text{-}SSet}_* \to \mathbf{Spectra}$ from the 1-category of global Γ-spaces X satisfying $X(0^+) = *$ to symmetric spectra via the $\mathbf{SSet}_*$-enriched coend

$$\mathcal{E}^\otimes X := \int^{T_+ \in \Gamma} \mathbb{S}^{\times T} \otimes X(T_+)$$

with the evident functoriality in X; here $\otimes$ denotes the pointwise smash product of a spectrum with a pointed $\mathcal{I}$-simplicial set, see [Len20, Construction 3.2.9].

For any finite group G, pulling through the G-actions yields a functor

$$\mathcal{E}^\otimes\colon \mathbf{\Gamma\text{-}\mathit{G}\text{-}\mathcal{I}\text{-}SSet}_* \to \mathbf{\mathit{G}\text{-}Spectra}$$

that we again denote by $\mathcal{E}^\otimes$.

Proposition 7.2.2. *For any finite G, there is a model structure on $\mathbf{\Gamma\text{-}\mathit{G}\text{-}\mathcal{I}\text{-}SSet}_*$ in which a map f is a weak equivalence or fibration if and only if $f(S_+)$ is a weak equivalence or fibration in the model structure on $\mathbf{(\mathit{G} \times \Sigma_{\mathit{S}})\text{-}\mathcal{I}\text{-}SSet}$ from Theorem 3.1.12 for every finite set S; in particular, its weak equivalences are precisely the G-global level weak equivalences.*

Moreover, the above functor $\mathcal{E}^\otimes$ is homotopical and part of a Quillen adjunction

$$\mathcal{E}^\otimes\colon \mathbf{\Gamma\text{-}\mathit{G}\text{-}\mathcal{I}\text{-}SSet}_* \rightleftarrows \mathbf{\mathit{G}\text{-}Spectra}_{G\text{-gl inj}} :\Phi^\otimes.$$

Proof. The existence of the model structure is part of [Len20, Theorem 2.2.31], while the remaining statements appear as Corollaries 3.4.20 and 3.4.24 of *op. cit.* □

Remark 7.2.3. While the precise form of the above right adjoint will not be relevant below, we record that there is a natural isomorphism $(\Phi^{\otimes}X)(1^+) \cong \Omega^\bullet X$, see [Len20, Construction 3.2.17]. Restricting to injectively fibrant objects, we in particular immediately obtain an equivalence $\mathbb{U}\mathbf{R}\Phi^{\otimes} \simeq \mathbf{R}\Omega^\bullet$ of derived functors for any fixed G.

Passing to localizations, $\mathcal{E}^{\otimes}$ induces a global functor $\underline{\Gamma\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I},*} \to \underline{\mathscr{S}p}^{\mathrm{gl}}$.

Lemma 7.2.4. *The global functor $\mathcal{E}^{\otimes}\colon \underline{\Gamma\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I},*} \to \underline{\mathscr{S}p}^{\mathrm{gl}}$ admits a parametrized right adjoint which is pointwise given by the $\mathbf{R}\Phi^{\otimes}$.*

We will denote this parametrized right adjoint simply by $\mathbf{R}\Phi^{\otimes}$ again.

Proof. It only remains to prove that for every $\alpha\colon H \to G$ the mate transformation $\alpha^*\mathbf{R}\Phi^{\otimes} \Rightarrow \mathbf{R}\Phi^{\otimes}\alpha^*$ at the level of homotopy categories is an equivalence. By the same computation as in Proposition 7.1.10 this reduces to showing that for any injectively fibrant G-global spectrum X and some (hence any) injectively fibrant replacement $\iota\colon \alpha^*X \to Y$ of G-global spectra the induced map $\Phi^{\otimes}\iota\colon \Phi^{\otimes}\alpha^*X \to \Phi^{\otimes}Y = \mathbf{R}\Phi^{\otimes}\alpha^*X$ is an H-global level weak equivalence. This is precisely the content of [Len20, claim in proof of Proposition 3.4.30]. □

Definition 7.2.5. A special G-global Γ-space $X \in \mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}_*$ is called *very special* if for every finite group H, every homomorphism $\varphi\colon H \to G$, and some (hence any) complete H-set universe $\mathcal{U}_H$ the induced monoid structure on $\pi_0^{\varphi}(X) := \pi_0\big((\varphi^*X)(1^+)(\mathcal{U}_H)\big)$ coming from the zig-zag

$$X(1^+) \times X(1^+) \xleftarrow[\sim]{\rho} X(2^+) \xrightarrow{X(\mu)} X(1^+),$$

where μ is defined by $\mu(1) = \mu(2) = 1$, is a group structure. We write $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ vspc}}_{\mathcal{I},*} \subset \underline{\Gamma\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I},*}$ for the full global subcategory of very special objects.

Remark 7.2.6. The above condition is equivalent to $\varphi^*X(\mathcal{U}_H)$ being very special as an H-equivariant Γ-space in the sense of [Ost16, Definition 4.5] for every H and φ as above, see [Len20, discussion after Definition 3.4.12].

We can now rephrase the G-global delooping theorem [Len20, Theorem 3.4.21] in our setting as follows:

Theorem 7.2.7. *The parametrized adjunction $\mathcal{E}^{\otimes} \dashv \mathbf{R}\Phi^{\otimes}$ restricts to an equivalence $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ vspc}}_{\mathcal{I},*} \simeq \underline{\mathscr{S}p}^{\mathrm{gl}}_{\geq 0}$.* □

Finally, we want to reinterpret this in terms of equivariant stabilizations, in the sense of Example 6.2.3

Theorem 7.2.8. *The global ∞-category $\underline{\mathscr{S}p}^{\mathrm{gl}}$ is equivariantly stable and the functor*

$$\mathbf{R}\Phi^{\otimes}\colon \underline{\mathscr{S}p}^{\mathrm{gl}} \to \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\ spc}}_{\mathcal{I},*} \tag{19}$$

is the universal equivariant stabilization.

For the proof of the theorem we will need two lemmas:

Lemma 7.2.9. *The adjunction* $\mathrm{incl}\colon (\mathscr{S}\!p_G^{\mathrm{gl}})_{\ge 0} \rightleftarrows \mathscr{S}\!p_G^{\mathrm{gl}} : \tau_{\ge 0}$ *is the universal stabilization in the world of presentable ∞-categories.*

Proof. By Theorem 7.1.12, $(\mathscr{S}\!p_G^{\mathrm{gl}})_{\ge 0}$ is the connective part of a right complete t-structure. As mentioned without proof in the introduction of [Lur18, Appendix C], this formally implies the statement of the lemma. Let us give the argument in this generality for completeness: given a right complete t-structure on a stable ∞-category $\mathscr{C}$ we consider the diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\Omega} & \mathscr{C}_{\ge 0} & \xrightarrow{\Omega} & \mathscr{C}_{\ge 0} & \xrightarrow[\simeq]{\Omega} & \mathscr{C}_{\ge 0} \\
 & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\Omega^2} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\Omega} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\mathrm{id}} \\
\cdots & \xrightarrow[\tau_{\ge -2}]{} & \mathscr{C}_{\ge -2} & \xrightarrow[\tau_{\ge -1}]{} & \mathscr{C}_{\ge -1} & \xrightarrow[\tau_{\ge 0}]{} & \mathscr{C}_{\ge 0}
\end{array}$$

where the little squares are filled by the total mates of the identity transformations $\Sigma^n \circ \mathrm{incl} = \Sigma^{n-1} \circ \Sigma$. Passing to row-wise homotopy limits we then get a commutative diagram

$$\begin{array}{ccc}
\mathrm{Sp}(\mathscr{C}_{\ge 0}) = \lim_n \mathscr{C}_{\ge 0} & \xrightarrow{\Omega^\infty = \mathrm{pr}_0} & \\
{\scriptstyle \lim_n \Omega^n}\big\downarrow & & \mathscr{C}_{\ge 0} \\
\lim_n \mathscr{C}_{\ge -n} & \xrightarrow[\mathrm{pr}_0]{} &
\end{array}$$

in which the vertical map on the left is an equivalence as a homotopy limit of equivalences. On the other hand, by right completeness the lower map agrees up to equivalence with $\tau_{\ge 0}\colon \mathscr{C} \to \mathscr{C}_{\ge 0}$; the claim follows immediately as $\Omega^\infty\colon \mathrm{Sp}(\mathscr{C}_{\ge 0}) \to \mathscr{C}_{\ge 0}$ is the universal stabilization by [Lur17, Remark 1.4.2.25]. □

Lemma 7.2.10. *Let T be an ∞-category and let $U\colon \mathcal{D} \to \mathcal{C}$ be a T-functor such that $\mathcal{D}$ is fiberwise stable, $\mathcal{C}$ has fiberwise finite limits, and each $U(A)\colon \mathcal{D}(A) \to \mathcal{C}(A)$ is a stabilization in the non-parametrized sense. Then U is a fiberwise stabilization.*

Put differently, if we already know fiberwise stability of the source, then fiberwise stabilizations can be checked pointwise without regards to any homotopies or higher structure.

Proof. In the naturality square

$$\begin{array}{ccc}
\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{D}) & \xrightarrow{\underline{\mathrm{Sp}}^{\mathrm{fib}}(U)} & \underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C}) \\
{\scriptstyle\Omega^\infty}\big\downarrow & & \big\downarrow{\scriptstyle\Omega^\infty} \\
\mathcal{D} & \xrightarrow[U]{} & \mathcal{C}
\end{array}$$

the left hand vertical arrow is an equivalence as $\mathcal{D}$ is fiberwise stable, and so is the top horizontal map as $\big(\underline{\mathrm{Sp}}^{\mathrm{fib}}(U)\big)(A) = \mathrm{Sp}\,\big(U(A)\big)$ and each $U(A)$ was assumed to be a stabilization. Finally, the right hand vertical map is a fiberwise stabilization by construction, so the claim follows immediately. □

Proof of Theorem 7.2.8. As each $\mathscr{S}p^{\mathrm{gl}}_G$ is stable and all restriction maps between them are exact (being right adjoints), it will suffice by the previous lemma that

$$\mathbf{R}\Phi^{\otimes}\colon \mathscr{S}p^{\mathrm{gl}}_G \to \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*}(G)$$

is a stabilization in the non-parametrized sense for every fixed G, for which it suffices by stability of the source that this induces an equivalence after applying spectrification. By Lemma 7.2.9, it suffices to show this for the restriction to $(\mathscr{S}p^{\mathrm{gl}}_G)_{\ge 0}$, for which it is then in turn enough by Theorem 7.2.7 that the inclusion $\mathrm{incl}\colon \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,vspc}}_{\mathcal{I},*}(G) \hookrightarrow \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*}(G)$ of very special G-global Γ-spaces induces an equivalence after spectrification.

For this we observe that the loop space functor $\Omega\colon \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*}(G) \to \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*}(G)$ factors through $\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,vspc}}_{\mathcal{I},*}(G)$ as for any special G-global Γ-space X the commutative monoid structure on $\pi_0^{\varphi}(\Omega X)$ coming from the Γ-space structure agrees with the group structure coming from Ω by the Eckmann-Hilton argument. It is then clear that for the induced functor $\mathrm{Sp}(\Omega)\colon \mathrm{Sp}(\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*}(G)) \to \mathrm{Sp}(\underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,vspc}}_{\mathcal{I},*}(G))$ the composites $\mathrm{Sp}(\mathrm{incl})\,\mathrm{Sp}(\Omega)$ and $\mathrm{Sp}(\Omega)\,\mathrm{Sp}(\mathrm{incl})$ are given by the respective loop functors, so they are equivalences by stability. The claim follows by 2-out-of-6. □

7.3. **Proof of Theorem C.**

Using the above we now easily get:

Theorem 7.3.1. *The functor* $\mathbf{R}\Omega^{\bullet}\colon \underline{\mathscr{S}p}^{\mathrm{gl}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ *exhibits* $\underline{\mathscr{S}p}^{\mathrm{gl}}$ *as the equivariantly stable envelope of* $\underline{\mathscr{S}}^{\mathrm{gl}}$*, i.e. for every equivariantly stable global ∞-category $\mathcal{C}$ postcomposition with* $\mathbf{R}\Omega^{\bullet}$ *induces an equivalence*

$$\underline{\mathrm{Fun}}^{\text{Orb-lex}}_{\mathrm{Glo}}(\mathcal{C},\mathbf{R}\Omega^{\bullet})\colon \underline{\mathrm{Fun}}^{\text{Orb-ex}}_{\mathrm{Glo}}(\mathcal{C},\underline{\mathscr{S}p}^{\mathrm{gl}}) \to \underline{\mathrm{Fun}}^{\text{Orb-lex}}_{\mathrm{Glo}}(\mathcal{C},\underline{\mathscr{S}}^{\mathrm{gl}}).$$

Moreover, the left adjoint $\Sigma^{\bullet}_{+}$ *exhibits* $\underline{\mathscr{S}p}^{\mathrm{gl}}$ *as the equivariantly stable completion in the following sense: for any globally cocomplete equivariantly stable global ∞-category $\mathcal{D}$ precomposition with* $\Sigma^{\bullet}_{+}$ *yields an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\Sigma^{\bullet}_{+},\mathcal{D})\colon \underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}p}^{\mathrm{gl}},\mathcal{D}) \to \underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}}^{\mathrm{gl}},\mathcal{D}).$$

Proof. By Theorem 6.2.13 and Corollary 6.2.18, respectively, together with Corollary 5.3.7 it will suffice to show that the diagrams

$$\begin{array}{ccc} \underline{\mathscr{S}p}^{\mathrm{gl}} & \xrightarrow{\mathbf{R}\Omega^{\bullet}} & \underline{\mathscr{S}}^{\mathrm{gl}} \\ & {\scriptstyle\mathbf{R}\Phi^{\otimes}}\searrow \quad \nearrow{\scriptstyle\mathbb{U}} & \\ & \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \underline{\mathscr{S}p}^{\mathrm{gl}} & \xleftarrow{\Sigma^{\bullet}_{+}} & \underline{\mathscr{S}}^{\mathrm{gl}} \\ & {\scriptstyle\mathcal{E}^{\otimes}}\nwarrow \quad \swarrow{\scriptstyle\mathbb{P}} & \\ & \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{\mathcal{I},*} & \end{array}$$

of global functors commute up to equivalence.

By uniqueness of adjoints, it suffices to prove this for the second diagram, for which it is enough by the universal property of global spaces to chase through $* \in \mathscr{S}^{\mathrm{gl}}_1$; in particular, it suffices to show that this commutes after evaluation at the trivial group. But by uniqueness of adjoints again, it is then enough to prove this for the diagram on the left instead, where this is immediate from Remark 7.2.3. □

Together with Theorem 3.3.2 we then immediately get Theorem C from the introduction:

Theorem 7.3.2. *Let $\mathcal{D}$ be any globally cocomplete equivariantly stable global ∞-category. Then evaluation at the global sphere spectrum $\mathbb{S} \cong \Sigma^\bullet_+(*) \in \mathscr{S}\!p^{\mathrm{gl}}_1$ defines an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathrm{Glo}}(\underline{\mathscr{S}\!p}^{\mathrm{gl}}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}.$$

Put differently, $\underline{\mathscr{S}\!p}^{\mathrm{gl}}$ is the free globally cocomplete (or presentable) Orb*-stable global ∞-category on one generator (namely, the global sphere spectrum $\mathbb{S}$).* □

Comparing universal properties we can also reformulate this as follows:

Corollary 7.3.3. *The essentially unique left adjoint functor $\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Glo}} \to \underline{\mathscr{S}\!p}^{\mathrm{gl}}$ sending $\Sigma^\infty_+(*)$ to $\mathbb{S}$ is an equivalence.* □

Appendix A. Slices of $(2,1)$-categories

In this short appendix we will prove that for a strict $(2,1)$-category the ∞-categorical and 2-categorical slices agree. More precisely:

Proposition A.1. *Let $\mathscr{C}$ be a strict $(2,1)$-category. Then the cocartesian fibration $\mathrm{ev}_1 \colon \mathrm{N}_\Delta(\mathscr{C})^{\Delta^1} \to \mathrm{N}_\Delta(\mathscr{C})$ classifies the homotopy coherent nerve of the composition*

$$\mathscr{C} \xrightarrow{\mathscr{C}_{/\bullet}} \mathbf{Cat}_{(2,1)} \xrightarrow{\mathrm{N}_\Delta} \mathbf{Cat}_\infty.$$

Proof. We begin by making the 2-categorical Grothendieck construction $\pi\colon \mathscr{G}r \to \mathscr{C}$ (Construction 5.2.5) of the functor $\mathscr{C}_{/\bullet}\colon \mathscr{C} \to \mathbf{Cat}_{(2,1)}$ explicit, which, upon passing to homotopy coherent nerves, will then yield a concrete model of the unstraightening:

(1) An object of $\mathscr{G}r$ is a morphism $f\colon X \to Y$ in $\mathscr{C}$.
(2) A morphism $f \to g$ is a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{x} & X_2 \\ {\scriptstyle f}\downarrow & \overset{\theta}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ Y_1 & \xrightarrow[y]{} & Y_2 \end{array} \tag{20}$$

(the pair (x,θ) being a morphism from the pushforward $\mathscr{C}_{/y}(f)$ to g in $\mathscr{C}_{/Y_2}$). Composition of morphisms is given by composition of 1-cells and pasting of 2-cells in $\mathscr{C}$.
(3) A 2-cell between two such morphisms (x,θ,y), (x',θ',y') is a pair of a 2-cell $\sigma\colon x \Rightarrow x'$ and a 2-cell $\tau\colon y \Rightarrow y'$ such that the pastings

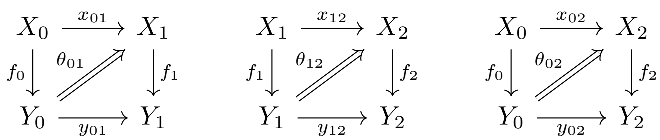
$$\begin{array}{ccc} X_1 & \overset{x'}{\underset{x}{\rightrightarrows}}\ {\scriptstyle \sigma\Uparrow} & X_2 \\ {\scriptstyle f}\downarrow & \underset{\theta}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ Y_1 & \xrightarrow[y]{} & Y_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} X_1 & \xrightarrow{x'} & X_2 \\ {\scriptstyle f}\downarrow & \overset{\theta'}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ Y_1 & \overset{y'}{\underset{y}{\rightrightarrows}}\ {\scriptstyle \tau\Uparrow} & Y_2 \end{array}$$

agree. Horizontal and vertical composition of 2-cells is given by horizontal and vertical composition, respectively, in $\mathscr{C}$.

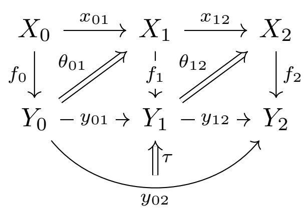
The projection $\pi\colon \mathscr{G}r \to \mathscr{C}$ sends an object $f\colon X \to Y$ to Y, a morphism (20) to y, and a 2-cell (σ, τ) to τ.

The homotopy coherent nerve $\mathrm{N}_\Delta(\mathscr{G}r)$ is then a strictly 3-coskeletal simplicial set, hence it suffices to describe the 2-truncation and to characterize which diagrams $\partial\Delta^3 \to \mathrm{N}_\Delta(\mathscr{G})$ extend over Δ^3. Unravelling the definitions, we get:

(1) A vertex of $\mathrm{N}_\Delta(\mathscr{G})$ is a morphism $f\colon X \to Y$ in $\mathscr{C}$.
(2) An edge $f \to g$ in $\mathrm{N}_\Delta(\mathscr{G})$ is a diagram (20).
(3) A 2-simplex with boundary



amounts to the data of a natural transformation $\sigma\colon x_{02} \Rightarrow x_{12}x_{01}$ and a natural transformation $\tau\colon y_{02} \Rightarrow y_{12}y_{01}$ such that the two pastings



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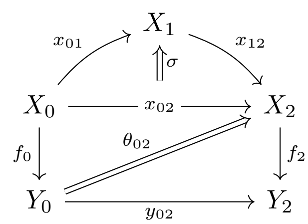
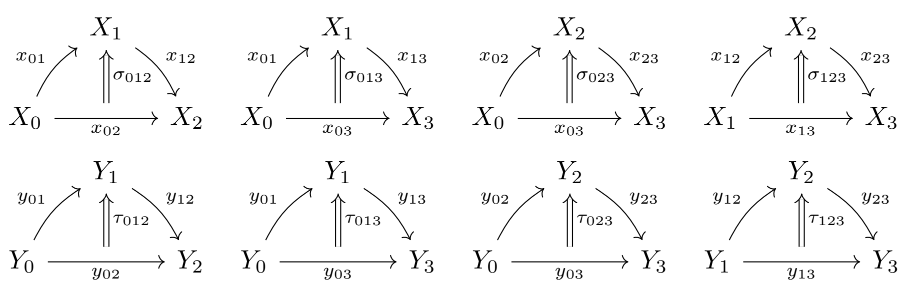
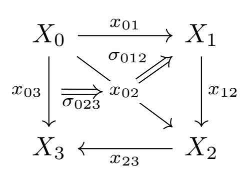


agree.
(4) A diagram $\partial\Delta^3 \to \mathrm{N}_\Delta(\mathscr{G}r)$ corresponding to

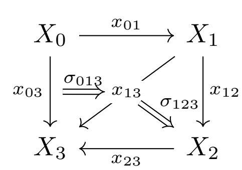


(21)

extends to Δ^3 if and only if the pastings



and



agree, and likewise for the τ's. Put differently, $\partial\Delta^3 \to \mathrm{N}_\Delta(\mathscr{G}r)$ extends over Δ^3 if and only if the two maps $\partial\Delta^3 \to \mathrm{N}_\Delta(\mathscr{C})$ defined by (21) extend over Δ^3.

The degeneracy map $\mathrm{N}_\Delta(\mathscr{G}r)_0 \to \mathrm{N}_\Delta(\mathscr{G}r)_1$ is given by sending $f\colon X \to Y$ to the square

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle f}\downarrow & \overset{\mathrm{id}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}]{} & Y \end{array}$$

and similarly the degeneracies $\mathrm{N}_\Delta(\mathscr{G}r)_1 \to \mathrm{N}_\Delta(\mathscr{G}r)_2$ are given by inserting identity arrows and identity 2-cells.

The map $\mathrm{N}_\Delta(\pi)\colon \mathrm{N}_\Delta(\mathscr{G}r) \to \mathrm{N}_\Delta(\mathscr{C})$ is the evident forgetful map. It then remains to construct an equivalence $\mathrm{N}_\Delta(\mathscr{G}r) \simeq \mathrm{N}_\Delta(\mathscr{C})^{\Delta^1}$ of cocartesian fibrations over $\mathrm{N}_\Delta(\mathscr{G}r)$.

For this we observe that $\mathrm{N}_\Delta(\mathscr{C})^{\Delta^1}$ is again strictly 3-coskeletal (as $\mathrm{N}_\Delta(\mathscr{C})$ is), and that unravelling definitions it can be described as follows:

(1) A vertex of $\mathrm{N}_\Delta(\mathscr{C})^{\Delta^1}$ is a morphism $f\colon X \to Y$ in $\mathscr{C}$.
(2) An edge $f \to g$ in $\mathrm{N}_\Delta(\mathscr{C})^{\Delta^1}$ is a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{x} & X_2 \\ {\scriptstyle f}\downarrow & \overset{\theta}{\nearrow\!\!\!\nearrow}\ \searrow\ \underset{\theta'}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle g} \\ Y_1 & \xrightarrow[y]{} & Y_2. \end{array} \tag{22}$$

(3) A 2-simplex in $\mathrm{N}_\Delta(\mathscr{C})^{\Delta^1}$ with boundary

$$\begin{array}{ccc} X_0 & \xrightarrow{x_{01}} & X_1 \\ {\scriptstyle f_0}\downarrow & \overset{\theta_{01}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle f_1} \\ Y_0 & \xrightarrow[y_{01}]{} & Y_1 \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{x_{12}} & X_2 \\ {\scriptstyle f_1}\downarrow & \overset{\theta_{12}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow[y_{12}]{} & Y_2 \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{x_{02}} & X_2 \\ {\scriptstyle f_0}\downarrow & \overset{\theta_{02}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle f_2} \\ Y_0 & \xrightarrow[y_{02}]{} & Y_2 \end{array}$$

(where we have pasted the two natural isomorphisms and omitted the middle arrow) amounts to the data of a natural transformation $\sigma\colon x_{02} \Rightarrow x_{12}x_{01}$ and a transformation $\tau\colon y_{02} \Rightarrow y_{12}y_{01}$ satisfying the same conditions as for $\mathrm{N}_\Delta(\mathscr{G})$.
(4) A diagram $\partial\Delta^3 \to \mathrm{N}_\Delta(\mathscr{C})^{\Delta^1}$ corresponding to (21) extends to Δ^3 if and only if it satisfies the same pasting condition as for $\mathrm{N}_\Delta(\mathscr{G})$, i.e. if and only if the two maps $\partial\Delta^3 \to \mathrm{N}_\Delta(\mathscr{C})$ defined by the above extend to Δ^3.

In each case, the degeneracy maps are again given by inserting identity arrows and 2-cells.

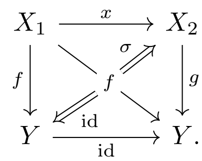
It is then straight-forward to check that we have a unique map $\Phi\colon \mathrm{N}_\Delta(\mathscr{C})^{\Delta^1} \to \mathrm{N}_\Delta(\mathscr{G}r)$ that is the identity on vertices, sends an edge (22) to the edge given by pasting of θ and $(\theta')^{-1}$, and that sends a 2-simplex of $\mathrm{N}_\Delta(\mathscr{C})^{\Delta^1}$ corresponding to $\sigma\colon x_{02} \Rightarrow x_{12}x_{01}$, $\tau\colon y_{02} \Rightarrow y_{12}y_{01}$ to the 2-simplex of $\mathrm{N}_\Delta(\mathscr{G}r)$ corresponding to the same transformations. This is clearly a map over $\mathrm{N}_\Delta(\mathscr{C})$ and so by [Lur09, Proposition 3.1.3.5] it only remains to show that it induces equivalences on fibers.

It is bijective on objects by definition, so it only remains to prove that for all $f\colon X_1 \to Y$, $g\colon X_2 \to Y$ the induced map

$$\mathrm{Hom}^{\mathrm{L}}_{(\mathrm{N}_\Delta(\mathscr{C})^{\Delta^1})_Y}(f,g) \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\Delta(\mathscr{G}r)_Y}(f,g) \tag{23}$$

is a weak homotopy equivalence. However, both sides are nerves of groupoids, so it is enough to show that it is surjective on vertices and that for any two vertices α, β on the left hand side it induces a bijection between edges $\alpha \to \beta$ and edges between their images.

For the first statement, it suffices to observe that by definition (23) is given on vertices by the effect of Φ on edges $f \to g$; thus, given any edge $(x, \mathrm{id}_Y, \sigma)$ of $\mathrm{N}_\Delta(\mathscr{G})_Y$, a preimage is given by



Similarly, the effect of (23) on edges is induced by the effect of Φ on 2-cells, so it follows immediately from the above description that it induces bijections between edges $\alpha \to \beta$ and edges between their images. □

Remark A.2. Let $\mathscr{I}$ be a (say, strict) $(2,1)$-category; as announced in [Dus01], the ∞-categorical functor category $\mathrm{N}_\Delta(\mathscr{C})^{\mathrm{N}_\Delta(\mathscr{I})}$ can be identified with the homotopy coherent nerve of the strict $(2,1)$-category $\mathrm{Fun}^{\mathrm{pseudo}}(\mathscr{I}, \mathscr{C})$ of normal (i.e. strictly unital) pseudofunctors $\mathscr{I} \to \mathscr{C}$, pseudonatural transformations, and modifications. If one is willing to take this for granted, the proof of the proposition can be significantly shortened, as the above Grothendieck construction $\mathscr{G}r$ is canonically *isomorphic* to $\mathrm{Fun}^{\mathrm{pseudo}}([1], \mathscr{C})$.

However, the authors are unaware of any place in the literature where such a comparison is actually proven: in particular, the announced sequel to [Dus01] apparently never appeared. On the level of objects (i.e. that maps $\mathrm{N}_\Delta(\mathscr{I}) \to \mathrm{N}_\Delta(\mathscr{C})$ correspond to normal pseudofunctors $\mathscr{I} \to \mathscr{C}$) a detailed proof is given as [Lur23, Tag `00AU`]. The statement that at least every pseudonatural transformation of functors $\mathscr{I} \to \mathscr{C}$ gives rise to a transformation of maps $\mathrm{N}_\Delta(\mathscr{I}) \to \mathrm{N}_\Delta(\mathscr{C})$ appears as [BFB05, Proposition 4.4], but its proof is left as an exercise.

References

[ABFJ22] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal, *Left-exact localizations of ∞-topoi. I: Higher sheaves*, Adv. Math. **400** (2022), 64, Id/No 108268.

[AS69] Michael F. Atiyah and Graeme B. Segal, *Equivariant K-theory and completion*, J. Differ. Geom. **3** (1969), 1–18.

[Bar21] Miguel Barrero, *Operads in unstable global homotopy theory*, arXiv:2110.01674 (2021).

[Bar23] ———, *Global transfer systems of abelian compact Lie groups*, Proc. Am. Math. Soc. **151** (2023), no. 7, 3169–3182.

[BD20] Paul Balmer and Ivo Dell'Ambrogio, *Mackey 2-functors and Mackey 2-motives*, EMS Monogr. Math., European Mathematical Society (EMS), 2020.

[BDG$^+$16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, *Parametrized higher category theory and higher algebra: Exposé I – Elements of parametrized higher category theory*, arXiv:1608.03657 (2016).

[BFB05] Manuel Bullejos, Emilio Faro, and Vicente Blanco, *A full and faithful nerve for 2-categories*, Appl. Categ. Struct. **13** (2005), no. 3, 223–233.

[Boh14] Anna Marie Bohmann, *Global orthogonal spectra*, Homology, Homotopy Appl. **16** (2014), no. 1, 313–332.

[Buc14] Mitchell Buckley, *Fibred 2-categories and bicategories*, J. Pure Appl. Algebra **218** (2014), no. 6, 1034–1074.

[Car84] Gunnar Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. Math. (2) **120** (1984), 189–224.

[Car91] ______, *Equivariant stable homotopy and Sullivan's conjecture*, Invent. Math. **103** (1991), no. 1, 497–525.

[DK84] W. G. Dwyer and D. M. Kan, *Singular functors and realization functors*, Indag. Math. **46** (1984), 147–153.

[Dus01] John W. Duskin, *Simplicial matrices and the nerves of weak n-categories I: Nerves of bicategories*, Theory Appl. Categ. **9** (2001), 198–308.

[Elm83] Anthony D. Elmendorf, *Systems of fixed point sets*, Trans. Am. Math. Soc. **277** (1983), 275–284.

[GH07] David Gepner and André Henriques, *Homotopy theory of orbispaces*, arXiv:math/0701916 (2007).

[GM20] David Gepner and Lennart Meier, *On equivariant topological modular forms*, arXiv:2004.10254 (2020).

[Gro60] Alexander Grothendieck, *Éléments de géométrie algébrique. I: Le langage des schémas*, Publ. Math., Inst. Hautes Étud. Sci. **4** (1960), 1–228.

[Hau17] Markus Hausmann, *G-symmetric spectra, semistability and the multiplicative norm*, J. Pure Appl. Algebra **221** (2017), no. 10, 2582–2632.

[Hau19] ______, *Symmetric spectra model global homotopy theory of finite groups*, Algebr. Geom. Topol. **19** (2019), no. 3, 1413–1452.

[Hau22] ______, *Global group laws and equivariant bordism rings*, Ann. Math. (2) **195** (2022), no. 3, 841–910.

[HHLN22a] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten, *Lax monoidal adjunctions, two-variable fibrations and the calculus of mates*, arXiv:2011.08808 (2022).

[HHLN22b] ______, *Two-variable fibrations, factorisation systems and ∞-categories of spans*, arXiv:2011.11042 (2022).

[HHR16] Michael A Hill, Michael J Hopkins, and Douglas C Ravenel, *On the nonexistence of elements of kervaire invariant one*, Ann. Math. (2) **184** (2016), 1–262.

[Hil22] Kaif Hilman, *Parametrised presentability over orbital categories*, arXiv:2202.02594 (2022).

[Hin16] Vladimir Hinich, *Dwyer-Kan localization revisited*, Homology Homotopy Appl. **18** (2016), no. 1, 27–48.

[HL13] Michael Hopkins and Jacob Lurie, *Ambidexterity in $K(n)$-local stable homotopy theory*, `https://people.math.harvard.edu/~lurie/papers/Ambidexterity.pdf` (2013).

[HNP19] Yonatan Harpaz, Joost Jakob Nuiten, and Matan Prasma, *Quillen cohomology of $(\infty,2)$-categories*, High. Struct. **3** (2019), no. 1, 17–66.

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra*, J. Am. Math. Soc. **13** (2000), no. 1, 149–208.

[HY17] Asaf Horev and Lior Yanovski, *On conjugates and adjoint descent*, Topology Appl. **232** (2017), 140–154.

[Jur20] Branko Juran, *Orbifolds, orbispaces and global homotopy theory*, arXiv:2006.12374 (2020).

[Len20] Tobias Lenz, *G-Global Homotopy Theory and Algebraic K-Theory*, to appear in *Mem. Amer. Math. Soc.*, arXiv:2012.12676 (2020).

[Len22] ______, *Genuine vs. naïve symmetric monoidal G-categories*, arXiv:2203.02277 (2022).

[LNP22] Sil Linskens, Denis Nardin, and Luca Pol, *Global homotopy theory via partially lax limits*, arXiv:2206.01556 (2022).

[Lur09] Jacob Lurie, *Higher topos theory*, Ann. Math. Stud., vol. 170, Princeton University Press, Princeton, NJ, 2009, updated version available at `https://people.math.harvard.edu/~lurie/papers/highertopoi.pdf`.

[Lur17] ______, *Higher algebra*, `https://www.math.ias.edu/~lurie/papers/HA.pdf` (2017).

[Lur18] ______, *Spectral Algebraic Geometry*, under construction (version dated February 2018), `www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf` (2018).

[Lur23] ______, *Kerodon*, online textbook, available at `https://kerodon.net` (2023).

[Mar21] Louis Martini, *Yoneda's lemma for internal higher categories*, arXiv:2103.17141 (2021).

[MW21] Louis Martini and Sebastian Wolf, *Limits and colimits in internal higher category theory*, arXiv:2111.14495 (2021).

[MW22] ———, *Presentable categories internal to an ∞-topos*, arXiv:2209.05103 (2022).

[Nar16] Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category*, arXiv:1608.07704 (2016).

[Nar17] ———, *Stability and distributivity over orbital ∞-categories*, Ph.D. thesis, Massachusetts Institute of Technology, 2017.

[NS18] Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), no. 2, 203–409.

[Ost16] Dominik Ostermayr, *Equivariant Γ-spaces*, Homology Homotopy Appl. **18** (2016), no. 1, 295–324.

[Par20] John Pardon, *Orbifold bordism and duality for finite orbispectra*, arXiv:2006.12702 (2020).

[Ram22] Maxime Ramzi, *The Yoneda embedding is natural*, arXiv:2209.12575 (2022).

[Rez14] Charles Rezk, *Global homotopy theory and cohesion*, `https://faculty.math.illinois.edu/~rezk/global-cohesion.pdf` (2014).

[Rie14] Emily Riehl, *Categorical Homotopy Theory*, New Math. Monogr., vol. 24, Cambridge: Cambridge University Press, 2014.

[Sch18] Stefan Schwede, *Global homotopy theory*, New Math. Monogr., vol. 34, Cambridge University Press, 2018.

[Sch20] ———, *Orbispaces, orthogonal spaces, and the universal compact Lie group*, Mathematische Zeitschrift **294** (2020), no. 1, 71–107.

[Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Sha21] Jay Shah, *Parametrized higher category theory*, to appear in *Algebr. Geom. Topol.*, arXiv:1809.05892 (2021).

[Shi89] Kazuhisa Shimakawa, *Infinite Loop G-Spaces Associated to Monoidal G-Graded Categories*, Publ. Res. Inst. Math. Sci. **25** (1989), no. 2, 239–262.

[SS12] Steffen Sagave and Christian Schlichtkrull, *Diagram spaces and symmetric spectra*, Adv. Math. **231** (2012), no. 3-4, 2116–2193.

[SS20] Hisham Sati and Urs Schreiber, *Proper orbifold cohomology*, arXiv:2008.01101 (2020).

[Wir74] Klaus Wirthmüller, *Equivariant homology and duality*, Manuscr. Math. **11** (1974), no. 4, 373–390.

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PARTIAL PARAMETRIZED PRESENTABILITY AND THE UNIVERSAL PROPERTY OF EQUIVARIANT SPECTRA

BASTIAAN CNOSSEN, TOBIAS LENZ, AND SIL LINSKENS

Abstract. We introduce a notion of *partial presentability* in parametrized higher category theory and investigate its interaction with the concepts of parametrized semiadditivity and stability from [CLL23]. In particular, we construct the free partially presentable T-categories in the unstable, semiadditive, and stable contexts and explain how to exhibit them as full subcategories of their fully presentable analogues.

Specializing our results to the setting of (global) equivariant homotopy theory, we obtain a notion of *equivariant presentability* for the global categories of [CLL23], and we show that the global category of genuine equivariant spectra is the free global category that is both equivariantly presentable and equivariantly stable. As a consequence, we deduce the analogous result about the G-category of genuine G-spectra for any finite group G, previously formulated by [Nar17].

Contents

1. Introduction

The term *equivariant mathematics* was coined by Balmer and dell'Ambrogio [BD20] to refer in a unified way to the study of objects with group actions across a wide range of mathematical disciplines, for example in representation theory or equivariant homotopy theory. Given a group homomorphism $\alpha\colon H \to G$, any G-action on an object X can naturally be restricted to an H-action, and accordingly most notions of 'equivariant objects' give rise to *global categories*: collections of (∞-)categories[1] $\mathcal{C}(G)$ for every finite group G equipped with suitably coherent restriction functors $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$, or more precisely categories parametrized over the 2-category Glo of finite connected groupoids.

Many fundamental concepts of (higher) category theory have analogues in the world of global categories, leading for instance to notions of *presentability*, *equivariant semiadditivity*, and *equivariant stability*. These properties were introduced and studied by the present authors in the previous article [CLL23], where we in particular showed that the universal presentable, presentable equivariantly semiadditive, and presentable equivariantly stable global categories all admit explicit models in terms of *global homotopy theory* in the sense of [Sch18, Hau19, Len20].

The presentability condition on a global category $\mathcal{C}$ used in these results is quite strong: in particular, it demands the existence of left adjoints to *all* restriction functors $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$. This is in fact too strong for certain applications: several interesting examples, like the global category sending G to the category of *genuine G-spectra*, only admit such adjoints for *injective* homomorphisms.

In this article, we will therefore introduce and study a weaker notion of presentability for global categories called *equivariant presentability*, which emphasizes the role of the subgroup inclusions among all group homomorphisms and allows one to capture these additional examples. As our main results, we will show that the universal examples of equivariantly presentable global categories in the unstable, semiadditive, and stable contexts are given by equivariant homotopy theory:

Theorem A (Universal property of equivariant spaces, Theorem 5.3)**.** The global category $\underline{\mathscr{S}}$ which associates to a finite group G the category $\mathscr{S}_G$ of G-spaces is the free equivariantly presentable global category on one generator: for every equivariantly presentable global category $\mathcal{D}$, evaluation at the 1-point space $* \in \underline{\mathscr{S}}(1)$ induces an equivalence of global categories

$$\underline{\mathrm{Fun}}^{\text{eq-cc}}_{\mathrm{Glo}}(\underline{\mathscr{S}}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

where the left hand side denotes a certain global category of 'equivariantly cocontinuous' functors.

Theorem B (Universal property of equivariant special Γ-spaces, Theorem 7.17)**.** The global category $\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_*$ which associates to each finite group G the category of special Γ-G-spaces in the sense of Shimakawa [Shi89] is the free equivariantly presentable equivariantly semiadditive global category on one generator: for every equivariantly presentable equivariantly semiadditive global category $\mathcal{D}$ evaluation at the free commutative monoid $\mathbb{P}(*)$ provides an equivalence of global categories

$$\underline{\mathrm{Fun}}^{\text{eq-cc}}_{\mathrm{Glo}}(\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_*, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Theorem C (Universal property of genuine equivariant spectra, Theorem 9.4)**.** The global category $\underline{\mathscr{S}p}$ which associates to a finite group G the category $\mathscr{S}p_G$ of genuine G-spectra is the free equivariantly presentable equivariantly stable global category on one generator: for any other such $\mathcal{D}$ evaluation at the sphere spectrum $\mathbb{S}$ defines an equivalence

$$\mathrm{Fun}^{\text{eq-cc}}_{\mathrm{Glo}}(\underline{\mathscr{S}p}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

[1]We work in the context of higher category theory throughout, and so we will refer to ∞-categories simply as 'categories.'

In this sense, the original, stronger notion of presentability from [CLL23] can be viewed as a characteristic feature of global homotopy theory, distinguishing it from classical equivariant homotopy theory, and we will accordingly use the term *global presentability* for it below.

Partial presentability in parametrized higher category theory. The above notions of equivariant presentability, semiadditivity, and stability are in fact instances of more general notions defined in the setting of *parametrized higher category theory* as introduced in [BDG+16]. Such parametrized notions usually come in various degrees of 'parametrized refinement': in particular, [CLL23] studied various levels of semiadditivity and stability that can exist in a parametrized category, encoded in the choice of a so-called *atomic orbital* subcategory of the parametrizing category T. Equivariant stability and semiadditivity of global categories correspond to the case of the wide subcategory $\mathrm{Orb} \subset \mathrm{Glo}$ of faithful functors.

To study the analogous situation for presentability of parametrized categories, we introduce *clefts* $S \subset T$ in the present article and associate to each of them a notion of presentability, interpolating between naïve, or 'fiberwise,' presentability and the full parametrized presentability considered e.g. in [MW22, Hil22, CLL23]. The aforementioned atomic orbital subcategories are examples of clefts, and equivariant presentability of global categories is again recovered from the case $\mathrm{Orb} \subset \mathrm{Glo}$.

Definition (Definition 4.3, Lemma 4.9)**.** A T-category $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}$ is said to be *S-presentable* if the following conditions are satisfied:

(1) $\mathcal{C}$ is fiberwise presentable, i.e. it factors through the non-full subcategory $\mathrm{Pr}^{\mathrm{L}} \subset \mathrm{Cat}$ of presentable categories and colimit-preserving functors.
(2) For every morphism $f\colon A \to B$ in S, the restriction $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $f_!\colon \mathcal{C}(A) \to \mathcal{C}(B)$, and these left adjoints satisfy base change for pullbacks along arbitrary maps in T (see Lemma 4.9 for a precise definition).

As one of our key technical results (Theorem 3.9), we moreover show how clefts give rise to *fractured ∞-topoi* in the sense of [Lur18, Definition 20.1.2.1], which allows us to investigate the behavior of partial presentability under changing the parametrizing category along a cleft. Using this 'change of parameter' yoga, we then establish analogues of the results from [MW21, CLL23] in the partial parametrized world by constructing the free unstable, semiadditive, and stable examples of S-presentable T-categories, and relating them both to the corresponding universal S-presentable S-categories as well as T-presentable T-categories:[2]

Theorem D (Theorem 8.11)**.** Let $S \subset T$ be a cleft, and let $P \subset T$ be an atomic orbital subcategory such that $P \subset S$. Then there exists an S-presentable P-stable T-category $\underline{\mathrm{Sp}}^P_{S \triangleright T}$ with the following universal property: for any S-presentable P-stable T-category $\mathcal{D}$, evaluation at a certain object $\mathbb{S}$ induces an equivalence

$$\underline{\mathrm{Fun}}_T^{S\text{-cc}}(\underline{\mathrm{Sp}}^P_{S \triangleright T}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Moreover, the underlying S-category of $\underline{\mathrm{Sp}}^P_{S \triangleright T}$ agrees with the free S-presentable P-stable S-category $\underline{\mathrm{Sp}}^P_S$.

[2]For brevity we only state the stable cases of these theorems here, and we refer the reader to Lemma 3.17 and Corollary 4.27 resp. Theorems 6.18 and 6.19 for the unstable and semiadditive versions.

Theorem E (Theorem 8.12)**.** Let $P \subset S \subset T$ be as above and consider the unique S-cocontinuous T-functor

$$\iota_! \colon \underline{\mathrm{Sp}}^P_{S \triangleright T} \to \underline{\mathrm{Sp}}^P_T$$

sending $\mathbb{S}$ to $\mathbb{S}$. Then $\iota_!$ is fully faithful, and its underlying S-functor sits in a sequence of S-adjoints $\iota_! \dashv \iota^* \dashv \iota_*$.

This then allows us to deduce Theorems A, B, and C from their global analogues established in [CLL23]: building on the model categorical results of [Len20], we show that the global categories $\underline{\mathscr{S}}$ of equivariant spaces, $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}$ of equivariant special Γ-spaces, and $\underline{\mathscr{S}\!p}$ of equivariant spectra likewise embed into their global counterparts, and furthermore that the images of these embeddings match up with those on the parametrized side.

Outlook. While Theorem E above (together with its unstable and semiadditive versions) explains how to obtain the S-presentable universal examples as full subcategories of their T-presentable analogues, it is sometimes also possible to go the other way round, and to actually reconstruct the universal fully presentable categories from the partially presentable ones: namely, as the third author will show in [Lin23], under somewhat more restrictive conditions on the pair $S \subset T$ the forgetful functor from T-presentable to S-presentable T-categories admits a left adjoint, which can be explicitly computed in terms of certain partially lax limits. Furthermore this left adjoint preserves the subcategories of P-stable T-categories for $P \subset S$. Specializing to the inclusion Orb $\subset$ Glo again, the main results of the present paper as well as its prequel [CLL23] then yield a description of *G-global spectra* as a partially lax limit of H-equivariant spectra over all homomorphisms $H \to G$, generalizing the result for $G = 1$ proven in [LNP22].

Organization. We begin by recalling the necessary background on parametrized higher category theory in Section 2. We then introduce the notion of a cleft in Section 3 and explain its connection to fractured ∞-topoi. We moreover show that any atomic orbital subcategory and any right class of a factorization system give rise to a cleft, in particular establishing our key example Orb $\subset$ Glo.

Section 4 explains how a cleft S of T yields a well-behaved theory of partial presentability for T-categories, and how general (co)limits behave under changing the parametrizing category along a cleft. This allows us to reinterpret and extend work of Martini and Wolf [MW21] on freely adding S-colimits, in particular identifying the free S-presentable T-category with a full subcategory of the free T-presentable T-category. In Section 5 we use this to describe the free equivariantly presentable global category as the underlying global category of a diagram of model categories of equivariant spaces, proving Theorem A.

In Section 6 we recall the notion of P-semiadditivity from [CLL23] for atomic orbital subcategories $P \subset T$. Given a cleft S with $P \subset S$, we then construct the free S-presentable P-semiadditive T-category as an extension of the corresponding S-category, and we once again exhibit it as a full subcategory of the free T-presentable P-semiadditive T-category. Combining this with results from [CLL23], we then prove Theorem B describing the free equivariantly presentable equivariantly semiadditive global category in terms of equivariant Γ-spaces in Section 7.

The final two sections are then devoted to the stable case: In Section 8 we construct the free S-presentable P-stable T-category, and relate it to the corresponding presentable S- and T-categories, proving Theorems D and E. From this we then deduce Theorem C in Section 9, giving an explicit model of the free equivariantly presentable equivariantly stable global category via equivariant stable homotopy theory.

Conventions. We work in the context of higher category theory throughout, and refer to ∞-categories as 'categories.' We fix a chain of Grothendieck universes $\mathfrak{U} \in \mathfrak{V} \in \mathfrak{W}$, and we will use the terms 'small category,' '(large) category,' and 'very large category' to refer to $\mathfrak{U}$-small, $\mathfrak{V}$-small, and $\mathfrak{W}$-small categories, respectively. A 'locally small category' will mean a $\mathfrak{V}$-small category such that all its mapping spaces have $\mathfrak{U}$-small homotopy groups.

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2. Preliminaries on parametrized higher categories

We begin by recalling the necessary background on parametrized higher category theory, as developed in [BDG$^+$16, Nar16, Sha21] and, from the perspective of categories internal to ∞-topoi, in [Mar21, MW21, MW22]. Throughout this section, let us fix a small category T.

Definition 2.1. A *T-category* is a functor $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}$ into the (very large) category of categories. If $\mathcal{C}$ and $\mathcal{D}$ are T-categories, then a *T-functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a natural transformation from $\mathcal{C}$ to $\mathcal{D}$. The category Cat_T of T-categories is defined as the functor category $\mathrm{Cat}_T \coloneqq \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat})$.

Example 2.2. Define Glo as the $(2,1)$-category of finite groups, group homomorphisms, and conjugations, i.e. a 2-morphism $h\colon f \Rightarrow f'$ in Glo between group homomorphisms $f, f'\colon G \to H$ is an element $h \in H$ such that $f'(g) = hf(g)h^{-1}$ for all $g \in G$. In particular, Glo comes with a fully faithful functor $B\colon \mathrm{Glo} \hookrightarrow \mathrm{Grpd}$ into the $(2,1)$-category of groupoids which sends a finite group G to the corresponding 1-object groupoid BG. We will use the term *global category* for a Glo-category, *global functor* for a Glo-functor, etc.

Example 2.3. For a finite group G, let $T = \mathrm{Orb}_G$ be the *orbit category of G*, the full subcategory of the 1-category of G-sets spanned by the transitive G-sets. Following [BDG$^+$16], we will refer to Orb_G-categories as *G-categories*.

Let us mention some common examples of T-categories:

Example 2.4. Every presheaf X on T gives rise to a T-category $\underline{X}\colon T^{\mathrm{op}} \to \mathrm{Cat}$ by postcomposing with the inclusion $\mathrm{Spc} \hookrightarrow \mathrm{Cat}$ of spaces into categories. In particular, every object $A \in T$ yields a T-category $\underline{A}$ via the Yoneda embedding.

Example 2.5. Every category $\mathcal{E}$ gives rise to a *T-category of T-objects* $\underline{\mathcal{E}}_T$, given by $\underline{\mathcal{E}}_T(B) = \mathrm{Fun}((T_{/B})^{\mathrm{op}}, \mathcal{E})$ where the functoriality of $T_{/B}$ is given by straightening the cocartesian target fibration $T^{[1]} \to T$.

Example 2.6. Any category $\mathcal{E}$ gives rise a *constant* T-category $\mathrm{const}_{\mathcal{E}}\colon A \mapsto \mathcal{E}$. The construction $\mathcal{E} \mapsto \mathrm{const}_{\mathcal{E}}$ is left adjoint to the *underlying category functor* $\Gamma\colon \mathrm{Cat}_T \to \mathrm{Cat}$ which sends $\mathcal{C}$ to $\Gamma(\mathcal{C}) \coloneqq \lim_{B \in T^{\mathrm{op}}} \mathcal{C}(B)$.

Convention 2.7 (cf. [CLL23, Convention 2.1.15])**.** Any T-category $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}$ admits a unique extension to a limit-preserving functor $\mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}$, which we will abusively denote by $\mathcal{C}$ again. By convention, all limits and colimits of objects in T are taken in the presheaf category $\mathrm{PSh}(T)$.

Example 2.8. Viewing $\mathcal{C}$ as a functor $\mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}$ as above, its value at the terminal presheaf 1 is given by the underlying category $\Gamma(\mathcal{C})$ of $\mathcal{C}$, in the sense of Example 2.6.

Example 2.9. The category Cat_T is cartesian closed, i.e. given T-categories $\mathcal{C}$ and $\mathcal{D}$, there is a T-category $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ of T-functors, characterized by the property that there is a natural equivalence

$$\mathrm{Hom}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \simeq \mathrm{Hom}(\mathcal{E}, \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D}))$$

for every third T-category $\mathcal{E}$. We let

$$\mathrm{Fun}_T(\mathcal{C}, \mathcal{D}) \coloneqq \Gamma(\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D}))$$

denote the underlying category of $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$. By adjunction, its objects can be identified with T-functors $\mathcal{C} \simeq \mathcal{C} \times \mathrm{const}_{[0]} \to \mathcal{D}$, while its morphisms are *natural transformations* of T-functors, i.e. functors $\mathcal{C} \times \mathrm{const}_{[1]} \to \mathcal{D}$.

To describe these functor categories more explicitly, we will use:

Lemma 2.10 (Categorical Yoneda lemma, [CLL23, Corollary 2.2.8])**.** *For every presheaf $B \in \mathrm{PSh}(T)$ and every T-category $\mathcal{C}$, there is an equivalence of categories*

$$\mathrm{Fun}_T(\underline{B}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}(B),$$

natural in both variables, determined by the fact that for $B \in T$ it is given by evaluation at the identity $\mathrm{id}_B \in \mathrm{Hom}_T(B, B) = \underline{B}(B)$. □

Combining this with the (internal) adjunction equivalence for $\underline{\mathrm{Fun}}_T$ we immediately get:

Corollary 2.11 (cf. [CLL23, Corollary 2.2.9])**.** *Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_T$ and $X \in \mathrm{PSh}(T)$. There are natural equivalences*

$$\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(X) \simeq \mathrm{Fun}_T(\mathcal{C} \times \underline{X}, \mathcal{D}) \simeq \mathrm{Fun}_T\big(\mathcal{C}, \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})\big). \qquad \square$$

In particular, we can (and will) identify objects of $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(X)$ with T-functors $\mathcal{C} \times \underline{X} \to \mathcal{D}$ or equivalently $\mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})$.

Example 2.12. As $\mathrm{PSh}(T)$ has pullbacks, the target map $\mathrm{PSh}(T)^{[1]} \to \mathrm{PSh}(T)$ is a *cartesian* fibration, so we can straighten it to a functor

$$\underline{\mathrm{Spc}}_T \coloneqq \mathrm{PSh}(T)_{/\bullet}\colon \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}.$$

Explicitly, this sends $X \in \mathrm{PSh}(T)$ to the slice $\mathrm{PSh}(T)_{/X}$ and a map $f\colon Y \to X$ to the pullback functor $f^*\colon \mathrm{PSh}(T)_{/Y} \to \mathrm{PSh}(T)_{/X}$. By [Lur09, Theorem 6.1.3.9 and Proposition 6.1.3.10], this functor preserves limits, so it defines a T-category via our convention.

As the notation suggests, this can be identified with the T-category of T-objects (Example 2.5) in Spc: [CLL23, Remark 2.1.16] constructs an equivalence between the two which is given in degree $A \in T$ by the colimit extension $\mathrm{PSh}(T_{/A}) \to \mathrm{PSh}(T)_{/A}$ of the slice of the Yoneda embedding $T \to \mathrm{PSh}(T)$ over A.

2.1. **Adjunctions.** In Cat_T there is a natural notion of (internal) *adjunctions*: a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$ if there are natural transformations $\eta\colon \mathrm{id} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow \mathrm{id}$ satisfying the triangle identities up to homotopy. We will frequently rely on the following 'pointwise criterion' for adjoints:

Proposition 2.13 (see [MW21, Proposition 3.2.8 and Corollary 3.2.10])**.** *A functor $F\colon \mathcal{C} \to \mathcal{D}$ of T-categories admits a right adjoint if and only if the following hold:*

(1) For every $A \in T$ the functor $F_A\colon \mathcal{C}(A) \to \mathcal{D}(A)$ admits a right adjoint G_A.
(2) For every $f\colon A \to B$ in T the Beck–Chevalley transformation $f^* \circ G_B \Rightarrow G_A \circ f^*$ given by the composite

$$f^*G_B \xRightarrow{\eta} G_AF_Af^*G_B \xRightarrow{\sim} G_Af^*F_BG_B \xRightarrow{\varepsilon} G_Af^*$$

is an equivalence.

Moreover, in this case the following hold:

(1′) For every $X \in \mathrm{PSh}(T)$ the functor $F_X\colon \mathcal{C}(X) \to \mathcal{D}(X)$ (cf. Convention 2.7) admits a right adjoint G_X.
(2′) For every $f\colon X \to Y$ in $\mathrm{PSh}(T)$ the Beck–Chevalley map $G_Xf^* \Rightarrow f^*G_Y$ is an equivalence.

Finally, the right adjoint G is given in degree $X \in \mathrm{PSh}(T)$ by G_X as above and the unit and counit are given pointwise by the unit and counit of $F_X \dashv G_X$. □

2.2. **Limits and colimits.** Next, we come to parametrized notions of limits and colimits. While this can be developed 'internally' using the notions of parametrized adjunctions and parametrized functor categories, we will instead take a purely 'pointwise' perspective in the spirit of the previous proposition in this paper.

Remark 2.14. Below we will for simplicity restrict ourselves to the case of colimits; the theory of limits is then formally dual.

Definition 2.15. A T-category $\mathcal{C}$ is called *fiberwise cocomplete* if $\mathcal{C}(A)$ is cocomplete for every $A \in T$ and the restriction $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ is cocontinuous for every $f\colon A \to B$. A T-functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *fiberwise cocontinuous* if $F_A\colon \mathcal{C}(A) \to \mathcal{D}(A)$ is cocontinuous for every $A \in T$.

Note that in the above situation $\mathcal{C}(X)$ is more generally cocomplete for any $X \in \mathrm{PSh}(T)$, and for any $f\colon X \to Y$ in $\mathrm{PSh}(T)$ the restriction $f^*\colon \mathcal{C}(Y) \to \mathcal{C}(X)$ is cocontinuous, see [Lur09, Corollary 5.1.2.3 and Lemma 5.4.5.5].

Definition 2.16. Let $\mathbf{U} \subset \underline{\mathrm{Spc}}_T$ be any T-subcategory. We say that a T-category $\mathcal{C}$ *admits* $\mathbf{U}$*-colimits* if the following conditions are satisfied:

(1) For every $D \in \mathrm{PSh}(T)$ and every $(f\colon C \to D) \in \mathbf{U}(D)$ the restriction $f^*\colon \mathcal{C}(D) \to \mathcal{C}(C)$ admits a left adjoint $f_!$.

(2) For any pullback

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle u}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ C & \xrightarrow[f]{} & D \end{array}$$

in $\mathrm{PSh}(T)$ such that $f \in \mathbf{U}(D)$ (and hence $g \in \mathbf{U}(B)$ as $\mathbf{U}$ is a T-subcategory), the Beck–Chevalley transformation $g_!u^* \Rightarrow t^*f_!$ is an equivalence.

If $\mathcal{D}$ is another $\mathbf{U}$-cocomplete T-category, then a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ is called $\mathbf{U}$-*cocontinuous* if for every $(f\colon C \to D) \in \mathbf{U}(D)$ the Beck–Chevalley map $f_!F_C \Rightarrow F_D f_!$ is an equivalence.

Remark 2.17. In the definition of a $\mathbf{U}$-cocomplete T-category, it suffices that the above conditions are satisfied whenever B and D are representable, see [CLL23, Remark 2.3.15], and likewise for $\mathbf{U}$-cocontinuity.

Definition 2.18. A T-category $\mathcal{C}$ is called *T-cocomplete* if it is fiberwise cocomplete (Definition 2.15) and $\underline{\mathrm{Spc}}_T$-cocomplete (Definition 2.16).

Similarly, a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ between T-cocomplete T-categories is called *T-cocontinuous* if it is fiberwise cocontinuous and $\underline{\mathrm{Spc}}_T$-cocontinuous.

Example 2.19. The T-category $\underline{\mathrm{Spc}}_T$ is T-cocomplete, see [MW21, Example 5.2.11].

Example 2.20. If $\mathcal{D}$ is $\mathbf{U}$-cocomplete for some $\mathbf{U} \subset \underline{\mathrm{Spc}}_T$, and $\mathcal{C}$ is any T-category, then $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ is again $\mathbf{U}$-cocomplete, see [CLL23, Corollary 2.3.25].

Example 2.21. Any left adjoint $F\colon \mathcal{C} \to \mathcal{D}$ of T-cocomplete T-categories is T-cocontinuous: indeed, it is clearly fiberwise cocontinuous, and the Beck–Chevalley map from Definition 2.16 is simply the total mate of the Beck–Chevalley map from Proposition 2.13. Conversely, a functor of T-cocomplete categories is a T-left adjoint if and only if it is T-cocontinuous and admits a pointwise right adjoint.

Definition 2.22. Let $\mathbf{U} \subset \underline{\mathrm{Spc}}_T$ be any T-subcategory. For any $\mathbf{U}$-cocomplete T-categories $\mathcal{C}, \mathcal{D}$ and any $X \in \mathrm{PSh}(T)$ we write $\underline{\mathrm{Fun}}_T^{\mathbf{U}\text{-cc}}(\mathcal{C}, \mathcal{D})(X) \subset \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})(X)$ for the full subcategory spanned by those functors $F\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})$ that are $\mathbf{U}$-cocontinuous.

Similarly, we define $\underline{\mathrm{Fun}}_T^{T\text{-cc}}(\mathcal{C}, \mathcal{D})(X)$ whenever $\mathcal{C}$ and $\mathcal{D}$ are T-cocomplete.

By [MW21, Remark 4.2.1 and Proposition 4.3.1] the above define T-subcategories of $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$.

Remark 2.23. The articles [MW21] and [CLL23] use an a priori different definition of $\underline{\mathrm{Fun}}_T^{\mathbf{U}\text{-cc}}$ and $\underline{\mathrm{Fun}}_T^{T\text{-cc}}$, see [CLL23, Proposition 2.3.26 and Remark 2.3.27] for the equivalence to the above.

With this terminology at hand, we can now formulate the universal property of T-spaces:

Theorem 2.24 ([MW21, Theorem 7.1.1])**.** *For any T-cocomplete $\mathcal{D}$, evaluation at the terminal object defines an equivalence of T-categories*

$$\underline{\mathrm{Fun}}_T^{T\text{-cc}}(\underline{\mathrm{Spc}}_T, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}\,. \qquad \square$$

2.3. **Presentability.** Finally, we come to the notion of *presentability* for T-categories from [MW22]:

Definition 2.25. A T-category $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}$ is called *fiberwise presentable* if it factors through the non-full subcategory $\mathrm{Pr}^{\mathrm{L}} \subset \mathrm{Cat}$ of presentable categories and left adjoint functors.

In this case the limit extension again factors through Pr^{L}, i.e. for any $X \in \mathrm{PSh}(T)$ the category $\mathcal{C}(X)$ is presentable, and for any map $f\colon X \to Y$ of presheaves the restriction $f^*\colon \mathcal{C}(Y) \to \mathcal{C}(X)$ is a left adjoint, see [Lur09, Proposition 5.5.3.13].

Definition 2.26. A T-category is called *T-presentable* if it is fiberwise presentable and T-cocomplete.

Remark 2.27. Any T-presentable category is also T-complete, see [MW22, Corollary 6.2.5].

Example 2.28. The T-category $\underline{\mathrm{Spc}}_T$ of T-spaces is T-presentable: clearly each $\mathrm{PSh}(T)_{/X}$ is presentable, each $f^*\colon \mathrm{PSh}(T)_{/Y} \to \mathrm{PSh}(T)_{/X}$ is a left adjoint by local cartesian closedness, and finally $\underline{\mathrm{Spc}}_T$ is T-cocomplete by Example 2.19.

Example 2.29. If $\mathcal{C}$ is small and $\mathcal{D}$ is T-presentable, then $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ is again T-presentable, see [MW22, Corollary 6.2.6].

Remark 2.30. Let $\mathcal{C}$ be T-presentable and $\mathcal{D}$ be locally small and T-cocomplete. Combining Example 2.21 with the usual non-parametrized Special Adjoint Functor Theorem [Lur09, Corollary 5.5.2.9(1)], we see that a T-functor $\mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it is T-cocontinuous.

3. Cleft categories

Let T be a small category and let $S \subset T$ be a (wide) subcategory. Associated to the inclusion $\iota\colon S \hookrightarrow T$ we have a natural restriction functor $\iota^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$, which admits both a left adjoint $\iota_!$ as well as a right adjoint ι_*, given by left and right Kan extension, respectively. One of the central questions of the present paper is under which conditions the adjunction $\iota^* \dashv \iota_*$ interacts nicely with parametrized concepts, and in particular with the notions of parametrized colimits for T-categories and S-categories discussed above.

To address this question, we make use of a more 'geometric' description of the adjunction $\iota^* \dashv \iota_*$. By identifying T-categories with limit-preserving functors on $\mathrm{PSh}(T)^{\mathrm{op}}$ as in Convention 2.7, we see that precomposition with any colimit-preserving functor $f\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ determines a functor $f^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$. Applying this to the left Kan extension functor $f = \iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ recovers $\iota^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$, and consequently the right adjoint $\iota_*\colon \mathrm{Cat}_S \to \mathrm{Cat}_T$ of ι^* is obtained by precomposition with $\iota^*\colon \mathrm{PSh}(T) \to \mathrm{PSh}(S)$, with the unit and counit of the adjunction $\iota^*\colon \mathrm{Cat}_T \rightleftarrows \mathrm{Cat}_S :\iota_*$ given by plugging in the unit and counit of the adjunction $\iota_!\colon \mathrm{PSh}(S) \rightleftarrows \mathrm{PSh}(T) :\iota^*$.

The above description suggests that we can understand the category theoretic behavior of the adjunction $\iota^*\colon \mathrm{Cat}_T \rightleftarrows \mathrm{Cat}_S :\iota_*$ in terms of the geometric, or topos-theoretic, behavior of the adjunction $\iota_!\colon \mathrm{PSh}(S) \rightleftarrows \mathrm{PSh}(T) :\iota^*$. As a concrete example, consider the question of whether $\iota^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ preserves cocompleteness. If $\mathcal{C} \in \mathrm{Cat}_T$ is T-cocomplete, it is easy to see that the S-category

$\iota^*\mathcal{C}$ is fiberwise cocomplete and that its restriction functors admit pointwise left adjoints, without any restrictions on ι. However, the Beck–Chevalley condition for these adjoints does not always hold: it translates to the requirement that $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ preserves pullbacks. Similarly, one can translate cocontinuity of the unit $\mathcal{C} \to \iota_*\iota^*\mathcal{C}$ into a pullback condition on the *counit* of $\iota_! \dashv \iota^*$. Upon closer inspection, it turns out that all the required conditions we will need for a well-behaved theory can be nicely summed up in Lurie's notion of a *fractured ∞-topos* [Lur18, Definition 20.1.2.1]:

Definition 3.1. Let $\mathcal{X}$ be an ∞-topos. A functor $j_!\colon \mathcal{Y} \to \mathcal{X}$ is called a *fracture subcategory* if the following conditions are satisfied:

(F0) The functor $j_!$ is a monomorphism of categories, i.e. it is faithful and the induced functor on groupoid cores is even fully faithful.
(F1) The functor $j_!\colon \mathcal{Y} \to \mathcal{X}$ preserves pullbacks.
(F2) The functor $j_!\colon \mathcal{Y} \to \mathcal{X}$ admits a right adjoint $j^*\colon \mathcal{X} \to \mathcal{Y}$ which is conservative and preserves colimits.
(F3) For every morphism $f\colon X \to Y$ in $\mathcal{Y}$, the naturality square

$$\begin{array}{ccc} j_!j^*j_!X & \xrightarrow{j_!j^*j_!f} & j_!j^*j_!Y \\ {\scriptstyle \varepsilon j_!}\downarrow & & \downarrow{\scriptstyle \varepsilon j_!} \\ j_!X & \xrightarrow[j_!f]{} & j_!Y \end{array}$$

of the counit transformation $\varepsilon\colon j_!j^* \to \mathrm{id}$ is a pullback square in $\mathcal{X}$.

An ∞-topos $\mathcal{X}$ equipped with a fracture subcategory $\mathcal{Y}$ is called a *fractured ∞-topos.*

However, these axioms are quite strong, making them somewhat hard to check directly. Accordingly, before coming to the parametrized applications of fractured ∞-topoi sketched above, we devote the present section to their construction from simpler, less geometric data. Namely, as in the introductory example we will be interested in the special case of functors $\mathrm{PSh}(S) \to \mathrm{PSh}(T)$ arising as left Kan extension along the inclusion $S \hookrightarrow T$ of a wide subcategory. It turns out that in this can case we can give a more explicit characterization in terms of the indexing categories S and T:

Definition 3.2. Let T be a small category. A wide subcategory $S \subset T$ is called a *cleft* of T if the following conditions are satisfied:

(C1) The subcategory S contains all equivalences of T and is left-cancellable, i.e. whenever f and g are composable maps in T with $g \in S$ and $gf \in S$, then $f \in S$.
(C2) For any map $f\colon A \to B$ in S and any map $g\colon B' \to B$ in T there exists a map $f'\colon X' \to B'$ in $\mathrm{PSh}(S)$ and a pullback square

$$\begin{array}{ccc} \iota_!X' & \xrightarrow{\iota_!f'} & \iota_!B' \\ \downarrow & & \downarrow{\scriptstyle g} \\ \iota_!A & \xrightarrow[\iota_!f]{} & \iota_!B \end{array}$$

in $\mathrm{PSh}(T)$, where $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ denotes left Kan extension along the inclusion $\iota\colon S \hookrightarrow T$.

(C3) If $\alpha\colon A \to B$, $\beta\colon B \to A$ are maps in T such that $\beta\alpha = \mathrm{id}_A$ and $\alpha\beta$ is a map in S, then also α belongs to S (whence so does β by left cancellability).

We call a small category T equipped with a cleft $S \subset T$ a *cleft category.*

Remark 3.3. As Axiom (C3) might look somewhat exotic, we record several more familiar properties that imply this axiom:

(C3′) Any idempotent $e\colon B \to B$ in S is the identity.
(C3″) The morphisms of S are closed under retracts in the arrow category of T.
(C3‴) The morphisms of S satisfy the *restricted* 2*-out-of-*6 *property*: given composable f, g, h in T such that hg and gf belong to S, so does f.

Indeed, to see that (C3′) implies (C3), note that the map $\alpha\beta\colon B \to B$ is an idempotent in S and thus the identity. It follows that α and β are (mutually inverse) equivalences, hence belong to S by (C1). In case of (C3″), it suffices to observe that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & A \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha\beta} & & \big\downarrow{\scriptstyle\alpha} \\ B & = & B & = & B \end{array}$$

expresses α as a retract of $\alpha\beta$. Finally, applying (C3‴) to the chain α, β, α also implies (C3).

Remark 3.4. Axiom (C2) is a relaxation of the following more familiar condition:

(C2′) Pullbacks of maps in S along maps in T exist in T and belong to S.

Wide subcategories $S \subset T$ satisfying axioms (C1), (C2′) and (C3″) are called *admissibility structures* in [Lur18, Definition 20.2.1.1]. In particular, every admissibility structure on T is also a cleft in the above sense.

Let us mention some examples of cleft categories:

Example 3.5 (Trivial clefts)**.** Every category T admits two extremal clefts: letting S consist of all maps in T constitutes the maximal cleft on T, while letting S consist of only the equivalences of T constitutes the minimal cleft on T.

Example 3.6 (Factorization systems)**.** Let (E, M) be a *factorization system* on T. We will prove in Proposition 3.33 below that the right class M is a cleft.

Example 3.7 (Atomic orbital subcategories)**.** Let $P \subset T$ be an *atomic orbital subcategory* in the sense of [CLL23, Definition 4.3.1]. We will prove in Proposition 3.36 below that $P \subset T$ is a cleft category.

Example 3.8. Recall the *global indexing category* Glo from Example 2.2. We define a wide subcategory $\mathrm{Orb} \subset \mathrm{Glo}$ spanned by the *injective* homomorphisms. Then $\mathrm{Orb} \subset \mathrm{Glo}$ is a cleft category: this follows either from Example 3.7 together with [CLL23, Example 4.3.3] or from Example 3.6 with [LNP22, Proposition 6.14].

3.1. **Clefts vs. fractures.** As promised, we will prove as the main result of this section:

Theorem 3.9. *For a wide subcategory $S \subset T$, the following are equivalent:*

(1) *The subcategory S is a cleft (Definition 3.2).*
(2) *The left Kan extension functor $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ along the inclusion $\iota\colon S \hookrightarrow T$ is a fracture subcategory (Definition 3.1).*

Remark 3.10. In the special case where $S \subset T$ defines an admissibility structure on T, cf. Remark 3.4, the implication (1) $\Rightarrow$ (2) was already proved by Lurie in [Lur18, Theorem 20.2.4.1]. In the examples we care about, and in particular for the inclusion $\mathrm{Orb} \subset \mathrm{Glo}$, the stronger Axiom (C2$'$) of an admissibility structure is not satisfied: the required pullbacks do *not* exist before passing to presheaves. The above strengthening of Lurie's result will therefore be crucial for our purposes.

The proof of Theorem 3.9 will occupy this whole subsection; it is somewhat involved and may be skipped on a first reading.

For the remainder of this subsection, we fix a cleft category $\iota\colon S \hookrightarrow T$. We start with some elementary consequences of the axioms.

Lemma 3.11. *The functor $(\iota_!)_{/A}\colon \mathrm{PSh}(S)_{/A} \to \mathrm{PSh}(T)_{/A}$ is fully faithful for any object $A \in S$.*

Proof. As recalled in Example 2.12, $(\iota_!)_{/A}$ may be identified with the functor $(\iota_{/A})_!\colon \mathrm{PSh}(S_{/A}) \to \mathrm{PSh}(T_{/A})$ given by left Kan extension along $\iota_{/A}\colon S_{/A} \to T_{/A}$. By Axiom (C1), S is left cancellable, so that $\iota_{/A}$ is fully faithful. Thus, also the Kan extension $(\iota_{/A})_!$ is fully faithful, whence so is $(\iota_!)_{/A}$. □

Lemma 3.12. *For every $g\colon A \to B$ in T, the pullback functor $g^*\colon \mathrm{PSh}(T)_{/B} \to \mathrm{PSh}(T)_{/A}$ sends the essential image of $(\iota_!)_{/B}$ to the essential image of $(\iota_!)_{/A}$.*

Proof. The functor $g^*\colon \mathrm{PSh}(T)_{/B} \to \mathrm{PSh}(T)_{/A}$ preserves colimits as $\mathrm{PSh}(T)$ is an ∞-topos. Since the essential image of the fully faithful left adjoint $(\iota_!)_{/A}$ is closed under colimits, it will be enough to show that g^* maps any element of the form $\iota_!f\colon \iota_!X \to \iota_!B$ for $f\colon X \to B$ a map in S into the essential image of $(\iota_!)_{/A}$. This is precisely Axiom (C2), finishing the proof. □

Construction 3.13. The functor $\iota^*\colon \mathrm{PSh}(T) \to \mathrm{PSh}(S)$ preserves pullbacks, so it induces a map

$$\mathrm{PSh}(T)^{[1]} \to \mathrm{PSh}(S)^{[1]} \times_{\mathrm{PSh}(S)} \mathrm{PSh}(T) \tag{1}$$

of cartesian fibrations over $\mathrm{PSh}(T)$.

We now define the S-functor $\iota^*\colon \iota^*\underline{\mathrm{Spc}}_T \to \underline{\mathrm{Spc}}_S$ as the composite

$$\iota^*\underline{\mathrm{Spc}}_T = \mathrm{PSh}(T)_{/\iota_!(\bullet)} \longrightarrow \mathrm{PSh}(S)_{/\iota^*\iota_!(\bullet)} \xrightarrow{\eta^*} \mathrm{PSh}(S)_{/\bullet} = \underline{\mathrm{Spc}}_S,$$

where the first map is obtained from the straightening of (1) by restricting along $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$, while the second map is obtained by pullback along the unit transformation $\eta\colon \mathrm{id} \Rightarrow \iota^*\iota_!$.

Lemma 3.14. *The left Kan extension functor $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ preserves pullbacks.*

Proof. For any $X \in \mathrm{PSh}(S)$, the above functor $\iota^*\colon \mathrm{PSh}(T)_{/\iota_! X} \to \mathrm{PSh}(S)_{/X}$ admits a left adjoint given by $(\iota_!)_{/X}$. The lemma then precisely amounts to saying that the Beck–Chevalley map $(\iota_!)_{/X} g^* \Rightarrow g^*(\iota_!)_{/Y}$ is an equivalence for any $g\colon X \to Y$ in $\mathrm{PSh}(S)$. By Propositions 2.13 it suffices to check this in the case that X and Y are representable, i.e. g is a map in S. Since the functors $(\iota_!)_{/X}$ and $(\iota_!)_{/Y}$ are fully faithful in this case by Lemma 3.11, the Beck–Chevalley condition is equivalent to the condition that g^* preserves their essential images, which is an instance of Lemma 3.12. □

Construction 3.15. As a consequence of the previous lemma, $\iota_!$ induces a map $\mathrm{PSh}(S)^{[1]} \to \mathrm{PSh}(T)^{[1]} \times_{\mathrm{PSh}(T)} \mathrm{PSh}(S)$ of cartesian fibrations, which we straighten to an S-functor $\iota_!\colon \underline{\mathrm{Spc}}_S \to \iota^*\underline{\mathrm{Spc}}_T$. For any presheaf X in $\mathrm{PSh}(S)$, this is given by $(\iota_!)_{/X}\colon \mathrm{PSh}(S)_{/X} \to \mathrm{PSh}(T)_{/\iota_! X}$.

Lemma 3.16. *The S-functor $\iota_!$ is left adjoint to the S-functor ι^* from Construction 3.13.*

Proof. We have already seen in the proof of Lemma 3.14 that ι^* admits a left adjoint L which agrees *pointwise* with $\iota_!$. In the same way, one shows that $\iota_!$ is indeed a left adjoint (with adjoint agreeing pointwise with ι^*). But then $L \simeq \iota_!$ because left adjoint functors out of $\underline{\mathrm{Spc}}_S$ are characterized by their value on the terminal presheaf by Theorem 2.24, so $\iota_!$ is left adjoint to ι^* as claimed. □

Lemma 3.17. *The fully faithful S-functor $\iota_!\colon \underline{\mathrm{Spc}}_S \hookrightarrow \iota^*\underline{\mathrm{Spc}}_T$ extends uniquely to a T-functor $\iota_!\colon \underline{\mathrm{Spc}}_{S\triangleright T} \hookrightarrow \underline{\mathrm{Spc}}_T$ (which is again fully faithful).*

Proof. The statement is equivalent to the claim that the essential image of the inclusion $\iota_!\colon \underline{\mathrm{Spc}}_S \hookrightarrow \iota^*\underline{\mathrm{Spc}}_T$ is in fact a T-subcategory of $\underline{\mathrm{Spc}}_T$, which is precisely the content of Lemma 3.12. □

As an upshot, Axiom (C2) holds without any representability assumptions on A, B, or B'.

Lemma 3.18. *For any presheaf $X \in \mathrm{PSh}(S)$, the unit map $\eta_X\colon X \to \iota^*\iota_! X$ is a monomorphism. Put differently, the functor $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ is faithful.*

Proof. This works in exactly the same way as for admissibility structures [Lur18, Proposition 20.2.4.5-(a)]: By Kan's pointwise formula, the presheaf $\iota^*\iota_! X$ is given in degree $A \in S$ by $\mathrm{colim}_{B \in (A/\iota)^{\mathrm{op}}} X(B)$ with $A/\iota := A/T \times_T S$, and the unit map $\eta\colon X(A) \to (\iota^*\iota_! X)(A)$ corresponds under this identification with the structure map of the term $\mathrm{id}_A \in A/\iota$. Since this term is contained in the full subcategory $A/S \subset A/\iota$ of maps in S, we may factor η as

$$X(A) \to \mathrm{colim}_{B \in (A/S)^{\mathrm{op}}} X(B) \to \mathrm{colim}_{B \in (A/\iota)^{\mathrm{op}}} X(B).$$

The first map is an equivalence (the object $\mathrm{id}_A \in A/S$ being terminal), and thus it remains to show that the second map is a monomorphism. For this, we claim that the category A/ι is a disjoint union of the full subcategory A/S and its complement (consisting of maps not in S), i.e. any object $t\colon A \to B$ in A/ι mapping to or from an object in A/S must itself be in A/S. Indeed, let $s\colon A \to B'$ be any map in S: if there is a map $s \to t$ in A/ι, then t belongs to S as the latter is a subcategory; on the other hand, if there is a map $t \to s$, then t belongs to S by left cancellability.

It follows that $\operatorname{colim}_{B\in(A/\iota)^{\mathrm{op}}} X(B)$ splits as a disjoint union $\operatorname{colim}_{B\in(A/S)^{\mathrm{op}}} X(B)\amalg Y$, finishing the proof. $\square$

Lemma 3.19. *Let $f\colon X\to Y$ be a map in $\mathrm{PSh}(S)$. Then the naturality square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \iota^*\iota_!X & \xrightarrow[\iota^*\iota_!f]{} & \iota^*\iota_!Y \end{array}$$

of the unit transformation $\eta\colon \mathrm{id}\Rightarrow\iota^*\iota_!$ is a pullback square.

Proof. Again, this is analogous to the proof for admissibility structures [Lur18, Proposition 20.2.4.5-(b)]. The proof of the previous lemma shows that after evaluating at $A\in T$ the naturality square is equivalent to a square of the form

$$\begin{array}{ccc} X(A) & \xrightarrow{f(A)} & Y(A) \\ \hookdownarrow & & \hookdownarrow \\ X(A)\amalg X' & \xrightarrow[f(A)\amalg f']{} & Y(A)\amalg Y', \end{array}$$

which is evidently a pullback. $\square$

Our next goal is to prove the following sharpening of Lemma 3.18:

Proposition 3.20. *The functor $\iota_!\colon \mathrm{PSh}(S)\to\mathrm{PSh}(T)$ is fully faithful on groupoid cores, and thus a monomorphism of categories.*

The proof of this proposition is surprisingly subtle and will require some further preparations.

Definition 3.21. Let $X,Y\in\mathrm{PSh}(S)$. We call $f\colon \iota_!X\to\iota_!Y$ in $\mathrm{PSh}(T)$ *admissible* if it lies in the image of the inclusion $\iota_!\colon \mathrm{Hom}_{\mathrm{PSh}(S)}(X,Y)\hookrightarrow\mathrm{Hom}_{\mathrm{PSh}(T)}(\iota_!X,\iota_!Y)$.

Beware that a priori this depends on the equivalence classes of X and Y in $\mathrm{PSh}(S)$, not only on the equivalence classes of their left Kan extensions in $\mathrm{PSh}(T)$, and only once we have proven Proposition 3.20 will we know that this independent of the choices of preimages.

Lemma 3.22. *Let $X,Y\in\mathrm{PSh}(S)$ and let $f\colon\iota_!X\to\iota_!Y$ be a map in $\mathrm{PSh}(T)$.*

(1) The map f is admissible if and only if its adjunct $\tilde f\colon X\to\iota^*\iota_!Y$ factors through the monomorphism $\eta\colon Y\to\iota^*\iota_!Y$.
(2) Let $(g_i)_{i\in I}\colon\coprod_{i\in I}X_i\twoheadrightarrow X$ be an effective epimorphism in $\mathrm{PSh}(S)$. Then f is admissible if and only if the composite $f\circ\iota_!(g_i)\colon\iota_!X_i\to\iota_!Y$ is admissible for every $i\in I$.
(3) Let $(h_i)_{i\in I}\colon\coprod_{i\in I}Y_i\twoheadrightarrow Y$ be an effective epimorphism in $\mathrm{PSh}(S)$. Then f is admissible if and only if for every $i\in I$ there exists a pullback diagram in $\mathrm{PSh}(T)$ of the form

$$\begin{array}{ccc} \iota_!X_i & \xrightarrow{\iota_!h_i'} & \iota_!X \\ {\scriptstyle f_i}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \iota_!Y_i & \xrightarrow[\iota_!h_i]{} & \iota_!Y \end{array}$$

such that f_i is admissible.

Proof. Part (1) is immediately clear from the definitions. Using (1), we see that part (2) is equivalent to the statement that the adjunct map $\tilde{f}\colon X \to \iota^*\iota_!Y$ factors through the unit $\eta\colon X \to \iota^*\iota_!Y$ if and only if each of the composites $\tilde{f} \circ g_i\colon X_i \to \iota^*\iota_!Y$ do, which is immediate. For part (3), the 'only if'-direction follows directly from Lemma 3.14. For the 'if'-direction, observe that the map $(h'_i)_{i\in I}\colon \coprod_{i\in I} X_i \to X$ from part (3) is an effective epimorphism in $\mathrm{PSh}(S)$: by Lemma 3.19 it is a pullback of the effective epimorphism $\iota^*\iota_!(h_j)_{j\in J}\colon \iota^*\iota_! \coprod_{j\in J} Y_j \to \iota^*\iota_!Y$. The claim thus follows from part (2), as for every $i \in I$ the composite $f \circ \iota_!(h'_i) = \iota_!h_i \circ f_i$ is admissible by assumption. □

Lemma 3.23. *Let $X, Y, Z \in \mathrm{PSh}(S)$ and let $f\colon \iota_!X \to \iota_!Y$ and $g\colon \iota_!Y \to \iota_!Z$ be maps in $\mathrm{PSh}(T)$ such that g and gf are admissible. Then also f is admissible.*

Proof. By the previous lemma, we have to show that the composite $\iota^*(f)\eta\colon X \to \iota^*\iota_!Y$ factors through $\eta\colon Y \to \iota^*\iota_!Y$. However, by Lemma 3.19 and admissibility of g the latter is pulled back from the unit $\eta\colon Z \to \iota^*\iota_!Z$ along $\iota^*(g)$. It therefore suffices to show that $\iota^*(g)\iota^*(f)\eta$ factors accordingly. However, this is immediate from admissibility of gf. □

We are now ready for the proof of Proposition 3.20:

Proof of Proposition 3.20. In light of the faithfulness of $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ from Lemma 3.18, it remains to show that $\iota_!$ is full on cores. Note that it suffices to prove that for presheaves $X, Y \in \mathrm{PSh}(S)$ any equivalence $f\colon \iota_!X \xrightarrow{\sim} \iota_!Y$ is admissible. We will prove this in two steps:

Step 1: We will first treat the special case where $X = A \in S$ is a representable presheaf. Consider the image $f_A(\mathrm{id}_A) \in (\iota_!Y)(A)$ of the identity $\mathrm{id}_A \in \iota_!(A)(A)$ under f. Because of the equivalence $(\iota_!Y)(A) \simeq \mathrm{colim}_{B\in(A/\iota)^{\mathrm{op}}} Y(B)$, we may represent $f(\mathrm{id}_A)$ by a class $[\alpha, y]$ for some morphism $\alpha\colon A \to B$ in T and some object $y \in Y(B)$. By Lemma 3.22, we have to prove that this class $[\alpha, y]$ lies in the image of the monomorphism

$$\eta_A\colon Y(A) \simeq \mathrm{colim}_{B\in(A/S)^{\mathrm{op}}} Y(B) \hookrightarrow \mathrm{colim}_{B\in(A/\iota)^{\mathrm{op}}} Y(B) = (\iota^*\iota_!Y)(A)$$

induced by the disjoint summand inclusion $A/S \hookrightarrow A/\iota$ (see the proof of Lemma 3.18). In other words, we have to show that α is a morphism in S.

Since the map $f_B\colon \mathrm{Hom}_T(B, A) = (\iota_!A)(B) \xrightarrow{\sim} (\iota_!Y)(B)$ is an equivalence, there exists a map $\beta\colon B \to A$ in T satisfying $f_B(\beta) = [\mathrm{id}_B, y]$. We thus have $f_A(\beta\alpha) = \alpha^* f_B(\beta) = [\alpha, y] = f_A(\mathrm{id})$, and since also f_A is an equivalence we deduce that $\beta\alpha = \mathrm{id}$. On the other hand, we have $[\mathrm{id}_B, y] = f_B(\beta) = \beta^* f_A(\mathrm{id}_A) = \beta^*[\alpha, y] = [\alpha\beta, y]$, and since $A/S \hookrightarrow A/\iota$ is a disjoint summand inclusion we see that $\alpha\beta$ belongs to S. It follows from Axiom (C3) that also α belongs to S, finishing Step 1.

Step 2: We will now deduce the statement for an arbitrary presheaf $X \in \mathrm{PSh}(S)$. Pick an effective epimorphism $(h_j)_{j\in J}\colon \coprod_{j\in J} Y_j \twoheadrightarrow Y$ in $\mathrm{PSh}(S)$ for representable

Y_j, and choose for each $j \in J$ a pullback

$$\begin{array}{ccc} \iota_! P_j & \xrightarrow{\iota_! h'_j} & \iota_! X \\ {\scriptstyle f_j}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \iota_! Y_j & \xrightarrow[\iota_! h_j]{} & \iota_! Y \end{array}$$

in $\mathrm{PSh}(T)$ using Axiom (C2). As f is an equivalence, so is each f_j. As Y_j is representable, it follows from Step 1 that $f_j^{-1} \colon \iota_! Y_j \xrightarrow{\sim} \iota_! P_j$ is admissible, and thus by Lemma 3.23 also f_j is admissible. It thus follows from Lemma 3.22 that also f is admissible, completing the proof of the proposition. □

Remark 3.24. Axiom (C3) is necessary for the previous proposition: every wide subcategory $\iota \colon S \hookrightarrow T$ for which the left Kan extension functor $\iota_! \colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ is a monomorphism of categories automatically satisfies (C3). To see this, consider morphisms α, β as in Axiom (C3), and define $X \in \mathrm{PSh}(S)$ to be the colimit of the diagram

$$A \xrightarrow{\alpha\beta} A \xrightarrow{\alpha\beta} \cdots .$$

Since $\iota_!$ preserves colimits, it follows that $\iota_! X$ is the colimit of the analogous diagram in $\mathrm{PSh}(T)$. But since α and β are maps in T, the maps $\alpha \colon A \to B$ exhibit B as another colimit of this diagram, and thus we get an equivalence $\iota_! X \simeq \iota_! B$ in $\mathrm{PSh}(T)$ compatible with the colimit structure maps. Assuming that $\iota_!$ is a monomorphism, it follows that $X \simeq B$ is a representable presheaf on S, and thus the map $\alpha \colon A \to B$ in T agrees up to equivalence in T with the structure maps $A \to X$, which belong to S by construction. As S contains all equivalences, this shows that also α belongs to S, finishing the argument.

Note moreover that (C3) is not implied by the remaining two axioms as the following example shows:

Example 3.25. Let R be a commutative ring. We let $T = \mathrm{Perf}(R)$ be the stable category of perfect R-chain complexes, and we let S consist of those $f \colon X \to Y$ such that $[X] = [Y] \in K_0(R)$, or equivalently (by the defining relations of K_0) such that the fiber of f vanishes in K_0.

The first description makes it clear that S is a subcategory, contains all equivalences, and even satisfies 2-out-of-3, proving (C1). On the other hand, the second description shows that S is closed under pullbacks, proving (C2′). However, (C3) does not hold: $0 \to R \to 0$ is the identity and $R \to 0 \to R$ belongs to S as $[R] = [R]$, but neither $0 \to R$ nor $R \to 0$ are contained in S as $[R] \neq 0$ in $K_0(R)$.

Definition 3.26. Following Lurie's notation and terminology for fractured ∞-topoi, we let $\mathrm{PSh}(T)^{\mathrm{corp}} \subset \mathrm{PSh}(T)$ denote the (non-full) essential image of the left Kan extension functor $\iota_! \colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$. A presheaf on T is called *corporeal* if it is an object of $\mathrm{PSh}(T)^{\mathrm{corp}}$, and a morphism between two corporeal presheaves on T is called *admissible* if it is a morphism in $\mathrm{PSh}(T)^{\mathrm{corp}}$.

Note that for two $X, Y \in \mathrm{PSh}(S)$ a map $f \colon \iota_! X \to \iota_! Y$ is admissible in the sense of Definition 3.26 if and only if it is admissible in the sense of Definition 3.21 above.

Lemma 3.27. *Let $X, Y, Z \in \mathrm{PSh}(T)$ be corporeal presheaves.*

(1) *Let* $f\colon X \to Z$ *be an admissible morphism, and let* $g\colon Y \to Z$ *be arbitrary. Then the base change* $g^*(f)\colon g^*(X) \to Y$ *of* f *along* g *is again an admissible morphism of corporeal presheaves.*
(2) *Let* $f\colon X \twoheadrightarrow Y$ *be an effective epimorphism, and let* $g\colon Y \to Z$ *be arbitrary. Assume that* f *and* gf *are admissible. Then also* g *is admissible.*
(3) *Let* $f\colon X \to Y$, $g\colon Y \to Z$ *be maps such that* g *and* gf *are admissible. Then* f *is admissible.*

Proof. The first statement is a consequence of Lemma 3.17, while the second statement follows from Lemma 3.22. Finally, the third statement follows from Lemma 3.23. ☐

We now come to the final missing ingredient of the proof of Theorem 3.9:

Proposition 3.28. *Let* $f\colon X \to Y$ *be a map in* $\mathrm{PSh}(T)^{\mathrm{corp}}$*. Then the naturality square*

$$\begin{array}{ccc} \iota_!\iota^*X & \xrightarrow{\iota_!\iota^*f} & \iota_!\iota^*Y \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback in $\mathrm{PSh}(T)$.

For the proof we will use:

Lemma 3.29. *Let* $Y \in \mathrm{PSh}(T)$ *be an arbitrary presheaf. Then the composite*

$$\mathrm{PSh}(S)_{/\iota^*Y} \xrightarrow{\iota_!} \mathrm{PSh}(T)_{/\iota_!\iota^*Y} \xrightarrow{\mathrm{PSh}(T)_{/\varepsilon}} \mathrm{PSh}(T)_{/Y} \qquad (2)$$

induces an equivalence onto the non-full subcategory $(\mathrm{PSh}(T)_{/Y})^{\mathrm{corp}}$ *whose objects are those* $X \to Y$ *where* X *is corporeal (but there is no condition on the map to* Y*) and whose morphisms are the admissible maps in* $\mathrm{PSh}(T)$.

Proof. It is clear that (2) factors through $(\mathrm{PSh}(T)_Y)^{\mathrm{corp}}$, so it only remains to show that the induced functor is essentially surjective and fully faithful. For this we observe that since $\iota_!$ and ι^* are adjoint, the map $\mathrm{Hom}(X, \iota^*Y) \to \mathrm{Hom}(\iota_!X, Y)$, $g \mapsto \varepsilon \circ \iota_!(g)$ is an equivalence for any $X \in \mathrm{PSh}(S)$. This immediately implies essential surjectivity, while for full faithfulness we observe that for objects $X, X' \in \mathrm{PSh}(S)_{/\iota^*Y}$ the induced map on mapping spaces fits in the following diagram of fiber sequences:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathrm{PSh}(S)_{/\iota^*Y}}(X, X') & \longrightarrow & \mathrm{Hom}_{\mathrm{PSh}(S)}(X, X') & \longrightarrow & \mathrm{Hom}_{\mathrm{PSh}(S)}(X, \iota^*Y) \\ \big\downarrow & & \big\downarrow{\scriptstyle\iota_!} & & \big\downarrow{\scriptstyle\simeq} \\ \mathrm{Hom}_{\mathrm{PSh}(T)_{/Y}}(\iota_!X, \iota_!X') & \longrightarrow & \mathrm{Hom}_{\mathrm{PSh}(T)}(\iota_!X, \iota_!X') & \longrightarrow & \mathrm{Hom}_{\mathrm{PSh}(T)}(\iota_!X, Y). \end{array}$$

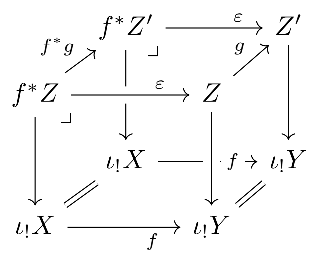
We now simply note that the middle map is a monomorphism by Lemma 3.18, with image the admissible maps. ☐

Proposition 3.30. *The* T*-functor* $\iota^*\colon \mathrm{PSh}(T)_{/\bullet} \to \mathrm{PSh}(S)_{/\iota^*(\bullet)}$ *admits an* S*-left adjoint (that is, the underlying* S*-functor admits a parametrized left adjoint) given pointwise by the composites* (2).

Proof. It is clear that the composites (2) yield a pointwise left adjoint, so it only remains to check the Beck–Chevalley condition. By the previous lemma, this amounts to saying that the adjunction $\mathrm{PSh}(T)_{/f}\colon \mathrm{PSh}(T)_{/\iota_!X} \rightleftarrows \mathrm{PSh}(T)_{/\iota_!Y} : f^*$ restricts to an adjunction $(\mathrm{PSh}(T)_{/\iota_!X})^{\mathrm{corp}} \rightleftarrows (\mathrm{PSh}(T)_{/\iota_!Y})^{\mathrm{corp}}$ for any admissible $f\colon \iota_!X \to \iota_!Y$, i.e.

(1) The right adjoint f^* restricts to $(\mathrm{PSh}(T)_{/\iota_!Y})^{\mathrm{corp}} \to (\mathrm{PSh}(T)_{/\iota_!X})^{\mathrm{corp}}$.
(2) For each $Z \in (\mathrm{PSh}(T)_{/\iota_!Y})^{\mathrm{corp}}$ the counit $\mathrm{PSh}(T)_{/f} f^*Z \to Z$ is admissible.
(3) For each $W \in (\mathrm{PSh}(T)_{/\iota_!X})^{\mathrm{corp}}$ the unit $W \to f^* \mathrm{PSh}(T)_{/f}W$ is admissible.

For this, let $g\colon Z \to Z'$ be a map in $(\mathrm{PSh}(T)_{/\iota_!Y})^{\mathrm{corp}}$ and consider the coherent cube



Lemma 3.27-(1) then shows that the objects f^*Z and f^*Z' are corporeal and that the maps $\varepsilon\colon f^*Z \to Z$ and $\varepsilon\colon f^*Z' \to Z'$ are admissible, proving the second claim and one half of the first claim. Together with Lemma 3.27-(3) we then conclude that f^*g is again admissible, proving the remaining half of of the first claim.

Finally, if $W \in (\mathrm{PSh}(T)_{/\iota_!X})^{\mathrm{corp}}$, then as a morphism in $\mathrm{PSh}(T)$ the unit $\eta\colon W \to f^* \mathrm{PSh}(T)_{/f}Y$ is right inverse to the counit ε. Thus, η is admissible by another application of Lemma 3.27-(3). ☐

Proof of Proposition 3.28. We may assume without loss of generality that f is of the form $\iota_!f'$ for some $f'\colon X' \to Y'$ in $\mathrm{PSh}(S)$. In this case, the previous proposition shows that the Beck–Chevalley transformation

$$\begin{array}{ccc} \mathrm{PSh}(S)_{/\iota^*Y} & \xrightarrow{(\iota^*f)^*} & \mathrm{PSh}(S)_{/\iota^*X} \\ {\scriptstyle \mathrm{PSh}(T)_{/\varepsilon}\circ\iota_!}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle \mathrm{PSh}(T)_{/\varepsilon}\circ\iota_!} \\ \mathrm{PSh}(T)_{/Y} & \xrightarrow[f^*]{} & \mathrm{PSh}(T)_{/X} \end{array}$$

is an equivalence. Chasing through the identity of ι^*Y precisely yields the claim. ☐

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. If $\iota\colon S \hookrightarrow T$ is a cleft category, then $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ is a fracture subcategory:

(F0) The functor $\iota_!$ is a monomorphism by Proposition 3.20.
(F1) The functor $\iota_!$ preserves pullbacks by Lemma 3.14.
(F2) The right adjoint ι^* of $\iota_!$ is clearly cocontinuous, and it is conservative as S contains all objects of T.
(F3) The pullback condition for the counit was verified in Proposition 3.28.

Conversely, assume that $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ is a fracture subcategory. Then $(\iota_!)_{/X}\colon \mathrm{PSh}(S)_{/X} \to \mathrm{PSh}(T)_{/\iota_! X}$ is fully faithful for any X by [Lur18, Proposition 20.1.3.1]; specializing to $X = A \in S$, we see that left Kan extension along $\iota_{/A}\colon S_{/A} \hookrightarrow T_{/A}$ is fully faithful, whence so is $\iota_{/A}$ itself by the Yoneda Lemma. Letting A vary, this precisely amounts to saying that S is left cancellable, proving (C1).

For (C2), consider a map $f\colon X \to Y$ in $\mathrm{PSh}(S)$ and a map $g\colon \iota_! Y' \to \iota_! Y$ in $\mathrm{PSh}(T)$. Write $\tilde{g}\colon Y' \to \iota^*\iota_! Y$ for the adjunct of g, and define X' via the following pullback square in $\mathrm{PSh}(S)$:

$$\begin{array}{ccc} X' & \longrightarrow & \iota^*\iota_! X \\ {\scriptstyle f'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \iota^*\iota_! f} \\ Y' & \xrightarrow[\tilde{g}]{} & \iota^*\iota_! Y. \end{array}$$

In the diagram

$$\begin{array}{ccccc} \iota_!(X') & \longrightarrow & \iota_!\iota^*\iota_!(X) & \xrightarrow{\varepsilon} & \iota_!(X) \\ {\scriptstyle \iota_!(f')}\big\downarrow & & \big\downarrow{\scriptstyle \iota_!\iota^*\iota_! f} & & \big\downarrow{\scriptstyle \iota_!(f)} \\ \iota_!(Y') & \xrightarrow[\iota_!(\tilde{g})]{} & \iota_!\iota^*\iota_!(Y) & \xrightarrow[\varepsilon]{} & \iota_!(Y), \end{array}$$

the left-hand square is a pullback as $\iota_!$ preserves pullbacks by (F1), while the right-hand square is a pullback square by (F3). Thus, the total square expresses $\iota_!(f')$ as a pullback of $\iota_!(f)$ along g, showing (C2).

Finally, Axiom (C3) holds because $\iota_!$ is a monomorphism, see Remark 3.24. $\square$

3.2. **Examples.** We close this section by establishing our two key examples of cleft categories. We begin with Example 3.6, for which we recall:

Definition 3.31. A *factorization system* on an category T consists of two wide subcategories $E, M \subset T$ satisfying the following conditions:

1. (1) Both E and M contain all equivalences.
2. (2) Every morphism in E is *left orthogonal* to every morphism in M in the following sense: for every pair of morphisms $e\colon A \to B$ in E and $m\colon X \to Y$ in M and every solid square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle e}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle m} \\ B & \longrightarrow & Y, \end{array}$$

there is a contractible space of dotted lifts making both triangles commute, i.e. the square

$$\begin{array}{ccc} \mathrm{Hom}_T(A, X) & \xrightarrow{m\circ -} & \mathrm{Hom}_T(A, Y) \\ {\scriptstyle -\circ e}\big\downarrow & & \big\downarrow{\scriptstyle -\circ e} \\ \mathrm{Hom}_T(B, X) & \xrightarrow[m\circ -]{} & \mathrm{Hom}_T(B, Y) \end{array}$$

is a pullback square in the category of spaces;
3. (3) Every morphism $f \in T$ admits a factorization $f = me$, with e in E and m in M.

Remark 3.32. The above definition follows [ABFJ22, Definition 3.1.6]. By Lemma 3.1.9 of *op. cit.*, the class E in a factorization system is *exactly* the class of morphisms in T which are left orthogonal to all morphisms in M, and vice-versa. In particular, this implies that both E and M are closed under retracts, so that the above is equivalent to [Lur09, Definition 5.2.8.8] (where this condition is assumed a priori).

Proposition 3.33. *Let (E, M) be a factorization system on T. Then the right class M is a cleft of T.*

If T has pullbacks, this proposition appears (with a rather different proof) as [Lur18, Proposition 20.2.2.1].

Proof. By assumption $M \subset T$ is a wide subcategory containing all equivalences, and it is left cancellable by [Lur09, Proposition 5.2.8.6-(3)], proving (C1). Moreover, Axiom (C3$''$) was noted in the previous remark.

It remains to verify (C2), i.e. that for every $f\colon A \to B$ in T the pullback functor $f^*\colon \operatorname{PSh}(T)_{/B} \to \operatorname{PSh}(T)_{/A}$ maps the image of $M_{/B}$ into $\operatorname{PSh}(M)_{/A}$. We will prove this more generally for $\operatorname{PSh}(M)_{/B}$. For this we observe that the diagram

$$\begin{array}{ccc} \operatorname{PSh}(M_{/A}) & \xrightarrow{\sim} & \operatorname{PSh}(M)_{/A} \\ {\scriptstyle (\iota_{/A})_!}\downarrow & & \downarrow{\scriptstyle (\iota_!)_{/A}} \\ \operatorname{PSh}(T_{/A}) & \xrightarrow[\sim]{} & \operatorname{PSh}(T)_{/A} \end{array} \tag{3}$$

with the horizontal equivalences as in Example 2.12 commutes up to equivalence since both paths are cocontinuous and agree on the Yoneda image. Arguing in the same way for B, it then suffices to show: *if $X \in \operatorname{PSh}(T_{/B})$ is left Kan extended from* $\operatorname{PSh}(M_{/B})$*, then its restriction to* $\operatorname{PSh}(T_{/A})$ *is left Kan extended from* $\operatorname{PSh}(M_{/A})$.

To this end, let $f\colon X \to A$ be any map in T, and fix a factorization

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array}$$

with e in E and m in M. Viewing this as a map in $T_{/A}$, [Lur09, Remark 5.2.8.3] shows that for every other other $t \in T_{/A}$ the map $e^*\colon \operatorname{Hom}(m,t) \to \operatorname{Hom}(f,t)$ is an equivalence. It follows that $\iota_{/A}\colon S_{/A} \hookrightarrow T_{/A}$ admits a left adjoint $\lambda_A\colon T_{/A} \to S_{/A}$ sending f to m with unit $f \to m$ given by the above triangle. In particular, all units live in the subcategory $T^E_{/A} \coloneqq E \times_T T_A$; conversely, an easy 2-out-of-3 argument shows that λ_A inverts all maps in $T^E_{/A}$. By abstract nonsense about Bousfield localizations, it follows that λ_A is a localization at $T^E_{/A}$, so that $X \in \operatorname{PSh}(T_{/A})$ is left Kan extended if and only if it inverts $T^E_{/A}$. Arguing in the same way for B, the proposition follows as $T_{/f}$ obviously restricts to $T^E_{/A} \to T^E_{/B}$. $\square$

Next, we recall atomic orbital subcategories from [CLL23, Definition 4.3.1]:

Definition 3.34. A wide subcategory $P \subset T$ containing all equivalences is called *atomic orbital* if the following conditions are satisfied:

(1) For every $p\colon C \to D$ in P and $t\colon B \to D$ in T there exists a pullback

$$\begin{array}{ccc} \coprod_{i=1}^n A_i & \xrightarrow{(p_i)_{i=1,\dots,n}} & B \\ \downarrow & \lrcorner & \downarrow t \\ C & \xrightarrow[p]{} & D \end{array}$$

in $\mathrm{PSh}(T)$ such that each $p_i\colon A_i \to B$ belongs to P.

(2) For every $p\colon A \to B$ in P the diagonal $A \to A \times_B A$ is a disjoint summand inclusion in $\mathrm{PSh}(T)$, i.e. it is equivalent to an inclusion of the form $A \hookrightarrow A \sqcup C$ for some $C \in \mathrm{PSh}(T)$.

Remark 3.35. By [CLL23, Lemma 4.3.2] we can equivalently replace (2) by the following axiom:

(2′) Every map in P that admits a section in T is an equivalence.

Atomic orbital subcategories were introduced in [CLL23] to encode different degrees of 'parametrized semiadditivity,' and we will revisit them from this perspective in Section 6. For now we are interested in them as examples of clefts:

Proposition 3.36. *Any atomic orbital subcategory $P \subset T$ is a cleft.*

For the proof we will use:

Lemma 3.37. *Let P be atomic orbital (say, as a subcategory of itself) and consider an object $A \in P$. Then any endomorphism in $P_{/A}$ is invertible.*

Proof. Let $B \in P_{/A}$ and fix a decomposition $B \times_A B = \coprod_{i=1}^n X_i$ into representables. We introduce the following terminology:

(A) Given any map $g\colon C \to B \times_A B$ from a representable, it factors through a unique X_i, and we call $i =: \mathrm{idx}(g)$ the *index* of g.

(B) An index $i \in \{1, \dots, n\}$ is called *good* if the projection $\mathrm{pr}_2\colon B \times_A B \to B$ to the second factor restricts to an equivalence $X_i \to B$.

Now let f be an endomorphism of B, inducing a map $(1, f)\colon B \to B \times_A B$. We claim that $\mathrm{idx}(1, f)$ is good, which will then imply the lemma as $(1, f)$ induces an equivalence onto $X_{\mathrm{idx}(1,f)}$, being a section to the map $\mathrm{pr}_1\colon X_{\mathrm{idx}(1,f)} \to B$ in P.

To prove the claim, we make the following basic observations:

(1) Given any endomorphism g of B, the index $\mathrm{idx}(g, 1)$ is good (arguing as above using that $\mathrm{pr}_2(g, 1) = 1$).

(2) Given any map $\alpha\colon X \to Y$ of representables and a map $\beta\colon Y \to B \times_A B$, we have $\mathrm{idx}(\beta\alpha) = \mathrm{idx}(\beta)$.

(3) If $\alpha, \beta\colon X \rightrightarrows B \times_A B$ are maps from a representable with $\mathrm{idx}(\alpha) = \mathrm{idx}(\beta)$ and γ is any endomorphism of $B \times_A B$, then $\mathrm{idx}(\gamma\alpha) = \mathrm{idx}(\gamma\beta)$.

By (2), we have

$$\mathrm{idx}(1, f) = \mathrm{idx}(f^k, f^{k+1})$$

for any $k \geq 0$. Now by the pigeonhole principle we find $\ell > k \geq 0$ with $\mathrm{idx}(f^k, 1) = \mathrm{idx}(f^\ell, 1)$ and hence also

$$\mathrm{idx}(f^k, f^{k+1}) = \mathrm{idx}(f^\ell, f^{k+1})$$

by (3) applied to $1 \times_A f^{k+1}$. However, by construction $\ell \geq k+1$, whence

$$\mathrm{idx}(f^\ell, f^{k+1}) = \mathrm{idx}(f^{\ell-k-1}, 1)$$

by another application of (2). Altogether we therefore get

$$\mathrm{idx}(1, f) = \mathrm{idx}(f^k, f^{k+1}) = \mathrm{idx}(f^\ell, f^{k+1}) = \mathrm{idx}(f^{\ell-k-1}, 1)$$

and the right hand side is good by (1), finishing the proof. □

Proof of Proposition 3.36. Axiom (C1) follows from [CLL23, Lemma 4.3.5], while (C2) is immediate from Definition 3.34-(1). To prove (C3′), we note that any idempotent $e\colon A \to A$ defines an endomorphism of itself considered as an object of $P_{/A}$. By the previous lemma, we conclude that e is invertible, hence homotopic to the identity. □

4. Partial presentability

Given a small category T, there is a natural notion of *T-presentability* for a T-category, recalled in Definition 2.26. This is quite a strong condition on $\mathcal{C}$: it in particular requires that the restriction functors $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admit left adjoints for *all* morphisms $f\colon A \to B$, which is unfortunately not satisfied in several naturally occurring examples, see for example Warning 9.8 about the global category of equivariant spectra.

The goal of this section is to introduce and study relaxations of the notion of presentability for a T-category $\mathcal{C}$. While we still demand that $\mathcal{C}$ be fiberwise presentable, we will weaken the cocompleteness assumption: more precisely, for any cleft $S \subset T$, we will introduce notions of *S-cocompleteness* and *S-presentability*, see Subsection 4.1. In Subsection 4.2 we discuss the relation between S-presentable T-categories and S-presentable S-categories. We end this section in Subsection 4.3 with a discussion of the *S-cocompletion* of a small T-category and the relation to the S-cocompletion of its underlying S-category.

4.1. S-(co)limits and S-presentability.

We fix a cleft category $S \subset T$ and we write $\iota\colon S \hookrightarrow T$ for the inclusion. In this subsection we study what it means for a T-category $\mathcal{C}$ to be *S-(co)complete* or *S-presentable*.

Definition 4.1. We define the T-subcategory $\mathbf{U}_S \subset \underline{\mathrm{Spc}}_T$ as the essential image of the fully faithful T-functor $\iota_!\colon \underline{\mathrm{Spc}}_{S \triangleright T} \hookrightarrow \underline{\mathrm{Spc}}_T$ from Lemma 3.17: for an object $A \in T$, the subcategory $\mathbf{U}_S(A) \subset \underline{\mathrm{Spc}}_T(A) = \mathrm{PSh}(T)_{/A}$ is the full subcategory spanned by the admissible maps.

Definition 4.2 (S-(co)completeness)**.** A T-category $\mathcal{C}$ is called *S-cocomplete* if it is fiberwise cocomplete and admits all $\mathbf{U}_S$-colimits in the sense of Definition 2.16. Dually, $\mathcal{C}$ is called *S-complete* if it is fiberwise complete and admits all $\mathbf{U}_S$-limits.

Definition 4.3 (S-presentability)**.** A T-category $\mathcal{C}$ is called *S-presentable* if it is S-cocomplete and fiberwise presentable (Definition 2.25).

Warning 4.4. As recalled in Remark 2.27 any T-presentable T-category is also T-complete. In contrast, there are interesting examples of T-categories that are S-presentable in the above sense, but *not* S-complete, see Warnings 9.6 and 9.8.

We will now provide a description of S-(co)completeness in terms of pointwise conditions. For this we first introduce:

Definition 4.5. A morphism $f\colon X \to Y$ in $\mathrm{PSh}(T)$ is called *admissible* if it defines an object in $\mathbf{U}(Y) \subset \underline{\mathrm{Spc}}_T$, i.e. for every $A \in T$ and $t\colon A \to Y$ in $\mathrm{PSh}(T)$ the pulled back map $t^*(f)\colon t^*(X) \to A$ is an admissible map of corporeal objects in the sense of Definition 3.26.

Remark 4.6. Note that for a corporeal object Y this recovers the previous definition by Lemma 3.27.

By the pasting law, the admissible maps are closed under composition, and it is clear that every equivalence is admissible; in particular, the admissible maps define a wide subcategory $\mathrm{PSh}(T)^{\mathrm{ad}} \subset \mathrm{PSh}(T)$. By another application of the pasting law, this is closed under pulling back along arbitrary maps in $\mathrm{PSh}(T)$.

Lemma 4.7. *Let $\mathcal{C}$ be a fiberwise cocomplete T-category. Then the following are equivalent:*

1. *For every $m\colon A \to B$ in S, the map $m^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $m_!$.*
2. *For every $B' \in T$ and any admissible $n\colon A' \to B'$ the functor n^* admits a left adjoint $n_!$.*

Proof. It is immediate that (2) implies (1). Conversely, let $B' \in T$ and consider an object $n\colon A' \to B'$ in $\mathbf{U}_S(B')$. Decomposing a preimage in $\mathrm{PSh}(S)_{/B}$ into representables, we get an equivalence $(k_i)_{i\in I}\colon \mathrm{colim}_{i\in I} A'_i \simeq A'$ for a functor $A'_\bullet\colon I \to T$ such that for every $i \in I$ the composite $n_i = nk_i\colon A'_i \to A' \to B'$ lies in S. Then n^* agrees up to equivalence with the functor $\mathcal{C}(B') \to \lim_{i\in I} \mathcal{C}(A'_i)$ induced by the n_i^*. Now each of these n_i^* admits a left adjoint by assumption and moreover $\mathcal{C}(B)$ is cocomplete; thus, also n^* admits a left adjoint by [HY17, Theorem B$^{\mathrm{op}}$]. □

Remark 4.8. For later use, we make the construction of the left adjoint given in *loc. cit.* semi-explicit, keeping the notation from the previous proof:

1. For $X \in \mathcal{C}(A')$, $n_!X$ is the colimit of a suitable I^{op}-diagram with $i \mapsto n_{i!}k_i^*(X)$.
2. The counit $n_!n^*X = \mathrm{colim}_{i\in I^{\mathrm{op}}} n_!k_i^*n^*X = \mathrm{colim}_{i\in I^{\mathrm{op}}} n_{i!}n_i^*X \to X$ is induced by a cocone given at $i \in I^{\mathrm{op}}$ by the counit of $n_{i!} \dashv n_i^*$.
3. The unit $Y \to n^*n_!Y$ is given after restricting along k_i by the composite $k_i^*Y \to n_i^*n_{i!}k_i^*Y \to n_i^* \mathrm{colim}_{j\in J} n_{j!}k_j^*Y$ of the unit and the structure map of the colimit.

Using this we can now prove:

Lemma 4.9. *Let $\mathcal{C}$ be a T-category. Then $\mathcal{C}$ is S-cocomplete if and only if the following conditions are satisfied:*

1. *The T-category $\mathcal{C}$ is fiberwise cocomplete,*
2. *For every morphism $m\colon A \to B$ in S, the restriction $m^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $m_!$,*

(3) For every pullback square

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ {\scriptstyle t}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ A' & \xrightarrow[n]{} & B' \end{array}$$

in $\mathrm{PSh}(T)$ *where* n *belongs to* S *and* u *is a map in* T*, the Beck–Chevalley map* $m_!t^* \to u^*n_!$ *is an equivalence (note that* $m_!$ *exists by Lemma 4.7).*

The dual characterization for S*-completeness also holds.*

Proof. By definition, S-cocompleteness implies all of the above conditions. Conversely, if these three conditions are satisfied, it only remains by the previous lemma together with Remark 2.17 to show that the Beck–Chevalley condition (3) actually holds without representability assumption on A'.

For this we fix a decomposition $(k_i)_{i\in I}\colon \operatorname{colim}_{i\in I} A'_i \simeq A'$ in $\mathrm{PSh}(T)$ into representables as before. We now pull back each individual $n_i = nk_i$ along u to an m_i, and then appeal to universality of colimits to obtain a pullback

$$\begin{array}{ccc} \operatorname{colim}_{i\in I}(A'_i \times_{B'} B) & \xrightarrow{m=(m_i)} & B \\ {\scriptstyle t=\operatorname{colim} t_i}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ \operatorname{colim}_{i\in I} A'_i & \xrightarrow[(n_i)]{} & B'. \end{array}$$

It then follows from cocontinuity of u^* and the above description of unit and counit, that the Beck–Chevalley map $m_!t^*X \to u^*n_!X$ is given for any $X \in \mathcal{C}(\operatorname{colim}_{i\in I} A_i)$ as a colimit (over I^{op}) of the Beck–Chevalley maps

$$m_{i!}t_i^*k_i^*X \to u^*n_{i!}k_i^*X,$$

each of which is an equivalence by assumption. □

Warning 4.10. Even for a fiberwise cocomplete T-category, being S-cocomplete is not just a property of the underlying S-category: the former includes more Beck–Chevalley conditions.

Lemma 4.11. *Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a* T*-functor of* S*-cocomplete* T*-categories. Then the following are equivalent:*

(1) The T*-functor* F *preserves fiberwise colimits and* $\mathbf{U}_S$*-colimits.*
(2) The T*-functor* F *preserves fiberwise colimits and for every map* m *in* S *the Beck–Chevalley map* $m_!F \to Fm_!$ *is an equivalence.*
(3) The S*-functor* ι^*F *is* S*-cocontinuous.*

The dual statement for S*-complete categories also holds.*

Proof. The equivalence between (1) and (2) follows just as in Lemma 4.9. Since the conditions in (2) only depend on the underlying S-functor ι^*F, the equivalence between (2) and (3) is clear. □

Definition 4.12. An S-functor F satisfying the above equivalent conditions is called *S-cocontinuous*. We write $\mathrm{Cat}_T^{S\text{-cc}} \subset \mathrm{Cat}_T$ for the very large category of

S-cocomplete T-categories and S-cocontinuous functors, and $\mathrm{Pr}^S_T \subset \mathrm{Cat}^{S\text{-cc}}_T$ for the full subcategory spanned by the S-presentable T-categories.

Lemma 4.13. *Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_T$ such that $\mathcal{D}$ is S-cocomplete. Then $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ is again S-cocomplete. Moreover, for any $F\colon \mathcal{C} \to \mathcal{C}'$ the restriction $\underline{\mathrm{Fun}}_T(\mathcal{C}', \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ is S-cocontinuous.*

Proof. This is a special case of [CLL23, Corollary 2.3.25]. □

Definition 4.14. We write $\underline{\mathrm{Fun}}^{S\text{-cc}}_T(\mathcal{C}, \mathcal{D}) \subset \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$ for the full subfunctor spanned in degree $X \in \mathrm{PSh}(T)$ by the S-cocontinuous functors $\mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})$.

Lemma 4.15. *$\underline{\mathrm{Fun}}^{S\text{-cc}}_T(\mathcal{C}, \mathcal{D})$ defines a T-subcategory of $\underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D})$.*

Proof. If $X \to Y$ is any map in $\mathrm{PSh}(T)$, then Lemma 4.13 shows that composing with the restriction $\underline{\mathrm{Fun}}_T(\underline{Y}, \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})$ preserves S-cocontinuous functors. To see that this subfunctor is limit preserving, it suffices to observe that the functors $\underline{\mathrm{Fun}}_T(\underline{Y}, \mathcal{D}) \to \underline{\mathrm{Fun}}_T(\underline{A}, \mathcal{D})$ for all $A \to Y$ with A representable are jointly conservative and hence detect S-cocontinuity, cf. the proof of [CLL23, Proposition 2.3.28]. □

4.2. **Colimits in Kan extensions.** Recall that for any functor $\alpha\colon S \to T$ the restriction $\alpha^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ admits a right adjoint α_*, which can be computed via restriction along $\alpha^*\colon \mathrm{PSh}(T) \to \mathrm{PSh}(S)$. We will now study the interplay of these adjoints with parametrized colimits and limits in the case that $\alpha = \iota$ is a cleft category.

Convention 4.16. For the rest of this subsection let us fix a cleft category $\iota\colon S \hookrightarrow T$ and a T-subcategory $\mathbf{V}^{(T)} \subset \mathbf{U}_S \subset \underline{\mathrm{Spc}}_T$. We will write $\mathbf{V}^{(S)}$ for the S-subcategory defined as the preimage of $\iota^*\mathbf{V}^{(T)}$ along the inclusion $\underline{\mathrm{Spc}}_S \hookrightarrow \iota^*\underline{\mathrm{Spc}}_T$.

Lemma 4.17. *Let $A \in T$. Then $\iota^*\colon \mathrm{PSh}(T)_{/A} \to \mathrm{PSh}(S)_{/\iota^* A}$ restricts to a map $\mathbf{V}^{(T)}(A) \to \mathbf{V}^{(S)}(\iota^* A)$.*

Proof. Let $(u\colon X \to A) \in \mathbf{V}^{(T)}(A)$ arbitrary. By assumption on $\mathbf{V}^{(T)}$, u is admissible, so we have a pullback

$$\begin{array}{ccc} \iota_!\iota^* X & \xrightarrow{\varepsilon} & X \\ {\scriptstyle \iota_!\iota^* u}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ \iota_!\iota^* A & \xrightarrow[\varepsilon]{} & A \end{array} \tag{4}$$

in $\mathrm{PSh}(T)$ by Proposition 3.28; in particular $\iota_!\iota^* u \in \mathbf{V}^{(T)}(\iota_!\iota^* A)$ as $\mathbf{V}^{(T)}$ is a T-subcategory of $\underline{\mathrm{Spc}}_T$. But then $\iota^* u \in \mathbf{V}^{(S)}(\iota^* A)$ as desired. □

From now on we will confuse $\mathbf{V}^{(S)}$ and $\mathbf{V}^{(T)}$ and simply write $\mathbf{V}$ for both of them.

Theorem 4.18. *The adjunction $\iota^*\colon \mathrm{Cat}_T \rightleftarrows \mathrm{Cat}_S :\iota_*$ restricts to an adjunction $\mathrm{Cat}^{\mathbf{V}\text{-cc}}_T \rightleftarrows \mathrm{Cat}^{\mathbf{V}\text{-cc}}_S$ between the categories of $\mathbf{V}$-cocomplete T- and S-categories, respectively, and $\mathbf{V}$-cocontinuous functors.*

For the proof of the theorem we will use:

Lemma 4.19 (See [CLL23, Lemma 2.3.17])**.** *Let $f\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ be a left adjoint functor that preserves pullbacks, let $\mathbf{V}' \subset \underline{\mathrm{Spc}}_S$, and let $\mathbf{V} \subset \underline{\mathrm{Spc}}_T$ such that for every $A \in S$ and every $v \in \mathbf{V}(A)$ also $f(v) \in \mathbf{V}'(f(A))$. Then $f^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ restricts to $\mathrm{Cat}_T^{\mathbf{V}\text{-cc}} \to \mathrm{Cat}_S^{\mathbf{V}'\text{-cc}}$.* □

Proof of Theorem 4.18. The functor $\iota_!\colon \mathrm{PSh}(S) \to \mathrm{PSh}(T)$ preserves pullbacks by Lemma 3.14, so the previous lemma shows that $\iota^*\colon \mathrm{Cat}_T \to \mathrm{Cat}_S$ preserves $\mathbf{V}$-cocomplete categories and $\mathbf{V}$-cocontinuous functors. In the same way, we deduce from Lemma 4.17 that ι_* restricts accordingly.

Now let $\mathcal{C}$ be a $\mathbf{V}$-cocomplete T-category. Then the unit $\mathcal{C} \to \iota_*\iota^*\mathcal{C}$ is given by restriction along the *co*unit $\varepsilon\colon \iota_!\iota^* \Rightarrow \mathrm{id}$ of the adjunction $\iota_!\colon \mathrm{PSh}(S) \rightleftarrows \mathrm{PSh}(T) :\iota^*$. Thus, if $A \in T$ and $(u\colon X \to A) \in \mathbf{V}(A)$ are arbitrary, then the Beck–Chevalley map $u_!\eta \to \eta u_!$ is given by the Beck–Chevalley map $(\iota_!\iota^*u)_!\varepsilon^* \to \eta^* u_!$ associated to the pullback (4) and hence is an equivalence by $\mathbf{V}$-cocompleteness of $\mathcal{C}$.

Similarly, if $\mathcal{D}$ is a $\mathbf{V}$-cocomplete S-category, then the counit $\varepsilon\colon \iota^*\iota_*\mathcal{D} \to \mathcal{D}$ is given by restricting along the unit of $\mathrm{PSh}(S) \rightleftarrows \mathrm{PSh}(T)$, and the Beck–Chevalley transformation $u_!\varepsilon \to \varepsilon u_!$ for $(u\colon X \to Y) \in \mathbf{V}(Y)$ is simply the Beck–Chevalley transformation for the pullback

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \iota^*\iota_!X \\ {\scriptstyle u}\downarrow & \lrcorner & \downarrow{\scriptstyle \iota^*\iota_!u} \\ Y & \xrightarrow[\eta]{} & \iota^*\iota_!Y \end{array}$$

in $\mathrm{PSh}(S)$ from Lemma 3.19, hence an equivalence as claimed. □

Corollary 4.20. *The adjunction $\iota^*\colon \mathrm{Cat}_T \rightleftarrows \mathrm{Cat}_S :\iota_*$ restricts to adjunctions $\mathrm{Cat}_T^{S\text{-cc}} \rightleftarrows \mathrm{Cat}_S^{S\text{-cc}}$ and $\mathrm{Pr}_T^S \rightleftarrows \mathrm{Pr}_S^S$.*

Proof. Clearly, ι^* and ι_* preserve fiberwise cocompleteness and cocontinuity; moreover, the unit $\mathcal{C} \to \iota_*\iota^*\mathcal{C}$ and counit $\iota^*\iota_*\mathcal{D} \to \mathcal{D}$ are simply given by restricting along suitable maps in $\mathrm{PSh}(T)$ or $\mathrm{PSh}(S)$ respectively, hence fiberwise cocontinuous.

The first claim now follows from the special case $\mathbf{V} = \mathbf{U}_S$ of the previous theorem. For the second one it then only remains to observe that ι^* and ι_* preserve fiberwise presentability by the same reasoning as for fiberwise cocompleteness. □

We close this discussion by giving an 'internal' version of the above adjunction, for which we introduce:

Construction 4.21. For any $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_T$ we get a natural map $\iota^*\colon \iota^*\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D}) \to \underline{\mathrm{Fun}}_S(\iota^*\mathcal{C}, \iota^*\mathcal{D})$ as the composite

$$\begin{aligned} \iota^*\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D}) &\xrightarrow{\eta} \underline{\mathrm{Fun}}_S(\iota^*\mathcal{C}, \iota^*\mathcal{C} \times \iota^*\underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})) \\ &\xrightarrow{\sim} \underline{\mathrm{Fun}}_S(\iota^*\mathcal{C}, \iota^*(\mathcal{C} \times \underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D}))) \\ &\xrightarrow{\iota^*\varepsilon} \underline{\mathrm{Fun}}_T(\iota^*\mathcal{C}, \iota^*\mathcal{D}), \end{aligned}$$

where the unlabelled equivalence is the canonical one. Put differently, for any fixed $\mathcal{C}$, this is the mate of the canonical natural equivalence filling the square

$$\begin{array}{ccc} \mathrm{Cat}_T & \xrightarrow{\mathcal{C}\times -} & \mathrm{Cat}_T \\ {\scriptstyle \iota^*}\downarrow & & \downarrow{\scriptstyle \iota^*} \\ \mathrm{Cat}_S & \xrightarrow[\iota^*\mathcal{C}\times -]{} & \mathrm{Cat}_S\,. \end{array} \tag{5}$$

Explicitly, this sends an object in degree $A \in T$ corresponding to $F\colon \underline{\iota(A)} \times \mathcal{C} \to \mathcal{D}$ to the composite

$$\underline{A} \times \iota^*\,\mathcal{C} \xrightarrow{\eta} \iota^*\underline{\iota(A)} \times \iota^*\,\mathcal{C} \simeq \iota^*(\underline{\iota(A)} \times \mathcal{C}) \xrightarrow{\iota^* F} \iota^* D,$$

where η now refers to the adjunction $\iota_! \dashv \iota^*$.

Passing to mates once more, we also obtain an equivalence $\Phi\colon \underline{\mathrm{Fun}}_T(\mathcal{C}, \iota_*\mathcal{D}) \simeq \iota_*\underline{\mathrm{Fun}}_S(\iota^*\mathcal{C}, \mathcal{D})$ natural in $\mathcal{C} \in \mathrm{Cat}_T$ and $\mathcal{D} \in \mathrm{Cat}_S$, given for any $A \in T$ by sending a functor $F\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{A}, \iota_*\mathcal{D})$ to the composite

$$\iota^*\mathcal{C} \xrightarrow{\iota^* F} \iota^*\underline{\mathrm{Fun}}_T(\underline{A}, \iota_*\mathcal{D}) \xrightarrow{\iota^*} \underline{\mathrm{Fun}}_T(\iota^*\underline{A}, \iota^*\iota_*\mathcal{D}) \xrightarrow{\varepsilon\circ -} \underline{\mathrm{Fun}}(\iota^*\underline{A}, \mathcal{D}).$$

The composition of the two rightmost arrows agrees with $\varepsilon \circ \iota^*\Phi$ by the triangle identity, i.e. $\Phi(F)$ is the adjunct of the composite

$$\mathcal{C} \xrightarrow{F} \underline{\mathrm{Fun}}_T(\underline{A}, \iota_*\mathcal{D}) \xrightarrow[\sim]{\Phi} \iota_*\underline{\mathrm{Fun}}_S(\iota^*\underline{A}, \mathcal{D}).$$

The equivalence Φ can accordingly be viewed as an 'internal' version of the adjunction equivalence for $\iota^*\colon \mathrm{Cat}_T \rightleftarrows \mathrm{Cat}_S :\iota_*$.

Corollary 4.22. *Let $\mathcal{C}$ be an S-cocomplete T-category and $\mathcal{D}$ an S-cocomplete S-category. Then the previous construction restricts to an equivalence*

$$\underline{\mathrm{Fun}}_T^{S\text{-cc}}(\mathcal{C}, \iota_*\,\mathcal{D}) \xrightarrow{\sim} \iota_*\underline{\mathrm{Fun}}_S^{S\text{-cc}}(\iota^*\,\mathcal{C}, \mathcal{D}).$$

Proof. It only remains to show that $F\colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(X, \iota_*\,\mathcal{D})$ is S-cocontinuous if and only if $\Phi(F)$ is so. However, by the above explicit description of $\Phi(F)$, this is precisely the statement of Corollary 4.20. □

Remark 4.23. One can deduce from the previous corollary that the functor $\iota^*\colon \mathrm{Cat}_T^{S\text{-cc}} \to \mathrm{Cat}_S^{S\text{-cc}}$ from Corollary 4.20 is symmetric monoidal with respect to the symmetric monoidal structures defined in [MW22, Section 8.2], applied to $\mathbf{U}_S$. It follows in particular that the subcategory $\mathrm{Pr}_T^S \subset \mathrm{Cat}_T^{S\text{-cc}}$ is closed under tensor products, being the preimage along ι^* of the symmetric monoidal subcategory $\mathrm{Pr}_S^{\mathrm{L}} \subset \mathrm{Cat}_S^{S\text{-cc}}$. Since we will not make use of these symmetric monoidal structures in this paper, we will leave the details to the interested reader.

4.3. ***S*-cocompletion.** As an application of the above theory we can now reinterpret and extend work of Martini and Wolf on parametrized cocompletions:

Theorem 4.24. *Let I be any small T-category. Then the unique S-cocontinuous S-functor $\iota_!\colon \underline{\mathrm{PSh}}_S(\iota^*I) = \underline{\mathrm{Fun}}_S(\iota^*I^{\mathrm{op}}, \underline{\mathrm{Spc}}_S) \to \iota^*\underline{\mathrm{Fun}}_T(I, \underline{\mathrm{Spc}}_T) = \iota^*\underline{\mathrm{PSh}}_T(I)$ compatible with the Yoneda embeddings is fully faithful. Moreover, its essential image is actually a T-subcategory, and this is the T-subcategory generated under S-colimits by the Yoneda image.*

Proof. Write $\mathcal{C}$ for the full T-subcategory of $\underline{\mathrm{PSh}}_T(I)$ generated under S-colimits by the Yoneda image. Then [MW21, Theorem 7.1.11] (for U the union of $\mathbf{U}_S$ and the constant T-categories) shows that restriction along the Yoneda embedding y defines an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_T^{S\text{-cc}}}(\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}_T}(I,\mathcal{D})$$

for any S-cocomplete $\mathcal{D}$. Specializing to $\mathcal{D}=\iota_*\mathcal{E}$ for $\mathcal{E}\in\mathrm{Cat}_S^{S\text{-cc}}$ and appealing to Corollary 4.20 we see that restricting along $\iota^*(y)$ defines an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}_S^{S\text{-cc}}}(\iota^*\mathcal{C},\mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}_S}(\iota^*I,\mathcal{E}).$$

However, the Yoneda embedding $\iota^*I \to \underline{\mathrm{PSh}}_S(\iota^*I)$ has the same property by [MW21, Theorem 7.1.1], so comparing corepresented functors shows that $\iota_!$ defines an equivalence $\underline{\mathrm{PSh}}_S(\iota^*I)\simeq\mathcal{C}$. □

Construction 4.25. We let $\iota_!\colon \underline{\mathrm{PSh}}_{S\triangleright T}(I)\to\underline{\mathrm{PSh}}_T(I)$ denote the unique extension of $\iota_!\colon \underline{\mathrm{PSh}}_S(\iota^*I)\to\iota^*\underline{\mathrm{PSh}}_T(I)$ to a T-functor obtained from Theorem 4.24. Note that for $I=1$ the terminal presheaf, this recovers the functor $\iota_!\colon \underline{\mathrm{Spc}}_{S\triangleright T}\to\underline{\mathrm{Spc}}_T$ from Lemma 3.17.

By full faithfulness of $\iota_!$, there is then a unique lift of the S-parametrized Yoneda embedding $\iota^*I\to\underline{\mathrm{PSh}}_S(I)$ to a T-functor $y\colon I\to\underline{\mathrm{PSh}}_{S\triangleright T}(I)$ together with an equivalence between $\iota_!y$ and the T-parametrized Yoneda embedding $I\to\underline{\mathrm{PSh}}_T(I)$.

Corollary 4.26. *In the above situation, $\underline{\mathrm{PSh}}_{S\triangleright T}(I)$ is S-presentable. For any S-cocomplete T-category $\mathcal{D}$, restriction along y defines an equivalence*

$$\underline{\mathrm{Fun}}_T^{S\text{-cc}}(\underline{\mathrm{PSh}}_{S\triangleright T}(I),\mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_T(I,\mathcal{D}).$$

Proof. For S-presentability, we observe that $\underline{\mathrm{PSh}}_{S\triangleright T}(I)$ is S-cocomplete as it is equivalent to a subcategory of $\underline{\mathrm{Spc}}_T$ closed under S-colimits, and that for any $A\in T$, $\underline{\mathrm{PSh}}_{S\triangleright T}(I)(A)=\mathrm{PSh}(S)_{/A}$ is clearly presentable.

The universal property is an instance of [MW21, Theorem 7.1.11] as before. □

Corollary 4.27. *The T-category $\underline{\mathrm{Spc}}_{S\triangleright T}$ is S-presentable. For any S-cocomplete T-category $\mathcal{D}$, evaluation at the terminal object defines an equivalence*

$$\underline{\mathrm{Fun}}_T^{S\text{-cc}}(\underline{\mathrm{Spc}}_{S\triangleright T},\mathcal{D}) \xrightarrow{\sim} \mathcal{D}. \qquad \square$$

In the situation of the previous corollary we actually have further right adjoints:

Proposition 4.28.
1. *The S-functor $\iota^*\colon \iota^*\underline{\mathrm{Spc}}_T\to\underline{\mathrm{Spc}}_S$ right adjoint to $\iota_!$ admits an S-right adjoint ι_*.*
2. *The adjunct $\tilde{\iota}^*\colon \underline{\mathrm{Spc}}_T\to\iota_*\underline{\mathrm{Spc}}_S$ of ι^* admits a T-right adjoint $\tilde{\iota}_*$.*

Proof. We will prove the second statement. Corollary 4.20 then implies that ι^* is S-cocontinuous, so that the first statement is an instance of the Special Adjoint Functor Theorem.

Recalling the definition (Construction 3.13), ι^* is given by the composite

$$\mathrm{PSh}(T)_{/\iota_!(\bullet)} \xrightarrow{\iota^*} \mathrm{PSh}(S)_{/\iota^*\iota_!(\bullet)} \xrightarrow{\eta^*} \mathrm{PSh}(S)_{/\bullet},$$

i.e. it is adjunct to $(\iota^*)_{/\bullet}\colon\ \mathrm{PSh}(T)_{/\bullet}\to\mathrm{PSh}(S)_{/\iota^*(\bullet)}$. The latter obviously preserves T-colimits and has a pointwise right adjoint given by the composites

$$\mathrm{PSh}(S)_{/\iota^*(\bullet)} \xrightarrow{\iota_*} \mathrm{PSh}(T)_{/\iota_*\iota^*(\bullet)} \xrightarrow{\eta^*} \mathrm{PSh}(T)_{/\bullet}. \qquad \square$$

Remark 4.29. We close this discussion by giving a different interpretation of the functor $\iota^*\colon \iota^*\underline{\mathrm{Spc}}_T \to \underline{\mathrm{Spc}}_S$. For this we recall once more the equivalences $\underline{\mathrm{Spc}}_T \simeq \mathrm{PSh}(T_{/\bullet})$, $\underline{\mathrm{Spc}}_S \simeq \mathrm{PSh}(S_{/\bullet})$ from Example 2.12; we claim that under these equivalences our functor ι^* is given by the restriction $f\colon \iota^*\,\mathrm{PSh}(T_{/\bullet}) \to \mathrm{PSh}(S_{/\bullet})$ along the S-natural transformation $\iota^* T_{/\bullet} \to S_{/\bullet}$.

While this can be carefully proven by hand, we will instead resort to a sequence of cheap tricks that avoids ever talking about coherences. Namely, by the universal property of S-spaces it suffices to show that f admits a left adjoint and that this preserves terminal objects. Indeed, f admits a pointwise left adjoint given by the left Kan extension functors $(\iota_{/A})_!\colon \mathrm{PSh}(S_{/A}) \to \mathrm{PSh}(T_{/A})$, and each of these preserves terminal objects (as they are simply represented by the respective identity maps). It remains to show that for every $f\colon A \to B$ in S the Beck–Chevalley transformation $(\iota_{/A})_! f^* \Rightarrow f^*(\iota_{/B})_!$ is an equivalence. By full faithfulness of $(\iota_{/A})_!$ and $(\iota_{/B})_!$, this is equivalent to demanding that f^* preserves the essential images, for which it is turn enough to show that there is *some* equivalence $(\iota_{/A})_! f^* \simeq f^*(\iota_{/B})_!$. This however follows simply from the equivalences (3) and the fact that ι^* has a left adjoint.

Note that this argument more generally shows that ι^* corresponds under *any pair of equivalences* to the above restriction functor $\iota^*\,\mathrm{PSh}(T_{/\bullet}) \to \mathrm{PSh}(S_{/\bullet})$.

5. The universal property of equivariant spaces

Building on the above, we will establish a universal property of equivariant unstable homotopy theory in this section. We begin by introducing the object of study:

Construction 5.1. Write **SSet** for the 1-category of simplicial sets. We define a strict 2-functor $\bullet$**-SSet**: $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_1$ into the $(2,1)$-category of 1-categories as the composite

$$\mathrm{Glo}^{\mathrm{op}} \xhookrightarrow{B} \mathrm{Grpd}^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathbf{SSet})} \mathrm{Cat}_1\,.$$

This lifts to a functor into the $(2,1)$-category RelCat of relative categories, homotopical functors, and natural isomorphisms by equipping $\boldsymbol{G}$**-SSet** $:= \mathrm{Fun}(BG, \mathbf{SSet})$ with the *G-equivariant weak equivalences*, i.e. the class of those maps f such that f^H is a weak equivalence for every subgroup $H \subset G$, or equivalently such that the geometric realization $|f|$ is a G-equivariant homotopy equivalence.

Postcomposing with the localization functor RelCat $\to$ Cat, we obtain a global category $\underline{\mathscr{S}}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}$. We call $\underline{\mathscr{S}}$ the *global category of equivariant spaces.*

Note that $\underline{\mathscr{S}}(G) =: \mathscr{S}_G$ is the usual category of G-spaces, and for any $\alpha\colon G \to G'$ the structure map $\alpha^*\colon \mathscr{S}_{G'} \to \mathscr{S}_G$ is the usual restriction functor.

Notation 5.2. Recall from Example 3.8 that Orb $\subset$ Glo is an example of a cleft category, giving rise to notions of Orb-cocompleteness and Orb-presentability. To emphasize the connections to equivariant homotopy theory obtained in this article we will refer to these as *equivariant cocompleteness* and *equivariant presentability.*

Similarly an Orb-cocontinuous functor $F\colon \mathcal{C} \to \mathcal{D}$ between equivariantly cocomplete global categories will be called *equivariantly cocontinuous*, and we will write $\underline{\mathrm{Fun}}^{\text{eq-cc}}_{\mathrm{Glo}}(\mathcal{C}, \mathcal{D})$ for the global category $\underline{\mathrm{Fun}}^{\mathrm{Orb}\text{-cc}}_{\mathrm{Glo}}(\mathcal{C}, \mathcal{D})$ of equivariantly cocontinuous functor.

We can now state the main result of this section:

Theorem 5.3. *The global category $\underline{\mathscr{S}}$ is equivariantly presentable. Moreover, it is the free equivariantly cocomplete global category in the following sense: for any equivariantly cocomplete global category $\mathcal{D}$, evaluating at the 1-point space provides an equivalence*

$$\underline{\mathrm{Fun}}^{\text{eq-cc}}_{\mathrm{Glo}}(\underline{\mathscr{S}}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Remark 5.4. While we will not prove this here, we remark that $\underline{\mathscr{S}}$ is in fact even globally presentable: this is a rather straightforward model categorical computation using that the left adjoints $\alpha_!\colon \boldsymbol{H}\textbf{-SSet} \to \boldsymbol{G}\textbf{-SSet}$ of the restrictions are again homotopical and that the Beck–Chevalley conditions hold on the pointset level by smooth and proper base change.

However, this 'extra presentability' should be considered as an anomaly for two reasons: firstly, it is something rather specific to $\mathrm{Orb} \subset \mathrm{Glo}$, and does not hold for general cleft categories $S \subset T$; secondly, it breaks down as soon as we pass to the semiadditive and stable world, cf. Warning 9.8.

In view of Corollary 4.27, the second half of the theorem can be reformulated as follows:

Theorem 5.5. *The essentially unique equivariantly cocontinuous global functor $\underline{\mathrm{Spc}}_{\mathrm{Orb} \triangleright \mathrm{Glo}} \to \underline{\mathscr{S}}$ preserving the terminal object is an equivalence.*

In fact, our proof of these two theorems below will proceed the other way round by first establishing an equivalence $\underline{\mathrm{Spc}}_{\mathrm{Orb} \triangleright \mathrm{Glo}} \simeq \underline{\mathscr{S}}$ and then deducing all the remaining statements from this.

5.1. $\boldsymbol{G}$-global spaces.

To do so, we begin by recalling the *global category of global spaces*:

Construction 5.6. We write I for the category of finite sets and injections and $\mathcal{I}$ for the simplicial category obtained by applying the right adjoint $E\colon \mathbf{Set} \to \mathbf{SSet}$ of the evaluation functor $\mathrm{ev}_0\colon \mathbf{SSet} \to \mathbf{Set}$ to all hom sets. We write $\boldsymbol{\mathcal{I}}\textbf{-SSet}$ for the category of enriched functors $\mathcal{I} \to \mathbf{SSet}$, and for any G we denote the category of G-objects in $\boldsymbol{\mathcal{I}}\textbf{-SSet}$ by $\boldsymbol{G}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet}$. Analogously to Construction 5.1 these assemble into a functor $\bullet\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_1$.

We can evaluate a G-$\mathcal{I}$-simplicial set X at any (not necessarily finite) set A via

$$X(A) \coloneqq \operatorname*{colim}_{B \subset A \text{ finite}} X(B),$$

and this acquires an action of the symmetric group Σ_A via permuting the factors. In particular, if A is a G-set, then we can equip $X(A)$ with the diagonal G-action, yielding a functor $\mathrm{ev}_A\colon \boldsymbol{G}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet} \to \boldsymbol{G}\textbf{-SSet}$.

We now call a map $f\colon X \to Y$ of G-$\mathcal{I}$-simplicial sets a *G-equivariant weak equivalence* if $f(\mathcal{U})$ is a G-equivariant weak equivalence in $\boldsymbol{G}\textbf{-SSet}$ for some, hence any complete G-set universe $\mathcal{U}$ (i.e. a countable G-set into which any other countable G-set embeds equivariantly). Finally, we call f a *G-global weak equivalence* if $\varphi^* f$ is an H-equivariant weak equivalence of H-$\mathcal{I}$-simplicial sets for any homomorphism $\varphi\colon H \to G$ from a finite group to G.

Clearly, for any $\alpha\colon G \to G'$ the restriction functor $\alpha^*\colon \boldsymbol{G'}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet} \to \boldsymbol{G}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet}$, sends G'-global weak equivalences to G-global weak equivalences, lifting $\bullet\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet}$ to $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{RelCat}$. Localizing, we then again get a global category, which we denote by $\underline{\mathscr{S}}^{\mathrm{gl}}$.

We write $\mathscr{S}^{\mathrm{gl}}_G := \underline{\mathscr{S}}^{\mathrm{gl}}(G)$ and call it the *category of G-global spaces.* Note that [CLL23] uses the notation '$\underline{\mathscr{S}}^{\mathrm{gl}}_{\mathcal{I}}$' for the above global category and reserves $\underline{\mathscr{S}}^{\mathrm{gl}}$ for a different, but equivalent, model based on actions of a certain 'universal finite group.' In the present paper, however, we will only be interested in the above approach.

Remark 5.7. The G-global weak equivalences are part of several model structures on $\boldsymbol{G}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet}$, see [Len20, Section 1.4]. We will not need them explicitly in this section, but they will make an indirect appearance in Section 7.

Our interest in $\underline{\mathscr{S}}^{\mathrm{gl}}$ comes from the following 'global' version of Theorem 5.5:

Theorem 5.8 (See [CLL23, Theorem 3.3.1 and Corollary 3.2.5])**.** *The global category $\underline{\mathscr{S}}^{\mathrm{gl}}$ is globally presentable. The unique globally cocontinuous functor $\underline{\mathrm{Spc}}_{\mathrm{Glo}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ preserving the terminal object is an equivalence.* □

On the other hand we can relate the global categories of global and equivariant spaces as follows:

Lemma 5.9. *There exists a global functor* $\mathrm{const}\colon \underline{\mathscr{S}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$ *with the following properties:*

(1) const *is fully faithful and sends the terminal object of $\mathscr{S}$ to the terminal object of $\mathscr{S}^{\mathrm{gl}}$,*
(2) it admits an Orb*-right adjoint* $\mathbf{R}\,\mathrm{ev}\colon \underline{\mathscr{S}}^{\mathrm{gl}}|_{\mathrm{Orb}} \to \underline{\mathscr{S}}|_{\mathrm{Orb}}$.

Once the above two theorems have been established, we will see that this adjunction is actually uniquely described by the requirement that the left adjoint preserve the terminal object.

Proof. The functor $\mathrm{const}\colon \boldsymbol{G}\textbf{-SSet} \to \boldsymbol{G}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet}$ is homotopical and strictly natural in G, so it induces a global functor $\mathrm{const}\colon \underline{\mathscr{S}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$. By [Len20, Corollary 1.4.56] this functor is fully faithful, and it admits a pointwise right adjoint (given by the right derived functor of $\mathrm{ev}_\varnothing\colon \boldsymbol{G}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet} \to \boldsymbol{G}\textbf{-SSet}$).

To complete the proof, it only remains to establish the Beck–Chevalley condition for the pointwise right adjoint, or equivalently that for any injective $\alpha\colon G \to G'$ the mate transform $\alpha_! \circ \mathrm{const} \Rightarrow \mathrm{const} \circ \alpha_!$ is an equivalence of functors $\mathscr{S}_G \to \mathscr{S}^{\mathrm{gl}}_{G'}$. But indeed, this holds on the pointset level by direct inspection, so the claim follows as $\alpha_!\colon \boldsymbol{G}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet} \to \boldsymbol{G'}\textbf{-}\boldsymbol{\mathcal{I}}\textbf{-SSet}$ is homotopical by [Len20, Lemma 1.4.42] while $\alpha_!\colon \boldsymbol{G}\textbf{-SSet} \to \boldsymbol{G'}\textbf{-SSet}$ is so by (a well-known special case of) Proposition 1.1.18 of *op. cit.* □

Proof of Theorems 5.3 and 5.5. Lemma 5.9 provides a fully faithful global functor $\mathrm{const}\colon \underline{\mathscr{S}} \to \underline{\mathscr{S}}^{\mathrm{gl}}$. Now the right hand side is globally cocomplete, hence in particular equivariantly cocomplete. Moreover, the essential image of the functor const is closed under all equivariant colimits as it is an Orb-left adjoint. Thus, also $\underline{\mathscr{S}}$ is equivariantly cocomplete.

Appealing to Corollary 4.27, we therefore see that there is an essentially unique equivariantly cocontinuous functor $F\colon \underline{\mathrm{Spc}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \to \underline{\mathscr{S}}$ preserving the terminal object, and we claim that this is an equivalence. For this, we consider the diagram

$$\begin{array}{ccc} \underline{\mathrm{Spc}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} & \xrightarrow{F} & \underline{\mathscr{S}} \\ {\scriptstyle\iota_!}\downarrow & & \downarrow{\scriptstyle\mathrm{const}} \\ \underline{\mathrm{Spc}}_{\mathrm{Glo}} & \xrightarrow[\sim]{} & \underline{\mathscr{S}}^{\mathrm{gl}} \end{array}$$

of global functors where the lower equivalence is as in Theorem 5.8; both paths through this diagram are equivariantly cocontinuous and preserve the terminal object, so this commutes by the universal property of $\underline{\mathrm{Spc}}_{\mathrm{Orb}\triangleright\mathrm{Glo}}$. Moreover, the vertical arrows are fully faithful by Theorem 4.24 and Lemma 5.9, respectively. It follows that also F is fully faithful.

To see that each $F_G\colon \underline{\mathrm{Spc}}_{\mathrm{Orb}}(G) \to \mathscr{S}_G$ is essentially surjective, we observe that F_G is a fully faithful left adjoint, so that its essential image is closed under all colimits. On the other hand, by Elmendorf's Theorem [Elm83] (or simply looking at the standard generating cofibrations), $\mathscr{S}_G$ is generated under colimits by the G/H's for subgroups $H \subset G$, so it is enough that each G/H is contained in the essential image. However, $G/H = i_!(*)$, where $i\colon H \hookrightarrow G$ denotes the inclusion, so $F_G(i_!(*)) \simeq i_!F_H(*) \simeq i_!(*) \simeq G/H$ by the defining properties of F.

Finally, the universal property of $\underline{\mathscr{S}}$ follows from combining the above with Corollary 4.27. □

5.2. **The universal property of G-spaces.** Fix a finite group G. Using the previous theorem, we can now give a model categorical description of the universal G-presentable G-category (Example 2.3).

Lemma 5.10. *The assignment* $\mathrm{Orb}_{/G} \to \mathrm{Orb}_G$ *sending an object* $\varphi\colon H \to G$ *to* $G/\mathrm{im}(\varphi)$ *and a morphism*

$$\begin{array}{ccc} H & \longrightarrow & K \\ & {\scriptstyle\varphi}\searrow \;\overset{\Rightarrow}{\scriptstyle g}\; \swarrow{\scriptstyle\psi} & \\ & G & \end{array} \tag{6}$$

to the map $G/\mathrm{im}(\varphi) \to G/\mathrm{im}(\psi)$ *given by right multiplication with* g*, is well-defined and an equivalence of categories.*

Proof. One easily checks that this is well-defined and an essentially surjective functor. To see that it is fully faithful, we may for ease of notation restrict to the essentially wide subcategory of Orb_G spanned by the honest inclusions $H \hookrightarrow G$. We now observe that the map $H \to K$ in a morphism (6) is necessarily given by $h \mapsto ghg^{-1}$; conversely, $g \in G$ defines a map $(H \hookrightarrow G) \to (K \hookrightarrow G)$ if and only if $ghg^{-1} \in K$ for every $h \in H$, i.e. $[g] \in (G/K)^H$. On the other hand, 2-cells $g \Rightarrow g'$ are in bijection with elements $k \in K$ such that $gk{=}g'$. Altogether, we see that $\mathrm{Hom}(H \hookrightarrow G, K \hookrightarrow G)$ is discrete and equivalent to $(G/K)^H$ by sending (6) to the class of g.

The claim then follows by observing that also $\mathrm{Hom}(G/H, G/K) \cong (G/K)^H$ via evaluation at the coset of the identity. □

Construction 5.11. We write υ_G for the composite $\mathrm{Orb}_G \simeq \mathrm{Orb}_{/G} \xrightarrow{\pi_G} \mathrm{Orb} \hookrightarrow \mathrm{Glo}$. Restricting along υ_G then yields a functor $\upsilon_G^*\colon \mathrm{Cat}_{\mathrm{Glo}} \to \mathrm{Cat}_{\mathrm{Orb}_G}$ sending a global category to its 'underlying G-category.'

Theorem 5.12. *(1) The G-category $\upsilon_G^*\underline{\mathscr{S}}$ is G-presentable, and the unique left adjoint $\underline{\mathrm{Spc}}_{\mathrm{Orb}_G} \to \upsilon_G^*\underline{\mathscr{S}}$ preserving the terminal object is an equivalence.*
(2) For any G-cocomplete $\mathcal{D}$, evaluation at the terminal object defines an equivalence $\underline{\mathrm{Fun}}^{G\text{-cc}}_{\mathrm{Orb}_G}(\upsilon_G^*\underline{\mathscr{S}}, \mathcal{D}) \simeq \mathcal{D}$.

More informally, $\upsilon_G^*\underline{\mathscr{S}}$ is given by sending G/H to the category $\mathscr{S}_H$ of H-spaces, $G/H \twoheadrightarrow G/K$ to the restriction $\mathscr{S}_K \to \mathscr{S}_H$ for any $K \supset H$, and $-\cdot g\colon G/H \to G/H$ for an element g of the normalizer $N_G H$ to the conjugation $c_g^*\colon \mathscr{S}_H \to \mathscr{S}_H$.

Proof. It suffices to construct an equivalence $\upsilon_G^*\underline{\mathscr{S}} \simeq \underline{\mathrm{Spc}}_{\mathrm{Orb}_G}$; the theorem will then follow from the universal property of the right hand side (Theorem 2.24).

But by Theorem 5.5 we have an equivalence of global categories $\underline{\mathscr{S}} \simeq \underline{\mathrm{Spc}}_{\mathrm{Orb}\triangleright\mathrm{Glo}}$, and hence in particular an equivalence $\underline{\mathscr{S}}|_{\mathrm{Orb}} \simeq \underline{\mathrm{Spc}}_{\mathrm{Orb}}$ of Orb-categories. To finish the proof it suffices now to observe that for any small T and any $A \in T$, there is an equivalence $\pi_A^*\underline{\mathrm{Spc}}_T \simeq \underline{\mathrm{Spc}}_{T/A}$ by [MW21, Lemma 7.1.9]. □

Remark 5.13. Evaluating at $H \subset G$, the above in particular shows $\mathscr{S}_H \simeq \mathrm{PSh}(\mathrm{Orb}_G)_{/(G/H)} \simeq \mathrm{PSh}(\mathrm{Orb}_H)$. In this sense, the theorems above can be viewed as a 'coherent' version of the classical Elmendorf Theorem [Elm83], additionally taking into account the restriction functors as well as all higher structure between them.

6. The semiadditive story

We continue to fix a cleft category $\iota\colon S \hookrightarrow T$. In this section we will give a description of the universal S-presentable T-category that is in addition *semiadditive* in a suitable sense.

6.1. *P*-semiadditivity and *P*-commutative monoids.

We begin with a recollection of the relevant material from [CLL23]. Throughout, we fix an *atomic orbital subcategory* $P \subset T$ in the sense of Definition 3.34.

Construction 6.1. We write $\mathbb{F}_T \subset \mathrm{PSh}(T)$ for the finite coproduct completion of T and $\mathbb{F}_T^P$ for the finite coproduct completion of P, viewed as a subcategory of $\mathbb{F}_T$.

We define a T-subcategory $\underline{\mathbb{F}}_T^P \subset \underline{\mathrm{Spc}}_T$ by letting $\underline{\mathbb{F}}_T^P(B)$ be the full subcategory of $\mathrm{PSh}(T)_{/B}$ spanned by objects of the form $(p_i)\colon \coprod_{i=1}^n A_i \to B$ such that each morphism $p_i\colon A_i \to B$ is in P; put differently, this is the slice $(\mathbb{F}_T^P)_{/B}$. Note that by atomic orbitality of P, $\underline{\mathbb{F}}_T^P$ indeed forms a T-subcategory of $\underline{\mathrm{Spc}}_T$.

Definition 6.2. We say a T-category has *finite P-products* or *finite P-coproducts* if it has $\underline{\mathbb{F}}_T^P$-limits or $\underline{\mathbb{F}}_T^P$-colimits, respectively, in the sense of Definition 2.16.

Definition 6.3. A T-category $\mathcal{C}$ is called *pointed* if it factors through the non-full subcategory $\mathrm{Cat}_* \subset \mathrm{Cat}$ of categories with zero objects and functors preserving the zero object.

Construction 6.4. Let $\mathcal{C}$ be a pointed T-category which has finite P-coproducts, and let $\mathcal{D}$ be a T-category with finite P-products. For any functor $F\colon \mathcal{C} \to \mathcal{D}$ and any $p\colon A \to B$ in $\mathbb{F}_T^P$, [CLL23, Construction 4.6.1] defines a *relative norm map*

$$\mathrm{Nm}_p^F\colon F_B \circ p_! \implies p_* \circ F_A.$$

If $\mathcal{C}$ also has finite P-products, we write $\mathrm{Nm}_p\colon p_! \Rightarrow p_*$ for the relative norm map of $\mathrm{id}_{\mathcal{C}}$, and simply call it the *norm map*.

Definition 6.5. A T-category $\mathcal{C}$ is called *P-semiadditive* if it is pointed, has finite P-products and P-coproducts, and they agree in the sense that for every p in $\mathbb{F}_T^P$ the norm map $\mathrm{Nm}_p\colon p_! \Rightarrow p_*$ is an equivalence.

Example 6.6. When $P \subset T$ equals $\mathrm{Orb} \subset \mathrm{Glo}$, the previous definition specializes to the notion of *equivariant semiadditivity* from [CLL23].

Example 6.7. When $P \subset T$ equals $\mathrm{Orb}_G \subset \mathrm{Orb}_G$, the notion of semiadditivity obtained agrees with G-semiadditivity as defined in [Nar16], see [CLL23, Proposition 4.6.4].

Definition 6.8. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of T-categories, such that $\mathcal{C}$ is pointed and has finite P-coproducts, while $\mathcal{D}$ has finite P-products. Then F is called *P-semiadditive* if it sends P-coproducts to P-products in the sense that the relative norm map $\mathrm{Nm}_p^F\colon Fp_! \Rightarrow p_*F$ is an equivalence for every p in $\mathbb{F}_T^P$.

Definition 6.9. We write $\mathrm{Cat}_T^{P\text{-}\oplus} \subset \mathrm{Cat}_T$ for the non-full subcategory of P-semiadditive categories and P-semiadditive T-functors.

By [CLL23, Proposition 4.6.14], the morphisms of $\mathrm{Cat}_T^{P\text{-}\oplus}$ are equivalently the $\underline{\mathbb{F}}_T^P$-cocontinuous or $\underline{\mathbb{F}}_T^P$-continuous T-functors.

Remark 6.10. Similarly to Warning 4.10, having finite P-(co)products is not just a property of the underlying P-category. On the other hand, if a T-category either has finite P-coproducts or finite P-products, then it is P-semiadditive if and only if its underlying P-category is so [CLL23, Lemma 4.5.2 and Lemma 4.6.4].

Definition 6.11. In the above situation, we write $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \mathcal{D})$ for the parametrized subcategory spanned in degree $X \in \mathrm{PSh}(T)$ by the P-semiadditive functors $\mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{X}, \mathcal{D})$.

Note that the above is indeed a T-subcategory by [CLL23, Corollary 4.6.10].

Definition 6.12. We define $\underline{\mathbb{F}}_{T,*}^P$, the T-category of *finite pointed P-sets*, to be the essential image of $\underline{\mathbb{F}}_T^P$ under the functor $(-)_+\colon \underline{\mathrm{Spc}}_T \to \underline{\mathrm{Spc}}_{T,*}$ which adds a disjoint basepoint.

Definition 6.13. Given a T-category $\mathcal{C}$ with P-products we define $\underline{\mathrm{CMon}}^P(\mathcal{C})$, the *$T$-category of P-commutative monoids in $\mathcal{C}$*, as $\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\underline{\mathbb{F}}_{T,*}^P, \mathcal{C})$. If $\mathcal{C} = \underline{\mathrm{Spc}}_T$, we write $\underline{\mathrm{CMon}}_T^P \coloneqq \underline{\mathrm{CMon}}_P(\underline{\mathrm{Spc}}_T)$.

This construction enjoys several universal properties. To express them we introduce:

Construction 6.14. Let $\mathcal{C}$ have finite P-products. Evaluation at the global section $S^0 \coloneqq (\mathrm{id})_+ \in \underline{\mathbb{F}}_{T,*}^P(1) \subset (\mathrm{PSh}(T)_{/1})_*$ gives a forgetful functor

$$\mathbb{U}\colon \underline{\mathrm{CMon}}_P(\mathcal{C}) \to \mathcal{C}\,.$$

Construction 6.15. Assume $\mathcal{C}$ is presentable. Then $\underline{\mathrm{CMon}}^P(\mathcal{C}) \hookrightarrow \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \mathcal{C})$ admits a left adjoint $L^{P\text{-}\oplus}$ by [CLL23, Proposition 4.6.15]. In particular, the functor $\mathbb{U}$ from the previous construction has a left adjoint given by composing the left Kan extension functor $(S^0)_! \colon \mathcal{C} \to \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \mathcal{C})$ [MW21, Corollary 6.3.7] with $L^{P\text{-}\oplus}$.

Theorem 6.16 (See [CLL23, Theorem 4.8.10])**.** *Let $\mathcal{C}$ be a T-category with finite P-products. The functor $\mathbb{U} \colon \underline{\mathrm{CMon}}^P(\mathcal{C}) \to \mathcal{C}$ exhibits $\underline{\mathrm{CMon}}^P(\mathcal{C})$ as the P-semiadditive envelope of $\mathcal{C}$ in the following sense: for every P-semiadditive T-category $\mathcal{T}$ post-composition with $\mathbb{U}$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{P\text{-}\times}(\mathcal{T}, \mathbb{U}) \colon \underline{\mathrm{Fun}}^{P\text{-}\oplus}(\mathcal{T}, \underline{\mathrm{CMon}}^P(\mathcal{C})) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{P\text{-}\times}(\mathcal{T}, \mathcal{C}).$$

Suppose now that $\mathcal{C}$ is moreover presentable. Then the left adjoint $\mathbb{P}$ of $\mathbb{U}$ exhibits $\underline{\mathrm{CMon}}^P(\mathcal{C})$ as the presentable P-semiadditive completion of $\mathcal{C}$ in the following sense: for any presentable P-semiadditive T-category $\mathcal{T}$ precomposition with $\mathbb{P}$ yields an equivalence

$$\underline{\mathrm{Fun}}^{T\text{-cc}}(\mathbb{P}, \mathcal{T}) \colon \underline{\mathrm{Fun}}^{T\text{-cc}}(\underline{\mathrm{CMon}}^P(\mathcal{C}), \mathcal{T}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{T\text{-cc}}(\mathcal{C}, \mathcal{T}). \qquad \square$$

Theorem 6.17 (See [CLL23, Theorem 4.8.11])**.** *The T-category $\underline{\mathrm{CMon}}^P_T$ is P-semiadditive and T-presentable. Moreover, it has the following universal property: for any locally small T-cocomplete P-semiadditive $\mathcal{D}$, evaluation at $\mathbb{P}(*) \simeq L^{P\text{-}\oplus} y(S^0)$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{T\text{-cc}}(\underline{\mathrm{CMon}}^P_T, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

6.2. **The free *P*-semiadditive *S*-presentable *T*-category.** Let now $P \subset S$ be atomic orbital *as a subcategory of T*. As the main results of this section, we will prove the following 'partially presentable' versions of the previous theorem:

Theorem 6.18. *There is a unique S-cocontinuous functor $\iota_! \colon \underline{\mathrm{CMon}}^P_S \to \iota^* \underline{\mathrm{CMon}}^P_T$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$. Moreover, $\iota_!$ is fully faithful, and it sits in a sequence of S-adjoints $\iota_! \dashv \iota^* \dashv \iota_*$.*

Theorem 6.19. *The S-functor $\iota_!$ uniquely extends to a T-functor $\underline{\mathrm{CMon}}^P_{S \triangleright T} \to \underline{\mathrm{CMon}}^P_T$. Moreover, $\underline{\mathrm{CMon}}^P_{S \triangleright T}$ is S-presentable, P-semiadditive, and it has the following universal property: for any S-cocomplete P-semiadditive T-category $\mathcal{D}$ evaluation at a certain global section $\mathbb{P}(*)$ defines an equivalence*

$$\underline{\mathrm{Fun}}^{S\text{-cc}}_T(\underline{\mathrm{CMon}}^P_{S \triangleright T}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Remark 6.20. Note that in contrast to Theorem 6.17 there is no local smallness condition on $\mathcal{D}$ here anymore; in particular, for $S = T$ this improves upon our result in [CLL23].

The proof of these two theorems will occupy the rest of this section.

6.2.1. *Construction of the universal example.* As our first step, we will construct *some* P-semiadditive S-cocomplete T-category $\mathcal{C}$ with the correct universal property; more precisely, we want to show:

Proposition 6.21. *Write $\mathcal{C} \subset \underline{\mathrm{CMon}}^P_T$ for the full subcategory generated under S-colimits by $\mathbb{P}(*)$. Then $\mathcal{C}$ is a P-semiadditive S-cocomplete T-category, and for any other such $\mathcal{D}$ evaluating at $\mathbb{P}(*)$ yields an equivalence $\underline{\mathrm{Fun}}^{T\text{-cc}}_T(\mathcal{C}, \mathcal{D}) \simeq \mathcal{D}$.*

The basic idea will be to deduce the universal property of $\mathcal{C}$ from the one for $\underline{\mathrm{CMon}}_T^P$. However, we only understand maps from the latter to T-*cocomplete* categories, so we will have to embed a general P-semiadditive S-cocomplete T-category into a T-cocomplete one first. However, in this process some size issues pop up; to avoid any ambiguities, we will therefore for once refer back to our chosen Grothendieck universes explcitly:

Lemma 6.22. *Let $\mathcal{D}$ be an S-cocomplete P-semiadditive $\mathfrak{V}$-small T-category. Then there exists a $\mathfrak{W}$-small, locally $\mathfrak{V}$-small P-semiadditive T-category $\mathcal{D}'$ having all $\mathfrak{V}$-small T-colimits together with a fully faithful functor $j\colon \mathcal{D} \to \mathcal{D}'$ preserving all $\mathfrak{U}$-small S-colimits.*

Proof. Write $\underline{\mathrm{SPC}}_T$ for the $\mathfrak{W}$-small and locally $\mathfrak{V}$-small T-category of $\mathfrak{V}$-small spaces. Then the Yoneda embedding $\mathcal{D}^{\mathrm{op}} \to \underline{\mathrm{Fun}}_T(\mathcal{D}, \underline{\mathrm{SPC}}_T)$ actually lands in the full subcategory $\mathcal{E} := \underline{\mathrm{Fun}}_T^{P\text{-}\times}(\mathcal{D}, \underline{\mathrm{SPC}}_T)$ by [MW21, Corollary 4.4.8]. Now $\mathcal{E}$ is closed under all $\mathfrak{V}$-small T-limits and the Yoneda embedding preserves all $\mathfrak{U}$-small S-limits by Proposition 4.4.7 of *op. cit.* Thus, the Yoneda embedding $\mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ is a fully faithful functor into a category with all $\mathfrak{V}$-small T-limits preserving $\mathfrak{U}$-small S-limits. Moreover, as $\mathcal{D}$ is P-semiadditive, so is $\mathcal{E}$ by [CLL23, Proposition 4.6.13], and hence so is $\mathcal{E}^{\mathrm{op}}$ by Lemma 4.5.4 of *op. cit.* The dual $\mathcal{D} \to \mathcal{E}^{\mathrm{op}}$ of the Yoneda embedding therefore has the required properties. □

As the lemma requires us to pass a larger universe, it is not clear a priori whether $\underline{\mathrm{CMon}}_T^P$ still has the correct universal property (we will see a posteriori that, as a matter of fact, it does). For locally small $\mathcal{D}$, one might try to avoid this issue by considering $\underline{\mathrm{Fun}}_T^{P\text{-}\times}(\mathcal{D}, \underline{\mathrm{Spc}}_T)$ instead or even by just closing up the Yoneda image under $\mathfrak{U}$-small T-limits in there, but even in this case it is not clear whether the result is still locally small—and said local smallness was crucial in the proof of Theorem 6.16 given in [CLL23], which relied on the Special Adjoint Functor Theorem. Accordingly, we will have to consider a $\mathfrak{W}$-version $\underline{\mathrm{CMON}}_T^P$ of $\underline{\mathrm{CMon}}_T^P$. The crucial technical lemma to relate these two to each other will be the following:

Lemma 6.23. *The functor $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathrm{SPC}}_T) \to \underline{\mathrm{CMON}}_T^P := \underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathrm{SPC}}_T)$ left adjoint to the inclusion preserves $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathrm{Spc}}_T)$.*

Accordingly, it restricts to a left adjoint $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathrm{Spc}}_T) \to \underline{\mathrm{CMon}}_T^P$ of the inclusion, and there is no harm in denoting both the localization functor in ordinary T-categories and in large T-categories by the same symbol $L^{P\text{-}\oplus}$.

Proof. Let $A \in T$ be arbitrary. [CLL23, Remark 2.2.14] provides for any $\mathfrak{W}$-small category $\mathcal{E}$ a natural equivalence

$$\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}_{T,*}^P, \underline{\mathcal{E}}_T)(A) \simeq \mathrm{Fun}(\textstyle\int \underline{\mathbb{F}}_{T,*}^P \times \underline{A}, \mathcal{E}) \tag{7}$$

where $\int$ denotes the usual Grothendieck construction over T^{op} and $(-)_T$ denotes the T-category of T-objects (Example 2.5). On the other hand, [CLL23, Remark 4.9.9] characterizes the essential image of $\underline{\mathrm{CMon}}^P(\underline{\mathcal{E}}_T)(A)$ under this—it consists precisely of the functors $F^\dagger\colon \int \underline{\mathbb{F}}_{T,*}^P \times \underline{A} \to \mathcal{E}$ satisfying the following:

(1) For every $f\colon B \to A$ in T the restriction of $F^\dagger$ to the non-full subcategory $\underline{\mathbb{F}}_{T,*}^P(B) \simeq \underline{\mathbb{F}}_{T,*}^P(B) \times \{f\} \subset \int \underline{\mathbb{F}}_{T,*}^P \times \underline{A}$ is semiadditive in the usual sense.

(2) For every $p\colon B \to B'$ in P and $f\colon B' \to A$ in T a certain natural *Segal map* $F^\dagger(B', B_+, f) \to F^\dagger(B, B_+, pf)$ is an equivalence; here we as usual denote objects in the Grothendieck construction by triples $(C \in T^{\mathrm{op}}, X \in \underline{\mathbb{F}}^P_{T,*}(C), g \in \mathrm{Hom}(C, A))$.

Specializing to $\mathcal{E} = \mathrm{SPC}$ and writing y for the (non-parametrized) Yoneda embedding of $\int \underline{\mathbb{F}}^P_{T,*} \times \underline{A}$, we see that F is P-semiadditive if and only if $F^\dagger$ is local with respect to the set $U^\dagger$ made up of suitable maps

(1) $\emptyset \to y(B, *, f)$ for every map $f\colon B \to A$ in T (so that the restriction to each $\underline{\mathbb{F}}^P_{T,*}(B) \times \{f\}$ is pointed)
(2) $y(B, X_+, f) \amalg y(B, Y_+, f) \to y(B, X_+ \vee Y_+, f)$ for all $f\colon B \to A$ in T and $X_+, Y_+ \in \underline{\mathbb{F}}^P_{T,*}(B)$ (so that each restriction sends coproducts to products)
(3) $y(B, B_+, pf) \to y(B', B_+, f)$ for every $p\colon A \to B$ in P and $f\colon B' \to B$ in T (ensuring that the Segal maps are equivalences).

Transporting $U^\dagger$ along the equivalence (7), we then get a set U such that F is P-semiadditive if and only if it is U-local. By direct inspection, each map in $U^\dagger$ actually lives in $\mathrm{Fun}(\int \underline{\mathbb{F}}^P_{T,*} \times \underline{A}, \mathrm{Spc})$ (as opposed to functors into SPC). By naturality of (7) we can therefore also take the set U to consist of maps in $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T)(A)$. We now write U_1 for the strongly saturated class generated by U in $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T)(A)$ (with respect to $\mathfrak{U}$-small colimits) and U_2 for the strongly saturated class generated in $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{SPC}}_T)(A)$ (with respect to $\mathfrak{V}$-small colimits). Clearly, $U_1 \subset U_2$.

By [Lur09, Proposition 5.4.5.15], there exists for any $F \in \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T)(A)$ a map $\eta\colon F \to F'$ into a U-local F' such that $\eta \in U_1$. But then also $\eta \in U_2$, so it qualifies as the adjunction unit in the larger category by the same result, and in particular the image of F under the localization functor to $\underline{\mathrm{CMON}}^P_T(A)$ is equivalent to $F' \in \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T)(A)$ as desired. □

Proof of Proposition 6.21. By the previous lemma, $\underline{\mathrm{CMon}}^P_T \subset \underline{\mathrm{CMON}}^P_T$ contains $\mathbb{P}(*)$. We claim that it is closed under $\mathfrak{U}$-small T-colimits: indeed, fiberwise colimits in $\underline{\mathrm{CMON}}^P_T$ are formed by first computing them in $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{SPC}}_T)$ and then reflecting via $L^{P\text{-}\oplus}$, so $\underline{\mathrm{CMon}}^P_T$ is closed under $\mathfrak{U}$-small fiberwise colimits by the lemma, and similarly the functor $f_!\colon \underline{\mathrm{CMON}}^P_T(A) \to \underline{\mathrm{CMON}}^P_T(B)$ factors for any map $f\colon A \to B$ in T as

$$\underline{\mathrm{CMON}}^P_T(A) \hookrightarrow \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{SPC}}_T)(A) \xrightarrow{f_!} \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{SPC}}_T)(B) \xrightarrow{L^{P\text{-}\oplus}} \underline{\mathrm{CMON}}^P_T(B).$$

In particular, $\mathcal{C} \subset \underline{\mathrm{CMon}}^P_T$ is also closed under all $\mathfrak{U}$-small S-colimits in $\underline{\mathrm{CMON}}^P_T$, and thus under finite P-coproducts. As $\underline{\mathrm{CMON}}^P_T$ is P-semiadditive, $\mathcal{C}$ is then also closed under finite P-products and moreover P-semiadditive itself. By [CLL23, Corollary 4.7.8$^{\mathrm{op}}$] there is then a unique functor $j\colon (\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}} \to \mathcal{C}$ that preserves finite P-products and sends S^0 to $\mathbb{P}(*)$. We moreover write k for the inclusion $\mathcal{C} \hookrightarrow \underline{\mathrm{CMON}}^P_T$; then k preserves $\mathfrak{U}$-small S-colimits, and hence in particular finite P-(co)products.

If now $\mathcal{D}'$ is a $\mathfrak{W}$-small and locally $\mathfrak{V}$-small P-semiadditive T-category which admits all $\mathfrak{V}$-small T-colimits, then [MW21, Theorem 6.3.5 and Corollary 6.3.7] show that the left Kan extension functors $j_!\colon \underline{\mathrm{Fun}}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}, \mathcal{D}') \to \underline{\mathrm{Fun}}_T(\mathcal{C}, \mathcal{D}')$ and

$k_!\colon \underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D}') \to \underline{\mathrm{Fun}}_T(\underline{\mathrm{CMON}}^P_T,\mathcal{D}')$ exist and that the latter is fully faithful, while [CLL23, Proposition 4.8.12] shows that $k_!j_!$ restricts to an equivalence $\underline{\mathrm{Fun}}^{P\text{-}\times}_T(\underline{\mathbb{F}}^P_{T,*},\mathcal{D}') \simeq \underline{\mathrm{Fun}}^{T\text{-CC}}_T(\underline{\mathrm{CMON}}^P_T,\mathcal{D}')$, where the right hand side denotes functors preserving all $\mathfrak{V}$-small T-colimits.

We claim that $j_!$ restricts to an equivalence $\underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\mathcal{D}') \simeq \underline{\mathrm{Fun}}^{S\text{-cc}}_T(\mathcal{C},\mathcal{D}')$, for which it is enough by 2-out-of-3 that for any $A\in T$ and any finite P-product preserving $f\colon (\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}} \to \underline{\mathrm{Fun}}_T(\underline{A},\mathcal{D}')$ the Kan extension $j_!f$ preserves $\mathfrak{U}$-small S-colimits, and that conversely any S-cocontinuous functor arises this way.

For the first statement, it is enough to observe that $k_!j_!f\colon \underline{\mathrm{CMON}}^P_T \to \underline{\mathrm{Fun}}_T(A,\mathcal{D}')$ is in particular S-cocontinuous, whence so is $k^*k_!j_!f \simeq j_!f$ as k is S-cocontinuous. Conversely, if $F\colon \mathcal{C}\to \underline{\mathrm{Fun}}_T(\underline{A},\mathcal{D}')$ is S-cocontinuous, then its restriction to $(\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}}$ preserves finite P-products. Consider now the subcategory of $\mathcal{C}$ of all objects for which the counit $\varepsilon\colon j_!j^*F\to F$ is an equivalence. Then this is closed under $\mathfrak{U}$-small S-colimits as both sides are S-cocontinuous, and it moreover contains $\mathbb{P}(*)$ as the unit $j^*F\to j^*j_!j^*F$ is an equivalence by full faithfulness of $j_!$. The claim then follows as $\mathcal{C}$ is generated by $\mathbb{P}(*)$ under $\mathfrak{U}$-small S-colimits by construction.

Let now $\mathcal{D}$ be a P-semiadditive S-cocomplete T-category, and use Lemma 6.22 to obtain an S-cocontinuous embedding into a large $\mathcal{D}'$ as above. Then the Kan extension $j_!\colon \underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\mathcal{D}') \to \underline{\mathrm{Fun}}^{T\text{-CC}}_T(\mathcal{C},\mathcal{D}')$ restricts to $\underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\mathcal{D}) \to \underline{\mathrm{Fun}}_T(\mathcal{C},\mathcal{D})$ as $\mathcal{D}\subset\mathcal{D}'$ is closed under $\mathfrak{U}$-small S-colimits and $\mathcal{C}$ is generated under them by $\mathbb{P}(*)$. In particular, $j^*\colon \underline{\mathrm{Fun}}^{S\text{-cc}}_T(\mathcal{C},\mathcal{D}) \to \underline{\mathrm{Fun}}^{P\text{-}\times}_T((\underline{\mathbb{F}}^P_{T,*})^{\mathrm{op}},\mathcal{D})$ is an equivalence. The proposition follows as the right hand side is further equivalent to $\mathcal{D}$ via evaluation at S^0 by [CLL23, Corollary 4.7.8$^{\mathrm{op}}$]. □

Remark 6.24. Running the same argument in an even larger universe $\mathfrak{X}$, the above proof (without the penultimate paragraph) shows that $\underline{\mathrm{Fun}}^{T\text{-cc}}_T(\underline{\mathrm{CMon}}^P_T,\mathcal{D})\simeq\mathcal{D}$ via evaluation at $\mathbb{P}(*)$ for any $\mathfrak{W}$-small P-semiadditive T-category $\mathcal{D}$ with $\mathfrak{U}$-small S-colimits.

As an upshot, we can now stop thinking about universes.

6.2.2. *Relation to* $\underline{\mathrm{CMon}}^P_S$. Next, we want to understand the underlying S-category of the universal P-semiadditive S-cocomplete T-category $\mathcal{C}$ constructed above. As in the unstable situation this will be some formal Yoneda yoga.

Proposition 6.25. *The adjunction* $\iota^*\colon \mathrm{Cat}_T \rightleftarrows \mathrm{Cat}_S :\iota_*$ *restricts to give adjunctions* $\mathrm{Cat}^{P\text{-}\oplus}_T \rightleftarrows \mathrm{Cat}^{P\text{-}\oplus}_S$, $\mathrm{Cat}^{P\text{-}\oplus,S\text{-cc}}_T \rightleftarrows \mathrm{Cat}^{P\text{-}\oplus,S\text{-cc}}_S$, *and* $\mathrm{Pr}^{S,P\text{-}\oplus}_T \rightleftarrows \mathrm{Pr}^{S,P\text{-}\oplus}_S$.

Proof. If will suffice to prove the first statement; the second one will then follow from Corollary 4.20.

By Lemma 3.14, $\iota_!\colon \mathrm{PSh}(S)\to\mathrm{PSh}(T)$ preserves pullbacks, while its right adjoint restricts to $\underline{\mathbb{F}}^P_T \to \iota_*\underline{\mathbb{F}}^P_S$ by Lemma 4.17 for $\mathbf{V}=\underline{\mathbb{F}}^P_T$. [CLL23, Lemma 4.6.5] therefore shows that both ι^* and ι_* restrict accordingly. Moreover, Theorem 4.18 shows that the unit and counit are P-cocontinuous and in particular P-semiadditive. □

In fact, the above argument also shows slightly more generally:

Proposition 6.26. *Let* $\mathcal{C}$ *be a pointed* T*-category with finite* P*-coproducts and let* $\mathcal{D}$ *be an* S*-category with finite* P*-products. Then a* T*-functor* $\mathcal{C}\to\iota_*\mathcal{D}$ *is* P*-semiadditive if and only if its adjunct* $\iota^*\mathcal{C}\to\mathcal{D}$ *is so.* □

Arguing as in Corollary 4.22 we immediately deduce:

Corollary 6.27. *Let $\mathcal{C}$ be a pointed T-category with finite P-coproducts and let $\mathcal{D}$ be an S-category with finite P-products. Then the equivalence $\Phi\colon \underline{\mathrm{Fun}}_T(\mathcal{C}, \iota_* \mathcal{D}) \xrightarrow{\sim} \iota_* \underline{\mathrm{Fun}}_S(\iota^* \mathcal{C}, \mathcal{D})$ from Construction 4.21 restricts to an equivalence*

$$\underline{\mathrm{Fun}}_T^{P\text{-}\oplus}(\mathcal{C}, \iota_* \mathcal{D}) \xrightarrow{\sim} \iota_* \underline{\mathrm{Fun}}_S^{P\text{-}\oplus}(\iota^* \mathcal{C}, \mathcal{D}). \qquad \square$$

On the other hand, we now easily get the following result subsuming Theorem 6.19 and one half of 6.18:

Theorem 6.28. *(1) There is a unique S-cocontinuous functor $\iota_!\colon \underline{\mathrm{CMon}}_S^P \to \iota^* \underline{\mathrm{CMon}}_T^P$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$.*

(2) $\iota_!$ is fully faithful and extends uniquely to a T-functor $\underline{\mathrm{CMon}}_{S \triangleright T}^P \to \underline{\mathrm{CMon}}_T^P$.

(3) $\underline{\mathrm{CMon}}_{S \triangleright T}^P$ is S-presentable and P-semiadditive.

(4) Let $\mathbb{P}(*) \in \Gamma(\underline{\mathrm{CMon}}_{S \triangleright T}^P)$ denote the preimage of the object of $\Gamma(\underline{\mathrm{CMon}}_T^P)$ of the same name. Then $\underline{\mathrm{CMon}}_{S \triangleright T}^P$ has the following universal property: for any P-semiadditive S-cocomplete T-category $\mathcal{D}$ evaluation at $\mathbb{P}(*)$ defines an equivalence $\underline{\mathrm{Fun}}_T(\underline{\mathrm{CMon}}_{S \triangleright T}^P, \mathcal{D}) \simeq \mathcal{D}$.

Proof. Let $\mathcal{C} \subset \underline{\mathrm{CMon}}_T^P$ again be generated under S-colimits by $\mathbb{P}(*)$. Arguing as in the proof of Theorem 4.24, Proposition 6.25 together with Proposition 6.21 shows that there is a unique S-cocontinuous functor $\underline{\mathrm{CMon}}_S^P \to \iota^* \mathcal{C}$ preserving $\mathbb{P}(*)$, and that this is an equivalence. Thus, $\iota_!\colon \underline{\mathrm{CMon}}_S^P \to \iota^* \underline{\mathrm{CMon}}_T^P$ extends uniquely to a fully faithful T-functor $\underline{\mathrm{CMon}}_{S \triangleright T}^P \to \underline{\mathrm{CMon}}_T^P$, and this induces an equivalence onto $\mathcal{C}$. The universal property then follows by another application of Proposition 6.21.

It only remains to show that the category $\mathcal{C}$ (and hence also $\underline{\mathrm{CMon}}_{S \triangleright T}^P$) is S-presentable. But indeed, $\mathcal{C}$ is S-cocomplete as $\underline{\mathrm{CMon}}_T^P$ is so, and $\mathcal{C}(A) \simeq \underline{\mathrm{CMon}}_S^P(A)$ is presentable for any $A \in T$. $\square$

6.3. An additional adjoint.

Our goal in this subsection will be to understand the right adjoint ι^* of the above S-functor $\iota_!\colon \underline{\mathrm{CMon}}_S^P \to \iota^* \underline{\mathrm{CMon}}_T^P$ better, and to use this to show that it in turn admits another right adjoint ι_*, finishing the proof of Theorem 6.18. We begin with the following observation:

Lemma 6.29. *The diagram*

$$\begin{array}{ccc} \iota^* \underline{\mathrm{CMon}}_T^P & \xrightarrow{\iota^*} & \underline{\mathrm{CMon}}_S^P \\ {\scriptstyle \iota^* \mathbb{U}} \downarrow & & \downarrow {\scriptstyle \mathbb{U}} \\ \iota^* \underline{\mathrm{Spc}}_T & \xrightarrow[\iota^*]{} & \underline{\mathrm{Spc}}_S. \end{array}$$

commutes up to natural equivalence.

Note that ι^* is P-semiadditive as it is right adjoint; by the universal property of $\underline{\mathrm{CMon}}_S^P$ from Theorem 6.16 the above then actually characterizes ι^* completely.

Proof. All functors in the diagram

$$\begin{array}{ccc} \underline{\mathrm{Spc}}_S & \xrightarrow{\iota_!} & \iota^* \underline{\mathrm{Spc}}_T \\ {\scriptstyle \mathbb{P}} \downarrow & & \downarrow {\scriptstyle \iota^* \mathbb{P}} \\ \underline{\mathrm{CMon}}_S^P & \xrightarrow[\iota_!]{} & \iota^* \underline{\mathrm{CMon}}_T^P \end{array} \tag{8}$$

are left adjoints, and both paths through this diagram send the terminal object to the same object by the defining property of the horizontal maps. Thus, the universal property of $\underline{\mathrm{Spc}}_S$ shows that (8) commutes up to equivalence. The claim follows by passing to total mates. □

This suggests a natural strategy to get a more explicit description of ι^* and to prove that it has a right adjoint: construct *some* left adjoint $\iota^*\underline{\mathrm{CMon}}^P_T \to \underline{\mathrm{CMon}}^P_S$ and then show that it is compatible with the forgetful functors. Indeed, this is precisely what we will do now, using the restriction functor from Construction 4.21:

Proposition 6.30. *The composite*

$$\iota^*\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T) \xrightarrow{\iota^*} \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \iota^*\underline{\mathrm{Spc}}_T) \xrightarrow{\underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \iota^*)} \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S) \quad (9)$$

restricts to the functor $\iota^*\underline{\mathrm{CMon}}^P_T \to \underline{\mathrm{CMon}}^P_S$ right adjoint to $\iota_!$. Moreover, (9) admits a right adjoint $\iota_*\colon \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S) \to \iota^*\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T)$, which again restricts to $\underline{\mathrm{CMon}}^P_S \to \iota^*\underline{\mathrm{CMon}}^P_T$.

Proof. For (9) to restrict as claimed, it will be enough to show that its adjunct $\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T) \to \iota_*\underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S)$ restricts to $\underline{\mathrm{CMon}}^P_T \to \iota_*\underline{\mathrm{CMon}}^P_S$. However, unravelling the definitions, the adjunct is precisely given by

$$\mathrm{Fun}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T) \xrightarrow{\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \tilde{\iota}^*)} \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \iota_*\underline{\mathrm{Spc}}_S) \xrightarrow[\sim]{\Phi} \iota_*\underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S), \quad (10)$$

where $\tilde{\iota}^*$ is the adjunct of ι^*, as in Proposition 4.28. The first functor restricts to semiadditive functors as $\tilde{\iota}^*$ is S-continuous by Corollary 4.20^{op}, and so does the second functor by Corollary 6.27.

Proposition 4.28 then shows that (10) has a right adjoint $\tilde{\iota}_*$ given by the composite

$$\iota_*\underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S) \xrightarrow[\sim]{\Phi^{-1}} \underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \iota_*\underline{\mathrm{Spc}}_S) \xrightarrow{\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \tilde{\iota}_*)} \mathrm{Fun}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T)$$

which restricts to $\iota_*\underline{\mathrm{CMon}}^P_S \to \underline{\mathrm{CMon}}^P_T$ by the same argument as before.

We can now show that (9) has a right adjoint ι_*: namely, as it is adjunct to (10), it factors as

$$\iota^*\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T) \xrightarrow{\iota^*(10)} \iota^*\iota_*\underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S) \xrightarrow{\varepsilon} \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S),$$

and the first map has a right adjoint given by $\iota^*(\tilde{\iota}_*)$ as ι^* obviously preserves adjunctions, while the second one has a right adjoint as it S-cocontinuous by (the proof of) Corollary 4.20.

Next, let us show that ι_* restricts to $\underline{\mathrm{CMon}}^P_S \to \iota^*\underline{\mathrm{CMon}}^P_T$. By the above, $\iota^*(\tilde{\iota}_*)$ restricts to $\iota^*\iota_*\underline{\mathrm{CMon}}^P_S \to \iota^*\underline{\mathrm{CMon}}^P_T$, so it only remains to show that the right adjoint of the counit $\varepsilon\colon \iota^*\iota_* \to \mathrm{id}$ at $\underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S)$ restricts to $\underline{\mathrm{CMon}}^P_S \to \iota^*\iota_*\underline{\mathrm{CMon}}^P_S$. But ε is simply given by restricting along the unit of $\iota_!\colon \mathrm{PSh}(S) \rightleftarrows \mathrm{PSh}(T) : \iota^*$, so the claim follows as $\underline{\mathrm{CMon}}^P_S \subset \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S)$ is closed under all S-limits.

It now only remains to show that the restriction of (9) to $\iota^*\underline{\mathrm{CMon}}^P_T \to \underline{\mathrm{CMon}}^P_S$ indeed agrees with the functor ι^* considered before. For this we consider the diagram

$$\begin{array}{ccccc} \iota^*\underline{\mathrm{Fun}}_T(\underline{\mathbb{F}}^P_{T,*}, \underline{\mathrm{Spc}}_T) & \xrightarrow{\iota^*} & \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \iota^*\underline{\mathrm{Spc}}_T) & \xrightarrow{\underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \iota^*)} & \underline{\mathrm{Fun}}_S(\underline{\mathbb{F}}^P_{S,*}, \underline{\mathrm{Spc}}_S) \\ \downarrow{\scriptstyle \iota^*\mathrm{ev}_{S^0}} & & \downarrow{\scriptstyle \mathrm{ev}_{S^0}} & & \downarrow{\scriptstyle \mathrm{ev}_{S^0}} \\ \iota^*\underline{\mathrm{Spc}}_T & = & \iota^*\underline{\mathrm{Spc}}_T & \xrightarrow[\iota^*]{} & \underline{\mathrm{Spc}}_S \end{array}$$

with top row (9). The right hand square commutes by naturality, as does the left hand square by a straightforward mate argument. Thus, the restricted functor $\iota^*\underline{\mathrm{CMon}}^P_T \to \underline{\mathrm{CMon}}^P_S$ lies over $\iota^*\colon \iota^*\underline{\mathrm{Spc}}_T \to \underline{\mathrm{Spc}}_S$. But it is also P-semiadditive (it is even an S-left adjoint by the above), so the claim follows from Lemma 6.29 and the universal property of $\underline{\mathrm{CMon}}^P_S$. □

Proof of Theorem 6.18. Combine Theorem 6.28 with the previous proposition. □

7. The universal property of equivariant special Γ-spaces

In this section we will identify the universal equivariantly semiadditive equivariantly presentable global category in terms of Shimakawa's *special* Γ-G-*spaces* [Shi89, Shi91].

7.1. Model categories of equivariant Γ-spaces.

We begin by introducing the main players, which will require a bit more model categorical sophistication than the unstable case.

Definition 7.1. We write Γ for the category of finite pointed sets and based maps. For any $n \geq 0$ we let $n^+ := \{0, \dots, n\}$ with basepoint 0.

Definition 7.2. Let G be a finite group. A Γ-G-*space* is a functor $\Gamma \to \boldsymbol{G}$**-SSet** that sends the singleton set 0^+ to the 1-point space. We write $\boldsymbol{\Gamma}$**-**$\boldsymbol{G}$**-SSet**$_*$ for the category of Γ-G-spaces.

By [MMO17, Lemma 1.17] we can equivalently think of a Γ-G-space as an **Set**$_*$-enriched functor $\Gamma \to \boldsymbol{G}$**-SSet**$_*$ into the category of *pointed* G-spaces, with the equivalence given by forgetting the basepoints and the enrichment.

7.1.1. *Level model structures.* Next, we will equip $\boldsymbol{\Gamma}$**-**$\boldsymbol{G}$**-SSet**$_*$ with a suitable *level model structure*. To put this into context, we recall the standard equivariant model structures on $\boldsymbol{G}$**-SSet**:

Proposition 7.3. *Let G be a finite group and let $\mathcal{F}$ be a* family of subgroups *of G, i.e. a non-empty collection of subgroups that is closed under taking subconjugates. Then* $\boldsymbol{G}$**-SSet** *carries a model structure with*

1. *weak equivalences the* $\mathcal{F}$-weak equivalences*, i.e. those maps f such that f^H is a weak homotopy equivalence for every $H \in \mathcal{F}$;*
2. *fibrations the* $\mathcal{F}$-fibrations*, i.e. those maps f such that f^H is a Kan fibration for every $H \in \mathcal{F}$;*
3. *cofibrations the* $\mathcal{F}$-cofibrations*: those maps f that are levelwise injective and such that the isotropy of any simplex outside the image of f belongs to $\mathcal{F}$.*

We call this the $\mathcal{F}$-model structure. It is combinatorial, simplicial, and proper.

Proof. See e.g. [Len20, Proposition 1.1.2] and [Ste16, Proposition 2.16]. □

Example 7.4. In the special case that $\mathcal{F} = \mathcal{A}\ell\ell$ consists of all subgroups, we call this the G-*equivariant model structure*; its weak equivalences are the G-*equivariant weak equivalences* considered before.

Proposition 7.5. *The category* $\mathbf{\Gamma}$-$\boldsymbol{G}$-$\mathbf{SSet}_*$ *admits a unique model structure with*

1. *weak equivalences those maps* $f\colon X \to Y$ *such that* $f(S_+)\colon X(S_+) \to Y(S_+)$ *is a* G*-weak equivalence for any finite* G-set S*; here we equip both sides with the* diagonal G*-action induced from the actions on* X, Y, *and* S*;*
2. *fibrations those* f *such that* $f(S_+)$ *is a* G*-fibration for any finite* G*-set* S.

We call this the G-equivariant level model structure *and its weak equivalences the* G-equivariant level weak equivalences*. It is simplicial, proper, and combinatorial with generating cofibrations the maps*

$$\Gamma(S_+,-) \wedge G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+$$

for all $n \geq 0$*, all* $H \subset G$*, and all finite* G*-sets* S*, while its generating acyclic cofibrations are similarly given by the maps* $\Gamma(S_+,-) \wedge G/H_+ \wedge (\Lambda^n_k \hookrightarrow \Delta^n)_+$.

Remark 7.6. By [Ost16, Remark 4.11], we could equivalently ask for $f(S_+)$ to be an H-weak equivalence or H-fibration for any $H \subset G$ and any finite H-set S. Put differently, if $\mathcal{G}_{G,\Sigma_S}$ denotes the family of *graph subgroups* of $G \times \Sigma_S$ (i.e. the subgroups of the form $\Gamma_{H,\varphi} \coloneqq \{(h,\varphi(h)) : h \in H\}$ for $H \subset G$ and $\varphi\colon H \to \Sigma_S$, or equivalently those subgroups intersecting $1 \times \Sigma_S$ trivially), then a map f is a weak equivalence or fibration in the above model structure if and only if $f(S_+)$ is a $\mathcal{G}_{G,\Sigma_S}$-weak equivalence or fibration, respectively, for any finite set S.

Proof of Proposition 7.5. The model structure appears without proof as [Ost16, Theorem 4.7]; see [Len20, Proposition 2.2.36] for a complete argument. □

Lemma 7.7. *Let* $\alpha\colon G \to G'$ *be a homomorphism of finite groups. Then the restriction* $\alpha^*\colon \mathbf{\Gamma}$-$\boldsymbol{G'}$-$\mathbf{SSet}_* \to \mathbf{\Gamma}$-$\boldsymbol{G}$-$\mathbf{SSet}_*$ *is left Quillen for the level model structures.*

Proof. It suffices that the right adjoint α_* preserves (acyclic) fibrations. As the latter are defined levelwise, this amounts to saying that

$$(\alpha \times \Sigma_S)^*\colon (\boldsymbol{G'} \times \boldsymbol{\Sigma_S})\text{-}\mathbf{SSet}_{\mathcal{G}_{G',\Sigma_S}} \rightleftarrows (\boldsymbol{G} \times \boldsymbol{\Sigma_S})\text{-}\mathbf{SSet}_{\mathcal{G}_{G,\Sigma_S}} :(\alpha \times \Sigma_S)_*$$

is a Quillen adjunction for every finite set S. But clearly $(\alpha \times \Sigma_S)^*$ preserves cofibrations and sends generating acyclic cofibrations to weak equivalences. □

Remark 7.8. If $\alpha\colon G \to G'$ is injective, then $\alpha^*\colon \mathbf{\Gamma}$-$\boldsymbol{G'}$-$\mathbf{SSet}_* \to \mathbf{\Gamma}$-$\boldsymbol{G}$-$\mathbf{SSet}$ is easily seen to preserve weak equivalences and fibrations; in particular, it is also right Quillen.

Beware that the previous remark does *not* hold for non-injective α, see e.g. [Len20, Example 2.2.15], and accordingly the composition

$$\mathbf{\Gamma}\text{-}\bullet\text{-}\mathbf{SSet}_*\colon \mathrm{Glo}^{\mathrm{op}} \hookrightarrow \mathrm{Grpd}^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathbf{\Gamma}\text{-}\mathbf{SSet}_*)} \mathrm{Cat}_1$$

does not lift to RelCat via the above weak equivalences. However, by Ken Brown's Lemma we can fix this by restricting to the subcategories of *cofibrant* objects:

Definition 7.9. We write $\mathbf{\Gamma}\text{-}\bullet\text{-}\mathbf{SSet}_*^{\mathrm{cof}}$ for the resulting functor $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{RelCat}$ and $\underline{\Gamma\mathscr{S}}_*$ for the global category obtained by pointwise localization.

We can (and will) equivalently think of $\underline{\Gamma\mathscr{S}}_*$ as sending a finite group G to the localization of $\mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_*$ and a homomorphism α to the left derived functor $\mathbf{L}\alpha^*$.

Lemma 7.10. *Let G be a finite group. Then we have a Quillen adjunction*

$$\Gamma(1^+, –) \wedge (–)_+ : \boldsymbol{G}\text{-}\mathbf{SSet} \rightleftarrows \mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_* : \mathrm{ev}_{1^+}$$

in which both adjoints are homotopical.

Proof. It is clear that both adjoints are homotopical, and that the right adjoint moreover preserves fibrations, so that it is in particular right Quillen. □

Thus, $\Gamma(1^+, –) \wedge (–)_+$ induces a natural transformation $\bullet\text{-}\mathbf{SSet} \to \mathbf{\Gamma}\text{-}\bullet\text{-}\mathbf{SSet}_*^{\mathrm{cof}}$, and hence a global functor $\underline{\mathscr{S}} \to \underline{\Gamma\mathscr{S}}_*$, which we denote by $\mathbb{P}$. It is not hard to check that $\mathbb{P}$ admits a global right adjoint (induced by ev_{1^+}); as we will not need this below, we leave the details to the interested reader.

7.1.2. *Specialness.* In order to study equivariant commutative monoids, we have to Bousfield localize the above level model structures. For this we recall:

Definition 7.11. A Γ-G-space is called *special* if for every finite G-set S the *Segal map* $X(S_+) \to X(1^+)^{\times S}$ induced by the characteristic maps $\chi_s : S_+ \to 1^+$ for varying $s \in S$, is a G-weak equivalence.

Similarly to the different characterizations of the G-equivariant level weak equivalences, specialness is equivalent to asking more generally for the Segal maps to be H-equivariant weak equivalences for all $H \subset G$ and all finite H-sets S, or for them to be $\mathcal{G}_{G,\Sigma_S}$-weak equivalences for every finite set S, see [Len20, Lemma 2.2.10].

Proposition 7.12 (See [Len20, Proposition 2.2.60])**.** *The G-equivariant level model structure on $\mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_*$ admits a Bousfield localization with fibrant objects precisely the level fibrant* special Γ-G*-spaces. We call this the* G-equivariant model structure *and its weak equivalences the* G-equivariant weak equivalences*. It is combinatorial, simplicial, and left proper.* □

Remark 7.13. The above model structure is obtained from the level model structure by localizing with respect to the maps $S_+ \wedge \Gamma(1^+, –) \wedge G/H_+ \to \Gamma(S_+, –) \wedge G/H_+$ induced by the map $S_+ \to \Gamma(S_+, 1^+)$ sending $s \in S$ to its characteristic map $\chi_S : S_+ \to 1^+$ for all finite G-sets S. In particular, all of these maps are G-equivariant weak equivalences.

Lemma 7.14. *Let $\alpha : G \to G'$ be a homomorphism. Then*

$$\alpha^* : \mathbf{\Gamma}\text{-}\boldsymbol{G'}\text{-}\mathbf{SSet}_* \rightleftarrows \mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\mathbf{SSet}_*$$

is left Quillen with respect to the above model structures. If α is injective, then α^* is also right Quillen.

Proof. For the first statement, it will suffice by [Lur09, Corollary A.3.7.2] that α^* preserves cofibrations and α_* preserves fibrant objects. The first statement is clear from Lemma 7.7, while for the second statement it is enough by adjunction to show that $\mathbf{L}\alpha^*$ sends the maps from the previous remark to weak equivalences.

As these are maps between cofibrant objects, it is enough to prove the same for α^*. However, decomposing $\alpha^*(G'/H)$ into G-orbits expresses $\alpha^*(S_+ \wedge \Gamma(1^+,\text{–}) \wedge G'/H_+ \to \Gamma(S_+,\text{–}) \wedge G'/H_+)$ as a coproduct of weak equivalences between cofibrant objects, so the claim follows.

The second statement follows similarly from Remark 7.8 as α^* clearly preserves specialness for injective α. $\square$

In particular, we get a functor $\mathbf{\Gamma\text{-}\bullet\text{-}SSet}_*^{\text{cof, spc}}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{RelCat}$ that sends G to $\mathbf{\Gamma\text{-}}\boldsymbol{G}\mathbf{\text{-}SSet}_*^{\text{cof}}$ with the above weak equivalences. The identity of underlying categories $\mathbf{\Gamma\text{-}\bullet\text{-}SSet}_*^{\text{cof}} \to \mathbf{\Gamma\text{-}\bullet\text{-}SSet}_*^{\text{cof, spc}}$ then induces a localization $L\colon \underline{\Gamma\mathscr{S}}_* \to \underline{\Gamma\mathscr{S}}_*^{\text{spc}}$. We will write $\mathbb{P}\colon \underline{\mathscr{S}} \to \underline{\Gamma\mathscr{S}}_*^{\text{spc}}$ for $L \circ \mathbb{P}$; note that this is again induced by the homotopical left Quillen functors $\Gamma(1^+,\text{–}) \wedge (\text{–})_+$.

Warning 7.15. The functors $\mathbf{L}\alpha^*$ do *not* preserve specialness for non-injective α, i.e. the pointwise right adjoints of L do not assemble into a global right adjoint. This is hard to see directly (as we know so few cofibrant objects in the above model structure, making it hard to compute $\mathbf{L}\alpha^*$), so we use a trick and a bit of equivariant infinite loop space theory instead:

Let $\Gamma(1^+,\text{–}) \to \mathcal{S}$ be an acyclic cofibration to a special Γ-space. In particular, $\mathcal{S}$ is cofibrant, so if $\mathbf{L}\alpha^*$ preserved specialness, then $\mathcal{S}$ with the trivial G-action would be a special Γ-G-space for any finite G. On the other hand, as restrictions are left Quillen by the above, it would be equivalent to $\Gamma(1^+,\text{–})$ with trivial G-action. We show that already for $G = \mathbb{Z}/2$ this is impossible: no special Γ-$\mathbb{Z}/2$-space equivalent to $\Gamma(1^+,\text{–})$ can have trivial action.

For this we use that the delooping of $\Gamma(1^+,\text{–})$ (and hence of any Γ-$\mathbb{Z}/2$-space equivalent to it) is the equivariant sphere spectrum. Now the zeroth stable homotopy groups of the latter are given by the Burnside ring, and hence in particular $\pi_0^1(\mathbb{S}) \cong \mathbb{Z} \ncong \mathbb{Z}^2 \cong \pi_0^{\mathbb{Z}/2}(\mathbb{S})$. However, for a *special* Γ-$\mathbb{Z}/2$-space the homotopy groups of its delooping are simply given as the group completions of the original homotopy monoids. In the case of a trivial G-action, the restriction homomorphisms between these homotopy monoids are clearly isomorphisms, and in particular their group completions are isomorphic, yielding the desired contradiction.

Note that the same argument shows that also the underived functors α^* do not preserve specialness, although there are much more concrete counterexamples available in this case.

We can now finally state the main results of this section.

Theorem 7.16. *The global category $\underline{\Gamma\mathscr{S}}_*^{\text{spc}}$ is equivariantly presentable and equivariantly semiadditive. Moreover, the unique equivariantly cocontinuous global functor $\underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \to \underline{\Gamma\mathscr{S}}_*^{\text{spc}}$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$ is an equivalence.*

Theorem 7.17. *The global category $\underline{\Gamma\mathscr{S}}_*^{\text{spc}}$ is the free equivariantly cocomplete equivariantly semiadditive global category on one generator in the following sense: for every other such $\mathcal{D}$ evaluation at $\mathbb{P}(*)$ provides an equivalence*

$$\underline{\mathrm{Fun}}^{\text{eq-cc}}_{\mathrm{Glo}}(\underline{\Gamma\mathscr{S}}_*^{\text{spc}}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

7.2. ***G*-global vs. *G*-equivariant Γ-spaces.** In order to prove the above theorems, we will again reduce to our identification of the universal *globally cocomplete* equivariantly semiadditive category from [CLL23].

7.2.1. *Model categories of G-global Γ-spaces.* We begin by introducing the relevant model categories.

Definition 7.18. A *G-global Γ-space* is a functor $X\colon \Gamma \to \boldsymbol{G\text{-}\mathcal{I}\text{-}SSet}$ such that $X(0^+) = *$.

Proposition 7.19 (See [Len20, Theorem 2.2.31])**.** *The category $\boldsymbol{\Gamma\text{-}G\text{-}\mathcal{I}\text{-}SSet_*}$ of G-global Γ-spaces carries a unique model structure with*

1. *weak equivalences those maps f such that $f(S_+)$ is a $(G\times\Sigma_S)$-global weak equivalence for every finite set S*
2. *acyclic fibrations those maps f such that $f(S_+)(A)$ is a $\mathcal{G}_{\Sigma_A, G\times\Sigma_S}$-acyclic fibration for all finite sets S and A.*

We call this the G-global level model structure *and its weak equivalences the* G-global level weak equivalences*. It is combinatorial, simplicial, and proper. Moreover, pushouts along injective cofibrations (i.e. levelwise injections) are homotopy pushouts in this model structure; in particular, they preserve weak equivalences.* □

For any $\alpha\colon G \to G'$ the functor $\alpha^*\colon \boldsymbol{\Gamma\text{-}G'\text{-}\mathcal{I}\text{-}SSet_*} \to \boldsymbol{\Gamma\text{-}G\text{-}\mathcal{I}\text{-}SSet_*}$ preserves weak equivalences, so the above yields a global category $\underline{\Gamma\mathscr{S}}^{\text{gl}}_*$ by the usual procedure. Note that [CLL23] uses the notation '$\underline{\Gamma\mathscr{S}}^{\text{gl}}_{\mathcal{I},*}$' instead; however, the above is equivalent to the category denoted by the same symbols in *op. cit.* by [Len20, Theorem 2.2.33].

Construction 7.20. For any G, we have a homotopical adjunction

$$\Gamma(1^+,-)\wedge(-)_+\colon \boldsymbol{G\text{-}\mathcal{I}\text{-}SSet} \rightleftarrows \boldsymbol{\Gamma\text{-}G\text{-}\mathcal{I}\text{-}SSet_*} :\mathrm{ev}_{1^+}.$$

As both adjoints are moreover strictly compatible with restriction, we obtain an induced adjunction $\mathbb{P}\colon \underline{\mathscr{S}}^{\text{gl}} \rightleftarrows \underline{\Gamma\mathscr{S}}^{\text{gl}}_* :\mathbb{U}$. We will refer to $\mathbb{U}$ as the *forgetful functor*.

Remark 7.21. We can also consider the category $\boldsymbol{\Gamma\text{-}G\text{-}\mathcal{I}\text{-}SSet}$ of all functors $\Gamma \to \boldsymbol{G\text{-}\mathcal{I}\text{-}SSet}$ and equip this with the analogue of the G-global level weak equivalence. For varying G, these again assemble into a global category, which we denote by $\underline{\Gamma\mathscr{S}}^{\text{gl}}$.

The inclusion $\boldsymbol{\Gamma\text{-}G\text{-}\mathcal{I}\text{-}SSet_*} \hookrightarrow \boldsymbol{\Gamma\text{-}G\text{-}\mathcal{I}\text{-}SSet}$ admits a left adjoint Λ given by taking the cofibers of the maps $X(0^+) \to X(S_+)$ induced by the unique maps $i\colon 0^+ \to S_+$ in Γ. As each $X(i)$ is an injective cofibration (as i admits a retraction), this is actually a homotopy cofiber and Λ is homotopical. It follows easily that the map $\underline{\Gamma\mathscr{S}}^{\text{gl}}_* \to \underline{\Gamma\mathscr{S}}^{\text{gl}}$ induced by the inclusions is fully faithful with essential image given in degree G by those X with $X(0^+)\simeq *$ in $\mathscr{S}^{\text{gl}}_G$.

Definition 7.22. A G-global Γ-space X is called *special* if the Segal map $X(S_+) \to X(1^+)^{\times S}$ is a $(G\times\Sigma_S)$-global weak equivalence for every finite set S.

Note that unlike their equivariant counterparts, these are stable under arbitrary restrictions, so they form a global subcategory $\underline{\Gamma\mathscr{S}}^{\text{gl, spc}}_*$.

Theorem 7.23 (See [CLL23, Corollary 5.3.6])**.** *There exists an equivalence of global categories $\Xi\colon \underline{\Gamma\mathscr{S}}^{\text{gl}} \simeq \underline{\mathrm{Fun}}_{\mathrm{Glo}}(\underline{\mathbb{F}}^{\mathrm{Orb}}_{\mathrm{Glo},*}, \underline{\mathrm{Spc}}_{\mathrm{Glo}})$ compatible with the forgetful functors and restricting to an equivalence $\underline{\Gamma\mathscr{S}}^{\text{gl, spc}}_* \simeq \underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Glo}}$.* □

Corollary 7.24. *The inclusions* $\underline{\Gamma\mathscr{S}}_*^{\text{gl, spc}} \hookrightarrow \underline{\Gamma\mathscr{S}}_*^{\text{gl}}$ *and* $\underline{\Gamma\mathscr{S}}_*^{\text{gl, spc}} \hookrightarrow \underline{\Gamma\mathscr{S}}^{\text{gl}}$ *admit global left adjoints.*

Proof. The second statement follows from the previous theorem as $\underline{\text{CMon}}_{\text{Glo}}^{\text{Orb}} \hookrightarrow \underline{\text{Fun}}_{\text{Glo}}(\mathbb{F}_{\text{Glo},*}^{\text{Orb}}, \underline{\text{Spc}}_{\text{Glo}})$ admits a left adjoint. The first one then follows from this together with Remark 7.21. □

Remark 7.25. One can also prove the corollary via purely model categorical arguments: by [Len20, Proposition 2.2.61], the G-global level model structure admits a Bousfield localization with fibrant objects the level fibrant *special* G-global Γ-spaces. In particular, we get a pointwise left adjoint, and the Beck–Chevalley condition then translates to demanding that each $\alpha^*\colon \underline{\Gamma\mathscr{S}}_*^{\text{gl}}(G') \to \underline{\Gamma\mathscr{S}}_*^{\text{gl}}(G)$ preserve the weak equivalences of these Bousfield localizations, or equivalently that the restriction functors

$$\alpha^*\colon \mathbf{\Gamma}\text{-}\boldsymbol{G'}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}_* \to \mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}_* \tag{11}$$

be homotopical for the localized model structures. While this is doable by careful inspection, it is actually more work than in the equivariant case (as the maps we localize at are more complicated), and hence deliberately avoided in [Len20], which is why we went via the above route instead.

Note however that conversely the above corollary now shows that the functor $\alpha^*\colon \underline{\Gamma\mathscr{S}}_*^{\text{gl}}(G') \to \underline{\Gamma\mathscr{S}}_*^{\text{gl}}(G)$ and hence also (11) is homotopical for any α, yielding an ∞-categorical proof of a model categorical statement.

Composing the above with the adjunction from Construction 7.20, we get an adjunction $\underline{\mathscr{S}}^{\text{gl}} \rightleftarrows \underline{\Gamma\mathscr{S}}_*^{\text{gl, spc}}$ that we again denote by $\mathbb{P} \dashv \mathbb{U}$. The (inverse) equivalence $\underline{\text{CMon}}_{\text{Glo}}^{\text{Orb}} \simeq \underline{\Gamma\mathscr{S}}_*^{\text{gl}}$ from Theorem 7.23 can then be described (by some easy mate yoga) as the unique left adjoint that sends $\mathbb{P}(*)$ to $\mathbb{P}(*)$.

7.2.2. *The comparison.* Finally, let us relate G-global and G-equivariant Γ-spaces to each other:

Proposition 7.26. *There is a global functor* $\mathbf{L}\,\text{const}\colon \underline{\Gamma\mathscr{S}}_*^{\text{spc}} \to \underline{\Gamma\mathscr{S}}_*^{\text{gl, spc}}$ *with the following properties:*

(1) It is fully faithful and sends $\mathbb{P}(*)$ *to* $\mathbb{P}(*)$.
(2) It admits an Orb*-right adjoint.*

Once again, after the universal property of $\underline{\Gamma\mathscr{S}}_*$ is established, we will see a posteriori that the above adjunction is actually unique.

For the proof of the proposition we will need another model structure:

Lemma 7.27 (See [Len20, Corollary 2.2.40 and proof of Proposition 2.2.42])**.** *The category* $\mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}_*$ *admits a model structure with*

(1) weak equivalences the G*-global level weak equivalences*
(2) cofibrations the injective cofibrations.

We call this the injective G-global level model structure. *It is combinatorial, simplicial, and proper. Moreover, if* $\alpha\colon G \to G'$ *is an* injective *homomorphism, then* $\alpha^*\colon (\mathbf{\Gamma}\text{-}\boldsymbol{G'}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}_*)_{\text{injective}} \to (\mathbf{\Gamma}\text{-}\boldsymbol{G}\text{-}\boldsymbol{\mathcal{I}}\text{-}\mathbf{SSet}_*)_{\text{injective}}$ *is right Quillen.* □

Proof of Proposition 7.26. For every G, we have a Quillen adjunction

$$\mathrm{const}\colon \mathbf{\Gamma\text{-}\mathit{G}\text{-}SSet}_* \rightleftarrows (\mathbf{\Gamma\text{-}\mathit{G}\text{-}\mathcal{I}\text{-}SSet}_*)_{\text{injective}} :\mathrm{ev}_\varnothing, \tag{12}$$

see [Len20, Proposition 2.2.25]. By Ken Brown's Lemma, we in particular see that const sends G-equivariant weak equivalences between cofibrant objects to G-global level weak equivalences, so we get an induced global functor $\mathbf{L}\,\mathrm{const}\colon \underline{\Gamma\mathscr{S}}_* \to \underline{\Gamma\mathscr{S}}_*^{\mathrm{gl}}$, which we can postcompose with the localization to $\underline{\Gamma\mathscr{S}}_*^{\mathrm{gl,\,spc}}$. Note that this sends $\Gamma(1^+,\text{–})$ to $\mathbb{P}(*)$ by direct inspection.

We now claim that this descends to $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}$, which amounts to saying that the left adjoint in (12) sends G-equivariant weak equivalences of cofibrant objects to G-global weak equivalences, for which it is in turn enough that the right derived functor $\mathbf{R}\mathrm{ev}_\varnothing$ preserve specialness. However, by *loc. cit.* this right adjoint is equivalent to $\mathrm{ev}_{\mathcal{U}}$ for our favourite complete G-set universe $\mathcal{U}$, and it is clear that the latter has the required property (also see Lemma 2.2.51 of *op. cit.*). Altogether, we therefore get a functor $\mathbf{L}\,\mathrm{const}\colon \underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}} \to \underline{\Gamma\mathscr{S}}_*^{\mathrm{gl,\,spc}}$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$; moreover, this is fully faithful as the right adjoint $\mathcal{R}$ of the right adjoint $\mathbf{R}\mathrm{ev}_\varnothing$ is fully faithful by Theorem 2.2.59 of *op. cit.*

It only remains to show that the pointwise right adjoints $\mathbf{R}\mathrm{ev}_\varnothing$ assemble into an Orb-right adjoint, i.e. that for any *injective* homomorphism $\alpha\colon G \to G'$ the Beck–Chevalley transformation $\mathbf{L}\alpha^* \circ \mathbf{R}\mathrm{ev}_\varnothing \Rightarrow \mathbf{R}\mathrm{ev}_\varnothing \circ \alpha^*$ is an equivalence.

However, the *pointset level* Beck–Chevalley map $\alpha^* \circ \mathrm{ev}_\varnothing \Rightarrow \mathrm{ev}_\varnothing \circ \alpha^*$ is clearly an isomorphism, and all functors in question are right Quillen by the above together with Lemmas 7.14 and 7.27, so this already models the derived Beck–Chevalley map when restricted to injectively fibrant objects. □

7.3. **Proof of Theorems 7.16 and 7.17.** Finally, we turn to the universal property of $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}$.

Lemma 7.28. *The category $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}(G)$ is generated under (non-parametrized) colimits by the G-equivariant Γ-spaces $\Gamma(1^+,\text{–}) \wedge G/H_+$ for $H \subset G$.*

Proof. Inspecting the generating cofibrations from Proposition 7.5 we see that $\underline{\Gamma\mathscr{S}}_*(G)$ is generated under colimits by the $\Gamma(S_+,\text{–}) \wedge G/H_+$ for finite G-sets S and subgroups $H \subset G$. Thus, these objects also generate $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}(G)$. However, in the latter $\Gamma(S_+,\text{–}) \wedge G/H_+ \simeq S_+ \wedge \Gamma(1^+,\text{–}) \wedge G/H_+$ by Remark 7.13. The claim follows by decomposing the G-set $G/H \times S$ into its orbits. □

Note that $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}(G) \ni \Gamma(1^+,\text{–}) \wedge G/H_+ \simeq i_! p^* \Gamma(1^+,\text{–})$ where $i\colon H \hookrightarrow G$ denotes the inclusion and $p\colon H \to 1$ the unique map. Thus, once we know that $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}$ is equivariantly cocomplete, the lemma will tell us that it is generated under equivariant colimits by $\mathbb{P}(*) = \Gamma(1^+,\text{–}) \in \underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}(1)$.

Proof of Theorems 7.16 and 7.17. The fully faithful functor $\mathbf{L}\,\mathrm{const}$ from Proposition 7.26 identifies $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}$ with a full subcategory of $\underline{\Gamma\mathscr{S}}_*^{\mathrm{gl,\,spc}}$, and the latter is globally presentable by Theorem 7.23. However, the essential image of $\mathbf{L}\,\mathrm{const}$ is closed under all equivariant colimits as $\mathbf{L}\,\mathrm{const}$ has an Orb-right adjoint, so $\underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}$ is equivariantly cocomplete.

In particular, there is a unique equivariantly cocontinuous functor $\underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \to \underline{\Gamma\mathscr{S}}_*^{\mathrm{spc}}$ sending $\mathbb{P}(*)$ to $\mathbb{P}(*)$. We claim that this is an equivalence, for which it will be enough to construct *some* equivalence preserving $\mathbb{P}(*)$. To this end, we will show that the composite

$$\underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \xrightarrow{\iota_!} \underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Glo}} \xrightarrow{\Xi} \underline{\Gamma\mathscr{S}}^{\mathrm{gl,\,spc}}_{*} \tag{13}$$

of the fully faithful functor from Theorem 6.18 and the equivalence from Theorem 7.23 (which sends $\mathbb{P}(*)$ to $\mathbb{P}(*)$ by construction) restricts to an equivalence $\underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \simeq \operatorname{ess\,im}(\mathbf{L}\,\mathrm{const}) =: \mathcal{E}$. On the one hand, the source of (13) is generated under equivariant colimits by $\mathbb{P}(*)$, so that (13) factors through $\mathcal{E}$ as both functors are in particular Orb-left adjoints. On the other hand, Lemma 7.28 shows that $\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_{*}$ and hence also $\mathcal{E}$ is generated by $\mathbb{P}(*)$, so this restriction is also essentially surjective, hence an equivalence.

Finally, the universal property of $\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_{*}$ follows immediately from this equivalence and the universal property of $\underline{\mathrm{CMon}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}}$ established in Theorem 6.19. ☐

7.4. The universal property of special Γ-*G*-spaces.

We close this section by similarly establishing a universal property of special Γ-G-spaces for a fixed finite group G:

Theorem 7.29. *Recall the functor $\upsilon_G\colon \mathrm{Orb}_G \to \mathrm{Glo}$ from Construction 5.11.*

1. *(1) The G-category $\upsilon_G^*\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_{*}$ (sending G/H to the category of special Γ-H-spaces) is G-presentable and G-semiadditive in the sense of Example 6.7. Moreover, the unique left adjoint $\underline{\mathrm{CMon}}_{\mathrm{Orb}_G} \to \upsilon_G^*\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_{*}$ preserving $\mathbb{P}(*)$ is an equivalence.*
2. *(2) For any G-cocomplete G-semiadditive $\mathcal{D}$, evaluation at $\mathbb{P}(*)$ induces an equivalence $\underline{\mathrm{Fun}}^{G\text{-cc}}_{\mathrm{Orb}_G}(\upsilon_G^*\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_{*}, \mathcal{D}) \simeq \mathcal{D}$.*

For the proof we will need:

Proposition 7.30. *Let $P \subset T$ be atomic orbital, let $A \in T$, and write $\pi_A\colon T_{/A} \to T$ for the forgetful functor. Then $\pi_A^*\underline{\mathrm{CMon}}^P_T$ is $T_{/A}$-cocomplete and $T^P_{/A}$-semiadditive, and the unique left adjoint $\underline{\mathrm{CMon}}^{T^P_{/A}}_{T_{/A}} \to \pi_A^*\underline{\mathrm{CMon}}^P_T$ preserving $\mathbb{P}(*)$ is an equivalence.*

Proof. By [CLL23, Proposition 2.3.26 and Corollary 4.6.9] $\pi_A^* \dashv \pi_{A*}$ restricts to an adjunction

$$\mathrm{Cat}_T^{P\text{-}\oplus,T\text{-cc}} \rightleftarrows \mathrm{Cat}_{T_{/A}}^{T^P_{/A}\text{-}\oplus,T_{/A}\text{-cc}},$$

so the claim follows as before by comparing corepresented functors. ☐

Proof of Theorem 7.29. As in the unstable case (Theorem 5.12), it will be enough to construct an equivalence $\underline{\mathrm{CMon}}_{\mathrm{Orb}_G} \simeq \upsilon_G^*\underline{\Gamma\mathscr{S}}^{\mathrm{spc}}_{*}$ preserving $\mathbb{P}(*)$, for which it in turn suffices to combine the previous proposition with Theorem 7.16. ☐

8. The stable story

As in the previous sections, we fix a cleft category $\iota\colon S \hookrightarrow T$. The goal of this section is to establish the stable analogues of the results from Section 6. We begin with the fiberwise (or naïve) version of stability:

Definition 8.1. A T-category $\mathcal{C}$ is called *fiberwise stable* if factors through the non-full subcategory $\mathrm{Cat}^{\mathrm{st}} \subset \mathrm{Cat}$ of stable categories and exact functors.

Construction 8.2. Recall [Lur17, Proposition 1.4.4.4, Example 4.8.1.23] that the inclusion $\mathrm{Pr}^{\mathrm{L,\,st}} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}$ of presentable stable categories and left adjoints into all presentable categories admits a left adjoint, given by tensoring with the category Sp of spectra.

If now $\mathcal{C}$ is a fiberwise presentable T-category, then we write $\mathrm{Sp}\otimes\mathcal{C}$ for the composite

$$T^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Pr}^{\mathrm{L}} \xrightarrow{\mathrm{Sp}\otimes-} \mathrm{Pr}^{\mathrm{L,\,st}} \subset \mathrm{Cat}$$

and call it the *left fiberwise stabilization* of $\mathcal{C}$. It comes with a functor $\Sigma^\infty\colon \mathcal{C} \to \mathrm{Sp}\otimes\mathcal{C}$ induced by the unit of the adjunction $\mathrm{Pr}^{\mathrm{L}} \rightleftarrows \mathrm{Pr}^{\mathrm{L,\,st}}$.

Remark 8.3. There is another way to fiberwise stabilize suitable T-categories, which we will refer to as *right fiberwise stabilization* below: if $\mathcal{C}$ factors through the non-full subcategory $\mathrm{Cat}^{\mathrm{lex}}$ of pointed categories with finite limits and left exact functors, then we can define $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ by composing with the *right* adjoint to the inclusion $\mathrm{Cat}^{\mathrm{st}} \hookrightarrow \mathrm{Cat}^{\mathrm{lex}}$ of stable categories. This is the perspective taken in [CLL23, Subsection 6.1].

For the T-categories which we would like to stabilize, such as $\underline{\mathrm{CMon}}^{P}_{S\triangleright T}$, it is not clear whether the restriction functors preserve finite limits (as a consequence of the example in Warning 9.8 below, they cannot preserve general limits). Therefore $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$ is not well-defined, and we cannot sensibly ask for $\mathrm{Sp}\otimes\mathcal{C}$ to agree with $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C})$.

However, on the category $\mathrm{Pr}^{\mathrm{L,\,lex}}$ of pointed presentable categories and left exact left adjoints, the two stabilization constructions agree [Lur17, Example 4.8.1.23]. Thus, whenever we are given some subcategory $T' \subset T$ such that $\mathcal{C}\,|_{T'}$ is pointed and restrictions in $\mathcal{C}$ along maps in T' *are* left exact, then $(\mathrm{Sp}\otimes\mathcal{C})|_{T'}$ agrees with $\underline{\mathrm{Sp}}^{\mathrm{fib}}(\mathcal{C}|_{T'})$. This will allow us below to still apply the results from [CLL23, Section 6] to the present situation.

Lemma 8.4. *Let $\mathcal{C}$ be a fiberwise presentable T-category. Then $\mathrm{Sp}\otimes\mathcal{C}$ is fiberwise presentable and fiberwise stable. Moreover, for every fiberwise cocomplete and fiberwise stable $\mathcal{D}$, restriction along Σ^∞ defines an equivalence*

$$\underline{\mathrm{Fun}}^{\text{fib-cc}}_T(\mathrm{Sp}\otimes\mathcal{C},\mathcal{D}) \to \underline{\mathrm{Fun}}^{\text{fib-cc}}_T(\mathcal{C},\mathcal{D}) \tag{14}$$

of T-categories of fiberwise cocontinuous functors.

Proof. It is clear that $\mathrm{Sp}\otimes\mathcal{C}$ is fiberwise presentable and fiberwise stable. Replacing $\mathcal{D}$ by $\underline{\mathrm{Fun}}_T(\mathcal{T},\mathcal{D})$ for small $\mathcal{T} \in \mathrm{Cat}_T$, it will suffice for the universal property to show that the induced map

$$\mathrm{Hom}_{\mathrm{Cat}^{\text{fib-cc}}_T}(\mathrm{Sp}\otimes\mathcal{C},\mathcal{D}) \to \mathrm{Hom}_{\mathrm{Cat}^{\text{fib-cc}}_T}(\mathcal{C},\mathcal{D})$$

of mapping spaces in the category $\mathrm{Cat}^{\text{fib-cc}}_T \coloneqq \mathrm{Fun}(T^{\mathrm{op}},\mathrm{Cat}^{\mathrm{cc}})$ of fiberwise cocomplete T-categories and fiberwise colimit-preserving functors is an equivalence.

Writing both sides as the ends of the mapping spaces in $\mathrm{Cat}^{\mathrm{cc}}$, it then suffices to consider the case $T = 1$, i.e. that for any cocomplete stable $\mathcal{D}$ restriction along $\mathcal{C} \to \mathrm{Sp}\otimes\mathcal{C}$ defines an equivalence $\mathrm{Hom}^{\mathrm{cc}}(\mathrm{Sp}\otimes\mathcal{C},\mathcal{D}) \simeq \mathrm{Hom}^{\mathrm{cc}}(\mathcal{C},\mathcal{D})$. Using that the tensor product of presentable categories agrees with the tensor product of cocomplete categories [Lur17, Proposition 4.8.1.15], the tensor-hom adjunction reduces to the case $\mathcal{C} = \mathrm{Spc}$, i.e. we want to show that evaluation at the sphere defines an equivalence $\mathrm{Hom}^{\mathrm{cc}}(\mathrm{Sp}, \mathcal{D}) \simeq \iota\mathcal{D}$. This however follows at once by exhibiting Sp as the Ind-completion of the Spanier–Whitehead category [Lur18, Remark C.1.1.6] and noting that right exact functors out of the latter classify objects by [Lur18, Proposition C.1.1.7] together with [Lur17, Proposition 1.4.2.21]. □

Lemma 8.5. *Assume $\mathcal{C}$ is S-presentable. Then $\mathrm{Sp} \otimes \mathcal{C}$ is again S-presentable, hence in particular S-cocomplete. Moreover, if also $\mathcal{D}$ is S-cocomplete, then* (14) *restricts to an equivalence*

$$\underline{\mathrm{Fun}}_T^{S\text{-cc}}(\mathrm{Sp} \otimes \mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_T^{S\text{-cc}}(\mathcal{C}, \mathcal{D}).$$

Proof. From the previous lemma we see that $\mathrm{Sp} \otimes \mathcal{C}$ is fiberwise presentable. If $s\colon A \to B$ is in S, then the adjunction $s_!\colon \mathcal{C}(A) \rightleftarrows \mathcal{C}(B) : s^*$ is an internal adjunction in Pr^{L} (as s^* is itself a left adjoint by fiberwise presentability), so we get an induced adjunction $\mathrm{Sp} \otimes s_! \dashv \mathrm{Sp} \otimes s^*$ by 2-functoriality of the tensor product. Moreover, the Beck–Chevalley conditions for $\mathrm{Sp} \otimes \mathcal{C}$ follow immediately from the ones for $\mathcal{C}$.

Finally, for the universal property it suffices by the previous lemma and replacing $\mathcal{D}$ by $\mathcal{D}^A$ to show that Σ^∞ preserves S-colimits and that for any S-cocontinuous $F\colon \mathcal{C} \to \mathcal{D}$ also the lift $\tilde{F}\colon \mathrm{Sp} \otimes \mathcal{C} \to \mathcal{D}$ is S-cocontinuous.

For the first statement, we observe that Σ^∞ is clearly fiberwise cocontinuous, and that for any admissible $f\colon X \to Y$ in $\mathrm{PSh}(T)$ the Beck–Chevalley maps are equivalences by the explicit description of the adjoints $f_!\colon (\mathrm{Sp} \otimes \mathcal{C})(X) \to (\mathrm{Sp} \otimes \mathcal{C})(Y)$ given above.

For the second statement, we first observe that $\tilde{F}$ is fiberwise cocontinuous by definition. Given now any admissible $f\colon X \to Y$, the mate of the total square in

$$\begin{array}{ccc} \mathcal{C}(Y) & \xleftarrow{\;f^*\;} & \mathcal{C}(X) \\ \Sigma^\infty \big\downarrow & & \big\downarrow \Sigma^\infty \\ \mathrm{Sp} \otimes \mathcal{C}(Y) & \xleftarrow{\;f^*\;} & \mathrm{Sp} \otimes \mathcal{C}(X) \\ \tilde{F} \big\downarrow & & \big\downarrow \tilde{F} \\ \mathcal{D}(Y) & \xleftarrow[\;f^*\;]{} & \mathcal{D}(X) \end{array}$$

is the Beck–Chevalley map $f_! F \Rightarrow F f_!$, hence an equivalence by S-cocontinuity of F, and similarly the mate of the top square is an equivalence by the above. By the compatibility of mates with pastings, we conclude that the Beck–Chevalley map $f_!\tilde{F} \Rightarrow \tilde{F} f_!$ becomes an equivalence after precomposition with $\Sigma^\infty\colon \mathcal{C}(X) \to \mathrm{Sp}(\mathcal{C}(X))$. However, both $f_!\tilde{F}$ and $\tilde{F} f_!$ are cocontinuous functors, so the claim follows from the universal property of $\mathrm{Sp} \otimes -$ (cf. the previous lemma). □

Let us restate the key step in the above proof separately for easy reference:

Corollary 8.6. *Let $\mathcal{C}, \mathcal{D}$ be as above. Then a fiberwise cocontinuous functor $F\colon \mathrm{Sp} \otimes \mathcal{C} \to \mathcal{D}$ is S-cocontinuous if and only if $F \circ \Sigma^\infty\colon \mathcal{C} \to \mathcal{D}$ is so.* □

Finally, let us move to the setting of *genuine* stability:

Definition 8.7. Let $P \subset T$ be atomic orbital. A T-category $\mathcal{C}$ is called *P-stable* if it is P-semiadditive (Definition 6.5) and fiberwise stable (Definition 8.1).

Lemma 8.8. *Let $P \subset S$ be atomic orbital in T and let $\mathcal{C}$ be a P-semiadditive S-cocomplete T-category. Then $\mathrm{Sp} \otimes \mathcal{C}$ is P-stable.*

Proof. We already know that $\mathrm{Sp}\otimes\mathcal{C}$ is S-cocomplete and fiberwise stable. Moreover, its underlying S-category is P-semiadditive by [CLL23, Lemma 6.2.6], so $\mathrm{Sp} \otimes \mathcal{C}$ is also P-semiadditive as a T-category by Remark 6.10. □

Definition 8.9. We define $\underline{\mathrm{Sp}}^P_{S \triangleright T} := \mathrm{Sp} \otimes \underline{\mathrm{CMon}}^P_{S \triangleright T}$, and we write Σ^∞_+ for the composite

$$\underline{\mathrm{Spc}}_{S\triangleright T} \xrightarrow{\mathbb{P}} \underline{\mathrm{CMon}}^P_{S\triangleright T} \xrightarrow{\Sigma^\infty} \mathrm{Sp} \otimes \underline{\mathrm{CMon}}^P_{S\triangleright T} = \underline{\mathrm{Sp}}^P_{S\triangleright T}.$$

Remark 8.10. Note that Σ^∞_+ is by construction an extension of the S-functor $\Sigma^\infty_+ := \Sigma^\infty \circ \mathbb{P} \colon \underline{\mathrm{Spc}}_S \to \mathrm{Sp} \otimes \underline{\mathrm{CMon}}^P_S = \underline{\mathrm{Sp}}^P_S$ from [CLL23, Definition 6.2.12].

Combining the above fiberwise results with the universal property of $\underline{\mathrm{CMon}}^P_{S\triangleright T}$ from Theorem 6.19 we get:

Theorem 8.11. *The T-category $\underline{\mathrm{Sp}}^P_{S\triangleright T}$ is S-presentable and P-stable. For any S-cocomplete P-stable T-category $\mathcal{D}$ evaluation at $\mathbb{S} := \Sigma^\infty_+(*)$ induces an equivalence $\underline{\mathrm{Fun}}^{S\text{-cc}}_T(\underline{\mathrm{Sp}}^P_{S\triangleright T}, \mathcal{D}) \simeq \mathcal{D}$.* □

We can also compare this to $\underline{\mathrm{Sp}}^P_T$:

Theorem 8.12. *The essentially unique S-cocontinuous functor $\iota_! \colon \underline{\mathrm{Sp}}^P_{S\triangleright T} \to \underline{\mathrm{Sp}}^P_T$ preserving $\mathbb{S}$ is fully faithful. Moreover, it admits an S-right adjoint ι^*, which in turn admits a further S-right adjoint ι_* (again fully faithful for formal reasons).*

Proof. The functor $\mathrm{Sp}\otimes\iota_! \colon \underline{\mathrm{Sp}}^P_{S\triangleright T} = \mathrm{Sp}\otimes\underline{\mathrm{CMon}}^P_{S\triangleright T} \to \mathrm{Sp}\otimes\underline{\mathrm{CMon}}^P_T = \underline{\mathrm{Sp}}^P_T$ admits an S-right adjoint given by $\mathrm{Sp}\otimes\iota^*$ (as ι^* is itself an S-left adjoint). For each $A \in T$ the unit $\mathrm{id} \to \mathrm{Sp}\otimes(\iota^*\iota_!)$ is then induced by the unit of $\iota_! \dashv \iota^*$, so it is an equivalence as $\iota_!$ is fully faithful (Theorem 6.18). Thus, also $\mathrm{Sp} \otimes \iota_!$ is fully faithful. Moreover, it sends $\Sigma^\infty_+(*)$ to $\Sigma^\infty_+(*)$ simply by naturality, so this is the functor $\underline{\mathrm{Sp}}^P_{S\triangleright T} \to \underline{\mathrm{Sp}}^P_T$ in question.

It only remains to show that also $\mathrm{Sp} \otimes \iota^*$ admits an S-right adjoint. However, by construction it admits a pointwise right adjoint, and it is moreover S-cocontinuous as a consequence of Corollary 8.6 (for $T = S$), so the claim follows. □

9. The universal property of equivariant spectra

In this section, we will describe the universal equivariantly presentable equivariantly stable (i.e. Orb-stable) global category in terms of classical equivariant stable homotopy theory.

9.1. *G*-equivariant spectra.

We start by introducing the global category of equivariant spectra, and state our main results.

Definition 9.1. We write **Spectra** for the 1-category of *symmetric spectra* [HSS00] in simplicial sets. For any finite G, we write ***G*-Spectra** for the category of G-objects; by slight abuse of language, we will refer to its objects simply as G-spectra.

We refer the reader to [Hau17, Definition 2.35] for the definition of the *G-stable weak equivalences* of G-spectra. Below, we will simply refer to these as *G-equivariant weak equivalences.*

Proposition 9.2 (See [Hau17, Theorem 4.8 and Proposition 4.9])**.** *The category* $\boldsymbol{G}$**-Spectra** *carries a model structure with*

(1) weak equivalences the G-equivariant weak equivalences
(2) acyclic fibrations those maps f such that f_n is a $\mathcal{G}_{G,\Sigma_n}$-acyclic fibration for every $n \geq 0$.

We call this the G-equivariant projective model structure*. It is combinatorial and stable.* □

All that we will need to know about this model structure below is that the sphere spectrum is cofibrant, which follows from [Hau17, discussion after Corollary 2.26] or by simply observing that the above acyclic fibrations are surjective in degree 0 and hence have the right lifting property against $0 \to \mathbb{S}$.

Lemma 9.3. *Let $\alpha\colon G \to G'$ be any homomorphism. Then* $\alpha^*\colon \boldsymbol{G'}$**-Spectra** $\to$ $\boldsymbol{G}$**-Spectra** *is left Quillen with respect to the above model structures.*

Proof. Factoring α, we may assume that it is either injective or surjective. In the first case, the claim is an instance of [Hau17, 5.2], while in the latter case it follows by combining 5.3 and 5.1 of *op. cit.* □

As before, we therefore get a global category $\underline{\mathscr{S}p}$ with $\mathscr{S}p_G := \underline{\mathscr{S}p}(G)$ the localization of (projectively cofibrant) G-spectra at the G-weak equivalences, and with structure maps given by the left derived functors $\mathbf{L}\alpha^*$. We will refer to this as the *global category of equivariant spectra*. It has a natural section $\mathbb{S}$ given by the equivariant sphere spectra (determined by the usual sphere in $\mathscr{S}p_1$).

Using this, we can now state our main results:

Theorem 9.4. *The global category $\underline{\mathscr{S}p}$ is equivariantly presentable and equivariantly stable. For any other equivariantly cocomplete equivariantly stable $\mathcal{D}$ evaluation at $\mathbb{S}$ defines an equivalence $\underline{\mathrm{Fun}}^{\text{eq-cc}}_{\mathrm{Glo}}(\underline{\mathscr{S}p}, \mathcal{D}) \simeq \mathcal{D}$.*

Theorem 9.5. *The essentially unique equivariantly cocontinuous global functor $\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \to \underline{\mathscr{S}p}$ sending $\mathbb{S}$ to $\mathbb{S}$ is an equivalence.*

The proof will be given at the end of this section. For now let us stop to observe that some pleasant properties one might have hoped for $\underline{\mathrm{Sp}}^{P}_{S\triangleright T}$ to satisfy do not hold even for $\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \simeq \underline{\mathscr{S}p}$:

Warning 9.6. For any $f\colon G \to G'$ the functor $\mathbf{L}f^*\colon \mathscr{S}p_{G'} \to \mathscr{S}p_G$ admits a right adjoint $\mathbf{R}f_*$ by Lemma 9.3. However, these do *not* satisfy the Beck–Chevalley condition in general (i.e. $\underline{\mathscr{S}p}$ does not have finite *global* products). To see this, consider the pullback

$$\begin{array}{ccc} \mathbb{Z}/2 \times \mathbb{Z}/2 & \xrightarrow{\mathrm{pr}_2} & \mathbb{Z}/2 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ \mathbb{Z}/2 & \xrightarrow[q]{} & 1 \end{array}$$

in Glo, giving rise to a map $\mathbf{L}q^*\mathbf{R}q_*X \to \mathbf{R}\mathrm{pr}_{2*}\mathbf{L}\mathrm{pr}_1^*X$ for any $X \in \mathscr{S}p_{\mathbb{Z}/2}$; we will now show that this cannot be an equivalence for $X = \mathbb{S}$ by computing the result of applying $\mathbf{R}q_*$ to both sides:

The functor $\mathbf{R}q_*$ is given by taking categorical $\mathbb{Z}/2$-fixed points, so the tom Dieck-splitting [tD75] tells us that

$$\mathbf{R}q_*\mathbb{S} \simeq \bigvee_{G \subset \mathbb{Z}/2} \Sigma^\infty_+ B\big((\mathbb{Z}/2)/G\big).$$

The right hand side is actually cofibrant, so $\mathbf{L}q^*\mathbf{R}q_*\mathbb{S}$ is simply given by equipping this with the trivial $\mathbb{Z}/2$-action. Accordingly, another application of the tom Dieck splitting shows

$$\mathbf{R}q_*\mathbf{L}q^*\mathbf{R}q_*\mathbb{S} \simeq \bigvee_{G \subset \mathbb{Z}/2} \bigvee_{H \subset \mathbb{Z}/2} \Sigma^\infty_+ \big(B\big((\mathbb{Z}/2)/G\big) \times B\big((\mathbb{Z}/2)/H\big)\big).$$

If we take π_0, then each wedge summand contributes a summand of $\mathbb{Z}$ (being the unreduced suspension of a connected space), so $\pi_0(\mathbf{R}q_*\mathbf{L}q^*\mathbf{R}q_*\mathbb{S})$ is free abelian of rank 4.

On the other hand, by uniqueness of adjoints $\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}$ agrees with $\mathbf{R}r_*$ for $r\colon \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$ the unique map, so $\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}\mathbf{L}\mathrm{pr}_1^*\mathbb{S}$ is given by the categorical $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-fixed points of $\mathbb{S}$. By another application of the tom Dieck splitting (or using the classical computation of the zeroth equivariant homotopy groups of $\mathbb{S}$ as the Burnside ring), we therefore see that $\pi_0(\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}\mathbf{L}\mathrm{pr}_1^*\mathbb{S}) \cong \pi_0^{\mathbb{Z}/2\times\mathbb{Z}/2}(\mathbb{S})$ is free abelian of rank the number of subgroups of $\mathbb{Z}/2 \times \mathbb{Z}/2$, which is 5 instead of 4.

Remark 9.7. The extra $\mathbb{Z}$-summand in $\pi_0(\mathbf{R}q_*\mathbf{R}\mathrm{pr}_{2*}\mathbf{L}\mathrm{pr}_1^*\mathbb{S})$ can be attributed to the fact that $\mathbb{Z}/2 \times \mathbb{Z}/2$ has a subgroup that is not given as a product of subgroups of its factors, namely the diagonal subgroup. A similar phenomenon appears for general G, and as observed in [Nic22] this is what prevents the tom Dieck map

$$\bigvee_{(H \subset G)/\mathrm{conj.}} \Sigma^\infty\big(E(W_GH) \wedge_{W_GH} X^H\big) \to F^G\Sigma^\infty X \tag{15}$$

for a pointed G-simplicial set X from being a *global weak equivalence* instead of just a non-equivariant weak equivalence: after taking categorical K-fixed points on both sides, the left hand side only contains the wedge summands of the tom Dieck splitting of $F^{K\times G}\Sigma^\infty X$ corresponding to subgroups of the form $L \times H \subset K \times G$ for $L \subset K, H \subset G$. In fact, this is the only obstruction to (15) being a global weak equivalence, see *op. cit.* for details.

Warning 9.8. $\underline{\mathscr{S}p}$ is neither globally cocomplete nor fiberwise complete, and hence neither is $\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}}$ by Theorem 9.5. In fact, already the restriction functor $\mathbf{L}q^*\colon \mathscr{S}p_1 \to \mathscr{S}p_{\mathbb{Z}/2}$ induced by the unique map $q\colon \mathbb{Z}/2 \to 1$ does not preserve all products, and in particular it does not admit a left adjoint. The third author learned the following argument for this fact from Denis Nardin: By [BDS16, Theorem 3.3], $\mathbf{L}q^*$ preserves all products if and only if $\mathbf{R}q_*$ preserves compact objects. However, as observed above $\mathbf{R}q_*\mathbb{S}$ contains $\Sigma^\infty_+ B(\mathbb{Z}/2)$ as a wedge summand. As the latter is not compact, neither is $\mathbf{R}q_*\mathbb{S}$, yielding the desired contradiction. A similar argument shows that $\mathbf{L}q^*$ does not have a left adjoint whenever q has a non-trivial kernel.

9.2. ***G*-global spectra.** As before, the proof of Theorems 9.4 and 9.5 will proceed via comparison with a model of the universal *globally* presentable equivariantly stable category.

Definition 9.9. A map $f\colon X \to Y$ in $\boldsymbol{G}$**-Spectra** is called a *G-global weak equivalence* if $\alpha^* f$ is an H-equivariant weak equivalence for every finite group H and every homomorphism $\alpha\colon H \to G$.

We emphasize that we are *not* deriving α^* here with respect to the equivariant model structures (as otherwise this would of course simply recover the G-weak equivalences again).

Proposition 9.10 (See [Len20, Corollary 3.1.46–Proposition 3.1.48])**.** *The category* $\boldsymbol{G}$**-Spectra** *admits a model structure with*

(1) weak equivalences the G-global weak equivalences
(2) cofibrations the injective cofibrations.

We call this the injective G-global model structure*. It is combinatorial, simplicial, proper, and stable.* □

Basically by definition, the restriction functors $\alpha^*\colon \boldsymbol{G'}$**-Spectra** $\to$ $\boldsymbol{G}$**-Spectra** are homotopical and left Quillen. In particular, we again obtain a global category $\underline{\mathscr{S}p}^{\mathrm{gl}}$.

Theorem 9.11 (See [CLL23, Corollary 7.3.3])**.** *$\underline{\mathscr{S}p}$ is globally presentable and equivariantly stable. The essentially unique globally cocontinuous functor $\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Glo}} \to \underline{\mathscr{S}p}$ sending $\Sigma^\infty_+(*)$ to the global sphere spectrum $\mathbb{S}$ is an equivalence.* □

9.3. **Proof of Theorems 9.4 and 9.5.** Let us begin with a comparison of the above models complementing Theorem 8.12:

Lemma 9.12. *There is a global functor $\mathbf{L}\,\mathrm{id}\colon \underline{\mathscr{S}p} \to \underline{\mathscr{S}p}^{\mathrm{gl}}$ with the following properties:*

(1) It is fully faithful and sends $\mathbb{S}$ to $\mathbb{S}$.
(2) It admits an Orb*-right adjoint.*

Proof. For any G, [Len20, Proposition 3.3.1] provides a Quillen adjunction

$$\mathrm{id}\colon \boldsymbol{G}\textbf{-Spectra}_{G\text{-equiv. proj.}} \rightleftarrows \boldsymbol{G}\textbf{-Spectra}_{G\text{-gl. inj.}} :\mathrm{id}\,. \tag{16}$$

In particular, G-equivariant weak equivalences between projectively cofibrant spectra are G-global weak equivalences (also see Lemma 9.3), so the inclusion of projectively cofibrant objects yields a functor $\mathbf{L}\,\mathrm{id}\colon \underline{\mathscr{S}p} \to \underline{\mathscr{S}p}^{\mathrm{gl}}$ sending $\mathbb{S}$ to $\mathbb{S}$. Moreover, the right adjoint in (16) evidently induces a localization, so that $\mathbf{L}\,\mathrm{id}$ is fully faithful.

It only remains that the right adjoints assemble into an Orb-right adjoint. However, the pointset level Beck–Chevalley maps $\alpha^*\circ\mathrm{id} \Rightarrow \mathrm{id}\circ\alpha^*$ are isomorphisms for trivial reasons, and for *injective* α, α^* is also homotopical in the equivariant world [Hau17, 5.2], so that this already models the derived Beck–Chevalley map. □

Proof of Theorems 9.4 and 9.5. By Theorem 8.11 it is enough to prove that $\underline{\mathscr{S}p}$ is equivariantly stable and equivariantly cocomplete, and that the preferred map $\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \to \underline{\mathscr{S}p}$ is an equivalence.

For this, let us write $\mathcal{E}$ for the essential image of $\mathbf{L}\,\mathrm{id}\colon \underline{\mathscr{S}\!p} \to \underline{\mathscr{S}\!p}^{\mathrm{gl}}$; this is then closed under equivariant colimits as $\mathbf{L}\,\mathrm{id}$ admits an Orb-right adjoint, and it is closed under desuspension as each $\mathscr{S}\!p_G$ is stable. It follows that $\mathcal{E}$ and hence also $\underline{\mathscr{S}\!p}$ is indeed equivariantly cocomplete and equivariantly stable.

Now let $F\colon \underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \to \underline{\mathscr{S}\!p}$ be the unique equivariantly cocontinuous functor preserving $\mathbb{S}$. Then $\mathbf{L}\,\mathrm{id}\circ F\colon \underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \to \underline{\mathscr{S}\!p}^{\mathrm{gl}}$ is an equivariantly cocontinuous functor sending $\mathbb{S}$ to $\mathbb{S}$. The same holds for the composite

$$\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}} \xrightarrow{\iota_!} \underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Glo}} \xrightarrow{\sim} \underline{\mathscr{S}\!p}^{\mathrm{gl}}$$

of the fully faithful functor from Theorem 7.16 with the equivalence from Theorem 9.11, so they have to agree by the universal property of $\underline{\mathrm{Sp}}^{\mathrm{Orb}}_{\mathrm{Orb}\triangleright\mathrm{Glo}}$. In particular, F is fully faithful. To see that it is also essentially surjective, it is by [Hau17, Proposition 4.9] enough to see that it hits the suspension spectra $\Sigma^\infty_+(G/H)$ for all $H \subset G$. However, as before we have $i_!\mathbb{S} \simeq \Sigma^\infty_+(G/H)$ for $i\colon H \hookrightarrow G$ the inclusion, so the claim follows from the defining properties of F. □

Again this immediately implies a variant for the G-category of G-spectra for any finite group G:

Theorem 9.13. *Recall the functor $\upsilon_G\colon \mathrm{Orb}_G \to \mathrm{Glo}$ from Construction 5.11.*

(1) The G-category $\upsilon_G^*\underline{\mathscr{S}\!p}$ (sending G/H to $\mathscr{S}\!p_H$) is G-presentable and G-stable. Moreover, the unique left adjoint $\underline{\mathrm{Sp}}_{\mathrm{Orb}_G} \to \upsilon_G^*\underline{\mathscr{S}\!p}$ preserving $\mathbb{S}$ is an equivalence.

(2) For any G-cocomplete G-stable $\mathcal{D}$, evaluation at $\mathbb{S}$ defines an equivalence $\underline{\mathrm{Fun}}^{G\text{-cc}}_{\mathrm{Orb}_G}(\upsilon_G^*\underline{\mathscr{S}\!p}, \mathcal{D}) \simeq \mathcal{D}$.

A proof of this has previously been sketched by Nardin as [Nar16, Theorem A.4].

Proof. Arguing as in the unstable (Theorem 5.12) and semiadditive case (Theorem 7.29), it only remains to show that there is for any atomic orbital $P \subset T$ and $A \in T$ an equivalence $\pi_A^*\underline{\mathrm{Sp}}^P_T \simeq \underline{\mathrm{Sp}}^{T^P_{/A}}_{T_{/A}}$ preserving $\mathbb{S}$. This however follows at once from Proposition 7.30 by applying $\mathrm{Sp}\otimes -$ to both sides. □

References

[ABFJ22] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal, *Left-exact localizations of ∞-topoi. I: Higher sheaves*, Adv. Math. **400** (2022), 64, Id/No 108268.

[BD20] Paul Balmer and Ivo Dell'Ambrogio, *Mackey 2-functors and Mackey 2-motives*, EMS Monogr. Math., European Mathematical Society (EMS), 2020.

[BDG+16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, *Parametrized higher category theory and higher algebra: Exposé I – Elements of parametrized higher category theory*, arXiv:1608.03657 (2016).

[BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders, *Grothendieck-Neeman duality and the Wirthmüller isomorphism*, Compos. Math. **152** (2016), no. 8, 1740–1776.

[CLL23] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens, *Parametrized stability and the universal property of global spectra*, arXiv:2301.08240 (2023).

[Elm83] Anthony D. Elmendorf, *Systems of fixed point sets*, Trans. Am. Math. Soc. **277** (1983), 275–284.

[Hau17] Markus Hausmann, *G-symmetric spectra, semistability and the multiplicative norm*, J. Pure Appl. Algebra **221** (2017), no. 10, 2582–2632.

[Hau19] ———, *Symmetric spectra model global homotopy theory of finite groups*, Algebr. Geom. Topol. **19** (2019), no. 3, 1413–1452.

[Hil22] Kaif Hilman, *Parametrised presentability over orbital categories*, arXiv:2202.02594 (2022).

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra*, J. Am. Math. Soc. **13** (2000), no. 1, 149–208.

[HY17] Asaf Horev and Lior Yanovski, *On conjugates and adjoint descent*, Topology Appl. **232** (2017), 140–154.

[Len20] Tobias Lenz, *G-Global Homotopy Theory and Algebraic K-Theory*, to appear in *Mem. Amer. Math. Soc.*, arXiv:2012.12676 (2020).

[Lin23] Sil Linskens, *Globalizing and stabilizing global ∞-categories*, in preparation, 2023.

[LNP22] Sil Linskens, Denis Nardin, and Luca Pol, *Global homotopy theory via partially lax limits*, arXiv:2206.01556 (2022).

[Lur09] Jacob Lurie, *Higher topos theory*, Ann. Math. Stud., vol. 170, Princeton University Press, Princeton, NJ, 2009, updated version available at `https://people.math.harvard.edu/~lurie/papers/highertopoi.pdf`.

[Lur17] ———, *Higher algebra*, `https://www.math.ias.edu/~lurie/papers/HA.pdf` (2017).

[Lur18] ———, *Spectral Algebraic Geometry*, under construction (version dated February 2018), `www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf` (2018).

[Mar21] Louis Martini, *Yoneda's lemma for internal higher categories*, arXiv:2103.17141 (2021).

[MMO17] J. Peter May, Mona Merling, and Angélica M. Osorno, *Equivariant Infinite Loop Space Theory, the Space Level Story*, to appear in *Mem. Amer. Math. Soc.*, available as arXiv:1704.03413 (2017).

[MW21] Louis Martini and Sebastian Wolf, *Limits and colimits in internal higher category theory*, arXiv:2111.14495 (2021).

[MW22] ———, *Presentable categories internal to an ∞-topos*, arXiv:2209.05103 (2022).

[Nar16] Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category*, arXiv:1608.07704 (2016).

[Nar17] ———, *Stability and distributivity over orbital ∞-categories*, Ph.D. thesis, Massachusetts Institute of Technology, 2017.

[Nic22] Jana Nickel, *The tom Dieck splitting in global equivariant stable homotopy theory*, Master's thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2022.

[Ost16] Dominik Ostermayr, *Equivariant Γ-spaces*, Homology Homotopy Appl. **18** (2016), no. 1, 295–324.

[Sch18] Stefan Schwede, *Global homotopy theory*, New Math. Monogr., vol. 34, Cambridge University Press, 2018.

[Sha21] Jay Shah, *Parametrized higher category theory*, to appear in *Algebr. Geom. Topol.*, arXiv:1809.05892 (2021).

[Shi89] Kazuhisa Shimakawa, *Infinite Loop G-Spaces Associated to Monoidal G-Graded Categories*, Publ. Res. Inst. Math. Sci. **25** (1989), no. 2, 239–262.

[Shi91] ———, *A note on Γ_G-spaces*, Osaka J. Math. **28** (1991), no. 2, 223–228.

[Ste16] Marc Stephan, *On equivariant homotopy theory for model categories*, Homology Homotopy Appl. **18** (2016), no. 2, 183–208.

[tD75] Tammo tom Dieck, *Orbittypen und äquivariante Homologie. II*, Arch. Math. **26** (1975), 650–662.

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PARAMETRIZED HIGHER SEMIADDITIVITY AND THE UNIVERSALITY OF SPANS

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Abstract. Using the framework of ambidexterity developed by Hopkins and Lurie, we introduce a parametrized analogue of higher semiadditivity called $\mathcal{Q}$-*semiadditivity*, depending on a chosen class of morphisms $\mathcal{Q}$. Our first main result identifies the free $\mathcal{Q}$-semiadditive parametrized category on a single generator with a certain parametrized span category $\underline{\mathrm{Span}}(\mathcal{Q})$, simultaneously generalizing a result of Harpaz in the non-parametrized setting and a result of Nardin in the equivariant setting. As a consequence, we deduce that the $\mathcal{Q}$-semiadditive completion of a parametrized category $\mathcal{C}$ consists of the $\mathcal{Q}$-*commutative monoids in* $\mathcal{C}$, defined as $\mathcal{Q}$-limit preserving parametrized functors from $\underline{\mathrm{Span}}(\mathcal{Q})$ to $\mathcal{C}$.

As our second main result, we provide an explicit 'Mackey sheaf' description of the free presentable $\mathcal{Q}$-semiadditive category. Using this, we reprove the Mackey functor description of global spectra first obtained by the second-named author and generalize it to G-global spectra. Moreover, we obtain universal characterizations of the categories of $\mathbb{Z}$-valued G-Mackey profunctors and of quasi-finitely genuine G-spectra as studied by Kaledin and Krause–McCandless–Nikolaus, respectively.

Contents

1. Introduction

The notion of *ambidexterity*, introduced by Hopkins and Lurie [HL13], is a vast generalization of the notion of semiadditivity in category theory.[1] Recall that a

[1]Throughout this article, we will say 'category' for '∞-category.'

category $\mathcal{C}$ is called *semiadditive* if it admits finite products and finite coproducts which canonically agree with each other. Thinking of finite products (resp. coproducts) as limits (resp. colimits) indexed by a finite set X, this can be expressed as the condition that a certain norm map $\mathrm{Nm}_X\colon \mathrm{colim}_X \to \lim_X$ between the X-indexed limit and colimit functors is a natural equivalence. The formalism of ambidexterity allows one to consider such properties for a broader class of indexing objects X, leading to a diverse range of generalized notions of semiadditivity, including *higher semiadditivity* [HL13, Har20, CSY22], *tempered ambidexterity* [Lur19], and *equivariant semiadditivity* [Nar16, QS21, CLL23a].

The concept of semiadditivity is closely related to the algebraic structure of commutative monoids. For example, a category with finite products is semiadditive if and only if it admits an enrichment in the category of commutative monoids. At the heart of this lies the fact that, when $\mathcal{C}$ admits finite products, the category $\mathrm{CMon}(\mathcal{C})$ of commutative monoids in $\mathcal{C}$ is the universal semiadditive category admitting a functor to $\mathcal{C}$ which preserves finite products. We might summarize this by saying that $\mathrm{CMon}(\mathcal{C})$ is the *semiadditive completion* of $\mathcal{C}$.

Each of the generalized notions of semiadditivity mentioned above comes with its own generalized notion of commutative monoid: in the case of higher semiadditivity these are known as *m-commutative monoids*, while for equivariant semiadditivity these usually go under the name of *Mackey functors*. It turns out that also in these two cases the semiadditive completion of a category $\mathcal{C}$ is given by the commutative monoids in $\mathcal{C}$. For m-commutative monoids this universal property was established by Harpaz [Har20], and for Mackey functors this was done by Nardin [Nar16].

The goal of this article is to show that the above phenomenon is not specific to these two examples and occurs for a large family of notions of (higher) semiadditivity in the context of parametrized category theory. Given an (∞-)topos $\mathcal{B}$, we refer to a limit-preserving functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ as a *$\mathcal{B}$-category*. The generalized notions of semiadditivity we consider depend on a choice of a so-called *locally inductible* subcategory $\mathcal{Q}$ of $\mathcal{B}$, meaning that $\mathcal{Q}$ is a local class of locally truncated morphisms in $\mathcal{C}$ which is closed under diagonals, see Definition 3.1. We then define:

Definition ($\mathcal{Q}$-semiadditivity)**.** Given a locally inductible subcategory $\mathcal{Q} \subseteq \mathcal{B}$, we say that a $\mathcal{B}$-category $\mathcal{C}$ is *$\mathcal{Q}$-semiadditive* if the following conditions are satisfied:

(1) It admits *$\mathcal{Q}$-colimits*: The functors $q^* \coloneqq \mathcal{C}(q)\colon \mathcal{C}(B) \to \mathcal{C}(A)$ for $q\colon A \to B$ in $\mathcal{Q}$ admit left adjoints $q_!\colon \mathcal{C}(A) \to \mathcal{C}(B)$ satisfying base change;
(2) It satisfies *ambidexterity* for $\mathcal{Q}$: For every $n \geq -2$ and any n-truncated morphism $q\colon A \to B$ in $\mathcal{Q}$ an inductively defined transformation $\widetilde{\mathrm{Nm}}_q\colon q^*q_! \to \mathrm{id}_{\mathcal{C}(A)}$ exhibits the left adjoint $q_!$ additionally as a *right* adjoint to q^*.

We are mostly interested in the case of presheaf topoi $\mathcal{B} = \mathrm{PSh}(T)$ for a small category T. In this case, the data of a $\mathcal{B}$-category is equivalent to that of a functor $T^{\mathrm{op}} \to \mathrm{Cat}$ by restricting to representables; the original functor $\mathrm{PSh}(T) \to \mathrm{Cat}$ is recovered via limit-extension. In this situation, it usually suffices to check conditions (1) and (2) only for a much smaller subcategory $Q \subseteq \mathrm{PSh}(T)$ that 'generates' $\mathcal{Q}$ in a suitable sense; we will speak of *Q-semiadditivity* in this case. For example:

- When $T = *$, a functor $T^{\mathrm{op}} \to \mathrm{Cat}$ is just a category. Taking Q to be the subcategory $\mathrm{Fin} \subseteq \mathrm{Spc} = \mathrm{PSh}(*)$ of *finite sets*, condition (1) demands the existence of finite coproducts, while condition (2) asks that finite coproducts are also exhibited as products (via some preferred map). In other words: Fin-semiadditivity is precisely ordinary semiadditivity for categories.

- More generally, taking Q to be the subcategory $\mathrm{Spc}_m \subseteq \mathrm{Spc}$ of *m-finite spaces*[2] for $-2 \leq m < \infty$, we recover the notion of m-semiadditivity: a category is m-semiadditive if and only if (1) it admits A-indexed colimits for every m-finite space A, and (2) if these colimits are also exhibited as limits via a preferred map.
- For a finite group G, contravariant functors $\mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Cat}$ from the orbit category of G are known as *G-categories* [BDG+16]. Taking Q to be the subcategory $\mathrm{Fin}_G \subseteq \mathrm{Spc}_G = \mathrm{PSh}(\mathrm{Orb}_G)$ of *finite G-sets* precisely recovers the notion of G-semiadditivity introduced by Nardin [Nar16].
- Various variations are possible, including p-typical m-semiadditivity, equivariant semiadditivity, global semiadditivity and very G-semiadditivity for an arbitrary group G; see Section 3.4.

Our first main result identifies the free $\mathcal{Q}$-semiadditive $\mathcal{B}$-category with a certain parametrized span category $\underline{\mathrm{Span}}(\mathcal{Q})$. Recall that the *span category* $\mathrm{Span}(\mathcal{C})$ of a category $\mathcal{C}$ with pullbacks is a category with the same objects as $\mathcal{C}$, but where a morphism from X to Y is given by a *span* $X \leftarrow Z \to Y$ in $\mathcal{C}$; composition is given via pullback. The assignment $A \mapsto \mathrm{Span}(\mathcal{Q}_{/A})$ defines a $\mathcal{B}$-category $\underline{\mathrm{Span}}(\mathcal{Q}) \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$, where the functoriality in A is given by the pullback functors $f^* \colon \mathrm{Span}(\mathcal{Q}_{/B}) \to \mathrm{Span}(\mathcal{Q}_{/A})$ for $f \colon A \to B$. This $\mathcal{B}$-category is $\mathcal{Q}$-semiadditive: the functor f^* admits a left adjoint given by applying $\mathrm{Span}(-)$ to the postcomposition functor $f_! \colon \mathcal{Q}_{/A} \to \mathcal{Q}_{/B}$, and by self-duality of span categories this is also a *right* adjoint of f^*. We then show:

Theorem A (Theorem 5.1)**.** *The $\mathcal{B}$-category $\underline{\mathrm{Span}}(\mathcal{Q})$ is the free $\mathcal{Q}$-semiadditive $\mathcal{B}$-category on a single generator.*

More precisely, this means that for every $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{D}$, evaluation at the identity maps $\mathrm{id}_A \in \mathrm{Span}(\mathcal{Q}_{/A})$ induces an equivalence of parametrized categories

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Here the left-hand side denotes the full subcategory of the parametrized functor category spanned by the functors which are *$\mathcal{Q}$-continuous*, meaning that they commute with the rights adjoints of q^*.

To appreciate the generality of Theorem A, let us explain how it recovers various results previously established in the literature. As before we take $\mathcal{B} = \mathrm{PSh}(T)$ and assume $\mathcal{Q}$ is generated by some smaller subcategory Q.

- For $T = *$ and $Q = \mathrm{Fin}$, we recover the fact that the span category $\mathrm{Span}(\mathrm{Fin})$ of finite sets is the free semiadditive category;
- For $T = *$ and $Q = \mathrm{Spc}_m$, we obtain the fact that the span category $\mathrm{Span}(\mathrm{Spc}_m)$ of m-finite spaces is the free m-semiadditive category, as was previously established by Harpaz [Har20, Theorem 1.1]. In fact, our proof strategy for Theorem A mostly parallels Harpaz's arguments.

[2] Recall that a space is m-finite if it is m-truncated, has finitely many path components, and has finite homotopy groups.

- Taking $T = Q = \mathrm{Spc}_m$, we get a strengthening of Harpaz's result: if we embed m-semiadditive categories into Spc_m-semiadditive Spc_m-categories by sending a category $\mathcal{C}$ to the functor $\mathrm{Fun}(-, \mathcal{C})\colon \mathrm{Spc}_m^{\mathrm{op}} \to \mathrm{Cat}$, then Theorem A shows that the image of $\mathrm{Span}(\mathrm{Spc}_m)$ is even free among *all* Spc_m-semiadditive Spc_m-categories, not just those coming from m-semiadditive categories. This strengthening will be crucial in forthcoming work by Ben-Moshe on transchromatic characters.
- Taking $T = \mathrm{Orb}_G$ and $Q = \mathrm{Fin}_G$ for a finite group G, we recover the statement that the G-category $G/H \mapsto \mathrm{Span}(\mathrm{Fin}_H)$ is the free G-semiadditive G-category, as was previously established by Nardin [Nar16, Theorem 6.5, Proposition 5.11].

If $\mathcal{C}$ is a $\mathcal{B}$-category admitting $\mathcal{Q}$-limits (the dual of condition (1) in the above definition), we define the $\mathcal{B}$-category of *$\mathcal{Q}$-commutative monoids in $\mathcal{C}$* as

$$\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}) \coloneqq \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{C}).$$

Building on Theorem A, we prove:

Theorem B (Theorem 7.4,Theorem 7.27)**.** For a $\mathcal{B}$-category $\mathcal{C}$ admitting $\mathcal{Q}$-limits, the forgetful functor $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}) \to \mathcal{C}$ is terminal among $\mathcal{Q}$-limit-preserving functors $\mathcal{D} \to \mathcal{C}$ from a $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{D}$:

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{D}, \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{D}, \mathcal{C}).$$

We will say that $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})$ is the *$\mathcal{Q}$-semiadditive completion* of $\mathcal{C}$.

Moreover, if $\mathcal{C}$ is presentable (and $\mathcal{Q}$ satisfies a mild smallness condition), then also $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})$ is presentable, and the forgetful functor admits a left adjoint $\mathcal{C} \to \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})$ which is initial among left adjoint functors $\mathcal{C} \to \mathcal{D}$ into a presentable $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{D}$:

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

$\mathcal{Q}$-commutative monoids as Mackey sheaves. Even though $\mathcal{Q}$-commutative monoids in $\mathcal{C}$ are defined as certain *parametrized* functors, we will show as our second main result that they admit a concrete non-parametrized description in terms of *Mackey sheaves* for suitable choices of $\mathcal{C}$. If $\mathcal{E}$ is a presentable category, we may consider the presentable $\mathcal{B}$-category $\underline{\mathrm{Shv}}(\mathcal{B}; \mathcal{E})$ of *$\mathcal{E}$-valued sheaves*, defined by assigning to $B \in \mathcal{B}$ the category $\mathrm{Shv}(\mathcal{B}_{/B}; \mathcal{E}) = \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}_{/B}, \mathcal{E})$ of $\mathcal{E}$-valued sheaves on the slice category $\mathcal{B}_{/B}$. We then show that a $\mathcal{Q}$-commutative monoid in $\underline{\mathrm{Shv}}(\mathcal{B}; \mathcal{E})$ is equivalently given by a *$\mathcal{E}$-valued Mackey sheaf on* $\mathcal{B}$, defined as a functor $F\colon \mathrm{Span}(\mathcal{B}, \mathcal{B}, \mathcal{Q}) \to \mathcal{E}$ whose restriction to the subcategory $\mathcal{B}^{\mathrm{op}} \simeq \mathrm{Span}(\mathcal{B}, \mathcal{B}, \iota\mathcal{B})$ preserves limits. In fact, assembling the $\mathcal{E}$-valued Mackey functors on all slices $\mathcal{B}_{/B}$ into a $\mathcal{B}$-category $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B}; \mathcal{E})$, we prove:

Theorem C (Theorem 8.2)**.** For every presentable category $\mathcal{E}$, there exists a natural equivalence

$$\underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{Shv}}(\mathcal{B}; \mathcal{E})) \simeq \underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B}; \mathcal{E}).$$

When $\mathcal{E}$ is the category Spc of ∞-groupoids, $\underline{\mathrm{Shv}}(\mathcal{B}; \mathrm{Spc})$ is the $\mathcal{B}$-category $\underline{\mathrm{Spc}}_{\mathcal{B}}$ of $\mathcal{B}$-groupoids, and combining Theorem C with Theorem B we deduce that every $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{C}$ is canonically enriched in Mackey sheaves: the parametrized Hom-functor $\underline{\mathrm{Hom}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \underline{\mathrm{Spc}}_{\mathcal{B}}$ uniquely lifts to $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B}; \mathrm{Spc})$.

In practice, the data of a Mackey sheaf can be significantly reduced, leading to simpler descriptions more in line with classical Mackey functors. We explain this reduction in Section 9.1.

Global Mackey functors. We then apply Theorem C to give Mackey functor descriptions of various categories considered in (global) equivariant homotopy theory. In particular, we use the universal property of global spectra established in [CLL23a] to deduce the following result which has also been concurrently proven by Pützstück [Pü24]:

Theorem D (Theorem 9.11)**.** Let G be any finite group. Then the category of G*-global spectra* from [Len20] is naturally equivalent to

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathscr{F}_{/BG}, \mathscr{F}_{/BG}, \mathscr{F}_{/BG}[\mathscr{F}_{\dagger}]), \mathrm{Sp})$$

where $\mathscr{F}_{/BG}$ is the $(2,1)$-category of finite groupoids over BG, and $\mathscr{F}_{/BG}[\mathscr{F}_{\dagger}]$ denotes the wide subcategory of faithful functors.

We further use our results in [CLL23b] to reprove the Mackey functor description of G-equivariant stable homotopy theory from [CMNN20], see Corollary 9.16.

Mackey profunctors. As a new application of our results, we further generalize the aforementioned Mackey functor description of equivariant stable homotopy theory to give for any discrete group G a parametrized interpretation of the category of *quasi-finitely genuine* G*-spectra* as defined by Krause–McCandless–Nikolaus [KMN23] building on work of Kaledin [Kal22].

For this, let $\mathrm{QFin}_G \subseteq \mathrm{Set}_G$ denote the full subcategory spanned by the *quasi-finite*[3] G-sets, i.e. those G-sets S for which all orbits are finite and for which the fixed point sets S^H are finite for all cofinite subgroups $H \subseteq G$. Following [KMN23], we define a *quasi-finitely genuine* G*-spectrum* to be a functor $M\colon \mathrm{Span}(\mathrm{QFin}_G) \to \mathrm{Sp}$ such that for every quasi-finite G-set S the canonical map

$$M(S) \to \prod_{\overline{s} \in S/G} M(\pi^{-1}(\overline{s}))$$

is an equivalence, where $\pi\colon S \to S/G$ denotes the quotient map. The category of all such functors is denoted $\mathrm{Sp}_G^{\mathrm{qfgen}}$.

Letting $\widehat{\mathrm{Orb}}_G \subseteq \mathrm{QFin}_G$ denote the full subcategory spanned by the finite orbits G/H, we will refer to $\widehat{\mathrm{Orb}}_G$-categories as G*-procategories*. We then define what it means for a G-procategory to be *very* G*-semiadditive* by applying our framework to $Q = \mathrm{QFin}_G$, and we prove:

Theorem E (Theorem 9.20)**.** The category $\mathrm{Sp}_G^{\mathrm{qfgen}}$ of quasi-finitely genuine G-spectra is the underlying category of the free presentable very G-semiadditive stable G-procategory.

We further provide a similar universal interpretation of Kaledin's category $\widehat{\mathcal{M}}(G, \mathbb{Z})$ of $\mathbb{Z}$*-valued Mackey profunctors*, see Theorem 9.21.

[3] We adopt the terminology from [KMN23]; Kaledin used the term 'admissible'.

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2. Recollections on parametrized category theory

The variants of higher semiadditivity we are interested in are most conveniently phrased using the language of *parametrized category theory*, which we will briefly recall now.

Definition 2.1 (Parametrized categories)**.** Throughout this article, we will use the following two notions of parametrized categories:

(1) For a small category T, we define a *T-category* to be a functor $\mathcal{C}\colon T^{\mathrm{op}} \to \mathrm{Cat}$. We write $\mathrm{Cat}_T := \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat})$ for the (very large) category of T-categories.
(2) For an (∞-)topos $\mathcal{B}$, we define a *$\mathcal{B}$-category* to be a limit-preserving functor $\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$. We write $\mathrm{Cat}(\mathcal{B}) := \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat})$ for the (very large) category of $\mathcal{B}$-categories.

For a morphism $f\colon A \to B$ in T or $\mathcal{B}$, we refer to the functor $f^* := \mathcal{C}(f)\colon \mathcal{C}(B) \to \mathcal{C}(A)$ as the *restriction functor* of f.

Remark 2.2. The formalism of $\mathcal{B}$-categories is more general than that of T-categories: restriction along the Yoneda embedding $T \hookrightarrow \mathrm{PSh}(T)$ defines an equivalence $\mathrm{Cat}(\mathrm{PSh}(T)) \xrightarrow{\sim} \mathrm{Cat}_T$ between $\mathrm{PSh}(T)$-categories and T-categories, with inverse given by limit-extension. Given a T-category $\mathcal{C}$, we will generally abuse notation and denote its limit-extension $\mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}$ again by $\mathcal{C}$.

While the general formalism of parametrized semiadditivity will be developed for $\mathcal{B}$-categories, most of our examples will come from T-categories for suitable T. It will occasionally be convenient to state definitions that apply both to $\mathcal{B}$-categories as well as to T-categories; in these cases we work with functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$ for some (either small or large) category $\mathcal{A}$.

Example 2.3 ($\mathcal{B}$-groupoid)**.** Every object B of $\mathcal{B}$ defines a $\mathcal{B}$-category $\underline{B}$ via the Yoneda embedding:

$$\underline{B} := \hom_{\mathcal{B}}(-, B)\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Spc} \hookrightarrow \mathrm{Cat}\,.$$

The $\mathcal{B}$-categories of this form are called *$\mathcal{B}$-groupoids*.

Example 2.4 (The $\mathcal{B}$-category of $\mathcal{B}$-groupoids)**.** Since $\mathcal{B}$ is a topos, the functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ given by $B \mapsto \mathcal{B}_{/B}$ (i.e. the cartesian unstraightening of the target map $\mathrm{Ar}(\mathcal{B}) \to \mathcal{B}$) preserves limits and thus defines a $\mathcal{B}$-category that we denote by $\underline{\mathrm{Spc}}_{\mathcal{B}}$ and refer to as the *$\mathcal{B}$-category of $\mathcal{B}$-groupoids*.

Definition 2.5 (Underlying category)**.** Every $\mathcal{B}$-category $\mathcal{C}$ has an *underlying category* $\Gamma\mathcal{C} := \mathcal{C}(1)$, where $1 \in \mathcal{B}$ is the terminal object.

Definition 2.6 (Parametrized functor category)**.** The category $\mathrm{Cat}(\mathcal{B})$ is cartesian closed by [Mar21, Proposition 3.2.11]. We denote the internal hom by $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$ (or by $\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$ if $\mathcal{B}$ is clear from the context), and denote its underlying category by $\mathrm{Fun}_{\mathcal{B}}(\mathcal{C},\mathcal{D}) \coloneqq \Gamma\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$.

Proposition 2.7 (Categorical Yoneda Lemma, cf. [CLL23a, Lemma 2.2.7, Corollary 2.2.9])**.** *For an object $B \in \mathcal{B}$, evaluation at $\mathrm{id}_B \in \underline{B}(B)$ defines a natural equivalence*

$$\mathrm{Fun}_{\mathcal{B}}(\underline{B},\mathcal{C}) \xrightarrow{\sim} \mathcal{C}(B).$$

As a consequence, there are natural equivalences

$$\underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{B},\mathcal{C}) \simeq \mathcal{C}(B\times -)$$

and

$$\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C},\mathcal{D})(B) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{C}\times\underline{B},\mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{C},\underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{B},\mathcal{D})).$$

Proof. In the special case $\mathcal{B} = \mathrm{PSh}(T)$ (whence $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Cat}_T$), this is the content of [CLL23a, Lemma 2.2.7 and Corollary 2.2.9]. For the general case, we claim that the embedding $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathrm{CAT})$ preserves internal homs, where we jump universes to ensure smallness of $\mathcal{B}$. For this, let $\mathcal{C},\mathcal{D} \in \mathrm{Cat}(\mathcal{B})$ arbitrary. By the above special case, the internal hom in the category on the right satisfies

$$\begin{aligned}\iota\big(\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})(A)^{[n]}\big) \simeq \iota\underline{\mathrm{Fun}}(\mathcal{C}\times[n],\mathcal{D})(A) &\simeq \iota\,\mathrm{Fun}(\mathcal{C}\times[n],\mathcal{D}(A\times -))\\ &= \hom(\mathcal{C}\times[n],\mathcal{D}(A\times -)),\end{aligned}$$

for every $A \in \mathcal{B}$ and $n \geq 0$; in particular, the complete Segal space associated to $\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})(A)$ is contained in the original universe, so $\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$ is contained in $\mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathrm{Cat})$. It remains to show that it is even contained in $\mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}},\mathrm{Cat})$. As $\mathrm{Fun}(\mathcal{C},-)$ is a right adjoint, it suffices that $\mathcal{B}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathrm{CAT}), A \mapsto \mathcal{D}(A\times -)$ preserves small limits. Since limits in functor categories are pointwise, this amounts to saying that $\mathcal{D}(-\times B) : \mathcal{B}^{\mathrm{op}} \to \mathrm{CAT}$ preserves small limits for every B, which directly follows from cartesian closure of $\mathcal{B}$ and the sheaf property of $\mathcal{D}$. □

Remark 2.8. In what follows, we will freely cite results from [CLL23a] for internal homs of T-categories even when working with general $\mathcal{B}$-categories; in each case the reduction step used in the proof of Proposition 2.7 applies.

Remark 2.9. Given an object $B \in \mathcal{B}$, every $\mathcal{B}$-category $\mathcal{C}$ canonically gives rise to a $\mathcal{B}_{/B}$-category $\pi_B^*\mathcal{C}$ by precomposing $\mathcal{C}$ with the (colimit-preserving) forgetful functor $\pi_B\colon \mathcal{B}_{/B} \to \mathcal{B}$. The resulting functor $\pi_B^*\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}_{/B})$ preserves internal homs by [CLL23a, Corollary 2.2.11], and as a result there is for all $\mathcal{C},\mathcal{D} \in \mathrm{Cat}(\mathcal{B})$ and every $B \in \mathcal{B}$ a natural equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C},\mathcal{D})(B) \xrightarrow{\sim} \mathrm{Fun}_{\mathcal{B}_{/B}}(\pi_B^*\mathcal{C},\pi_B^*\mathcal{D}).$$

Under this equivalence, restriction along $f\colon A \to B$ corresponds to restriction along $\mathcal{B}_{/f}\colon \mathcal{B}_{/A} \to \mathcal{B}_{/B}$ and conjugating by the evident equivalence, see [CLL23a, Lemma 2.2.12].

2.1. **Parametrized colimits.** In parametrized category theory, there is a notion of 'groupoid-indexed colimit' that we will now recall. To this end, recall that a class of morphisms $\mathcal{Q}$ in a category $\mathcal{A}$ is said to be *closed under base change* if base changes (= pullbacks) of morphisms in $\mathcal{Q}$ along morphisms in $\mathcal{A}$ exist and are again in $\mathcal{Q}$.

Definition 2.10 ($\mathcal{Q}$-colimits)**.** Let $\mathcal{A}$ be a category and let $\mathcal{Q}$ be a class of morphisms in $\mathcal{A}$ closed under base change. Given a functor $\mathcal{C}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$, we say that $\mathcal{C}$ *admits $\mathcal{Q}$-colimits* or *is $\mathcal{Q}$-cocomplete* if the following conditions are satisfied:

(1) For every morphism $q\colon A \to B$ in $\mathcal{Q}$, the functor $q^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a left adjoint $q_!\colon \mathcal{C}(A) \to \mathcal{C}(B)$.
(2) For every pullback square

$$\begin{array}{ccc} A' & \xrightarrow{g} & A \\ {\scriptstyle q'}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ B' & \xrightarrow{f} & B \end{array}$$

in $\mathcal{A}$ with q in $\mathcal{Q}$, the Beck–Chevalley transformation $\mathrm{BC}_!\colon q'_! g^* \to f^* q_!$ of functors $\mathcal{C}(A) \to \mathcal{C}(B')$ is an equivalence.

Dually, we define what it means for $\mathcal{C}$ to *admit $\mathcal{Q}$-limits*.

Remark 2.11. By [MW21, Corollary 3.2.11], the above amounts to saying that $q^*\colon \pi_B^*\mathcal{C} \to \underline{\mathrm{Fun}}(\underline{A}, \pi_B^*\mathcal{C}) \simeq \mathcal{C}(A \times_B -)$ has a *parametrized left adjoint* $q_!$, i.e. a left adjoint in the homotopy 2-category of $\mathrm{Fun}\big((\mathcal{A}_{/B})^{\mathrm{op}}, \mathrm{Cat}\big)$.

We will mostly use this notion in the case of $\mathcal{B}$-categories for a topos $\mathcal{B}$. In this case, we will further assume that the class of morphisms $\mathcal{Q}$ in $\mathcal{B}$ is *local*, meaning that a morphism $q\colon A \to B$ is in $\mathcal{Q}$ whenever there exists an effective epimorphism $\coprod_{i\in I} B_i \twoheadrightarrow B$ in $\mathcal{B}$ such that each of the base change maps $A \times_B B_i \to B_i$ is in $\mathcal{Q}$.

Remark 2.12. Let T be small and let $Q \subseteq T$ be closed under base change. We define $\mathcal{Q} \coloneqq Q_{\mathrm{loc}}$ as the collection of all maps $q\colon X \to Y$ in $\mathrm{PSh}(T)$ such that for every map $A \to Y$ from a representable the base change $A \times_Y X \to A$ belongs to Q. Given an effective epimorphism $\coprod_i Y_i \to Y$, the Yoneda lemma shows that any map $A \to Y$ from a representable factors through one of the Y_i. Thus, Q_{loc} is a local class in $\mathrm{PSh}(T)$, and we will refer to it as the *local class generated by* Q. By [CLL23a, Remark 2.3.15], a T-category is then Q-cocomplete if and only if its limit extension is Q_{loc}-cocomplete.

Definition 2.13. Let $\mathcal{C}, \mathcal{D}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$ be $\mathcal{Q}$-cocomplete. A natural transformation $F\colon \mathcal{C} \to \mathcal{D}$ is said to *preserve $\mathcal{Q}$-colimits* if for every morphism $q\colon A \to B$ in $\mathcal{Q}$ the Beck–Chevalley map $q_! F_A \to F_B q_!$ is an equivalence; alternatively we say that *F is $\mathcal{Q}$-cocontinuous.*

If $\mathcal{A} = \mathcal{B}$ is a topos, we denote by $\mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\amalg} \subseteq \mathrm{Cat}(\mathcal{B})$ the (non-full) subcategory spanned by those $\mathcal{B}$-categories admitting $\mathcal{Q}$-colimits and those $\mathcal{B}$-functors preserving $\mathcal{Q}$-colimits. Dually, we define the non-full subcategory $\mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\times} \subseteq \mathrm{Cat}(\mathcal{B})$.

Remark 2.14. If $\mathcal{A} = T$ is small and $Q \subseteq T$ is closed under base change, [CLL23a, Lemma 2.3.16] shows that a functor $F\colon \mathcal{C} \to \mathcal{D}$ in Cat_T preserves Q-colimits if and only if it preserves Q_{loc}-colimits when viewed as a map in $\mathrm{Cat}(\mathrm{PSh}(T))$.

Example 2.15. In the case $\mathcal{B} = \mathrm{Spc}$, the condition of being $\mathcal{Q}$-cocomplete reduces to a non-parametrized cocompleteness condition. Recall that taking global sections defines an equivalence $\mathrm{Cat}(\mathrm{Spc}) \xrightarrow{\sim} \mathrm{Cat}$, with inverse given by sending a category $\mathcal{C}$ to $\mathrm{Fun}(-,\mathcal{C})$. For local $\mathcal{Q} \subseteq \mathrm{Spc}$, a category $\mathcal{C}$ then has $\mathcal{Q}$-colimits if and only if $q^*\colon \mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(A,\mathcal{C})$ has a left adjoint (satisfying base change) for every $q\colon A \to B$ in $\mathcal{Q}$. Specializing to $B = 1$, we see that $\mathcal{C}$ has A-indexed colimits for all $A \in \mathcal{Q}_{/1} \subseteq \mathrm{Spc} \subseteq \mathrm{Cat}$; conversely, if $\mathcal{C}$ admits such colimits, then Kan's pointwise formula and the closure of $\mathcal{Q}$ under base change show that all the required adjoints exist and satisfy base change, i.e. $\mathcal{C}$ is $\mathcal{Q}$-cocomplete as a $\mathcal{B}$-category.

In the same way, we see that a Spc-functor is $\mathcal{Q}$-cocontinuous if and only if it preserves $\mathcal{Q}_{/1}$-colimits as a functor of non-parametrized categories.

Construction 2.16. For a local class of morphisms $\mathcal{Q}$ in $\mathcal{B}$ and an object $B \in \mathcal{B}$, we denote by

$$\underline{\mathbf{U}}_{\mathcal{Q}}(B) \subseteq \mathcal{B}_{/B}$$

the full subcategory spanned by those morphisms $q\colon A \to B$ which are contained in $\mathcal{Q}$. Since $\mathcal{Q}$ is closed under base change, pullback along a morphism $f\colon A \to B$ restricts to a functor $f^*\colon \underline{\mathbf{U}}_{\mathcal{Q}}(B) \to \underline{\mathbf{U}}_{\mathcal{Q}}(A)$, and since $\mathcal{Q}$ is local we obtain a $\mathcal{B}$-subcategory $\underline{\mathbf{U}}_{\mathcal{Q}} \subseteq \underline{\mathrm{Spc}}_{\mathcal{B}}$.

Remark 2.17. The $\mathcal{B}$-category $\underline{\mathbf{U}}_{\mathcal{Q}}$ is a *class of $\mathcal{B}$-groupoids* in the terminology of [MW21], and thus determines a notion of *$\underline{\mathbf{U}}_{\mathcal{Q}}$-colimits* in a $\mathcal{B}$-category. By Proposition 5.4.2 of *op. cit.* this precisely recovers the above definitions of $\mathcal{Q}$-colimits and $\mathcal{Q}$-cocontinuity.

Proposition 2.18 ([MW21, Proposition 5.2.7])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{B}$-categories, and assume that $\mathcal{D}$ has $\mathcal{Q}$-colimits. Then:*

1. *The $\mathcal{B}$-category $\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$ again has all $\mathcal{Q}$-colimits.*
2. *For any $\mathcal{C} \to \mathcal{C}'$ the restriction $\underline{\mathrm{Fun}}(\mathcal{C}',\mathcal{D}) \to \underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$ is $\mathcal{Q}$-cocontinuous.*
3. *For any $\mathcal{Q}$-cocontinuous functor $\mathcal{D} \to \mathcal{D}'$ the induced functor $\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D}) \to \underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D}')$ is again $\mathcal{Q}$-cocontinuous.* □

Construction 2.19. Let $\mathcal{C},\mathcal{D}$ be $\mathcal{Q}$-cocomplete $\mathcal{B}$-categories. We define a full $\mathcal{B}$-subcategory $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\amalg}_{\mathcal{B}}(\mathcal{C},\mathcal{D}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$ spanned in degree $B \in \mathcal{B}$ by the objects corresponding to $\pi_B^{-1}\mathcal{Q}$-cocontinuous functors $\pi_B^*\mathcal{C} \to \pi_B^*\mathcal{D}$ under the equivalence from Remark 2.9; see [MW21, Remark 5.2.4] for a proof that this is indeed a $\mathcal{B}$-category.

By [CLL23a, Remark 2.3.27], $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\amalg}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$ can equivalently be described as the full subcategory spanned in degree $B \in \mathcal{B}$ by those objects that correspond to $\mathcal{Q}$-cocontinuous functors $\mathcal{C} \to \underline{\mathrm{Fun}}(\underline{B},\mathcal{D})$ under the final equivalence of Proposition 2.7.

Assume now that the morphisms in $\mathcal{Q}$ are closed under composition and contain all equivalences, so that $\mathcal{Q} \subseteq \mathcal{B}$ defines a wide subcategory. In this case, the $\mathcal{B}$-category $\underline{\mathbf{U}}_{\mathcal{Q}}$ admits $\mathcal{Q}$-colimits. In fact, it is universal with this property:

Proposition 2.20 ([MW21, Theorem 7.1.13])**.** *The $\mathcal{B}$-category $\underline{\mathbf{U}}_{\mathcal{Q}}$ is the free $\mathcal{Q}$-cocomplete $\mathcal{B}$-category with $\mathcal{Q}$-colimits: for every $\mathcal{Q}$-cocomplete $\mathcal{B}$-category $\mathcal{D}$, evaluation at the point* $\mathrm{pt}\colon \underline{1} \to \underline{\mathbf{U}}_{\mathcal{Q}}$ *defines an equivalence of $\mathcal{B}$-categories*

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\amalg}(\underline{\mathbf{U}}_{\mathcal{Q}},\mathcal{D}) \xrightarrow{\sim} \mathcal{D},$$

whose inverse is given by left Kan extension along $\mathrm{pt}\colon \underline{1} \to \underline{\mathbf{U}}_{\mathcal{Q}}$. □

While most of our paper only refers to the above 'groupoid indexed colimits,' we will on some rare occasions need the complementary notion of *fiberwise colimits*:

Definition 2.21. Let K be a (non-parametrized) category. We say that a $\mathcal{B}$-category $\mathcal{C}$ has *fiberwise K-shaped colimits* if the category $\mathcal{C}(A)$ has K-shaped colimits for every $A \in \mathcal{B}$ and the restriction functor $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ preserves K-shaped colimits for each $f\colon A \to B$ in $\mathcal{B}$.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ of $\mathcal{B}$-categories with fiberwise K-shaped colimits, we say that F *preserves fiberwise K-shaped colimits* if each $F_A\colon \mathcal{C}(A) \to \mathcal{D}(A)$ preserves K-shaped colimits.

Remark 2.22. [MW21, Proposition 5.7.2] also proves the analogue of Proposition 2.18 for fiberwise colimits in $\mathcal{B}$-categories and functors preserving them, making precise that all colimits in functor categories are pointwise.

Combining the above two notions of colimits we define:

Definition 2.23. A $\mathcal{B}$-category is called *cocomplete* if it is $\mathcal{B}$-cocomplete in the sense of Definition 2.10 and moreover *fiberwise cocomplete*, i.e. has all small fiberwise colimits in the sense of Definition 2.21.

A $\mathcal{B}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ of cocomplete $\mathcal{B}$-categories is called *cocontinous* if it is $\mathcal{B}$-cocontinuous and preserves all small fiberwise colimits.

Warning 2.24. If $\mathcal{B} = \mathrm{PSh}(T)$, we referred to the above notion as *T-cocompleteness* in [CLL23a, CLL23b], which clashes with the terminology in Definition 2.10 above.

Remark 2.25. If $\mathcal{C}$ is cocomplete, then the inclusion of constant diagrams $\pi_A^*\mathcal{C} \to \underline{\mathrm{Fun}}(\mathcal{K}, \pi_A^*\mathcal{C})$ has a left adjoint for every $A \in \mathcal{B}$ and every small $\mathcal{B}_{/A}$-category $\mathcal{K}$ by [MW21, Corollary 5.4.7]. In particular, it makes sense to talk about $\mathcal{K}$-shaped colimits in $\pi_A^*\mathcal{C}$ for any such $\mathcal{K}$.

3. Parametrized semiadditivity

In this section, we introduce a wide range of generalized notions of semiadditivity for parametrized categories, using the framework of *ambidexterity* by Hopkins and Lurie [HL13].

3.1. **Ambidexterity.** We start with a recollection on ambidexterity.

Definition 3.1 (Inductible subcategory)**.** Let $\mathcal{A}$ be a category and let $\mathcal{Q}$ be a wide subcategory of $\mathcal{A}$ closed under base change. We say that $\mathcal{Q}$ is *inductible* if the following conditions are satisfied:

(1) $\mathcal{Q}$ is *closed under diagonals*: for every morphism $q\colon A \to B$ in $\mathcal{Q}$, the diagonal map $\Delta_q\colon A \to A \times_B A$ is again in $\mathcal{Q}$;
(2) $\mathcal{Q}$ is *truncated*: every morphism $q\colon A \to B$ in $\mathcal{Q}$ is truncated (i.e. n_q-truncated for some natural number n_q).

The assumptions on $\mathcal{Q}$ allow us to make inductive definitions for morphisms in $\mathcal{Q}$ by iteratively passing to diagonals, explaining our terminology. The condition that $\mathcal{Q}$ is closed under diagonals in $\mathcal{A}$ admits various alternative characterizations:

Lemma 3.2. *For a wide subcategory $\mathcal{Q} \subseteq \mathcal{A}$ closed under base change, the following conditions are equivalent:*

(1) $\mathcal{Q}$ is closed under diagonals;
(2) $\mathcal{Q}$ is left-cancelable*: for morphisms $p\colon A \to B$ and $q\colon B \to C$ in $\mathcal{A}$, if both q and qp are in $\mathcal{Q}$ then also p is in $\mathcal{Q}$;*
(3) $\mathcal{Q}$ admits pullbacks and the inclusion $\mathcal{Q} \hookrightarrow \mathcal{A}$ preserves pullbacks.

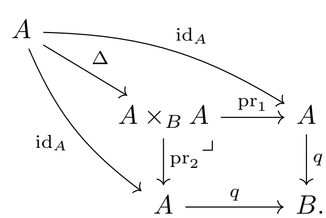
Proof. For (1) $\implies$ (2), observe that with p and q as in (2) we may factor p as the composite of $(1, p)\colon A \to A \times_C B$ and $\mathrm{pr}_B\colon A \times_C B \to B$. The first map is a base change of $\Delta_q\colon B \to B \times_C B$ and the second map is a base change of qp, hence by assumption both lie in $\mathcal{Q}$ and thus so does p. For (2) $\implies$ (3), consider morphisms $A \to B$ and $A' \to B$ in $\mathcal{Q}$. It follows from (2) that a map $C \to A \times_B A'$ is in $\mathcal{Q}$ if and only if the two components $C \to A$ and $C \to A'$ are, from which (3) is an immediate consequence. The implication (3) $\implies$ (1) is clear. $\square$

Consider an inductible subcategory $\mathcal{Q}$ of a category $\mathcal{A}$, and let $\mathcal{C}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$ be a functor which is $\mathcal{Q}$-cocomplete in the sense of Definition 2.10. The restriction of $\mathcal{C}$ to $\mathcal{Q}^{\mathrm{op}}$ admits a cartesian unstraightening $\int(\mathcal{C}|_{\mathcal{Q}^{\mathrm{op}}}) \to \mathcal{Q}$, which due to $\mathcal{Q}$-cocompleteness of $\mathcal{C}$ is a *Beck–Chevalley fibration* in the sense of [HL13, Definition 4.1.3] and thus gives rise to a notion of *$\mathcal{C}$-ambidexterity*:

Construction 3.3 (Ambidexterity, [HL13, Construction 4.1.8])**.** Let $\mathcal{Q}$ be an inductible subcategory of a category $\mathcal{A}$ and let $\mathcal{C}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$ be a functor which is $\mathcal{Q}$-cocomplete in the sense of Definition 2.10. We will inductively define what it means for an n-truncated morphism $q\colon A \to B$ in $\mathcal{Q}$ to be *$\mathcal{C}$-ambidextrous*, in which case we will construct a transformation $\mu_q^{(n)}\colon \mathrm{id}_{\mathcal{C}(B)} \to q_! q^*$ exhibiting $q_!$ as a right adjoint to q^*.

The induction starts at $n = -2$, in which case any (-2)-truncated morphism q is declared to be $\mathcal{C}$-ambidextrous. Since q is an equivalence, the counit map $q_! q^* \to \mathrm{id}_{\mathcal{C}(B)}$ is an equivalence, and we define $\mu_q^{(-2)}\colon \mathrm{id}_{\mathcal{C}(B)} \to q_! q^*$ as its inverse.

Assume now that we have defined the n-truncated $\mathcal{C}$-ambidextrous morphisms for some $n \geq -2$ and have assigned to them the required transformations $\mu_q^{(n)}$. We say that an $(n+1)$-truncated morphism $q\colon A \to B$ in $\mathcal{Q}$ is *weakly $\mathcal{C}$-ambidextrous* if its diagonal $\Delta_q\colon A \to A \times_B A$ is $\mathcal{C}$-ambidextrous (which is well-defined since Δ_q is n-truncated). Consider the following commutative diagram:



We define the *adjoint norm map* $\widetilde{\mathrm{Nm}}_q\colon q^* q_! \to \mathrm{id}$ as the following composite:

$$\widetilde{\mathrm{Nm}}_q\colon q^* q_! \xrightarrow{\mathrm{BC}_!^{-1}} \mathrm{pr}_{1!}\mathrm{pr}_2^* \xrightarrow{\mu_\Delta^{(n)}} \mathrm{pr}_{1!}\Delta_!\Delta^*\mathrm{pr}_2^* \simeq \mathrm{id}.$$

An $(n+1)$-truncated morphism $q\colon A \to B$ is called *$\mathcal{C}$-ambidextrous* if every base change q' of q is weakly $\mathcal{C}$-ambidextrous and the adjoint norm map $\widetilde{\mathrm{Nm}}_{q'}\colon q'^* q'_* \to \mathrm{id}_{\mathcal{C}(A')}$ exhibits $q'_!$ as a right adjoint of q'^*. In this case, we let $\mu_q^{(n+1)}\colon \mathrm{id}_{\mathcal{C}(B)} \to q_! q^*$ denote the corresponding unit for the resulting adjunction $q^* \dashv q_!$.

Remark 3.4. The norm map is independent of the choice of pullback. In particular, taking the same object $A \times_B A$ but with the two projection maps swapped, we see that we can equivalently define the adjoint norm map as the composite

$$q^* q_! \simeq \mathrm{pr}_{2!}\mathrm{pr}_1^* \xrightarrow{\mu} \mathrm{pr}_{2!}\Delta_!\Delta^*\mathrm{pr}_1^* \simeq \mathrm{id}.$$

Remark 3.5. Let $f\colon \mathcal{A}' \to \mathcal{A}$ be a functor and let $\mathcal{Q}' \subseteq \mathcal{A}'$ be inductible such that $f(\mathcal{Q}') \subseteq \mathcal{Q}$ and f preserves pullbacks along maps in $\mathcal{Q}'$. Given any $\mathcal{C}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$, we define $f^*\mathcal{C} := \mathcal{C} \circ f\colon \mathcal{A}'^{\mathrm{op}} \to \mathrm{Cat}$. It then follows straight from the definition that $f^*\mathcal{C}$ is $\mathcal{Q}'$-cocomplete if $\mathcal{C}$ is $\mathcal{Q}$-cocomplete, and that $q \in \mathcal{Q}'$ is (weakly) $f^*\mathcal{C}$-ambidextrous if $f(q)$ is (weakly) $\mathcal{C}$-ambidextrous. Moreover, the adjoint norm map for q agrees with the adjoint norm map for $f(q)$ in $\mathcal{C}$.

Remark 3.6 (Norm map)**.** In the situation of Construction 3.3, consider a weakly $\mathcal{C}$-ambidextrous morphism $q\colon A \to B$. If the functor $q^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ admits a right adjoint $q_*\colon \mathcal{C}(A) \to \mathcal{C}(B)$, then the adjoint norm map $\widetilde{\mathrm{Nm}}_q\colon q^*q_! \to \mathrm{id}$ corresponds to a transformation $\mathrm{Nm}_q\colon q_! \to q_*$ that we call the *norm map* associated to q. In this case, it follows that q is $\mathcal{C}$-ambidextrous if and only if for each base change q' the restriction functor q'^* admits a right adjoint q'_* and the norm map $\mathrm{Nm}_{q'}\colon q'_! \to q'_*$ is an equivalence.

The above construction interacts with natural transformations as one would expect:

Proposition 3.7 (cf. [CSY22, Theorem 3.2.3])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a natural transformation of $\mathcal{Q}$-cocomplete functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$. Assume that for every $(n-1)$-truncated map p in $\mathcal{Q}$ at least one of the Beck–Chevalley maps $\mathrm{BC}_!\colon p_!F \to Fp_!$ and $\mathrm{BC}_*\colon Fp_* \to p_*F$ is invertible.*

(1) *Let q be an n-truncated map that is both weakly $\mathcal{C}$-ambidextrous and weakly $\mathcal{D}$-ambidextrous. Then the following diagram commutes:*

$$\begin{array}{ccccc} q^*q_!F & \xrightarrow{\mathrm{BC}_!} & q^*Fq_! & \xrightarrow{\simeq} & Fq^*q_! \\ {\scriptstyle \widetilde{\mathrm{Nm}}_q F}\downarrow & & & & \downarrow{\scriptstyle F\,\widetilde{\mathrm{Nm}}_q} \\ F & & = & & F. \end{array}$$

(2) *Assume in addition that $q^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ and $q^*\colon \mathcal{D}(B) \to \mathcal{D}(A)$ admit right adjoints q_*. Then the following diagram commutes:*

$$\begin{array}{ccc} q_!F & \xrightarrow{\mathrm{Nm}_q} & q_*F \\ {\scriptstyle \mathrm{BC}_!}\downarrow & & \uparrow{\scriptstyle \mathrm{BC}_*} \\ Fq_! & \xrightarrow[F(\mathrm{Nm}_q)]{} & Fq_*. \end{array}$$

(3) *Assume that q is $\mathcal{C}$-ambidextrous and $\mathcal{D}$-ambidextrous and that at least one of the Beck–Chevalley maps $q_!F \to Fq_!$ and $Fq_* \to q_*F$ is invertible. Then also the following diagram commutes:*

$$\begin{array}{ccc} F & \xrightarrow{F(\mu_q)} & Fq_!q^* \\ {\scriptstyle \mu_q}\downarrow & & \uparrow{\scriptstyle \mathrm{BC}_!} \\ q_!q^*F & = & q_!Fq^*. \end{array}$$

Proof. First fix n and q and observe that (2) follows from (1) via adjoining over, also see [CSY22, Lemma 2.2.11]. We will now show that this in turn implies (3): indeed, in the diagram

$$\begin{tikzcd} F \arrow[r, "F(\eta)"] \arrow[d, "\eta"'] & Fq_*q^* \arrow[r, "F(\mathrm{Nm}^{-1})"] \arrow[d, "\mathrm{BC}_*"] & Fq_!q^* \\ q_*q^*F \arrow[r, equal] & q_*Fq^* \arrow[r, "\mathrm{Nm}^{-1}"'] & q_!Fq^* \arrow[u, "\mathrm{BC}_!"'] \end{tikzcd}$$

the right-hand square commutes by (2) and the assumption that at least one of the two Beck–Chevalley maps is invertible, while the left-hand square commutes by direct inspection.

Using this, we will now prove (1) by induction on n. For $n = -2$, $\widetilde{\mathrm{Nm}}_q$ is simply the inverse of the unit $\mathrm{id} \to q^*q_!$, and the statement follows by a standard mate argument, also see [CSY22, Lemma 2.2.3(3)]. If we already know the statement for $n-1$, then we consider the diagram

$$\begin{tikzcd} q^*q_!F \arrow[r, "\mathrm{BC}_!^{-1}"] \arrow[dd, "\mathrm{BC}_!"'] & (\mathrm{pr}_1)_!\mathrm{pr}_2^*F \arrow[r, "\mu_\Delta"] \arrow[d, equal] & (\mathrm{pr}_1)_!\Delta_!\Delta^*\mathrm{pr}_2^*F \arrow[r, "\sim"] \arrow[d, equal] & F \arrow[dd, equal] \\ & (\mathrm{pr}_1)_!F\mathrm{pr}_2^* \arrow[r, "\mu_\Delta"] \arrow[dd, equal] & (\mathrm{pr}_1)_!\Delta_!\Delta^*F\mathrm{pr}_2^* \arrow[d, equal] & \\ q^*Fq_! \arrow[dd, "\sim"'] & (*) \qquad\qquad (\dagger) & (\mathrm{pr}_1)_!\Delta_!F\Delta^*\mathrm{pr}_2^* \arrow[r, "\sim"] \arrow[d, "\mathrm{BC}_!"'] & F \arrow[dd, equal] \\ & (\mathrm{pr}_1)_!F\mathrm{pr}_2^* \arrow[r, "\mu_\Delta"] \arrow[d, "\mathrm{BC}_!"'] & (\mathrm{pr}_1)_!F\Delta_!\Delta^*\mathrm{pr}_2^* \arrow[d, "\mathrm{BC}_!"'] & (*) \\ Fq^*q_! \arrow[r, "\mathrm{BC}_!^{-1}"] & F(\mathrm{pr}_1)_!(\mathrm{pr}_2)^* \arrow[r, "\mu_\Delta"] & F(\mathrm{pr}_1)_!\Delta_!\Delta^*\mathrm{pr}_2^* \arrow[r, "\sim"] & F \end{tikzcd}$$

whose top and bottom row spell out $\widetilde{\mathrm{Nm}}_q$ and $F(\widetilde{\mathrm{Nm}}_q)$, respectively; here and in what follows, we will simply denote the naturality constraints of an $\mathcal{A}^{\mathrm{op}}$-natural transformation by equality signs to streamline notation.

The two subdiagrams marked $(*)$ commute by basic mate arguments, cf. [CSY22, Lemma 2.2.4(1)], while the subdiagram $(\dagger)$ commutes by the induction hypothesis and the above implication (1) $\Rightarrow$ (3). As all the remaining subdiagrams commute simply by naturality, this completes the inductive step. $\square$

As an immediate consequence, we can now describe the interaction of the norm with base change, cf. [HL13, Proposition 4.2.1 and Remark 4.2.3]:

Corollary 3.8. *Let*

$$\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "q"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

be a pullback in $\mathcal{A}$ such that q is a map in $\mathcal{Q}$ (whence so is q').

(1) If $\mathcal{C}$ is $\mathcal{Q}$-cocomplete and q is weakly $\mathcal{C}$-ambidextrous, then we have a commutative diagram

$$\begin{array}{ccc} q'^* f^* q_! & = & g^* q^* q_! \\ {\scriptstyle \mathrm{BC}_!}\uparrow & & \downarrow{\scriptstyle g^* \widetilde{\mathrm{Nm}}_q} \\ q'^* q'_! g^* & \xrightarrow[\widetilde{\mathrm{Nm}}_{q'}]{} & g^*. \end{array}$$

(2) Assume in addition that q^* and q'^* admit right adjoints. Then also

$$\begin{array}{ccc} f^* q_! & \xrightarrow{f^* \mathrm{Nm}_q} & f^* q_* \\ {\scriptstyle \mathrm{BC}_!}\uparrow & & \downarrow{\scriptstyle \mathrm{BC}_*} \\ q'_! g^* & \xrightarrow[\mathrm{Nm}_{q'}]{} & q'_* g^* \end{array}$$

commutes.

Proof. For the first statement, let $\pi_B \colon \mathcal{A}_{/B} \to \mathcal{A}$ denote the projection. It then suffices to apply Proposition 3.7(1) to the $\mathcal{A}_{/B}$-natural transformation $f^* \colon \pi_B^* \mathcal{C} \to \mathcal{C}(A \times_B -)$, using Remark 3.5 to identify the adjoint norms on both sides.

The second statement follows in the same way from Proposition 3.7(2). □

3.2. **Parametrized semiadditivity.** The notion of ambidexterity leads to a variety of notions of parametrized semiadditivity for $\mathcal{B}$-categories. These varieties are most naturally indexed on *locally inductible subcategories*, which we introduce now.

Definition 3.9. A wide local subcategory $\mathcal{Q}$ of a topos $\mathcal{B}$ is *locally inductible* if

(1) every morphism $q \colon A \to B$ in $\mathcal{Q}$ *locally truncated*: there exists a covering $(B_i \to B)_{i \in I}$ (i.e. the induced map $\coprod_{i \in I} B_i \to B$ is an effective epimorphism) such that each base change $q_i \colon B_i \times_B A \to B_i$ is truncated, *and*
(2) $\mathcal{Q}$ is closed under diagonals.

Definition 3.10 ($\mathcal{Q}$-semiadditivity)**.** Let $\mathcal{B}$ be a topos equipped with a local inductible subcategory $\mathcal{Q}$. We say that a $\mathcal{B}$-category $\mathcal{C}$ is $\mathcal{Q}$-*semiadditive* if it admits $\mathcal{Q}$-colimits and if every truncated map $q \colon A \to B$ in $\mathcal{Q}$ is $\mathcal{C}$-ambidextrous.

Remark 3.11. Let $f \colon \mathcal{B}' \to \mathcal{B}$ be a left adjoint functor that preserves pullbacks, and let $\mathcal{Q}' \subseteq \mathcal{B}', \mathcal{Q} \subseteq \mathcal{B}$ be locally inductible with $f(\mathcal{Q}') \subseteq \mathcal{Q}$. Specializing Remark 3.5, we see that for any $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{C}$ the restriction $f^*\mathcal{C}$ is a $\mathcal{Q}'$-semiadditive $\mathcal{B}'$-category, with the evident (adjoint) norms for truncated maps.

In particular, if $A \in \mathcal{B}$ is arbitrary, we can apply this to the forgetful functor $\pi_A \colon \mathcal{B}_{/A} \to \mathcal{B}$ and the locally inductible subcategory $\mathcal{Q}' = \mathcal{B}_{/A}[\mathcal{Q}] \coloneqq \pi_A^{-1}(\mathcal{Q})$. This will in various proofs allow us to restrict to slices, simplifying notation.

Remark 3.12. Suppose that $\mathcal{C}$ is $\mathcal{Q}$-semiadditive. Because parametrized (co)limits in functor categories are computed pointwise, one easily checks by induction that $\underline{\mathrm{Fun}}(\mathcal{I}, \mathcal{C})$ is again $\mathcal{Q}$-semiadditive for every small $\mathcal{B}$-category $\mathcal{I}$, with (adjoint) norm maps given pointwise by the norms in $\mathcal{C}$.

Remark 3.13. Note that the definition of $\mathcal{Q}$-semiadditivity for a locally inductible class $\mathcal{Q}$ only requires that truncated maps in $\mathcal{Q}$ are $\mathcal{C}$-ambidextrous, because only in this case does the inductive procedure of Construction 3.3 terminate. Nevertheless, we will show in Theorem 6.11 that there are natural units and counits witnessing an adjunction $q^* \dashv q_!$ for any map $q \in \mathcal{Q}$.

Conversely, it suffices to check $\mathcal{Q}$-cocompleteness on the classes $\mathcal{Q}_{\leq n}$ of n-truncated maps for every finite n:

Lemma 3.14. *Let $\mathcal{Q} \subseteq \mathcal{B}$ be a locally truncated local class. Then a $\mathcal{B}$-category $\mathcal{C}$ is $\mathcal{Q}$-cocomplete if and only if it is $\mathcal{Q}_{\leq n}$-cocomplete for every $n \geq -2$.*

Proof. The 'only if' part is clear. For the other direction, fix $q\colon A \to B$ and consider the full subcategory $\Sigma \subseteq \mathcal{B}_{/B}$ of all $f\colon B' \to B$ such that the pullback $q' := f^*(q)\colon A \times_B B' \to B'$ is truncated. This is a sieve as truncated maps are stable under pullback, and it is covering by the assumption that q be locally truncated. Moreover, $\mathcal{Q}_{\leq n}$-cocompleteness for all $n \geq -2$ shows that q'^* admits a left adjoint $q'_!$ satisfying base change along maps in Σ. Letting q vary, the lemma is therefore an instance of Corollary A.4. □

In the same way Lemma A.5 specializes to:

Lemma 3.15. *Let $\mathcal{Q} \subseteq \mathcal{B}$ be local and locally truncated. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ of $\mathcal{Q}$-cocomplete $\mathcal{B}$-categories is $\mathcal{Q}$-cocontinuous if and only if it is $\mathcal{Q}_{\leq n}$-cocontinuous for all $n \geq -2$.* □

While the definition of $\mathcal{Q}$-semiadditivity only refers to $\mathcal{Q}$-colimits, we in fact also have all $\mathcal{Q}$-limits:

Corollary 3.16. *Every $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{C}$ admits $\mathcal{Q}$-limits.*

Proof. By Lemma 3.14$^{\mathrm{op}}$ it is enough to show that it has $\mathcal{Q}_{\leq n}$-limits for all $n \geq -2$. Let $q\colon A \to B$ be a map in $\mathcal{Q}_{\leq n}$. By $\mathcal{Q}$-semiadditivity, we know that q^* has a right adjoint q_*, so it only remains to verify the Beck–Chevalley condition, i.e. that for every pullback

$$\begin{array}{ccc} A' & \xrightarrow{f'} & A \\ {\scriptstyle q'}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ B' & \xrightarrow[f]{} & B \end{array}$$

the Beck–Chevalley map $\mathrm{BC}_*\colon f^*q_* \to q'_*f'^*$ is an equivalence. However, this follows immediately from Corollary 3.8 by 2-out-of-3. □

In the same way one shows (using Lemma 3.15 and its dual):

Corollary 3.17. *A functor between $\mathcal{Q}$-semiadditive $\mathcal{B}$-categories preserves $\mathcal{Q}$-limits if and only if it preserves $\mathcal{Q}$-colimits.* □

Definition 3.18. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of $\mathcal{Q}$-semiadditive $\mathcal{B}$-categories is called *$\mathcal{Q}$-semiadditive* if it preserves $\mathcal{Q}$-colimits or, equivalently, $\mathcal{Q}$-limits. We write $\mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\oplus}$ for the category of $\mathcal{Q}$-semiadditive $\mathcal{B}$-categories and $\mathcal{Q}$-semiadditive functors. Given $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\oplus}$, we write $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\oplus}(\mathcal{C}, \mathcal{D}) := \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\amalg}(\mathcal{C}, \mathcal{D})$.

3.3. **Presheaf topoi.** For the applications we have in mind, we are mainly interested in the case where the topos $\mathcal{B}$ is a presheaf topos $\mathrm{PSh}(T)$ on some small category T, so that $\mathcal{B}$-categories correspond to T-categories $T^{\mathrm{op}} \to \mathrm{Cat}$ by Remark 2.2. In this case, the local classes $\mathcal{Q}$ that appear in practice are usually generated by a much smaller collection of morphisms, and the condition of $\mathcal{Q}$-semiadditivity of a T-category simplifies accordingly. We suggestively refer to these smaller classes as 'pre-inductible':

Definition 3.19 (Pre-inductible subcategory)**.** Let T be a small category and let $Q \subseteq \mathrm{PSh}(T)$ be a replete subcategory containing all representable presheaves. We say that Q is *pre-inductible* if the following conditions are satisfied:

(1) (Locality) Consider a morphism $q\colon A \to B$ in $\mathrm{PSh}(T)$ with $B \in Q$. Then q lies in Q if and only if for every pullback square

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ {\scriptstyle q'}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ B' & \xrightarrow{g} & B \end{array}$$

in $\mathrm{PSh}(T)$ with $B' \in T$ the base change q' lies in Q.

(2) (Diagonals) For every morphism q in Q, also its diagonal Δ_q lies in Q.

(3) (Truncation) Every morphism in Q with target in T is truncated.

Remark 3.20. The first axiom together with the pasting law implies that Q is closed under base change along maps $f\colon A \to B$ such that $A, B \in Q$ (but f need not be a map in Q).

There are two 'extreme' cases of pre-inductible subcategories:

Example 3.21. Every inductible subcategory $Q \subseteq T$ is pre-inductible when regarded as a subcategory of $\mathrm{PSh}(T)$: condition (3) holds by assumption and conditions (1) and (2) are a consequence of the fact that the Yoneda embedding preserves pullbacks.

Example 3.22. Every locally inductible subcategory $\mathcal{Q} \subseteq \mathrm{PSh}(T)$ is in particular pre-inductible: conditions (1) and (2) hold by assumption and condition (3) follows from the fact that every locally truncated map with representable target $A \in T$ is already truncated: any cover $(A_i \to A)_{i \in I}$ of A has to contain a map $A_i \to A$ hitting the component of id_A, so that already $A_i \to A$ itself is an effective epimorphism, implying the claim.

Definition 3.23 (Q-semiadditivity)**.** Let $Q \subseteq \mathrm{PSh}(T)$ be a pre-inductible subcategory, and let $\mathcal{A} \subseteq \mathrm{PSh}(T)$ be the full subcategory spanned by the objects of Q. For a T-category $\mathcal{C}$, we denote by $\mathcal{C}|_{\mathcal{A}^{\mathrm{op}}} \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$ its right Kan extension along $T^{\mathrm{op}} \hookrightarrow \mathcal{A}^{\mathrm{op}}$, or equivalently the restriction to $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{C}\colon \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}$.

(1) We say that $\mathcal{C}$ is *Q-cocomplete* if $\mathcal{C}|_{\mathcal{A}^{\mathrm{op}}}$ is Q-cocomplete in the sense of Definition 2.10;

(2) We say that $\mathcal{C}$ is *Q-semiadditive* if in addition every truncated morphism q in Q is $\mathcal{C}|_{\mathcal{A}^{\mathrm{op}}}$-ambidextrous.

Remark 3.24. If $Q = \mathcal{Q} \subseteq \mathrm{PSh}(T)$ is in fact locally inductible then $\mathcal{A} = \mathrm{PSh}(T)$ and one observes that Definition 3.23 specializes to Definition 3.10.

Remark 3.25. If $T = \mathcal{B}$ happens to be a topos and $Q = \mathcal{Q} \subset \mathcal{B}$ is an inductible local class, then after passing to a larger universe we may regard $\mathcal{Q}$ as a pre-inductible subcategory of $\mathrm{PSh}(\mathcal{B})$ by Example 3.21. In this case we get $\mathcal{A} = \mathcal{B}$, and we see that a $\mathcal{B}$-category $\mathcal{C}$ is $\mathcal{Q}$-semiadditive in the sense of Definition 3.10 if and only if its underlying functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ is $\mathcal{Q}$-semiadditive in the sense of Definition 3.23.

The main reason for introducing Q-semiadditivity for pre-inductible Q is the flexibility of this setup: essentially all examples of parametrized semiadditivity provided in Section 3.4 below will be of this form. We will now show that this setup is indeed a special case of our general formalism of $\mathcal{Q}$-semiadditivity for locally inductible $\mathcal{Q}$.

Construction 3.26. Let $Q \subseteq \mathrm{PSh}(T)$ be a pre-inductible subcategory. A morphism $q\colon A \to B$ in $\mathrm{PSh}(T)$ is said to be *locally in* Q if for every morphism $B' \to B$ in $\mathrm{PSh}(T)$ with $B' \in Q$ we have that the base change map $A \times_B B' \to B'$ lies in Q. Since such morphisms are clearly closed under composition and contain all equivalences, they determine a wide subcategory Q_{loc} of $\mathrm{PSh}(T)$. We refer to Q_{loc} as the *locally inductible subcategory generated by* Q.

Remark 3.27. If $Q = \mathcal{Q}$ is already locally inductible, then we have $\mathcal{Q}_{\mathrm{loc}} = \mathcal{Q}$.

Lemma 3.28. *For every pre-inductible subcategory $Q \subseteq \mathrm{PSh}(T)$, the wide subcategory $Q_{\mathrm{loc}} \subseteq \mathrm{PSh}(T)$ is locally inductible.*

Proof. It is easy to check that Q_{loc} is closed under base change and composition, and it is local by the same argument as in Remark 2.12.

For a morphism $q\colon A \to B$ in Q_{loc} we may cover B by representable objects so that assumption (3) immediately implies that q is locally truncated. It remains to show that Q_{loc} is closed under diagonals. By Lemma 3.2, we may equivalently show that Q_{loc} is left-cancellable: if $p\colon A \to B$ and $q\colon B \to C$ are morphisms of presheaves on T such that q and qp are in Q_{loc}, then also p must be in Q_{loc}. In other words, given a morphism $b\colon B' \to B$ in $\mathrm{PSh}(T)$ with $B' \in Q$, we have to show that the base change $p'\colon A \times_B B' \to B'$ is in Q. To this end, consider the following commutative pullback diagram:

$$\begin{array}{ccccc}
A \times_B B' & \xrightarrow{1 \times_q 1} & A \times_C B' & \xrightarrow{\mathrm{pr}_1} & A \\
{\scriptstyle p'}\downarrow & \lrcorner & {\scriptstyle p''}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
B' & \xrightarrow{(b,1)} & B \times_C B' & \xrightarrow{\mathrm{pr}_1} & B \\
 & \searrow{\scriptstyle =} & {\scriptstyle q'}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\
 & & B' & \xrightarrow{qb} & C.
\end{array}$$

Since q and qp are locally in Q, the morphisms q' and $q'p''$ are in Q. As Q is closed under diagonals, Lemma 3.2 implies that also p'' is in Q, and hence p' is in Q by Remark 3.20. This finishes the proof. $\square$

The following is the main result of this subsection:

Proposition 3.29. *Let $Q \subseteq \mathrm{PSh}(T)$ be a pre-inductible subcategory.*

(1) A T-category $\mathcal{C}$ is Q-cocomplete in the sense of Definition 3.23 if and only if its limit-extension $\mathcal{C}\colon \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}$ is Q_{loc}-cocomplete in the sense of Definition 2.10.

(2) A T-category $\mathcal{C}$ is Q-semiadditive in the sense of Definition 3.23 if and only if its limit-extension $\mathcal{C}\colon \mathrm{PSh}(T)^{\mathrm{op}} \to \mathrm{Cat}$ is Q_{loc}-semiadditive in the sense of Definition 3.10.

Proof. For the first statement, the 'if'-part is clear; for the other direction, we note that it even suffices to check the existence of adjoints and the Beck–Chevalley conditions after restricting to maps in Q_{loc} with representable target by [CLL23a, Remark 2.3.15].

For the second statement, the 'if'-direction is again clear. For the 'only if'-direction, we will argue by induction that $\mathcal{C}$ is $(Q_{\mathrm{loc}})_{\leq n}$-semiadditive for all $n \geq -2$.

For $n = -2$ there is nothing to show. Now assume that we already know that $\mathcal{C}$ is $(Q_{\mathrm{loc}})_{\leq n-1}$-semiadditive. By assumption, the restriction along any n-truncated $q \in Q$ has a right adjoint q_*, and $\widetilde{\mathrm{Nm}}\colon q^*q_! \to \mathrm{id}$ adjoins to an equivalence $q_! \to q_*$. Arguing as in Corollary 3.16, we deduce from Corollary 3.8 that q_* satisfies base change along maps in $\mathcal{A}$, so [CLL23a, Remark 2.3.15$^{\mathrm{op}}$] shows that $\mathcal{C}$ is $(Q_{\mathrm{loc}})_{\leq n}$-complete. Given now a general map $q\colon A \to B$ in $\mathcal{Q}$, Corollary 3.8 shows that $f^*\,\mathrm{Nm}_q$ agrees up to equivalence with the norm along $q' := f^*(q)$; in particular, $f^*\,\mathrm{Nm}_q$ is invertible whenever B' is representable (so that $q' \in Q$). Covering B by representables, we see that Nm_q itself is invertible, as desired. □

3.4. **Examples.** We will now provide various examples of pre-inductible subcategories and discuss their associated notion of semiadditivity. Let us start with the non-parametrized examples:

Example 3.30 (Ordinary semiadditivity)**.** The subcategory $\mathrm{Fin} \subseteq \mathrm{Spc}$ of finite sets is pre-inductible. A category $\mathcal{C}$ is Fin-semiadditive if and only if it is semiadditive in the classical sense.

Example 3.31 (m-semiadditivity)**.** Given an integer $-2 \leq m < \infty$, recall that a space is called *m-finite* for $-2 \leq m < \infty$ if it is m-truncated, has finitely many path components, and all its homotopy groups are finite. The subcategory $\mathrm{Spc}_m \subseteq \mathrm{Spc}$ by the m-finite spaces is pre-inductible, and a category $\mathcal{C}$ is Spc_m-semiadditive if and only if $\mathcal{C}$ is m-semiadditive in the sense of [HL13, Definition 4.4.2].

Example 3.32 (∞-semiadditivity)**.** Recall that a space is called *π-finite* if it is m-finite for some integer m. The subcategory $\mathrm{Spc}_\pi \subseteq \mathrm{Spc}$ of π-finite spaces is pre-inductible, and the associated notion of semiadditivity is that of *∞-semiadditivity* [CSY22, Definition 3.1.10]: a category $\mathcal{C}$ is ∞-semiadditive if and only if it is m-semiadditive for all $m \geq -2$.

Example 3.33 (p-typical m-semiadditivity)**.** As a variation on the previous two examples, let p be a prime and let $\mathrm{Spc}_m^{(p)} \subseteq \mathrm{Spc}_m$ be the full subcategory consisting of the m-finite p-spaces, i.e. those m-finite spaces all of whose homotopy groups are p-groups. Then $\mathrm{Spc}_m^{(p)}$ is pre-inductible, and the corresponding notion of semiadditivity is that of *p-typical m-semiadditivity* [CSY21, Definition 3.1.1]. Working with $\mathrm{Spc}_\pi^{(p)}$, the category of π-finite p-spaces, similarly gives the notion of *p-typical ∞-semiadditivity.*

Example 3.34. As the common generalization of the previous examples, let $Q \subseteq \mathrm{Spc}$ be a full subcategory of truncated spaces which is closed under base change and extensions and which satisfies $1 \in Q$. Then Q is pre-inductible, giving rise to a notion of Q-semiadditivity for categories $\mathcal{C}$. A non-parametrized category is Q-semiadditive in our sense if and only if it admits A-shaped colimits and A-shaped limits for every $A \in Q$, and the norm $\mathrm{Nm}_A \colon \mathrm{colim}_A \to \lim_A$ is an equivalence for each such A.

In fact, this is the most general form of semiadditivity our formalism provides in the non-parametrized setting: given an arbitrary locally inductible subcategory $\mathcal{Q} \subseteq \mathrm{Spc}$, the full subcategory $Q := \mathcal{Q}_{/1} \subseteq \mathrm{Spc}_{/1} = \mathrm{Spc}$ satisfies the above assumptions, and since $\mathcal{Q} = Q_{\mathrm{loc}}$ we see that $\mathcal{Q}$-semiadditivity agrees with Q-semiadditivity by Proposition 3.29.

It turns out that the individual categories of a parametrized semiadditive category inherit some degree of non-parametrized semiadditivity.

Lemma 3.35 (Fiberwise semiadditivity)**.** *Let $\mathcal{Q}$ be a locally inductible subcategory of a topos $\mathcal{B}$ and consider the full subcategory $Q_{\mathrm{fib}} \subseteq \mathrm{Spc}$ consisting of those spaces A which are truncated and for which the map $\mathrm{colim}_A 1 \to 1$ in $\mathcal{B}$ is contained in $\mathcal{Q}$.*

(1) The subcategory $Q_{\mathrm{fib}} \subseteq \mathrm{Spc}$ is pre-inductible;
(2) Every $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{C} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ is fiberwise Q_{fib}-semiadditive, i.e. factors through the (non-full) subcategory $\mathrm{Cat}^{Q_{\mathrm{fib}}\text{-}\oplus}$ of Q_{fib}-semiadditive categories.

Proof. Denote by $L \colon \mathrm{Spc} \to \mathcal{B}$ be the unique left exact left adjoint, given on objects by sending a space A to $\mathrm{colim}_A 1 \in \mathcal{B}$. Note that Q_{fib} consists precisely of those truncated spaces A such that the canonical map $L(A) \to L(\mathrm{pt}) = 1$ is in $\mathcal{Q}$. Since L preserves colimits and finite limits, it follows that Q_{fib} contains the point and is closed under base change and extensions, hence it is pre-inductible by Example 3.34.

Given now any object $X \in \mathcal{B}$, the functor $X \times L(-)$ is again a pullback-preserving left adjoint. Since $\mathcal{Q}$ is closed under base change, we see that $X \times L(A) \to X$ is in $\mathcal{Q}$ for all $A \in Q_{\mathrm{fib}}$, and thus by left-cancelability and locality of $\mathcal{Q}$ we deduce that $X \times L(-)$ maps all morphisms of $(Q_{\mathrm{fib}})_{\mathrm{loc}} \subseteq \mathrm{Spc}$ to morphisms in $\mathcal{Q}$. By Remark 3.11 we conclude that the category $\mathcal{C}(X)$ is Q_{fib}-semiadditive. Moreover, if $f \colon X \to Y$ is any map in $\mathcal{B}$, then the base change condition for $\mathcal{C}$ shows that $f^* \colon \mathcal{C}(Y) \to \mathcal{C}(X)$ preserves A-indexed (co)limits for $A \in Q_{\mathrm{fib}}$. It follows that $\mathcal{C}$ factors through $\mathrm{Cat}^{Q_{\mathrm{fib}}\text{-}\oplus}$, finishing the proof. □

Example 3.36. If $\mathcal{Q} \subseteq \mathcal{B}$ is a locally inductible subcategory containing the map $1 \amalg 1 \to 1$ (hence all fold maps $X \amalg X \to X$), then each $\mathcal{C}(X)$ is semiadditive in the usual sense, and each $f^* \colon \mathcal{C}(Y) \to \mathcal{C}(X)$ is a semiadditive functor.

We now come to the examples of semiadditivity that are truly parametrized.

Example 3.37 (G-semiadditivity)**.** For a finite group G the subcategory $\mathrm{Fin}_G \subseteq \mathrm{Spc}_G = \mathrm{PSh}(\mathrm{Orb}_G)$ of finite G-sets is pre-inductible. A *G-category* $\mathcal{C} \colon \mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Cat}$ is Fin_G-semiadditive if and only if $\mathcal{C}$ is G-semiadditive in the sense of [Nar16, QS21].

Example 3.38 (Equivariant semiadditivity)**.** Consider the subcategory $\mathrm{Glo} \subseteq \mathscr{F}$ of the $(2,1)$-category $\mathscr{F}$ of finite groupoids spanned by the *connected* finite groupoids, i.e. groupoids of the form BG for a finite group G. Consider also the wide subcategory $\mathscr{F}_{\dagger}$ of $\mathscr{F}$ spanned by the faithful functors between finite groupoids. Identifying $\mathscr{F}$ with the full subcategory of $\mathrm{PSh}(\mathrm{Glo})$ spanned by the finite disjoint unions of representable objects, the resulting subcategory $\mathscr{F}_\dagger$ of PSh(Glo) is pre-inductible. A *global category* $\mathcal{C}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}$ is $\mathscr{F}_\dagger$-semiadditive if and only if it is *equivariantly semiadditive* in the sense of [CLL23a, Example 4.5.2].

Example 3.39 (Global semiadditivity)**.** In fact, also the full subcategory $\mathscr{F} \subseteq \mathrm{PSh}(\mathrm{Glo})$ is pre-inductible. We will refer to the associated notion of semiadditivity as *global semiadditivity*. Informally, the difference to the notion from the previous example is that we now require that for *any* homomorphism $\alpha\colon H \to G$ of finite groups the restriction functor α^* admits both adjoints and that they agree, instead of just requiring this for subgroup inclusions.

The notion of global semiadditivity may be seen as a generalization of the notion of 1-semiadditivity from Example 3.31 as follows: Given a (non-parametrized) category $\mathcal{C}$ we may form its *Borelification* $\mathcal{C}^{\mathrm{Bor}}$, i.e. the global category defined via $\mathcal{C}^{\mathrm{Bor}}(BG) \coloneqq \mathrm{Fun}(BG, \mathcal{C})$; here we use the canonical embedding $\mathrm{Glo} \subseteq \mathscr{F} \hookrightarrow \mathrm{Spc}$. Since the essential image of the inclusion functor $\mathscr{F} \hookrightarrow \mathrm{Spc}$ is precisely the subcategory of 1-finite spaces, one observes that a category $\mathcal{C}$ is 1-semiadditive if and only if its Borelification is globally semiadditive, also cf. [CLL23c, Lemma 5.9]. In this sense, global semiadditivity generalizes 1-semiadditivity (see also Remark 9.14).

Example 3.40 (P-semiadditivity)**.** As a common generalization of Examples 3.30, 3.37, and 3.38 (but *not* of the previous example), let T be a small category and let $P \subseteq T$ be an atomic orbital subcategory, in the sense of [CLL23a, Definition 4.3.1]. Let $\mathbb{F}_T \subseteq \mathrm{PSh}(T)$ be the full subcategory of $\mathrm{PSh}(T)$ spanned by finite disjoint unions of representable presheaves, and let $\mathbb{F}_T^P \subseteq \mathbb{F}_T$ be the wide subcategory consisting of finite disjoint unions of morphisms of the form $\coprod_{i=1}^n p_i\colon \coprod_{i=1}^n A_i \to B$, where each morphism $p_i\colon A_i \to B$ lies in P. Then the subcategory $\mathbb{F}_T^P \subseteq \mathrm{PSh}(T)$ is pre-inductible. A T-category $\mathcal{C}$ is $\mathbb{F}_T^P$-semiadditive if and only if it is P-semiadditive in the sense of [CLL23a, Definition 4.5.1].

Example 3.41 (Very G-semiadditive G-procategories)**.** Let G be an arbitrary group. We denote by $\widehat{\mathrm{Orb}}_G \subseteq \mathrm{Orb}_G$ the full subcategory spanned by the orbits of the form G/H where H is a finite-index subgroup of G.[4] We refer to functors $\mathcal{C}\colon \widehat{\mathrm{Orb}}_G^{\mathrm{op}} \to \mathrm{Cat}$ as *G-procategories*.

In [Kal22, Definition 3.1], Kaledin considers G-sets S satisfying the following two conditions:

(1) For every $s \in S$ the stabilizer subgroup $G_s \subseteq G$ is cofinite.
(2) Every cofinite subgroup $H \subseteq G$ the fixed point set S^H is finite.

Following [KMN23], we will call such G-sets *quasi-finite*, and write QFin_G for the the full subcategory of Set_G spanned by them. Assigning to S the presheaf $G/H \mapsto S^H$ determines a fully faithful functor $\mathrm{QFin}_G \hookrightarrow \mathrm{PSh}(\widehat{\mathrm{Orb}}_G)$ which exhibits QFin_G as a pre-inductible subcategory. We say a profinite G-category $\mathcal{C}$ is *very G-semiadditive* if it is QFin_G-semiadditive.

Example 3.42 (Tempered ambidexterity)**.** Let $\mathcal{T}$ be a subcategory of Glo containing the final object 1, and consider the category $\mathrm{PSh}(\mathcal{T})$. We write $R\colon \mathrm{Spc} \to \mathrm{PSh}(\mathcal{T})$ for the fully faithful right adjoint of $\mathrm{ev}_1\colon \mathrm{PSh}(\mathcal{T}) \to \mathrm{Spc}$. We observe that $R(\mathrm{Spc}_\pi)$ is a pre-inductible subcategory of $\mathrm{PSh}(\mathcal{T})$. To see this, note that the category $\mathcal{T}$ is a full subcategory of Spc, and so by Yoneda's lemma $R(BG)$ is equivalent

[4] This subcategory is denoted Orb_G by [KMN23].

to the representable object associated to $G \in \mathcal{T}$. The remaining properties of a pre-inductible subcategory are inherited from π-finite spaces, using that R preserves limits. In [Lur19], Lurie considers the case where $\mathcal{T}$ is the full subcategory of Glo spanned by the groupoids with abelian isotropy. The main result of [Lur19] shows that the $\mathcal{T}$-category of tempered local systems associated to an oriented $\mathbf{P}$-divisible group is $R(\mathrm{Spc}_\pi)$-semiadditive.

Example 3.43. In [Sch23, Lecture 6], Scholze defined for every six-functor formalism $\mathcal{D}$ notions of *cohomologically proper* and *cohomologically étale* morphisms f: roughly speaking, this condition demands that the functor $f_!$ given by the covariant functoriality of the six-functor formalism is *right* (resp. *left*) adjoint to the morphism f^* coming from the contravariant functoriality in some preferred way.

Only remembering the contravariant functoriality, every six-functor formalism $\mathcal{D}$ forgets to a category parametrized by some category T. As we will show in future work, the class Q of maps in T that are both cohomologically étale and cohomology proper form an inductible subcategory of T, and $\mathcal{D}$ is Q-semiadditive in the sense of Definition 3.23.

3.5. Alternative characterizations of $\mathcal{Q}$-semiadditivity.

Let us close this section by discussing various equivalent definitions of $\mathcal{Q}$-semiadditivity:

Proposition 3.44. *Let $\mathcal{C}$ be a $\mathcal{Q}$-complete and $\mathcal{Q}$-cocomplete $\mathcal{B}$-category. The following are equivalent:*

(1) *The category $\mathcal{C}$ is $\mathcal{Q}$-semiadditive.*
(2) *The category $\mathcal{C}^{\mathrm{op}}$ is $\mathcal{Q}$-semiadditive.*
(3) *For every truncated $q\colon A \to B$ in $\mathcal{Q}$, the functor $q_!\colon \underline{\mathrm{Fun}}(\underline{A}, \pi_B^*\mathcal{C}) \to \pi_B^*\mathcal{C}$ preserves $\mathcal{Q}$-limits.*
(4) *For every truncated $q\colon A \to B$ in $\mathcal{Q}$, the functor $q_*\colon \underline{\mathrm{Fun}}(\underline{A}, \pi_B^*\mathcal{C}) \to \pi_B^*\mathcal{C}$ preserves $\mathcal{Q}$-colimits.*
(5) *For every pullback square*

$$\begin{array}{ccc} A' & \xrightarrow{p'} & A \\ {\scriptstyle q'}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ B' & \xrightarrow{p} & B \end{array}$$

consisting of truncated maps in $\mathcal{Q}$ the double Beck–Chevalley transformation $\mathrm{BC}_{!,*}\colon p_!q'_* \to q_*p'_!$ is an equivalence.
(6) *For every truncated map $q\colon A \to B$ in $\mathcal{Q}$ the double Beck–Chevalley transformation $\mathrm{BC}_{!,*}\colon q_!\mathrm{pr}_{1*} \to q_*\mathrm{pr}_{2!}$ associated to the pullback square*

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ A & \xrightarrow{q} & B. \end{array}$$

in $\mathcal{Q}$ is an equivalence.

Proof. We will prove that (1) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1). Dually, we then have (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (2), so that all statements are indeed equivalent.

For (1) ⇒ (3), note that $q_!$ is a left adjoint, so it preserves $\mathcal{Q}$-colimits by [MW21, Proposition 5.2.5], whence $\mathcal{Q}$-limits by Corollary 3.17. The implication (3) ⇒ (5) simply amounts to spelling out the definition of preserving $\mathcal{Q}$-limits, while (5) ⇒ (6) is immediate. Finally, for (6) ⇒ (1), we will be done by induction if we show that for any truncated q the norm map $\mathrm{Nm}_q\colon q_! \to q_*$ can be factored as the composite

$$q_! \simeq q_! \mathrm{pr}_{1*}\Delta_* \xrightarrow{\mathrm{Nm}_\Delta^{-1}} q_! \mathrm{pr}_{1*}\Delta_! \xrightarrow{\mathrm{BC}_{!,*}} q_* \mathrm{pr}_{2!}\Delta_! \simeq q_*.$$

Up to replacing $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$, the proof of this claim is identical to that of [CLL23a, Lemma 4.4.2] and will hence be omitted. □

We will later prove that one can equivalently drop all the truncatedness assumptions, see Corollary 6.12.

4. Parametrized span categories

Fix a topos $\mathcal{B}$ equipped with a locally inductible subcategory $\mathcal{Q} \subseteq \mathcal{B}$. The goal of this section is to construct the $\mathcal{B}$-category $\underline{\mathrm{Span}}(\mathcal{Q})$ and show it is $\mathcal{Q}$-semiadditive.

Definition 4.1. Let $\mathcal{Q}' \subseteq \mathcal{Q} \subseteq \mathcal{B}$ be another locally inductible subcategory. For every $A \in \mathcal{B}$, we denote by

$$\mathcal{Q}_{/A}[\mathcal{Q}'] \subseteq \mathcal{Q}_{/A}$$

the wide subcategory consisting of those morphisms in $\mathcal{Q}_{/A}$ whose underlying map in $\mathcal{Q}$ lies in $\mathcal{Q}'$. As $\mathcal{Q}'$ is a local class of morphisms, this defines a $\mathcal{B}$-subcategory $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}'] \subseteq \underline{\mathbf{U}}_{\mathcal{Q}}$.

Lemma 4.2. *For every object $A \in \mathcal{B}$, the category $\mathcal{Q}_{/A}$ admits finite limits and the wide subcategory $\mathcal{Q}_{/A}[\mathcal{Q}'] \subseteq \mathcal{Q}_{/A}$ is closed under base change.*

Proof. Note that $\mathcal{Q}_{/A}$ admits pullbacks by Lemma 3.2. It also admits a terminal object, given by the identity map on A, and thus admits all finite limits. The second claim is immediate from the fact that morphisms in $\mathcal{Q}'$ are closed under base change. □

Construction 4.3 (Parametrized span category)**.** Let $\mathcal{Q}_L, \mathcal{Q}_R \subseteq \mathcal{Q}$ be locally inductible subcategories of $\mathcal{B}$. By Lemma 4.2, the triple $(\mathcal{Q}_{/A}, \mathcal{Q}_{/A}[\mathcal{Q}_L], \mathcal{Q}_{/A}[\mathcal{Q}_R])$ is an adequate triple for every $A \in \mathcal{B}$, in the sense of [Bar17, Definition 5.2]. Since $\mathcal{Q}$, $\mathcal{Q}_L$ and $\mathcal{Q}_R$ are local classes, this defines a limit-preserving functor

$$(\underline{\mathbf{U}}_{\mathcal{Q}}, \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_L], \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R])\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{AdTrip}$$

to the category of adequate triples by [HHLN23, Lemma 2.4]. We define the *parametrized span category* $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ as the composite

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{(\underline{\mathbf{U}}_{\mathcal{Q}}, \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_L], \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R])} \mathrm{AdTrip} \xrightarrow{\mathrm{Span}} \mathrm{Cat}\,.$$

Since the functor Span: AdTrip → Cat is a right adjoint [HHLN23, Theorem 2.18] and hence preserves limits, it follows that $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ is indeed a $\mathcal{B}$-category. For simplicity, we will write $\underline{\mathrm{Span}}(\mathcal{Q})$ for $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}, \mathcal{Q})$, so that for $A \in \mathcal{B}$ we have

$$\underline{\mathrm{Span}}(\mathcal{Q})(A) = \mathrm{Span}(\mathcal{Q}_{/A}).$$

For a morphism $f\colon A \to B$ in $\mathcal{B}$, the restriction functor $f^*\colon \mathrm{Span}(\mathcal{Q}_{/B}) \to \mathrm{Span}(\mathcal{Q}_{/A})$ is given by pullback along f.

Warning 4.4. We warn the reader that the underlying category of $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ is *not* equivalent to $\mathrm{Span}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$, because $\mathcal{Q}_{/1} \neq \mathcal{Q}$.

The contravariant and covariant parts of the span categories give rise to canonical inclusions

$$\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_L]^{\mathrm{op}} \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \hookleftarrow \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R]$$

of $\mathcal{B}$-categories. For later reference, let us record the following property of these inclusions:

Following [Sha22, Definition 3.1] we define a *factorization system* on a $\mathcal{B}$-category $\mathcal{C}$ to be a pair (E, M) of wide $\mathcal{B}$-subcategories $E, M \subseteq \mathcal{C}$ such that for every $B \in \mathcal{B}$ the wide subcategories $E(B)$ and $M(B)$ of $\mathcal{C}(B)$ define a factorization system on $\mathcal{C}(B)$ in the sense of [Lur09, Definition 5.2.8.8]. We denote maps in E with the symbol $\twoheadrightarrow$ and maps in M with the symbol $\rightarrowtail$.

Proposition 4.5. *Let $\mathcal{Q}_L, \mathcal{Q}_R \subseteq \mathcal{Q}$ be locally inductible subcategories. The inclusions $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_L]^{\mathrm{op}}, \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R] \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ define the left and right class of a factorization system.*

Proof. This follows by applying [HHLN23, Proposition 4.9] levelwise. □

The parametrized span category $\underline{\mathrm{Span}}(\mathcal{Q})$ can be seen to be both $\mathcal{Q}$-complete and $\mathcal{Q}$-cocomplete, with the relevant adjoints given by applying $\mathrm{Span}(-)$ to the post-composition functor $q_!\colon \mathcal{Q}_{/A} \to \mathcal{Q}_{/B}$ for q in $\mathcal{Q}$. More generally, we have:

Proposition 4.6. *Let $\mathcal{Q}_L, \mathcal{Q}_R \subseteq \mathcal{Q} \subseteq \mathcal{B}$ be locally inductible subcategories.*

(1) The $\mathcal{B}$-category $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ admits $\mathcal{Q}_R$-colimits, and the inclusion of $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R]$ into $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ preserves $\mathcal{Q}_R$-colimits;
(2) Dually, $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ admits $\mathcal{Q}_L$-limits, and the inclusion of $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_L]^{\mathrm{op}}$ into $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ preserves $\mathcal{Q}_L$-limits.

Proof. Since $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)^{\mathrm{op}} \simeq \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_R, \mathcal{Q}_L)$, part (2) is dual to part (1). To prove (1), consider a morphism $q\colon A \to B$ in $\mathcal{Q}_R$, and consider the adjunction

$$q_!\colon \mathcal{Q}_{/A} \rightleftarrows \mathcal{Q}_{/B} :q^*,$$

where $q_!(f\colon A' \to A) = q \circ f$. Note that both functors preserve pullbacks and preserve morphisms in $\mathcal{Q}_L$ and $\mathcal{Q}_R$, so that they are morphisms of adequate triples.

We will apply [BH21, Corollary C.21] to show that this adjunction induces the required adjunctions at the level of span categories. In order to do so, let us spell out the unit and counit maps of this adjunction. For an object $(f\colon A' \to A) \in \mathcal{Q}_{/A}$, the unit of the adjunction is given by the map

$$A' \xrightarrow{(f,1)} A \times_B A' = q^* q_! A'$$

in $\mathcal{Q}_{/A}$. Observe that this map is a base change of the diagonal $\Delta_q\colon A \to A \times_B A$ of q, and hence lies in $\mathcal{Q}_R$ since $\mathcal{Q}_R$ is locally inductible. As a direct consequence of the pasting law of pullback squares, we see that for any other map $g\colon A'' \to A$

and any map $h\colon A'' \to A'$ over A, the naturality square

$$\begin{array}{ccc} A'' & \xrightarrow{(g,1)} & A \times_B A'' \\ {\scriptstyle h}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 1\times h} \\ A' & \xrightarrow{(f,1)} & A \times_B A' \end{array}$$

is a pullback square. Similarly, for an object $(g\colon B' \to B) \in \mathcal{Q}_{/B}$, the counit of the adjunction is provided by the projection map

$$q_!q^*B' = A \times_B B' \xrightarrow{\mathrm{pr}_2} B'$$

in $\mathcal{Q}_{/B}$, where the source lives over B via the composite $A \times_B B' \xrightarrow{\mathrm{pr}_1} A \xrightarrow{q} B$. This map is a base change of q, and thus lies in $\mathcal{Q}_R$. Again it follows from the pasting law of pullback squares that for any morphism $h\colon B'' \to B'$ over B the naturality square

$$\begin{array}{ccc} A \times_B B'' & \xrightarrow{\mathrm{pr}_2} & B'' \\ {\scriptstyle 1\times h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ A \times_B B' & \xrightarrow{\mathrm{pr}_2} & B' \end{array}$$

is a pullback square. Thus, [BH21, Corollary C.21] implies that the adjunction $q_!\colon \mathcal{Q}_{/A} \leftrightarrows \mathcal{Q}_{/B} : q^*$ induces an adjunction at the level of spans of the form

$$\mathrm{Span}(q_!)\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)(A) \rightleftarrows \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)(B) : \mathrm{Span}(q^*)$$

with unit and counit given by the right-pointing arrows associated to the original unit and counit. The Beck–Chevalley conditions for the adjunctions $q_! \dashv q^*$ then immediately imply the Beck–Chevalley conditions for the adjunctions $\mathrm{Span}(q_!) \dashv \mathrm{Span}(q^*)$. We conclude that $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ admits $\mathcal{Q}_R$-colimits and that the inclusion $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R] \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ preserves $\mathcal{Q}_R$-colimits. □

Corollary 4.7. *A functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \to \mathcal{C}$ preserves $\mathcal{Q}_R$-colimits if and only if the composite $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R] \overset{\iota}{\hookrightarrow} \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \xrightarrow{F} \mathcal{C}$ preserves $\mathcal{Q}_R$-colimits. The dual statement for $\mathcal{Q}_L$-limits also holds.*

Proof. Since ι preserves $\mathcal{Q}_R$-colimits by Proposition 4.6, the 'only if'-direction is clear. For the converse, assume that $F \circ \iota$ preserves $\mathcal{Q}_R$-colimits. We have to show that for any map $q\colon A \to B$ in $\mathcal{Q}_R$, the Beck–Chevalley transformation $q_!F_A \Rightarrow F_Bq_!$ filling the right square of the following diagram is an equivalence:

$$\begin{array}{ccccc} \mathcal{Q}_{/A}[\mathcal{Q}_R] & \overset{\iota_A}{\hookrightarrow} & \mathrm{Span}(\mathcal{Q}_{/A}, \mathcal{Q}_{/A}[\mathcal{Q}_L], \mathcal{Q}_{/A}[\mathcal{Q}_R]) & \xrightarrow{F_A} & \mathcal{C}_A \\ \big\downarrow{\scriptstyle q_!} & \overset{\simeq}{\Leftarrow} & {\scriptstyle q_!}\big\downarrow & \Leftarrow & \big\downarrow{\scriptstyle q_!} \\ \mathcal{Q}_{/B}[\mathcal{Q}_R] & \overset{\iota_B}{\hookrightarrow} & \mathrm{Span}(\mathcal{Q}_{/B}, \mathcal{Q}_{/B}[\mathcal{Q}_L], \mathcal{Q}_{/B}[\mathcal{Q}_R]) & \xrightarrow{F_B} & \mathcal{C}_B. \end{array}$$

Since the inclusion $\iota_A\colon \mathcal{Q}_{/A}[\mathcal{Q}_R] \hookrightarrow \mathrm{Span}(\mathcal{Q}_{/A}, \mathcal{Q}_{/A}[\mathcal{Q}_L], \mathcal{Q}_{/A}[\mathcal{Q}_R])$ is essentially surjective, it suffices to test this after precomposing by ι_A. But since Beck–Chevalley transformations compose and $\iota\colon \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R] \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ preserves $\mathcal{Q}_R$-colimits, this follows from the assumption that $F \circ \iota$ preserves $\mathcal{Q}_R$-colimits. This finishes the proof of the first statement. Once again the second statement is formally dual to the first. □

By Proposition 4.6, the restriction functors f^* of $\underline{\mathrm{Span}}(\mathcal{Q})$ admit both left and right adjoints which canonically agree, suggesting that it is $\mathcal{Q}$-semiadditive. This is indeed the case:

Lemma 4.8. *Let $\mathcal{Q}_L, \mathcal{Q}_R \subseteq \mathcal{Q} \subseteq \mathcal{B}$ be locally inductible subcategories. Then the $\mathcal{B}$-category $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ is $(\mathcal{Q}_L \cap \mathcal{Q}_R)$-semiadditive.*

Proof. Let $q\colon B \to C$ be a morphism in $\mathcal{Q}_R$. By Proposition 4.6, the functor $q_!\colon \mathrm{Span}(\mathcal{Q}_{/B}, \mathcal{Q}_{/B}[\mathcal{Q}_L], \mathcal{Q}_{/B}[\mathcal{Q}_R]) \to \mathrm{Span}(\mathcal{Q}_{/C}, \mathcal{Q}_{/C}[\mathcal{Q}_L], \mathcal{Q}_{/C}[\mathcal{Q}_R])$ is given on objects by $q_!(f) = qf$ for any $(f\colon A \to C) \in \mathcal{Q}_{/B}$, with unit

$$A \xleftarrow{=} A \xrightarrow{(\mathrm{id},f)} A \times_C B.$$

The associated counit is then given on any $g\colon A \to C$ by

$$A \times_C B \xleftarrow{=} A \times_C B \xrightarrow{\mathrm{pr}_1} A.$$

Dually, if q is in $\mathcal{Q}_L$, then $q_*f = qf$ with counit and unit given by the flipped spans

$$A \times_C B \xleftarrow{(\mathrm{id},f)} A \xrightarrow{=} A \qquad \text{and} \qquad A \xleftarrow{\mathrm{pr}_1} A \times_C B \xrightarrow{=} A \times_C B,$$

respectively. Thus, the duality equivalence $\underline{\mathrm{Span}}(\mathcal{Q}) \simeq \underline{\mathrm{Span}}(\mathcal{Q})^{\mathrm{op}}$ flipping spans exhanges the adjunctions $q_! \dashv q^*$ and $q^* \dashv q_*$ for any $q \in \mathcal{Q}_L \cap \mathcal{Q}_R$, and hence maps the Beck–Chevalley maps $q_!\mathrm{pr}_1^* \to \mathrm{pr}_2^* q_!$ and $\mathrm{pr}_2^* q_* \to q_*\mathrm{pr}_1^*$ to each other (note $q_! = q_*$). As the above equivalence is given on groupoid cores by inverting, we conclude that these two Beck–Chevalley maps are just inverse to each other, and so the double Beck–Chevalley transformation from part (6) of Proposition 3.44 is simply the canonical equivalence $q_!\mathrm{pr}_{1*} = q_!\mathrm{pr}_{1!} \xrightarrow{\sim} q_!\mathrm{pr}_{2!} = q_*\mathrm{pr}_{2!}$. In particular, $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ is $(\mathcal{Q}_L \cap \mathcal{Q}_R)$-semiadditive by Proposition 3.44 as claimed. □

Remark 4.9. Using the inductive description of the norm maps in terms of the double Beck–Chevalley maps given in the proof of Proposition 3.44, we see that with respect to the above choices the norm map $q_! \to q_*$ is simply the identity for every truncated $q \in \mathcal{Q}_L \cap \mathcal{Q}_R$. We immediately get that the adjoint norm map $\widetilde{\mathrm{Nm}}\colon q^*q_! \to \mathrm{id}$ and the corresponding map $\mu\colon \mathrm{id} \to q_!q^*$ are just the counit and unit $q^*q_! \to \mathrm{id}$ and $\mathrm{id} \to q_!q^*$ constructed in the above proof, i.e. they are given by flipping the spans representing the unit and counit, respectively, of $q_! \dashv q^*$.

5. The universal property of parametrized spans

Throughout this section, we fix a topos $\mathcal{B}$ together with a locally inductible subcategory $\mathcal{Q} \subseteq \mathcal{B}$. The goal of this section is to show that the parametrized span category $\underline{\mathrm{Span}}(\mathcal{Q})$ admits a universal property: it is the free $\mathcal{Q}$-semiadditive $\mathcal{B}$-category on a single generator. More precisely, if we denote by $\mathrm{pt}\colon \underline{1} \to \underline{\mathrm{Span}}(\mathcal{Q})$ the functor given in degree $B \in \mathcal{B}$ by the object $\mathrm{id}_B \in \mathcal{Q}_{/B} \subseteq \mathrm{Span}(\mathcal{Q}_{/B})$, we have:

Theorem 5.1. *For every $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{D}$, restriction along the functor $\mathrm{pt}\colon \underline{1} \to \underline{\mathrm{Span}}(\mathcal{Q})$ induces an equivalence of $\mathcal{B}$-categories.*

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\oplus}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

As we will explain at the end of this section, this can be seen as a parametrized generalization of a theorem of Harpaz [Har20] identifying the free non-parametrized m-semiadditive category. Our proof of Theorem 5.1 is inspired by an alternative approach to this non-parametrized result due to Lior Yanovski, and we would like to thank him for sharing his ideas and notes with us.

The first key technical ingredient needed for the proof of the universal property is an extension result for functors out of parametrized span categories, letting us increase the number of right-pointing arrows on which the functor is defined. We will present a general form of this result in Section 5.1 and specialize to span categories in Section 5.2. As the second key ingredient, we prove in Section 5.3 that for functors into a semiadditive category this result can be dualized, allowing us to increase the number of *left*-pointing arrows. Combining these ingredients, we then establish the universal property in Section 5.4.

5.1. **The coSegal condition.** This section contains one of the main technical ingredient needed for the proof of the universal property of parametrized spans: the existence of unique extensions for so-called *coSegal functors*, see Proposition 5.18 below. We start with various definitions.

Definition 5.2 (Distinguished object)**.** Let (E, M) be a factorization system on a $\mathcal{B}$-category $\mathcal{C}$. We say an object $X \in \Gamma M \subseteq \Gamma\mathcal{C}$ is *distinguished* if the corresponding $\mathcal{B}$-functor $X\colon \underline{1} \to M$ is fully faithful. We will frequently denote a distinguished object by pt, and denote the corresponding inclusion by $\{\mathrm{pt}\} \hookrightarrow M$. By restriction we also obtain an object $\mathrm{pt}_B \coloneqq B^*\mathrm{pt} \in \mathcal{C}(B)$ for every $B \in \mathcal{B}$. Given an object $X \in \mathcal{C}(B)$, we refer to maps in M of the form $\mathrm{pt}_B \rightarrowtail X$ as *coSegal maps*.

Given the inclusion $\mathcal{C} \subseteq \mathcal{D}$ of a subcategory and an object $X \in \Gamma\mathcal{C}$, we write $\mathcal{C}_{/X}$ for the pullback $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/X}$.

Definition 5.3. Let $F\colon \mathcal{C}' \to \mathcal{D}$ be a functor, and let $\mathcal{C} \subseteq \mathcal{C}'$ be a full subcategory. We say F is (pointwise) *left Kan extended from* $\mathcal{C}$ if for every object $A \in \mathcal{B}$ and every object $X \in \mathcal{C}'(A)$, the $\mathcal{B}_{/A}$-parametrized colimit of the composite $\pi_A^*\mathcal{C}_{/X} \to \pi_A^*\mathcal{C} \to \pi_A^*\mathcal{D}$ exists and the canonical map

$$\mathrm{colim}_{\pi_A^*\mathcal{C}_{/X}}\, \pi_A^* F \to F_A(X)$$

is an equivalence.

Remark 5.4. It follows from [MW21, Remark 6.3.6] that a pointwise left Kan extension admits the universal property of a left Kan extension.

Definition 5.5 (CoSegal functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{B}$-categories, let (E, M) be a factorization system on $\mathcal{C}$, and let pt be a distinguished object. We say a $\mathcal{B}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is *coSegal* if $F|_M$ is left Kan extended from pt. We let $\mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of coSegal $\mathcal{B}$-functors.

Example 5.6. As we will show in the next subsection, pt is a distinguished object for the standard factorization system on $\underline{\mathrm{Span}}(\mathcal{Q})$, and a functor $\underline{\mathrm{Span}}(\mathcal{Q}) \to \mathcal{D}$ into a $\mathcal{Q}$-cocomplete category is coSegal if and only if it is $\mathcal{Q}$-cocontinuous.

Given a general subcategory $\mathcal{C}^\circ \subseteq \mathcal{C}$, it does not even make sense to ask whether the restriction of a coSegal functor on $\mathcal{C}$ to $\mathcal{C}^\circ$ is again coSegal. We therefore introduce:

Definition 5.7 (Adapted subcategory)**.** Let (E, M) be a factorization system on an non-parametrized category $\mathcal{C}$. We call a subcategory $\mathcal{C}^\circ \subseteq \mathcal{C}$ *adapted* if the following two conditions hold for every object $X \in \mathcal{C}^\circ$:

(1) Every morphism $X \twoheadrightarrow Y$ in $\mathcal{C}$ belongs to $\mathcal{C}^\circ$.
(2) For a morphism $e\colon X \twoheadrightarrow X'$ in $\mathcal{C}$, a morphism $f\colon X' \to Y$ belongs to $\mathcal{C}^\circ$ if and only if the composite $fe\colon X \to Y$ belongs to $\mathcal{C}^\circ$.

Given a parametrized factoriation system (E, M) on some $\mathcal{B}$-category $\mathcal{C}$, we call a parametrized subcategory $\mathcal{C}^\circ \subseteq \mathcal{C}$ *adapted* if each $\mathcal{C}^\circ(A) \subseteq \mathcal{C}(A)$ is adapted in the above sense.

Example 5.8. A *full* subcategory $\mathcal{C}^\circ \subseteq \mathcal{C}$ is adapted if and only if it satisfies the following (a priori weaker) version of the first axiom: *given $X \twoheadrightarrow Y$ with $X \in \mathcal{C}^\circ$, also $Y \in \mathcal{C}^\circ$.*

Corollary 5.9. *If $\mathcal{C}^\circ \subseteq \mathcal{C}$ is adapted, then $E^\circ \coloneqq E \cap \mathcal{C}^\circ$ and $M^\circ \coloneqq M \cap \mathcal{C}^\circ$ form a factorization system on $\mathcal{C}^\circ$.*

Proof. It suffices to prove the non-parametrized statement. First note that if f is a morphism in $\mathcal{C}^\circ$ with factorization $f = me$ in $(\mathcal{C}, E, M)$, then e belongs to E° by the first axiom of an adapted subcategory, so m belongs to M° by the second axiom.

It then only remains to show that E° and M° are orthogonal to each other, which amounts to saying that for every commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow \\ X' & \longrightarrow & Y' \end{array}$$

in $\mathcal{C}^\circ$ the essentially unique lift $X' \to Y$ in $\mathcal{C}$ already belongs to $\mathcal{C}^\circ$. This is again immediate from part (2) of the definition. $\square$

Definition 5.10 (Good subcategory)**.** Let $\mathcal{C}$ be equipped with a factorization system (E, M) and a distinguished object pt. We say that a subcategory $\mathcal{C}^\circ$ is *good* if it adapted and contains the coSegal maps $\mathrm{pt}_B \rightarrowtail X$ for all $X \in \mathcal{C}^\circ(B)$.

Note that for a good subcategory $\mathcal{C}^\circ$ we may now again talk about coSegal functors: Corollary 5.9 shows that (E°, M°) is a parametrized factorization system on $\mathcal{C}^\circ$, while the fact that $\mathcal{C}^\circ$ contains all coSegal maps ensures that $\mathcal{C}^\circ$ inherits a distinguished object $\mathrm{pt}^\circ = \mathrm{pt}$. The goal for the rest of this section is to show that every coSegal functor on $\mathcal{C}^\circ$ uniquely extends to $\mathcal{C}$ and that all coSegal functors on $\mathcal{C}$ are of this form. We start with the following preliminary result:

Lemma 5.11. *Restriction along a good inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ takes coSegal functors to coSegal functors, and the resulting functor $\mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}^\circ, \mathcal{D})$ is conservative.*

Proof. For the first claim, we must show that for a coSegal functor $F\colon \mathcal{C} \to \mathcal{D}$ the restriction $F|_{M^\circ}$ is pointwise left Kan extended from $\{\mathrm{pt}\}$. By definition, this requires the comparison of $F_A X$ with a certain colimit indexed by a category of morphisms $\mathrm{pt}_B \rightarrowtail X$ in M° for $X \in C^\circ(B)$. But by assumption this indexing category agrees with the category of *all* morphisms $\mathrm{pt}_B \rightarrowtail X$ in M, hence the claim is immediate from the coSegal property of F.

For the second claim, it suffices to observe that further composition with evaluation at $\mathrm{pt} \in \mathcal{C}$ is conservative: it equals the composition

$$\mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{coSeg}}(M, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\{\mathrm{pt}\}, \mathcal{D}) \simeq \mathcal{D},$$

where the first functor is conservative since $M \subseteq \mathcal{C}$ is wide, and the second functor is an equivalence by the definition of being coSegal. □

Next, we show that the coSegal condition behaves well with respect to left Kan extension along the inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$. For an object $X \in \mathcal{C}$ and an adapted subcategory $\mathcal{C}^\circ \subseteq \mathcal{C}$, we use the notation

$$\mathcal{C}^\circ_{/X} := \mathcal{C}^\circ \times_{\mathcal{C}} \mathcal{C}_{/X} \qquad \text{and} \qquad M^\circ_{/X} := M^\circ \times_M M_{/X}.$$

Lemma 5.12. *For every global section $X \in \Gamma\mathcal{C}$, the inclusion $M_{/X} \hookrightarrow \mathcal{C}_{/X}$ admits a parametrized left adjoint, and the resulting adjunction $\mathcal{C}_{/X} \rightleftarrows M_{/X}$ restricts to $\mathcal{C}^\circ_{/X} \rightleftarrows M^\circ_{/X}$.*

Proof. Let $A \in \mathcal{B}$. Then the inclusion $M_{/X}(A) \hookrightarrow \mathcal{C}_{/X}(A)$ can be identified with the inclusion $M(A)_{/X_A} \to \mathcal{C}(A)_{/X_A}$. By [CLL23b, proof of Proposition 3.33] the latter is fully faithful and it admits a left adjoint λ_A such that the unit consists of maps in $\mathcal{C}(A)_{/X_A} \times_{\mathcal{C}(A)} E(A)$. To see that the λ_A's assemble into a parametrized functor, it will be enough to check the Beck–Chevalley condition. By full faithfulness of the inclusions this just amounts to saying that for every $f\colon A \to A'$ there is *some* dashed arrow filling

$$\begin{array}{ccc} \mathcal{C}(A')_{/X_{A'}} & \xrightarrow{f^*} & \mathcal{C}(A)_{/X_A} \\ {\scriptstyle \lambda_{A'}}\downarrow & & \downarrow{\scriptstyle \lambda_A} \\ M(A')_{/X_{A'}} & \dashrightarrow & M(A)_{/X_A}. \end{array}$$

This follows at once from the fact that the vertical arrows are localizations at $\mathcal{C}(A')_{/X_{A'}} \times_{\mathcal{C}(A')} E(A')$ and $\mathcal{C}(A)_{/X_A} \times_{\mathcal{C}(A)} E(A)$, respectively, by *loc. cit.*

Next we show the second statement, which entails showing that the left adjoint λ as well as the unit and counit restrict accordingly. The fact that λ_A sends objects of $\mathcal{C}^\circ_{/X}$ to objects of $M^\circ_{/X}$ and that the unit lies pointwise in $\mathcal{C}^\circ_{/X}$ is immediate from the first axiom of an adapted subcategory, while the second axiom guarantees that it maps morphisms of $\mathcal{C}^\circ_{/X}$ to morphisms in $M^\circ_{/X}$. Finally, the counit is an equivalence, whence lies pointwise in $M^\circ_{/X}$ as claimed. □

Corollary 5.13. *The inclusion $M^\circ_{/X} \hookrightarrow \mathcal{C}^\circ_{/X}$ is final in the sense of* [MW21, Proposition 4.6.1]*, i.e. $\mathcal{C}^\circ_{/X}$-shaped colimits can be computed after restricting to $M^\circ_{/X}$.*

Proof. We will prove more generally that any parametrized right adjoint $R\colon \mathcal{C} \to \mathcal{D}$ is final. By Quillen's Theorem A for parametrized categories, see [Mar21, Corollary 4.4.8], it suffices to show that the comma category $\pi_A^*\mathcal{C}_{X/}$ is weakly contractible for all $X \in \mathcal{D}(A)$. However by [MW21, Corollary 3.3.5] this category even admits an initial object, and so is clearly weakly contractible. □

Proposition 5.14. *Let $\iota\colon \mathcal{C}^\circ \hookrightarrow \mathcal{C}$ be a good inclusion such that $\mathcal{C}$ is small, and suppose that $\mathcal{D}$ admits all colimits. Then the Beck–Chevalley transformation filling* *the square*

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}^\circ, \mathcal{D}) & \xrightarrow{\iota_!} & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \\ {\scriptstyle -|_{M^\circ}}\downarrow & & \downarrow{\scriptstyle -|_M} \\ \mathrm{Fun}(M^\circ, \mathcal{D}) & \xrightarrow{\iota_!} & \mathrm{Fun}(M, \mathcal{D}) \end{array}$$

is an equivalence.

Note that in the above situation the two left Kan extension functors indeed exist by [MW21, Corollary 6.3.7].

Proof. Consider a functor $F\colon \mathcal{C}^\circ \to \mathcal{D}$ and let $X \in M(A)$. Then a quick computation shows that the composite

$$\mathrm{colim}_{\pi_A^* M^\circ_{/X}} F \simeq \iota_!(F|_{M^\circ})(X) \to (\iota_! F)|_M(X) \simeq \mathrm{colim}_{\pi_A^* \mathcal{C}^\circ_{/X}} F$$

is induced on colimits by the map $\pi_A^* M^\circ_{/X} \to \pi_A^* \mathcal{C}^\circ_{/X}$ of $\mathcal{B}_{/A}$-categories. However we note that $\pi_A^*\mathcal{C}^\circ \hookrightarrow \pi_A^*\mathcal{C}$ is again a good inclusion, and so this is an equivalence by the previous corollary. □

Definition 5.15. Let $\mathcal{C}$ be a $\mathcal{B}$-category equipped with a factorization system and a distinguished object. We say a $\mathcal{B}$-category $\mathcal{D}$ *admits $\mathcal{C}$-coSegal colimits* if for every object $Y \in \Gamma\mathcal{D}$ the pointwise left Kan extension of $Y\colon \{\mathrm{pt}\} \to \mathcal{D}$ along the inclusion $\{\mathrm{pt}\} \hookrightarrow M$ exists.

Remark 5.16. Suppose $\mathcal{D}$ admits $\mathcal{C}$-coSegal colimits and suppose $\mathcal{C}^\circ \subseteq \mathcal{C}$ is a good inclusion. Then $\mathcal{D}$ also admits $\mathcal{C}^\circ$-coSegal colimits.

Lemma 5.17. *Let $\mathcal{C}^\circ \subseteq \mathcal{C}$ be a good subcategory and suppose $\mathcal{D}$ is a $\mathcal{B}$-category with $\mathcal{C}$-coSegal colimits. For every coSegal $\mathcal{B}$-functor $F^\circ\colon \mathcal{C}^\circ \to \mathcal{D}$, there exists a left Kan extension $F\colon \mathcal{C} \to \mathcal{D}$ along the inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$. Furthermore, the $\mathcal{B}$-functor F is again coSegal, and the canonical map $F^\circ(\mathrm{pt}) \to F(\mathrm{pt})$ is an equivalence.*

Proof. By changing universe we may assume that $\mathcal{C}$ is small. Let us first assume that $\mathcal{D}$ is cocomplete, so that we may apply Proposition 5.14 to deduce the restriction $F|_M\colon M \to \mathcal{D}$ is the left Kan extension along $M^\circ \hookrightarrow M$ of the restriction $F^\circ|_{M^\circ}$. Because F° is coSegal, $F^\circ|_{M^\circ}$ is itself left Kan extended from $\{\mathrm{pt}\}$, and thus it follows from transitivity of left Kan extension that also $F|_M$ is left Kan extended from $\{\mathrm{pt}\}$, i.e. that F is coSegal. The final claim follows from the assumption that the inclusions $\{\mathrm{pt}\} \hookrightarrow M^\circ$ and $\{\mathrm{pt}\} \hookrightarrow M$ are fully faithful.

For arbitrary $\mathcal{D}$, pick an embedding $\mathcal{D} \hookrightarrow \mathcal{D}'$ into a cocomplete $\mathcal{B}$-category $\mathcal{D}'$ which preserves all colimits that exists in $\mathcal{D}$ (e.g. the coYoneda embedding). By the previous paragraph, the left Kan extension F of F° exists as a functor into $\mathcal{D}'$ and has the required properties. Furthermore as we have seen, once F is restricted to M it is pointwise left Kan extended from $\{\mathrm{pt}\}$ and hence lands in $\mathcal{D}$ by the assumption that $\mathcal{D}$ has coSegal colimits. However $M \subseteq \mathcal{C}$ is a wide subcategory, and so F itself lands in $\mathcal{D}$. □

Putting everything together, we obtain the main result of this subsection:

Proposition 5.18. *Consider a good inclusion $\iota\colon \mathcal{C}^\circ \hookrightarrow \mathcal{C}$, and let $\mathcal{D}$ be a $\mathcal{B}$-category with $\mathcal{C}$-coSegal colimits. Then restriction along $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ induces an equivalence of categories*

$$\mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}^\circ, \mathcal{D}).$$

Proof. By Lemma 5.11 the restriction functor $\iota^*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^\circ, \mathcal{D})$ restricts to a conservative functor $\mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}^\circ, \mathcal{D})$. By Lemma 5.17, this restriction admits a left adjoint

$$\iota_!\colon \mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}^\circ, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{coSeg}}(\mathcal{C}, \mathcal{D}),$$

given by left Kan extension along ι. Since ι^* is conservative when restricted to the coSegal functors, it remains to show that the unit $\mathrm{id} \to \iota^*\iota_!$ of this adjunction is an equivalence. This follows again from Lemma 5.17 and the fact that evaluation at the point is conservative. □

5.2. **Good inclusions of span categories.** Our main interest in Proposition 5.18 is in the case where $\mathcal{C}$ is a parametrized span category. In this subsection, we show that various inclusions of parametrized span categories are good inclusions.

Convention 5.19. Throughout this section, we fix locally inductible subcategories $\mathcal{Q}_L, \mathcal{Q}_R \subseteq \mathcal{Q} \subseteq \mathcal{B}$. We will always equip the span category $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ with the canonical factorization system from Proposition 4.5, in which the left class consists of the left-pointing maps and the right class consists of the right-pointing maps:

$$A \twoheadrightarrow B = (A \leftarrow B = B),$$
$$A \rightarrowtail B = (A = A \to B).$$

We further take the distinguished object of the span category to always be the identity map $\mathrm{pt} := (1 \to 1) \in \iota(\mathcal{Q}_{/1}) \subseteq \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)(1)$.

As a first step, we show that the $\mathcal{C}$-coSegal colimits for $\mathcal{C} = \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q})$ can be expressed in terms of $\mathcal{Q}$-colimits.

Proposition 5.20. *Let $\mathcal{D}$ be $\mathcal{Q}$-cocomplete. Then $\mathcal{D}$ has $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q})$-coSegal colimits and a functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}) \to \mathcal{D}$ is coSegal if and only if it preserves $\mathcal{Q}$-colimits.*

Proof. By Proposition 2.20, the left Kan extension of any $\underline{1} \to \mathcal{D}$ along $\{\mathrm{pt}\} \hookrightarrow \underline{\mathbf{U}}_{\mathcal{Q}}$ exists exists, and a functor $\underline{\mathbf{U}}_{\mathcal{Q}} \to \mathcal{D}$ arises this way if and only if it is $\mathcal{Q}$-cocontinuous. The claim follows as by Corollary 4.7 a functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}) \to \mathcal{D}$ preserves $\mathcal{Q}$-colimits if and only if the composite $G\colon \underline{\mathbf{U}}_{\mathcal{Q}} \to \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}) \to \mathcal{D}$ preserves $\mathcal{Q}$-colimits. □

We now provide two examples of good inclusions of parametrized span categories.

Lemma 5.21. *The inclusion*

$$\iota\colon \underline{\mathrm{Span}}(\mathcal{Q}_L) \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q})$$

is good. In particular, for a $\mathcal{B}$-category $\mathcal{D}$ admitting $\mathcal{Q}$-colimits, restriction along ι induces an equivalence of ∞-categories

$$\iota^*\colon \mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}_L), \mathcal{D}).$$

Proof. Note that $\underline{\mathrm{Span}}(\mathcal{Q}_L)$ is a *full* subcategory of $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q})$ by left-cancelability, and obviously contains pt. Therefore $\underline{\mathrm{Span}}(\mathcal{Q}_L)$ is clearly good if it is adapted. Condition (1) follows from left-cancelability of $\mathcal{Q}_L$, while Condition (2) is automatic. The final claim follows immediately from Proposition 5.18 and Proposition 5.20. $\square$

Notation 5.22. Given any collection of maps $\mathcal{Q} \subseteq \mathcal{B}$, we write $\Delta(\mathcal{Q})$ for the collection of all maps in $\mathcal{B}$ of the form $\Delta_q \colon A \to A \times_B A$ for morphisms $q \colon A \to B$ in $\mathcal{Q}$.

Lemma 5.23. *Assume that $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_R$. Then the inclusion*

$$\iota \colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q})$$

is good. In particular, for any $\mathcal{B}$-category $\mathcal{D}$ admitting $\mathcal{Q}$-colimits, restriction along ι induces an equivalence of ∞-categories

$$\mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R), \mathcal{D}).$$

Proof. Note that $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ contains all left-pointing maps in $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q})$, and thus condition (1) of an adapted subcategory is immediate. Condition (2) follows from a simple calculation of the composition in the relevant span categories, showing that $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ is an adapted subcategory. To see it is even a good subcategory, it remains to show that it contains the coSegal maps $s \colon \mathrm{pt}_B \rightarrowtail A$ for all $A \in \mathcal{Q}_{/B}$, which boils down to showing that for every span

$$\begin{array}{ccccc} B & = & B & \xrightarrow{s} & A \\ & \searrow\!\!\!\searrow & \| & \swarrow_{f} & \\ & & B & & \end{array}$$

in $\mathcal{Q}_{/B}$ the morphism s is contained in $\mathcal{Q}_R$. To this end, consider the following diagram in $\mathcal{Q}$

$$\begin{array}{ccccc} B & \xrightarrow{s} & A & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ A & \xrightarrow{\Delta} & A \times_B A & \longrightarrow & A \\ & & \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ & & A & \xrightarrow[f]{} & B \end{array}$$

in which all squares are pullbacks. Since the diagonal map $\Delta \colon A \to A \times_B A$ lies in $\mathcal{Q}_R$ by assumption, also its base change $s \colon B \to A$ lies in $\mathcal{Q}_R$ as desired. We conclude that $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ is a good subcategory. The final claim follows immediately from Proposition 5.18 and Proposition 5.20. $\square$

Using this, we can give a more concrete description of coSegal functors out of $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$ when $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_R$, generalizing Proposition 5.20.

Construction 5.24 (CoSegal map)**.** Let $\mathcal{D}$ be a $\mathcal{B}$-category admitting $\mathcal{Q}$-colimits, and let $F \colon \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R] \to \mathcal{D}$ be a $\mathcal{B}$-functor. Assume moreover that $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_R$. We will construct for every morphism $q \colon A \to B$ in $\mathcal{Q}$ a *coSegal map*

$$\mathrm{coSegal} \colon q_! F(\mathrm{id}_A) \to F(q).$$

For this, let $F'\colon \underline{\mathbf{U}}_{\mathcal{Q}} \to \mathcal{D}$ be the left Kan extension of $F(\mathrm{id}_1)\colon \underline{1} \to \mathcal{D}$. By Lemma 5.23 (for $\mathcal{Q}_L = \iota\mathcal{Q}$), the restriction of F' to $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R]$ is still left Kan extended along $\{\mathrm{id}_1\} \hookrightarrow \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R]$, and hence there exists a unique natural transformation

$$\mathrm{coSegal}\colon F'|_{\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R]} \to F$$

which results in the identity of $F(\mathrm{id}_1)$ when evaluated on id_1. Since F' preserves $\mathcal{Q}$-colimits by Proposition 2.20, we have $F'(q) \simeq q_!F'(\mathrm{id}_A) = q_!F(\mathrm{id}_A)$ for every morphism $q\colon A \to B$ in $\mathcal{Q}$, resulting in the desired map

$$q_!F(\mathrm{id}_A) \simeq F'(q) \xrightarrow{\mathrm{coSegal}} F(q).$$

Construction 5.25. We continue to assume that $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_R$. Consider any $\mathcal{B}$-functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \to \mathcal{D}$. For every morphism $q\colon A \to B$ in $\mathcal{Q}$, we may apply Construction 5.24 to the restriction of F to $\underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R]$ to obtain a coSegal map

$$\mathrm{coSegal}\colon q_!F(\mathrm{id}_A) \to F(q).$$

Unwinding definitions, we see that F is a coSegal functor if and only if the coSegal map is an equivalence for every $q \in \mathcal{Q}$.

5.3. **CoSegal functors as Segal functors.** While Lemma 5.23 allows us to extend the *covariant* functoriality of a functor out of a parametrized span category, we will also need an analogous result which lets us extend the *contravariant* functoriality, at least under suitable semiadditivity assumptions on $\mathcal{D}$. We will accomplish this by showing that in this case a functor out of a parametrized span category is coSegal if and only if it is *Segal*, defined dually. We continue using the notations from Convention 5.19.

Definition 5.26 (Segal functor)**.** A functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \to \mathcal{D}$ is called *Segal* if the composite

$$\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_R, \mathcal{Q}_L) \simeq \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}}$$

is coSegal. We denote by

$$\mathrm{Fun}_{\mathrm{Seg}}(\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R), \mathcal{D}) \subseteq \mathrm{Fun}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R), \mathcal{D})$$

the full subcategory spanned by the Segal functors.

The following is the main result of this subsection:

Proposition 5.27. *Let $\mathcal{D}$ be a $\mathcal{Q}$-semiadditive $\mathcal{B}$-category. If $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_L \cap \mathcal{Q}_R$, then a $\mathcal{B}$-functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \to \mathcal{D}$ is Segal if and only if it is coSegal.*

Before discussing the proposition, let us record the main corollary:

Corollary 5.28. *Let $\mathcal{D}$ be a $\mathcal{Q}$-semiadditive $\mathcal{B}$-category and assume that $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_L$. Then restricting along the inclusion $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}) \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q})$ defines an equivalence of categories*

$$\mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}), \mathcal{D}).$$

Proof. By Proposition 5.27, we may equivalently show the claim for Segal functors. Unwinding definitions, this reduces to the claim that restriction along the inclusion $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}, \mathcal{Q}_L) \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q})$ defines an equivalence

$$\mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}^{\mathrm{op}}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}, \mathcal{Q}_L), \mathcal{D}^{\mathrm{op}}).$$

This is a special case of Lemma 5.23. □

The proposition should not be too surprising: as explained above, the coSegal condition amounts to demanding that for every $q \in \mathcal{Q}$ a certain map $q_! F(\mathrm{pt}) \to F(q)$ is an equivalence, while the Segal condition amounts to saying that the dually defined map $F(q) \to q_* F(\mathrm{pt})$ is an equivalence. The equivalence $\mathrm{Nm}_q \colon q_! F(\mathrm{pt}) \simeq q_* F(\mathrm{pt})$ coming from $\mathcal{Q}$-semiadditivity of $\mathcal{D}$ thus strongly suggests that these two conditions are equivalent. While this is true, relating the above two maps (defined basically in terms of maps in $\underline{\mathrm{Span}}(\mathcal{Q})$) to the norm map Nm_q of $\mathcal{D}$ turns out to be somewhat subtle and will take up the remainder of this subsection.

We begin by describing the coSegal map explicitly in the above situation:

Lemma 5.29. *Let $q \colon A \to B$ be a morphism in $\mathcal{Q}$ and assume $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_R$. Then the coSegal map $q_! F(\mathrm{id}_A) \to F(q)$ is adjoint to*

$$F(\mathrm{id}_A) \xrightarrow{F(\Delta_q)} F(q^*(q)) = q^* F(q).$$

Here the map $\Delta_q \colon \mathrm{id}_A \to q^*(q) = (\mathrm{pr}_1 \colon A \times_B A \to A)$ is the morphism in the slice $\mathcal{Q}_{/A}[\mathcal{Q}_R]$ corresponding to the diagonal map $\Delta_q \colon A \to A \times_B A$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
F'(q) & \xleftarrow{\ \sim\ } & q_! F'(\mathrm{id}_A) & \xrightarrow[\sim]{\text{coSegal}} & q_! F(\mathrm{id}_A) \\
\downarrow{\scriptstyle F'(q_!\Delta_q)} & & \downarrow{\scriptstyle q_! F'(\Delta_q)} & & \downarrow{\scriptstyle q_! F(\Delta_q)} \\
F'(q_! q^*(q)) & \xleftarrow{\ \sim\ } & q_! F'(q^*(q)) & \xrightarrow{\text{coSegal}} & q_! F(q^*(q)) \\
\| & & \| & & \| \\
F'(q_! q^*(q)) & \xleftarrow{\ \sim\ } & q_! q^* F'(q) & \xrightarrow{\text{coSegal}} & q_! q^* F(q) \\
\downarrow{\scriptstyle F'(\mathrm{pr}_1)} & & \downarrow{\scriptstyle \varepsilon_q} & & \downarrow{\scriptstyle \varepsilon_q} \\
F'(q) & = & F'(q) & \xrightarrow{\text{coSegal}} & F(q).
\end{array}$$

The right half of the diagram commutes by naturality of the coSegal map $F' \to F$. In the left half of the diagram we use that $F' \colon \underline{\mathbf{U}}_{\mathcal{Q}} \to \mathcal{D}$ preserves $\mathcal{Q}$-colimits; the bottom left square commutes because the counit $\varepsilon_q \colon q_! q^*(q) \to q$ in $\underline{\mathbf{U}}_{\mathcal{Q}}$ is given by the projection map $\mathrm{pr}_1 \colon A \times_B A \to A$. The left vertical composite is the identity as $\Delta_q \colon A \to A \times_B A$ is a section of pr_1. As the top of the diagram is the canonical identification $F'(q) = q_!(\mathrm{id}_A)$, the claim follows. □

Corollary 5.30. *Assume again that $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_R$. Let $\mathcal{D}$ be a $\mathcal{B}$-category admitting $\mathcal{Q}$-colimits and let $F \colon \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R] \to \mathcal{D}$ be a $\mathcal{B}$-functor. Consider a pullback square*

$$\begin{array}{ccc}
A & \xrightarrow{q'} & B \\
{\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
C & \xrightarrow[q]{} & D
\end{array}$$

in $\mathcal{Q}$, expressing the relation $q' = p^*(q)$ in $\underline{\mathbf{U}}_{\mathcal{Q}}$. Then the diagram

$$\begin{array}{ccccc} q'_!F(\mathrm{id}_A) & = & q'_!(p')^*F(\mathrm{id}_C) & \xrightarrow{\mathrm{BC}_!} & p^*q_!F(\mathrm{id}_C) \\ {\scriptstyle \mathrm{coSegal}}\downarrow & & & & \downarrow{\scriptstyle p^*(\mathrm{coSegal})} \\ F(q') & & = & & p^*F(q) \end{array}$$

commutes up to homotopy.

Proof. By Lemma 5.29, this is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccccc} F(\mathrm{id}_A) & & = & & {p'}^*F(\mathrm{id}_C) \\ {\scriptstyle F(\Delta_{q'})}\downarrow & & & & \downarrow{\scriptstyle {p'}^*F(\Delta_q)} \\ {q'}^*F(q') & = & {q'}^*p^*F(q) & = & {p'}^*q^*F(q). \end{array}$$

But this is immediate from the fact that the image of the map $\Delta_q \colon C \to C \times_D C$ under the pullback functor ${p'}^* \colon \mathcal{Q}_{/C} \to \mathcal{Q}_{/A}$ is the diagonal $\Delta_{q'} \colon A \to A \times_B A$. □

The description of the coSegal map from Lemma 5.29 naturally leads us to consider the following more general coSegal maps:

Construction 5.31. Let $F \colon \underline{\mathbf{U}}_{\mathcal{Q}}[\mathcal{Q}_R] \to \mathcal{D}$ be a $\mathcal{B}$-functor and assume that $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_R$. For morphisms $p \colon A \to B$ and $q \colon B \to C$ in $\mathcal{Q}$, we define a *coSegal map*

$$\mathrm{coSegal} \colon q_!F(p) \to F(qp)$$

as the map adjoint to $F(p) \xrightarrow{F(1,p)} F(q^*(qp)) = q^*F(qp)$. Here $(1, p) \colon p \to q^*(qp)$ is the morphism in the slice $\mathcal{Q}_{/B}[\mathcal{Q}_R]$ corresponding to the map $(1, p) \colon A \to A \times_C B$.

Remark 5.32. On $\underline{\mathrm{Span}}(\mathcal{Q})$, the functor $q_\circ \colon p \mapsto qp$ is simply the left adjoint $q_!$ of q^*, and the maps $(1, p)$ form the unit, see Proposition 4.6. In particular, we see that the above generalized coSegal map 'is' natural in p.

Proposition 5.33. *Let $\mathcal{Q}_L, \mathcal{Q}_R \subseteq \mathcal{Q}$ be subclasses with $\Delta(\mathcal{Q}) \subseteq \mathcal{Q}_L \cap \mathcal{Q}_R$ and let $\mathcal{D}$ be a $\mathcal{Q}$-semiadditive $\mathcal{B}$-category. For a $\mathcal{B}$-functor $F \colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \to \mathcal{D}$ that is coSegal the composite*

$$q_!F(\mathrm{id}_A) \xrightarrow{\mathrm{coSegal}} F(q) \xrightarrow{\mathrm{Segal}} q_*F(\mathrm{id}_A)$$

is homotopic to the norm map $\mathrm{Nm}_q \colon q_!F(\mathrm{id}_A) \to q_*F(\mathrm{id}_A)$ in $\mathcal{D}$.

Proof. As $\mathcal{Q}_L$ contains the diagonal $\Delta_q \colon A \to A \times_B A$ of q by assumption, the span

$$A \times_B A \xleftarrow{\Delta_q} A \xrightarrow{=} A$$

defines a map $q^*q \to \mathrm{id}_A$ in $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)(A) = \mathrm{Span}(\mathcal{Q}_{/A}, \mathcal{Q}_{/A}[\mathcal{Q}_L], \mathcal{Q}_{/A}[\mathcal{Q}_R])$; we will denote this span by ∇_q. The dual of Lemma 5.29 shows that the Segal map $F(q) \to q_*F(\mathrm{id}_A)$ is adjoint to the composite

$$q^*F(q) = F(q^*q) \xrightarrow{F(\nabla_q)} F(\mathrm{id}_A).$$

The proposition is thus equivalent to the claim that the composite

$$q^*q_!F(\mathrm{id}_A) \xrightarrow{\mathrm{coSegal}} q^*F(q) = F(q^*q) \xrightarrow{F(\nabla_q)} F(\mathrm{id}_A)$$

is homotopic to the adjoint norm map $\widetilde{\mathrm{Nm}}_q$ of $\mathcal{D}$.

To prove this, we consider the following diagram, where the top row spells out the definition of $\widetilde{\mathrm{Nm}}_q\colon q^*q_!F(\mathrm{id}_A) \to F(\mathrm{id}_A)$:

$$\begin{array}{ccccccccc}
q^*q_!F(\mathrm{id}_A) & \xrightarrow{\mathrm{BC}_!^{-1}} & \mathrm{pr}_{1!}\mathrm{pr}_2^*F(\mathrm{id}_A) & \xrightarrow{\eta} & \mathrm{pr}_{1!}\Delta_*\Delta^*\mathrm{pr}_2^*F(\mathrm{id}_A) & \xrightarrow{\mathrm{Nm}_\Delta^{-1}} & \mathrm{pr}_{1!}\Delta_!\Delta^*\mathrm{pr}_2^*F(\mathrm{id}_A) & \xrightarrow{\sim} & F(\mathrm{id}_A)\\
{\scriptstyle\mathrm{coSegal}}\downarrow & & \| & & \| & & \| & & \downarrow\\
F(q^*q) & \xrightarrow[\mathrm{coSegal}^{-1}]{} & \mathrm{pr}_{1!}F(\mathrm{id}_{A\times_B A}) & \xrightarrow[\eta]{} & \mathrm{pr}_{1!}\Delta_*\Delta^*F(\mathrm{id}_{A\times_B A}) & (*) & \mathrm{pr}_{1!}\Delta_!\Delta^*F(\mathrm{id}_{A\times_B A}) & & \\
 & & & \searrow^{F(\eta)} & \uparrow{\scriptstyle \mathrm{BC}_*=\mathrm{Segal}} & & \downarrow{\scriptstyle \mathrm{BC}_!=\mathrm{coSegal}} & & \sim\\
 & & & & \mathrm{pr}_{1!}F(\Delta_*\mathrm{id}_A) & \xrightarrow[F(\mathrm{Nm}_\Delta^{-1})]{} & \mathrm{pr}_{1!}F(\Delta_!\mathrm{id}_A) & & \\
 & & & & \| & & \| & & \downarrow\\
 & & & & \mathrm{pr}_{1!}F(\Delta) & = & \mathrm{pr}_{1!}F(\Delta) & \xrightarrow[\mathrm{coSegal}]{} & F(\mathrm{pr}_1\Delta)
\end{array}$$

Here the square on the left commutes by Corollary 5.30. Moreover, as Δ is a map in $\mathcal{Q}_L \cap \mathcal{Q}_R$ by assumption, Proposition 3.7 shows that the rectangle $(*)$ commutes before inverting the norm equivalences; as the right hand vertical map is invertible by assumption, we conclude that also the rectangle with the inverted norm maps commutes. The rightmost rectangle commutes by direct inspection, and so does the triangle in the second column. To finish the proof it will therefore suffice to show that the bottom composite $F(q^*q) \to F(\mathrm{id}_A)$ is simply the map $F(\nabla)$. However by Remark 5.32, the subcomposite $F(\mathrm{pr}_1) = F(q^*q) \to F(\mathrm{pr}_1\Delta)$ agrees with $F(\mathrm{pr}_1 \circ \eta)$, so this is a straight-forward computation in $\underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R)$, using that the unit map $\eta\colon \mathrm{id}_{A\times_B A} \to \Delta$ is given by the analogous span in $\mathcal{Q}_{/A\times_B A}$:

$$A \times_B A \xleftarrow{\Delta_q} A \xrightarrow{=} A. \qquad \square$$

Proof of Proposition 5.27. By symmetry, it suffices to show that any coSegal functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}, \mathcal{Q}_L, \mathcal{Q}_R) \to \mathcal{D}$ is also Segal. But this is immediate from Proposition 5.33 since Nm_q is an equivalence. $\square$

5.4. **The universal property.** Combining all results of this section, we will now prove the universal property of $\underline{\mathrm{Span}}(\mathcal{Q})$ from Theorem 5.1: for every $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{D}$ evaluation at the global section pt restricts to an equivalence of $\mathcal{B}$-categories

$$\mathrm{ev}_{\mathrm{pt}}\colon \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\oplus}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

The main part will be an induction proving the analogous statement for the truncations $\mathcal{Q}_{\leq n}$. In order to pass to the full span category, we will use:

Lemma 5.34. *Let $\mathcal{V}$ be a $\mathcal{B}$-category. Assume we have an increasing chain $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}$ of full $\mathcal{B}$-subcategories such that each $X \in \mathcal{V}(A)$ is* locally in the $\mathcal{V}_n$'s *in the following sense: there exists a cover $(f_i\colon A_i \to A)_{i\in I}$ and for each $i \in I$ a natural number $n_i \in \mathbb{N}$ such that $f_i^*X \in \mathcal{V}_{n_i}(A_i)$.*

Then the inclusions exhibit $\mathcal{V}$ as the colimit in $\mathrm{Cat}(\mathcal{B})$ *of the $\mathcal{V}_n$'s.*

Proof. Let us write $\mathcal{B}' := \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Spc}) \simeq \mathcal{B}$. Identifying Cat with complete Segal spaces we then obtain a fully faithful functor

$$\mathrm{Cat}(\mathcal{B}) = \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Spc})) \simeq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{B}')$$

given in degree $[k] \in \Delta$ by $\mathcal{C} \mapsto \iota\left(\mathcal{C}^{[k]}\right)$, also see [Mar21, Proposition 3.5.1]. As fully faithful functors reflect colimits and since colimits in functor categories are pointwise, it will be enough to show that $\iota(\mathcal{V}^{[k]})$ is the colimit of the $\iota(\mathcal{V}_n^{[k]})$'s. Clearly, each $\iota(\mathcal{V}_n^{[k]}) \to \iota(\mathcal{V}^{[k]})$ is fully faithful (i.e. a monomorphism of $\mathcal{B}$-groupoids). Given now an object $X_\bullet = (X_0 \to \cdots \to X_k) \in \mathcal{V}^{[k]}(A)$ we can find covers $(f_i^{(j)}\colon A_i^{(j)} \to A)_{i\in I_j}$ such that each $(f_i^{(j)})^* X_j$ is contained in some $\mathcal{V}_{n_{i,j}}^{[k]}$. Passing to a common refinement and setting $n_i = \max\{n_{i,0}, \dots, n_{i,k}\}$ we see that $X_\bullet$ is locally in the $\mathcal{V}_n^{[k]}$'s. Replacing $\mathcal{V}$ by $\iota(\mathcal{V}^{[k]})$ we are therefore altogether reduced to proving the analogous statement in $\mathcal{B}'$.

In Spc, transfinite compositions of monomorphisms are monomorphisms; exhibiting $\mathcal{B}' \simeq \mathcal{B}$ as a left exact localization of a presheaf topos, we therefore see that the same holds true in $\mathcal{B}'$, and it only remains to show that the subgroupoid inclusion $\mathrm{colim}_n \mathcal{V}_n \to \mathcal{V}$ is essentially surjective. This follows immediately from the assumption that each $X \in \mathcal{V}(A)$ be locally in the $\mathcal{V}_n$'s. □

Proof of Theorem 5.1. The map $\mathrm{ev}_{\mathrm{pt}}$ is given in degree $B \in \mathcal{B}$ by

$$\mathrm{Fun}^{\pi_B^{-1}\mathcal{Q}\text{-}\times}(\pi_B^*\underline{\mathrm{Span}}(\mathcal{Q}), \pi_B^*\mathcal{D}) \simeq \mathrm{Fun}^{\pi_B^{-1}\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\pi_B^{-1}\mathcal{Q}), \pi_B^*\mathcal{D}) \to (\pi_B^*\mathcal{D})(\mathrm{id}_B) = \mathcal{D}(B).$$

Replacing $\mathcal{B}$ by $\mathcal{B}_{/B}$, it therefore suffices to prove the statement on underlying functor categories.

By the previous lemma, the inclusions exhibit $\underline{\mathrm{Span}}(\mathcal{Q})$ as a colimit of the truncations $\underline{\mathrm{Span}}(\mathcal{Q}_{\le n})$, which by Lemma 3.15 induces an equivalence

$$\mathrm{Fun}^{\mathcal{Q}\text{-}\oplus}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \xrightarrow{\sim} \lim_{n\ge -2} \mathrm{Fun}^{\mathcal{Q}_{\le n}\text{-}\oplus}(\underline{\mathrm{Span}}(\mathcal{Q}_{\le n}), \mathcal{D}).$$

Thus, it suffices to prove the theorem for $\mathcal{Q}$ replaced by $\mathcal{Q}_{\le n}$ for all $n \ge -2$. As a functor $\mathrm{Span}(\mathcal{Q}_{\le n}) \to \mathcal{D}$ is $\mathcal{Q}_{\le n}$-semiadditive if and only if it is coSegal (Proposition 5.20), we are altogether reduced to prove that evaluation at the identity defines an equivalence

$$\mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}_{\le n}), \mathcal{D}) \xrightarrow{\sim} \Gamma\mathcal{D}.$$

We proceed by induction on n. When $n = -2$, we get that $\underline{\mathrm{Span}}(\mathcal{Q}_{\le n}) = \underline{1}$ is the terminal $\mathcal{B}$-category and every functor is coSegal, so the claim holds trivially. Assume that the result holds for $n-1$, and consider the inclusions

$$\underline{\mathrm{Span}}(\mathcal{Q}_{\le n-1}) \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}_{\le n}, \mathcal{Q}_{\le n-1}, \mathcal{Q}_{\le n}) \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q}_{\le n}).$$

As $\Delta(\mathcal{Q}_{\le n}) \subseteq \mathcal{Q}_{\le n-1}$, Lemma 5.21 and Corollary 5.28 show that restriction along these inclusions induce equivalences

$$\mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}_{\le n}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}_{\le n}, \mathcal{Q}_{\le n-1}, \mathcal{Q}_{\le n}), \mathcal{D})$$

and

$$\mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}_{\le n}, \mathcal{Q}_{\le n-1}, \mathcal{Q}_{\le n}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{coSeg}}(\underline{\mathrm{Span}}(\mathcal{Q}_{\le n-1}), \mathcal{D}).$$ □

5.5. **Examples.** We will now specialize the universal property of parametrized spans to the examples given in Section 3.4. Recall that in each of these examples the topos $\mathcal{B}$ is a presheaf topos $\mathrm{PSh}(T)$ on a small category T and that the locally inductible subcategory $\mathcal{Q}$ is of the form Q_{loc} for some pre-inductible subcategory $Q \subseteq \mathrm{PSh}(T)$. For easier reference, we explicitly spell out this special case of the theorem:

Theorem 5.35. *For a pre-inductible subcategory $Q \subseteq \mathrm{PSh}(T)$, the free Q-semiadditive T-category is the T-category $\underline{\mathrm{Span}}(Q)$ given by*

$$\underline{\mathrm{Span}}(Q)\colon T^{\mathrm{op}} \to \mathrm{Cat}, \quad B \mapsto \mathrm{Span}(Q_{/B}).$$

Proof. Taking $\mathcal{B} = \mathrm{PSh}(T)$ and $\mathcal{Q} = Q_{\mathrm{loc}}$, Theorem 5.1 says that the free Q_{loc}-semiadditive $\mathcal{B}$-category is given by the $\mathcal{B}$-category $\underline{\mathrm{Span}}(Q_{\mathrm{loc}})\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ sending B to $\mathrm{Span}((Q_{\mathrm{loc}})_{/B})$. By Proposition 3.29, it follows that its underlying T-category is the free Q-semiadditive T-category. Since for $B \in T$ we have $(Q_{\mathrm{loc}})_{/B} = Q_{/B}$, the claim follows. □

Example 5.36 (Ordinary semiadditivity)**.** Taking $Q = \mathrm{Fin} \subseteq \mathrm{Spc}$ recovers the well-known fact that the category $\mathrm{Span}(\mathrm{Fin})$ of spans of finite sets is the free semiadditive category on a single generator.

Example 5.37 (m-semiadditivity)**.** For $-2 \leq m < \infty$, taking $Q = \mathrm{Spc}_m \subseteq \mathrm{Spc}$ recovers the fact that the category $\mathrm{Span}(\mathrm{Spc}_m)$ of spans of m-finite spaces is the free m-semiadditive category on a single generator, as was previously established by Harpaz [Har20, Theorem 1.1]. Similarly, taking $Q = \mathrm{Spc}_\pi$ shows that $\mathrm{Span}(\mathrm{Spc}_\pi)$ is the free ∞-semiadditive category on a single generator. Analogous results hold for $\mathrm{Span}(\mathrm{Spc}_m^{(p)})$ and $\mathrm{Span}(\mathrm{Spc}_\pi^{(p)})$ in the p-typical setting.

Example 5.38 (Equivariant semiadditivity)**.** For $Q = \mathscr{F}_\dagger \subseteq \mathrm{PSh}(\mathrm{Glo})$, [CLL23a, Lemma 5.2.3] provides a natural equivalence $Q_{/BG} \simeq \mathrm{Fin}_G$. Thus, we deduce that the assignment $G \mapsto \mathrm{Span}(\mathrm{Fin}_G)$ determines the free equivariantly semiadditive global category.

Example 5.39 (G-semiadditivity)**.** For a finite group G, taking $Q = \mathrm{Fin}_G \subseteq \mathrm{Spc}_G$ shows that the G-category $\underline{\mathrm{Span}}(\mathrm{Fin}_G)\colon G/H \mapsto \mathrm{Span}(\mathrm{Fin}_H)$ of spans of finite G-sets is the free G-semiadditive G-category.

Example 5.40 (P-semiadditivity)**.** More generally when $P \subseteq T$ is an atomic orbital subcategory of a small category T, we deduce that the T-category $\underline{\mathrm{Span}}(\underline{\mathbb{F}}_T^P)$ is the free P-semiadditive T-category.

Example 5.41. If $Q \subseteq T$ is an inductible subcategory, it is in particular a pre-inductible subcategory of $\mathrm{PSh}(T)$ by Example 3.21, and hence $\underline{\mathrm{Span}}(Q)$ is the free Q-semiadditive T-category.

Remark 5.42. The previous example can be used to provide a strengthening of Harpaz's result from Example 5.37. If we take T to be the category Spc_m of m-finite spaces, then assigning to a category $\mathcal{C}$ the T-category $\mathrm{Fun}(-, \mathcal{C})\colon \mathrm{Spc}_m^{\mathrm{op}} \to \mathrm{Cat}$ provides a fully faithful inclusion $\mathrm{Cat} \hookrightarrow \mathrm{Cat}_T$, whose essential image consists of those functors $\mathrm{Spc}_m^{\mathrm{op}} \to \mathrm{Cat}$ that preserve m-finite limits. Under this inclusion, the span category $\mathrm{Span}(\mathrm{Spc}_m)$ gets sent to the functor $A \mapsto \mathrm{Span}(\mathrm{Spc}_m)^A \simeq \mathrm{Span}((\mathrm{Spc}_m)_{/A})$, which is precisely the parametrized span category $\underline{\mathrm{Span}}(Q)$ for $T = Q = \mathrm{Spc}_m$. The previous example then tells us that $\underline{\mathrm{Span}}(Q)$ is free among all Spc_m-semiadditive T-categories, strengthening Harpaz's statement that it is free among just those contained in the essential image of the inclusion $\mathrm{Cat} \hookrightarrow \mathrm{Cat}_T$.

This strengthening of Harpaz's result will be crucially used in forthcoming work of Shay Ben-Moshe on transchromatic characters.

6. The $\underline{\mathrm{Span}}(\mathcal{Q})$-tensoring on $\mathcal{Q}$-semiadditive $\mathcal{B}$-categories

In this section, we will show that the property for a $\mathcal{B}$-category to be $\mathcal{Q}$-semiadditive can be characterized via a suitable notion of $\underline{\mathrm{Span}}(\mathcal{Q})$-tensorings, generalizing the analogous result of [Har20, Section 5.1] in the m-semiadditive situation. We will further discuss various useful consequences of this characterization.

Definition 6.1. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be $\mathcal{B}$-categories. We will use the term *bifunctor* for a $\mathcal{B}$-functor $- \boxtimes -\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$.

If $\mathcal{C}$ and $\mathcal{E}$ are $\mathcal{Q}$-cocomplete, then we say such a bifunctor preserves *$\mathcal{Q}$-colimits in the first variable* if the curried functor $\mathcal{C} \to \underline{\mathrm{Fun}}(\mathcal{D}, \mathcal{E})$ is $\mathcal{Q}$-cocontinuous, or equivalently if the curried functor $\mathcal{D} \to \underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{E})$ factors through the full subcategory $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\amalg}(\mathcal{C}, \mathcal{E})$. Analogously, we define what it means for a bifunctor to preserve $\mathcal{Q}$-colimits in the second variable (if $\mathcal{D}$ and $\mathcal{E}$ have $\mathcal{Q}$-colimits), or in both variables (if all three of them have $\mathcal{Q}$-colimits).

Of course, there is a dual notion of *preserving $\mathcal{Q}$-limits* in some or all of the variables.

Remark 6.2. Unraveling the definitions, a functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserves $\mathcal{Q}$-colimits in the first variable if and only if for every $q\colon A \to B$ in $\mathcal{Q}$, $X \in \mathcal{C}(A)$, and $Y \in \mathcal{D}(B)$ the *projection map*

$$q_!(X \boxtimes q^*Y) \to q_!X \boxtimes Y,$$

defined as the mate of the naturality equivalence $q^*(- \boxtimes Y) = q^*(-) \boxtimes q^*Y$ is an equivalence.

Example 6.3. Let $\mathcal{C}$ be $\mathcal{Q}$-cocomplete. By Proposition 2.20 the evaluation functor $\mathrm{ev}_{\mathrm{pt}}\colon \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\amalg}(\underline{\mathbf{U}}_{\mathcal{Q}}, \mathcal{C}) \to \mathcal{C}$ is an equivalence, so it has a unique section (automatically an equivalence, hence in particular $\mathcal{Q}$-cocontinuous). In other words, there is a unique bifunctor $- \otimes -\colon \underline{\mathbf{U}}_{\mathcal{Q}} \times \mathcal{C} \to \mathcal{C}$ that preserves $\mathcal{Q}$-colimits in each variable and restricts to the identity on $\{\mathrm{pt}\} \times \mathcal{C}$.

Using the universal property of spans we can extend this tensoring in the case that $\mathcal{C}$ is *$\mathcal{Q}$-semiadditive*:

Corollary 6.4. *Let $\mathcal{C}$ be $\mathcal{Q}$-semiadditive. Then the above tensoring $\underline{\mathbf{U}}_{\mathcal{Q}} \times \mathcal{C} \to \mathcal{C}$ extends uniquely to a bifunctor $\underline{\mathrm{Span}}(\mathcal{Q}) \times \mathcal{C} \to \mathcal{C}$. Moreover, this tensoring again preserves $\mathcal{Q}$-colimits in each variable separately.*

Proof. Note that any such extension is necessarily $\mathcal{Q}$-cocontinuous in each variable by Corollary 4.7. Conversely, Theorem 5.1 gives by the same argument as in the previous example a bifunctor $\underline{\mathrm{Span}}(\mathcal{Q}) \times \mathcal{C} \to \mathcal{C}$ preserving $\mathcal{Q}$-colimits in each variable. Its restriction to $\underline{\mathbf{U}}_{\mathcal{Q}} \times \mathcal{C}$ then again preserves $\mathcal{Q}$-colimits in each variable by another application of Corollary 4.7, so it necessarily agrees with the canonical tensoring. □

The main goal for the rest of this section will be to prove the following converse:

Theorem 6.5. *Let $\mathcal{D}$ be $\mathcal{Q}$-cocomplete and let $\mathcal{V}$ be $\mathcal{Q}$-semiadditive. Assume there exists a bifunctor $-\boxtimes-\colon \mathcal{V} \times \mathcal{D} \to \mathcal{D}$ preserving $\mathcal{Q}$-colimits in each variable separately together with a global section $\mathbb{I} \in \Gamma\mathcal{V}$ such that $- \boxtimes -$ restricts to the identity on $\{\mathbb{I}\} \times \mathcal{D}$. Then $\mathcal{D}$ is $\mathcal{Q}$-semiadditive.*

In practice one applies the previous theorem in the case that $\mathcal{V}$ is $\underline{\mathrm{Span}}(\mathcal{Q})$; however, the notation in the proof is simplified by assuming $\mathcal{V}$ is arbitrary.

6.1. **Bifunctors and the adjoint norm.** The key idea to prove the theorem will be to use the maps $\mu\colon \mathrm{id} \to q_! q^*$ in $\mathcal{V}$ to construct analogous transformations in $\mathcal{C}$ that together with the adjoint norm maps will exhibit $q_!$ as right adjoint to q^*. To do so, we will first have to understand the interaction of general bifunctors with these sorts of maps better.

Lemma 6.6. *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be $\mathcal{B}$-categories and assume that $\mathcal{C}$ and $\mathcal{E}$ are $\mathcal{Q}$-cocomplete and $\mathcal{Q}_{\leq n-1}$-semiadditive. Let $-\boxtimes- \colon \mathcal{C}\times\mathcal{D} \to \mathcal{E}$ be a bifunctor preserving $\mathcal{Q}$-colimits in the first variable and let $q\colon A \to B$ be an n-truncated map in $\mathcal{Q}$.*

(1) The diagram

$$\begin{array}{ccc} q^*q_!(\mathrm{id}\boxtimes q^*) & \xrightarrow{\widetilde{\mathrm{Nm}}} & \mathrm{id}\boxtimes q^* \\ {\scriptstyle \mathrm{proj}}\big\downarrow{\scriptstyle \sim} & & \big\uparrow{\scriptstyle \widetilde{\mathrm{Nm}}\boxtimes q^*} \\ q^*(q_!\boxtimes \mathrm{id}) & \xrightarrow[=]{} & q^*q_!\boxtimes q^* \end{array}$$

of natural transformations between functors $\mathcal{C}(A)\times\mathcal{D}(B) \to \mathcal{E}(A)$ commutes.

(2) Assume that $\mathcal{C}$ is even $\mathcal{Q}_{\leq n}$-semiadditive. Then the composite

$$q^*\boxtimes q^* = q^*(\mathrm{id}\boxtimes\mathrm{id}) \xrightarrow{q^*(\mu\boxtimes\mathrm{id})} q^*(q_!q^*\boxtimes\mathrm{id}) \xrightarrow{\mathrm{proj}^{-1}} q^*q_!(q^*\boxtimes q^*) \xrightarrow{\widetilde{\mathrm{Nm}}} q^*\boxtimes q^*$$

is the identity.

Proof. Unravelling definitions, the first part is an instance of Proposition 3.7-(1) applied to the composite $\pi_A^*\mathcal{C} \to \underline{\mathrm{Fun}}(\pi_A^*\mathcal{D}, \pi_A^*\mathcal{E}) \to (\pi_A^*\mathcal{E})^{\mathcal{D}(A)}$. The second part then follows immediately from this together with the triangle identity for the adjunction $q^* \dashv q_!$ in $\mathcal{C}$. $\square$

We will also need the following complementary result:

Proposition 6.7. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be $\mathcal{Q}$-cocomplete $\mathcal{Q}_{\leq n-1}$-semiadditive $\mathcal{B}$-categories, and let $-\boxtimes-\colon \mathcal{C}\times\mathcal{D} \to \mathcal{E}$ be a bifunctor preserving $\mathcal{Q}$-colimits in each variable. Moreover, let q be any n-truncated map in $\mathcal{Q}$. Then:*

(1) The following diagram commutes:

$$\begin{array}{ccc} q_!(q^*\boxtimes q^*q_!) & \xrightarrow{\mathrm{proj}} & q_!q^*\boxtimes q_! \\ {\scriptstyle q_!(q^*\boxtimes\widetilde{\mathrm{Nm}})}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{proj}^{-1}} \\ q_!(q^*\boxtimes\mathrm{id}) & \xleftarrow[q_!(q^*\widetilde{\mathrm{Nm}}\boxtimes\mathrm{id})]{} & q_!(q^*q_!q^*\boxtimes\mathrm{id}). \end{array} \tag{1}$$

(2) Assume that $\mathcal{C}$ is even $\mathcal{Q}_{\leq n}$-semiadditive. Then the following composite is the identity:

$$\mathrm{id}\boxtimes q_! \xrightarrow{\mu\boxtimes\mathrm{id}} q_!q^*\boxtimes q_! \xrightarrow{\mathrm{proj}^{-1}} q_!(q^*\boxtimes q^*q_!) \xrightarrow{q_!(q^*\boxtimes\widetilde{\mathrm{Nm}})} q_!(q^*\boxtimes\mathrm{id}) \xrightarrow{\mathrm{proj}} \mathrm{id}\boxtimes q_!.$$

Unlike the previous result, this does not seem to directly translate to a statement about non-parametrized functors, making its proof a bit more involved computationally. We start with the following Beck–Chevalley lemma whose proof we leave to the reader:

Lemma 6.8. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be $\mathcal{Q}$-cocomplete $\mathcal{B}$-categories and let $-\boxtimes-\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$ be any bifunctor. Assume moreover we have a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array} \tag{2}$$

in $\mathcal{Q}$. Then the diagram

$$\begin{array}{ccccccc} f_!g'_!(f'^*g^*\boxtimes g'^*) & \xrightarrow{\text{proj}} & f_!(g'_!f'^*g^*\boxtimes \mathrm{id}) & \xrightarrow{\mathrm{BC}_!\boxtimes\mathrm{id}} & f_!(f^*g_!g^*\boxtimes\mathrm{id}) & \xrightarrow{\text{proj}} & g_!g^*\boxtimes f_! \\ \sim\downarrow & & & & & & \| \\ g_!f'_!(f'^*g^*\boxtimes g'^*) & \xrightarrow[\text{proj}]{} & g_!(g^*\boxtimes f'_!g'^*) & \xrightarrow[g^*\boxtimes\mathrm{BC}_!]{} & g_!(g^*\boxtimes g^*f_!) & \xrightarrow[\text{proj}]{} & g_!g^*\boxtimes f_! \end{array}$$

commutes, where the unlabeled equivalence is induced by (2). □

Proof of Proposition 6.7. Consider the following diagram:

$$\begin{array}{ccccc} q_!(q^*\boxtimes q^*q_!) & \xrightarrow{\text{proj}} & q_!q^*\boxtimes q_! & \xrightarrow{\text{proj}^{-1}} & q_!(q^*q_!q^*\boxtimes\mathrm{id}) \\ {\scriptstyle q_!(q^*\boxtimes\mathrm{BC}_!^{-1})}\downarrow & & & & \downarrow{\scriptstyle q_!(\mathrm{BC}_!^{-1}\boxtimes\mathrm{id})} \\ q_!(q^*\boxtimes \mathrm{pr}_{2!}\mathrm{pr}_1^*) & \xrightarrow[\text{proj}^{-1}]{} & q_!\mathrm{pr}_{2!}(\mathrm{pr}_2^*q^*\boxtimes\mathrm{pr}_1^*)\simeq q_!\mathrm{pr}_{1!}(\mathrm{pr}_2^*q^*\boxtimes\mathrm{pr}_1^*) & \xrightarrow[\text{proj}]{} & q_!(\mathrm{pr}_{1!}\mathrm{pr}_2^*q^*\boxtimes\mathrm{id}) \\ {\scriptstyle q_!(q^*\boxtimes\mathrm{pr}_{2!}\mu)}\downarrow & & & & \downarrow{\scriptstyle q_!(\mathrm{pr}_{1!}\mu\boxtimes\mathrm{id})} \\ q_!(q^*\boxtimes\mathrm{pr}_{2!}\Delta_!\Delta^*\mathrm{pr}_1^*) & & & & q_!(\mathrm{pr}_{1!}\Delta_!\Delta^*\mathrm{pr}_2^*q^*\boxtimes\mathrm{id}) \\ \sim\downarrow & & & & \downarrow\sim \\ q_!(q^*\boxtimes\mathrm{id}) & = & = & = & q_!(q^*\boxtimes\mathrm{id}). \end{array}$$

Note that the right-hand vertical column spells out the definition of $q_!(\widetilde{\mathrm{Nm}}\boxtimes\mathrm{id})$, while the left-hand column agrees with $q_!(q^*\boxtimes\widetilde{\mathrm{Nm}})$ by virtue of Remark 3.4. Thus, the first statement amounts to saying that the total rectangle commutes.

To prove this we first note that the top rectangle commutes by the previous lemma. To show that the bottom rectangle commutes, we expand it as follows:

$$\begin{array}{ccccccc} q_!(\mathrm{pr}_{1!}\mathrm{pr}_2^*q^*\boxtimes\mathrm{id}) & \xleftarrow{\text{proj}} & q_!\mathrm{pr}_{1!}(\mathrm{pr}_2^*q^*\boxtimes\mathrm{pr}_1^*) & \xrightarrow{\sim} & q_!\mathrm{pr}_{2!}(\mathrm{pr}_2^*q^*\boxtimes\mathrm{pr}_1^*) & \xrightarrow{\text{proj}} & q_!(q^*\boxtimes\mathrm{pr}_{2!}\mathrm{pr}_1^*) \\ {\scriptstyle q_!(\mathrm{pr}_{1!}\mu\boxtimes\mathrm{id})}\downarrow & & {\scriptstyle q_!\mathrm{pr}_{1!}\mu}\downarrow & & \downarrow{\scriptstyle q_!\mathrm{pr}_{2!}\mu} & & \downarrow{\scriptstyle q_!(q^*\boxtimes\mathrm{pr}_{2!}\mu)} \\ q_!(\mathrm{pr}_{1!}\Delta_!\Delta^*\mathrm{pr}_2^*q^*\boxtimes\mathrm{id}) & & q_!\mathrm{pr}_{1!}\Delta_!(\Delta^*\mathrm{pr}_2^*q^*\boxtimes\Delta^*\mathrm{pr}_1^*) & \xrightarrow{\sim} & q_!\mathrm{pr}_{2!}\Delta_!(\Delta^*\mathrm{pr}_2^*q^*\boxtimes\Delta^*\mathrm{pr}_1^*) & & q_!(q^*\boxtimes\mathrm{pr}_{2!}\Delta_!\Delta^*\mathrm{pr}_1^*) \\ & \searrow\sim & \sim\downarrow & & \downarrow\sim & \sim\swarrow & \\ & & q_!(q^*\boxtimes\mathrm{id}) & = & q_!(q^*\boxtimes\mathrm{id}) & & \end{array}$$

The top square in the middle column commutes by naturality, and the bottom middle square commutes since the composite homotopy $q\mathrm{pr}_1\Delta\simeq q\mathrm{pr}_2\Delta\simeq q$ agrees with the homotopy induced by $\mathrm{pr}_1\Delta\simeq\mathrm{id}$. To see that the two pentagons commute we begin by observing that they are symmetric, and so it suffices to spell out the left-hand one. For this we expand it once again:

$$\begin{array}{ccccc}
q_!(\mathrm{pr}_{1!}\mathrm{pr}_2^*q^* \boxtimes \mathrm{id}) & \xleftarrow{\mathrm{proj}} & q_!\mathrm{pr}_{1!}(\mathrm{pr}_2^*q^* \boxtimes \mathrm{pr}_1^*) & \xrightarrow{\mu} & q_!\mathrm{pr}_{1!}\Delta_!\Delta^*(\mathrm{pr}_2^*q^* \boxtimes \mathrm{pr}_1^*) \\
{\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} & & \| \\
q_!(\mathrm{pr}_{1!}\Delta_!\Delta^*\mathrm{pr}_2^*q^* \boxtimes \mathrm{id}) & \xleftarrow{\mathrm{proj}} & q_!\mathrm{pr}_{1!}(\Delta_!\Delta^*\mathrm{pr}_2^*q^* \boxtimes \mathrm{pr}_1^*) & \xleftarrow{\mathrm{proj}} & q_!\mathrm{pr}_{1!}\Delta_!(\Delta^*\mathrm{pr}_2^*q^* \boxtimes \Delta^*\mathrm{pr}_1^*) \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
q_!(\mathrm{pr}_{1!}\Delta_!q^* \boxtimes \mathrm{id}) & \xleftarrow[\mathrm{proj}]{} & q_!\mathrm{pr}_!(\Delta_!q^* \boxtimes \mathrm{pr}_1^*) & \xleftarrow[\mathrm{proj}]{} & q_!\mathrm{pr}_{1!}\Delta_!(q^* \boxtimes \Delta^*\mathrm{pr}_1^*) \\
{\scriptstyle\sim}\downarrow & & & & \downarrow{\scriptstyle\sim} \\
q_!(q^* \boxtimes \mathrm{id}) & = & = & = & q_!(q^* \boxtimes \mathrm{id}).
\end{array}$$

Here the top left square as well as the two squares in the middle row commute by naturality, the top right square commutes by Lemma 6.6 (note that Δ is $(n-1)$-truncated), and the bottom rectangle commutes by direct inspection. This finishes the proof of the first statement.

For the second statement, we consider the diagram

$$\begin{array}{ccccc}
q_!(q^* \boxtimes \mathrm{id}) & \xrightarrow{(q^*\mu\boxtimes\mathrm{id})} & q_!(q^*q_!q^* \boxtimes \mathrm{id}) & \xrightarrow{q_!(\widetilde{\mathrm{Nm}}\boxtimes\mathrm{id})} & q_!(q^* \boxtimes \mathrm{id}) \\
{\scriptstyle\mathrm{proj}}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle\mathrm{proj}} & & \uparrow{\scriptstyle q_!(q^*\boxtimes\widetilde{\mathrm{Nm}})} \\
\mathrm{id} \boxtimes q_! & \xrightarrow[\mu\boxtimes\mathrm{id}]{} & q_!q^* \boxtimes \mathrm{id} & \xrightarrow[\mathrm{proj}^{-1}]{\sim} & q_!(q^* \boxtimes q^*q_!)
\end{array}$$

where the left square commutes by naturality while the right-hand square commutes by part (1). As the composite of the top row is the identity by the triangle identity in $\mathcal{C}$, the claim follows. $\square$

Using this, we can now easily prove the theorem:

Proof of Theorem 6.5. It suffices by induction to show that if $\mathcal{C}$ is $\mathcal{Q}_{\leq n}$-semiadditive and if $q\colon A \to B$ is any $(n+1)$-truncated map, then $\widetilde{\mathrm{Nm}}\colon q^*q_! \to \mathrm{id}$ is the counit of an adjunction.

We claim that the natural map $m\colon \mathrm{id} \to q_!q^*$ defined as the composite

$$\mathrm{id} = \mathbb{I} \boxtimes \mathrm{id} \xrightarrow{\mu\boxtimes\mathrm{id}} q_!q^*\mathbb{I} \boxtimes \mathrm{id} \xrightarrow{\mathrm{proj}^{-1}} q_!(q^*\mathbb{I} \boxtimes q^*) = q_!q^*$$

provides a compatible unit, i.e. we have to verify the triangle identities. The identity $(\widetilde{\mathrm{Nm}}\, q^*) \circ (q^*m) = \mathrm{id}$ is simply a special case of Lemma 6.6-(2), while $(q_!\,\widetilde{\mathrm{Nm}}) \circ (mq_!) = \mathrm{id}$ follows from Proposition 6.7-(2) as the projection map $q_!(q^*\boxtimes\mathrm{id}) \to \mathrm{id}\boxtimes q_!$ is simply the identity when restricted to $\mathbb{I}$ in the first component. $\square$

6.2. **Applications.** The characterization of semiadditivity in terms of the existence of a $\underline{\mathrm{Span}}(\mathcal{Q})$-tensoring has various interesting consequences:

Theorem 6.9. *Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{B}$-categories such that $\mathcal{C}$ is $\mathcal{Q}$-semiadditive and $\mathcal{D}$ is $\mathcal{Q}$-complete. Then $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D})$ is again $\mathcal{Q}$-semiadditive.*

Proof. By Proposition 2.18$^{\mathrm{op}}$, $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$ is $\mathcal{Q}$-complete. If $F\colon \mathcal{C} \to \underline{\mathrm{Fun}}(\underline{A}, \mathcal{D})$ defines any object of $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})(A)$ and $q\colon A \to B$ is any map in $\mathcal{Q}$, then q_*F is simply given by the composition

$$\mathcal{C} \xrightarrow{F} \underline{\mathrm{Fun}}(\underline{A}, \mathcal{D}) \xrightarrow{q_*} \underline{\mathrm{Fun}}(\underline{B}, \mathcal{D}),$$

hence $\mathcal{Q}$-continuous as a composition of $\mathcal{Q}$-continuous functors. In other words, $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D})$ is closed under $\mathcal{Q}$-limits and hence in particular itself $\mathcal{Q}$-complete. It will therefore suffice by Theorem 6.5$^{\mathrm{op}}$ to construct a functor

$$\underline{\mathrm{Span}}(\mathcal{Q}) \times \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D}) \to \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D})$$

preserving $\mathcal{Q}$-limits in each variable and restricting to the identity on $\{\mathrm{pt}\} \times \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D})$. Adjoining over, this amounts to constructing a $\mathcal{Q}$-continuous section

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D}) \to \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D})) \tag{3}$$

of the evaluation functor.

For this we observe that we have by Corollary 6.4$^{\mathrm{op}}$ a functor $-\boxtimes-\colon \underline{\mathrm{Span}}(\mathcal{Q}) \times \mathcal{C} \to \mathcal{C}$ preserving $\mathcal{Q}$-limits in each variable and restricting to the identity on $\{\mathrm{pt}\} \times \mathcal{C}$. Using this, we consider the composite

$$\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\boxtimes^*} \underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}) \times \mathcal{C}, \mathcal{D}) \xrightarrow[\sim]{\text{adjunction}} \underline{\mathrm{Fun}}\big(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})\big).$$

We claim that this restricts to the desired section (3). Indeed, one easily checks that this is $\mathcal{Q}$-continuous (in fact, it preserves all limits that exist in $\mathcal{D}$) and a section. The claim that it lands in $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D}))$ amounts to saying that for every $\mathcal{Q}$-continuous $F\colon \mathcal{C} \to \mathcal{D}^{\underline{A}}$ the composite $F(-\boxtimes-)\colon \underline{\mathrm{Span}}(\mathcal{Q}) \times \mathcal{C} \to \mathcal{D}^{\underline{A}}$ preserves $\mathcal{Q}$-limits in each variable. This is clear by assumption on $\boxtimes$. $\square$

Finally, let us use the tensoring to show that the left and right adjoints $q_!$ and q_* also agree for non-truncated q, and to moreover upgrade this to a parametrized comparison.

Construction 6.10. Fix $q\colon A \to B$ in $\mathcal{Q}$, and let $\mathcal{C}$ be $\mathcal{Q}$-semiadditive. We define a natural transformation m from the identity of $\pi_B^*\mathcal{C}$ to the composite

$$\pi_B^*\mathcal{C} \xrightarrow{q^*} \mathcal{C}(A \times_B -) \xrightarrow{q_!} \pi_B^*\mathcal{C}$$

as follows: write $\boxtimes$ for the (essentially unique) $\underline{\mathrm{Span}}(\mathcal{Q})$-tensoring of $\mathcal{C}$ and let $\mathrm{pt} = (\mathrm{id}_B\colon B \to B)$ denote the prefered global section of $\pi_B^*\underline{\mathrm{Span}}(\mathcal{Q})$; then we define m as the composite

$$\mathrm{id}_{\pi_B^*\mathcal{C}} = \mathrm{pt} \boxtimes \mathrm{id} \xrightarrow{p \boxtimes \mathrm{id}} q_!q^*\mathrm{pt} \boxtimes \mathrm{id} \xrightarrow{\sim} q_!(q^*\mathrm{pt} \boxtimes q^*) = q_!q^*$$

where $p\colon \mathrm{pt} \to q_!q^*\mathrm{pt}$ is represented by the span $B \xleftarrow{q} A \xrightarrow{=} B$ and the unlabeled equivalence comes from the projection formula, i.e. it is the adjunct of the map $q^*\mathrm{pt} \boxtimes q^* \to q^*q_!q^*\mathrm{pt} \boxtimes q^*$ induced by the unit.

Unravelling definitions, we see that for any $f\colon B' \to B$ and $q' := f^*(q)\colon A' \to B'$, the value of m on f is precisely the transformation $m_{q'}\colon \mathrm{id} \to q'_!q'^*$ considered in the proof of Theorem 6.5. In particular, we have shown that it is the unit of an adjunction $q'^* \dashv q'_!$ whenever q' is truncated, with corresponding counit $\widetilde{\mathrm{Nm}}$.

Theorem 6.11. *Let $\mathcal{C}$ be $\mathcal{Q}$-semiadditive and let $q\colon A \to B$ be any map in $\mathcal{Q}$, not necessarily truncated. Then the above transformation $m\colon \mathrm{id} \to q_!q^*$ is the unit of a parametrized adjunction $q^* \dashv q_!$ of $\mathcal{B}_{/B}$-categories. In particular, there is a natural equivalence $\mathrm{Nm}_q\colon q_! \xrightarrow{\sim} q_*$ of $\mathcal{B}_{/B}$-functors.*

As explained before, the parametrized transformation Nm_q recovers the usual inductively defined norm on underlying categories whenever q is truncated.

Proof. Fix $q\colon A \to B$ in $\mathcal{Q}$; to simplify notation, we will replace $\mathcal{B}$ by $\mathcal{B}_{/B}$, so that B is terminal. We have to show that m is the unit of an adjunction, or equivalently that $q_!$ admits a left adjoint $q^!$ and that the adjoined map $q^! \to q^*$ is an equivalence.

For the construction of $q^!$, we once more consider the covering sieve $\Sigma \subseteq \mathcal{B}$ of the terminal object given by those $f\colon B' \to 1$ such that the pulled back map $q' := f^*(q)\colon B' \times A \to B'$ is truncated. By the above, we then have for each such q' an adjunction $(q')^! = q'^* \dashv q'_!$ with unit $m_{q'}$; the fact that m is a natural transformation of $\mathcal{B}$-functors then translates to saying that for every map $g\colon B'' \to B'$ in Σ the Beck–Chevalley transformation $(q'')^!(A\times g)^* \to g^*(q')^!$ of $(A\times g)^*$ is just the naturality equivalence $(q'')^*(A \times g)^* \to g^*(q')^*$ again, in particular invertible. Thus, Proposition A.3 shows that the parametrized left adjoint $q^!$ exists.

Consider now the natural transformation of $\mathcal{B}$-functors $\widetilde{m}\colon q^! \to q^*$ induced by m. By construction of $q^!$, $\tilde{m}_{B'}$ is an equivalence (even the identity) whenever $B' \in \Sigma$. Given an arbitrary $B \in \mathcal{B}$, we can now cover it by objects of Σ (for example via the projections $C \times B \to B$ with $C \in \Sigma$). It therefore follows immediately from naturality and descent that $\widetilde{m}$ is an equivalence as claimed. □

Corollary 6.12. *Let $\mathcal{C}$ be a $\mathcal{Q}$-complete and $\mathcal{Q}$-cocomplete $\mathcal{B}$-category. The following are equivalent:*

(1) $\mathcal{C}$ is $\mathcal{Q}$-semiadditive.
(2) For every $q\colon A \to B$ in $\mathcal{Q}$ there exists an equivalence of parametrized functors $q_! \simeq q_*\colon \underline{\mathrm{Fun}}(\underline{A}, \pi_B^*\mathcal{C}) \to \pi_B^*\mathcal{C}$ (a priori unrelated to the norms).
(3) For every $q\colon A \to B$ in $\mathcal{Q}$, the functor $q_!\colon \underline{\mathrm{Fun}}(\underline{A}, \pi_B^*\mathcal{C}) \to \pi_B^*\mathcal{C}$ admits a parametrized left adjoint.
(4) For every $q\colon A \to B$ in $\mathcal{Q}$, the functor $q_!\colon \underline{\mathrm{Fun}}(\underline{A}, \pi_B^*\mathcal{C}) \to \pi_B^*\mathcal{C}$ preserves $\mathcal{Q}$-limits.

Proof. Clearly, (2) ⇒ (3) ⇒ (4). The implication (1) ⇒ (2) is the content of the previous theorem, while (4) ⇒ (1) follows from Proposition 3.44. □

7. $\mathcal{Q}$-commutative monoids and their universal property

In this section, we introduce the $\mathcal{B}$-category $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ of *$\mathcal{Q}$-commutative monoids* in a $\mathcal{Q}$-complete $\mathcal{B}$-category $\mathcal{D}$, and show that the forgetful functor $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \to \mathcal{D}$ exhibits it as the *$\mathcal{Q}$-semiadditive completion* of $\mathcal{D}$.

7.1. $\mathcal{Q}$-commutative monoids.

The following is the key definition of this section:

Definition 7.1 (Commutative monoids)**.** Given a $\mathcal{Q}$-complete $\mathcal{B}$-category $\mathcal{D}$, we define its $\mathcal{B}$-category of *$\mathcal{Q}$-commutative monoids* as

$$\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) := \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}).$$

We let $\mathbb{U} := \mathrm{ev}_{\mathrm{pt}}\colon \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \to \mathcal{D}$ denote the evaluation functor at the point.

Informally speaking, we may think of a $\mathcal{Q}$-commutative monoid in $\mathcal{D}$ as a global section M of $\mathcal{D}$ equipped with certain 'parametrized addition/transfer maps.' Recall from Proposition 2.20$^{\mathrm{op}}$ that every global section M uniquely extends to a $\mathcal{Q}$-continuous functor $\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}} \to \mathcal{D}$ given at level $B \in \mathcal{B}$ by sending a map $q\colon A \to B$ in $\mathcal{Q}$ to the q-indexed product $(M_B)^A := q_* q^* M_B \in \mathcal{D}(B)$. Enhancing M to a $\mathcal{Q}$-commutative monoid in $\mathcal{D}$ is then equivalent to providing an extension of this functor along the inclusion $\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}} \hookrightarrow \underline{\mathrm{Span}}(\mathcal{Q})$, which we may interpret as providing a suitably coherent collection of 'addition/transfer maps' $\int_q\colon (M_B)^A \to M_B$.

Remark 7.2. For $A \in \mathcal{B}$ a $\mathcal{B}_{/A}$-functor $F\colon \underline{\mathrm{Span}}(\pi_A^{-1}\mathcal{Q}) \simeq \pi_A^* \underline{\mathrm{Span}}(\mathcal{Q}) \to \pi_A^* \mathcal{D}$ belongs to $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})(A)$ if and only if it is a Segal functor in the sense of Definition 5.26, see Proposition 5.20$^{\mathrm{op}}$.

Let us note the following immediate consequence of Theorem 6.9:

Corollary 7.3. *In the above situation, $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ is $\mathcal{Q}$-semiadditive.* ☐

In fact, it is the universal $\mathcal{Q}$-semiadditive completion of $\mathcal{D}$ in the following sense:

Theorem 7.4. *We have an adjunction* $\mathrm{incl}\colon \mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\oplus} \rightleftarrows \mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\times} : \underline{\mathrm{CMon}}^{\mathcal{Q}}$*, with counit given by the evaluation functor $\mathbb{U} = \mathrm{ev}_{\mathrm{pt}}\colon \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \to \mathcal{D}$.*

Proof. First observe that $\underline{\mathrm{CMon}}^{\mathcal{Q}}$ indeed lands in $\mathcal{Q}$-semiadditive categories by the previous corollary. Moreover, Proposition 2.18$^{\mathrm{op}}$ shows that it preserves $\mathcal{Q}$-continuous functors and that $\mathbb{U}$ is $\mathcal{Q}$-continuous. Thus, it only remains to show that $\mathbb{U}\colon \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \to \mathcal{D}$ is an equivalence for every $\mathcal{Q}$-semiadditive $\mathcal{D}$ and that $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathbb{U})\colon \underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})) \to \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ is an equivalence for every $\mathcal{Q}$-complete $\mathcal{D}$.

The first statement is precisely the content of Theorem 5.1. Similarly, appealing to the previous corollary once more, we know that $\mathbb{U}_{\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})}$ is an equivalence for every $\mathcal{Q}$-complete $\mathcal{D}$. Thus, it will suffice for the second statement that the automorphism of $\underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}))$ exchanging the two span-factors induces an automorphism of $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}))$. But this follows immediately from the observation that the adjunction equivalence $\underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})) \simeq \underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}) \times \underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})$ identifies $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}))$ with the full subcategory of functors preserving $\mathcal{Q}$-limits in each variable separately. ☐

We can further refine this universal property to a statement about parametrized functor categories, generalizing previous results due to Nardin for equivariant semiadditivity [Nar16, Corollary 5.11.1 and Theorem 6.5] and due to Harpaz for higher non-parametrized semiadditivity [Har20, Corollary 5.15]:

Theorem 7.5. *Let $\mathcal{C}$ be $\mathcal{Q}$-semiadditive and let $\mathcal{D}$ be $\mathcal{Q}$-complete. Then postcomposition with $\mathbb{U}$ defines an equivalence of $\mathcal{B}$-categories*

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D}).$$

Proof. It suffices to show that for any $\mathcal{Q}$-complete $\mathcal{B}$-category $\mathcal{T}$ the induced map

$$\hom_{\mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\times}}\big(\mathcal{T}, \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}))\big) \to \hom_{\mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\times}}\big(\mathcal{T}, \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{C}, \mathcal{D})\big) \quad (4)$$

is an equivalence. However, using the adjunction equivalence for $\underline{\mathrm{Fun}}$ and keeping track of $\mathcal{Q}$-limit conditions as in the proof of the previous theorem, this map agrees up to equivalence with the map

$$\hom_{\mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\times}}\big(\mathcal{C}, \underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{T}, \mathcal{D}))\big) \to \hom_{\mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\times}}\big(\mathcal{C}, \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{T}, \mathcal{D})\big)$$

induced by $\mathbb{U}$, so this is a consequence of the previous theorem. □

We close this subsection by giving two variants of the above universal property. Both of these rely on the following observation:

Lemma 7.6. *Let $\mathcal{R} \subseteq \mathcal{B}$ be any local class. If $\mathcal{D}$ is $\mathcal{R}$-complete and $\mathcal{Q}$-complete, then $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ is $\mathcal{R}$-complete and $\mathbb{U}\colon \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \to \mathcal{D}$ preserves and reflects $\mathcal{R}$-limits.*

Similarly, if K is any non-parametrized category and $\mathcal{D}$ admits fiberwise K-shaped limits, then so does $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$, and $\mathbb{U}$ preserves and reflects K-shaped limits.

Proof. Proposition 2.18 shows that the full functor category $\mathrm{Fun}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})$ has all $\mathcal{R}$-limits if $\mathcal{D}$ has them and that the evaluation functor $\mathrm{ev}_{\mathrm{pt}}$ preserves $\mathcal{R}$-limits in this case. As $\mathrm{CMon}^{\mathcal{Q}}(\mathcal{D}) \subseteq \underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})$ is closed under $\mathcal{R}$-limits by the same argument as in the beginning of the proof of Theorem 6.9, we then get the same statement for $\mathbb{U}\colon \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \to \mathcal{D}$.

The existence and preservation of fiberwise limits follows in the same way from Remark 2.22. To finish the proof it then suffices to prove that $\mathbb{U}$ is conservative for every $\mathcal{Q}$-complete $\mathcal{D}$. But $\mathbb{U}$ factors as the composite

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \xrightarrow{\mathrm{res}} \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}}, \mathcal{D}) \xrightarrow{\mathrm{ev}} \mathcal{D}$$

and the first functor is conservative as $\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}} \subseteq \underline{\mathrm{Span}}(\mathcal{Q})$ is a wide subcategory, while the second functor is even an equivalence by Proposition 2.20^{op}. □

Corollary 7.7. *Let $\mathcal{C}$ be $\mathcal{Q}$-semiadditive, let $\mathcal{D}$ be $\mathcal{Q}$-complete, and let $F\colon \mathcal{C} \to \mathcal{D}$ be $\mathcal{Q}$-continuous. Then F lifts uniquely to a functor $\mathcal{C} \to \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$.*

Proof. By Theorem 7.4 there is a unique such lift that is in addition $\mathcal{Q}$-continuous, while the previous lemma shows that in fact *any* lift is $\mathcal{Q}$-continuous. □

Corollary 7.8. *Let $\mathcal{C}, \mathcal{D}$ be complete $\mathcal{B}$-categories and assume $\mathcal{C}$ is $\mathcal{Q}$-semiadditive. Then postcomposition with the forgetful functor induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{R}}(\mathcal{C}, \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})) \to \underline{\mathrm{Fun}}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$$

of $\mathcal{B}$-categories of continuous functors.

Proof. This follows from Theorem 7.5, observing that by Lemma 7.6 a functor $\mathcal{C} \to \underline{\mathrm{Fun}}(\underline{A}, \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}))$ is continuous if and only if the induced functor $\mathcal{C} \to \underline{\mathrm{Fun}}(\underline{A}, \mathcal{D})$ is so. □

7.2. **Non-parametrized (higher) semiadditivity.** Let us make explicit how Theorem 7.5 recovers various results from non-parametrized higher category theory, and in particular Harpaz's result alluded to above. Recall once more that taking global section defines an equivalence $\mathrm{Cat}(\mathrm{Spc}) \xrightarrow{\sim} \mathrm{Cat}$, and hence Spc-parametrized functor categories are just non-parametrized functor categories between the underlying categories. By Example 2.15, the subcategory of $\mathcal{Q}$-limit preserving functors then consists precisely of those functors that preserve A-shaped limits in the usual sense for every $A \in \mathcal{Q}_{/1} \subseteq \mathrm{Spc} \subseteq \mathrm{Cat}$.

Example 7.9 (Commutative monoids)**.** Applying the theorem to the pre-inductible subcategory $\mathrm{Fin} \subseteq \mathrm{Spc}$ of finite sets recovers the well-known result that for every category $\mathcal{C}$ with finite products the forgetful functor

$$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathrm{pt}}} \mathcal{C}$$

exhibits its source as the universal semiadditive category equipped with a finite-product-preserving functor to $\mathcal{C}$. In other words, we have an equivalence of categories $\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathcal{C})$, as was previously established by Cranch [Cra09, Theorem 5.4] (for an ad-hoc construction of Span(Fin)) or in [BH21, Proposition C.1] (using Barwick's construction of Span).

Example 7.10 (m-commutative monoids, [Har20, Corollary 5.14])**.** More generally, consider the pre-inductible subcategory $\mathrm{Spc}_m \subseteq \mathrm{Spc}$ of m-finite spaces for some $-2 \leq m < \infty$, and let $\mathcal{C}$ be a category with m-finite limits. Then Theorem 7.5 translates to saying that the forgetful functor

$$\mathrm{CMon}^m(\mathcal{C}) := \mathrm{Fun}^{m\text{-fin}}(\mathrm{Span}(\mathrm{Spc}_m), \mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathrm{pt}}} \mathcal{C}$$

exhibits $\mathrm{CMon}^m(\mathcal{C})$ as the universal m-semiadditive category equipped with an m-finite limit-preserving functor to $\mathcal{C}$, a fact previously proven by Harpaz [Har20, Corollary 5.14]. If we instead consider the subcategory $\mathrm{Spc}_\pi \subseteq \mathrm{Spc}$, we also obtain the analogous statement for $m = \infty$, previously proven by Carmeli, Schlank, and Yanovski [CSY21, Proposition 2.1.16].

Example 7.11 (p-typical m-commutative monoids)**.** As a new variant of the previous example, we may consider for $-2 \leq m < \infty$ the pre-inductible subcateory $\mathrm{Spc}_m^{(p)} \subseteq \mathrm{Spc}$ of *p-typical* m-finite spaces from Example 3.33. For a category $\mathcal{C}$ admitting $\mathrm{Spc}_m^{(p)}$-indexed limits, we define the category $\mathrm{CMon}^m_{(p)}(\mathcal{C})$ of *p-typically m-commutative monoids in* $\mathcal{C}$ as the full subcategory

$$\mathrm{CMon}^m_{(p)}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Span}(\mathrm{Spc}_m^{(p)}), \mathcal{C}),$$

of functors which preserve $\mathrm{Spc}_m^{(p)}$-indexed limits. The evaluation functor to $\mathcal{C}$ then enjoys the 'p-typical analogue' of the previous universal property.

7.3. **$\mathcal{Q}$-stability.** Building on the result of Section 7.1, we can now introduce a notion of *$\mathcal{Q}$-stability* generalizing our work in [CLL23a]. For this let us first recall the notion of *fiberwise stability* from [Nar16, Definition 3.5] and [MW22, Definition 7.3.4]:

Definition 7.12. A $\mathcal{B}$-category $\mathcal{D}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ is called *fiberwise stable* if it factors through the non-full subcategory $\mathrm{Cat}^{\mathrm{ex}} \subseteq \mathrm{Cat}$ of stable categories and exact functors. We write $\mathrm{Cat}(\mathcal{B})^{\mathrm{fib\,ex}} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{ex}})$ and call its maps *fiberwise exact.*[5]

Remark 7.13. Write $\mathrm{Cat}^{\mathrm{lex}}$ for the category of left exact functors between categories with finite limits, and set $\mathrm{Cat}(\mathcal{B})^{\mathrm{fib\,lex}} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{lex}})$. Then the inclusion $\mathrm{Cat}(\mathcal{B})^{\mathrm{fib\,ex}} \hookrightarrow \mathrm{Cat}(\mathcal{B})^{\mathrm{fib\,lex}}$ admits a right adjoint $\mathrm{Sp}^{\mathrm{fib}}$ given by post-composing with the right adjoint Sp of $\mathrm{Cat}^{\mathrm{lex}} \hookrightarrow \mathrm{Cat}^{\mathrm{ex}}$.

[5]Note that $\mathrm{Cat}^{\mathrm{ex}}$ admits all limits, which are computed in Cat, by [Lur17, Theorem 1.1.4.4].

Definition 7.14. We say that $\mathcal{D}$ is *$\mathcal{Q}$-stable* if it is both $\mathcal{Q}$-semiadditive and fiberwise stable. We write $\mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-ex}} := \mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\oplus} \cap \mathrm{Cat}(\mathcal{B})^{\mathrm{fib\,ex}}$ for the category whose objects are the $\mathcal{Q}$-stable categories and whose morphisms are the functors that are both fiberwise exact and $\mathcal{Q}$-semiadditive.

Lemma 7.15. *The adjunction* $\mathrm{incl}\colon \mathrm{Cat}(\mathcal{B})^{\mathrm{fib\,ex}} \rightleftarrows \mathrm{Cat}(\mathcal{B})^{\mathrm{fib\,lex}} : \mathrm{Sp}^{\mathrm{fib}}$ *restricts to an adjunction* $\mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-ex}} \rightleftarrows \mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\oplus,\,\mathrm{lex}}$.

Proof. Observe that the functor $\mathrm{Sp} = \mathrm{Fun}_*^{\mathrm{exc}}(\mathrm{Spc}_*^{\mathrm{fin}}, -)$ can be extended to an $(\infty, 2)$-functor; all that we will need below is that it induces a functor on homotopy $(2, 2)$-categories. Observe now that each $q^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ has a *left exact* left adjoint $q_! \simeq q_*$ by $\mathcal{Q}$-semiadditivity, so 2-functoriality of Sp implies that also $\mathrm{Sp}(q^*) = q^*\colon \mathrm{Sp}^{\mathrm{fib}}(\mathcal{C})(B) \to \mathrm{Sp}^{\mathrm{fib}}(\mathcal{C})(A)$ has a left adjoint given by $\mathrm{Sp}(q_!)$ with the induced unit and counit. The Beck–Chevalley maps for these left adjoints are then again obtained from the Beck–Chevalley maps in $\mathcal{C}$ via applying Sp, and in particular the left adjoints again satisfy basechange, i.e. $\mathrm{Sp}^{\mathrm{fib}}(\mathcal{C})$ is $\mathcal{Q}$-cocomplete. In the same way, one shows that $\mathrm{Sp}^{\mathrm{fib}}(\mathcal{C})$ is $\mathcal{Q}$-complete and that $\mathrm{Sp}^{\mathrm{fib}}$ sends $\mathcal{Q}$-continuous functors to $\mathcal{Q}$-continuous functors.

It will then be enough to show by Proposition 3.44 that for any pullback

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\mathrm{pr}_1} & A \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ A & \xrightarrow[q]{} & B \end{array}$$

in $\mathcal{Q}$ the double Beck–Chevalley map $q_! \mathrm{pr}_{1*} \to q_* \mathrm{pr}_{2!}$ for $\mathrm{Sp}^{\mathrm{fib}}(\mathcal{C})$ is an equivalence. This however follows again immediately from 2-functoriality of $\mathrm{Sp}^{\mathrm{fib}}$ and the corresponding statement for $\mathcal{C}$. $\square$

Definition 7.16. Let $\mathcal{C}$ be a $\mathcal{B}$-category with $\mathcal{Q}$-limits and finite fiberwise limits. We define $\underline{\mathrm{Sp}}^{\mathcal{Q}}(\mathcal{C}) := \underline{\mathrm{Sp}}^{\mathrm{fib}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}))$, and we write $\Omega^\infty\colon \underline{\mathrm{Sp}}^{\mathcal{Q}}(\mathcal{C}) \to \mathcal{C}$ for the composite

$$\underline{\mathrm{Sp}}^{\mathrm{fib}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})) \xrightarrow{\Omega^\infty} \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}) \xrightarrow{\mathbb{U}} \mathcal{C}.$$

Proposition 7.17. *This defines a functor* $\underline{\mathrm{Sp}}^{\mathcal{Q}}\colon \mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\times,\,\mathrm{fib\,lex}} \to \mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-ex}}$ *right adjoint to the inclusion, with counit given by* Ω^∞.

Proof. It suffices by Lemma 7.15 to show that the adjunction $\mathrm{incl}\colon \mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\oplus} \rightleftarrows \mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\times} : \underline{\mathrm{CMon}}^{\mathcal{Q}}$ from Theorem 7.4 restricts to $\mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\oplus,\,\mathrm{fib\,lex}} \rightleftarrows \mathrm{Cat}_{\mathcal{B}}^{\mathcal{Q}\text{-}\times,\,\mathrm{fib\,lex}}$. This is immediate from Lemma 7.6. $\square$

7.4. **A presentable universal property for $\mathcal{Q}$-commutative monoids.** If $\mathcal{D}$ has $\mathcal{Q}$-limits, then Theorem 7.5 identifies (certain) functors *into* $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$; the goal of this subsection is to give (under a mild smallness assumption) a similar property for functors *out of* $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ whenever $\mathcal{D}$ is a presentable $\mathcal{B}$-category. For this we first recall:

Definition 7.18. A $\mathcal{B}$-category $\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ is called *presentable* if it is $\mathcal{B}$-cocomplete and factors through the non-full subcategory $\mathrm{Pr}^{\mathrm{L}} \subseteq \mathrm{Cat}$, i.e. each $\mathcal{C}(A)$ is presentable and each $f^*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ is a left adjoint.

We write $\mathrm{Pr}(\mathcal{B})^{\mathrm{L}}$ for the category of presentable $\mathcal{B}$-categories and left adjoint functors, and $\mathrm{Pr}(\mathcal{B})^{\mathrm{R}}$ for the category of presentable $\mathcal{B}$-categories and right adjoint functors.

Remark 7.19. [MW22] instead defines presentable $\mathcal{B}$-categories as accessible Bousfield localization of presheaf categories. This is equivalent to the above definition by Theorem 6.2.4 of *op. cit.*

Remark 7.20. By definition, every presentable $\mathcal{B}$-category is in particular cocomplete. Moreover, an easy application of the non-parametrized Special Adjoint Functor Theorem shows that every presentable $\mathcal{B}$-category is also complete, also see [MW22, Corollary 6.2.5].

Because we have not bounded the size of $\mathcal{Q}$, the $\mathcal{B}$-category $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})$ does not necessarily have to be presentable, even if $\mathcal{C}$ was presentable. To fix this, we introduce:

Definition 7.21. We say that a wide local subcategory $\mathcal{Q} \subseteq \mathcal{B}$ is *slicewise small* if the category $\mathcal{Q}_{/A}$ is (essentially) small for every $A \in \mathcal{B}$, i.e. if $\underline{\mathbf{U}}_{\mathcal{Q}}$ is a small $\mathcal{B}$-category.

Example 7.22. If $Q \subseteq \mathrm{PSh}(T)$ is a small pre-inductible category, then the corresponding locally inductible subcategory Q_{loc} of $\mathrm{PSh}(T)$ is slicewise small. This follows immediately from the fact that $\underline{\mathbf{U}}_{\mathcal{Q}}$ is the limit extension of the small T-category $A \mapsto Q_{/A}$.

In particular, all examples of locally inductible categories from Section 3.4 are slicewise small.

Proposition 7.23. *Assume that $\mathcal{Q}$ is slicewise small and let $\mathcal{D}$ be presentable. Then the inclusion $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})$ is an accessible Bousfield localization. In particular, $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ is again presentable.*

Proof. As seen in the proof of Lemma 7.6, $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ is complete and the inclusion is continuous. It will therefore suffice to show that we have an accessible Bousfield localization in each individual level $A \in \mathcal{B}$: the pointwise left adjoints will then assemble into a $\mathcal{B}$-left adjoint by Remark A.2.

For this recall the description $\underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})(A) \simeq \mathrm{Fun}_{\mathcal{B}_{/A}}(\pi_A^*\underline{\mathrm{Span}}(\mathcal{Q}), \pi_A^*\mathcal{D})$, under which $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})(A)$ corresponds precisely to the $\pi_A^*\mathcal{Q}$-continuous functors (see Construction 2.19). In other words, after replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$ it will suffice to give a set of maps S such that a functor $F \in \mathrm{Fun}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})$ preserves $\mathcal{Q}$-limits if and only if it is S-local.

We now observe that since each $\mathcal{D}(B)$ for $B \in \mathcal{B}$ is presentable, we can find a set $\mathcal{T}_B \subseteq \mathcal{D}(B)$ of objects jointly detecting equivalences. Moreover, observe that for any $X \in \underline{\mathrm{Span}}(\mathcal{Q})(B)$ the evaluation functor $\mathrm{Fun}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \to \mathcal{D}(B), F \mapsto F_A(B)$ agrees up to the equivalence from the Yoneda lemma with restriction along the map $\underline{B} \to \mathcal{C}$ classifying X, so it is a right adjoint (with left adjoint given by parametrized Kan extension).

Fix now any map $q\colon A \to B$ in $\mathcal{Q}$ and $X \in \underline{\mathrm{Span}}(\mathcal{Q})(A)$. Then the comparison map $q_*F_A(X) \to F_B q_*(X)$ in $\mathcal{D}(B)$ is natural in F and hence so is the induced map $\hom(T, q_*F_A(X)) \to \hom(T, F_B q_*(X))$ of spaces for any $T \in \mathcal{T}_B$.

The source and target of this map are corepresentable by the above, so this map agrees by the Yoneda lemma with $\hom(t, F)$ for some suitable map $t = t_{q,X,T}$ in $\mathrm{Fun}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D})$. By choice of $\mathcal{T}_B$ we see that $q_* F_A \to F_B q_*$ is an equivalence if and only if F is local with respect to the set of all $t_{q,X,T}$ with $T \in \mathcal{T}_B$ and X running through objects of $\underline{\mathrm{Span}}(\mathcal{Q})$ (up to equivalence).

Pick now a small category $\mathcal{M}$ together with a left exact localization $L\colon \mathrm{PSh}(\mathcal{M}) \to \mathcal{B}$, yielding a set $\mathcal{B}_0 \coloneqq L(\mathcal{M})$ of objects of $\mathcal{B}$ such that every $B \in \mathcal{B}$ can be covered by elements of $\mathcal{B}_0$. By Lemma A.5, we then see that F is $\mathcal{Q}$-continuous if and only if the Beck–Chevalley map $Fq_* \to q_* F$ is an equivalence for all $q\colon A \to B$ in $\mathcal{Q}$ such that $B \in \mathcal{B}_0$. By choice of $\mathcal{B}_0$ and assumption on $\mathcal{Q}$, there is only a set worth of such maps (up to equivalence). Thus, we may take S to be the *set* of all $t_{q,X,T}$ for such q and for X and T as before. □

Corollary 7.24. *Assume that $\mathcal{Q}$ is slicewise small and let $\mathcal{D}$ be presentable. Then the forgetful functor $\mathbb{U}\colon \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \to \mathcal{D}$ admits a left adjoint $\mathbb{P}$.*

Proof. We may factor $\mathbb{U}$ as the composite

$$\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \xrightarrow{\mathrm{ev}_{\mathrm{pt}}} \underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{1}, \mathcal{D}) \simeq \mathcal{D}.$$

The first functor admits a left adjoint by the previous proposition, while the second one admits a left adjoint via parametrized left Kan extension. □

Proposition 7.25. *If $\mathcal{Q}$ is slicewise small, then the adjunction*

$$\mathrm{incl}\colon \mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\oplus} \leftrightarrows \mathrm{Cat}(\mathcal{B})^{\mathcal{Q}\text{-}\times} : \underline{\mathrm{CMon}}^{\mathcal{Q}}$$

restricts to an adjunction $\mathrm{Pr}(\mathcal{B})^{\mathrm{R}} \rightleftarrows \mathrm{Pr}(\mathcal{B})^{\mathrm{R},\, \mathcal{Q}\text{-}\oplus}$.

Proof. The previous proposition and corollary show that $\underline{\mathrm{CMon}}^{\mathcal{Q}}$ restricts on objects accordingly and that the counit $\mathbb{U}$ lies in Pr^{R}. Moreover, the unit is even an equivalence as the inclusion is fully faithful, so it only remains to show that for any adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ of presentable $\mathcal{B}$-categories, $\underline{\mathrm{CMon}}^{\mathcal{Q}}(G)$ is again a right adjoint. But indeed, the composition of $\underline{\mathrm{Fun}}(\mathrm{Span}(\mathcal{Q}), F)$ with the localization $\underline{\mathrm{Fun}}_{\mathcal{B}}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{D}) \to \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ is easily seen to restrict to the desired left adjoint. □

Dualizing we get:

Corollary 7.26. *If $\mathcal{Q}$ is slicewise small, the inclusion $\mathrm{Pr}(\mathcal{B})^{\mathrm{L},\, \mathcal{Q}\text{-}\oplus} \hookrightarrow \mathrm{Pr}(\mathcal{B})^{\mathrm{L}}$ admits a left adjoint given on objects by $\mathcal{D} \mapsto \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{D})$ and with unit given by the left adjoints $\mathbb{P}$ of the forgetful maps.* □

The usual argument internalizes this to the following equivalence of $\mathcal{B}$-categories:

Theorem 7.27. *Assume $\mathcal{Q}$ is slicewise small, let $\mathcal{C}$ be presentable, and let $\mathcal{D}$ be $\mathcal{Q}$-semiadditive and presentable. Then restriction along $\mathbb{P}\colon \mathcal{C} \to \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathcal{C}, \mathcal{D}).$$ □

7.5. **Fiberwise modules.** We can also prove a presentable universal property for $\underline{\mathrm{Sp}}^{\mathcal{Q}}(\mathcal{C})$ when $\mathcal{C}$ is presentable. In fact the only thing relevant about the property of stability is that it is equivalent to being a module over the idempotent object Sp in Pr^{L}. We present the argument in this generality.

Definition 7.28 (See [CSY21, Definition 5.2.4])**.** A *mode* is an idempotent object in Pr^{L}, i.e. a pair $(\mathcal{E}, E)$ of a presentable category $\mathcal{E}$ together with an object $E \in \mathcal{E}$ such that the map $\mathcal{E} \simeq \mathrm{Spc} \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ induced by E is an equivalence.

Given any mode $(\mathcal{E}, E)$, a *module* over it is a presentable category $\mathcal{F}$ such that the map $\mathcal{F} \simeq \mathrm{Spc} \otimes \mathcal{F} \to \mathcal{E} \otimes \mathcal{F}$ induced by E is an equivalence. We write $\mathrm{Mod}_{\mathcal{E}} \subseteq \mathrm{Pr}^{\mathrm{L}}$ for the full subcategory of $\mathcal{E}$-modules.

As usual, we will just refer to $\mathcal{E}$ as a mode when the object $E \in \mathcal{E}$ is understood.

Remark 7.29. In the above setting, $\mathcal{E}$ actually admits a (unique) commutative algebra structure with unit E [Lur17, Proposition 4.8.2.9], and for this algebra structure the forgetful functor from $\mathcal{E}$-modules (in the usual sense) to presentable categories is fully faithful with essential image $\mathrm{Mod}_{\mathcal{E}}$, see [Lur17, Proposition 4.8.2.10], justifying the terminology.

As a direct consequence, the inclusion $\mathrm{Mod}_{\mathcal{E}} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}$ admits a left adjoint given by $\mathcal{E} \otimes -$; in particular, $\mathrm{Mod}_{\mathcal{E}}$ is closed under limits.

Remark 7.30. As a left adjoint, $\mathcal{E} \otimes -\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ preserves all colimits; we will need below that it also preserves certain limits, namely limits of diagrams $X\colon K \to \mathrm{Pr}^{\mathrm{L}}$ such that all structure maps $X(k \to \ell)$ are also *right* adjoints. Indeed, on $\mathrm{Pr}^{\mathrm{L}} \cap \mathrm{Pr}^{\mathrm{R}}$ the functoriality of the Lurie tensor product $\mathcal{E} \otimes \mathcal{C} = \mathrm{Fun}^{\mathrm{R}}(\mathcal{E}^{\mathrm{op}}, \mathcal{C})$ is simply given by postcomposition, so the statement is clear.

Example 7.31. The pair $(\mathrm{Spc}, 1)$ is a mode, and every presentable category is a Spc-module.

Example 7.32. The pair $(\mathrm{Sp}, \mathbb{S})$ is a mode, and the Sp-modules are precisely the *stable* presentable categories, see [Lur17, Proposition 4.8.2.18].

Example 7.33. The pair $(\mathrm{Set}, 1)$ is a mode, and the Set-modules are precisely the presentable 1-categories, see [Lur17, Proposition 4.8.2.15].

Example 7.34. The pair $(\mathrm{Ab}, \mathbb{Z})$ is a mode, and the Ab-modules are precisely the presentable additive 1-categories; this is immediate from the previous example together with [GGN15, Theorem 4.6].

Example 7.35 (cf. [Har20, Corollary 5.21])**.** Let $\mathcal{Q} \subseteq \mathrm{Spc}$ be locally inductible. We claim that $(\mathrm{CMon}^{\mathcal{Q}}(\mathrm{Spc}), \mathbb{P}(1))$ is a mode whose modules are precisely the $\mathcal{Q}$-semiadditive categories.

Indeed, if $\mathcal{D}$ is presentable, then the map $\mathcal{D} \to \mathrm{CMon}^{\mathcal{Q}}(\mathrm{Spc}) \otimes \mathcal{D}$ is left adjoint to the forgetful map $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathrm{CMon}^{\mathcal{Q}}(\mathrm{Spc})) \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Spc}) \simeq \mathcal{D}$, and Corollary 7.8 shows that this is an equivalence whenever $\mathcal{D}$ is $\mathcal{Q}$-semiadditive and cocomplete (so that $\mathcal{D}^{\mathrm{op}}$ is complete). It follows immediately that $\mathrm{CMon}^{\mathcal{Q}}(\mathrm{Spc})$ is a mode and that every presentable $\mathcal{Q}$-semiadditive category is a module over it. On the other hand, one easily checks that $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{op}}, \mathrm{CMon}^{\mathcal{Q}}(\mathrm{Spc}))$ is $\mathcal{Q}$-semiadditive for every presentable $\mathcal{D}$, so conversely every $\mathrm{CMon}^{\mathcal{Q}}(\mathrm{Spc})$-module is $\mathcal{Q}$-semiadditive.

Definition 7.36. A fiberwise presentable $\mathcal{B}$-category $\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$ is called a *fiberwise $\mathcal{E}$-module* if it factors through the full subcategory $\mathrm{Mod}_{\mathcal{E}} \subseteq \mathrm{Pr}^{\mathrm{L}}$, i.e. if every $\mathcal{C}(A)$ is an $\mathcal{E}$-module. We write $\mathrm{Mod}_{\mathcal{E}}(\mathcal{B}) \subseteq \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ for the full subcategory spanned by those presentable categories that are in addition fiberwise $\mathcal{E}$-modules.

Example 7.37. By Example 7.32, a (fiberwise) presentable $\mathcal{B}$-category is fiberwise stable if and only if it is a fiberwise Sp-module.

Example 7.38. By Lemma 3.35 and Example 7.35, a (fiberwise) presentable $\mathcal{Q}$-semiadditive $\mathcal{B}$-category is always a fiberwise $\mathrm{CMon}^{Q_{\mathrm{fib}}}(\mathrm{Spc})$-module.

Lemma 7.39. *Let $\mathcal{E}$ be any mode. Then the inclusion $\mathrm{Mod}_{\mathcal{E}}(\mathcal{B}) \subseteq \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ admits a left adjoint given by applying $\mathcal{E}\otimes -$ pointwise, with unit $\mathcal{C} \to \mathcal{E}\otimes\mathcal{C}$ induced by the map $\mathrm{Spc} \to \mathcal{E}$.*

Moreover, this restricts to an adjunction $\mathrm{Mod}_{\mathcal{E}}^{\mathcal{Q}\text{-}\oplus}(\mathcal{B}) \rightleftarrows \mathrm{Pr}^{\mathrm{L},\,\mathcal{Q}\text{-}\oplus}(\mathcal{B})$.

Proof. First observe that the pointwise tensor product $\mathcal{E}\otimes\mathcal{C}$ is indeed a $\mathcal{B}$-category for any presentable $\mathcal{B}$-category $\mathcal{C}$ by Remark 7.30. Next, let us show that $\mathcal{E}\otimes\mathcal{C}$ is again $\mathcal{B}$-cocomplete, whence presentable. For this we recall that the tensor product lifts to an $(\infty,2)$-functor; all we will need below is that it is a functor on the homotopy 2-category of Pr^{L} (which also follows immediately from its construction as a functor category), and hence sends adjunctions to adjunctions. Given now any $f\colon A \to B$ in $\mathcal{B}$, both f^* and $f_!$ are left adjoints, so they form an internal adjunction in Pr^{L} and hence induce an adjunction $\mathcal{E}\otimes f_! \dashv \mathcal{E}\otimes f^*$. Given any $g\colon B' \to B$, the base change map $(\mathcal{E}\otimes f'_!)(\mathcal{E}\otimes g'^*) \to (\mathcal{E}\otimes g^*)(\mathcal{E}\otimes f_!)$ is then induced via 2-functoriality from the base change map $f'_! g'^* \to g^* f_!$, so it is invertible as the latter one is. Finally, the same 2-functoriality argument together with Proposition A.1 shows that the canonical map $\mathcal{C} \to \mathcal{E}\otimes\mathcal{C}$ is indeed a parametrized left adjoint.

If $\mathcal{C}$ is now $\mathcal{Q}$-semiadditive, then the functor q_* is itself a map in Pr^{L} for any $q\colon A \to B$ in $\mathcal{Q}$ (as $q_* \simeq q_!$), so the right adjoint of $\mathcal{E}\otimes q^*$ is given by $\mathcal{E}\otimes q_*$, with the induced unit and counit. Arguing as before, we see that the double Beck–Chevalley map $(\mathcal{E}\otimes \mathrm{pr}_{1!})(\mathcal{E}\otimes q_*) \to (\mathcal{E}\otimes \mathrm{pr}_{2*})(\mathcal{E}\otimes q_!)$ is invertible, so that $\mathcal{E}\otimes\mathcal{C}$ is $\mathcal{Q}$-semiadditive, proving the second statement. ☐

Let us specialize this to the stable case (Example 7.37):

Corollary 7.40. *Assume $\mathcal{Q}$ is slicewise small. The inclusion $\mathrm{Pr}(\mathcal{B})^{\mathrm{L},\,\mathcal{Q}\text{-ex}} \hookrightarrow \mathrm{Pr}(\mathcal{B})^{\mathrm{L}}$ admits a left adjoint $\underline{\mathrm{Sp}}^{\mathcal{Q}} := \mathrm{Sp}\otimes\underline{\mathrm{CMon}}^{\mathcal{Q}}$. For every presentable $\mathcal{C}$, the unit $\Sigma^\infty_+\colon \mathcal{C} \to \underline{\mathrm{Sp}}^{\mathcal{Q}}(\mathcal{C})$ is given by the composite*

$$\mathcal{C} \xrightarrow{\mathbb{P}} \underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}) \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}\otimes\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C}) = \underline{\mathrm{Sp}}^{\mathcal{Q}}(\mathcal{C}).$$

Proof. Combine Proposition 7.25 with Lemma 7.39. ☐

As before, we formally deduce the following internal version:

Theorem 7.41. *Assume $\mathcal{Q}$ is slicewise small. Let $\mathcal{C}$ be any presentable $\mathcal{B}$-category, and let $\mathcal{D}$ be presentable and $\mathcal{Q}$-stable. Then restriction along $\Sigma^\infty_+\colon \mathcal{C} \to \underline{\mathrm{Sp}}^{\mathcal{Q}}(\mathcal{C})$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathrm{Sp}}^{\mathcal{Q}}(\mathcal{C}),\mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathcal{C},\mathcal{D}).$$ ☐

8. $\mathcal{Q}$-Mackey sheaves

Our definition of $\mathcal{Q}$-commutative monoids in a general $\mathcal{B}$-category $\mathcal{C}$ makes heavy use of the language of parametrized category theory. In this section, we will see that $\mathcal{Q}$-commutative monoids admit a concrete *non-parametrized* description whenever $\mathcal{C}$ is obtained in a suitable way from a non-parametrized category.

Recall from [MW22, Section 8.3] that every presentable category $\mathcal{E}$ gives rise to a presentable $\mathcal{B}$-category $\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E})$ of '$\mathcal{E}$-valued sheaves,' given on objects by assigning to $B \in \mathcal{B}$ the category $\mathrm{Shv}(\mathcal{B}_{/B};\mathcal{E}) := \mathrm{Fun}^{\mathrm{R}}((\mathcal{B}_{/B})^{\mathrm{op}},\mathcal{E})$ of $\mathcal{E}$-valued sheaves on the slice topos $\mathcal{B}_{/B}$, and on morphisms by sending a map $f\colon A \to B$ in $\mathcal{B}$ to the functor $f^*\colon \mathrm{Shv}(\mathcal{B}_{/B};\mathcal{E}) \to \mathrm{Shv}(\mathcal{B}_{/A};\mathcal{E})$ given by precomposition with $f_!\colon \mathcal{B}_{/A} \to \mathcal{B}_{/B}$. The goal of this section is to explicitly describe the $\mathcal{B}$-category $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E}))$ in terms of *Mackey sheaves*:

Definition 8.1 ($\mathcal{Q}$-Mackey sheaves)**.** An *$\mathcal{E}$-valued $\mathcal{Q}$-Mackey sheaf on $\mathcal{B}$* is a (non-parametrized) functor $M\colon \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}) \to \mathcal{E}$ for which the restriction

$$\mathcal{B}^{\mathrm{op}} \simeq \mathrm{Span}(\mathcal{B},\mathcal{B},\iota\mathcal{B}) \hookrightarrow \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}) \xrightarrow{M} \mathcal{E}$$

is an $\mathcal{E}$-valued sheaf on $\mathcal{B}$, i.e. preserves limits. We let $\mathrm{Mack}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}),\mathcal{E})$ spanned by the $\mathcal{Q}$-Mackey sheaves.

The assignment $A \mapsto (\mathcal{B}_{/A},\mathcal{B}_{/A},\mathcal{B}_{/A}[\mathcal{Q}])$ extends to a functor $\mathcal{B} \to \mathrm{AdTrip}$ via pushforward, hence giving rise to a $\mathcal{B}$-presheaf of categories

$$A \mapsto \mathrm{Fun}(\mathrm{Span}(\mathcal{B}_{/A},\mathcal{B}_{/A},\mathcal{B}_{/A}[\mathcal{Q}]),\mathcal{E}).$$

We denote by $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})$ the full subfunctor given in degree A by the subcategory $\mathrm{Mack}^{\mathcal{B}_{/A}[\mathcal{Q}]}(\mathcal{B}_{/A};\mathcal{E})$. It comes equipped with a forgetful map $\mathbb{U}\colon \underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E}) \to \underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E})$ induced by the inclusions $(\mathcal{B}_{/A},\mathcal{B}_{/A},\iota\,\mathcal{B}_{/A}) \hookrightarrow (\mathcal{B}_{/A},\mathcal{B}_{/A},\mathcal{B}_{/A}[\mathcal{Q}])$.

Theorem 8.2. *Let $\mathcal{E}$ be a presentable category. Then there is a unique equivalence*

$$\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E}) \xrightarrow{\sim} \underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E}))$$

of $\mathcal{B}$-presheaves over $\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E})$. In particular, $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})$ is a $\mathcal{B}$-category.

Before moving to the proof of the theorem, let us explain how it allows us to describe the free presentable $\mathcal{Q}$-semiadditive and $\mathcal{Q}$-stable $\mathcal{B}$-categories in non-parametrized terms:

Corollary 8.3. *Assume $\mathcal{Q}$ is slicewise small. Then the $\mathcal{B}$-category $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc})$ is the free presentable $\mathcal{Q}$-semiadditive $\mathcal{B}$-category, i.e. for every other presentable $\mathcal{Q}$-semiadditive $\mathcal{D}$ evaluation at a certain 'free $\mathcal{Q}$-Mackey sheaf' $\mathbb{P}(1) \in \mathrm{Mack}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc})$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc}),\mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

We will give a concrete description of $\mathbb{P}(1)$ in Corollary 8.27.

Proof. Note that for $\mathcal{E} = \mathrm{Spc}$ we have $\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E}) = \underline{\mathrm{Spc}}_{\mathcal{B}}$. The claim thus follows from the string of equivalences

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc}),\mathcal{D}) \overset{8.2}{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathrm{CMon}}^{\mathcal{Q}}(\underline{\mathrm{Spc}}_{\mathcal{B}}),\mathcal{D}) \overset{7.27}{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathrm{Spc}}_{\mathcal{B}},\mathcal{D}) \xrightarrow{\sim} \mathcal{D},$$

where the last equivalence holds by [MW21, Theorem 7.1.1]. □

Corollary 8.4. *Assume $\mathcal{Q}$ is slicewise small. Then the $\mathcal{B}$-category $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Sp})$ is the free presentable $\mathcal{Q}$-stable $\mathcal{B}$-category, i.e. for any other such $\mathcal{D}$ evaluation at a certain global section $\mathbb{S} \in \mathrm{Mack}^{\mathcal{Q}}(\mathcal{B};\mathrm{Sp})$ induces an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Sp}),\mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Proof. Combining Corollaries 7.40 and 8.3, the free presentable $\mathcal{Q}$-stable category is given by $\mathrm{Sp}\otimes\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc})$. Using the explicit description of $\mathrm{Sp}\otimes -$ as $\mathrm{Fun}^{\mathrm{R}}(\mathrm{Sp}^{\mathrm{op}},-)$, this is immediately seen to be equivalent to $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Sp})$. □

Remark 8.5. More generally, if $\mathcal{E}$ is any mode, then Lemma 7.39 shows that $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc})\otimes\mathcal{E} \simeq \underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})$ is the free presentable $\mathcal{Q}$-semiadditive fiberwise $\mathcal{E}$-module.

Remark 8.6 (Fiberwise semiadditivity, redux)**.** Let $\mathcal{F} \subseteq \mathrm{Spc}$ be locally inductible such that the unique left exact left adjoint $\mathrm{Spc} \to \mathcal{B}$ maps $\mathcal{F}$ into $\mathcal{Q}$; for example, we could take the maximal such class $\mathcal{Q}_{\mathrm{fib}} \subseteq \mathrm{Spc}$ from Lemma 3.35. Combining said remark with Example 7.35, we see that every presentable $\mathcal{Q}$-semiadditive category $\mathcal{C}$ is a fiberwise $\mathrm{CMon}^{\mathcal{F}}(\mathrm{Spc})$-module, so that $\mathrm{CMon}^{\mathcal{F}}(\mathrm{Spc})\otimes\mathcal{C} \xrightarrow{\sim} \mathcal{C}$. Specializing $\mathcal{C}$, we obtain an equivalence $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{CMon}^{\mathcal{F}}(\mathcal{E})) \xrightarrow{\sim} \underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})$.

In particular, we see that if $\mathcal{Q}$ contains the map $1 \amalg 1 \to 1$, then the free presentable $\mathcal{Q}$-semiadditive $\mathcal{B}$-category can be equivalently described as $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{CMon})$.

We also record the following useful corollary:

Corollary 8.7. *Given a $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{D}$, every $\mathcal{Q}$-continuous functor $\mathcal{D} \to \underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E})$ admits a unique lift to $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})$: the forgetful functor*

$$\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{D},\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})) \to \underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\mathcal{D},\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E}))$$

is an equivalence.

Proof. This is immediate from Theorem 8.2 and the universal property of $\underline{\mathrm{CMon}}^{\mathcal{Q}}$ from Theorem 7.5. □

Corollary 8.8. *For every $\mathcal{Q}$-semiadditive $\mathcal{B}$-category $\mathcal{D}$, there is a unique functor $\underline{\mathrm{Hom}}^{\mathcal{Q}}_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}}\times\mathcal{D} \to \underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc})$ that lifts the parametrized hom-functor $\underline{\mathrm{Hom}}_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}}\times\mathcal{D} \to \underline{\mathrm{Spc}}_{\mathcal{B}}$.* □

8.1. **Reduction to presheaves.**

It will be technically convenient to deduce the theorem from a more general result about 'Mackey presheaves.' Throughout, let $\mathcal{A}$ be any category, not necessarily small, and let $\mathcal{Q} \subseteq \mathcal{A}$ be left-cancelable and closed under base change.

Definition 8.9. For any category $\mathcal{E}$, we write $\underline{\mathrm{MackPSh}}^{\mathcal{Q}}(\mathcal{A};\mathcal{E})$ for the functor

$$\mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}, A \mapsto \mathrm{Fun}(\mathrm{Span}(\mathcal{A}_{/A},\mathcal{A}_{/A},\mathcal{A}_{/A}[\mathcal{Q}]),\mathcal{E})$$

with functoriality via pushforward. We will refer to global sections of this as *Mackey presheaves.*

We further define $\underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E}) \coloneqq \underline{\mathrm{MackPSh}}^{\iota\mathcal{A}}(\mathcal{A};\mathcal{E})\colon A \mapsto \mathrm{Fun}((\mathcal{A}_{/A})^{\mathrm{op}},\mathcal{E})$, and we write $\mathbb{U}\colon \underline{\mathrm{MackPSh}}^{\mathcal{Q}}(\mathcal{A};\mathcal{E}) \to \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ for the evident forgetful map.

Remark 8.10. If $\mathcal{A} = T$ is small, the category $\underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ is denoted $\underline{\mathcal{E}}_T$ and called the *category of T-objects* in [CLL23a, Example 2.1.11].

Lemma 8.11. *The* $\mathcal{A}$*-category* $\underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ *has all* $\mathcal{Q}$*-limits.*

Proof. Let $q\colon A \to B$ be any map in $\mathcal{Q}$. As $\mathcal{Q}$ is closed under base change, $\mathcal{A}_{/q}\colon \mathcal{A}_{/A} \to \mathcal{A}_{/B}$ has a right adjoint given by pullback along q, and this satisfies base change with respect to pushforward along maps in $\mathcal{A}$ by the pasting law for pullbacks. The claim now follows simply from 2-functoriality of PSh. □

Construction 8.12. If $\mathcal{A}$ has all pullbacks, then the target map $t\colon \mathrm{Ar}(\mathcal{A}) \to \mathcal{A}$ is a cartesian fibration classifying the functor $\mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}, A \mapsto \mathcal{A}_{/A}$. If $\mathcal{Q} \subseteq \mathcal{A}$ is closed under base change, then this restricts to a cartesian fibration $t\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \to \mathcal{A}$ where the source denotes the full subcategory spanned by the maps in $\mathcal{Q}$; this then classifies the functor $\underline{\mathbf{U}}_{\mathcal{Q}}\colon A \mapsto \mathcal{Q}_{/A}$ considered before if $\mathcal{A} = \mathcal{B}$ is a topos.

If now $\mathcal{A}$ is arbitrary, then embedding it in a pullback-preserving way into a category with all pullbacks, we see that $t\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \to \mathcal{A}$ is still a cartesian fibration. We denote the straightening $\mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}$ again by $A \mapsto \mathcal{Q}_{/A}$; note that this agrees with the previous functor of the same name if $\mathcal{A} = \mathcal{B}$ is a topos. Taking spans levelwise then as before gives us a functor $\underline{\mathrm{Span}}(\mathcal{Q})\colon A \mapsto \mathrm{Span}(\mathcal{Q}_{/A})$.

Proposition 8.13. *There is an equivalence*

$$\Phi\colon \underline{\mathrm{MackPSh}}^{\mathcal{Q}}(\mathcal{A};\mathcal{E}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_{\mathcal{A}}^{\mathcal{Q}\text{-}\times}\big(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})\big)$$

of $\mathcal{A}$*-categories over* $\underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$*.*

The proof of the proposition will take up most of this section; for now let us record that it immediately implies the theorem:

Proof of Theorem 8.2, assuming Proposition 8.13. Applying the previous proposition for $\mathcal{A} = \mathcal{B}$ (and using the comparison of internal homs from the proof of Proposition 2.7), it only remains to show that $F\colon \mathrm{Span}(\mathcal{B}_{/A}, \mathcal{B}_{/A}, \mathcal{B}_{/A}[\mathcal{Q}]) \to \mathcal{E}$ restricts to a sheaf $(\mathcal{B}_{/A})^{\mathrm{op}} \to \mathcal{E}$ if and only if $\Phi(F)\colon \pi_A^*\underline{\mathrm{Span}}(\mathcal{Q}) \to \pi_A^*\underline{\mathrm{PSh}}(\mathcal{B};\mathcal{E})$ factors through $\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E})$. As Φ is a functor over $\underline{\mathrm{PSh}}(\mathcal{B};\mathcal{E})$, it is clear that F restricts to a sheaf if and only if $\Phi(F)(\mathrm{id}_A)\colon \mathcal{B}_{/A}^{\mathrm{op}} \to \mathcal{E}$ is a sheaf. However, in this case we have for any $f\colon B \to A$ in $\mathcal{B}$ and any $q\colon C \to B$ in $\mathcal{Q}$ equivalences

$$\Phi(F)(q) \simeq \Phi(F)(q_*(fq)^*\mathrm{id}_A) \simeq q_*(fq)^*\Phi(F)(\mathrm{id}_A)$$

by $\mathcal{Q}$-continuity; the claim follows as $\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E}) \subseteq \underline{\mathrm{PSh}}(\mathcal{B};\mathcal{E})$ is closed under $\mathcal{Q}$-limits by the above description of limits and local cartesian closure of $\mathcal{B}$. □

8.2. **Comparison of underlying categories.**

Before establishing the full parametrized equivalence from Proposition 8.13, we will prove in this subsection that there exists an equivalence on global sections:

$$\mathrm{Fun}(\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{Q}),\mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})).$$

The outline of the proof is as follows:

(1) As we will recall below, there is a 1:1-correspondence between $\mathcal{A}$-functors $F\colon \underline{\mathrm{Span}}(\mathcal{Q}) \to \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ and non-parametrized functors $\widetilde{F}\colon \int\underline{\mathrm{Span}}(\mathcal{Q}) \to \mathcal{E}$ from the total category of the cocartesian unstraightening of $\underline{\mathrm{Span}}(\mathcal{Q})$. Following [HHLN23], we describe this unstraightening $\int\underline{\mathrm{Span}}(\mathcal{Q})$ explicitly in terms of certain spans in the arrow category $\mathrm{Ar}(\mathcal{A})$ of $\mathcal{A}$.

(2) Next, we prove that a parametrized functor $F\colon \underline{\mathrm{Span}}(\mathcal{Q}) \to \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ preserves $\mathcal{Q}$-limits if and only if its associated functor $\widetilde{F}\colon \int \underline{\mathrm{Span}}(\mathcal{Q}) \to \mathcal{E}$ inverts a certain explicit class of maps $\mathcal{W}$, see Proposition 8.16. In particular, $\mathcal{Q}$-continuous functor $\underline{\mathrm{Span}}(\mathcal{Q}) \to \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ correspond to functors out of the localization of $\int \underline{\mathrm{Span}}(\mathcal{Q})$ at $\mathcal{W}$.
(3) Finally, we show in Proposition 8.19 that this localization is given by the span category $\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{Q})$.

Let us start by making the unstraightening $\int \underline{\mathrm{Span}}(\mathcal{Q})$ explicit:

Proposition 8.14. *Let* $\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \subseteq \mathrm{Ar}(\mathcal{A})$ *again denote the full subcategory spanned by the maps of* $\mathcal{Q}$ *and write* $\mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}} \subseteq \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}$ *for the wide subcategory of all maps inverted by* $t\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \to \mathcal{A}$.
Then $(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}})$ *is an adequate triple, and*

$$t\colon \mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}) := \mathrm{Span}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}}) \to \mathrm{Span}(\mathcal{A},\mathcal{A},\iota\mathcal{A}) \simeq \mathcal{A}^{\mathrm{op}}$$

is a cocartesian fibration classifying the functor $\underline{\mathrm{Span}}(\mathcal{Q})$ *from Construction 8.12.*

Proof. This is an instance of [HHLN23, Theorem 3.9], using that $t\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \to \mathcal{A}$ is (by definition) the cartesian fibration classifying $A \mapsto \mathcal{Q}_{/A}$. □

Lemma 8.15. *For every presentable category* $\mathcal{E}$*, there is a natural equivalence*

$$\mathrm{Fun}_{\mathcal{A}}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}), \mathcal{E}).$$

Proof. Applying [CLL23a, Lemma 2.2.13] in a larger universe, there is a natural equivalence

$$\mathrm{Fun}_{\mathcal{A}}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})) \simeq \mathrm{Fun}(\textstyle\int \underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{E});$$

The claim is now immediate from the above explicit description of $\int \underline{\mathrm{Span}}(\mathcal{Q})$. □

Now that we have obtained a description of $\mathcal{A}$-functors $\underline{\mathrm{Span}}(\mathcal{Q}) \to \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ as non-parametrized functors out of an explicit span category, we would like to identify which of them correspond to $\mathcal{Q}$-continuous parametrized functors. This is addressed by the following result:

Proposition 8.16. *Consider an* $\mathcal{A}$*-functor* $F\colon \underline{\mathrm{Span}}(\mathcal{Q}) \to \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$*, and let*

$$\widetilde{F}\colon \mathrm{Span}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}}) \to \mathcal{E}$$

denote the associated functor from Lemma 8.15. Then F *preserves* $\mathcal{Q}$*-limits if and only if* $\widetilde{F}$ *inverts the collection* $\mathcal{W}_s^{\mathrm{Span}}$ *of all maps of the form*

$$\begin{array}{ccccc} C & = & C & = & C \\ {\scriptstyle qf}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ B & \xleftarrow[q]{} & A & = & A \end{array} \tag{5}$$

for composable morphisms f *and* q *in* $\mathcal{Q}$.

The proof relies on the following simple observation:

Lemma 8.17. *Let $\mathcal{A}$ be any category and let $\mathcal{Q}$ be a wide subcategory. Then the source map $s\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \to \mathcal{A}$ is a localization at the class $\mathcal{W}_s$ of maps of the form*

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle qf} \\ B & \xrightarrow[q]{} & C \end{array} \tag{6}$$

with q and f in $\mathcal{Q}$. Moreover, s admits a left adjoint const *given by the inclusion of constant arrows.*

Proof. It is clear that the inclusion of constant arrows is left adjoint and right inverse to s. To complete the proof it now suffices to observe that s inverts the maps (6) and that the counit $\mathrm{const} \circ s \to \mathrm{id}$ is levelwise of this form. □

Proof of Proposition 8.16. We first recall from Corollary 4.7 and Proposition 2.20 that F preserves $\mathcal{Q}$-limits if and only if its restriction to $\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}}$ is right Kan extended from the point. Similarly, the invertibility condition on $\tilde{F}$ only depends on its restriction to the subfibration $t\colon (\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}})^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$ classifying $\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}}$; by naturality, we are therefore reduced to proving that a functor $F\colon \underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}} \to \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})$ is right Kan extended from the point if and only if the associated functor $\widetilde{F}\colon (\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}})^{\mathrm{op}} \to \mathcal{E}$ inverts the maps $(\mathcal{W}_s)^{\mathrm{op}}$ from (6).

For this let us consider the naturality square

$$\begin{array}{ccc} \mathrm{Fun}(\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}}, \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})) & \xrightarrow{\sim} & \mathrm{Fun}((\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}})^{\mathrm{op}}, \mathcal{E}) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}=(\mathrm{const}^{\mathrm{op}})^*} \\ \mathrm{Fun}(\underline{1}, \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})) & \xrightarrow[\sim]{} & \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{E}) \end{array}$$

associated to the map $\underline{1} \to \underline{\mathbf{U}}_{\mathcal{Q}}$ classifying the point.

The horizontal maps are equivalences and the vertical maps admit right adjoints; it then follows formally that the top horizontal map restricts to an equivalence between the essential images of these adjoints. The right adjoint of the vertical arrow on the left is precisely right Kan extension. On the other hand, by Lemma 8.17 we have an adjunction $s^{\mathrm{op}} \dashv \mathrm{const}^{\mathrm{op}}$, so that the right adjoint of $(\mathrm{const}^{\mathrm{op}})^*\colon \mathrm{Fun}((\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}})^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{E})$ is given by $(s^{\mathrm{op}})^*$. Appealing to the lemma once more, the essential image of this functor is precisely characterized by the above invertibility condition. □

As a consequence of the previous result, a functor $\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}) \to \mathcal{E}$ preserves $\mathcal{Q}$-limits if and only if it factors through the localization of $\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}})$ at the maps of the form (5). We will now give an explicit description of this localization:

Construction 8.18. Consider the source functor $s\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \to \mathcal{A}$ once more. By left-cancelability of $\mathcal{Q}$, this maps $\mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}}$ into $\mathcal{Q}$. As $\mathcal{Q}$ is closed under base change, this then further shows that the pullback of a map in $\mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}}$ along a map in $\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}$ is just computed pointwise, so that s preserves all requisite pullbacks. Altogether, we see that s defines a map of adequate triples $(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}, \mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}}) \to (\mathcal{A}, \mathcal{A}, \mathcal{Q})$, and thus induces a functor

$$s\colon \mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}) \to \mathrm{Span}(\mathcal{A}, \mathcal{A}, \mathcal{Q}).$$

Proposition 8.19. *This is a localization at the class of maps* $\mathcal{W}_s^{\mathrm{Span}}$ *from* (5).

Proof. By the localization criterion from [CHLL24, Theorem 4.1.1], it will be enough to show that $s\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \to \mathcal{A}$ is a localization at the maps $\mathcal{W}_s$ and that $s\colon \mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}} \to \mathcal{Q}$ is a right fibration. However, the first statement is an instance of Lemma 8.17, while for the second statement it is enough to observe that $\mathrm{Ar}(\mathcal{Q})^{\mathrm{fw}} = \mathrm{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}}$ consists precisely of the cartesian edges of the cartesian fibration $s\colon \mathrm{Ar}(\mathcal{Q}) \to \mathcal{Q}$. □

Combining the above results, we can now prove the equivalence from Proposition 8.13 on underlying non-parametrized categories:

Proposition 8.20. *There is a natural equivalence of non-parametrized categories* $\mathrm{Fun}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})) \simeq \mathrm{Fun}(\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{Q}),\mathcal{E})$.

Proof. Combining Lemma 8.15 and Proposition 8.16, the left hand side is equivalent to the full subcategory $\mathcal{F} \subseteq \mathrm{Fun}(\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}),\mathcal{E})$ spanned by the functors inverting $\mathcal{W}_s^{\mathrm{Span}}$. On the other hand, Proposition 8.19 shows that precomposing with s induces an equivalence between $\mathrm{Fun}(\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{Q}),\mathcal{E})$ and the same $\mathcal{F}$. □

8.3. **Proof of Proposition 8.13.** We will now show how one can upgrade the non-parametrized equivalence of Proposition 8.20 to a parametrized equivalence, yielding a proof of Proposition 8.13 and thus completing the proof of Theorem 8.2. The basic idea will be to reduce this to the unparametrized statement with $\mathcal{A}$ replaced by $\mathcal{A}_{/A}$ for all $A \in \mathcal{A}$; however, some care has to be taken to get all coherences straight.

Observation 8.21. Let $\mathcal{C}$ be an $\mathcal{A}$-category and let $\mathcal{E}$ be presentable. Combining the categorical Yoneda lemma with [CLL23a, Lemma 2.2.13], we obtain an equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{A}}(\mathcal{C}, \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E}))(A) \xrightarrow{\sim} \mathrm{Fun}(\textstyle\int(\mathcal{C}\times \underline{A}),\mathcal{E}) = \mathrm{Fun}(\textstyle\int\mathcal{C}\times_{\mathcal{A}^{\mathrm{op}}}(\mathcal{A}_{/A})^{\mathrm{op}},\mathcal{E})$$

natural in $\mathcal{C}$, $\mathcal{E}$, and in $A \in \mathcal{A}^{\mathrm{op}}$.

Below we will apply this to $\mathcal{C} = \underline{\mathrm{Span}}(\mathcal{Q})$, in which case we have the same explicit description of the cocartesian unstraightening as before. Let us also describe the resulting pullback explicitly:

Lemma 8.22. *Let $\mathcal{A}$ be any category and let $A \in \mathcal{A}$. Then* $(\mathrm{Ar}(\pi_A), t)\colon \mathrm{Ar}(\mathcal{A}_{/A}) \to \mathrm{Ar}(\mathcal{A}) \times_{\mathcal{A}} \mathcal{A}_{/A}$ *is an equivalence of cartesian fibrations over $\mathcal{A}$. Moreover, this can be made natural in $A \in \mathcal{A}$ (with respect to the functoriality via postcomposition).*

Proof. It is clear that this is a map of cartesian fibrations, so it is enough to show that it underlies an equivalence in $\mathrm{Fun}(\mathcal{A},\mathrm{Cat}_{/\mathcal{A}}) \simeq \mathrm{Fun}(\mathcal{A},\mathrm{Cat})_{/\mathrm{const}\,\mathcal{A}}$.

We begin by making some cocartesian unstraightenings explicit. The cocartesian unstraightening of $\mathcal{A}_{/\bullet}\colon \mathcal{A} \to \mathrm{Cat}$ is the fibration $t\colon \mathrm{Ar}(\mathcal{A}) = \mathrm{Fun}([1],\mathcal{A}) \to \mathcal{A}$. As unstraightening commutes with Cat-tensors it also commutes with Cat-cotensors, so the unstraightening $\mathcal{X} \to \mathcal{A}$ of $\mathrm{Ar}(\mathcal{A}_{/\bullet})$ is given by the cotensor $t^{[1]}$ in $\mathrm{Cat}_{/\mathcal{A}}^{\mathrm{cocart}}$, i.e. by the pullback

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \operatorname{Fun}([1]\times[1],\mathcal{A}) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (-,1)^*} \\ \mathcal{A} & \xrightarrow[\text{const}]{} & \operatorname{Fun}([1],\mathcal{A}) \end{array}$$

where $(-,1)\colon [1] \to [1]\times[1]$ denotes the map classifying the edge $(0,1) \to (1,1)$. The composite

$$\mathcal{X} \to \operatorname{Fun}([1]\times[1],\mathcal{A}) \xrightarrow{((-,0)^*,(1,1)^*)} \operatorname{Fun}([1],\mathcal{A}) \times \mathcal{A}$$

then straightens to a natural transformation given pointwise by $\operatorname{Ar}(\pi_A)$, while the target map $\operatorname{Ar}(\mathcal{A}_{/\bullet}) \to \mathcal{A}_{/\bullet}$ unstraightens to the map $\mathcal{X} \to \operatorname{Fun}([1],\mathcal{A})$ induced by restricting to the edge $(1,0) \to (1,1)$. Altogether, we get a commutative square of maps of cocartesian fibrations

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{((-,0)^*,(1,1)^*)} & \operatorname{Fun}([1],\mathcal{A}) \times \mathcal{A} \\ {\scriptstyle (1,-)^*}\big\downarrow & \searrow^{(1,1)^*} \quad \mathcal{A} \quad {}^{\mathrm{pr}_2}\swarrow & \big\downarrow{\scriptstyle t\times\mathcal{A}} \\ & \nearrow_{t} \qquad \nwarrow_{\mathrm{pr}_2} & \\ \operatorname{Fun}([1],\mathcal{A}) & \xrightarrow[(s,t)]{} & \mathcal{A}\times\mathcal{A} \end{array}$$

such that the induced map on pullbacks pointwise straightens to the map $(\operatorname{Ar}(\pi_A), t)$. Moreover, the diagonal composite $\mathcal{X} \to \mathcal{A}\times\mathcal{A}$ straightens to the structure map $\operatorname{Ar}(\mathcal{A}_{/\bullet}) \to \operatorname{const}\mathcal{A}$, so it only remains to show that this is a pullback square in Cat.

By direct inspection, the pullback is given by $\operatorname{Fun}(\Lambda^2_1,\mathcal{A}) \simeq \operatorname{Fun}([2],\mathcal{A})$ with the comparison map $\mathcal{X} \to \operatorname{Fun}([2],\mathcal{A})$ induced by restriction along the map $f\colon [2] \to [1]\times[1]$ classifying $(0,0) \to (1,0) \to (1,1)$. The claim therefore amounts to saying that f induces an equivalence $[2] \to \big([1]\times[1]\big)/\big([1]\times\{1\}\big)$. However, one immediately checks that an inverse equivalence is induced by the map $[1]\times[1] \to [2], (a,b) \mapsto \min\{2, a+42b\}$. □

Observation 8.23. The equivalence from the previous lemma restricts to natural equivalences of cartesian fibrations

$$\operatorname{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}} := \operatorname{Ar}(\mathcal{A}_{/A})^{\mathcal{A}_{/A}[\mathcal{Q}]} \xrightarrow{\sim} \operatorname{Ar}(\mathcal{A})^{\mathcal{Q}} \times_{\mathcal{A}} \mathcal{A}_{/A}$$

for all $A \in \mathcal{A}$. By direct inspection, this identifies the weak equivalences $\mathcal{W}_s \subseteq \operatorname{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}}$ from Lemma 8.17 with $\mathcal{W}_s \times_{\mathcal{A}} \mathcal{A}_{/A}$.

Similarly, one checks that it restricts to an equivalence

$$\operatorname{Ar}(\mathcal{A}_{/A})^{\mathcal{Q},\mathrm{fw}} \xrightarrow{\sim} \operatorname{Ar}(\mathcal{A})^{\mathcal{Q},\mathrm{fw}} \times_{\iota\mathcal{A}} \iota(\mathcal{A}_{/A}).$$

Using that Span preserves pullbacks of adequate triples, we therefore get a natural commutative diagram

$$\begin{array}{ccc} \big(\operatorname{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}}\big)^{\mathrm{op}} & \xrightarrow{\sim} & \big(\operatorname{Ar}(\mathcal{A})^{\mathcal{Q}}\big)^{\mathrm{op}} \times_{\mathcal{A}^{\mathrm{op}}} (\mathcal{A}_{/A})^{\mathrm{op}} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Span}_{\mathrm{fw}}\big(\operatorname{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}}\big) & \xrightarrow[\sim]{} & \operatorname{Span}_{\mathrm{fw}}\big(\operatorname{Ar}(\mathcal{A})^{\mathcal{Q}}\big)^{\mathrm{op}} \times_{\mathcal{A}^{\mathrm{op}}} (\mathcal{A}_{/A})^{\mathrm{op}} \end{array}$$

where the horizontal maps are equivalences of *co*cartesian fibrations, and the lower one identifies $\mathcal{W}_s^{\mathrm{Span}}$ with $\mathcal{W}_s^{\mathrm{Span}} \times_{\mathcal{A}^{\mathrm{op}}} (\mathcal{A}_{/A})^{\mathrm{op}}$.

Combining this with Observation 8.21 we get a natural equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{A}}\big(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})\big)(A) \xrightarrow{\sim} \mathrm{Fun}\big(\mathrm{Span}_{\mathrm{fw}}\big(\mathrm{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}}\big), \mathcal{E}\big).$$

Corollary 8.24. *Consider any* $F \in \underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E}))(A)$*, with associated functor* $\widetilde{F}\colon \mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}}) \to \mathcal{E}$*. Then* F *belongs to the full subcategory* $\underline{\mathrm{Fun}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(Q), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E}))(A)$ *if and only if* $\widetilde{F}$ *inverts all maps in* $\mathcal{W}_s^{\mathrm{Span}}$.

Proof. As in the proof of Proposition 8.16, both conditions only rely on the restriction to backwards arrows. We now have a commutative square

$$\begin{array}{ccc} \mathcal{A}_{/A} & = & \mathcal{A} \times_{\mathcal{A}} \mathcal{A}_{/A} \\ {\scriptstyle\mathrm{const}}\downarrow & & \downarrow{\scriptstyle\mathrm{const}\times_{\mathcal{A}}\mathcal{A}_{/A}} \\ \mathrm{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}} & \xrightarrow[\sim]{} & \mathrm{Ar}(\mathcal{A})^{\mathcal{Q}} \times_{\mathcal{A}} \mathcal{A}_{/A} \end{array}$$

and hence altogether a commutative square

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{\mathcal{A}}\big(\underline{\mathbf{U}}_{\mathcal{Q}}^{\mathrm{op}}, \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})\big)(A) & \xrightarrow{\sim} & \mathrm{Fun}\left((\mathrm{Ar}(\mathcal{A}_{/A})^{\mathcal{Q}})^{\mathrm{op}}, \mathcal{E}\right) \\ {\scriptstyle\mathrm{pt}^*}\downarrow & & \downarrow{\scriptstyle(\mathrm{const}^{\mathrm{op}})^*} \\ \underline{\mathrm{Fun}}_{\mathcal{A}}\big(\underline{1}, \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})\big)(A) & \xrightarrow[\sim]{} & \mathrm{Fun}((\mathcal{A}_{/A})^{\mathrm{op}}, \mathcal{E}). \end{array}$$

By the same formal Beck–Chevalley yoga as before, an object of the top left corner is $\mathcal{Q}$-continuous if and only if the top horizontal equivalence maps it into the essential image of the right adjoint of the right-hand vertical map. Replacing $\mathcal{A}$ by $\mathcal{A}_{/A}$, this essential image was identified in the proof of Proposition 8.16 as precisely those functors satisfying the above invertibility condition. □

Proof of Proposition 8.13. By the above, we have a map of $\mathcal{A}$-categories

$$\underline{\mathrm{Fun}}_{\mathcal{A}}^{\mathcal{Q}\text{-}\times}\big(\underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})\big) \to \mathrm{Fun}\big(\,\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A}_{/\bullet})^{\mathcal{Q}}), \mathcal{E}\big)$$

that induces an equivalence onto the full subcategory $\mathcal{F}$ spanned in degree $A \in \mathcal{A}$ by those functors that invert $\mathcal{W}_s^{\mathrm{Span}}$. On the other hand, we have a natural map $s\colon \mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A}_{/\bullet})^{\mathcal{Q}}) \to \mathrm{Span}(\mathcal{A}_{/\bullet}, \mathcal{A}_{/\bullet}, \mathcal{A}_{/\bullet}[\mathcal{Q}])$, and using Proposition 8.19 with $\mathcal{A}$ replaced by $\mathcal{A}_{/A}$ this likewise induces an equivalence onto $\mathcal{F}$, yielding an equivalence $\underline{\mathrm{MackPSh}}^{\mathcal{Q}}(\mathcal{A};\mathcal{E}) \simeq \underline{\mathrm{Fun}}_{\mathcal{A}}^{\mathcal{Q}\text{-}\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E}))$. It remains to show that this is equivalence is compatible with the forgetful functors.

We will show more generally that our equivalence is compatible with passing to a smaller left-cancelable $\mathcal{Q}' \subseteq \mathcal{Q}$ closed under base change. This is clear for restriction along s. For the map $\underline{\mathrm{Fun}}_{\mathcal{A}}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})) \to \mathrm{Fun}(\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A}_{/\bullet})^{\mathcal{Q}}), \mathcal{E})$ note that this holds for the intermediate composite $\underline{\mathrm{Fun}}_{\mathcal{A}}(\underline{\mathrm{Span}}(\mathcal{Q}), \underline{\mathrm{PSh}}(\mathcal{A};\mathcal{E})) \to \underline{\mathrm{Fun}}(\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}) \times_{\mathcal{A}^{\mathrm{op}}} (\mathcal{A}_{/\bullet})^{\mathrm{op}}, \mathcal{E})$ simply by naturality. Finally, the equivalence $\mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A})^{\mathcal{Q}}) \times_{\mathcal{A}^{\mathrm{op}}} (\mathcal{A}_{/\bullet})^{\mathrm{op}} \simeq \mathrm{Span}_{\mathrm{fw}}(\mathrm{Ar}(\mathcal{A}_{/\bullet})^{\mathcal{Q}})$ was construced as restriction of a fixed equivalence $\mathrm{Span}(\mathrm{Ar}(\mathcal{A})) \times_{\mathcal{A}^{\mathrm{op}}} (\mathcal{A}_{/\bullet})^{\mathrm{op}} \simeq \mathrm{Span}(\mathrm{Ar}(\mathcal{A}_{/\bullet}))$, so it is again compatible with passing to a subclass. □

8.4. **The free Mackey sheaf.** Classically, an easy application of the Yoneda lemma shows that the free Mackey functor $\mathrm{Span}(\mathrm{Fin}_G) \to \mathrm{Ab}$ is corepresented by the 1-point set. The analogue holds in our setting, except that proving that the corepresented functor actually is a $\mathcal{Q}$-Mackey sheaf is not entirely trivial:

Lemma 8.25. *The functor* $\hom(1,-)\colon \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}) \to \mathrm{Spc}$ *is a* $\mathcal{Q}$*-Mackey sheaf. Its restriction to* $\mathcal{B}^{\mathrm{op}}$ *agrees with the functor* $\iota\,\underline{\mathbf{U}}_{\mathcal{Q}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Spc}$ *which sends* A *to the groupoid core of* $\mathcal{Q}_{/A}$*.*

Proof. As $\underline{\mathbf{U}}_{\mathcal{Q}}$ is a $\mathcal{B}$-category, it will be enough to prove the second statement. We will prove this by computing the cocartesian unstraightening of the restriction of $\hom(1,-)$ to $\mathcal{B}^{\mathrm{op}}$, and show it agrees with the unstraightening of $\iota\underline{\mathbf{U}}_{\mathcal{Q}}$. We first give an explicit description of the forgetful functor $\pi\colon \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})_{1/} \to \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})$, which is the unstraightening of $\hom(1,-)\colon \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}) \to \mathrm{Spc}$. To this end, consider the adequate triple $Q_{[1]}$ from [HHLN23, Lemma 2.5 and Definition 2.16]: the underlying category is the full subcategory of $\mathrm{Fun}(\Lambda^2_0,\mathcal{B})$ spanned by the functors sending $0 \to 2$ to a map in $\mathcal{Q}$. The backward maps consist of all diagrams

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_0 & \longrightarrow & X_2 \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ Y_1 & \longleftarrow & Y_0 & \longrightarrow & Y_2 \end{array}$$

such that the right-hand square is a pullback. The forward maps are given by those natural transformations that are pointwise in $\mathcal{Q}$ and for which the *left-hand* square is a pullback.

By Corollary 2.22 of *op. cit.*, we may identify the functor $(s,t)\colon \mathrm{Ar}(\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})) \to \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})^{\times 2}$ with the map $(\mathrm{ev}_1,\mathrm{ev}_2)\colon \mathrm{Span}(Q_{[1]}) \to \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})^{\times 2}$. Pulling back to $\{1\}$ in the first factor and using that Span preserves limits, we obtain the following description of $\pi\colon \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})_{1/} \to \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})$: the source is the category of spans in $\mathrm{Ar}(\mathcal{B})^{\mathcal{Q}}$ of the form

$$\begin{array}{ccccc} Y_0 & \longleftarrow & X_0 & = & Z_0 \\ \downarrow & \llcorner & \downarrow & & \downarrow \\ Y_2 & \longleftarrow & X_2 & \xrightarrow[q]{} & Z_2 \end{array}$$

where the left-hand square is a pullback and q belongs to $\mathcal{Q}$ (note that compared to the previous diagram this has been rotated by $\frac{3}{2}\pi$ radians); the forgetful map is then given by the target map.

We thus obtain the unstraightening of $\hom(1,-)|_{\mathcal{B}^{\mathrm{op}}}$ by restricting this forgetful functor to $\mathcal{B}^{\mathrm{op}} \simeq \mathrm{Span}(\mathcal{B},\mathcal{B},\iota\mathcal{Q})$ in the target. The resulting functor is the target map $t^{\mathrm{op}}\colon (\mathrm{Ar}(\mathcal{B})^{\mathcal{Q}}_{\mathrm{cart}})^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$, where $\mathrm{Ar}(\mathcal{B})^{\mathcal{Q}}_{\mathrm{cart}}$ is the wide subcategory of $\mathrm{Ar}(\mathcal{B})^{\mathcal{Q}}$ spanned by the cartesian squares. As these are precisely the cartesian morphisms for the cartesian fibration $t\colon \mathrm{Ar}(\mathcal{B})^{\mathcal{Q}} \to \mathcal{B}$ classified by $\underline{\mathbf{U}}_{\mathcal{Q}}$, we conclude that this resulting functor is indeed the cocartesian unstraightening of $\iota\underline{\mathbf{U}}_{\mathcal{Q}}$, finishing the proof. □

Remark 8.26. The above proof actually allows to describe $\hom(1,-)$ as a functor on all of $\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})$: it is obtained from Barwick's *unfurling* of $\underline{\mathbf{U}}_{\mathcal{Q}}$ (see

[HHLN23, Example 3.4]) by passing to groupoid cores pointwise. In particular, the covariant functoriality in $\mathcal{Q}$ is given by postcomposition.

Corollary 8.27. *The free $\mathcal{Q}$-Mackey sheaf* $\mathbb{P}(1)\colon \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}) \to \mathrm{Spc}$ *is corepresented by* 1.

Proof. By the non-parametrized Yoneda lemma, $\hom_{\mathrm{Span}}(1,-)$ corepresents evaluation at 1 on the category of all functors $\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}) \to \mathrm{Spc}$. As the same holds true on $\mathrm{Mack}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc})$ for $\mathbb{P}(1)$ by adjointness, and since $\hom_{\mathrm{Span}}(1,-) \in \mathrm{Mack}^{\mathcal{Q}}(\mathcal{B};\mathrm{Spc})$ by Lemma 8.25, the claim follows. □

9. Examples and applications

In this section, we discuss various examples and applications of our results. We start in Section 9.1 by proving a general result which lets us in many practical situations reduce the big Mackey sheaf descriptions obtained in the previous section to much more manageable descriptions. In the remainder of the section we then specialize this result to the contexts of higher semiadditivity, equivariant semiadditivity, and 'very G-semiadditivity,' and discuss various interesting consequences and applications.

9.1. **Smaller spans.** As indicated above, the information encoded in a Mackey sheaf $F\colon \mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}) \to \mathcal{E}$ is most of the time highly redundant. For example, if $\mathcal{B}$ is the topos of ∞-groupoids and $\mathcal{Q}$ is the class of finite covering maps, then the above does not immediately recover the definition of the category of commutative monoids as $\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}),\mathcal{E})$ but instead describes it in a somewhat bloated way as a subcategory of $\mathrm{Fun}(\mathrm{Span}(\mathrm{Spc},\mathrm{Spc},\mathrm{Fin}_{\mathrm{loc}}),\mathcal{E})$. As the most extreme case, consider an arbitrary topos $\mathcal{B}$ with $\mathcal{Q} = \iota\mathcal{B}$ (no semiadditivity conditions), i.e. of a continuous functor $F\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{E}$. If $\mathcal{B} = \mathrm{PSh}(T)$, such a functor is completely determined by its restriction along the Yoneda embedding. More generally, if $\mathcal{B}$ is given by sheaves on some site $\mathcal{A}$ we may equivalently describe F as a functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{E}$ satisfying descent. We will now give a similar sheaf description for non-trivial $\mathcal{Q}$ as long as the latter is defined via the site $\mathcal{A}$.

Definition 9.1 (Mackey sheaves on a site)**.** Let $\mathcal{A}$ be a small category equipped with a Grothendieck topology τ, and let $Q \subseteq \mathcal{A}$ be a wide τ-local subcategory closed under base change and diagonals. Given a complete category $\mathcal{E}$, we define an *$\mathcal{E}$-valued Q-Mackey τ-sheaf* on $(\mathcal{A},\tau)$ to be a functor $M\colon \mathrm{Span}(\mathcal{A},\mathcal{A},Q) \to \mathcal{E}$ whose restriction $M|_{\mathcal{A}^{\mathrm{op}}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{E}$ is a τ-sheaf. The categories of Mackey sheaves over $(\mathcal{A}_{/A},\tau)$ for varying $A \in \mathcal{A}$ then assemble into a functor $\underline{\mathrm{Mack}}^{Q}_{\tau}(\mathcal{A};\mathcal{E})\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{E}$.

Example 9.2. If $\mathcal{Q} \subseteq \mathrm{Shv}_{\tau}(\mathcal{A})$ is locally inductible, then its preimage in $\mathcal{A}$ satisfies the above assumptions.

Proposition 9.3. *Let $\mathcal{Q} \subseteq \mathcal{B}$ be a locally inductible subcategory. Assume we have a small full subcategory $\mathcal{A} \subseteq \mathcal{B}$ equipped with a subcanonical Grothendieck topology τ such that the following conditions are satisfied:*

1. *The inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ extends to an equivalence* $\mathrm{Shv}_{\tau}(\mathcal{A}) \xrightarrow{\sim} \mathcal{B}$.
2. *$\mathcal{A}$ is closed under maps in $\mathcal{Q}$ in the following sense: for every $B \in \mathcal{A}$ and $A \to B$ in $\mathcal{Q}$, also $A \in \mathcal{A}$.*

Then the inclusion $(\mathcal{A},\mathcal{A},\mathcal{A}\cap\mathcal{Q})\hookrightarrow(\mathcal{B},\mathcal{B},\mathcal{Q})$ *is a map of adequate triples and the restriction functor* $\mathrm{Fun}(\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}),\mathcal{E})\to\mathrm{Fun}(\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{A}\cap\mathcal{Q}),\mathcal{E})$ *admits a right adjoint, restricting to an adjoint equivalence*

$$\mathrm{Mack}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})\simeq\mathrm{Mack}_{\tau}^{\mathcal{A}\cap\mathcal{Q}}(\mathcal{A};\mathcal{E}).$$

Proof. Consider a pullback square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & A \\ {\scriptstyle q'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle q\in\mathcal{Q}\cap\mathcal{A}} \\ B' & \xrightarrow[f\in\mathcal{A}]{} & B \end{array}$$

in $\mathcal{B}$ with $f\in\mathcal{A}$ and $q\in\mathcal{Q}\cap\mathcal{A}$ as indicated. Since $\mathcal{Q}$ is closed under base change, q belongs again to $\mathcal{Q}$, so the second assumption implies that all four objects belong to $\mathcal{A}$. It follows immediately that $(\mathcal{A},\mathcal{A},\mathcal{A}\cap\mathcal{Q})$ is an adequate triple and that $(\mathcal{A},\mathcal{A},\mathcal{A}\cap\mathcal{Q})\hookrightarrow(\mathcal{B},\mathcal{B},\mathcal{Q})$ is indeed a map of adequate triples.

We now observe that the second assumption on $\mathcal{Q}$ guarantees that the induced map $\iota\colon\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{A}\cap\mathcal{Q})\to\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})$ is fully faithful. We claim that we have an adjunction

$$\iota^*\colon\mathrm{Fun}(\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q}),\mathcal{E})\rightleftarrows\mathrm{Fun}(\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{A}\cap\mathcal{Q}),\mathcal{E}):\iota_*$$

with fully faithful right adjoint and such that $\iota_*X|_{\mathcal{B}^{\mathrm{op}}}$ is right Kan extended from $X|_{\mathcal{A}^{\mathrm{op}}}$. Indeed, after embedding $\mathcal{E}$ in a limit preserving way into a very large category $\widehat{\mathcal{E}}$ with large limits, this is an instance of Proposition 5.14$^{\mathrm{op}}$ (for $\mathcal{B}=\mathrm{Spc}$) as the second assumption on $\mathcal{A}$ guarantees that $\mathrm{Span}(\mathcal{A},\mathcal{A},\mathcal{A}\cap\mathcal{Q})\subseteq\mathrm{Span}(\mathcal{B},\mathcal{B},\mathcal{Q})$ is an adapted subcategory with respect to the canonical factorization systems; by the above explicit description of $\iota_*X|_{\mathcal{B}^{\mathrm{op}}}$ this right adjoint then actually restricts accordingly.

It is then clear that ι^* restricts to a functor $\mathrm{Mack}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})\to\mathrm{Mack}_{\tau}^{\mathcal{A}\cap\mathcal{Q}}(\mathcal{A};\mathcal{E})$. On the other hand, appealing to the above description of $\iota_*X|_{\mathcal{B}^{\mathrm{op}}}$ once more shows that ι_* restricts to an essentially surjective functor in the other direction since a functor $\mathcal{B}^{\mathrm{op}}\to\mathcal{E}$ is continuous if and only if it is right Kan extended from an $\mathcal{A}$-sheaf by [Lur18, Proposition 1.3.1.7]. ☐

Corollary 9.4. *In the above situation, restriction along* $\mathcal{A}\hookrightarrow\mathcal{B}$ *induces an equivalence*

$$\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})|_{\mathcal{A}^{\mathrm{op}}}\xrightarrow{\sim}\underline{\mathrm{Mack}}_{\tau}^{\mathcal{A}\cap\mathcal{Q}}(\mathcal{A};\mathcal{E}).$$

Proof. It is clear that the inclusion induces an $\mathcal{A}^{\mathrm{op}}$-natural map $\underline{\mathrm{Mack}}^{\mathcal{Q}}(\mathcal{B};\mathcal{E})|_{\mathcal{A}^{\mathrm{op}}}\to\underline{\mathrm{Mack}}_{\tau}^{\mathcal{A}\cap\mathcal{Q}}(\mathcal{A};\mathcal{E})$, and the previous proposition with $\mathcal{A}$ replaced by $\mathcal{A}_{/A}$ for varying $A\in\mathcal{A}$ shows that is indeed an equivalence. ☐

Corollary 9.5. *Let* $\mathcal{A}\subseteq\mathcal{B}$ *be equipped with a topology* τ *satisfying the assumptions of Proposition 9.3. Identifying* $\mathcal{B}$*-categories with* τ*-sheaves of categories on* $\mathcal{A}$*, we have:*

1. *The free presentable* $\mathcal{Q}$*-semiadditive* $\mathcal{B}$*-category is given by* $\underline{\mathrm{Mack}}_{\tau}^{\mathcal{A}\cap\mathcal{Q}}(\mathcal{A};\mathrm{Spc})$.
2. *The free presentable* $\mathcal{Q}$*-stable* $\mathcal{B}$*-category is given by* $\underline{\mathrm{Mack}}_{\tau}^{\mathcal{A}\cap\mathcal{Q}}(\mathcal{A};\mathrm{Sp})$.

Proof. In light of Corollary 8.3 and Corollary 8.4, these are direct consequences of the previous corollary applied to the two cases $\mathcal{E}=\mathrm{Spc}$ and $\mathcal{E}=\mathrm{Sp}$. ☐

Example 9.6 (Presheaf topoi)**.** The conditions of Proposition 9.3 are in particular satisfied in the case where $\mathcal{B} = \mathrm{PSh}(T)$ is a presheaf topos on some small category T and where $\mathcal{Q} = Q_{\mathrm{loc}}$ is obtained from a small pre-inductible subcategory $Q \subseteq \mathrm{PSh}(T)$. In this case, we let $\mathcal{A} \subseteq \mathrm{PSh}(T)$ denote the essential image of the inclusion $Q \hookrightarrow \mathrm{PSh}(T)$, and equip it with the Grothendieck topology τ in which collection $\{f_i \colon A_i \to A\}$ generates a covering sieve if and only if the map $\coprod_{i \in I} A_i \to A$ is an effective epimorphism in $\mathrm{PSh}(T)$, or equivalently if every morphism $B \to A$ from a representable object $B \in T$ factors through one of the morphisms f_i.

To see that the conditions of Proposition 9.3 are satisfied, first note that condition (2) is clear. For condition (1), notice that the image of the full inclusion $T \hookrightarrow \mathcal{A} \hookrightarrow \mathrm{Shv}_\tau(\mathcal{A})$ consists of completely compact objects which generate $\mathrm{Shv}_\tau(\mathcal{A})$ under colimits, so that restriction along this functor defines an equivalence $\mathrm{Shv}_\tau(\mathcal{A}) \xrightarrow{\sim} \mathrm{PSh}(T)$ by [Lur09, Corollary 5.1.6.11]. It is clear that this equivalence restricts on $\mathcal{A}$ to the inclusion $\mathcal{A} \hookrightarrow \mathrm{PSh}(T)$, showing condition (1).

Example 9.7 (Trivial descent)**.** As an extreme special case of Example 9.6, assume that T is a small category equipped with an inductible subcategory $Q \subseteq T$. Then $\mathcal{Q} = Q_{\mathrm{loc}} \subset \mathrm{PSh}(T)$ and $\mathcal{A} = T \subset \mathrm{PSh}(T)$, equipped with the trivial Grothendieck topology, satisfy the assumptions of Corollary 9.5. We conclude that the free presentable Q-semiadditive T-category $\underline{\mathrm{CMon}}^Q(\underline{\mathrm{Spc}}_T)$ is given by the functor

$$A \mapsto \mathrm{Fun}(\mathrm{Span}(T_{/A}, T_{/A}, T_{/A}[Q]), \mathrm{Spc}).$$

Note that the functors are not required to satisfy any sort of descent or limit-preservation condition. In particular, we obtain:

Corollary 9.8. *Let T be a small category and let $Q \subseteq T$ be an inductible subcategory. For every Q-semiadditive T-category $\mathcal{C}$, there is a unique collection of functors*

$$\mathrm{Hom}^Q_{\mathcal{C}(A)} \colon \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(A) \to \mathrm{Fun}(\mathrm{Span}(T_{/A}, T_{/A}, T_{/A}[Q]), \mathrm{Spc})$$

which are natural in $A \in T^{\mathrm{op}}$ and whose underlying functors $\mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(A) \to \mathrm{Spc}$ given by evaluation at id_A are the Hom-functors.

Proof. Given the identification of the previous example, the corollary follows immediately from the fact that the parametrized hom functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \underline{\mathrm{Spc}}_T$ uniquely lifts through the functor $\mathbb{U} \colon \underline{\mathrm{CMon}}^Q(\underline{\mathrm{Spc}}_T) \to \underline{\mathrm{Spc}}_T$, which holds by Corollary 7.7. □

9.2. **Equivariant and global homotopy theory.** We will now explain how to use this to prove, in a unified way, Mackey functor descriptions of various categories classically studied in equivariant homotopy theory, and to conversely establish universal properties of some categories of Mackey functors considered previously.

Throughout, we will work with T-categories, i.e. $\mathcal{B} = \mathrm{PSh}(T)$. Let us first consider the case of P-semiadditivity for $P \subseteq T$ atomic orbital (see Example 3.40). We write $\mathbb{F}_T$ for the finite coproduct completion of T, and $\mathbb{F}_T^P$ for the wide subcategory whose maps are finite coproducts of maps $\coprod_{i=1}^n A_i \to B$ with each $A_i \to B$ in P.

Corollary 9.9. *The free P-semiadditive presentable T-category is given by*

$$\underline{\mathrm{Mack}}_T^P \colon A \mapsto \mathrm{Fun}^\times(\mathrm{Span}((\mathbb{F}_T)_{/A}, (\mathbb{F}_T)_{/A}, (\mathbb{F}_T)_{/A}[\mathbb{F}_T^P]), \mathrm{Spc}).$$

More precisely, for every P-semiadditive presentable T-category $\mathcal{D}$, evaluation at $\hom(1,-)\colon \operatorname{Span}(\mathbb{F}_T,\mathbb{F}_T,\mathbb{F}_T^P)\to \mathrm{Spc}$ *defines an equivalence*

$$\underline{\mathrm{Fun}}_T^{\mathrm{L}}(\underline{\mathrm{Mack}}_T^P,\mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Similarly, the free P-stable presentable T-category is given by

$$A\mapsto \mathrm{Fun}^{\times}(\operatorname{Span}((\mathbb{F}_T)_{/A},(\mathbb{F}_T)_{/A},(\mathbb{F}_T)_{/A}[\mathbb{F}_T^P]),\mathrm{Sp}).$$

More generally, if $P\subseteq T$ is any wide subcategory such that $\mathbb{F}_T^P$ is pre-inductible, then these define the free presentable $(\mathbb{F}_T^P)_{\mathrm{loc}}$-semiadditive and -stable T-categories, respectively.

An independent proof of this corollary (excluding the last sentence) has been given concurrently by Pützstück [Pü24] using the theory of *cartesian patterns* of [CH22].

Proof. We will focus on the semiadditive case, the proof of the stable statement being analogous.

The topology from Lemma 9.6 on $\mathcal{A}=\mathbb{F}_T\subseteq \mathrm{PSh}(T)$ is just the disjoint union topology. Corollary 9.5 thus shows that the free $(\mathbb{F}_T^P)_{\mathrm{loc}}$-semiadditive presentable T-category is the full subcategory of $\mathrm{Fun}(\operatorname{Span}((\mathbb{F}_T)_{/\bullet},(\mathbb{F}_T)_{/\bullet},(\mathbb{F}_T^P)_{/\bullet}),\mathrm{Spc})$ given at $A\in T$ by the functors whose restriction to $(\mathbb{F}_T)_{/A}^{\mathrm{op}}$ preserves products. However, by Corollary 4.7 (or direct inspection), each $\operatorname{Span}((\mathbb{F}_T)_{/A},(\mathbb{F}_T)_{/A},(\mathbb{F}_T^P)_{/A})$ has finite products, and a functor out of it preserves finite products if and only if its restriction to $(\mathbb{F}_T)_{/A}^{\mathrm{op}}$ does so, verifying the above description.

Finally, Corollary 8.27 shows that $\hom(1,-)$ is the universal element, and so the equivalence $\underline{\mathrm{Fun}}_T^{\mathrm{L}}(\underline{\mathrm{Mack}}_T^P,\mathcal{D})\xrightarrow{\sim}\mathcal{D}$ is given by evaluation at $\hom(1,-)$ as stated. □

Remark 9.10. We can also describe the universal element of the above model of the P-stable presentable T-category as follows:

As $\mathbb{F}_T^P$ contains all fold maps $X\amalg X\to X$, Remark 8.6 implies that $\mathrm{Mack}_T^P(\mathrm{Spc})$ is equivalent to $\mathrm{Fun}^{\oplus}(\operatorname{Span}(\mathbb{F}_T,\mathbb{F}_T,\mathbb{F}_T^P),\mathrm{CMon})$ via the forgetful map.

As the delooping functor $\mathrm{CMon}\to\mathrm{Sp}$ is left adjoint to the forgetful functor (in particular semiadditive), it induces a functor $\mathrm{Fun}^{\oplus}(\operatorname{Span}(\mathbb{F}_T,\mathbb{F}_T,\mathbb{F}_T^P),\mathrm{CMon})\to \mathrm{Fun}^{\oplus}(\operatorname{Span}(\mathbb{F}_T,\mathbb{F}_T,\mathbb{F}_T^P),\mathrm{Sp})$ left adjoint to the forgetful functor. By adjointness, this then sends (the lift of) $\hom(1,-)$ to the universal element $\mathbb{S}$; in other words, $\mathbb{S}$ is given by pointwise delooping the unique E_∞-monoid structure on $\hom(1,-)$.

Let us make one special case of the above result explicit:

Theorem 9.11. *There exists an equivalence, natural in $G\in$ Glo, between the (∞-)category of G-global special Γ-spaces in the sense of* [Len20, Definition 2.2.50] *and* $\mathrm{Fun}^{\oplus}(\operatorname{Span}(\mathscr{F}_{/BG},\mathscr{F}_{/BG},\mathscr{F}_{/BG}[\mathscr{F}_\dagger]),\mathrm{CMon})$*, where as before $\mathscr{F}=\mathbb{F}_{\mathrm{Glo}}$ is the $(2,1)$-category of 1-groupoids and $\mathscr{F}_\dagger$ denotes the wide subcategory of faithful functors.*

Similarly, there exists a natural equivalence between the category of G-global spectra [Len20, Theorem 3.1.40] *and* $\mathrm{Fun}^{\oplus}(\operatorname{Span}(\mathscr{F}_{/BG},\mathscr{F}_{/BG},\mathscr{F}_{/BG}[\mathscr{F}_\dagger]),\mathrm{Sp})$.

Proof. By [CLL23a, Theorem 5.3.1] the categories of G-global special Γ-spaces make up the free presentable equivariantly semiadditive global category. The same holds for the categories of Mackey functors by Corollary 9.9, proving the first statement. The second statement follows similarly from [CLL23a, Theorem 7.3.2]. □

Remark 9.12. In the special case $G = 1$ the above models recover Schwede's ultra-commutative monoids and global spectra [Sch18] for the global family of finite groups. In this setting they first appeared as [Len22, Theorems 4.22 and 5.17].

We can further use this to describe the free presentable globally semiadditive and globally stable global categories (Example 3.39):

Corollary 9.13. *The assignment* $\mathrm{Mack}^{\mathrm{Glo}}_{\mathrm{Glo}}\colon G \mapsto \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathscr{F}_{/BG}), \mathrm{CMon})$ *defines the free presentable globally semiadditive global category. Similarly, the free presentable globally stable global category is given by* $G \mapsto \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathscr{F}_{/BG}), \mathrm{Sp})$*.* □

Remark 9.14. Note that compared to the category $\mathrm{Mack}^{\mathrm{Spc}_1}_{\mathrm{Spc}}$ of 1-commutative monoids, we have fewer limit conditions in $\mathrm{Mack}^{\mathrm{Glo}}_{\mathrm{Glo}}$, i.e. the two notions do *not* agree. Instead, the above descriptions tell us that 1-commutative monoids embed fully faithfully into 'fully globally commutative monoids' as those objects whose underlying global space is in the image of the fully faithful right adjoint of the forgetful functor $U = \mathrm{ev}_1\colon \mathrm{PSh}(\mathrm{Glo}) \to \mathrm{Spc}$. Such global spaces are called *cofree* in [Sch18, Definition 1.2.28] or *Borel complete* in [CLL23c].

Remark 9.15. Compared to the objects of classical global homotopy theory, the above 'fully global' versions come with extra structure in the form of 'deflations,' additive transfers along surjective group homomorphisms.

In addition to the non-equivariant examples arising via the previous remark, several interesting ultra-commutative monoids like the infinite orthogonal, unitary, and symplectic groups $\mathbf{O}$, $\mathbf{U}$, and $\mathbf{Sp}$ [Sch18, Examples 2.3.6, 2.3.7, and 2.3.9], as well as various global spectra occuring in nature like the sphere, the global algebraic K-theory of any $\mathbb{Q}$-algebra [Sch22, Definition 10.2 and Remark 10.7], or global complex topological K-theory $\mathbf{KU}$ [Sch18, Construction 6.4.9] and its real analogue $\mathbf{KO}$ are expected to enhance accordingly, making these fully global categories interesting objects of study. As another example, fully global Mackey functors arising from K-theoretic constructions have recently been applied to height 1 chromatic homotopy theory, see [Yua24] and [CY23]. We moreover remark that objects of the category $\mathrm{Mack}^{\mathrm{Glo}}_{\mathrm{Glo}}(\mathrm{Ab})$ (which can be viewed as a decategorification of $\mathrm{Mack}^{\mathrm{Glo}}_{\mathrm{Glo}}(\mathrm{Sp})$), or more generally $\mathrm{Mack}^{\mathrm{Glo}}_{\mathrm{Glo}}(\mathrm{Mod}_R)$ for an ordinary commutative ring R, have been well-studied in representation theory under the name *biset functors*, see e.g. [Bou10].

As another application, we can reprove the (by now classical) Mackey functor description of G-equivariant spectra [CMNN20, Theorem A.1] for a finite group G as well as its refinement to equivariantly commutative monoids recently established by Marc [Mar24]:

Corollary 9.16. *There is an equivalence, natural in* $G \in \mathrm{Orb}$, *between* $\mathrm{Mack}^G := \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathbb{F}_G), \mathrm{CMon})$ *and Shimakawa's G-equivariant special Γ-spaces* [Shi89]*.*

Similarly, there is a natural equivalence between genuine G-spectra (*say, in the incarnation of symmetric G-spectra* [Hau17]) *and* $\mathrm{Mack}^G(\mathrm{Sp}) := \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathbb{F}_G), \mathrm{Sp})$*.*

Proof. These follow as before as these models make up the universal presentable equivariantly semiadditive and equivariantly stable Orb-categories, respectively, by [CLL23b, Theorems 7.17 and 9.5]. $\square$

9.3. **Mackey profunctors and quasi-finitely genuine G-spectra.** As a new application of our result, we obtain universal characterizations for the category $\widehat{\mathcal{M}}(G,\mathbb{Z})$ of *$\mathbb{Z}$-valued G-Mackey profunctors* introduced by Kaledin [Kal22] and the category $\mathrm{Sp}_G^{\mathrm{qfin}}$ of *quasi-finitely genuine G-spectra* of Krause–McCandless–Nikolaus [KMN23].

Let G be an arbitrary group. Recall from Example 3.41 the pre-inductible subcategory $\mathrm{QFin}_G \subseteq \mathrm{PSh}(\widehat{\mathrm{Orb}}_G)$ of quasi-finite G-sets.

Definition 9.17 (Mackey profunctors, cf. [Kal22, Definition 3.2], [KMN23, Definition 4.5])**.** Let G be a group and let $\mathcal{E}$ be a presentable category. A functor $M\colon \mathrm{Span}(\mathrm{QFin}_G) \to \mathcal{E}$ is called *very additive* if for every quasi-finite G-set S the canonical map

$$M(S) \to \prod_{\overline{s}\in S/G} M(\pi^{-1}(\overline{s}))$$

is an equivalence in $\mathcal{E}$, where $\pi\colon S \to S/G$ denotes the quotient map. We write

$$\mathrm{Mack}_G^{\mathrm{pro}}(\mathcal{E}) := \mathrm{Fun}^{\mathrm{vadd}}(\mathrm{Span}(\mathrm{QFin}_G), \mathcal{E})$$

for the full subcategory of the functor category spanned by the very additive functors, and refer to its objects as *$\mathcal{E}$-valued G-Mackey profunctors*. The assignment $G/H \mapsto \mathrm{Mack}_H^{\mathrm{pro}}(\mathcal{E})$ naturally defines a G-procategory $\underline{\mathrm{Mack}}_G^{\mathrm{pro}}(\mathcal{E})\colon \widehat{\mathrm{Orb}}_G^{\mathrm{op}} \to \mathrm{Cat}$.

In order to apply our main results to this situation, we have to understand the Grothendieck topology τ on QFin_G provided by Example 9.6.

Lemma 9.18. *A functor $M\colon \mathrm{Span}(\mathrm{QFin}_G) \to \mathcal{E}$ is a τ-sheaf if and only if it is very additive.*

Proof. For the 'only if'-direction, note that for every quasi-finite G-set S the canonical map $\coprod_{\overline{s}\in S/G} \pi^{-1}(\overline{s}) \twoheadrightarrow S$ is a surjection on H-fixed points for all finite-index $H \leqslant G$, and thus becomes an effective epimorphism in $\mathrm{PSh}(\widehat{\mathrm{Orb}}_G)$. In particular, the inclusions $\{\pi^{-1}(\overline{s}) \hookrightarrow S\}_{\overline{s}\in S/G}$ define a τ-cover, showing that M is very additive whenever it is a τ-sheaf.

Conversely, assume that M is very additive. We have to show that M is a τ-sheaf, or equivalently that its restriction $M' := M|_{\mathrm{QFin}_G^{\mathrm{op}}}\colon \mathrm{QFin}_G^{\mathrm{op}} \to \mathcal{E}$ extends to a continuous functor $\mathrm{PSh}(\widehat{\mathrm{Orb}}_G)^{\mathrm{op}} \to \mathcal{E}$ Define $N\colon \mathrm{PSh}(\widehat{\mathrm{Orb}}_G)^{\mathrm{op}} \to \mathcal{E}$ as the limit-extension of the restriction $M'|_{\widehat{\mathrm{Orb}}_G^{\mathrm{op}}}\colon \widehat{\mathrm{Orb}}_G^{\mathrm{op}} \to \mathcal{E}$. Because QFin_G is a full subcategory of $\mathrm{PSh}(\widehat{\mathrm{Orb}}_G)$, the restriction of N to $\mathrm{QFin}_G^{\mathrm{op}}$ is precisely the right Kan extension of $M'|_{\widehat{\mathrm{Orb}}_G^{\mathrm{op}}}$ along the inclusion, so that there is a canonical map $M' \to N|_{\mathrm{QFin}_G^{\mathrm{op}}}$ extending the identity on $\widehat{\mathrm{Orb}}_G$. As both sides are very additive (for N by the first paragraph), we see that this an equivalence, finishing the proof. $\square$

Corollary 9.19. *For a presentable category $\mathcal{E}$ there is a canonical equivalence*

$$\underline{\mathrm{CMon}}_{\mathcal{B}}^{\mathrm{QFin}_G}(\underline{\mathrm{Shv}}(\mathcal{B};\mathcal{E})) \simeq \underline{\mathrm{Mack}}_G^{\mathrm{pro}}(\mathcal{E}),$$

where $\mathcal{B} := \mathrm{PSh}(\widehat{\mathrm{Orb}}_G)$.

Proof. Both sides are canonically equivalent to $\underline{\mathrm{Mack}}_{\mathcal{B}}^{\mathrm{QFin}_G}(\mathcal{E})$: for the left-hand side this is by Theorem 8.2 while for the right-hand side this is a combination of the previous lemma with Corollary 9.4. ☐

In the case $\mathcal{E} = \mathrm{Sp}$, the category $\mathrm{Mack}_G^{\mathrm{pro}}(\mathrm{Sp})$ is precisely the category $\mathrm{Sp}_G^{\mathrm{qfin}}$ of *quasi-finitely genuine G-spectra* of Krause–McCandless–Nikolaus [KMN23, Definition 4.5]. Corollary 8.4 therefore specializes to:

Theorem 9.20. *The category $\mathrm{Sp}_G^{\mathrm{qfin}}$ is the underlying category of the free presentable very G-semiadditive stable G-procategory.* ☐

On the other hand, for $\mathcal{E} = \mathrm{Ab}$ the category $\mathrm{Mack}_G^{\mathrm{pro}}(\mathrm{Ab})$ is precisely the category $\widehat{\mathcal{M}}(G, \mathbb{Z})$ of *$\mathbb{Z}$-valued Mackey profunctors* introduced by Kaledin [Kal22, Definition 3.2]. Combining Remark 8.5 with Example 7.34 we therefore similarly get:

Theorem 9.21. *The category $\widehat{\mathcal{M}}(G, \mathbb{Z})$ of G-Mackey profunctors in abelian groups is the underlying category of the free presentable 1-truncated very G-additive G-procategory.* ☐

Appendix A. A criterion for adjoints

In this short appendix we will recall a criterion from [MW21] for the existence of adjoints of parametrized functors and specialize it to a statement about parametrized colimits. We begin with the following characterization:

Proposition A.1 (See [MW21, Proposition 3.2.9])**.** *A $\mathcal{B}$-functor $G\colon \mathcal{C} \to \mathcal{D}$ admits a left adjoint if and only if the following conditions are satisfied:*

(1) For each $A \in \mathcal{B}$, the functor $G_A\colon \mathcal{C}(A) \to \mathcal{D}(A)$ admits a left adjoint F_A.
(2) For each $f\colon A \to B$ the Beck–Chevalley transformation $F_A f^* \to f^* F_B$ is an equivalence.

In this case, the left adjoint F is given at any object $A \in \mathcal{B}$ by the pointwise left adjoint F_A, and for any morphism $f\colon A \to B$ by the Beck–Chevalley square. ☐

Remark A.2. If the restriction functor f^* has a right adjoint f_*, the second condition is equivalent to demanding that the Beck–Chevalley map $G_B f_* \to f_* G_A$ be invertible. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{B}$-complete, then G has a left adjoint if and only if it preserves $\mathcal{B}$-limits and each G_A has a left adjoint.

The following proposition allows us to significantly reduce the amount of conditions we have to check:

Proposition A.3. *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{B}$-functor. Assume there exists a covering sieve $\Sigma \subseteq \mathcal{B}$ of the terminal object $1 \in \mathcal{B}$ such that for every $A \in \Sigma$ the functor G_A admits a left adjoint F_A and such that for every $f\colon A \to B$ in Σ the Beck–Chevalley map $F_A f^* \to f^* F_B$ is invertible. Then G admits a left adjoint.*

Proof. As Σ is a sieve, the assumptions imply via the previous proposition that for every $A \in \Sigma$ the $\mathcal{B}_{/A}$-functor $\pi_A^* G\colon \pi_A^*\mathcal{C} \to \pi_A^*\mathcal{D}$ is a right adjoint. As the objects of Σ cover $1 \in \mathcal{B}$, [MW21, Remark 3.3.6] then implies that also G itself is a right adjoint as claimed. ☐

Corollary A.4. *Let $\mathcal{Q} \subseteq \mathcal{B}$ local and let $\mathcal{C}$ be any $\mathcal{B}$-category. Assume that for every $q\colon A \to B$ there exists a covering sieve $\Sigma \subseteq \mathcal{B}_{/B}$ such that for every $(f\colon B' \to B) \in \Sigma$ restriction functor $q'^*\colon \mathcal{C}(B') \to \mathcal{C}(A \times_B B')$ along $q' := q^*(f)$ admits a left adjoint $q'_!$, and such that these left adjoint satisfy base change along maps in Σ. Then $\mathcal{C}$ is $\mathcal{Q}$-cocomplete.*

Proof. We have to show that for each q the $\mathcal{B}_{/B}$-functor $q^*\colon \pi_B^*\mathcal{C} \to \underline{\mathrm{Fun}}(\underline{A}, \pi_B^*\mathcal{C})$ admits a left adjoint. This is however simply an instance of the previous proposition (with $\mathcal{B}_{/B}$ in place of $\mathcal{B}$). $\square$

We also note the following result complementing this corollary:

Lemma A.5. *Let $\mathcal{Q} \subseteq \mathcal{B}$ be local and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{Q}$-cocomplete $\mathcal{B}$-categories. Assume that for every $q\colon A \to B$ in $\mathcal{Q}$ there exists a cover $(f_i\colon B_i \to B)_{i\in I}$ (not necessarily a sieve) such that for every $i \in I$ the Beck–Chevalley map $q'_! F_{A\times_B B'_i} \to F_{B'_i} q'_!$ is an equivalence, where $q' = f_i^*(q)$ denotes the pullback of q along f_i. Then F is $\mathcal{Q}$-cocontinuous.*

Proof. Fix $q\colon A \to B$ together with such a covering; we have to show that the Beck–Chevalley map $\mathrm{BC}_!\colon q_!F_A \to F_B q_!$ is an equivalence. As the f_i form a cover, it will be enough to show that $f_i^*\mathrm{BC}_!$ is an equivalence for every $i \in I$, i.e. that the pasting

$$\begin{array}{ccccc}
\mathcal{C}(A) & \xrightarrow{q_!} & \mathcal{C}(B) & \xrightarrow{f_i^*} & \mathcal{C}(B_i) \\
{\scriptstyle F}\downarrow & \overset{\mathrm{BC}_!}{\Nearrow} & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
\mathcal{D}(A) & \xrightarrow[q_!]{} & \mathcal{D}(B) & \xrightarrow[f_i^*]{} & \mathcal{D}(B_i)
\end{array}$$

is invertible. However, pasting with the equivalences $f_i^* q_! \to q'_! f_i^*$ coming from $\mathcal{Q}$-cocompleteness and appealing to the compatibility of mates with pastings this is equivalent to saying that the pasting

$$\begin{array}{ccccc}
\mathcal{C}(A) & \xrightarrow{(A\times_B f_i)^*} & \mathcal{C}(A \times_B B_i) & \xrightarrow{q'_!} & \mathcal{C}(B_i) \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} & \overset{\mathrm{BC}_!}{\Nearrow} & \downarrow{\scriptstyle F} \\
\mathcal{D}(A) & \xrightarrow[(A\times_B f_i)^*]{} & \mathcal{D}(A \times_B B_i) & \xrightarrow[q'_!]{} & \mathcal{D}(B_i)
\end{array}$$

is invertible, which holds by assumption on f_i. $\square$

References

[Bar17] Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Adv. Math. **304** (2017), 646–727.

[BDG+16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah, *Parametrized higher category theory and higher algebra: Exposé I – Elements of parametrized higher category theory*, arXiv:1608.03657 (2016).

[BH21] Tom Bachmann and Marc Hoyois, *Norms in motivic homotopy theory*, Astérisque, vol. 425, Paris: Société Mathématique de France (SMF), 2021.

[Bou10] Serge Bouc, *Biset Functors for Finite Groups*, Lect. Notes Math., vol. 1990, Berlin: Springer, 2010.

[CH22] Hongyi Chu and Rune Haugseng, *Free algebras through Day convolution*, Algebr. Geom. Topol. **22** (2022), no. 7, 3401–3458.

[CHLL24] Bastiaan Cnossen, Rune Haugseng, Tobias Lenz, and Sil Linskens, *Homotopical commutative rings and bispans*, arXiv:2403.06911 (2024).

[CLL23a] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens, *Parametrized stability and the universal property of global spectra*, arXiv:2301.08240 (2023).

[CLL23b] ———, *Partial parametrized presentability and the universal property of equivariant spectra*, arXiv:2307.11001 (2023).

[CLL23c] ———, *The Adams isomorphism revisited*, arXiv:2311.04884 (2023).

[CMNN20] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel, *Descent and vanishing in chromatic algebraic K-theory via group actions*, to appear in *Ann. Sci. Éc. Norm. Supér. (4)*, arXiv:2011.08233 (2020).

[Cra09] James Cranch, *Algebraic theories and $(\infty,1)$-categories*, Ph.D. Thesis, 2009. arXiv:1011.3243.

[CSY21] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski, *Ambidexterity and height*, Adv. Math. **385** (2021), 90.

[CSY22] ———, *Ambidexterity in chromatic homotopy theory*, Invent. Math. **228** (2022), no. 3, 1145–1254.

[CY23] Shachar Carmeli and Allen Yuan, *Higher semiadditive Grothendieck-Witt theory and the $K(1)$-local sphere*, Commun. Am. Math. Soc. **3** (2023), 65–111.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153.

[Har20] Yonatan Harpaz, *Ambidexterity and the universality of finite spans*, Proceedings of the London Mathematical Society **121** (2020), no. 5, 1121–1170.

[Hau17] Markus Hausmann, *G-symmetric spectra, semistability and the multiplicative norm*, J. Pure Appl. Algebra **221** (2017), no. 10, 2582–2632.

[HHLN23] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten, *Two-variable fibrations, factorisation systems and ∞-categories of spans*, Forum Math. Sigma **11** (2023), 70. Id/No e111.

[HL13] Michael Hopkins and Jacob Lurie, *Ambidexterity in $K(n)$-local stable homotopy theory*, `https://people.math.harvard.edu/~lurie/papers/Ambidexterity.pdf` (2013).

[Kal22] Dmitry Kaledin, *Mackey profunctors*, Mem. Amer. Math. Soc., vol. 1385, Providence, RI: American Mathematical Society (AMS), 2022.

[KMN23] Achim Krause, Jonas McCandless, and Thomas Nikolaus, *Polygonic spectra and TR with coefficients*, arXiv:2302.07686 (2023).

[Len20] Tobias Lenz, *G-Global Homotopy Theory and Algebraic K-Theory*, to appear in *Mem. Amer. Math. Soc.*, arXiv:2012.12676 (2020).

[Len22] ———, *Global algebraic K-theory is Swan K-theory*, arXiv:2202.07272 (2022).

[Lur09] Jacob Lurie, *Higher topos theory*, Ann. Math. Stud., vol. 170, Princeton University Press, Princeton, NJ, 2009. updated version available at `https://people.math.harvard.edu/~lurie/papers/highertopoi.pdf`.

[Lur17] ———, *Higher algebra*, `https://www.math.ias.edu/~lurie/papers/HA.pdf` (2017).

[Lur18] ———, *Spectral Algebraic Geometry*, under construction (version dated February 2018), `www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf` (2018).

[Lur19] ———, *Elliptic Cohomology III: Tempered cohomology*, `https://www.math.ias.edu/~lurie/papers/Elliptic-III.pdf` (2019).

[Mar21] Louis Martini, *Yoneda's lemma for internal higher categories*, arXiv:2103.17141 (2021).

[Mar24] Gregoire Marc, *A higher Mackey functor description of algebras over an N_∞-operad*, arXiv:2402.12447 (2024).

[MW21] Louis Martini and Sebastian Wolf, *Colimits and cocompletions in internal higher category theory*, arXiv:2111.14495 (2021).

[MW22] ———, *Presentable categories internal to an ∞-topos*, arXiv:2209.05103 (2022).

[Nar16] Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category*, arXiv:1608.07704 (2016).

[Pü24] Phil Pützstück, *Parametrized Higher Algebra and Global Picard Spectra*, Master's Thesis, 2024. Rheinische Friedrich-Wilhelms-Universität Bonn.

[QS21] J.D. Quigley and Jay Shah, *On the parametrized Tate construction*, arXiv:2110.07707 (2021).

[Sch18] Stefan Schwede, *Global homotopy theory*, New Math. Monogr., vol. 34, Cambridge University Press, 2018.

[Sch22] ———, *Global algebraic K-theory*, J. Topol. **15** (2022), no. 3, 1325–1454.

[Sch23] Peter Scholze, *Six-functor formalisms*, https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf (2023).

[Sha22] Jay Shah, *Parametrized higher category theory II*, arXiv:2109.11954 (2022).

[Shi89] Kazuhisa Shimakawa, *Infinite loop G-spaces associated to monoidal G-graded categories*, Publ. Res. Inst. Math. Sci. **25** (1989), no. 2, 239–262.

[Yua24] Allen Yuan, *The sphere of semiadditive height 1*, Int. Math. Res. Not. **2024** (2024), no. 1, 675–697.

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Part V

Algebraic patterns

HOMOTOPY-COHERENT ALGEBRA VIA SEGAL CONDITIONS

HONGYI CHU AND RUNE HAUGSENG

Abstract. Many homotopy-coherent algebraic structures can be described by Segal-type limit conditions determined by an "algebraic pattern", by which we mean an ∞-category equipped with a factorization system and a collection of "elementary" objects. Examples of structures that occur as such "Segal $\mathcal{O}$-spaces" for an algebraic pattern $\mathcal{O}$ include ∞-categories, (∞, n)-categories, ∞-operads (including symmetric, non-symmetric, cyclic, and modular ones), ∞-properads, and algebras for a (symmetric) ∞-operad in spaces.

In the first part of this paper we set up a general framework for algebraic patterns and their associated Segal objects, including conditions under which the latter are preserved by left and right Kan extensions. In particular, we obtain necessary and sufficent conditions on a pattern $\mathcal{O}$ for free Segal $\mathcal{O}$-spaces to be described by an explicit colimit formula, in which case we say that $\mathcal{O}$ is "extendable".

In the second part of the paper we explore the relationship between extendable algebraic patterns and polynomial monads, by which we mean cartesian monads on presheaf ∞-categories that are accessible and preserve weakly contractible limits. We first show that the free Segal $\mathcal{O}$-space monad for an extendable pattern $\mathcal{O}$ is always polynomial. Next, we prove an ∞-categorical version of Weber's Nerve Theorem for polynomial monads, and use this to define a canonical extendable pattern from any polynomial monad, whose Segal spaces are equivalent to the algebras of the monad. These constructions yield functors between polynomial monads and extendable algebraic patterns, and we show that these exhibit full subcategories of "saturated" algebraic patterns and "complete" polynomial monads as localizations, and moreover restrict to an equivalence between the ∞-categories of saturated patterns and complete polynomial monads.

Contents

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1. Introduction

Homotopy-coherent algebraic structures, where identities between operations are replaced by an infinite hierarchy of compatible coherence equivalences, have played an important role in algebraic topology since the 1960s[1], when they were first introduced in the special case of A_∞-spaces by Stasheff [Sta63], and have since found a variety of applications in many fields of mathematics. From a modern perspective, homotopy-coherent algebraic structures can be considered as the natural algebraic structures in the setting of ∞-categories (which are themselves the homotopy-coherent analogues of categories).

It turns out that many interesting homotopy-coherent algebraic structures can be described by "Segal conditions", i.e. they can be described as functors satisfying a specific type of limit condition. The canonical (and original) example is Segal's [Seg74] description of homotopy-coherently commutative monoids in spaces (or E_∞*-spaces*) as "special Γ-spaces". In ∞-categorical language, these are functors $F\colon \mathbb{F}_* \to \mathcal{S}$, where $\mathbb{F}_*$ is a skeleton of the category of pointed finite sets, with objects $\langle n\rangle := (\{0,1,\ldots,n\},0)$, and $\mathcal{S}$ is the ∞-category of spaces (or ∞-groupoids), which are required to satisfy the following condition:

> For all n, the map
> $$F(\langle n\rangle) \to \prod_{i=1}^{n} F(\langle 1\rangle),$$
> induced by the morphisms $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ given by
> $$\rho_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i, \end{cases}$$
> is an equivalence.

Other key examples of structures described by Segal conditions include:

- associative (or A_∞- or E_1-)monoids, using the simplex category $\mathbf{\Delta}^{\mathrm{op}}$ (in unpublished work of Segal),
- ∞-categories, again using $\mathbf{\Delta}^{\mathrm{op}}$, in the form of Rezk's Segal spaces [Rez01],
- (∞,n)-categories, using Joyal's categories $\mathbf{\Theta}_n^{\mathrm{op}}$, also in work of Rezk [Rez10],
- ∞-operads, using the dendroidal category $\mathbf{\Omega}^{\mathrm{op}}$ of Moerdijk–Weiss [MW07], in work of Cisinski and Moerdijk [CM13],
- algebras for an ∞-operad $\mathcal{O}$ (in the sense of [Lur17]) in $\mathcal{S}$, using the "category of operators" $\mathcal{O}$ itself.

Given these and other examples (many of which we will discuss below in §3), we might wonder why so many different algebraic structures can be described by Segal conditions. Our main results in this paper provide an explanation of this situation, by answering the following question:

Question 1.1. *Which homotopy-coherent algebraic structures can be described (in a reasonable way) by Segal conditions, and how canonical is this description?*

Before we describe our answer, we need to formulate a more precise version of this question, by defining the terms that appear. First of all, we will consider algebraic structures on (families of) spaces, which we take to mean algebras for monads on functor ∞-categories $\mathrm{Fun}(\mathcal{I},\mathcal{S})$ (where $\mathcal{I}$ is any small ∞-category). Next, let us specify what precisely we mean by "Segal conditions". Returning to the example of special Γ-spaces, the category $\mathbb{F}_*$ has the following features that we wish to abstract:

- A morphism $\phi\colon \langle n\rangle \to \langle m\rangle$ is called *inert* if $|\phi^{-1}(j)| = 1$ for $j \neq 0$, and *active* if $\phi^{-1}(0) = \{0\}$. The inert and active morphisms form a factorization system on $\mathbb{F}_*$: every morphism factors as an inert morphism followed by an active morphism, and this decomposition is unique up to isomorphism.

[1] More general frameworks for homotopy-coherent algebra, such as operads, arose out of work on infinite loop spaces by Boardman–Vogt [BV73] and May [May72] in the early 1970s.

- The morphisms ρ_i are precisely the inert morphisms $\langle n\rangle \to \langle 1\rangle$.
- If $\mathbb{F}_*^{\mathrm{int}}$ denotes the subcategory of $\mathbb{F}_*$ with only inert morphisms, then the special Γ-spaces are precisely the functors $F\colon \mathbb{F}_* \to \mathcal{S}$ such that the restriction $F|_{\mathbb{F}_*^{\mathrm{int}}}$ is a right Kan extension of $F|_{\{\langle 1\rangle\}}$.

These features recur in our other examples, which suggests that the input data for a class of "Segal conditions" should consist of an ∞-category $\mathcal{O}$ equipped with a factorization system (whereby every morphism factors as an "active" morphism followed by an "inert" morphism) and a class of "elementary" objects (or generators). From this data, which we will refer to as an *algebraic pattern*[2], we obtain the relevant Segal-type limit condition on a functor $F\colon \mathcal{O} \to \mathcal{S}$ by imposing the requirement that for every $O \in \mathcal{O}$ the object $F(O)$ is the limit over all inert morphisms to elementary objects,

$$F(O) \xrightarrow{\sim} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E);$$

we say that such a functor F is a *Segal $\mathcal{O}$-space*.[3] If $\mathcal{O}$ is any algebraic pattern, and $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{S})$ on the Segal $\mathcal{O}$-spaces, then the restriction functor

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$$

has a left adjoint. This adjunction is always monadic, and we write $T_{\mathcal{O}}$ for the corresponding monad on $\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$. The monad $T_{\mathcal{O}}$ is then "described by" the algebraic pattern $\mathcal{O}$. In general, however, it is not possible to describe this monad explicitly, because the left adjoint involves an abstract localization. We only want to consider a pattern to be "reasonable" if this localization is unnecessary, in which case $T_{\mathcal{O}}$ is given by a concrete formula, namely as

$$T_{\mathcal{O}}F(E) \simeq \operatorname*{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(E)} \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E'),$$

where $\mathrm{Act}_{\mathcal{O}}(E)$ is the space of active morphisms to E in $\mathcal{O}$. We call such patterns $\mathcal{O}$ *extendable*, and give explicit necessary and sufficient conditions for a pattern to be extendable in Proposition 8.8.

We can now state the precise version of the previous question that we will address:

Question 1.2. *Which monads on presheaf ∞-categories can be described as the free Segal $\mathcal{O}$-space monad for an extendable algebraic pattern $\mathcal{O}$, and how canonical is this description?*

We will characterize these monads as a certain class of *polynomial*[4] monads, by which we mean the monads on presheaf ∞-categories that are *cartesian*[5] and whose underlying endofunctors are accessible and preserve weakly contractible limits. Our first main result provides functors in both directions between ∞-categories of extendable patterns and of polynomial monads:

Theorem 1.3.

(i) If $\mathcal{O}$ is an extendable algebraic pattern then the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$ is polynomial. This determines a functor $\mathfrak{M}$ from extendable patterns to polynomial monads.

(ii) If T is a polynomial monad on $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ then there exists a canonical extendable algebraic pattern $\mathcal{W}(T)$ such that $\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ is equivalent to the ∞-category of T-algebras. This determines a functor $\mathfrak{P}$ from polynomial monads to extendable patterns.

We prove part (i) in §10 and part (ii) in §13. Part (ii) depends on an ∞-categorical version of Weber's *nerve theorem* [Web07], which we prove in §11 and use to construct a factorization system on the Kleisli ∞-category of a polynomial monad in §12.

Our second main result characterizes the images of these functors:

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[2] This terminology is inspired by Lurie's *categorical patterns* [Lur17, §B], the key examples of which all arise from algebraic patterns in our sense, and should not be confused with the notion of "pattern" considered by Getzler [Get09].

[3] Here we write $\mathcal{O}^{\mathrm{int}}$ for the subcategory of $\mathcal{O}$ containing only the inert morphisms, $\mathcal{O}^{\mathrm{el}}$ for the full subcategory of $\mathcal{O}^{\mathrm{int}}$ spanned by the elementary objects, and define $\mathcal{O}^{\mathrm{el}}_{O/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{O/}$.

[4] The analogous monads on ordinary categories are sometimes called *strongly cartesian* monads.

[5] The cartesian monads are those whose multiplication and unit transformations are *cartesian* natural transformations, which in turn means that their naturality squares are all cartesian, i.e. are pullback squares.

Theorem 1.4.

(i) Restricting to slim[6] *extendable patterns, there is a natural transformation* $\sigma\colon \mathrm{id} \to \mathfrak{PM}$*, and the component* $\sigma_{\mathcal{O}}$ *is an equivalence if and only if the pattern* $\mathcal{O}$ *is* saturated*, meaning that it is a slim extendable pattern such that the functors*

$$\mathrm{Map}_{\mathcal{O}}(O, \text{–})\colon \mathcal{O} \to \mathcal{S}$$

are Segal $\mathcal{O}$*-spaces for* $O \in \mathcal{O}$*. The pattern* $\mathcal{W}(T)$ *for a polynomial monad* T *is always saturated, and the transformation* σ *exhibits the full subcategory of saturated patterns as a localization of the* ∞*-category of slim extendable patterns.*

(ii) There is a natural transformation $\tau\colon \mathrm{id} \to \mathfrak{MP}$*, and* τ_T *is an equivalence for a polynomial monad* T *on* $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ *if and only if* T *is* complete*, meaning that the essentially surjective functor* $\mathcal{I} \to \mathcal{W}(T)^{\mathrm{el}}$ *is an equivalence. The monad* $T_{\mathcal{O}}$ *for an extendable pattern* $\mathcal{O}$ *is always complete, and the transformation* τ *exhibits the full subcategory of complete polynomial monads as a localization of the* ∞*-category of polynomial monads.*

(iii) The functors $\mathfrak{P}$ *and* $\mathfrak{M}$ *restrict to an equivalence between the* ∞*-categories of saturated patterns and complete polynomial monads.*

We will prove part (i) in §14 and parts (ii) and (iii) in §15.

The answer to our question above is thus that the monads of the form $T_{\mathcal{O}}$ for an extendable pattern $\mathcal{O}$ are precisely the *complete* polynomial monads, and there is a *unique* extendable pattern describing this monad that is *saturated*, namely the canonical pattern $\mathcal{W}(T_{\mathcal{O}})$. For example, returning to our initial example of commutative monoids described by an algebraic pattern structure on $\mathbb{F}_*$, this pattern is extendable, with free commutative monoids describe by the expected formula

$$X \mapsto \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n},$$

but it is *not* saturated. The corresponding saturated pattern is instead the ∞-category of free commutative monoids on finite sets (i.e. the *Lawvere theory* for commutative monoids), which by work of Cranch [Cra11] can be identified with the $(2,1)$-category $\mathrm{Span}(\mathbb{F})$ of finite sets with spans (or correspondences) as morphisms; see 14.22 for more details.

1.1. **Overview.** In the first part of the paper we set up a general categorical framework for algebraic patterns and Segal objects. In §2 we introduce these objects more formally and prove some of their basic properties, before we look at examples of algebraic patterns and their Segal objects in §3. We then introduce morphisms of algebraic patterns in §4 and construct an ∞-category of algebraic patterns in §5, where we also prove that this has limits and filtered colimits. Next, we provide conditions under which Segal objects are preserved by right and left Kan extensions in §6 and §7, respectively.

In §8 we apply our work on left Kan extensions to analyze free Segal objects; in particular, we obtain necessary and sufficient conditions for a pattern $\mathcal{O}$ to be extendable, meaning that free Segal $\mathcal{O}$-spaces are described by a colimit formula. In §9 we study (weak) Segal fibrations, which generalize Lurie's definitions of symmetric monoidal ∞-categories and symmetric ∞-operads. We show that any weak Segal fibration over an extendable base is again extendable, and moreover left Kan extension along any morphism of weak Segal fibrations preserves Segal objects; this recovers, for example, the formula of [Lur17] for operadic left Kan extensions of ∞-operad algebras in cartesian monoidal ∞-categories.

In §10 we introduce polynomial monads, and prove that the free Segal $\mathcal{O}$-space monad for any extendable pattern is polynomial. We then prove an ∞-categorical version of Weber's Nerve Theorem for presheaf ∞-categories in §11, and apply this to define a factorization system on the Kleisli ∞-category of a polynomial monad in §12. This gives a canonical algebraic pattern for every polynomial monad, which we study in §13. Next, we study the relationship between an extendable

[6]This is a mild technical hypothesis; it is satisfied in almost all examples, and the patterns $\mathcal{W}(T)$ are always slim. Moreover, any extendable pattern can be replaced by a full subcategory that is slim and determines the same monad.

pattern and the canonical pattern of its free Segal space monad; under a mild hypothesis there is a functor between these, and we show that this is an equivalence precisely when the pattern is saturated. Finally, in §15 we study complete polynomial monads, and prove that there is an equivalence between these and saturated patterns.

1.2. **Related Work.** There is an extensive literature on using (finite) limit conditions to describe algebraic structures in category theory, going back at least to Lawvere's thesis [Law68], where he introduced algebraic theories. Our work is in particular closely related to the "nerve theorem", one version of which *almost* says that a strongly cartesian monad on a presheaf category is described by Segal conditions; this version was first proved in unpublished work of Leinster (though his proof did not use the factorization system), and later extended by Weber [Web07] to a description of certain *weakly* cartesian monads.[7] We were particularly inspired by the simpler proof given by Berger, Melliès, and Weber [BMW12]. Their work has more recently been extended by Bourke and Garner [BG19], who study general classes of monads that can be described by some notion of "theories with arities", including in the enriched context.

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2. Algebraic Patterns and Segal Objects

In this section we introduce the basic structures we will study in this paper, namely algebraic patterns and their Segal objects.

Definition 2.1. An *algebraic pattern* $\mathfrak{O}$ is an ∞-category $\mathcal{O}$ equipped with:

- a factorization system $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$, the morphisms in which we refer to as the *inert* and *active* morphisms in $\mathfrak{O}$,
- a full subcategory $\mathcal{O}^{\mathrm{el}} \subseteq \mathcal{O}^{\mathrm{int}}$ whose objects we call the *elementary* objects of $\mathfrak{O}$.

Unless stated otherwise, we will assume by default that algebraic patterns are essentially small.

Remark 2.2. Here a *factorization system* on an ∞-category $\mathcal{C}$ means a pair of subcategories $(\mathcal{C}^L, \mathcal{C}^R)$ such that both contain all objects of $\mathcal{C}$, and for every morphism $f\colon X \to X'$ in $\mathcal{C}$, the space of factorizations

$$\left\{ \begin{array}{ccc} & Y & \\ {\scriptstyle l}\nearrow & & \searrow{\scriptstyle r} \\ X & \xrightarrow{\quad f \quad} & X' \end{array} \quad : l \in \mathcal{C}^L, r \in \mathcal{C}^R \right\}$$

is contractible.

Remark 2.3. We will often abuse notation and conflate an algebraic pattern with its underlying ∞-category $\mathcal{O}$, i.e. we will simply say that $\mathcal{O}$ is an algebraic pattern.

Notation 2.4. If $\mathcal{O}$ is an algebraic pattern, we will often indicate an inert map between objects O, O' of $\mathcal{O}$ as $O \rightarrowtail O'$ and an active map as $O \rightsquigarrow O'$. These symbols are not meant to suggest any intuition about the nature of inert and active maps.

[7] Weakly cartesian monads are of interest in the case of ordinary categories, as many "algebraic" monads that involve symmetries, such as the free commutative monoid monad, are not cartesian. This issue disappears if we replace sets by groupoids, and so weakly cartesian monads are not relevant in our ∞-categorical setting.

Notation 2.5. If $\mathcal{O}$ is an algebraic pattern and X is an object of $\mathcal{O}$, then we write $\mathcal{O}^{\mathrm{el}}_{X/}$ for the fibre product of ∞-categories $\mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{X/}$. Thus the objects of $\mathcal{O}^{\mathrm{el}}_{X/}$ are inert morphisms $X \rightarrowtail E$ where E is elementary, and the morphisms are commutative triangles

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ E & \rightarrowtail & E' \end{array}$$

where all morphisms are inert, and E and E' are elementary.

Definition 2.6. Let $\mathcal{O}$ be an algebraic pattern. We say an ∞-category is $\mathcal{O}$-*complete* if it has limits of shape $\mathcal{O}^{\mathrm{el}}_{X/}$ for all $X \in \mathcal{O}$.

Definition 2.7. Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-complete ∞-category. A *Segal* $\mathcal{O}$-*object* in $\mathcal{C}$ is a functor $F\colon \mathcal{O} \to \mathcal{C}$ such that for every $X \in \mathcal{O}$ the canonical map

$$F(X) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E)$$

is an equivalence. We write $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the Segal $\mathcal{O}$-objects.

Notation 2.8. We will often refer to Segal $\mathcal{O}$-objects in the ∞-category $\mathcal{S}$ of spaces as *Segal* $\mathcal{O}$-*spaces*, and to Segal $\mathcal{O}$-objects in the ∞-category Cat_∞ of ∞-categories as *Segal* $\mathcal{O}$-∞-*categories.*

Lemma 2.9. *Let $\mathcal{C}$ be an $\mathcal{O}$-complete ∞-category. Then $F\colon \mathcal{O} \to \mathcal{C}$ is a Segal $\mathcal{O}$-object if and only if the restriction $F|_{\mathcal{O}^{\mathrm{int}}}$ is a right Kan extension of $F|_{\mathcal{O}^{\mathrm{el}}}$.*

Proof. Since $\mathcal{C}$ is $\mathcal{O}$-complete, $F|_{\mathcal{O}^{\mathrm{int}}}$ is a right Kan extension of $F|_{\mathcal{O}^{\mathrm{el}}}$ if and only if for all $X \in \mathcal{O}^{\mathrm{int}}$, the natural map

$$F(X) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E)$$

is an equivalence. □

Definition 2.10. Let $\mathcal{O}$ be an algebraic pattern. For $O \in \mathcal{O}$ we write $y(O)_{\mathrm{Seg}}$ for the colimit $\mathrm{colim}_{E \in (\mathcal{O}^{\mathrm{el}}_{O/})^{\mathrm{op}}}\, y(E)$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{S})$, where y denotes the Yoneda embedding $\mathcal{O}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$. If $\mathcal{C}$ is a cocomplete ∞-category, and thus is tensored over $\mathcal{S}$, then we can consider $C \otimes y(O)$ and $C \otimes y(O)_{\mathrm{Seg}}$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ for $C \in \mathcal{C}$.

Lemma 2.11. *Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ a cocomplete ∞-category.*

(i) $F \in \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ *is a Segal $\mathcal{O}$-object if and only if F is local with respect to the canonical maps $C \otimes y(O)_{\mathrm{Seg}} \to C \otimes y(O)$ for all $O \in \mathcal{O}$.*
(ii) If $\mathcal{C}$ is κ-presentable, then F is a Segal $\mathcal{O}$-object if and only if F is local with respect to these maps where C is κ-compact.
(iii) If $\mathcal{C}$ is presentable, then the full subcategory $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ *is an accessible localization of* $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$.
(iv) If $\mathcal{C}$ is presentable, then so is the ∞-category $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$.

Proof. The object F is local with respect to $C \otimes y(O)_{\mathrm{Seg}} \to C \otimes y(O)$ precisely when the morphism of spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C \otimes y(O), F) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C \otimes y(O)_{\mathrm{Seg}}, F)$$

is an equivalence. Here we have equivalences

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C \otimes y(O), F) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y(O), \mathrm{Map}_{\mathcal{C}}(C, F)) \simeq \mathrm{Map}_{\mathcal{C}}(C, F(O)),$$

using the Yoneda Lemma, and similarly

$$\begin{aligned}\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C\otimes y(O)_{\mathrm{Seg}},F) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y(O)_{\mathrm{Seg}},\mathrm{Map}_{\mathcal{C}}(C,F))\\ &\simeq \lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y(E),\mathrm{Map}_{\mathcal{C}}(C,F))\\ &\simeq \mathrm{Map}_{\mathcal{C}}(C,\lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}} F(E)).\end{aligned}$$

Thus F is local with respect to this morphism for all C and O if and only if $F(O)\xrightarrow{\sim}\lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}}F(E)$ for all O, i.e. F is a Segal object. This proves (i). If $\mathcal{C}$ is κ-presentable, then to conclude that the Segal map $F(O)\to\lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}}F(E)$ is an equivalence it suffices to consider C in $\mathcal{C}^{\kappa}$, which proves (ii).

It follows that if $\mathcal{C}$ is presentable, then $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ is the full subcategory of objects in $\mathrm{Fun}(\mathcal{O},\mathcal{C})$ that are local with respect to a *set* of morphisms. Parts (iii) and (iv) then follow from [Lur09, Proposition 5.5.4.15]. □

3. Examples of Algebraic Patterns

In this section we will briefly describe some examples of algebraic patterns and their associated Segal objects.

Example 3.1. We write $\mathbb{F}_*^{\flat}$ for the algebraic pattern structure on $\mathbb{F}_*$ given by the inert–active factorization system we discussed above in the introduction, with $\mathbb{F}_*^{\flat,\mathrm{el}}$ containing the single object $\langle 1\rangle$. Then a Segal $\mathbb{F}_*^{\flat}$-space is precisely a commutative monoid, or equivalently a special Γ-space in the sense of [Seg74].

Example 3.2. We can also consider another pattern structure on $\mathbb{F}_*$: We define $\mathbb{F}_*^{\natural}$ by the same factorization system, but now $\mathbb{F}_*^{\natural,\mathrm{el}}$ contains the two objects $\langle 0\rangle$ and $\langle 1\rangle$, with the unique inert morphism $\langle 1\rangle\to\langle 0\rangle$. Segal $\mathbb{F}_*^{\natural}$-objects are functors $F\colon \mathbb{F}_*^{\natural}\to\mathcal{C}$ such that

$$F(\langle n\rangle)\simeq F(\langle 1\rangle)^{\times_{F(\langle 0\rangle)} n},$$

where the right-hand side denotes an iterated fibre product over $F(\langle 0\rangle)$; this is equivalently a commutative monoid in the slice $\mathcal{C}_{/F(\langle 0\rangle)}$.

Example 3.3. We write $\boldsymbol{\Delta}$ for the simplex category, i.e. the category of non-empty finite ordered sets $[n]:=\{0,\dots,n\}$ and order-preserving maps between them. A morphism $f\colon[n]\to[m]$ is *inert* if it is the inclusion of a sub-interval, i.e. $f(i)=f(0)+i$ for all i, and *active* if f preserves the end-points, i.e. $f(0)=0$ and $f(n)=m$. Every morphism in $\boldsymbol{\Delta}$ factors uniquely as an active morphism followed by an inert one, so this determines an inert–active factorization system on $\boldsymbol{\Delta}^{\mathrm{op}}$. Using this factorization system we can define two interesting algebraic pattern structures on $\boldsymbol{\Delta}^{\mathrm{op}}$:

- $\boldsymbol{\Delta}^{\mathrm{op},\natural}$ denotes the pattern where $\boldsymbol{\Delta}^{\mathrm{op},\natural,\mathrm{el}}$ contains the two objects $[0]$ and $[1]$, and the two inert morphisms $[1]\rightrightarrows[0]$,
- $\boldsymbol{\Delta}^{\mathrm{op},\flat}$ denotes the pattern where $\boldsymbol{\Delta}^{\mathrm{op},\flat,\mathrm{el}}:=\{[1]\}$.

A Segal $\boldsymbol{\Delta}^{\mathrm{op},\natural}$-object is a functor $F\colon\boldsymbol{\Delta}^{\mathrm{op}}\to\mathcal{C}$ such that

$$F([n])\xrightarrow{\sim}F([1])\times_{F([0])}\cdots\times_{F([0])}F([1]).$$

In particular, a Segal $\boldsymbol{\Delta}^{\mathrm{op},\natural}$-space is precisely a *Segal space* in the sense of Rezk [Rez01], which describes the algebraic structure of an ∞-category. On the other hand, a Segal $\boldsymbol{\Delta}^{\mathrm{op},\flat}$-object F satisfies

$$F([n])\simeq F([1])^{\times n},$$

and describes an associative monoid.

Example 3.4. For any integer n the product $\mathbf{\Delta}^{n,\mathrm{op}} := (\mathbf{\Delta}^{\mathrm{op}})^{\times n}$ has a coordinate-wise factorization system (i.e. a morphism is active or inert precisely when all of its components are). Using this we can define two algebraic pattern structures $\mathbf{\Delta}^{n,\mathrm{op},\natural}$ and $\mathbf{\Delta}^{n,\mathrm{op},\flat}$, where

$$\mathbf{\Delta}^{n,\mathrm{op},\natural,\mathrm{el}} := (\mathbf{\Delta}^{\mathrm{op},\natural,\mathrm{el}})^{n}$$

consists of all objects $([i_1],\ldots,[i_n])$ with $i_s = 0$ or 1 for all s, while

$$\mathbf{\Delta}^{n,\mathrm{op},\flat,\mathrm{el}} := \{([1],\ldots,[1])\}.$$

These are both special cases of products of algebraic patterns (Corollary 5.5). Segal $\mathbf{\Delta}^{n,\mathrm{op},\natural}$-spaces are *$n$-uple Segal spaces*, which describe internal ∞-categories in internal ∞-categories in ... in ∞-categories. A special class of these was first introduced by Barwick [Bar05] as a model for (∞,n)-categories. On the other hand, the Dunn–Lurie additivity theorem [Lur17, Theorem 5.1.2.2] implies that Segal $\mathbf{\Delta}^{n,\mathrm{op},\flat}$-objects are equivalent to $\mathbb{E}_n$-algebras, i.e. algebras for the little n-disc operad.

Example 3.5. Let $\mathbf{\Theta}_n$ be defined inductively by $\mathbf{\Theta}_0 := *$ and $\mathbf{\Theta}_n := \mathbf{\Delta} \wr \mathbf{\Theta}_{n-1}$, where for any category $\mathbf{C}$ the wreath product $\mathbf{\Delta} \wr \mathbf{C}$ has objects $[n](C_1,\ldots,C_n)$ with $C_i \in \mathbf{C}$, and morphisms $[n](C_1,\ldots,C_n) \to [m](C'_1,\ldots,C'_m)$ given by morphisms $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$ together with maps $\psi_{ij}\colon C_i \to C_j$ in $\mathbf{C}$ whenever $\phi(i-1) < j \leq \phi(i)$. (This category was first considered in unpublished work of Joyal; the "wreath product" definition is due to Berger [Ber07].) Then $\mathbf{\Theta}_n$ has an inductively defined factorization system (first defined in [Ber02, Lemma 1.11]): the morphism above is *inert* (or *active*) if ϕ is inert (active) and each ψ_{ij} is inert (active). We can again use this to define two algebraic patterns. To do so we need some notation: We inductively define objects $C_0,\ldots,C_n$ in $\mathbf{\Theta}_n$ by $C_0 := [0]()$ and $C_n := [1](C_{n-1})$, starting with C_0 being the unique object of $\mathbf{\Theta}_0$. Then

- $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ is defined by taking $\mathbf{\Theta}_n^{\mathrm{op},\natural,\mathrm{el}}$ to contain the objects $C_0,\ldots,C_n$; we can depict this category as
$$C_n \rightrightarrows C_{n-1} \rightrightarrows \cdots \rightrightarrows C_0.$$
- $\mathbf{\Theta}_n^{\mathrm{op},\flat}$ is defined by taking $\mathbf{\Theta}_n^{\mathrm{op},\flat,\mathrm{el}}$ to contain the single object C_n.

Segal $\mathbf{\Theta}_n^{\mathrm{op},\natural}$-spaces are then precisely Rezk's *$\mathbf{\Theta}_n$-spaces* [Rez10], which describe the algebraic structure of (∞,n)-categories. On the other hand, Segal $\mathbf{\Theta}_n^{\mathrm{op},\flat}$-objects are again equivalent to $\mathbb{E}_n$-algebras — this follows from [Bar18, Theorem 8.12] together with the Dunn–Lurie additivity theorem.

Example 3.6. All the examples considered so far are special cases of the following construction, due to Barwick: Suppose Φ is a *perfect operator category* in the sense of [Bar18], and let $\Lambda(\Phi)$ be its Leinster category, which is the Kleisli category of a certain monad on Φ. This has an active-inert factorization system by [Bar18, Lemma 7.3], where the active morphisms are the free morphisms on morphisms of Φ. Using this factorization system we can define two natural algebraic patterns:

- $\Lambda(\Phi)^{\flat}$ is defined by taking $\Lambda(\Phi)^{\flat,\mathrm{el}}$ to consist only of the terminal object $* \in \Phi$,
- $\Lambda(\Phi)^{\natural}$ is defined by taking $\Lambda(\Phi)^{\natural,\mathrm{el}}$ to contain all objects E such that there is an inert map $* \rightarrowtail E$ in $\Lambda(\Phi)$.

If $\mathbb{O}$ denotes the category of (possibly empty) ordered finite sets then $\Lambda(\mathbb{O}) \simeq \mathbf{\Delta}^{\mathrm{op}}$, while if $\mathbb{F}$ denotes the category of finite sets then $\Lambda(\mathbb{F}) \simeq \mathbb{F}_*$, and these pattern structures agree with those defined above. The same holds for $\mathbf{\Theta}_n^{\mathrm{op}}$, which can be described as the Leinster category of a wreath product $\mathbb{O}^{\wr n}$ of operator categories.

Example 3.7. Let $\mathbf{\Omega}$ be the *dendroidal category* of Moerdijk and Weiss [MW07, §3]; this can be defined as the category of free operads on trees. This has a natural active-inert factorization system, described for example in [Koc11] (where the inert maps are called "free" and the active ones "boundary-preserving"). Using this we can define an algebraic pattern $\mathbf{\Omega}^{\mathrm{op},\natural}$ where $\mathbf{\Omega}^{\mathrm{op},\natural,\mathrm{el}}$ consists of the *corollas* C_n (i.e. trees with one vertex) and the plain edge η. Segal $\mathbf{\Omega}^{\mathrm{op},\natural}$-spaces are the dendroidal Segal spaces introduced by Cisinski and Moerdijk [CM13], which describe the algebraic structure of ∞-operads. The Segal condition says that the value of a Segal object at a tree decomposes as a limit over the corollas and edges of the tree. (We can also consider a pattern $\mathbf{\Omega}^{\mathrm{op},\flat}$ where the elementary objects are just the corollas; then Segal $\mathbf{\Omega}^{\mathrm{op},\flat}$-spaces describe ∞-operads with a single object.)

Example 3.8. If Φ is an operator category, let $\mathbf{\Delta}_\Phi$ be the category defined in [Bar18, Definition 2.4]. This has pairs $([m], f\colon [m] \to \Phi)$ as objects, and morphisms $([m], f) \to ([n], g)$ are given by morphisms $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ together with certain natural transformations $\eta\colon f \to g \circ \phi$. We define a morphism $(\phi, \eta)\colon ([m], f) \to ([n], g)$ in $\mathbf{\Delta}_\Phi$ to be *inert* if ϕ is inert in $\mathbf{\Delta}$, and *active* if ϕ is active and $\eta_i\colon f(i) \to g(\phi(i))$ is an isomorphism for every $0 \leq i \leq m$. This gives an inert–active factorization system on $\mathbf{\Delta}_\Phi^{\mathrm{op}}$, and we define an algebraic pattern $\mathbf{\Delta}_\Phi^{\mathrm{op},\natural}$ by taking the elementary objects to be $([0], *)$ and $([1], I \to *)$ (where $*$ denotes the terminal object). Then Segal $\mathbf{\Delta}_\Phi^{\mathrm{op},\natural}$-spaces are precisely the Segal Φ-operads of [Bar18, §2], which describe Φ-∞-*operads*. (When Φ is $\mathbb{F}$ these agree with ∞-operads in the sense of Lurie by [Bar18, Theorem 10.16], and with dendroidal Segal spaces by [CHH18, Theorem 1.1].)

Example 3.9. Let $\mathbf{\Gamma}$ be the category of acyclic connected finite directed graphs defined by Hackney, Robertson, and Yau in [HRY15]. Then $\mathbf{\Gamma}^{\mathrm{op}}$ has an inert–active factorization system described in [Koc16, 2.4.14] (where the active maps are called "refinements" and the inert maps are called "convex open inclusions"). Using this we can define an algebraic pattern structure $\mathbf{\Gamma}^{\mathrm{op},\natural}$ by taking the elementary objects to be the elementary graphs with at most one vertex. Segal $\mathbf{\Gamma}^{\mathrm{op},\natural}$-spaces are equivalent to the model of ∞-*properads* as "graphical spaces" satisfying a Segal condition that is briefly discussed in [HR18]; this is presumably equivalent (after imposing a completeness condition) to the model of ∞-properads as certain presheaves of sets on $\mathbf{\Gamma}$ constructed in [HRY15].

Example 3.10. Let $\mathbf{\Xi}$ denote the category of unrooted trees defined in [HRY19]. Then $\mathbf{\Xi}^{\mathrm{op}}$ has an inert–active factorization system, described in [HRY19, §4], and using this we can give $\mathbf{\Xi}^{\mathrm{op}}$ an algebraic pattern structure $\mathbf{\Xi}^{\mathrm{op},\natural}$ where the elementary objects are the stars and the plain edge. Segal $\mathbf{\Xi}^{\mathrm{op},\natural}$-spaces are then precisely the model for cyclic ∞-operads considered by Hackney, Robertson, and Yau [HRY19].

Example 3.11. Let $\mathbf{U}$ denote the category of connected graphs defined in [HRY20]. Then $\mathbf{U}^{\mathrm{op}}$ has an inert–active factorization system, described in [HRY20, §2.1], and we can use this to equip $\mathbf{U}^{\mathrm{op}}$ with an algebraic pattern structure $\mathbf{U}^{\mathrm{op},\natural}$ where the elementary objects are the stars and the plain edge. We can also consider an algebraic pattern $\mathbf{U}^{\mathrm{op},\flat}$ where the elementary objects are just the stars; Segal $\mathbf{U}^{\mathrm{op},\flat}$-objects are then the *Segal modular operads* defined by Hackney, Robertson, and Yau [HRY20].

Remark 3.12. Below in §9 we will define (weak) Segal fibrations over an algebraic pattern, which give general classes of examples of algebraic patterns. As a special case, we will see that every ∞-operad $\mathcal{O}$ in the sense of Lurie [Lur17] has an algebraic pattern structure $\mathcal{O}^\flat$ such that a Segal $\mathcal{O}^\flat$-object in an ∞-category $\mathcal{C}$ with finite products is precisely an $\mathcal{O}$-monoid in $\mathcal{C}$.

4. Morphisms of Algebraic Patterns

In this section we define morphisms of algebraic patterns, and consider when they are compatible with Segal objects. We then discuss some examples of such morphisms.

Definition 4.1. Let $\mathcal{O}$ and $\mathcal{P}$ be algebraic patterns. A *morphism of algebraic patterns* from $\mathcal{O}$ to $\mathcal{P}$ is a functor $f\colon \mathcal{O} \to \mathcal{P}$ such that f preserves both active and inert maps, and takes elementary object in $\mathcal{O}$ to elementary objects in $\mathcal{P}$.

In general, morphisms of algebraic patterns do not necessarily interact well with Segal objects. We therefore isolate the class of morphisms that preserve Segal objects under restriction:

Definition 4.2. A morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ is called a *Segal morphism* if it satisfies the following condition:

(∗) For all $X \in \mathcal{O}$ the induced functor $\mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$ induces an equivalence

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} F \xrightarrow{\sim} \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F \circ f^{\mathrm{el}}$$

for every Segal $\mathcal{P}$-space $F\colon \mathcal{P} \to \mathcal{S}$.

Remark 4.3. The condition depends only on the restriction of F to $\mathcal{P}^{\mathrm{el}}$, so we could equivalently have considered functors $\mathcal{P}^{\mathrm{el}} \to \mathcal{S}$ that occur as restrictions of Segal $\mathcal{P}$-spaces.

Remark 4.4. In practice, a morphism f is a Segal morphism because the functor $\mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$ is coinitial, in which case we say that f is a *strong Segal morphism*. However, the more general definition allows for the following characterization:

Lemma 4.5. *The following are equivalent for a morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$:*

(1) f is a Segal morphism.
(2) The functor $f^*\colon \mathrm{Fun}(\mathcal{P}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$ restricts to a functor $\mathrm{Seg}_{\mathcal{P}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$.
(3) For every ∞-category $\mathcal{C}$, the functor $f^*\colon \mathrm{Fun}(\mathcal{P}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ restricts to a functor $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$.

Proof. It is immediate from the definition that (1) is equivalent to (2) and that (3) implies (2). It remains to check that (2) implies (3). Suppose $F\colon \mathcal{P} \to \mathcal{C}$ is a Segal $\mathcal{P}$-object; we need to show that f^*F is a Segal $\mathcal{O}$-object, i.e. that for all $X \in \mathcal{O}$ the natural map

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} F \to \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F \circ f^{\mathrm{el}}$$

is an equivalence in $\mathcal{C}$. Equivalently, we must show that for any $C \in \mathcal{C}$, the map of spaces

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} \mathrm{Map}(C, F) \to \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}(C, F) \circ f^{\mathrm{el}}$$

is an equivalence, which is true since $\mathrm{Map}(C, F)$ is a Segal $\mathcal{P}$-space. □

Remark 4.6. One might feel that the Segal property is sufficiently fundamental that it should be included as part of the notion of a morphism of algebraic patterns. However, more general morphisms also turn out to be occasionally useful. For example, the identity functor of $\mathbb{F}_*$ viewed as a functor $\mathbb{F}^\flat_* \to \mathbb{F}^\natural_*$ is a morphism of patterns, but is not a Segal morphism, and we will see later in §6 that it induces a functor from Segal $\mathbb{F}^\flat_*$-objects to Segal $\mathbb{F}^\natural_*$-object that can be viewed as a right Kan extension along $\mathrm{id}_{\mathbb{F}_*}$.

Proposition 4.7. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a Segal morphism of algebraic patterns, and $\mathcal{C}$ is a presentable ∞-category. Then there is an adjunction*

$$L_{\mathrm{Seg}} f_! \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) : f^*$$

where L_{Seg} is the localization functor left adjoint to the inclusion $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{O}, \mathcal{C})$, and $f_!$ is the functor of left Kan extension along f.

Proof. Since f^* restricts to a functor on Segal objects, for $F \in \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ and $G \in \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ we have a natural equivalence

$$\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})}(L_{\mathrm{Seg}} f_! F, G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(f_! F, G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{P},\mathcal{C})}(F, f^* G) \simeq \mathrm{Map}_{\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})}(F, f^* G),$$

which implies the claim. □

Remark 4.8. Below in §7 we will give conditions on a morphism f such that the left Kan extension functor $f_!$ preserves Segal objects, and so gives a left adjoint to f^* without localizing.

We now consider some examples of morphisms of patterns:

Example 4.9. There is a functor $|{-}|\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbb{F}_*$ which takes an object $[n]$ to $|[n]| := \langle n \rangle$ and a morphism $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$ to $|\alpha|\colon |[m]| \to |[n]|$ given by

$$|\alpha|(i) = \begin{cases} j & \text{if } \alpha(j-1) < j \leq \alpha(j) \\ 0 & \text{otherwise} \end{cases}$$

This functor gives a Segal morphism of algebraic patterns $\mathbf{\Delta}^{\mathrm{op},\natural} \to \mathbb{F}_*^{\natural}$ as well as $\mathbf{\Delta}^{\mathrm{op},\flat} \to \mathbb{F}_*^{\flat}$.

Example 4.10. There is a functor $\tau_n\colon \mathbf{\Delta}^{n,\mathrm{op}} \to \mathbf{\Theta}_n^{\mathrm{op}}$, defined inductively by setting $\tau_0 := \mathrm{id}$ and

$$\tau_n([i_1], \ldots, [i_n]) := [i_1](\tau_{n-1}([i_2], \ldots, [i_n]), \ldots, \tau_{n-1}([i_2], \ldots, [i_n])).$$

This functor gives a Segal morphism of algebraic patterns $\mathbf{\Delta}^{n,\mathrm{op},\natural} \to \mathbf{\Theta}_n^{\mathrm{op},\natural}$ as well as $\mathbf{\Delta}^{n,\mathrm{op},\flat} \to \mathbf{\Theta}_n^{\mathrm{op},\flat}$.

Example 4.11. The previous examples are special cases of the following: Let $f\colon \Phi \to \Psi$ be an *operator morphism* between perfect operator categories, as defined in [Bar18, Definition 1.10]. As discussed in [Bar18, §7] this induces a functor $\Lambda(f)\colon \Lambda(\Phi) \to \Lambda(\Psi)$ between the corresponding Leinster categories, and it is easy to check that this preserves the inert and active morphisms. Since operator morphisms preserve terminal objects by definition, it follows from Example 3.6 that $\Lambda(f)$ preserves elementary objects, and hence gives morphisms of algebraic patterns $\Lambda(\Phi)^{\natural} \to \Lambda(\Psi)^{\natural}$ and $\Lambda(\Phi)^{\flat} \to \Lambda(\Psi)^{\flat}$. The latter is evidently a Segal morphism, since

$$\Lambda(\Phi)^{\flat,\mathrm{el}}_{I/} \cong \{* \to I\} \cong \{* \to f(I)\} \cong \Lambda(\Psi)^{\flat,\mathrm{el}}_{f(I)/},$$

where the seond isomorphism is part of the definition of an operator morphism.

Example 4.12. Every operator category Φ has a unique operator morphism $|{-}|\colon \Phi \to \mathbb{F}$, which gives a Segal morphism $\Lambda(\Phi)^{\flat} \to \mathbb{F}_*^{\flat}$. This is also a Segal morphism $\Lambda(\Phi)^{\natural} \to \mathbb{F}_*^{\natural}$ provided the category $\Lambda(\Phi)^{\natural,\mathrm{el}}_{I/}$ is weakly contractible for all $I \in \Phi$.

Example 4.13. By [HRY19, Definition 1.20], the category $\mathbf{\Omega}$ of trees can be identified with a subcategory of the category $\mathbf{\Xi}$ of unrooted trees, and [HRY19, Definition 4.2] and [HRY19, Remark 4.3] show that this inclusion gives a morphism of algebraic patterns $\iota\colon \mathbf{\Omega}^{\mathrm{op},\natural} \to \mathbf{\Xi}^{\mathrm{op},\natural}$. The description of morphisms in $\mathbf{\Omega}^{\mathrm{op}}$ in [HRY19, Definition 1.20] implies that for every $X \in \mathbf{\Omega}^{\mathrm{op}}$ and every $\alpha \in \mathbf{\Xi}^{\mathrm{op,el}}_{\iota X/}$, the ∞-category $\mathbf{\Omega}^{\mathrm{op,el}}_{X/} \times_{\mathbf{\Xi}^{\mathrm{op,el}}_{\iota X/}} (\mathbf{\Xi}^{\mathrm{op,el}}_{\iota X/})_{/\alpha}$ has a terminal object. In particular, the functor $\mathbf{\Omega}^{\mathrm{op}}_{X/} \to \mathbf{\Xi}^{\mathrm{op}}_{\iota X/}$ is coinitial, and hence ι is a strong Segal morphism. The resulting functor

$$\iota^*\colon \mathrm{Seg}_{\mathbf{\Xi}^{\mathrm{op},\natural}}(\mathcal{S}) \to \mathrm{Seg}_{\mathbf{\Omega}^{\mathrm{op},\natural}}(\mathcal{S})$$

is the forgetful functor from cyclic ∞-operads to ∞-operads.

5. The ∞-Category of Algebraic Patterns

In this section we construct the ∞-category of algebraic patterns, and describe limits and filtered colimits in this ∞-category. As a first step, we consider the ∞-category of ∞-categories equipped with a factorization system:

Definition 5.1. We define Fact to be the full subcategory of $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty)$ (where Λ^2_2 denotes the category $0 \to 2 \leftarrow 1$) spanned by those cospans

$$\mathcal{C}_L \to \mathcal{C} \leftarrow \mathcal{C}_R$$

that describe factorization systems, i.e. those such that the functors $\mathcal{C}_L, \mathcal{C}_R \to \mathcal{C}$ are essentially surjective subcategory inclusions, and $\mathrm{Fun}_{L,R}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^{\{0,2\}}, \mathcal{C})$ is an equivalence, where the domain is defined as the pullback

$$\begin{array}{ccc} \mathrm{Fun}_{L,R}(\Delta^2, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\Delta^2, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\Delta^1, \mathcal{C}_L) \times \mathrm{Fun}(\Delta^1, \mathcal{C}_R) & \longrightarrow & \mathrm{Fun}(\Delta^{\{0,1\}}, \mathcal{C}) \times \mathrm{Fun}(\Delta^{\{1,2\}}, \mathcal{C}). \end{array}$$

Proposition 5.2. *The ∞-category* Fact *is closed under limits and filtered colimits in* $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty)$*. In particular, the ∞-category* Fact *has limits and filtered colimits, and the forgetful functor to* Cat_∞ *preserves these.*

This will follow from the following observation:

Lemma 5.3. *In the ∞-category* $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$*, the full subcategories of subcategory inclusions*[8]*, essentially surjective subcategory inclusions, and full subcategory inclusions, are all closed under limits and filtered colimits.*

Proof. A functor $F: \mathcal{C} \to \mathcal{D}$ is a subcategory inclusion precisely when $\mathcal{C}^\simeq \to \mathcal{D}^\simeq$ is a monomorphism of spaces, and $\mathrm{Map}_\mathcal{C}(x, y) \to \mathrm{Map}_\mathcal{D}(Fx, Fy)$ is a monomorphism of spaces for all $x, y \in \mathcal{C}$. A subcategory inclusion F is essentially surjective if the map $\mathcal{C}^\simeq \to \mathcal{D}^\simeq$ is an equivalence, and a full subcategory inclusion if the maps $\mathrm{Map}_\mathcal{C}(x, y) \to \mathrm{Map}_\mathcal{D}(Fx, Fy)$ are equivalences for all $x, y \in \mathcal{C}$. Since mapping spaces and the underlying space of a limit (or filtered colimit) in Cat_∞ are computed as limits (or filtered colimits) of spaces, it suffices to observe that equivalences and monomorphisms are closed under limits and filtered colimits in $\mathcal{S}$. This is obvious for equivalences, and for monomorphisms it follows from the characterization of these by [Lur09, Lemma 5.5.6.15] as the morphisms $f: X \to Y$ such that the diagonal $X \to X \times_Y X$ is an equivalence, since filtered colimits commute with finite limits and limits commute. □

Proof of Proposition 5.2. It follows from Lemma 5.3 that cospans of subcategory inclusions are closed under limits and filtered colimits in $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty)$. Since limits commute, the ∞-category $\mathrm{Fun}_{L,R}(\Delta^2, –)$, viewed as a functor $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$, preserves limits, which implies that objects such that the natural map $\mathrm{Fun}_{L,R}(\Delta^2, –) \to \mathrm{Fun}(\Delta^{\{0,2\}}, –)$ is an equivalence are also closed under limits. The same holds for filtered colimits, since the objects mapped out of in the definition of $\mathrm{Fun}_{L,R}(\Delta^2, –)$ are compact, and filtered colimits commute with finite limits in Cat_∞. □

Definition 5.4. We now define the ∞-category AlgPatt of algebraic patterns as the full subcategory of the fibre product $\mathrm{Fact} \times_{\mathrm{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ (where the pullback is over $\mathrm{ev}_0: \mathrm{Fact} \to \mathrm{Cat}_\infty$ and $\mathrm{ev}_1: \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$) containing the objects

$$\mathcal{C}' \to \mathcal{C}_L \to \mathcal{C} \leftarrow \mathcal{C}_R$$

where $\mathcal{C}' \to \mathcal{C}_L$ is a full subcategory inclusion.

Applying Lemma 5.3 again, now in the case of full subcategory inclusions, we get:

Corollary 5.5. *The full subcategory* AlgPatt *is closed under limits and filtered colimits in*

$$\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty) \times_{\mathrm{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty).$$

In particular, AlgPatt *has limits and filtered colimits, and the forgetful functor to* Cat_∞ *preserves these.* □

Remark 5.6. The ∞-category AlgPatt contains *all* morphisms of algebraic patterns; restricting these to Segal morphisms gives a (wide) subcategory $\mathrm{AlgPatt}^{\mathrm{Seg}}$. However, note that Segal morphisms do not seem to be closed under filtered colimits or general pullbacks, though by Lemma 4.5 and the next example they *are* closed under finite products.

[8] Note that we use "subcategory inclusion" in the equivalence-invariant sense — in other words, a subcategory in our sense must include all equivalences between its objects.

Example 5.7. For any pair of algebraic patterns $\mathcal{O}$, $\mathcal{P}$ we have a cartesian product pattern $\mathcal{O} \times \mathcal{P}$. For this we have an equivalence

$$\mathrm{Seg}_{\mathcal{O}\times\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathcal{O}}(\mathrm{Seg}_{\mathcal{P}}(\mathcal{C}))$$

for any $\mathcal{O} \times \mathcal{P}$-complete ∞-category $\mathcal{C}$. To see this, observe that a right Kan extension along $\mathcal{O}^{\mathrm{el}} \times \mathcal{P}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{int}} \times \mathcal{P}^{\mathrm{int}}$ can be computed in two stages in two ways, by first doing the right Kan extension to either $\mathcal{O}^{\mathrm{el}} \times \mathcal{P}^{\mathrm{int}}$ or $\mathcal{O}^{\mathrm{int}} \times \mathcal{P}^{\mathrm{el}}$; this shows that $F\colon \mathcal{O} \times \mathcal{P} \to \mathcal{C}$ is a Segal object if and only if $F(O, -)$ is a $\mathcal{P}$-Segal object for all $O \in \mathcal{O}$ and $F(-, P)$ is an $\mathcal{O}$-Segal object for all $P \in \mathcal{P}$.

Example 5.8. The pattern $\mathbf{\Delta}^{\mathrm{op},\flat}$ can be described as the pullback $\mathbf{\Delta}^{\mathrm{op},\natural} \times_{\mathbb{F}_*^{\natural}} \mathbb{F}_*^{\flat}$ using the map $\mathbf{\Delta}^{\mathrm{op},\natural} \to \mathbb{F}_*^{\natural}$ from Example 4.9 and the identity of $\mathbb{F}_*$ viewed as a morphism of patterns $\mathbb{F}_*^{\flat} \to \mathbb{F}_*^{\natural}$. (Similarly, for the other pairs of patterns $\mathcal{O}^{\flat}, \mathcal{O}^{\natural}$ mentioned in §3 the pattern $\mathcal{O}^{\flat}$ is the pullback $\mathcal{O}^{\natural} \times_{\mathbb{F}_*^{\natural}} \mathbb{F}_*^{\flat}$ for a morphism of patterns $\mathcal{O}^{\natural} \to \mathbb{F}_*^{\natural}$.)

Example 5.9. Let $\mathbf{\Theta}^{\mathrm{op},\natural}$ be the colimit $\mathrm{colim}_{n\geq 0}\, \mathbf{\Theta}_n^{\mathrm{op},\natural}$ induced by the sequence of natural inclusions $\mathbf{\Theta}_n^{\mathrm{op},\natural} \hookrightarrow \mathbf{\Theta}_{n+1}^{\mathrm{op},\natural}$, $n \geq 0$, where $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ is the algebraic pattern defined in Example 3.5. The underlying category $\mathbf{\Theta}$ is equivalent to that introduced by Joyal [Joy97] to give a definition of weak higher categories. It is easy to see that in this case we have an equivalence

$$\mathrm{Seg}_{\mathbf{\Theta}^{\mathrm{op},\natural}}(\mathcal{S}) \simeq \lim_{n\geq 0} \mathrm{Seg}_{\mathbf{\Theta}_n^{\mathrm{op},\natural}}(\mathcal{S}),$$

so that Segal $\mathbf{\Theta}^{\mathrm{op},\natural}$-spaces model (∞,∞)-categories (in the inductive sense). In particular, the canonical functor $\mathrm{Seg}_{\mathbf{\Theta}^{\mathrm{op},\natural}}(\mathcal{S}) \to \mathrm{Seg}_{\mathbf{\Theta}_n^{\mathrm{op},\natural}}(\mathcal{S})$ gives the underlying (∞, n)-category of an (∞,∞)-category.

6. Right Kan Extensions and Segal Objects

Our goal in this section is to give a sufficient criterion on a morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ such that right Kan extension along f preserves Segal objects.

Definition 6.1. We say that a morphism $f\colon \mathcal{O} \to \mathcal{P}$ of algebraic patterns has *unique lifting of active morphisms* if for every active morphism $\phi\colon P \to f(O)$ in $\mathcal{P}$, the ∞-groupoid of lifts of ϕ to an active morphism $O' \to O$ in $\mathcal{O}$ is contractible. More precisely, the fibre $(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq}_{\phi}$ of the morphism

$$(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq} \to (\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$$

at ϕ is contractible. Equivalently, f has unique lifting of active morphisms if this morphism of ∞-groupoids is an equivalence for all $O \in \mathcal{O}$.

Lemma 6.2. *A morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ has unique lifting of active morphisms if and only if it satisfies the following condition:*

($*$) *For all $P \in \mathcal{P}$ the functor*

$$\mathcal{O}^{\mathrm{int}}_{P/} \to \mathcal{O}_{P/}$$

is coinitial.

Proof. By [Lur09, Theorem 4.1.3.1], the functor $\mathcal{O}^{\mathrm{int}}_{P/} \to \mathcal{O}_{P/}$ is coinitial if and only if for every morphism $\phi\colon P \to f(O)$ in $\mathcal{P}$, the ∞-category $(\mathcal{O}^{\mathrm{int}}_{P/})_{/\phi}$ is weakly contractible. This ∞-category has objects pairs

$$\left(O' \xrightarrow{\alpha} O, \quad \begin{array}{ccc} & P & \\ {\scriptstyle\iota} \swarrow & & \searrow {\scriptstyle\phi} \\ f(O') & \xrightarrow{f(\alpha)} & f(O) \end{array} \right),$$

where ι is inert. The morphism α has an essentially unique inert–active factorization, and since f is compatible with this factorization we see that the full subcategory of objects where α is active is cofinal. By uniqueness of factorizations a morphism in this subcategory is required to be an equivalence, hence this is an ∞-groupoid, and so $(*)$ is equivalent to this ∞-groupoid being contractible. But an object in this subcategory gives an inert–active factorization of ϕ, and we see that it is equivalent to the ∞-groupoid of lifts of the active part of ϕ to an active morphism in $\mathcal{O}$. □

Proposition 6.3. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a morphism of algebraic patterns that has unique lifting of active morphisms and $\mathcal{C}$ is an $\mathcal{O}$- and $\mathcal{P}$-complete ∞-category such that the pointwise right Kan extension*

$$f_*\colon \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$$

exists. Then f_* restricts to a functor

$$f_*\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}).$$

Remark 6.4. We emphasize that the condition of unique lifting of active morphisms is far from a *necessary* one. Indeed, the functor f_* will preserve Segal objects if and only if its left adjoint f^* preserves Segal equivalences. In [CHH18] the latter condition was checked for a certain morphism $\tau\colon \mathbf{\Delta}^{1,\mathrm{op}}_{\mathbb{F}} \to \mathbf{\Omega}^{\mathrm{op}}$, which clearly does *not* have unique lifting of active morphisms.

Proof of Proposition 6.3. By Lemma 6.2, the condition that f has unique lifting of active morphisms implies that for any functor $F\colon \mathcal{O} \to \mathcal{C}$, the Beck–Chevalley transformation

$$(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \to f^{\mathrm{int}}_*(F|_{\mathcal{O}^{\mathrm{int}}})$$

is an equivalence. If F is a Segal $\mathcal{O}$-object, then $F|_{\mathcal{O}^{\mathrm{int}}} \simeq i_{\mathcal{O},*}F|_{\mathcal{O}^{\mathrm{el}}}$, where $i_{\mathcal{O}}$ is the inclusion $\mathcal{O}^{\mathrm{el}} \hookrightarrow \mathcal{O}^{\mathrm{int}}$, so in this case we have $(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \simeq f^{\mathrm{int}}_* i_{\mathcal{O},*}F|_{\mathcal{O}^{\mathrm{el}}}$. By naturality of right Kan extensions in the commutative square

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{el}} & \xrightarrow{f^{\mathrm{el}}} & \mathcal{P}^{\mathrm{el}} \\ {\scriptstyle i_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle i_{\mathcal{P}}} \\ \mathcal{O}^{\mathrm{int}} & \xrightarrow{f^{\mathrm{int}}} & \mathcal{P}^{\mathrm{int}} \end{array}$$

this can in turn be identified with $i_{\mathcal{P},*}f^{\mathrm{el}}_*F|_{\mathcal{O}^{\mathrm{el}}}$. Moreover, since $\mathcal{P}^{\mathrm{el}}$ is a full subcategory of $\mathcal{P}^{\mathrm{int}}$, we have

$$f^{\mathrm{el}}_*F|_{\mathcal{O}^{\mathrm{el}}} \simeq i^*_{\mathcal{P}} i_{\mathcal{P},*} f^{\mathrm{el}}_* F|_{\mathcal{O}^{\mathrm{el}}} \simeq i^*_{\mathcal{P}} f^{\mathrm{int}}_* i_{\mathcal{O},*}F|_{\mathcal{O}^{\mathrm{el}}} \simeq i^*_{\mathcal{P}} f^{\mathrm{int}}_* F|_{\mathcal{O}^{\mathrm{int}}}.$$

Combining these equivalences, we see that $(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \simeq i_{\mathcal{P},*}(i^*_{\mathcal{P}} f^{\mathrm{int}}_* F|_{\mathcal{O}^{\mathrm{int}}}) \simeq i_{\mathcal{P},*}(f_*F)|_{\mathcal{P}^{\mathrm{el}}}$, where the second equivalence is given by $i^*_{\mathcal{P}} f^{\mathrm{int}}_*(F|_{\mathcal{O}^{\mathrm{int}}}) \simeq i^*_{\mathcal{P}}(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \simeq (f_*F)|_{\mathcal{P}^{\mathrm{el}}}$. Hence f_*F is a Segal $\mathcal{P}$-object. □

Remark 6.5. If f in Proposition 6.3 is moreover a Segal morphism, we get an adjunction

$$f^*\colon \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) : f_*$$

by restricting the adjunction $f^* \dashv f_*$ on functor ∞-categories.

Example 6.6. Suppose we have two categorical patterns $\mathfrak{O}_1$ and $\mathfrak{O}_2$ with the same underlying ∞-category $\mathcal{O}$ and the same inert–active factorization system, and $\mathfrak{O}^{\mathrm{el}}_1$ is a full subcategory of $\mathfrak{O}^{\mathrm{el}}_2$. Then the identity functor of $\mathcal{O}$ gives a morphism of algebraic patterns $\mathfrak{O}_1 \to \mathfrak{O}_2$ for which unique lifting of active morphisms holds trivially. In this case, this just means that the Segal condition for $\mathfrak{O}_1$ is stronger than that for $\mathfrak{O}_2$. For example, this holds for the identity morphism of $\mathbb{F}_*$ viewed as a morphism $\mathbb{F}^{\flat}_* \to \mathbb{F}^{\natural}_*$. On the other hand, the identity functor would typically *not* be a Segal morphism.

Example 6.7. The inclusion $i\colon \{[0]\} \to \mathbf{\Delta}^{\mathrm{op},\natural}$ clearly has unique lifting of active morphisms, since the only active morphism to $[0]$ in $\mathbf{\Delta}^{\mathrm{op}}$ is the identity. In this case, the right Kan extension functor

$$i_*\colon \mathcal{C} \simeq \mathrm{Fun}(\{[0]\}, \mathcal{C}) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$$

takes an object $C \in \mathcal{C}$ to the simplicial object i_*C given by $(i_*C)_n \simeq \prod_{i=0}^n C$, with face maps corresponding to projections and degeneracies given by diagonal maps. This clearly satisfies the

Segal condition. More generally, the inclusion $\mathbf{\Theta}_{n-1}^{\mathrm{op},\natural} \hookrightarrow \mathbf{\Theta}_{n}^{\mathrm{op},\natural}$ has unique lifting of active morphisms for all $n \geq 1$.

Example 6.8. Let $\iota\colon \mathbf{\Omega}^{\mathrm{op},\natural} \to \mathbf{\Xi}^{\mathrm{op},\natural}$ be the Segal morphism of Example 4.13. Since the active morphisms in $\mathbf{\Xi}^{\mathrm{op}}$ are the boundary-preserving ones, it is easy to see that ι has unique lifting of active morphisms. Then Proposition 6.3 and Remark 6.5 give an adjunction

$$\iota^*\colon \mathrm{Seg}_{\mathbf{\Xi}^{\mathrm{op}}}(\mathcal{S}) \rightleftarrows \mathrm{Seg}_{\mathbf{\Omega}^{\mathrm{op},\natural}}(\mathcal{S}) :\iota_*,$$

where ι_* is a right adjoint to the forgetful functor ι^* from cyclic ∞-operads to ∞-operads. According to [DCH19, §2.15] the analogue of this right adjoint for ordinary cyclic operads was first considered in the unpublished thesis of J. Templeton.

7. Left Kan Extensions and Segal Objects

In this section we will give conditions under which left Kan extension along a morphism f preserves Segal objects in $\mathcal{C}$. In contrast to the case of right Kan extensions, this requires strong assumptions on both f and the target ∞-category $\mathcal{C}$. Part of the condition is a uniqueness requirement on lifts of inert morphisms, which we consider first:

Definition 7.1. A morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ is said to have *unique lifting of inert morphisms* if for every inert morphism $f(O) \to P$ the ∞-groupoid of lifts to inert morphisms $O \to O'$ is contractible. More precisely, the fibre $(\mathcal{O}^{\mathrm{int}}_{O/})^{\simeq}_{\phi}$ of the morphism

$$(\mathcal{O}^{\mathrm{int}}_{O/})^{\simeq} \to (\mathcal{P}^{\mathrm{int}}_{f(O)/})^{\simeq}$$

at ϕ is contractible. Equivalently, f has unique lifting of inert morphisms if this morphism of ∞-groupoids is an equivalence for all $O \in \mathcal{O}$.

Lemma 7.2. *A morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms if and only if it satisfies the following condition:*

($*$) *For all $P \in \mathcal{P}$ the functor*

$$\mathcal{O}^{\mathrm{act}}_{/P} \to \mathcal{O}_{/P}$$

is cofinal.

Proof. This follows by the same argument as for Lemma 6.2, with the roles of active and inert morphisms reversed. □

Unique lifting of inert morphisms allows us to functorially transport active morphisms along inert morphisms, in the following sense:

Proposition 7.3. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms. Let*

$$\mathcal{X} \subseteq \mathcal{O} \times_{\mathcal{P}} \mathcal{P}^{\Delta^1}$$

be the full subcategory of the fibre product over evaluation at 0, with objects those pairs

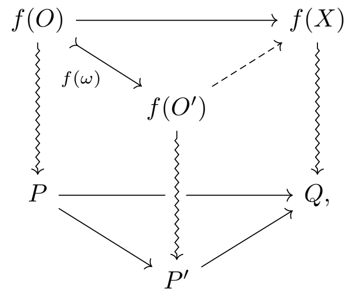
$$(O, f(O) \xrightarrow{\phi} P)$$

where ϕ is active. Then the projection $\mathcal{X} \to \mathcal{P}$ given by evaluation at $1 \in \Delta^1$ is a cocartesian fibration, and a morphism

$$\left(\begin{array}{c} O \\ \omega\downarrow \\ O' \end{array} , \begin{array}{ccc} f(O) & \rightsquigarrow & P \\ {\scriptstyle f(\omega)}\downarrow & & \downarrow \\ f(O') & \rightsquigarrow & P' \end{array} \right)$$

is cocartesian if and only if ω is inert.

Proof. We first show that such a morphism with ω inert is cocartesian. This means that given a morphism $O \to X$ in $\mathcal{O}$ and a commutative diagram



there exists a unique lift $O' \to X$ making the diagram commute.

The morphism $O \to X$ has a unique inert–active factorization as $O \rightarrowtail O'' \rightsquigarrow X$. Since f is compatible with the factorization system, we see that the unique inert–active factorization of $f(O) \to Q$ is $f(O) \rightarrowtail f(O'') \rightsquigarrow f(X) \rightsquigarrow Q$.

On the other hand, the inert–active factorization of $f(O') \to Q$ gives another factorization $f(O) \rightarrowtail f(O') \rightarrowtail Q' \rightsquigarrow Q$, where by uniqueness we must have $Q' \simeq f(O'')$. Since f has unique lifts of inert morphisms, the map $f(O') \rightarrowtail f(O'')$ lifts to a unique morphism $O' \rightarrowtail O''$, and moreover by uniqueness the composite $O \rightarrowtail O' \rightarrowtail O''$ must be the inert map $O \rightarrowtail O''$ arising from the factorization of $O \to X$.

Thus, there are unique diagrams

$$O \rightarrowtail O' \rightarrowtail O'' \rightsquigarrow X,$$

$$\begin{array}{ccccccc} f(O) & \rightarrowtail & f(O') & \rightarrowtail & f(O'') & \rightsquigarrow & f(X) \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & \swarrow & \\ P & \longrightarrow & P' & \longrightarrow & Q, & & \end{array}$$

which give the required unique factorization (since any other factorization through $(O', f(O') \rightsquigarrow P')$ must induce these by uniqueness of inert–active factorizations).

We next check that $\mathcal{X} \to \mathcal{P}$ is a cocartesian fibration. This amounts to showing that cocartesian morphisms exist, and by the first part of the proof it suffices to check that given $(O, f(O) \overset{\phi}{\rightsquigarrow} P)$ with ϕ active and a morphism $P \to P'$, there exists a morphism

$$\left(\begin{array}{c} O \\ \omega \downarrow \\ O' \end{array}, \begin{array}{ccc} f(O) & \rightsquigarrow & P \\ f(\omega)\downarrow & & \downarrow \\ f(O') & \rightsquigarrow & P' \end{array} \right)$$

with ω inert. This again follows from unique lifting of inert morphisms, which ensures that the inert–active factorization of $f(O) \rightsquigarrow P \to P'$ gives such a diagram.

It remains to show that ω must be inert for any cocartesian morphism. Since cocartesian morphisms are unique when they exist, this follows from the existence of the cocartesian morphisms we just described. □

Straightening this cocartesian fibration, we get:

Corollary 7.4. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms. Then there is a functor $\mathcal{P} \to \mathrm{Cat}_\infty$ that takes P to $\mathcal{O}^{\mathrm{act}}_{/P}$. The functor $\mathcal{O}^{\mathrm{act}}_{/P} \to \mathcal{O}^{\mathrm{act}}_{/P'}$ assigned to a morphism $P \to P'$ takes a pair $(O, f(O) \rightsquigarrow P)$ to $(O', f(O') \rightsquigarrow P')$ where $f(O) \rightarrowtail f(O') \rightsquigarrow P'$ is the inert–active factorization of $f(O) \rightsquigarrow P \to P'$.* □

Remark 7.5. Let $\mathcal{O}$ be an algebraic pattern, and write $\mathrm{Fun}(\Delta^1, \mathcal{O})_{\mathrm{act}}$ for the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O})$ spanned by the active morphisms. As a simple special case of the previous result (taking f to be $\mathrm{id}_{\mathcal{O}}$) we see that

$$\mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, \mathcal{O})_{\mathrm{act}} \to \mathcal{O}$$

is a cocartesian fibration. This corresponds to a functor $\mathcal{O} \to \mathrm{Cat}_\infty$ that takes O to $\mathcal{O}^{\mathrm{act}}_{/O}$ and a morphism $\phi\colon O \to O'$ to a functor $\mathcal{O}^{\mathrm{act}}_{/O} \to \mathcal{O}^{\mathrm{act}}_{/O'}$ that takes $X \rightsquigarrow O$ to $X' \rightsquigarrow O'$, where $X \rightarrowtail X' \rightsquigarrow O'$ is the inert–active factorization of the composite $X \rightsquigarrow O \to O'$.

Remark 7.6. Suppose $f\colon \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms, and let $\mathcal{X}^{\mathrm{int}} \to \mathcal{P}^{\mathrm{int}}$ be the pullback of the cocartesian fibration $\mathcal{X} \to \mathcal{P}$ of Proposition 7.3 to the subcategory $\mathcal{P}^{\mathrm{int}}$. Then for every active morphism $\phi\colon f(O) \rightsquigarrow P$ in $\mathcal{P}$ we can define a functor $\mathcal{P}^{\mathrm{int}}_{P/} \to \mathcal{O}^{\mathrm{int}}_{O/}$ as the composite

$$\mathcal{P}^{\mathrm{int}}_{P/} \to \mathcal{X}^{\mathrm{int}}_{(O,\phi)/} \to \mathcal{O}^{\mathrm{int}}_{O/}$$

where the first functor takes $\alpha\colon P \rightarrowtail P'$ to the cocartesian morphism $(O, \phi) \to (\alpha_! O, \alpha_! \phi)$ for the cocartesian fibration $\mathcal{X}$ (where $\alpha_!\phi$ is the active part of the map $\alpha \circ \phi$), and the second is induced by the forgetful functor $\mathcal{X} \to \mathcal{O}$. In particular, we can restrict to $\mathcal{P}^{\mathrm{el}}_{P/}$ and compose with the functor $\mathcal{O}^{\mathrm{int,op}}_{O/} \to \mathcal{O}^{\mathrm{op}} \xrightarrow{\mathcal{O}^{\mathrm{el}}_{-/}} \mathrm{Cat}_\infty$ to get a functor $\mathcal{P}^{\mathrm{el,op}}_{P/} \rightarrowtail \mathrm{Cat}_\infty$ that takes $\alpha\colon P \to E$ to $\mathcal{O}^{\mathrm{el}}_{\alpha_! O/}$. We write $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{P}^{\mathrm{el}}_{P/}$ for the corresponding cartesian fibration.

Using this functoriality we can now state the conditions we require of a morphism of algebraic patterns:

Definition 7.7. A morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ is *extendable* if the following conditions are satisfied:

(1) The morphism f has unique lifting of inert morphisms.
(2) For $P \in \mathcal{P}$, let $\mathcal{L}_P$ denote the limit of the composite functor $\epsilon_P\colon \mathcal{P}^{\mathrm{el}}_{P/} \to \mathcal{P}^{\mathrm{int}} \to \mathrm{Cat}_\infty$ taking E to $\mathcal{O}^{\mathrm{act}}_{/E}$ (where the second functor is that of Corollary 7.4). Then the canonical functor

$$\mathcal{O}^{\mathrm{act}}_{/P} \to \mathcal{L}_P$$

is cofinal.
(3) For every active morphism $\phi\colon f(O) \rightsquigarrow P$, the canonical functor

$$\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$$

induces an equivalence

$$\lim_{\mathcal{O}^{\mathrm{el}}_{O/}} F \xrightarrow{\sim} \lim_{\mathcal{O}^{\mathrm{el}}(\phi)} F$$

for every functor $F\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{S}$.

Remark 7.8. We have used the limit in condition (2) as this seems the most natural choice in Definition 7.11; we could also have used the lax limit instead, provided the same change is made in Definition 7.11. In the cases of interest the lax limit actually agrees with the usual limit, as it will either be a finite product or a limit of ∞-groupoids, so the distinction turns out not to matter in practice.

Remark 7.9. In practice, condition (3) holds because the map $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$ is coinitial.

Remark 7.10. Condition (3) implies that for a functor $\Phi\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{C}$, we have an equivalence

$$\lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \Phi(E) \simeq \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{\alpha_! O/}} \Phi(E)$$

whenever either limit exists in $\mathcal{C}$. If Φ is a Segal $\mathcal{O}$-object, this implies that the following "relative Segal condition" holds:

$$\Phi(O) \simeq \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \Phi(\alpha_! O).$$

We now turn to the requirements we must make of our target category, for which we need the following notion:

Definition 7.11. Consider a functor $K\colon \mathcal{I} \to \mathrm{Cat}_\infty$ with corresponding cocartesian fibration $\pi\colon \mathcal{K} \to \mathcal{I}$. Let $\mathcal{L}$ be the limit of K, which we can identify with the ∞-category of cocartesian sections $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I}, \mathcal{K})$. We then have a functor $p\colon \mathcal{I} \times \mathcal{L} \to \mathcal{K}$ adjoint to the forgetful functor $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I}, \mathcal{K}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{K})$; the composite $\pi \circ p$ is moreover the projection $\mathcal{L} \times \mathcal{I} \to \mathcal{I}$. This gives a commutative diagram

$$\begin{array}{ccccc} \mathcal{L} \times \mathcal{I} & \xrightarrow{p} & \mathcal{K} & \xrightarrow{\pi} & \mathcal{I} \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & & & \downarrow{\scriptstyle \iota} \\ \mathcal{L} & & \xrightarrow{\lambda} & & *, \end{array}$$

which for any ∞-category $\mathcal{C}$ (with appropriate limits and colimits) determines an equivalence of functors between functor ∞-categories

$$p^*\pi^*\iota^* \simeq \mathrm{pr}_1^*\lambda^*.$$

This induces a mate transformation

$$\lambda^*\iota_* \to \mathrm{pr}_{1,*}\mathrm{pr}_2^* \simeq \mathrm{pr}_{1,*}p^*\pi^*,$$

and this is an equivalence: for $\Phi\colon \mathcal{I} \to \mathcal{C}$, $\lambda^*\iota_*\Phi$ is the constant functor with value $\lim_{\mathcal{I}} \Phi$ while the right Kan extension $\mathrm{pr}_{1,*}$ takes limits over $\mathcal{I}$ fibrewise so that $\mathrm{pr}_{1,*}\mathrm{pr}_2^*\Phi$ is also the constant functor with value $\lim_{\mathcal{I}} \Phi$. From this equivalence we in turn obtain, by moving adjoints around, a natural transformation

$$\lambda_!\mathrm{pr}_{1,*}p^* \to \lambda_!\mathrm{pr}_{1,*}p^*\pi^*\pi_! \simeq \lambda_!\lambda^*\iota_*\pi_! \to \iota_*\pi_!.$$

For a functor $F\colon \mathcal{K} \to \mathcal{C}$ we can interpret this as a natural morphism

$$\operatorname*{colim}_{\mathcal{L}} \lim_{\mathcal{I}} p^*F \to \lim_{i \in \mathcal{I}} \operatorname*{colim}_{\mathcal{K}_i} F|_{\mathcal{K}_i}.$$

We say that $\mathcal{I}$-*limits distribute over* K-*colimits* in $\mathcal{C}$ if this morphism is an equivalence for any functor F.

Definition 7.12. Let $f\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of algebraic patterns. We say that an ∞-category $\mathcal{C}$ is f-*admissible* if $\mathcal{C}$ is $\mathcal{O}$- and $\mathcal{P}$-complete, the pointwise left Kan extension $f_!\colon \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$ exists, and $\mathcal{P}^{\mathrm{el}}_{P/}$-limits distribute over ϵ_P-colimits for all $P \in \mathcal{P}$, where ϵ_P is the functor from Definition 7.7(2). In other words, if $\mathcal{C}$ is f-admissible then for every $P \in \mathcal{P}$ and every functor Φ, the natural map

$$\operatorname*{colim}_{(O_E)_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \in \mathcal{L}_P} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \Phi(O_E) \to \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \operatorname*{colim}_{O_E \in \mathcal{O}^{\mathrm{act}}_{/E}} \Phi(O_E)$$

is an equivalence.

Having made these definitions, we can now state our result on left Kan extensions:

Proposition 7.13. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is an extendable morphism of algebraic patterns, and $\mathcal{C}$ is an f-admissible ∞-category. Then left Kan extension along f restricts to a functor*

$$f_!\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}),$$

given by $f_!\Phi(P) \simeq \mathrm{colim}_{O \in \mathcal{O}^{\mathrm{act}}_{/P}} \Phi(O)$.

Proof. Given $\Phi \in \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$, we must show that $f_!\Phi$ is a Segal object, i.e. that the natural map

$$(f_!\Phi)(P) \to \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} (f_!\Phi)(E)$$

is an equivalence. We have a sequence of equivalences

$$\begin{aligned} f_!\Phi(P) &\simeq \operatorname*{colim}_{O \in \mathcal{O}_{/P}} \Phi(O) && \\ &\simeq \operatorname*{colim}_{O \in \mathcal{O}^{\mathrm{act}}_{/P}} \Phi(O) && \text{(by 7.2)} \\ &\simeq \operatorname*{colim}_{O \in \mathcal{O}^{\mathrm{act}}_{/P}} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \Phi(O_E) && \text{(by 7.7(3))} \\ &\simeq \operatorname*{colim}_{(O_E)_E \in \mathcal{L}_p} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \Phi(O_E) && \text{(by 7.7(2))} \\ &\simeq \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \operatorname*{colim}_{O_E \in \mathcal{O}^{\mathrm{act}}_{/E}} \Phi(O_E) && \text{(by 7.12)} \\ &\simeq \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} (f_!\Phi)(E), && \end{aligned}$$

which completes the proof. □

Having identified conditions under which $f_!$ preserves Segal objects, we now turn to the question of when these conditions hold. For extendability, we will see some general classes of examples below in §9; here, we will discuss two classes of examples where f-admissibility holds. The starting point is the following examples of distributivity of limits over colimits:

Definition 7.14. We say an ∞-category $\mathcal{C}$ is $\times$-admissible if it has finite products and the cartesian product preserves colimits in each variable.

Lemma 7.15. *Suppose $\mathcal{C}$ is $\times$-admissible. Then finite products distribute over all colimits in $\mathcal{C}$.*

Proof. For any functors $F_i\colon \mathcal{I}_i \to \mathcal{C}$ $(i = 1, \ldots, n)$ whose colimits exist we have

$$\operatorname*{colim}_{\mathcal{I}_1 \times \cdots \times \mathcal{I}_n} F_1 \times \cdots \times F_n \simeq \operatorname*{colim}_{\mathcal{I}_1} \cdots \operatorname*{colim}_{\mathcal{I}_n} F_1 \times \cdots \times F_n \simeq \operatorname*{colim}_{\mathcal{I}_1} F_1 \times \cdots \times \operatorname*{colim}_{\mathcal{I}_n} F_n.$$

□

Proposition 7.16. *Let $\mathcal{C}$ be a presentable ∞-category and write $t\colon \mathcal{S} \to \mathcal{C}$ for the unique colimit-preserving functor taking $*$ to the terminal object $*_{\mathcal{C}}$ of $\mathcal{C}$. Consider a functor $K\colon \mathcal{I} \to \mathcal{S}$ and suppose the following conditions hold:*

(1) t preserves $\mathcal{I}$-limits.
(2) The functor $\mathcal{C}_{/t(S)} \to \lim_S \mathcal{C} \simeq \mathrm{Fun}(S, \mathcal{C})$ induced by taking pullbacks along $*_{\mathcal{C}} \simeq t(*) \to t(S)$, is an equivalence for $S = \lim_{\mathcal{I}} K(i)$ and $S = K_i$ for all $i \in \mathcal{I}$.

Then $\mathcal{I}$-limits distribute over K-colimits in $\mathcal{C}$.

Proof. Condition (2) implies that we have a commutative diagram of right adjoints

$$\begin{array}{ccc} \mathcal{C}_{/t(S)} & \xrightarrow{\ \sim\ } & \mathrm{Fun}(S, \mathcal{C}) \\ & {\scriptstyle -\times t(S)}\nwarrow \quad \nearrow{\scriptstyle \mathrm{const}} & \\ & \mathcal{C} & \end{array}$$

Passing to left adjoints, we get the commutative triangle

$$\begin{array}{ccc} \mathrm{Fun}(S, \mathcal{C}) & \xrightarrow{\sim} & \mathcal{C}_{/t(S)} \\ & \searrow^{\mathrm{colim}} \quad \swarrow_{\mathrm{src}} & \\ & \mathcal{C}, & \end{array}$$

from which we see that under the equivalence of (2) the colimit of a diagram $S \to \mathcal{C}$ is given by the source of the corresponding morphism to $t(S)$. Given $F\colon S \to \mathcal{C}$, it follows that we have pullback squares

$$\begin{array}{ccc} F(s) & \longrightarrow & \mathrm{colim}_S F \\ \downarrow & & \downarrow \\ *_{\mathcal{C}} & \xrightarrow{t(s)} & t(S) \end{array}$$

for $s \in S$.

Now consider a functor $F\colon \mathcal{K} \to \mathcal{C}$, where $\mathcal{K} \to \mathcal{I}$ is the left fibration corresponding to K. We have a commutative square

$$\begin{array}{ccc} \mathrm{colim}_L \lim_{\mathcal{I}} F & \longrightarrow & \lim_{i\in\mathcal{I}} \mathrm{colim}_{K_i} F \\ \downarrow & & \downarrow \\ \mathrm{colim}_L \lim_{\mathcal{I}} *_{\mathcal{C}} & \longrightarrow & \lim_{i\in\mathcal{I}} \mathrm{colim}_{K_i} *_{\mathcal{C}}, \end{array}$$

where $L := \lim_{\mathcal{I}} K(i)$. Here the bottom horizontal map can be identified with the natural map

$$t(L) \simeq \operatorname*{colim}_L *_{\mathcal{C}} \to \lim_{i\in\mathcal{I}} \operatorname*{colim}_{K_i} *_{\mathcal{C}} \simeq \lim_{i\in\mathcal{I}} t(K_i).$$

This is an equivalence by assumption (1). The equivalence of assumption (2) then implies that the top horizontal map is an equivalence if and only if it induces an equivalence on fibres over each map $t(l)\colon *_{\mathcal{C}} \to t(L)$ for $l \in L$. Using the pullback squares above and the fact that limits commute, we see that the map on fibres at $(k_i)_i \in L$ is the identity

$$\lim_{\mathcal{I}} F(k_i) \to \lim_{\mathcal{I}} F(k_i). \qquad \square$$

This argument applies to $\mathcal{C}$ being $\mathcal{S}$, or more generally any ∞-topos, giving:

Corollary 7.17. *Given any functor $K\colon \mathcal{I} \to \mathcal{S}$ we have that:*

(i) $\mathcal{I}$-limits distribute over K-colimits in $\mathcal{S}$,
(ii) $\mathcal{I}$-limits distribute over K-colimits in any ∞-topos provided $\mathcal{I}$ is a finite ∞-category.

Proof. Condition (2) of Proposition 7.16 holds in ∞-topoi by descent, [Lur09, Theorem 6.1.3.9], while condition (1) holds for finite limits since t is the left adjoint of a geometric morphism by [Lur09, Proposition 6.3.4.1] and so preserves finite limits. In the case of $\mathcal{S}$ the finiteness condition is unnecessary since t is an equivalence and so preserves all limits. $\square$

Corollary 7.18. *Let $f\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of algebraic patterns such that $\mathcal{P}^{\mathrm{el}}_{P/}$ is a finite set for all $P \in \mathcal{P}$. Suppose $\mathcal{C}$ is a $\times$-admissible ∞-category, and assume the pointwise left Kan extension $f_!\colon \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$ exists. Then $\mathcal{C}$ is f-admissible, and the left Kan extension along f restricts to a functor*

$$f_!\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}). \qquad \square$$

Remark 7.19. The assumption of $\times$-admissibility can be slightly weakened: It is enough to assume that the cartesian product in $\mathcal{C}$ preserves colimits of shape $\mathcal{O}^{\mathrm{act}}_{/E}$ in each variable for all $E \in \mathcal{P}^{\mathrm{el}}$.

Corollary 7.20. *Suppose $\mathcal{X}$ is an ∞-topos, and $f\colon \mathcal{O} \to \mathcal{P}$ is an extendable morphism of algebraic patterns such that*

(1) $\mathcal{O}^{\mathrm{act}}_{/E}$ is an ∞-groupoid for all $E \in \mathcal{P}^{\mathrm{el}}$,

(2) the ∞-category $\mathcal{P}^{\mathrm{el}}_{P/}$ is finite for all $P \in \mathcal{P}$ (or arbitrary if $\mathcal{X}$ is the ∞-topos $\mathcal{S}$).

Then $\mathcal{X}$ is f-admissible, and the left Kan extension restricts to a functor

$$f_! \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{X}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{X}).$$ □

8. Free Segal Objects

Suppose $\mathcal{O}$ is an algebraic pattern, and $\mathcal{C}$ an $\mathcal{O}$-complete ∞-category. Restricting Segal objects to the subcategory $\mathcal{O}^{\mathrm{el}}$ gives a functor

$$U_{\mathcal{O}} \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}).$$

We think of *free* Segal $\mathcal{O}$-objects as being given by a left adjoint $F_{\mathcal{O}}$ to this functor, when this exists.

The subcategory $\mathcal{O}^{\mathrm{int}}$ has a canonical pattern structure restricted from $\mathcal{O}$ (so only equivalences are active morphisms and the elementary objects are still those of $\mathcal{O}^{\mathrm{el}}$), and using this the inclusion $j_{\mathcal{O}} \colon \mathcal{O}^{\mathrm{int}} \to \mathcal{O}$ is a Segal morphism. The ∞-category $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C})$ is by definition the full subcategory of $\mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{C})$ spanned by the functors that are right Kan extensions along the fully faithful inclusion $i_{\mathcal{O}} \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{int}}$, which means that the restriction functor $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$ is an equivalence. The functor $U_{\mathcal{O}}$ thus factors as the composite

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{j^*_{\mathcal{O}}} \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \xrightarrow{i^*_{\mathcal{O}}} \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}),$$

where the second functor is an equivalence with inverse the right Kan extension functor $i_{\mathcal{O},*}$. If $\mathcal{C}$ is presentable, using Proposition 4.7 this means the left adjoint $F_{\mathcal{O}}$ is given by

$$\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}) \xrightarrow{i_{\mathcal{O},*}} \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \xrightarrow{L_{\mathrm{Seg}} f_!} \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}).$$

In this section we will first show that this adjunction is monadic, and then specialize the results of the previous section to $j_{\mathcal{O}}$ to get conditions under which the free Segal objects are described by a formula in terms of limits and colimits.

Monadicity is a special case of the following observation:

Proposition 8.1. *Suppose $f \colon \mathcal{O} \to \mathcal{P}$ is an essentially surjective Segal morphism and $\mathcal{C}$ is a presentable ∞-category. Then:*

(i) A functor $F \colon \mathcal{P} \to \mathcal{C}$ is a Segal object if and only if f^*F is a Segal $\mathcal{O}$-object.

(ii) The adjunction

$$Lf_! \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) : f^*$$

is monadic.

Proof. We first prove (i). One direction amounts to f being a Segal morphism, which is true by assumption. To prove the non-trivial direction, observe that for $\Phi \colon \mathcal{O} \to \mathcal{C}$ we have for every $O \in \mathcal{O}$ canonical morphisms

$$\Phi(f(O)) \to \lim_{E \in \mathcal{P}^{\mathrm{el}}_{f(O)/}} \Phi(E) \to \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{O/}} \Phi(f(E')).$$

Here the second morphism is an equivalence since f is a Segal morphism, and if $f^*\Phi$ is a Segal $\mathcal{O}$-object then the composite morphism is an equivalence. Thus the first morphism is an equivalence, and so Φ satisfies the Segal condition at every object of $\mathcal{P}$ in the image of f; since f is essentially surjective this completes the proof.

Using the monadicity theorem for ∞-categories [Lur17, Theorem 4.7.3.5], to prove (ii) it suffices to show that f^* detects equivalences, that $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ has colimits of f^*-split simplicial objects, and these colimits are preserved by f^*. Since f is essentially surjective it is immediate that f^* detects equivalences. Consider an f^*-split simplicial object $p \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$. Let $\overline{p} \colon (\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$ denote the colimit of p in $\mathrm{Fun}(\mathcal{P}, \mathcal{C})$. Since f^*, viewed as a functor $\mathrm{Fun}(\mathcal{P}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$, is a left adjoint, $f^*\overline{p}$ is the colimit of f^*p in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$. On the other hand, since f^*p extends to a split simplicial diagram, and all functors preserve colimits of split simplicial diagrams, we see that the colimit of f^*p in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ is also the colimit in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$. In particular, $f^*\overline{p}(\infty)$ lies in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$. By

(i) this implies that $\overline{p}(\infty)$ is in $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$. This completes the proof, since the colimit of p in $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ is the localization of $\overline{p}(\infty)$, which is already local. □

Applying this to $j_{\mathcal{O}}$, we get:

Corollary 8.2. *Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ a presentable ∞-category. Then the free-forgetful adjunction*

$$F_{\mathcal{O}} \colon \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}) \simeq \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) :U_{\mathcal{O}}$$

is monadic. □

Now we apply the results of the previous section to $j_{\mathcal{O}}$ to understand when the free algebras are simply given by the left Kan extension $j_{\mathcal{O},!}$. It is convenient to first introduce some notation:

Notation 8.3. Let $\mathcal{O}$ be an algebraic pattern. For $O \in \mathcal{O}$ we write $\mathrm{Act}_{\mathcal{O}}(O)$ for the ∞-groupoid of active morphisms to O in $\mathcal{O}$; this is equivalent to $(\mathcal{O}^{\mathrm{int}})^{\mathrm{act}}_{/O}$ since the only active morphisms in $\mathcal{O}^{\mathrm{int}}$ are the equivalences.

Remark 8.4. By Remark 7.5 the ∞-categories $\mathcal{O}^{\mathrm{act}}_{/O}$ are functorial in $O \in \mathcal{O}$. Passing to the underlying ∞-groupoids this means the ∞-groupoids $\mathrm{Act}_{\mathcal{O}}(O)$ are functorial in $O \in \mathcal{O}$, via the factorization system.

Definition 8.5. We say an algebraic pattern $\mathcal{O}$ is *extendable* if the inclusion $j_{\mathcal{O}} \colon \mathcal{O}^{\mathrm{int}} \to \mathcal{O}$ is extendable in the sense of Definition 7.7. This is equivalent to the following pair of conditions:

(1) The morphism
$$\mathrm{Act}_{\mathcal{O}}(O) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(E)$$
is an equivalence for all $O \in \mathcal{O}$. In other words, $\mathrm{Act}_{\mathcal{O}}$ is a Segal $\mathcal{O}$-space.

(2) For every active map $O \xrightarrow{\phi} O'$ in $\mathcal{O}$, the canonical functor $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$ induces an equivalence on limits
$$\lim_{\mathcal{O}^{\mathrm{el}}_{O/}} F \to \lim_{\mathcal{O}^{\mathrm{el}}(\phi)} F$$
for every functor $F \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{S}$.

Remark 8.6. Condition (2) implies that

$$\lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \Phi(E) \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{\alpha_! O/}} \Phi(E)$$

is an equivalence for any functor $\Phi \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{C}$, provided either limit exists, and $O \to \alpha_! O \to E$ is the inert–active factorization of $O \to O' \xrightarrow{\alpha} E$. This in particular implies the following "generalized Segal condition": If Φ is a Segal object, then for any active morphism $\phi \colon O \to O'$, we have

$$\Phi(O) \simeq \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \Phi(\alpha_! O).$$

Remark 8.7. In practice, condition (2) holds because the map $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$ is coinitial. However, with the more general formulation we get the following characterization of the extendable patterns:

Proposition 8.8. *The following are equivalent for an algebraic pattern $\mathcal{O}$:*

(1) $\mathcal{O}$ is extendable.

(2) $j_{\mathcal{O},!} \colon \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$ restricts to a functor $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$.

Proof. Suppose (1) holds. Since $\mathrm{Act}_{\mathcal{O}}(O)$ is an ∞-groupoid for all O, the ∞-category $\mathcal{S}$ is $j_{\mathcal{O}}$-admissible by Corollary 7.20, and so (2) follows from Proposition 7.13.

We now show that (2) implies the two conditions in Definition 8.5. To prove condition (1), consider the terminal object $* \in \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S})$. For this we have

$$j_{\mathcal{O},!} * (O) \simeq \operatorname*{colim}_{\mathrm{Act}_{\mathcal{O}}(O)} * \simeq \mathrm{Act}_{\mathcal{O}}(O),$$

so since $*$ is a Segal $\mathcal{O}^{\mathrm{int}}$-space, assumption (2) implies that $\mathrm{Act}_{\mathcal{O}}(-)$ is a Segal $\mathcal{O}$-space. To prove condition (2), consider $F\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{S}$ and its right Kan extension $F' := i_{\mathcal{O},*}F$, which is a Segal $\mathcal{O}^{\mathrm{int}}$-space. Then $j_{\mathcal{O},!}F'$ is a Segal $\mathcal{O}$-space, which means that in the commutative square

$$\begin{array}{ccc} \mathrm{colim}_{X\in \mathrm{Act}_{\mathcal{O}}(O)} F'(X) & \longrightarrow & \lim_{E\in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{colim}_{Y\in \mathrm{Act}_{\mathcal{O}}(E)} F'(Y) \\ \downarrow & & \downarrow \\ \mathrm{Act}_{\mathcal{O}}(O) & \xrightarrow{\sim} & \lim_{E\in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(E), \end{array}$$

the top horizontal morphism is an equivalence. Hence we get an equivalence on fibres at each active morphism $(\phi\colon X \to O) \in \mathrm{Act}_{\mathcal{O}}(O)$, which we can identify with the natural map

$$F'(X) \xrightarrow{\sim} \lim_{\alpha\in \mathcal{O}^{\mathrm{el}}_{O/}} F'(\alpha_! X).$$

Using the description of F' as a right Kan extension we get

$$\lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F \xrightarrow{\sim} \lim_{\alpha\in \mathcal{O}^{\mathrm{el}}_{O/}} \lim_{\mathcal{O}^{\mathrm{el}}_{\alpha_! X/}} F \simeq \lim_{\mathcal{O}^{\mathrm{el}}(\phi)} F,$$

as required. □

Definition 8.9. We say an ∞-category $\mathcal{C}$ is $\mathcal{O}$-*admissible* if $\mathcal{O}^{\mathrm{el}}_{O/}$-limits distribute over colimits indexed by the functor $\mathcal{O}^{\mathrm{el}}_{O/} \to \mathcal{S}$ taking E to $\mathrm{Act}_{\mathcal{O}}(E)$ for all $O \in \mathcal{O}$.

From Corollaries 7.18 and 7.20 we get:

Example 8.10. Let $\mathcal{O}$ be an extendable algebraic pattern. Then:

(i) $\mathcal{S}$ is $\mathcal{O}$-admissible.
(ii) Any ∞-topos is $\mathcal{O}$-admissible if the ∞-categories $\mathcal{O}^{\mathrm{el}}_{O/}$ are all finite.
(iii) Any $\times$-admissible ∞-category is $\mathcal{O}$-admissible if the ∞-categories $\mathcal{O}^{\mathrm{el}}_{O/}$ are all finite sets.

Corollary 8.11. *Let $\mathcal{O}$ be an extendable algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-admissible ∞-category. Then left Kan extension along $j_{\mathcal{O}}\colon \mathcal{O}^{\mathrm{int}} \to \mathcal{O}$ restricts to a functor*

$$j_{\mathcal{O},!}\colon \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}),$$

left adjoint to the restriction $j^*_{\mathcal{O}}\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C})$. This functor is given by

$$j_{\mathcal{O},!}\Phi(O) \simeq \operatorname*{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(O)} \Phi(O').$$

Combining this with the equivalence $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$ given by right Kan extension along $i_{\mathcal{O}}$, we can reformulate this as:

Corollary 8.12. *Let $\mathcal{O}$ be an extendable algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-admissible ∞-category. Then the restriction*

$$U_{\mathcal{O}}\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$$

has a left adjoint $F_{\mathcal{O}}$, which is given by

$$F_{\mathcal{O}}(\Phi)(O) \simeq j_{\mathcal{O},!}i_{\mathcal{O},*}\Phi(O) \simeq \operatorname*{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(O)} \lim_{E\in \mathcal{O}^{\mathrm{el}}_{O'/}} \Phi(E).$$

We end this section with some examples of extendable patterns:

Example 8.13. The algebraic patterns $\mathbb{F}_*^{\flat}$ and $\mathbb{F}_*^{\natural}$ are extendable. In the former case, we recover the familiar formula for free commutative monoids:

$$U_{\mathbb{F}_*^{\flat}} F_{\mathbb{F}_*^{\flat}}(X) \simeq \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n}.$$

In the latter case, we get

$$U_{\mathbb{F}_*^\natural} F_{\mathbb{F}_*^\natural}(X \to Y) \simeq \coprod_{n=0}^{\infty} X_{h\Sigma_n}^{\times_Y n} \to Y,$$

which describes a free commutative monoid on $X \to Y$ in the slice over Y.

Example 8.14. The algebraic patterns $\mathbf{\Delta}^{\mathrm{op},\flat}$ and $\mathbf{\Delta}^{\mathrm{op},\natural}$ are extendable. In the former case, we get the expected formula for free associative monoids:

$$U_{\mathbf{\Delta}^{\mathrm{op},\flat}} F_{\mathbf{\Delta}^{\mathrm{op},\flat}}(X) \simeq \coprod_{n=0}^{\infty} X^{\times n},$$

while in the latter case we get the formula for free ∞-categories:

$$U_{\mathbf{\Delta}^{\mathrm{op},\natural}} F_{\mathbf{\Delta}^{\mathrm{op},\natural}} \left(\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & & Y \end{array} \right) \simeq \left(\begin{array}{ccc} & \coprod_{n=0}^{\infty} X \times_Y \cdots \times_Y X & \\ \swarrow & & \searrow \\ Y & & Y \end{array} \right).$$

Example 8.15. More generally, the algebraic pattern $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ is extendable for every n; the conditions are checked in [Hau18], giving a formula for free (∞, n)-categories. (On the other hand, the pattern $\mathbf{\Theta}_n^{\mathrm{op},\flat}$ is *not* extendable for $n > 1$.)

Example 8.16. The algebraic pattern $\mathbf{\Omega}^{\mathrm{op},\natural}$ is extendable; the conditions are checked in [GHK17, §5.3], giving a formula for free ∞-operads. (On the other hand, the pattern $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural}$ is *not* extendable.)

9. Segal Fibrations and Weak Segal Fibrations

In this section we first consider *Segal fibrations* over an algebraic pattern, which are the cocartesian fibrations corresponding to Segal objects in Cat_∞, and then generalize these to the class of *weak Segal fibrations*; for the pattern $\mathbb{F}_*^\flat$, these objects are respectively symmetric monoidal ∞-categories and symmetric ∞-operads in the sense of [Lur17]. Our main goal is to show that extendability can be lifted from a base pattern to morphisms between (weak) Segal fibrations. Combined with our previous results this allows us, for example, to reproduce (in the cartesian setting) Lurie's formula for operadic Kan extensions along morphisms of symmetric ∞-operads.

Definition 9.1. Let $\mathcal{O}$ be an algebraic pattern. A *Segal $\mathcal{O}$-fibration* is a cocartesian fibration $\mathcal{E} \to \mathcal{O}$ whose corresponding functor $\mathcal{O} \to \mathrm{Cat}_\infty$ is a Segal $\mathcal{O}$-∞-category.

Examples 9.2.

(i) A Segal $\mathbb{F}_*^\flat$-fibration is a symmetric monoidal ∞-category.

(ii) A Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-fibration is a monoidal ∞-category, and a Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-fibration is a double ∞-category.

(iii) Segal $\mathbf{\Delta}^{n,\mathrm{op},\flat}$-fibrations and Segal $\mathbf{\Theta}_n^{\mathrm{op},\flat}$-fibrations both describe $\mathbb{E}_n$-monoidal ∞-categories.

Definition 9.3. Suppose $\mathcal{O}$ is an algebraic pattern, and $\pi\colon \mathcal{E} \to \mathcal{O}$ is a Segal $\mathcal{O}$-fibration. We say a morphism in $\mathcal{E}$ is *inert* if it is cocartesian and lies over an inert morphism in $\mathcal{O}$, and *active* if it lies over an active morphism in $\mathcal{O}$; moreover, we say an object of $\mathcal{E}$ is *elementary* if it lies over an elementary object of $\mathcal{O}$.

Lemma 9.4. *Equipped with this data, $\mathcal{E}$ is an algebraic pattern, and $\pi\colon \mathcal{E} \to \mathcal{O}$ is a Segal morphism.*

Proof. The inert and active morphisms form a factorization system by [Lur17, Proposition 2.1.2.5], so we have defined an algebraic pattern structure on $\mathcal{E}$. To see that π is a Segal morphism it suffices to show that for $\overline{X} \in \mathcal{E}_X$ the induced functor

$$\mathcal{E}^{\mathrm{el}}_{\overline{X}/} \to \mathcal{O}^{\mathrm{el}}_{X/}$$

is coinitial. But this functor is an equivalence since for each inert morphism $X \to E$ with E elementary there is a unique cocartesian morphism with source $\overline{X}$ lying over it. $\square$

We now show that we can lift extendability along Segal fibrations:

Proposition 9.5. *Consider a commutative square*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{O} & \xrightarrow{f} & \mathcal{P}, \end{array}$$

where f is an extendable morphism of algebraic patterns, $p\colon \mathcal{E} \to \mathcal{O}$ and $q\colon \mathcal{F} \to \mathcal{P}$ are Segal fibrations, and F preserves cocartesian morphisms. Then F is extendable. Moreover, if $\mathcal{C}$ is f-admissible and either

(i) $\mathcal{P}^{\mathrm{el}}_{P/}$-limits distribute over η-colimits in $\mathcal{C}$ for all functors $\eta\colon \mathcal{P}^{\mathrm{el}}_{P/} \to \mathrm{Cat}_\infty$ and all $P \in \mathcal{P}$, or

(ii) p and q are left fibrations, and $\mathcal{P}^{\mathrm{el}}_{/P}$-limits distribute over η-colimits in $\mathcal{C}$ for all functors $\eta\colon \mathcal{P}^{\mathrm{el}}_{/P} \to \mathcal{S}$ and all $P \in \mathcal{P}$,

then $\mathcal{C}$ is F-admissible.

Proof. It is immediate from the definitions that F preserves inert and active morphisms. We now observe that F has unique lifting of inert morphisms. Given $\overline{O} \in \mathcal{E}$ lying over $O \in \mathcal{O}$, and an inert morphism $\overline{\epsilon}\colon F(\overline{O}) \rightarrowtail \overline{P}$ in $\mathcal{F}$, lying over $\epsilon\colon f(O) \rightarrowtail P$ in $\mathcal{P}$, there exists a unique inert morphism $\gamma\colon O \rightarrowtail O'$ such that $f(\gamma) \simeq \epsilon$, since f is extendable. Since inert morphisms in $\mathcal{E}$ are cocartesian, there exists a unique inert morphism $\overline{\gamma}\colon \overline{O} \rightarrowtail \overline{O}'$ lying over γ. Moreover, as F preserves cocartesian morphisms, the morphism $F(\overline{\gamma})$ is the unique inert morphism over ϵ with source $F(\overline{O})$, i.e. $F(\overline{\gamma}) \simeq \overline{\epsilon}$, and since cocartesian morphisms are unique, $\overline{\gamma}$ is the unique inert morphism that maps to $\overline{\epsilon}$.

For every active morphism $\overline{\phi}\colon F(\overline{O}) \rightsquigarrow \overline{P}$ lying over $\phi\colon f(O) \rightsquigarrow P$, equivalences of the type $\mathcal{E}^{\mathrm{el}}_{\overline{X}/} \simeq \mathcal{O}^{\mathrm{el}}_{X/}$ imply that $\mathcal{E}^{\mathrm{el}}(\overline{\phi}) \to \mathcal{E}^{\mathrm{el}}_{\overline{O}/}$ is equivalent to $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$, hence condition (3) in Definition 7.7 follows immediately from f being extendable. It remains to prove condition (2). For $\overline{P} \in \mathcal{F}$ lying over $P \in \mathcal{P}$ and $\overline{\epsilon}\colon \overline{P} \rightarrowtail \overline{P}'$ an inert morphism in $\mathcal{F}$ lying over $\epsilon\colon P \rightarrowtail P'$, we have a functor

$$\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \to \mathcal{E}^{\mathrm{act}}_{/\overline{P}'},$$

which fits in a commutative square

$$\begin{array}{ccc} \mathcal{E}^{\mathrm{act}}_{/\overline{P}} & \longrightarrow & \mathcal{E}^{\mathrm{act}}_{/\overline{P}'} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{act}}_{/P} & \longrightarrow & \mathcal{O}^{\mathrm{act}}_{/P'}. \end{array}$$

We claim that here the vertical functors are cocartesian fibrations, and the top horizontal functor preserves cocartesian morphisms. The functor

$$\mathcal{E}_{/\overline{P}} := \mathcal{E} \times_{\mathcal{F}} \mathcal{F}_{/\overline{P}} \to \mathcal{O} \times_{\mathcal{P}} \mathcal{P}_{/P} =: \mathcal{O}_{/P}$$

is a fibre product of cocartesian fibrations along functors that preserve cocartesian morphisms, hence it is again a cocartesian fibration. We can write $\mathcal{E}^{\mathrm{act}}_{/\overline{P}}$ as a pullback $\mathcal{E}_{/\overline{P}} \times_{\mathcal{O}_{/P}} \mathcal{O}^{\mathrm{act}}_{/P}$, hence $\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \to \mathcal{O}^{\mathrm{act}}_{/P}$ is a pullback of a cocartesian fibration and so is itself cocartesian. Moreover, a morphism in $\mathcal{E}^{\mathrm{act}}_{/\overline{P}}$ is cocartesian if and only if its image in $\mathcal{E}$ is cocartesian (since the functor $\mathcal{F}_{/\overline{P}} \to \mathcal{F}$ detects cocartesian morphisms, by [Lur09, Proposition 2.4.3.2]). Since inert morphisms are cocartesian, this implies that the top horizontal functor preserves cocartesian morphisms by the 3-for-2 property of cocartesian morphisms ([Lur09, Proposition 2.4.1.7]).

For $\overline{P} \in \mathcal{F}$ we therefore have a commutative square

$$\begin{array}{ccc} \mathcal{E}^{\mathrm{act}}_{/\overline{P}} & \longrightarrow & \lim_{\alpha\colon P \rightarrowtail E \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{act}}_{/P} & \longrightarrow & \lim_{\alpha\colon P \rightarrowtail E \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E}. \end{array}$$

where the vertical functors are cocartesian fibrations, the top horizontal functor preserves cocartesian morphisms, and the bottom horizontal functor is cofinal, since f is extendable. Our goal is to show that the top horizontal functor is cofinal. Since pullbacks of cofinal functors along cocartesian fibrations are cofinal by [Lur09, Proposition 4.1.2.15], it suffices to show that the square is cartesian, which in this situation is equivalent to the functor on fibres being an equivalence.

Since the fibration $\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \to \mathcal{O}^{\mathrm{act}}_{/P}$ is a fibre product, its fibre at $(O, \psi\colon f(O) \rightsquigarrow P)$ is the fibre product $\mathcal{E}_O \times_{\mathcal{F}_{f(O)}} (\mathcal{F}^{\mathrm{act}}_{/\overline{P}})_\psi$; since $\mathcal{F}$ is cocartesian over $\mathcal{P}$, we can use the cocartesian pushforward over ψ to identify this with a fibre product $\mathcal{E}_O \times_{\mathcal{F}_P} \mathcal{F}_{P/\overline{P}}$ over the composite functor $\mathcal{E}_O \to \mathcal{F}_{f(O)} \xrightarrow{\psi_!} \mathcal{F}_P$.

If ψ is active, then as f is extendable and $\mathcal{E} \to \mathcal{O}$ is a Segal fibration we have an equivalence

$$\mathcal{E}_O \xrightarrow{\sim} \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}_{\alpha_! O}$$

by Remark 7.10. Putting this together with the equivalence $\mathcal{F}_{P/\overline{P}} \xrightarrow{\sim} \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{F}_{E/\alpha_!\overline{P}}$ (and similarly for $\mathcal{F}_P$) we get

$$(\mathcal{E}^{\mathrm{act}}_{/\overline{P}})_{(O,\psi)} \xrightarrow{\sim} \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}})_{(E,\psi_\alpha)},$$

i.e. the functor we get on fibres is indeed an equivalence, which completes the proof that F is extendable.

For admissibility, observe that since $\lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}} \to \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E}$ is a cocartesian fibration, if we compute the colimit of a functor Φ over its source in two stages using the left Kan extension along this functor, we get

$$\operatorname*{colim}_{\lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}}} \Phi \simeq \operatorname*{colim}_{(\omega_\alpha) \in \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E}} \operatorname*{colim}_{\lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}})_{\omega_\alpha}} \Phi,$$

from which we see that F-admissibility follows from f-admissibility plus either (i) or (ii). □

Definition 9.6. Let $\mathcal{O}$ be an algebraic pattern. A *weak Segal $\mathcal{O}$-fibration* is a functor $p\colon \mathcal{E} \to \mathcal{O}$ such that:

(1) For every object $\overline{X}$ in $\mathcal{E}$ lying over $X \in \mathcal{O}$ and every inert morphism $i\colon X \to Y$ in $\mathcal{O}$ there exists a p-cocartesian morphism $\bar{\imath}\colon \overline{X} \to \overline{Y}$ lying over i.

(2) For every object $X \in \mathcal{O}$, the functor

$$\mathcal{E}_X \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathcal{E}_E,$$

induced by the cocartesian morphisms over inert maps, is an equivalence.

(3) Given $\overline{X}$ in $\mathcal{E}_X$, choose a cocartesian lift $\xi\colon (\mathcal{O}^{\mathrm{el}}_{X/})^{\triangleleft} \to \mathcal{E}$ of the diagram of inert morphisms from X in $\mathcal{O}$, taking $-\infty$ to $\overline{X}$. Then for any $Y \in \mathcal{O}$ and $\overline{Y} \in \mathcal{E}_Y$, the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(\overline{Y}, \overline{X}) & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}_{\mathcal{E}}(\overline{Y}, \xi(E)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{O}}(Y, X) & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}_{\mathcal{O}}(Y, E) \end{array}$$

is cartesian.

Remark 9.7. Condition (3) in the definition can be rephrased as: For every map $\phi\colon Y \to X$ in $\mathcal{O}$, the natural map

$$\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y}, \overline{X}) \to \lim_{\alpha\colon X \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}^{\alpha\phi}_{\mathcal{E}}(\overline{Y}, \alpha_!\overline{X})$$

is an equivalence, where $\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y}, \overline{X})$ denotes the fibre at ϕ of $\mathrm{Map}_{\mathcal{E}}(\overline{Y}, \overline{X}) \to \mathrm{Map}_{\mathcal{O}}(Y, X)$. If ϕ is active, let $Y \overset{\alpha_Y}{\rightarrowtail} Y_\alpha \overset{\phi_\alpha}{\rightsquigarrow} E$ denote the inert–active factorization of $Y \overset{\phi}{\rightsquigarrow} X \overset{\alpha}{\rightarrowtail} E$, then combining this equivalence with the cocartesian morphisms $\overline{Y} \rightarrowtail \alpha_{Y,!}\overline{Y}$ over α_Y we obtain an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y}, \overline{X}) \simeq \lim_{\alpha\colon X \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}^{\phi_\alpha}_{\mathcal{E}}(\alpha_{Y,!}\overline{Y}, \alpha_!\overline{X}).$$

Examples 9.8.

(i) A weak Segal $\mathbb{F}_*^{\flat}$-fibration is a symmetric ∞-operad, and a weak Segal $\mathbb{F}_*^{\natural}$-fibration is a generalized ∞-operad, in the sense of [Lur17].

(ii) A weak Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-fibration is a non-symmetric ∞-operad, and a weak Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-fibration is a generalized non-symmetric ∞-operad, as considered in [GH15].

(iii) If Φ is a perfect operator category and $\Lambda(\Phi)$ is its Leinster category, then a weak Segal $\Lambda(\Phi)^{\flat}$-fibration is a Φ-∞-operad, in the sense of [Bar18], and weak Segal $\Lambda(\Phi)^{\natural}$-fibrations are the natural extension of generalized ∞-operads to *generalized* Φ-∞-*operads*.

(iv) Weak Segal $\mathbf{\Theta}_n^{\mathrm{op},\natural}$-fibrations can be viewed as an ∞-categorical analogue of the n-*operads* of Batanin [Bat98].

Definition 9.9. Suppose $\mathcal{O}$ is an algebraic pattern, and $\pi\colon \mathcal{E} \to \mathcal{O}$ is a weak Segal $\mathcal{O}$-fibration. We say a morphism in $\mathcal{E}$ is *inert* if it is cocartesian and lies over an inert morphism in $\mathcal{O}$, and *active* if it lies over an active morphism in $\mathcal{O}$; moreover, we say an object of $\mathcal{E}$ is *elementary* if it lies over an elementary object of $\mathcal{O}$.

Lemma 9.10. *Equipped with this data, $\mathcal{E}$ is an algebraic pattern, and $\pi\colon \mathcal{E} \to \mathcal{O}$ is a Segal morphism.*

Proof. As Lemma 9.4. □

Remark 9.11. A cocartesian fibration $\mathcal{E} \to \mathcal{O}$ is a Segal fibration if and only if it is a weak Segal fibration.

Remark 9.12. Suppose $\mathcal{E} \to \mathcal{O}$ and $\mathcal{F} \to \mathcal{O}$ are weak Segal fibrations. Then a morphism $\mathcal{E} \to \mathcal{F}$ over $\mathcal{O}$ is a Segal morphism if and only if it preserves inert morphisms.

Remark 9.13. Let $\mathrm{Cat}^{\mathrm{WSF}}_{\infty/\mathcal{O}}$ denote the subcategory of $\mathrm{Cat}_{\infty/\mathcal{O}}$ whose objects are the weak Segal fibrations and whose morphisms are those that preserve inert morphisms. This ∞-category is described by a *categorical pattern* in the sense of [Lur17, §B], and so arises from a combinatorial model category by [Lur17, Theorem B.0.20]. It follows that $\mathrm{Cat}^{\mathrm{WSF}}_{\infty/\mathcal{O}}$ is a presentable ∞-category.

For weak Segal fibrations we can prove a weaker version of Proposition 9.5; for this we need the following consequence of extendability, which we learned from Roman Kositsyn:

Lemma 9.14. *Let $\mathcal{O}$ be an extendable pattern. Then the functor $\mathcal{O} \to \mathrm{Cat}_\infty$ taking O to $\mathcal{O}^{\mathrm{act}}_{/O}$ from Remark 7.5 is a Segal $\mathcal{O}$-∞-category. In particular, for any active maps $\phi\colon X \rightsquigarrow O$, $\psi\colon Y \rightsquigarrow O$ in $\mathcal{O}$, the morphism of mapping spaces*

$$\mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\phi, \psi) \to \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\phi_\alpha, \psi_\alpha)$$

is an equivalence.

Proof. We must show that for any $O \in \mathcal{O}$, the functor

$$\mathcal{O}^{\mathrm{act}}_{/O} \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}$$

is an equivalence; to see this it suffices to check that it is an equivalence on underlying ∞-groupoids and is fully faithful. The map on underlying ∞-groupoids is the map $\mathrm{Act}_{\mathcal{O}}(O) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(E)$, which is an equivalence by assumption since $\mathcal{O}$ is extendable. Given active maps $\phi\colon X \rightsquigarrow O$, $\psi\colon Y \rightsquigarrow O$, the morphism of mapping spaces

$$\mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\phi, \psi) \to \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\phi_\alpha, \psi_\alpha)$$

fits in a commutative cube

$$\begin{array}{ccccccc}
\mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\phi, \psi) & & \longrightarrow & & \mathrm{Act}_{\mathcal{O}}(Y) & & \\
 & \searrow & & & \downarrow & \searrow & \\
\downarrow & & \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\phi_\alpha, \psi_\alpha) & & \longrightarrow & & \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(\alpha_! Y) \\
\{\phi\} & & \longrightarrow & & \mathrm{Act}_{\mathcal{O}}(X) & & \downarrow \\
 & \searrow & \downarrow & & & \searrow & \\
 & & \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \{\phi_\alpha\} & & \longrightarrow & & \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(\alpha_! X),
\end{array}$$

where the back and front faces are cartesian. Since $\mathcal{O}$ is extendable, we can apply the "extended Segal condition" of Remark 7.10 to $\mathrm{Act}_{\mathcal{O}}(-)$ and conclude the horizontal morphisms in the right-hand square are equivalences. It follows that the map on fibres in the left square is also an equivalence, as required. □

Using this we can prove the following key observation:

Proposition 9.15. *Suppose $\mathcal{O}$ is an extendable algebraic pattern. Consider a commutative triangle*

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{f} & \mathcal{F} \\
 & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\
 & \mathcal{O}, &
\end{array}$$

where p and q are weak Segal fibrations and f preserves inert morphisms. Then for any $F \in \mathcal{F}$ the functor

$$\mathcal{E}^{\mathrm{act}}_{/F} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_! F}$$

is an equivalence.

Proof. For any active morphisms $\phi\colon Y \rightsquigarrow X$, $\psi\colon X \rightsquigarrow q(F)$ in $\mathcal{O}$ and $\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}$ the inert-active factorization gives a commutative diagram

$$\begin{array}{ccccc}
Y & \overset{\phi}{\rightsquigarrow} & X & \rightsquigarrow & q(F) \\
{\scriptstyle \alpha_Y}\downarrow & & \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha} \\
Y_\alpha & \overset{\phi_\alpha}{\rightsquigarrow} & X_\alpha & \rightsquigarrow & E.
\end{array}$$

By combining Remark 7.10 (the "generalized Segal condition") with the argument of Remark 9.7 we then get an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y}, \overline{X}) \xrightarrow{\sim} \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathrm{Map}^{\phi_\alpha}_{\mathcal{E}}(\alpha_{Y,!}\overline{Y}, \alpha_{X,!}\overline{X}).$$

Thus in the commutative square

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{E}^{\mathrm{act}}_{/F}}(\overline{Y}, \overline{X}) & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathrm{Map}_{\mathcal{E}^{\mathrm{act}}_{/\alpha_! F}}(\alpha_{Y,!}\overline{Y}, \alpha_{X,!}\overline{X}) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/qF}}(Y, X) & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(Y_\alpha, X_\alpha),
\end{array}$$

the map on fibres is an equivalence for all $\phi\colon Y \rightsquigarrow X$, which means the square is cartesian. The bottom horizontal morphism is an equivalence by Lemma 9.14 since $\mathcal{O}$ is extendable. Hence we see that the functor $\mathcal{E}^{\mathrm{act}}_{/F} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_! F}$ induces equivalences on mapping spaces, and so is fully faithful. To see that this functor is also essentially surjective, consider the commutative square of ∞-groupoids

$$\begin{array}{ccc} (\mathcal{E}^{\mathrm{act}}_{/F})^{\simeq} & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_! F})^{\simeq} \\ \downarrow & & \downarrow \\ (\mathcal{O}^{\mathrm{act}}_{/q(F)})^{\simeq} & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} (\mathcal{O}^{\mathrm{act}}_{/E})^{\simeq}; \end{array}$$

we want to show that the top horizontal morphism is an equivalence. The bottom horizontal morphism is an equivalence by assumption, since $\mathcal{O}$ is extendable; it therefore suffices to show the map on fibres over $\phi\colon O \to q(F)$ is an equivalence. The fibre $(\mathcal{E}^{\mathrm{act}}_{/F})^{\simeq}_{\phi}$ we can identify with $\mathcal{E}^{\simeq}_{O} \times_{\mathcal{F}^{\simeq}_{O}} (\mathcal{F}_{/F})^{\simeq}_{\phi}$. By condition (2) in Definition 9.6 we have an equivalence $\mathcal{E}^{\simeq}_{O} \simeq \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{E}^{\simeq}_{E}$, and similarly for $\mathcal{F}$. Moreover, condition (3) implies that in the commutative square

$$\begin{array}{ccc} (\mathcal{F}_{/F})^{\simeq}_{\phi} & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} (\mathcal{F}_{/\alpha_! F})^{\simeq}_{\phi_\alpha} \\ \downarrow & & \downarrow \\ \mathcal{F}^{\simeq}_{O} & \xrightarrow{\sim} & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{F}^{\simeq}_{E}, \end{array}$$

the map on fibres over each object of $\mathcal{F}^{\simeq}_{O}$ is an equivalence, hence the top horizontal morphism is an equivalence. Since limits commute, it follows that we have an equivalence

$$(\mathcal{E}^{\mathrm{act}}_{/F})^{\simeq}_{\phi} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_! F})^{\simeq}_{\phi_\alpha},$$

which completes the proof. □

Corollary 9.16. *Suppose $\mathcal{O}$ is an extendable algebraic pattern. Then any morphism between weak Segal fibrations over $\mathcal{O}$ that preserves inert morphisms is extendable.*

Proof. Suppose $\mathcal{E}$ and $\mathcal{F}$ are weak Segal fibrations over $\mathcal{O}$. Then any morphism of algebraic patterns $f\colon \mathcal{E} \to \mathcal{F}$ over $\mathcal{O}$ has unique lifting of inert morphisms, as an inert morphism is uniquely determined by its source and its image in $\mathcal{O}$. Moreover, f satisfies condition (2) in Definition 7.7 by Proposition 9.15, and condition (3) reduces to the extendability of $\mathcal{O}$. □

Corollary 9.17. *Suppose $\mathcal{O}$ is an extendable algebraic pattern, and $\mathcal{E} \to \mathcal{O}$ is a weak Segal fibration. Then $\mathcal{E}$ is extendable.*

Proof. The restriction $\mathcal{E}^{\mathrm{int}} \to \mathcal{O}$ is also a weak Segal fibration, hence we can apply Corollary 9.16 to the inclusion $\mathcal{E}^{\mathrm{int}} \to \mathcal{E}$. □

Example 9.18. The pattern $\mathbb{F}^{\flat}_{*}$ is extendable. Our previous results therefore specialize to tell us that any morphism $f\colon \mathcal{O} \to \mathcal{P}$ of symmetric ∞-operads is extendable. If $\mathcal{C}$ is a cocomplete $\times$-admissible ∞-category, we conclude that left Kan extension along f restricts to a functor $f_!\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$, given by the formula

$$(f_! F)(P) \simeq \operatorname*{colim}_{O \in \mathcal{O}^{\mathrm{act}}_{/P}} F(O).$$

Note that this agrees with the formula for operadic left Kan extensions from [Lur17, §3.1.2], though in our case the target must be a cartesian symmetric monoidal ∞-category.

Example 9.19. Let us spell out the description of free Segal $\mathcal{O}$-objects for a symmetric ∞-operad $\mathcal{O} \to \mathbb{F}_*$ in a bit more detail. We can identify $\mathcal{O}^{\mathrm{el}}$ with the ∞-groupoid $\mathcal{O}^{\simeq}_{\langle 1 \rangle}$, and for $X \in \mathcal{O}_{\langle 1 \rangle}$ the space $\mathrm{Act}_{\mathcal{O}}(X)$ decomposes as $\coprod_{n=0}^{\infty} \mathrm{Act}_{\mathcal{O}}(X)_n$, where $\mathrm{Act}_{\mathcal{O}}(X)_n$ is the space of morphisms to X

in $\mathcal{O}$ lying over the unique active morphism $\langle n\rangle \to \langle 1\rangle$ in $\mathbb{F}_*$. If $\mathcal{C}$ is a cocomplete $\times$-admissible ∞-category, then for $F \in \operatorname{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C})$ our formula for the free Segal $\mathcal{O}$-object monad $T_{\mathcal{O}}$ gives:

$$(T_{\mathcal{O}}F)(X) \simeq \coprod_{n=0}^{\infty} \operatorname*{colim}_{(Y_1,\ldots,Y_n)\in \operatorname{Act}_{\mathcal{O}}(X)_n} F(Y_1)\times\cdots\times F(Y_n).$$

If $\mathcal{O}^{\simeq}_{\langle 1\rangle}$ is contractible, we can identify the space $\mathcal{O}(n)$ of n-ary operations with the fibre of $\operatorname{Act}_{\mathcal{O}}(X) \to \operatorname{Act}_{\mathbb{F}_*}(\langle 1\rangle) \simeq B\Sigma_n$, and so rewrite this as the familiar formula

$$T_{\mathcal{O}}C \simeq \coprod_{n=0}^{\infty} \operatorname*{colim}_{B\Sigma_n} \operatorname*{colim}_{\mathcal{O}(n)} C\times\cdots\times C \simeq \coprod_{n=0}^{\infty} \left(\mathcal{O}(n)\times C^{\times n}\right)_{h\Sigma_n}$$

for $C \in \mathcal{C} \simeq \operatorname{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C})$.

Remark 9.20. Our description of free algebras differs from what Lurie calls "free algebras" in [Lur17, Section 3.1.3], because Lurie defines these to be given by operadic Kan extension along the inclusion $\mathcal{O}\times_{\mathbb{F}_*}\mathbb{F}_*^{\text{int}} \to \mathcal{O}$ where the source is the subcategory containing *all* morphisms in $\mathcal{O}$ lying over inert morphisms in $\mathbb{F}_*$, not just the cocartesian ones. Lurie's construction amounts to specifying the unary operations in advance and freely adding the n-ary operations for $n > 1$, while our version adds all the operations freely.

Example 9.21. The pattern $\mathbf{\Delta}^{\text{op},\flat}$ is also extendable. The analogues of Examples 9.18 and 9.19 hence also hold for non-symmetric ∞-operads.

Example 9.22. The patterns $\mathbb{F}_*^{\natural}$ and $\mathbf{\Delta}^{\text{op},\natural}$ are also extendable. Hence any morphism of *generalized* symmetric or non-symmetric ∞-operads is extendable.

Remark 9.23. Suppose

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\quad f\quad} & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & \mathbb{F}_* & \end{array}$$

is a morphism of generalized symmetric ∞-operads. Then the previous example does *not* say that we can compute free Segal $\mathcal{P}^{\flat}$-objects on Segal $\mathcal{O}^{\flat}$-objects, as $f_!$ generally will not restrict to a functor between these. In the definition of extendability, condition (1) is still automatic (as the inert morphisms in $\mathbb{F}_*^{\natural}$ and $\mathbb{F}_*^{\flat}$ are the same), while condition (3) reduces to $\mathbb{F}_*^{\flat}$ being extendable. Thus the morphism $f^{\flat}\colon \mathcal{O}^{\flat} \to \mathcal{P}^{\flat}$ is extendable if and only if for all P over $\langle n\rangle$ in $\mathbb{F}_*$ the functor

$$\mathcal{O}^{\text{act}}_{/P} \to \prod_{i=1}^{n} \mathcal{O}^{\text{act}}_{/\rho_{i,!}P}$$

is cofinal, where $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ is as in the introduction.

10. Polynomial Monads from Patterns

In this section we introduce the notion of *polynomial monad* on an ∞-category of presheaves, and prove that the free Segal $\mathcal{O}$-space monad for an extendable pattern $\mathcal{O}$ is polynomial. Moreover, we show that this is compatible with Segal morphisms of algebraic patterns, yielding a functor

$$\mathfrak{M}\colon \operatorname{AlgPatt}^{\text{Seg}}_{\text{ext}} \to \operatorname{PolyMnd}$$

between the subcategory of AlgPatt consisting of extendable patterns and Segal morphisms, and an ∞-category of polynomial monads. We start by introducing some terminology:

Definition 10.1. A natural transformation $\phi\colon F \to G$ is *cartesian* if the naturality squares

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \phi_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

are all cartesian.

Definition 10.2. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *local right adjoint* if for every $c \in \mathcal{C}$ the induced functor $\mathcal{C}_{/c} \to \mathcal{D}_{/Fc}$ is a right adjoint.

Lemma 10.3. *If $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, then the following are equivalent for a functor $F\colon \mathcal{C} \to \mathcal{D}$:*

(1) F is accessible and preserves weakly contractible limits.
(2) F is a local right adjoint.
(3) The functor $F_{/*}\colon \mathcal{C} \to \mathcal{D}_{/F(*)}$ has a left adjoint.

Proof. The equivalence of (1) and (2) was proved as [GHK17, Proposition 2.2.8]. Since (3) is a special case of (2), it remains to prove that (3) implies (1). By the adjoint functor theorem [Lur09, Corollary 5.5.2.9], it follows from (3) that $F_{/*}$ is accessible and preserves limits. The forgetful functor $\mathcal{D}_{/F(*)} \to \mathcal{D}$ preserves and creates all colimits, as well as weakly contractible limits, by [GHK17, Lemma 2.2.7], so this implies that F itself is accessible and preserves weakly contractible limits. □

Definition 10.4. A monad T is *cartesian* if its multiplication and unit are cartesian natural transformations, and is *polynomial* if it is cartesian and the underlying endofunctor is a local right adjoint.

Remark 10.5. For ordinary categories, our notion of polynomial monads is the same as the *strongly cartesian* monads considered in [BMW12]. For monads on ∞-categories of the form $\mathcal{S}_{/X}$ for $X \in \mathcal{S}$, we recover the polynomial monads studied in [GHK17] (see Theorem 2.2.3 there), which is our reason for adopting this terminology.

Proposition 10.6. *If $\mathcal{O}$ is an extendable algebraic pattern, then the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$ on $\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$ is a polynomial monad.*

Proof. Since $\mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S})$ is an accessible localization of $\mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S})$, the inclusions $\mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S})$ are accessible and preserve limits. The endofunctor $T_{\mathcal{O}}$ of $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ factors as a composite

$$\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}) \xrightarrow{j_{\mathcal{O},!}} \mathrm{Fun}(\mathcal{O}, \mathcal{S}) \xrightarrow{j_{\mathcal{O}}^{*}} \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}),$$

where the composite lands in the subcategory $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$. To see that $T_{\mathcal{O}}$ is a local right adjoint it suffices to show that the three functors in this composition are accessible and preserve weakly contractible limits. All three functors are clearly accessible and except for $j_{\mathcal{O},!}$ they preserve limits. It therefore remains to show that $j_{\mathcal{O},!}$ preserves weakly contractible limits. By Lemma 7.2 for $O \in \mathcal{O}$ and $F \in \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S})$, the value of $j_{\mathcal{O},!}F$ at O is $\mathrm{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} F(X)$. Since $\mathrm{Act}_{\mathcal{O}}(O) = (\mathcal{O}^{\mathrm{int}})^{\mathrm{act}}_{/O}$ is an ∞-groupoid, this factors through the forgetful functor $\mathcal{S}_{/\mathrm{Act}_{\mathcal{O}}(O)} \to \mathcal{S}$, which detects weakly contractible limits by [GHK17, Lemma 2.2.7]. It therefore suffices to show that the functor $\mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}) \to \mathcal{S}_{/\mathrm{Act}_{\mathcal{O}}(O)}$ taking F to $\mathrm{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} F(X) \to \mathrm{Act}_{\mathcal{O}}(O)$ preserves weakly contractible limits. But this factors as restriction along $\mathrm{Act}_{\mathcal{O}}(O) \to \mathcal{O}^{\mathrm{int}}$, which certainly preserves limits, followed by the colimit functor $\mathrm{Fun}(\mathrm{Act}_{\mathcal{O}}(O), \mathcal{S}) \to \mathcal{S}_{/\mathrm{Act}_{\mathcal{O}}(O)}$, which is an equivalence.

Next, we show that the multiplication transformation $T^2_{\mathcal{O}} \to T_{\mathcal{O}}$ is cartesian. For $O \in \mathcal{O}$, we have an equivalence

$$(T^2_{\mathcal{O}}F)(O) \simeq \operatorname*{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} (T_{\mathcal{O}}F)(O) \simeq \operatorname*{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} \operatorname*{colim}_{Y \in \mathrm{Act}_{\mathcal{O}}(X)} F(Y) \simeq \operatorname*{colim}_{(Y \rightsquigarrow X \rightsquigarrow O) \in \mathrm{Act}^2_{\mathcal{O}}(O)} F(Y),$$

where $\mathrm{Act}^2_{\mathcal{O}}(O) \to \mathrm{Act}_{\mathcal{O}}(O)$ is the left fibration for the functor taking $X \rightsquigarrow O$ to $\mathrm{Act}_{\mathcal{O}}(X)$. We then have an identification

$$\mathrm{Act}^2_{\mathcal{O}}(O) \simeq \{Y \overset{g}{\rightsquigarrow} X \overset{f}{\rightsquigarrow} O : f, g \text{ active}\}$$

under which the multiplication transformation $T^2_{\mathcal{O}}F(X) \to T_{\mathcal{O}}F(X)$ is the morphism induced on colimits by the map $\mathrm{Act}^2_{\mathcal{O}}(O) \to \mathrm{Act}_{\mathcal{O}}(O)$ given by composition of active morphisms. Given $F \to G$, we want to show that the square

$$\begin{array}{ccc}
\mathrm{colim}_{(Y \rightsquigarrow X \rightsquigarrow O) \in \mathrm{Act}^2_{\mathcal{O}}(O)} F(Y) & \longrightarrow & \mathrm{colim}_{(Y \rightsquigarrow X \rightsquigarrow O) \in \mathrm{Act}^2_{\mathcal{O}}(O)} G(Y) \\
\downarrow & & \downarrow \\
\mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} F(Y) & \longrightarrow & \mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} G(Y)
\end{array}$$

is cartesian. To see this it suffices to show that the square on fibres over $(Y \overset{f}{\rightsquigarrow} O) \in \mathrm{Act}_{\mathcal{O}}(O)$ is cartesian. The fibre $(T^2_{\mathcal{O}}F(X))_f$ we can identify with the colimit over the fibre

$$\mathrm{Act}^2_{\mathcal{O}}(O)_f \simeq \left\{ \begin{array}{ccc} & X & \\ \nearrow & & \searrow \\ Y & \xrightarrow{\quad f \quad} & O \end{array} \right\}$$

of the *constant* functor with value $F(Y)$. The square of fibres is therefore

$$\begin{array}{ccc}
\mathrm{Act}^2_{\mathcal{O}}(O)_f \times F(Y) & \longrightarrow & \mathrm{Act}^2_{\mathcal{O}}(O)_f \times G(Y) \\
\downarrow & & \downarrow \\
F(Y) & \longrightarrow & G(Y),
\end{array}$$

which is indeed cartesian.

The value of the unit transformation $F(O) \to T_{\mathcal{O}}F(O)$ is similarly induced by the map $\{\mathrm{id}_O\} \to \mathrm{Act}_{\mathcal{O}}(O)$. To see that the unit transformation is cartesian we must show that for $F \to G$ the square

$$\begin{array}{ccc}
F(O) & \longrightarrow & G(O) \\
\downarrow & & \downarrow \\
\mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} F(Y) & \longrightarrow & \mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} G(Y)
\end{array}$$

is cartesian. It again suffices to consider the square of fibres over $(X \overset{f}{\rightsquigarrow} O) \in \mathrm{Act}_{\mathcal{O}}(O)$. The fibre of $\{\mathrm{id}_O\} \to \mathrm{Act}_{\mathcal{O}}(O)$ at f is the space

$$P_f := \mathrm{Map}_{\mathrm{Act}_{\mathcal{O}}(O)}(\mathrm{id}_O, f)$$

of paths from $\mathrm{id}_{\mathcal{O}}$ to f in $\mathrm{Act}_{\mathcal{O}}(O)$ (which is empty if id_O and f are not equivalent), and the square of fibres is

$$\begin{array}{ccc} P_f \times F(O) & \longrightarrow & P_f \times G(O) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & G(X), \end{array}$$

which is cartesian as required. □

Remark 10.7. We can regard polynomial monads as being the monads in an $(\infty, 2)$-category whose objects are presheaf ∞-categories, whose morphisms are local right adjoints, and whose 2-morphisms are cartesian transformations. The natural morphisms between polynomial monads are then the lax morphisms of monads in this $(\infty, 2)$-category. If T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$ and S is a polynomial monad on $\mathcal{S}^{\mathcal{J}}$, then by the results of [Hau20] these correspond to commutative squares

$$\begin{array}{ccc} \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) & \xrightarrow{\Phi} & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \\ {\scriptstyle U_S}\downarrow & & \downarrow{\scriptstyle U_T} \\ \mathcal{S}^{\mathcal{J}} & \xrightarrow{f^*} & \mathcal{S}^{\mathcal{I}}, \end{array}$$

for some functor $f\colon \mathcal{I} \to \mathcal{J}$, such that the mate transformation

$$F_T f^* \to \Phi F_S$$

is cartesian. Noting the contravariance here, this motivates the following definition of an ∞-category of polynomial monads:

Definition 10.8. Consider the pullback

$$\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \times_{\widehat{\mathrm{Cat}}_\infty} \mathrm{Cat}_\infty$$

along $\mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \to \widehat{\mathrm{Cat}}_\infty$ and $\mathcal{S}^{(-)}\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. We write $\mathrm{PolyMnd}^{\mathrm{op}}$ for the subcategory of this pullback whose objects are the monadic right adjoints of polynomial monads, and whose morphisms are commutative squares whose mate transformations are cartesian.

Remark 10.9. Note that since U_T detects pullbacks, the mate transformation above is cartesian if and only if the transformation

$$Tf^* \to f^* S$$

obtained by composing with U_T is cartesian.

Next, we observe that any Segal morphism between extendable patterns gives a morphism of polynomial monads:

Proposition 10.10. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a Segal morphism between extendable patterns. Then the mate transformation*

$$j_{\mathcal{O},!} f^{\mathrm{int},*} \to f^* j_{\mathcal{P},!}$$

of functors $\mathrm{Seg}_{\mathcal{P}^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ is cartesian.

Proof. We have to show that for every morphism $\Phi \to \Psi$ the commutative square

$$\begin{array}{ccc} j_{\mathcal{O},!} f^{\mathrm{int},*}\Phi & \longrightarrow & f^* j_{\mathcal{P},!}\Phi \\ \downarrow & & \downarrow \\ j_{\mathcal{O},!} f^{\mathrm{int},*}\Psi & \longrightarrow & f^* j_{\mathcal{P},!}\Psi \end{array}$$

is cartesian in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$. Since $\mathrm{Seg}_{\mathcal{P}^{\mathrm{int}}}(\mathcal{S})$ has a terminal object it suffices to consider $\Psi \simeq *$, in which case we obtain the commutative square

$$\begin{array}{ccc} \mathrm{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(E)} \Phi(fX) & \longrightarrow & \mathrm{colim}_{Y \in \mathrm{Act}_{\mathcal{P}}(f(E))} \Phi(Y) \\ \downarrow & & \downarrow \\ \mathrm{Act}_{\mathcal{O}}(E) & \longrightarrow & \mathrm{Act}_{\mathcal{P}}(f(E)) \end{array}$$

after evaluating at an object $E \in \mathcal{O}^{\mathrm{el}}$. To show that this square is cartesian, it now suffices to observe that for every point $(X \to E) \in \mathrm{Act}_{\mathcal{O}}(E)$, the map on fibres is the identity $\Phi(fX) \to \Phi(fX)$. □

Definition 10.11. We let $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$ denote the subcategory of AlgPatt whose objects are the extendable patterns and whose morphisms are the Segal morphisms.

Corollary 10.12. *The functor* $\mathrm{AlgPatt}^{\mathrm{Seg}} \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)^{\mathrm{op}}$ *taking a pattern* $\mathcal{O}$ *to the monadic right adjoint* $U_{\mathcal{O}} \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$ *restricts to a functor* $\mathfrak{M} \colon \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}} \to \mathrm{PolyMnd}$. □

11. Generic Morphisms and the Nerve Theorem

In the previous section we saw that the free Segal space monad for any extendable pattern was a polynomial monad. Our next goal is to extract an extendable pattern from any polynomial monad. As a first step towards this, in this section we prove an ∞-categorical version of Weber's nerve theorem [Web07]; our proof was particularly inspired by that of Berger, Melliès, and Weber [BMW12].

We begin by defining *generic morphisms* with respect to a local right adjoint functor, and extend some basic observations about them from [Web04] to the ∞-categorical setting.

Definition 11.1. Suppose $F \colon \mathcal{C} \to \mathcal{D}$ is a local right adjoint functor between presentable ∞-categories. Let $L_* \colon \mathcal{D}_{/F(*)} \to \mathcal{C}$ be the left adjoint to $F_{/*} \colon \mathcal{C} \to \mathcal{D}_{/F(*)}$; we will abusively write L_*D for the value of L_* at an object $D \to F(*)$. For any morphism $D \xrightarrow{\phi} F(C)$ in $\mathcal{D}$, we can view ϕ as a morphism in $\mathcal{D}_{/F(*)}$ via the map $F(q) \colon F(C) \to F(*)$, where q is the unique morphism $C \to *$. We say ϕ is F-*generic* (or just *generic* if F is clear from context) if the adjoint morphism

$$L_*D \simeq L_*(F(q) \circ \phi) \to C$$

is an equivalence. (In other words, the generic morphisms are precisely the unit morphisms $D \to F_{/*}L_*D$.)

Remark 11.2. Using the universal property of the left adjoint, we can rephrase this definition purely in terms of F as follows: $\phi \colon D \to F(B)$ is F-generic if for every commutative square

$$\begin{array}{ccc} D & \xrightarrow{\psi} & F(A) \\ {\scriptstyle\phi}\downarrow & \nearrow & \downarrow{\scriptstyle F(\alpha)} \\ F(B) & \xrightarrow{F(\beta)} & F(*) \end{array}$$

there exists a unique morphism $\gamma \colon B \to A$ such that $F(\gamma) \circ \phi \simeq \psi$ and the equivalence in the square arises by combining this with the canonical equivalence $F(\alpha) \circ F(\gamma) \simeq F(\alpha\gamma) \simeq F(\beta)$ induced by $*$ being terminal. This is the version of the definition considered in [Web04].

Lemma 11.3. *Let* $\phi \colon D \to F(B)$ *be an* F*-generic morphism. Then given a commutative square*

$$\begin{array}{ccc} D & \xrightarrow{\psi} & F(A) \\ {\scriptstyle\phi}\downarrow & \nearrow & \downarrow{\scriptstyle F(\alpha)} \\ F(B) & \xrightarrow{F(\beta)} & F(X), \end{array}$$

there exists a unique commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & X & \end{array}$$

such that $F(\gamma)\circ\phi \simeq \psi$ and the equivalence in the square arises by combining this with the equivalence $F(\alpha) \circ F(\gamma) \simeq F(\alpha\gamma) \simeq F(\beta)$ given by applying F to the triangle.

Proof. The existence of a unique filler in the original square is equivalent to the existence of such a filler in the adjoint square

$$\begin{array}{ccc} L_X D & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{\beta} & X. \end{array}$$

Since F preserves pullbacks, if ξ denotes the unique morphism $X \to *$ we have a commutative square of right adjoints

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F_{/*}} & \mathcal{D}_{/F(*)} \\ {\scriptstyle\xi^*}\downarrow & & \downarrow{\scriptstyle F(\xi)^*} \\ \mathcal{C}_{/X} & \xrightarrow{F_{/X}} & \mathcal{D}_{/F(X)}. \end{array}$$

This induces a corresponding square of left adjoints

$$\begin{array}{ccc} \mathcal{D}_{/F(X)} & \xrightarrow{L_X} & \mathcal{C}_{/X} \\ {\scriptstyle F(\xi)_!}\downarrow & & \downarrow{\scriptstyle\xi_!} \\ \mathcal{D}_{/F(*)} & \xrightarrow{L_*} & \mathcal{C}. \end{array}$$

Thus $\xi_! L_X \simeq L_* F(\xi)_!$; since $\xi_!$ detects equivalences, we see that for $D \xrightarrow{\phi} F(B) \xrightarrow{F(\beta)} F(X)$ the adjoint morphism $L_X D \to B$ over X is equivalent to $L_* X \to B$ computed using the morphism $F(B) \to F(*)$ that is the image of $B \to *$, as this is the composite $F(B) \xrightarrow{F(\beta)} F(X) \xrightarrow{F(\xi)} F(*)$. Since ϕ is generic, it therefore follows that the map $L_X D \to B$ is also an equivalence, hence the unique filler arises from the composite $B \simeq L_X D \to A$. □

Remark 11.4. For any morphism $\phi\colon D \to F(C)$, if $\psi\colon L_* D \to C$ is the adjoint morphism, we can write ϕ as a composite

$$D \xrightarrow{\eta_D} F(L_* D) \xrightarrow{F(\psi)} F(C),$$

where η_D is the unit of the adjunction $L_* \dashv F_{/*}$. This is the *unique* factorization of ϕ as a generic morphism followed by a morphism in the image of F; we will often refer to this as the *generic–free factorization* of ϕ.

Lemma 11.5 (Cf. [Web04, Proposition 5.10])**.** *Suppose $F, G\colon \mathcal{C} \to \mathcal{D}$ are local right adjoint functors between presentable ∞-categories and $\phi\colon F \to G$ is a cartesian natural transformation. Then a morphism $f\colon D \to F(C)$ is F-generic if and only if the composite $D \to F(C) \to G(C)$ is G-generic.*

Proof. Since ϕ is a cartesian transformation, we have natural cartesian squares

$$\begin{array}{ccc} F(X) & \longrightarrow & G(X) \\ \downarrow & & \downarrow \\ F(*) & \xrightarrow{\phi(*)} & G(*). \end{array}$$

This means we can write $F_{/*}$ as the composite

$$\mathcal{C} \xrightarrow{G_{/*}} \mathcal{D}_{/G(*)} \xrightarrow{\phi(*)^*} \mathcal{D}_{/F(*)}.$$

But then the left adjoint $L_{*,F}$ of $F_{/*}$ is the composite

$$\mathcal{D}_{/F(*)} \xrightarrow{\phi(*)_!} \mathcal{D}_{/G(*)} \xrightarrow{L_{*,G}} \mathcal{C},$$

where $L_{*,G}$ denotes the left adjoint to $G_{/*}$. Given $f\colon D \to F(C)$, this means the adjoint morphism $L_{*,F}D \to C$ is the same as the adjoint morphism $L_{*,G}D \to C$ for the composite $D \to F(C) \to G(C)$. □

Lemma 11.6 (Cf. [Web04, Lemma 5.14])**.** *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are local right adjoint functors between presentable ∞-categories. If $f\colon D \to F(C)$ is F-generic and $g\colon E \to G(D)$ is G-generic, then the composite*

$$E \xrightarrow{g} G(D) \xrightarrow{G(f)} GF(C)$$

is GF-generic.

Proof. The functor $(GF)_{/*}$ factors in two steps as

$$\mathcal{C} \xrightarrow{F_{/*}} \mathcal{D}_{/F(*)} \xrightarrow{G_{/F(*)}} \mathcal{E}_{/GF(*)}.$$

The left adjoint is therefore also computed in two steps; to find the morphism adjoint to $G(f)g$ we first get the commutative diagram

$$\begin{array}{ccccc} L_{*,G}E & \xrightarrow{\sim} & D & \xrightarrow{f} & F(C) \\ & \searrow & \downarrow & \swarrow & \\ & & F(*), & & \end{array}$$

and then $L_{*,F}L_{*,G}E \xrightarrow{\sim} L_{*,F}D \xrightarrow{\sim} C$, which is an equivalence as required. □

Definition 11.7. Suppose $\mathcal{I}$ is a small ∞-category and T is a polynomial monad on the functor ∞-category $\mathcal{S}^{\mathcal{I}}$. We define $\mathcal{U}(T)^{\mathrm{op}}$ to be the full subcategory of $\mathcal{S}^{\mathcal{I}}$ spanned by the objects X that admit a generic morphism $I \to TX$ with $I \in \mathcal{I}^{\mathrm{op}}$ (regarded as an object of $\mathcal{S}^{\mathcal{I}}$ through the Yoneda embedding). We write $\mathcal{W}(T)^{\mathrm{op}}$ for the full subcategory of $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ spanned by the free T-algebras on the objects of $\mathcal{U}(T)$.

Remark 11.8. From the definition of generic morphisms it follows that we can equivalently describe the objects of $\mathcal{U}(T)^{\mathrm{op}}$ as those of the form L_*I for some $I \in \mathcal{I}^{\mathrm{op}}$ and some morphism $I \to T*$ in $\mathcal{S}^{\mathcal{I}}$.

Lemma 11.9. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$.*

(i) For any object $X \in \mathcal{S}^{\mathcal{I}}$, the unit map $X \to T(X)$ is generic.

(ii) If $X \xrightarrow{\phi} T(Y)$ and $Y \xrightarrow{\psi} T(Z)$ are generic morphisms, then the composite

$$X \xrightarrow{\phi} TY \xrightarrow{T\psi} T^2Z \xrightarrow{\mu_Z} TZ$$

is generic, where μ denotes the multiplication transformation of the monad.

Proof. Since T is a polynomial monad, the unit transformation $\mathrm{id} \to T$ is cartesian and so by Lemma 11.5 the unit map $X \to TX$ is generic for all X (since an id-generic map is precisely an equivalence).

The composite $X \xrightarrow{\phi} TY \xrightarrow{T\psi} T^2Z$ is T^2-generic by Lemma 11.6, and as the multiplication μ is a cartesian transformation this implies the composite of this with $\mu_Z\colon T^2Z \to TZ$ is T-generic by Lemma 11.5. □

Proposition 11.10. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$.*

(i) The full subcategory $\mathcal{U}(T)^{\mathrm{op}}$ contains $\mathcal{I}^{\mathrm{op}}$.

(ii) For any generic morphism $X \to TY$ with $X \in \mathcal{U}(T)^{\mathrm{op}}$, the object Y also lies in $\mathcal{U}(T)^{\mathrm{op}}$.

Proof. The unit map $I \to TI$ is generic by Lemma 11.9(i). Hence $I \to TI \to T*$ is a generic–free factorization, where the second map is the image under T of the unique map $I \to *$. This shows that I is in $\mathcal{U}(T)^{\mathrm{op}}$, which proves (i).

To prove (ii), observe that since X is in $\mathcal{U}(T)^{\mathrm{op}}$, we have a generic morphism $I \to TX$ with I in $\mathcal{I}^{\mathrm{op}}$. Then by Lemma 11.9(ii) the composite

$$I \to TX \to T^2Y \xrightarrow{\mu_Y} TY$$

is also generic, which means that Y is also in $\mathcal{U}(T)^{\mathrm{op}}$. □

Remark 11.11. Note that the functor $\mathcal{U}(T) \to \mathcal{W}(T)$ need not exhibit $\mathcal{U}(T)$ as a subcategory of $\mathcal{W}(T)$.

Our goal is now to show that the algebras for the polynomial monad T can be described in terms of the ∞-categories $\mathcal{U}(T)$ and $\mathcal{W}(T)$ — this is the content of the nerve theorem. The next proposition gives the key input needed to prove this.

Notation 11.12. Given a functor $j\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}^{\mathcal{I}}$, we let

$$\nu_{\mathcal{A}}\colon \mathcal{S}^{\mathcal{I}} \to \mathrm{Fun}((\mathcal{S}^{\mathcal{I}})^{\mathrm{op}}, \mathcal{S}) \xrightarrow{j^*} \mathcal{S}^{\mathcal{A}}$$

denote the composition of the Yoneda embedding and j^*. Thus $\nu_{\mathcal{A}}$ takes $\Phi\colon \mathcal{I} \to \mathcal{S}$ to $\mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(j(-), \Phi)$.

Proposition 11.13. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$.*

(i) The functor $\nu_{\mathcal{U}(T)}\colon \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{U}(T)}$ is fully faithful, and given by right Kan extension along the inclusion $\mathcal{I} \hookrightarrow \mathcal{U}(T)$.

(ii) For every $\Phi \in \mathcal{S}^{\mathcal{I}}$, the diagram

$$(\mathcal{U}(T)^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$$

is a colimit diagram.

(iii) For every Φ in $\mathcal{S}^{\mathcal{I}}$ the composite diagram

$$(\mathcal{U}(T)^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}} \xrightarrow{T} \mathcal{S}^{\mathcal{I}} \xrightarrow{\nu_{\mathcal{U}(T)}} \mathcal{S}^{\mathcal{U}(T)}$$

is a colimit diagram. (In other words, the colimit diagram in (ii) is preserved by the functor $\nu_{\mathcal{U}(T)}T$.)

The proof uses the following technical observation:

Lemma 11.14. *Suppose $j\colon \mathcal{A}^{\mathrm{op}} \hookrightarrow \mathcal{S}^{\mathcal{I}}$ is a full subcategory of a presheaf ∞-category $\mathcal{S}^{\mathcal{I}}$ such that $\mathcal{I}^{\mathrm{op}}$ (viewed as a full subcategory of $\mathcal{S}^{\mathcal{I}}$ via the Yoneda embedding) is contained in $\mathcal{A}^{\mathrm{op}}$, so that we have a fully faithful functor $i\colon \mathcal{I} \to \mathcal{A}$. Then:*

(i) $\nu_{\mathcal{A}}$ is equivalent to the functor $i_*\colon \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{A}}$ given by right Kan extension along i.

(ii) $\nu_{\mathcal{A}}$ is fully faithful.

(iii) For every Φ in $\mathcal{S}^{\mathcal{I}}$, the diagram

$$(\mathcal{A}^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$$

is a colimit diagram, and this colimit is preserved by $\nu_{\mathcal{A}}$.

Proof. For any $\Phi \in \mathcal{S}^{\mathcal{I}}$, the diagram $(\mathcal{I}^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$ is a colimit, so we have a natural equivalence

$$\nu_{\mathcal{A}}\Phi(a) \simeq \mathrm{Map}(j(a), \Phi) \simeq \mathrm{Map}(\operatorname*{colim}_{x \in (\mathcal{I}^{\mathrm{op}})_{/j(a)}} y(x), \Phi) \simeq \lim_{x \in \mathcal{I}_{a/}} \Phi(x) \simeq (i_*\Phi)(a).$$

This proves (i). Since $i\colon \mathcal{I} \to \mathcal{A}$ is fully faithful, it follows that i_* is also fully faithful, which proves (ii). To prove (iii), since $\nu_{\mathcal{A}}$ is fully faithful it suffices to show that the composite

$$(\mathcal{A}^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}} \xrightarrow{\nu_{\mathcal{A}}} \mathcal{S}^{\mathcal{A}}$$

is a colimit diagram. But this is now a Yoneda cocone for $\mathcal{A}^{\mathrm{op}}$, which is always a colimit in $\mathcal{S}^{\mathcal{A}}$. □

Proof of Proposition 11.13. (i) and (ii) follow from Proposition 11.10(i) and Lemma 11.14. To prove (iii), since colimits in functor categories are computed objectwise, it suffices to show that for every $X \in \mathcal{U}(T)$ and $\Phi \in \mathcal{S}^{\mathcal{I}}$, the morphism

$$\operatorname*{colim}_{Y \in (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, TY) \to \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, T\Phi)$$

is an equivalence. Let $\mathcal{E} \to (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}$ be the left fibration for the functor $(\mathcal{U}(T)^{\mathrm{op}})_{/\Phi} \to \mathcal{S}$ taking Y to $\mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, TY)$; then we have a pullback square

$$\begin{array}{ccccc} \mathcal{E} & & \longrightarrow & & \mathcal{S}^{\mathcal{I}}_{X/} \\ \downarrow & & & & \downarrow \\ (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi} & \longrightarrow & \mathcal{S}^{\mathcal{I}} & \xrightarrow{T} & \mathcal{S}^{\mathcal{I}}, \end{array}$$

so that an object of $\mathcal{E}$ is a pair $(Y \to \Phi, X \to TY)$. By [Lur09, Proposition 3.3.4.5], the space $\mathrm{colim}_{Y \in (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, TY)$ is equivalent to the space $\|\mathcal{E}\|$ obtained by inverting all morphisms in $\mathcal{E}$, and the morphism we are interested in is the map of spaces induced by the functor of ∞-categories $\mathcal{E} \to \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, T\Phi)$ taking $(Y \xrightarrow{\alpha} \Phi, X \to TY)$ to the composite $X \to TY \xrightarrow{T\alpha} T\Phi$. By [Lur09, Proposition 4.1.1.3] a morphism of spaces that arises from a cofinal functor of ∞-categories is an equivalence, so it suffices to show that the functor $\mathcal{E} \to \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, T\Phi)$ is cofinal. Since every functor to an ∞-groupoid is a cartesian fibration, to prove this we may apply [Lur09, Lemma 4.1.3.2], which says that a cartesian fibration with weakly contractible fibres is cofinal. It thus suffices to check that the fibres $\mathcal{E}_\phi$ at $\phi \colon X \to T\Phi$ are weakly contractible. But the fibre $\mathcal{E}_\phi$ is the ∞-category of factorizations of ϕ of the form $X \to TY \xrightarrow{T\alpha} T\Phi$. Since T is a local right adjoint, this ∞-category has an initial object, corresponding to the generic-free factorization $X \to TY \to T\Phi$, as Y also lies in $\mathcal{U}(T)$ by Proposition 11.10(ii); hence $\mathcal{E}_\phi$ is indeed weakly contractible, as required. □

Theorem 11.15 (Nerve Theorem)**.** *Suppose T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$, and let j_T denote the restriction of F_T^{op} to a functor $\mathcal{U}(T) \to \mathcal{W}(T)$. Then the commutative square*

$$\begin{array}{ccc} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \xrightarrow{\nu_{\mathcal{W}(T)}} & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \\ {\scriptstyle U_T}\downarrow & & \downarrow{\scriptstyle j_T^*} \\ \mathcal{S}^{\mathcal{I}} & \xrightarrow{\nu_{\mathcal{U}(T)}} & \mathrm{Fun}(\mathcal{U}(T), \mathcal{S}) \end{array}$$

is cartesian, and the mate transformation

$$j_{T,!}\nu_{\mathcal{U}(T)} \to \nu_{\mathcal{W}(T)} F_T$$

is an equivalence. In particular, $\nu_{\mathcal{W}(T)} \colon \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Fun}(\mathcal{W}(T), \mathcal{S})$ is fully faithful, and the left adjoint $j_{T,!}$ restricts to F_T.

Proof. We want to apply [GHK17, Proposition 5.3.5] to conclude that the square is cartesian. All the requirements for this are clearly satisfied, with one exception: We must show that the mate transformation

$$j_{T,!}\nu_{\mathcal{U}(T)} \to \nu_{\mathcal{W}(T)} F_T$$

is an equivalence, i.e. is given by an equivalence when evaluated at every object $\Phi \in \mathcal{S}^{\mathcal{I}}$. We first consider the case of $X \in \mathcal{U}(T)^{\mathrm{op}} \subseteq \mathcal{S}^{\mathcal{I}}$. Then $\nu_{\mathcal{U}(T)} X$ is the presheaf on $\mathcal{U}(T)$ represented by X, hence $j_{T,!}\nu_{\mathcal{U}(T)} X$ is represented by $j_T X \simeq F_T X$, and so $j_{T,!}\nu_{\mathcal{U}(T)} X \xrightarrow{\sim} \nu_{\mathcal{W}(T)} F_T X$, as required.

Now let $\Phi \in \mathcal{S}^{\mathcal{I}}$ be a general object. Since j_T^* detects equivalences, it suffices to show that the evaluation of the transformation

$$j_T^* j_{T,!}\nu_{\mathcal{U}(T)} \to j_T^* \nu_{\mathcal{W}(T)} F_T \simeq \nu_{\mathcal{U}(T)} T$$

at Φ is an equivalence. We know from Lemma 11.14(iii) and Proposition 11.13(iii) that Φ is the colimit of the diagram $(\mathcal{U}(T)^{\mathrm{op}})_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$ taking $X \to \Phi$ to X, and this colimit is preserved by the functors $\nu_{\mathcal{U}(T)}$ and $\nu_{\mathcal{U}(T)}T$. Since $j_T^* j_{T,!}$ preserves colimits (being itself a left adjoint), we have a commutative square

$$\begin{array}{ccc} \operatorname{colim}_{X \in (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} j_T^* j_{T,!} \nu_{\mathcal{U}(T)} X & \longrightarrow & \operatorname{colim}_{X \in (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} \nu_{\mathcal{U}(T)} TX \\ \wr\downarrow & & \downarrow\wr \\ j_T^* j_{T,!} \nu_{\mathcal{U}(T)} \Phi & \longrightarrow & \nu_{\mathcal{U}(T)} T\Phi, \end{array}$$

where the vertical morphisms are equivalences. Moreover, the top horizontal morphism is an equivalence, since it is the colimit of equivalences $j_T^* j_{T,!} \nu_{\mathcal{U}(T)} X \xrightarrow{\sim} \nu_{\mathcal{U}(T)} TX$ for $X \in \mathcal{U}(T)^{\mathrm{op}}$. The bottom horizontal morphism is therefore also an equivalence, which completes the proof. □

Corollary 11.16. $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ *is equivalent to the full subcategory of* $\mathrm{Fun}(\mathcal{W}(T), \mathcal{S})$ *spanned by functors that are local with respect to the morphisms*

$$j_{T,!}(\operatorname*{colim}_{I \in (\mathcal{I}_{X/})^{\mathrm{op}}} y(I)) \to j_{T,!} y(X)$$

for $X \in \mathcal{U}(T)$. *In particular* $\mathrm{Alg}_T(\mathcal{S})^{\mathcal{I}}$ *is an accessible localization of* $Fun(\mathcal{W}(T), \mathcal{S})$ *and so a presentable* ∞*-category.* □

We now want to show that the ∞-categories $\mathcal{U}(T)$ and $\mathcal{W}(T)$ are compatible with morphisms of polynomial monads.

Proposition 11.17. *Let* T *be a polynomial monad on* $\mathcal{S}^{\mathcal{I}}$ *and* S *a polynomial monad on* $\mathcal{S}^{\mathcal{J}}$*, and suppose we have a commutative square*

$$\begin{array}{ccc} \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) & \xrightarrow{\Phi} & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \\ {\scriptstyle U_S}\downarrow & & \downarrow{\scriptstyle U_T} \\ \mathcal{S}^{\mathcal{J}} & \xrightarrow{f^*} & \mathcal{S}^{\mathcal{I}}, \end{array}$$

such that the mate transformation $F_T f^* \to \Phi F_S$ *is cartesian.*

(i) If $X \to TY$ *is* T*-generic, then the composite* $f_! X \to f_! TY \to S f_! Y$ *is* S*-generic, where* $f_! \colon \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{J}}$ *is the left adjoint to* f^**, given by left Kan extension along* f*, and the natural transformation* $f_! T \to S f_!$ *is obtained from the mate by applying* U_T *and moving adjoints around.*

(ii) The functor $f_!$ *restricts to a functor* $\mathcal{U}(T)^{\mathrm{op}} \to \mathcal{U}(S)^{\mathrm{op}}$*.*

(iii) The functor $\Phi \colon \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}})$ *has a left adjoint* Ψ*.*

(iv) The functor Ψ *restricts to a functor* $\mathcal{W}(S)^{\mathrm{op}} \to \mathcal{W}(T)^{\mathrm{op}}$*, and we have a commutative square*

$$\begin{array}{ccc} \mathcal{U}(T) & \xrightarrow{f_!^{\mathrm{op}}} & \mathcal{U}(S) \\ {\scriptstyle F_T^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle F_S^{\mathrm{op}}} \\ \mathcal{W}(T) & \xrightarrow{\Psi^{\mathrm{op}}} & \mathcal{W}(S). \end{array}$$

Proof. We first prove (i). Let u denote the map $T* \simeq Tf^** \to f^*S*$. Since the transformation $T_{/*}f^* \simeq Tf^* \to f^*S$ is cartesian, the functor $(Tf^*)_{/*} \colon \mathcal{S}^{\mathcal{J}} \to \mathcal{S}^{\mathcal{I}}_{/T*}$ is equivalent to the composite

$$\mathcal{S}^{\mathcal{J}} \xrightarrow{S_{/*}} \mathcal{S}^{\mathcal{J}}_{/S*} \xrightarrow{f^*} \mathcal{S}^{\mathcal{J}}_{/f^*S*} \xrightarrow{u^*} \mathcal{S}^{\mathcal{I}}_{/T*}.$$

This means we have a corresponding equivalence of left adjoints

$$f_! L_*^T \simeq L_*^S f_! u_!,$$

Since $X \to TY$ is T-generic, the adjoint map $L_*^T X \to Y$ is an equivalence, hence so is $f_! L_*^T X \to f_! Y$. But under the equivalence of left adjoints this map $L_*^S f_! u_! X \xrightarrow{\sim} f_! Y$ is adjoint to $f_! X \to f_! TY \to S f_! Y$, as required.

To prove (ii), we must show that if X is in $\mathcal{U}(T)^{\mathrm{op}}$, so that there is a generic morphism $I \to TX$ with $I \in \mathcal{I}^{\mathrm{op}}$, then $f_!X$ is in $\mathcal{U}(S)^{\mathrm{op}}$. By (i), the composite $f(I) \simeq f_!I \to f_!TX \to Sf_!X$ is T-generic. Since $f(I)$ is in $\mathcal{J}$, this implies that $f_!X$ is in $\mathcal{U}(T)^{\mathrm{op}}$.

To show part (iii), note that by Corollary 11.16 the ∞-categories $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ and $\mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}})$ are presentable. Since U_S detects equivalences, preserves limits, and is accessible, and f^* preserves both limits and colimits, it follows that Φ is accessible and preserves limits. By the adjoint functor theorem this implies that Φ has a left adjoint Ψ, as required.

From our commutative square of right adjoints we now get an equivalence $\Psi F_T \simeq F_S f_!$. By definition the ∞-categories $\mathcal{W}(T)^{\mathrm{op}}$ and $\mathcal{W}(S)^{\mathrm{op}}$ consist of free algebras on objects of $\mathcal{U}(T)^{\mathrm{op}}$ and $\mathcal{U}(S)^{\mathrm{op}}$, respectively, so it follows from (ii) that Ψ takes $\mathcal{W}(T)^{\mathrm{op}}$ to $\mathcal{W}(S)^{\mathrm{op}}$, and gives the required commutative square. □

Lemma 11.18. *The functor $\Psi\colon \mathcal{W}(T)^{\mathrm{op}} \to \mathcal{W}(S)^{\mathrm{op}}$ of the previous proposition preserves free maps and takes morphisms which are adjoint to T-generic maps to morphisms which are adjoint to S-generic maps.*

Proof. The commutativity of the square of Proposition 11.17.(iv) shows that Ψ preserves free maps. Suppose $\alpha\colon F_TX \to F_TY$ is a morphism in $\mathcal{W}(T)^{\mathrm{op}}$ which is adjoint to a T-generic map $X \to TY$, we want to see that $\Psi\alpha$ is adjoint to an S-generic morphism. By the equivalence $\Psi F_T \simeq F_S f_!$ of Proposition 11.17 and the construction of the generic–free factorization the map $\Psi\alpha$ is adjoint to the composite

$$f_!X \xrightarrow{\eta^S f_!} Sf_!X \to Sf_!Y,$$

where η^S is the unit of the monad S. We claim that there is a commutative diagram

$$\begin{array}{ccccc} & & f_!TX & \longrightarrow & f_!TY \\ & \nearrow^{f_!\eta^T} & \downarrow & & \downarrow \\ f_!X & \xrightarrow[\eta^S f_!]{} & Sf_!X & \longrightarrow & Sf_!Y \end{array}$$

where η^T is the unit of T, the right horizontal maps are induced by α and the vertical maps are induced by the equivalence $Sf_! \simeq U_S\Psi F_T$ together with the natural transformation $\tau\colon f_!U_T \to U_S\Psi$ adjoint to the unit $\eta^S_{f_!}\colon f_! \to Sf_!$. The square in the diagram commutes by naturality. To see that the triangle commutes we first observe that τ is also adjoint to the counit map $\epsilon_\Psi\colon F_SU_S\Psi \to \Psi$. Using this it is easy to see that left triangle is adjoint to a triangle

$$\begin{array}{ccc} F_Sf_!X & \longrightarrow & F_Sf_!TX \\ & \searrow^{\mathrm{id}} & \downarrow \\ & & F_Sf_!X \end{array}$$

which is equivalent to the commutative triangle

$$\begin{array}{ccc} \Psi F_TX & \longrightarrow & \Psi F_TU_TF_TX \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \Psi\epsilon_{F_TX}} \\ & & \Psi F_TX \end{array}$$

obtained from the adjunction identities. This shows that the diagram above commutes, and hence $\Psi\alpha$ is adjoint to the composite $f_!X \to f_!TX \to f_!TY \to Sf_!Y$, which is S-generic by Proposition 11.17(i). □

Combining the preceding results, we get the following:

Corollary 11.19. *In the situation of Proposition 11.17, we have a commutative cube*

$$\begin{array}{ccccc}
\mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) & & \hookrightarrow & & \mathrm{Fun}(\mathcal{W}(S),\mathcal{S}) & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
& & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \hookrightarrow & & & \mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \\
\mathcal{S}^{\mathcal{J}} & & \hookrightarrow & & \mathrm{Fun}(\mathcal{U}(S),\mathcal{S}) & & \downarrow \\
& \searrow & \downarrow & & & \searrow & \\
& & \mathcal{S}^{\mathcal{I}} & \hookrightarrow & & & \mathrm{Fun}(\mathcal{U}(T),\mathcal{S}),
\end{array}$$

which exhibits the morphism of polynomial monads $T \to S$ as arising from the commutative square in Proposition 11.17(iv). □

Proof. Taking left adjoints, the morphism of polynomial monads $T \to S$ gives a commutative square

$$\begin{array}{ccc}
\mathcal{S}^{\mathcal{I}} & \xrightarrow{f_!} & \mathcal{S}^{\mathcal{J}} \\
{\scriptstyle F_T}\downarrow & & \downarrow{\scriptstyle F_S} \\
\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \xrightarrow{\Psi} & \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}),
\end{array}$$

and we have shown that this restricts to a commutative square

$$\begin{array}{ccc}
\mathcal{U}(T)^{\mathrm{op}} & \longrightarrow & \mathcal{U}(S)^{\mathrm{op}} \\
\downarrow & & \downarrow \\
\mathcal{W}(T)^{\mathrm{op}} & \longrightarrow & \mathcal{W}(S)^{\mathrm{op}}
\end{array}$$

relating these full subcategories. Thus we have a commutative cube

$$\begin{array}{ccccc}
\mathcal{U}(T)^{\mathrm{op}} & & \hookrightarrow & & \mathcal{S}^{\mathcal{I}} & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
& & \mathcal{U}(S)^{\mathrm{op}} & \hookrightarrow & & & \mathcal{S}^{\mathcal{J}} \\
\mathcal{W}(T)^{\mathrm{op}} & & \hookrightarrow & & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & & \downarrow \\
& \searrow & \downarrow & & & \searrow & \\
& & \mathcal{W}(S)^{\mathrm{op}} & \hookrightarrow & & & \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}).
\end{array}$$

The right-hand square consists of cocomplete ∞-categories and colimit-preserving functors, so this canonically extends to presheaves on the left-hand square, giving a commutative cube

$$\begin{array}{ccccc}
\mathrm{Fun}(\mathcal{U}(T),\mathcal{S}) & & \longrightarrow & & \mathcal{S}^{\mathcal{I}} & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
& & \mathrm{Fun}(\mathcal{U}(S),\mathcal{S}) & \longrightarrow & & & \mathcal{S}^{\mathcal{J}} \\
\mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) & & \longrightarrow & & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & & \downarrow \\
& \searrow & \downarrow & & & \searrow & \\
& & \mathrm{Fun}(\mathcal{W}(S),\mathcal{S}) & \longrightarrow & & & \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}})
\end{array}$$

This consists entirely of left adjoints, and passing to right adjoints we get the cube we want. □

12. Factorization Systems from Polynomial Monads

Suppose T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. Then a morphism $F_T X \to F_T Y$ in the Kleisli ∞-category $\mathcal{K}(T)$ has a canonical factorization of the form

$$F_T X \to F_T L_* X \to F_T Y$$

adjoint to the generic-free factorization of $X \to TY$ as $X \to TL_*X \to TY$ through the unit of the local left adjoint L_*. Our first goal in this section is to show that this canonical factorization is well-defined, in the sense that if we have equivalences $F_TX \simeq F_TX'$, $F_TY \simeq F_TY'$ in $\mathcal{K}(T)$ (which need not come from morphisms in $\mathcal{S}^{\mathcal{J}}$), then there is a commutative diagram

$$\begin{array}{ccccc} F_TX & \longrightarrow & F_TL_*X & \longrightarrow & F_TY \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ F_TX' & \longrightarrow & F_TL_*X' & \longrightarrow & F_TY' \end{array}$$

where the middle vertical map is again an equivalence. We can then say that a morphism $\phi\colon F_TX \to F_TY$ is

- *inert* if the map $F_TX \to F_TL_*X$ in the canonical factorization is an equivalence,
- *active* if the map $F_TL_*X \to F_TY$ in the canonical factorization is an equivalence,

as this does not depend on the choice of the objects X and Y. We will see that the inert morphisms are obtained by closing the free morphisms under equivalences (which need not all be free), while the active morphisms are precisely those that are adjoint to generic morphisms. Our main goal in this section is to prove that these classes give a factorization system:

Theorem 12.1. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{J}}$. Then the active and inert morphisms give a factorization system on $\mathcal{K}(T)$, whereby every morphism factors as an active morphism followed by an inert morphism; this factorization is precisely the canonical factorization, up to equivalence.*

This factorization system restricts to the full subcategory $\mathcal{W}(T)^{\mathrm{op}}$, which induces a canonical pattern structure on $\mathcal{W}(T)$; in the next section we will discuss how this relates to the original monad T.

We start with some observations relating the local left adjoint of T to the Kleisli ∞-category:

Notation 12.2. For $X \in \mathcal{S}^{\mathcal{J}}$, we write $L_X\colon \mathcal{S}^{\mathcal{J}}_{/T(X)} \to \mathcal{S}^{\mathcal{J}}_{/X}$ for the left adjoint of the functor $T_X\colon \mathcal{S}^{\mathcal{J}}_{/X} \to \mathcal{S}^{\mathcal{J}}_{/T(X)}$ induced by T.

Proposition 12.3. *Let $\mathcal{K}(T)$ denote the Kleisli category of T, i.e. the full subcategory of $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{J}})$ spanned by the free algebras. For $\phi\colon FY \to FX$ in $\mathcal{K}(T)$, the ∞-category*

$$(\mathcal{S}^{\mathcal{J}}_{/X})_{\phi/} := \mathcal{S}^{\mathcal{J}}_{/X} \times_{\mathcal{K}(T)_{/FX}} (\mathcal{K}(T)_{/FX})_{\phi/}$$

has an initial object.

Proof. An object in this ∞-category is a morphism $f\colon Z \to X$ together with a commutative triangle

$$\begin{array}{ccc} FY & \longrightarrow & FZ \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle F(f)} & \\ & FX. & \end{array}$$

This corresponds to a commutative triangle

$$\begin{array}{ccc} Y & \longrightarrow & TZ \\ & {\scriptstyle\phi'}\searrow \quad \swarrow{\scriptstyle T(f)} & \\ & TX, & \end{array}$$

which in turn corresponds to

$$\begin{array}{ccc} L_XY & \longrightarrow & Z \\ & {\scriptstyle\phi''}\searrow \quad \swarrow{\scriptstyle f} & \\ & X. & \end{array}$$

Thus $L_XY \xrightarrow{\phi''} X$ gives an initial object, as required. ∎

Corollary 12.4. *The functor $F_X \colon \mathcal{S}^{\mathcal{J}}_{/X} \to \mathcal{K}(T)_{/FX}$ given by F has a left adjoint $\mathcal{L}_X$, which takes $\phi\colon FY \to FX$ to the corresponding map $\phi'' \colon L_X Y \to X$.*

Proof. By a standard argument the functor F_X is a right adjoint if and only if $(\mathcal{S}^{\mathcal{J}}_{/X})_{\phi/}$ has an initial object, which is the statement of Proposition 12.3. □

Remark 12.5. For $f\colon Y \to X$, the counit map $\mathcal{L}_X F_X(f) \to f$ is given by the commutative triangle

$$\begin{array}{ccc} L_X Y & \xrightarrow{\sim} & Y \\ & \searrow \quad \swarrow_{f} & \\ & X, & \end{array}$$

where the map $L_X Y \to Y$ is the map adjoint to the unit $Y \to TY$ which is an equivalence by Lemma 11.9. It follows that F_X is fully faithful, and so $\mathcal{L}_X$ exhibits $\mathcal{S}^{\mathcal{J}}_{/X}$ as a localization of $\mathcal{K}(T)_{/FX}$.

Remark 12.6. The functor $F_X \colon \mathcal{S}^{\mathcal{J}}_{/X} \to \mathcal{K}(T)_{/FX}$ also has a right adjoint U_X, which takes $\phi\colon FY \to FX$ to the morphism obtained as the pullback of $U\phi\colon TY \to TX$ along the unit map $\epsilon_X \colon X \to TX$. Note that the unit $\mathrm{id} \to U_X F_X$ is an equivalence since ϵ is a cartesian transformation, which also implies that F_X is fully faithful.

Remark 12.7. For $\phi\colon FY \to FX$, the unit map $\phi \to F_X\mathcal{L}_X(\phi)$ is the commutative triangle

$$\begin{array}{ccc} FY & \longrightarrow & FL_X Y \\ & {}_{\phi}\searrow \quad \swarrow & \\ & FX, & \end{array}$$

i.e. the canonical factorization of ϕ. By naturality, this means we can extend any commutative triangle

$$\begin{array}{ccc} FY & \xrightarrow{\psi} & FY' \\ & \searrow \quad \swarrow & \\ & FX & \end{array}$$

to a commutative diagram

$$\begin{array}{ccc} FY & \xrightarrow{\psi} & FY' \\ \downarrow & & \downarrow \\ FL_X Y & \xrightarrow{F_X \mathcal{L}_X \psi} & FL_X Y' \\ & \searrow \quad \swarrow & \\ & FX & \end{array}$$

relating the canonical factorizations of the two maps to FX. The next observations will allow us to prove that the canonical factorization is also natural when we vary FX.

Proposition 12.8. *For every object $C \in \mathcal{S}^{\mathcal{J}}$ we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{S}^{\mathcal{J}}_{/T^2C} & \xrightarrow{\mu_{C,!}} & \mathcal{S}^{\mathcal{J}}_{/TC} \\ {\scriptstyle L_{TC}}\downarrow & & \downarrow{\scriptstyle L_C} \\ \mathcal{S}^{\mathcal{J}}_{/TC} & \xrightarrow{L_C} & \mathcal{S}^{\mathcal{J}}_{/C}, \end{array}$$

where the top horizontal map is given by composition with the component at C of the multiplication $\mu\colon T^2 \to T$ at C.

Proof. It suffices to show that the diagram

$$\begin{array}{ccc} \mathcal{S}^{\mathcal{I}}_{/T^2C} & \xleftarrow[\mu_C^*]{} & \mathcal{S}^{\mathcal{I}}_{/TC} \\ {\scriptstyle T_{TC}}\uparrow & & \uparrow{\scriptstyle T_C} \\ \mathcal{S}^{\mathcal{I}}_{/TC} & \xleftarrow[T_C]{} & \mathcal{S}^{\mathcal{I}}_{/C} \end{array}$$

of the corresponding right adjoints commutes. Given an object $\alpha\colon B \to C$ in $\mathcal{S}^{\mathcal{I}}_{/C}$, its image under the composite of the right vertical and the upper horizontal map is the left vertical map of the pullback square

$$\begin{array}{ccc} A & \longrightarrow & TB \\ \downarrow & & \downarrow{\scriptstyle T\alpha} \\ T^2C & \xrightarrow{\mu_C} & TC. \end{array}$$

Since the multiplication μ of the polynomial monad T is a cartesian natural transformation, the map $A \to T^2C$ can be identified with $T^2\alpha\colon T^2B \to T^2C$ which is the same as the image of α under the composite of T_C and T_{TC}. □

Corollary 12.9. *Given morphisms $\alpha\colon A \to TB$ and $\beta\colon B \to TC$, we have adjoint morphisms $L_BA \to B$ and $L_CB \to C$; we also have the composites $A \xrightarrow{\alpha} TB \xrightarrow{T\beta} T^2C \xrightarrow{\mu} TC$ and $L_BA \to B \to TC$ with adjoints $L_CA \to C$ and $L_CL_BA \to C$. These are equivalent, i.e. $L_CA \simeq L_CL_BA$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathcal{S}^{\mathcal{I}}_{/TB} & \xrightarrow{(T\beta)_!} & \mathcal{S}^{\mathcal{I}}_{/T^2C} & \xrightarrow{\mu_{C,!}} & \mathcal{S}^{\mathcal{I}}_{/TC} \\ {\scriptstyle L_B}\downarrow & & \downarrow{\scriptstyle L_{TC}} & & \downarrow{\scriptstyle L_C} \\ \mathcal{S}^{\mathcal{I}}_{/B} & \xrightarrow{\beta_!} & \mathcal{S}^{\mathcal{I}}_{/TC} & \xrightarrow{L_C} & \mathcal{S}^{\mathcal{I}}_{/C}. \end{array}$$

Here the left square commutes since it is the square of left adjoints corresponding to the square

$$\begin{array}{ccc} \mathcal{S}^{\mathcal{I}}_{/TC} & \xrightarrow{\beta^*} & \mathcal{S}^{\mathcal{I}}_{/B} \\ \downarrow{\scriptstyle T_{TC}} & & \downarrow{\scriptstyle T_B} \\ \mathcal{S}^{\mathcal{I}}_{/T^2C} & \xrightarrow{(T\beta)^*} & \mathcal{S}^{\mathcal{I}}_{/TB}, \end{array}$$

which commutes since T preserves pullbacks, and the right square commutes by Proposition 12.8. By construction the morphisms $L_CA \to C$ and $L_CL_BA \to C$ are given by $L_C\mu_{C,!}T\beta_!(\alpha)$ and $L_C\beta_!L_B(\alpha)$, and so are equivalent by the commutativity of the outer square. □

Proposition 12.10. *Given a commutative square*

$$\begin{array}{ccc} F_TA & \longrightarrow & F_TB \\ \downarrow & & \downarrow \\ F_TC & \longrightarrow & F_TD, \end{array}$$

in $\mathcal{K}(T)$, there exists a canonical commutative diagram

$$\begin{array}{ccccc} F_TA & \longrightarrow & F_TL_BA & \longrightarrow & F_TB \\ \downarrow & & \downarrow & & \downarrow \\ F_TC & \longrightarrow & F_TL_DC & \longrightarrow & F_TD. \end{array}$$

Here the diagram associated to the degenerate square

$$\begin{array}{ccc} F_TA & \longrightarrow & F_TB \\ \| & & \| \\ F_TA & \longrightarrow & F_TB \end{array}$$

is the degenerate diagram

$$\begin{array}{ccccc} F_TA & \longrightarrow & F_TL_BA & \longrightarrow & F_TB \\ \| & & \| & & \| \\ F_TA & \longrightarrow & F_TL_BA & \longrightarrow & F_TB. \end{array}$$

Moreover, we have compatibility with composition, in the sense that if we have a commutative diagram

$$\begin{array}{ccc} F_TA & \longrightarrow & F_TB \\ \downarrow & & \downarrow \\ F_TC & \longrightarrow & F_TD \\ \downarrow & & \downarrow \\ F_TX & \longrightarrow & F_TY, \end{array}$$

then the vertical composite of the associated diagrams

$$\begin{array}{ccccc} F_TA & \longrightarrow & F_TL_BA & \longrightarrow & F_TB \\ \downarrow & & \downarrow & & \downarrow \\ F_TC & \longrightarrow & F_TL_DC & \longrightarrow & F_TD \\ \downarrow & & \downarrow & & \downarrow \\ F_TX & \longrightarrow & F_TL_YX & \longrightarrow & F_TY, \end{array}$$

is the diagram associated to the composite square

$$\begin{array}{ccc} F_TA & \longrightarrow & F_TB \\ \downarrow & & \downarrow \\ F_TX & \longrightarrow & F_TY. \end{array}$$

Proof. We can view the original square as a pair of morphisms

$$F_TC \longleftarrow F_TA \longrightarrow F_TB$$

in $\mathcal{K}(T)_{/F_TD}$. Adding the canonical factorization of the arrow $F_TA \to F_TB$, the naturality of the unit for the adjunction $\mathcal{L}_D \dashv F_D$ gives a commutative diagram

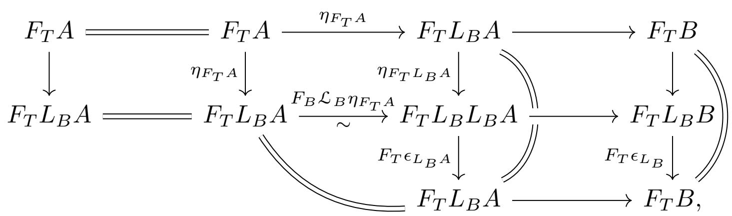
$$\begin{array}{ccccccc} F_TC & \longleftarrow & F_TA & \longrightarrow & F_TL_BA & \longrightarrow & F_TB \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_TL_DC & \longleftarrow & F_TL_DA & \xrightarrow{\sim} & F_TL_DL_BA & \longrightarrow & F_TL_DB \end{array}$$

over F_TD, where the second arrow in the bottom row is an equivalence by Corollary 12.9. If we invert this equivalence we can contract the diagram to

$$\begin{array}{ccccc} F_TA & \longrightarrow & F_TL_BA & \longrightarrow & F_TB \\ \downarrow & & \downarrow & & \\ F_TC & \longrightarrow & F_TL_DC & & \end{array}$$

over F_TD; adding F_TD back in now gives the desired diagram.

From the degenerate square we can make the extended diagram

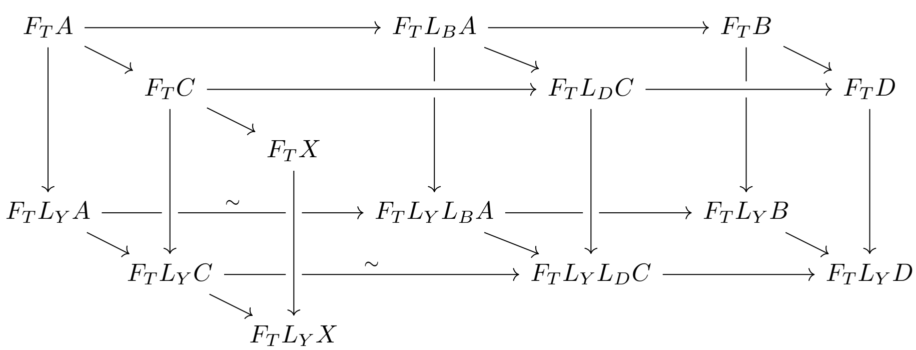


all over $F_T B$. Here the adjunction identities for $\mathcal{L}_B \dashv F_B$ imply that the two composites $F_T L_B A \to F_T L_B A$ are identities, as indicated; this means the general definition indeed specializes to give the degenerate diagram in this case.

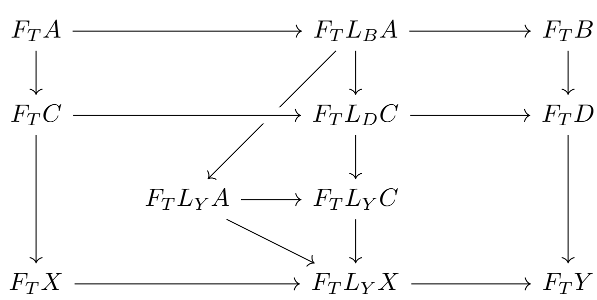
To see we have compatibility with composition, consider the diagram

$$\begin{array}{ccccc} F_T A & \longrightarrow & F_T L_B A & \longrightarrow & F_T B \\ \downarrow & & \downarrow & & \downarrow \\ F_T C & \longrightarrow & F_T L_D C & \longrightarrow & F_T D \\ \downarrow & & & & \\ F_T X & & & & \end{array}$$

in $\mathcal{K}(T)_{/F_T Y}$. Using the unit for the adjunction $\mathcal{L}_Y \dashv F_Y$ this extends to a commutative diagram



over $F_T Y$, where the two indicated morphisms are equivalences by Corollary 12.9. Inverting these, we can contract the diagram to



where we see both the composite of the diagrams for the two squares and the diagram for the composite square, as required. □

Corollary 12.11. *A commutative square*

$$\begin{array}{ccc} F_T A & \longrightarrow & F_T B \\ \wr\big\downarrow & & \big\downarrow\wr \\ F_T C & \longrightarrow & F_T D \end{array}$$

in $\mathcal{K}(T)$, where the vertical morphisms are equivalences, can be extended to a commutative diagram

$$\begin{array}{ccccc} F_T A & \longrightarrow & F_T L_B A & \longrightarrow & F_T B \\ \wr\big\downarrow & & \big\downarrow\wr & & \big\downarrow\wr \\ F_T C & \longrightarrow & F_T L_D C & \longrightarrow & F_T D \end{array}$$

where the middle vertical map is also an equivalence.

Proof. This follows immediately from the compatibility of the diagrams in Proposition 12.10 with composition and identities. □

In other words, the canonical factorizations in $\mathcal{K}(T)$ are invariant under equivalences. This means the following conditions on morphisms are well-defined:

Definition 12.12. We say a morphism $\phi\colon F_T A \to F_T B$ in $\mathcal{K}(T)$ with canonical factorization

$$F_T A \to F_T L_B A \to F_T B$$

is *inert* if the morphism $F_T A \to F_T L_B A$ in the canonical factorization is an equivalence, and *active* if the morphism $F_T L_B A \to F_T B$ in the canonical factorization is an equivalence.

Our goal in the rest of this section is to show that the active and inert morphisms form a factorization system on $\mathcal{K}(T)$. We start with some observations about equivalences in $\mathcal{K}(T)$ that will lead to a simpler characterization of the active maps.

Lemma 12.13. *If T is a polynomial monad then the free functor F_T is conservative: if $F_T(\phi)$ is an equivalence for some $\phi\colon X \to Y$ in $\mathcal{S}^{\mathcal{I}}$ then ϕ is an equivalence.*

Proof. The functor $F_Y\colon \mathcal{S}^{\mathcal{I}}_{/Y} \to \mathcal{K}(T)_{/F_T Y}$ is fully faithful by Remark 12.5. The inverse of $F_T\phi$ gives a morphism

$$\begin{array}{ccc} F_T Y & \xrightarrow{(F_T\phi)^{-1}} & F_T X \\ & \searrow\!\!\!= \quad \swarrow_{F_T\phi} & \\ & F_T Y & \end{array}$$

in $\mathcal{K}(T)_{/F_T Y}$ between objects in the image of F_Y, hence it is also in the image of F_Y and lifts to an equivalence in $\mathcal{S}^{\mathcal{I}}_{/Y}$ by faithfulness. □

Lemma 12.14. *Given a commutative triangle*

$$\begin{array}{ccc} FA & \xrightarrow{\phi} & FB \\ & {}_{\alpha}\searrow \quad \swarrow_{\beta} & \\ & FC, & \end{array}$$

and morphisms $a\colon A \to C$ and $b\colon B \to C$ with equivalences $\alpha \simeq F(a)$ and $\beta \simeq F(b)$, then the triangle lifts to a unique commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {}_{a}\searrow \quad \swarrow_{b} & \\ & C. & \end{array}$$

Proof. The functor $F_C \colon \mathcal{S}^{\mathcal{I}}_{/C} \to \mathcal{K}(T)_{/F_T C}$ is fully faithful by Remark 12.5. This immediately implies the result, since the first triangle is precisely a morphism in $\mathcal{K}(T)_{/F_T C}$ between objects in the image of F_C. ☐

Lemma 12.15. *Every equivalence $\phi \colon F_T X \xrightarrow{\sim} F_T Y$ is adjoint to a generic map.*

Proof. Regarding ϕ as a morphism in $\mathcal{K}(T)_{/F_T Y}$, we apply $\mathcal{L}_Y$ to get a commutative triangle

$$\begin{array}{ccc} L_Y X & \xrightarrow[\sim]{\mathcal{L}_Y \phi} & L_Y Y \\ & \searrow \quad \swarrow_{\sim} & \\ & Y, & \end{array}$$

where the right diagonal map is an equivalence by Lemma 11.9(i) and the horizontal map is an equivalence by the functoriality of $\mathcal{L}_Y$. Hence the left diagonal map is also an equivalence, which is precisely the condition for the map $X \to TY$ adjoint to ϕ to be generic. ☐

Lemma 12.16. *A morphism $\phi \colon F_T X \to F_T Y$ is active if and only if the adjoint morphism $\phi' \colon X \to TY$ is generic.*

Proof. Let $\phi'' \colon L_Y X \to Y$ be the map adjoint to ϕ'. By definition, ϕ is active if and only if in the canonical factorization

$$F_T X \xrightarrow{\phi_a} F_T L_Y X \xrightarrow{\phi_i} F_T Y,$$

the map $\phi_i = F_T(\phi'')$ is an equivalence. By Lemma 12.13 this happens if and only if ϕ'' is an equivalence, which is precisely the condition for ϕ' to be generic. ☐

Lemma 12.17. *For any morphism $\phi \colon X \to Y$ in $\mathcal{S}^{\mathcal{I}}$, the free morphism $F_T(\phi) \colon F_T X \to F_T Y$ is inert.*

Proof. The commutative triangle

$$\begin{array}{ccc} F_T X & & \\ \| & \searrow^{F_T \phi} & \\ F_T X & \xrightarrow[F_T \phi]{} & F_T Y \end{array}$$

is adjoint to

$$\begin{array}{ccc} X & & \\ {\scriptstyle \eta_X}\downarrow & \searrow & \\ TX & \xrightarrow{T\phi} & TY \end{array}$$

which is in turn adjoint to

$$\begin{array}{ccc} L_* X & & \\ {\scriptstyle \lambda}\downarrow{\scriptstyle \wr} & \searrow & \\ X & \xrightarrow{\phi} & Y, \end{array}$$

where λ is an equivalence since the unit map η_X is generic by Lemma 11.9(i). By adjointness we see that F_T takes the last triangle to the right triangle in the commutative diagram

$$\begin{array}{cccc} F_T X & \longrightarrow & F_T L_* X & \\ & \diagdown\!\!\diagdown & {\scriptstyle F_T \lambda}\downarrow{\scriptstyle \wr} & \searrow \\ & & F_T X & \xrightarrow{F_T \phi} F_T Y, \end{array}$$

where the upper horizontal map is adjoint to the unit $X \to TL_* X$; this is an equivalence as $F_T \lambda$ is one. By definition the upper horizontal map and the right diagonal map give the canonical factorization of $F_T \phi$, hence $F_T \phi$ is inert. ☐

Warning 12.18. Note, however, that it is *not* necessarily true that every inert map is of the form $F_T(\phi)$ for ϕ a morphism in $\mathcal{S}^{\mathcal{I}}$: The equivalences in $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ need not all be in the image of $\mathcal{S}^{\mathcal{I}}$.

Remark 12.19. By Lemmas 12.16 and 12.17, the canonical factorization of a morphism $\phi\colon F_T A \to F_T B$ as $F_T A \to F_T L_* A \to F_T B$ is a factorization of ϕ as an active morphism followed by an inert morphism.

Lemma 12.20. *Given a commutative square*

$$\begin{tikzcd} F_T A \arrow[r, "\phi"] \arrow[d, "\psi"'] & F_T B \arrow[d, "F_T\beta"] \\ F_T C \arrow[r, "F_T\gamma"'] \arrow[ur, dashed] & F_T D \end{tikzcd}$$

where ψ is active, there exists a unique diagonal filler, which is of the form $F_T(\alpha)$ for a unique commutative triangle

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & B \arrow[dl, "\beta"] \\ & D. & \end{tikzcd}$$

Proof. As in Lemma 11.3, the square is adjoint to

$$\begin{tikzcd} A \arrow[r, "\phi'"] \arrow[d, "\psi'"'] & TB \arrow[d, "T\beta"] \\ TC \arrow[r, "T\gamma"'] & TD, \end{tikzcd}$$

which in turn is adjoint to

$$\begin{tikzcd} L_* A \arrow[r, "\phi''"] \arrow[d, "\wr"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\gamma"'] & D, \end{tikzcd}$$

where the left vertical morphism is an equivalence since ψ is active and this implies that ψ' is generic by Lemma 12.16. This square has a unique filler, which in turn corresponds to a unique filler in the original square, since we saw in Lemma 12.14 that all fillers are uniquely of this form. □

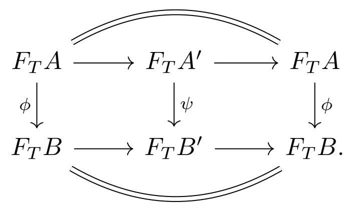
Proof of Theorem 12.1. We check the requirements of [Lur09, Definition 5.2.8.8] (which are equivalent to our previous definition of a factorization system by [Lur09, Proposition 5.2.8.17]). We must thus check:

(1) The classes of inert and active maps are closed under retracts.
(2) The active maps are left orthogonal to the inert maps, i.e. for every commutative square

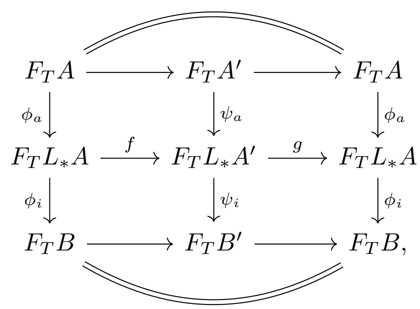
$$\begin{tikzcd} F_T A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & F_T B \arrow[d, "\gamma"] \\ F_T C \arrow[r, "\delta"'] \arrow[ur, dashed] & F_T D \end{tikzcd}$$

where β is active and γ is inert, there exists a unique filler.
(3) Every morphism can be factored as an active map followed by an inert map.

Condition (3) is by now clear, since by Remark 12.19 the canonical factorization gives an active-inert factorization. For condition (1), suppose we have a retract diagram

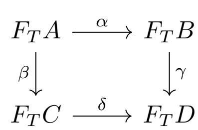


By applying Proposition 12.10 to the two squares, we obtain a commutative diagram

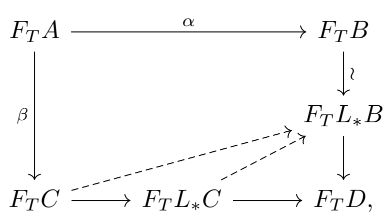


relating the canonical factorizations of ϕ and ψ, where the compatibility with composition and identities in Proposition 12.10 implies that $gf \simeq \mathrm{id}$. If ψ is active then by definition the map labelled ψ_i is an equivalence, and so ϕ_i is a retract of an equivalence; hence ϕ_i is also an equivalence, which means ϕ is active. The same argument shows that inert morphisms are also closed under retracts.

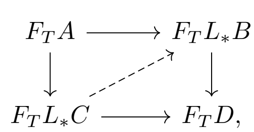
It remains to prove (2). Consider a commutative square



with β active and γ inert. Including the canonical factorizations of γ and δ, we get a diagram



and since the map $F_TB \to F_TL_*B$ is an equivalence (since γ is inert), a lift in the original square corresponds to a lift $F_TC \to F_TL_*B$ here. Applying Lemma 12.20 to the square



we see that there is a unique diagonal filler $F_T L_* C \to F_T L_* B$, which comes from a unique commutative triangle

$$\begin{array}{ccc} L_*C & \longrightarrow & L_*B \\ & \searrow \quad \swarrow & \\ & D. & \end{array}$$

This gives in particular a lift in the original square, but now applying Lemma 12.20 to a square

$$\begin{array}{ccc} F_T C & \longrightarrow & F_T L_* B \\ \downarrow & \nearrow & \downarrow \\ F_T L_* C & \longrightarrow & F_T D, \end{array}$$

we see that any lift $F_T C \to F_T L_* B$ must factor through $F_T L_* C$ and so must be the lift we just constructed. □

Let us say that a morphism in $\mathcal{W}(T)$ is inert or active if it corresponds to an inert or active morphisms in $\mathcal{K}(T)$ under the inclusion $\mathcal{W}(T)^{\mathrm{op}} \hookrightarrow \mathcal{K}(T)$. Then the factorization system we constructed restricts to one on $\mathcal{W}(T)$:

Corollary 12.21. *The inert and active morphisms restrict to a factorization system on $\mathcal{W}(T)$.*

Proof. It is enough to show that for a morphism $F_T I \to F_T J$ in $\mathcal{W}(T)^{\mathrm{op}}$, if $F_T I \to F_T X \to F_T J$ is its active-inert factorization in $\mathcal{K}(T)$, then $F_T X$ also lies in $\mathcal{W}(T)^{\mathrm{op}}$. Since the canonical factorization is an active-inert factorization, this follows from Proposition 11.10(ii). □

13. Patterns from Polynomial Monads

Suppose T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. In the previous section we saw that the ∞-category $\mathcal{W}(T)$ has a canonical inert–active factorization system. Using this we can define a natural algebraic pattern structure on $\mathcal{W}(T)$ by taking the elementary objects to be those of the form $F_T(I)$ with $I \in \mathcal{S}^{\mathcal{I}}$ in the image of $\mathcal{I}^{\mathrm{op}}$ under the Yoneda embedding.

In this section we will study these algebraic patterns. We will see that $\mathcal{W}(T)$ is always an extendable pattern, and that the free Segal $\mathcal{W}(T)$-space monad is closely related to the original monad T: there is a canonical morphism $T \to T_{\mathcal{W}(T)}$ in PolyMnd, which induces an equivalence on ∞-categories of algebras. Moreover, the patterns $\mathcal{W}(T)$ are natural in T, and so determine a functor

$$\mathfrak{P}\colon \mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}};$$

the morphisms $T \to T_{\mathcal{W}(T)}$ then give a natural transformation $\mathrm{id} \to \mathfrak{M}\mathfrak{P}$.

Notation 13.1. In the first part of this section we fix a polynomial monad T on $\mathcal{S}^{\mathcal{I}}$. From our work in the previous two sections we then have the following commutative diagram, where it will be convenient to name the various functors as indicated:

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{e} & \mathcal{W}(T)^{\mathrm{el}} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ \mathcal{U}(T) & \xrightarrow{u} & \mathcal{W}(T)^{\mathrm{int}} \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle j'} \\ & & \mathcal{W}(T). \end{array}$$

Proposition 13.2. *Let $\mathcal{K}(T)^{\mathrm{int}}$ denote the subcategory of $\mathcal{K}(T)$ containing only the inert morphisms. Then the slice $\mathcal{K}(T)^{\mathrm{int}}_{/F_T X}$ is equivalent to the full subcategory of $\mathcal{K}(T)_{/F_T X}$ spanned by the inert morphisms to FX. The functor $F_X \colon \mathcal{S}^{\mathcal{I}}_{/X} \to \mathcal{K}(T)_{/F_T X}$ restricts to an equivalence*

$$\mathcal{S}^{\mathcal{I}}_{/X} \xrightarrow{\sim} \mathcal{K}(T)^{\mathrm{int}}_{/F_T X}$$

with inverse $\mathcal{L}_X$*.*

Proof. It follows from the existence of the active–inert factorization system on $\mathcal{K}(T)$ that if we have a commutative triangle

$$\begin{array}{ccc} F_T A & \longrightarrow & F_T B \\ & \searrow \quad \swarrow & \\ & F_T X & \end{array}$$

where the two diagonal morphisms are inert, then the horizontal morphism is also inert. This implies that $\mathcal{K}(T)^{\mathrm{int}}_{/F_T X}$ is the full subcategory of $\mathcal{K}(T)_{/F_T X}$ spanned by the inert morphisms. Moreover, every inert morphism to $F_T X$ is equivalent to a free morphism in $\mathcal{K}(T)_{/F_T X}$, so this full subcategory consists precisely of the objects in the image of F_X. Since F_X is fully faithful by Remark 12.5, it follows that the adjunction $\mathcal{L}_X \dashv F_X$ restricts to an equivalence between $\mathcal{S}^{\mathcal{I}}_{/X}$ and $\mathcal{K}(T)^{\mathrm{int}}_{/F_T X}$. ☐

Restricting this equivalence, we get the following:

Corollary 13.3. *The functor* F_X^{op} *restricts to an equivalence*

$$\mathcal{I}_{X/} \xrightarrow{\sim} \mathcal{W}(T)^{\mathrm{el}}_{F_T X/}$$

for every $X \in \mathcal{U}(T)$*.* ☐

Corollary 13.4. *The top commutative square in Notation 13.1 induces a commutative square of functors to* $\mathcal{S}$*. Taking the mate of this square gives a commutative square*

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) & \xrightarrow{i'_*} & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S}) \\ \downarrow e^* & & \downarrow u^* \\ \mathrm{Fun}(\mathcal{I}, \mathcal{S}) & \xrightarrow{i_*} & \mathrm{Fun}(\mathcal{U}(T), \mathcal{S}), \end{array}$$

and this is moreover cartesian.

Proof. To see that there is such a commutative square amounts to checking that the mate transformation

$$u^* i'_* \Phi \to i_* e^* \Phi$$

is an equivalence for $\Phi \colon \mathcal{W}(T)^{\mathrm{el}} \to \mathcal{S}$. Evaluated at $X \in \mathcal{U}(T)$, this is the map on limits

$$\lim_{\mathcal{W}(T)^{\mathrm{el}}_{F_T X/}} \Phi \to \lim_{\mathcal{I}_{X/}} \Phi e,$$

induced by the functor $\mathcal{I}_{X/} \to \mathcal{W}(T)^{\mathrm{el}}_{F_T X/}$. Since this functor is an equivalence by Corollary 13.3, the mate transformation is indeed an equivalence. The functors i_* and i'_* are fully faithful, since they are given by right Kan extensions along the fully faithful functors i and i'. To see that the square is cartesian it therefore suffices to check that an object $\Phi \in \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S})$ is in the image of i'_* if and only if $u^*\Phi$ is in the image of i_*. Here Φ is in the image of i'_* if and only if the unit map $\Phi \to i'_* i'^* \Phi$ is an equivalence. The functor u^* is conservative, because u is essentially surjective, and so this holds if and only if $u^*\Phi \to u^* i'_* i'^* \Phi$ is an equivalence. We can identify the composite

$$u^*\Phi \to u^* i'_* i'^* \Phi \xrightarrow{\sim} i_* e^* i'^* \Phi \simeq i_* i^* u^* \Phi$$

with the unit map for $i^* \dashv i_*$, and since the mate transformation is an equivalence this means that the latter is an equivalence if and only if Φ is in the image of i'_*. As i_* is also fully faithful, this condition holds precisely when $u^*\Phi$ is in the image of i_*, as required. ☐

Corollary 13.5. *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \xhookrightarrow{\nu_{\mathcal{W}(T)}} & \mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \\ \downarrow & & \downarrow{\scriptstyle j'^*} \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}},\mathcal{S}) & \xhookrightarrow{i'_*} & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S}) \\ {\scriptstyle e^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathrm{Fun}(\mathcal{I},\mathcal{S}) & \xhookrightarrow{i_*} & \mathrm{Fun}(\mathcal{U}(T),\mathcal{S}), \end{array}$$

$$U_T\colon \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Fun}(\mathcal{I},\mathcal{S}), \qquad j^*\colon \mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \to \mathrm{Fun}(\mathcal{U}(T),\mathcal{S}),$$

where both squares are cartesian.

Proof. By the Nerve Theorem 11.15, we have a cartesian square

$$\begin{array}{ccc} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \xrightarrow{\nu_{\mathcal{W}(T)}} & \mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \\ {\scriptstyle U_T}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathrm{Fun}(\mathcal{I},\mathcal{S}) & \xhookrightarrow{i_*} & \mathrm{Fun}(\mathcal{U}(T),\mathcal{S}). \end{array}$$

Here the right vertical functor j^* factors as $\mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \xrightarrow{j'^*} \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S}) \xrightarrow{u^*} \mathrm{Fun}(\mathcal{U}(T),\mathcal{S})$. The left vertical functor therefore factors uniquely through the pullback of i_* along u^*, which we can identify with $\mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}},\mathcal{S})$ by Corollary 13.4. This gives the desired commutative diagram. Here the bottom and outer squares are cartesian, and so the top square is also cartesian. □

Corollary 13.6. *We have a commutative square*

$$\begin{array}{ccc} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \xrightarrow{\sim} & \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}},\mathcal{S}) & \xrightarrow{\sim} & \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \end{array}$$

where the horizontal functors are equivalences.

Proof. By definition, $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$ is the essential image of the fully faithful functor i'_* in $\mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S})$, and $\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ is the full subcategory of $\mathrm{Fun}(\mathcal{W}(T),\mathcal{S})$ spanned by the functors whose restriction along j' lies in this full subcategory; we thus have a pullback square

$$\begin{array}{ccc} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) & \hookrightarrow & \mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \\ \downarrow & & \downarrow{\scriptstyle j'^*} \\ \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) & \hookrightarrow & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S}). \end{array}$$

The top cartesian square in the diagram of Corollary 13.5 factors through this, giving a commutative diagram

$$\begin{array}{ccccc} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \longrightarrow & \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) & \hookrightarrow & \mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle j'^*} \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}},\mathcal{S}) & \xrightarrow[\sim]{i_*} & \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) & \hookrightarrow & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S}). \end{array}$$

Here the left-hand square is cartesian, since the outer and right-hand squares are cartesian, and so the induced functor $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ is indeed an equivalence. □

Proposition 13.7. *For $X \in \mathcal{U}(T)$, the functor*

$$\mathcal{U}(T)_{X/} \to \mathcal{U}(T)_{F_T X/} := \mathcal{U}(T) \times_{\mathcal{W}(T)^{\mathrm{int}}} \mathcal{W}(T)^{\mathrm{int}}_{F_T X/}$$

is coinitial.

Proof. By [Lur09, Theorem 4.1.3.1] it suffices to check that for $Y, \phi \colon FY \to FX$, the slice ∞-category $(\mathcal{U}(T)_{X/})_{/\phi}$ is weakly contractible. Here the canonical factorization of ϕ determines a terminal object, as in the proof of Proposition 12.3. □

Corollary 13.8. *There are natural equivalences of functors*

$$\mathrm{id} \xrightarrow{\sim} u_*u^*,$$
$$u_!u^* \xrightarrow{\sim} \mathrm{id},$$
$$j_!u^* \xrightarrow{\sim} j'_!.$$

Proof. For $\Phi \colon \mathcal{W}(T)^{\mathrm{int}} \to \mathcal{S}$ the unit map $\Phi \to u_*u^*\Phi$ evaluates at $F_TX \in \mathcal{W}(T)^{\mathrm{int}}$ as

$$\Phi(F_TX) \to \lim_{\mathcal{U}(T)_{F_TX/}} \Phi \circ u,$$

which is an equivalence by Corollary 13.7. This gives the first equivalence, which implies the second by passing to left adjoints. Applying $j'_!$ this gives the third equivalence, since $j'_!u_! \simeq (j'u)_! \simeq j_!$. □

Corollary 13.9. *The algebraic pattern $\mathcal{W}(T)$ is extendable.*

Proof. We must show that $j'_!$ restricts to a functor $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$. Thus for $\Phi \in \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$ we must show that $j'_!\Phi$ is a Segal object. By Corollary 13.8 the functor $j'_!\Phi$ is equivalent to $j_!u^*\Phi$. But since Φ is by assumption in $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$, we know by Corollary 13.4 that $u^*\Phi$ is right Kan extended from $\mathcal{I}$. Hence $j_!u^*\Phi$ is in $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \simeq \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ by Theorem 11.15, as required. □

Corollary 13.10. *Inverting the equivalence of Corollary 13.6, we have a commutative square*

$$\begin{array}{ccc} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) & \xrightarrow[\sim]{\phi} & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \\ {\scriptstyle U_{\mathcal{W}(T)}}\downarrow & & \downarrow{\scriptstyle U_T} \\ \mathcal{S}^{\mathcal{W}(T)^{\mathrm{el}}} & \xrightarrow{e^*} & \mathcal{S}^{\mathcal{I}}. \end{array}$$

This square is a morphism of polynomial monads $T \to T_{\mathcal{W}(T)}$.

Proof. Since $\mathcal{W}(T)$ is extendable, we know that the free Segal $\mathcal{W}(T)$-space monad $T_{\mathcal{W}(T)}$ is polynomial by Proposition 10.6. For the square to be a morphism of polynomial monads, it remains to show that the mate transformation $F_Te^* \to \phi F_{\mathcal{W}(T)}$ is cartesian. The equivalence $U_T\phi \simeq e^*U_{\mathcal{W}(T)}$ gives an equivalence of left adjoints $\phi^{-1}F_T \simeq F_{\mathcal{W}(T)}e_!$ under which the mate transformation corresponds to the transformation

$$\phi F_{\mathcal{W}(T)}e_!e^* \to \phi F_{\mathcal{W}(T)}$$

induced by the counit $e_!e^* \to \mathrm{id}$. This counit is easily seen to be cartesian (as in [GHK17, Lemma 2.1.5]), and since $U_{\mathcal{W}(T)}$ is conservative and preserves limits, it suffices to check this implies the transformation

$$T_{\mathcal{W}(T)}e_!e^* \to T_{\mathcal{W}(T)}$$

is cartesian, which is true since $T_{\mathcal{W}(T)}$ preserves pullbacks. □

We now show that the pattern $\mathcal{W}(T)$ is natural with respect to morphisms of polynomial monads:

Theorem 13.11. *There is a functor*

$$\mathfrak{P} \colon \mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$$

that takes a polynomial monad T on $\mathcal{S}^{\mathcal{I}}$ to the algebraic pattern $\mathcal{W}(T)$, and a natural transformation

$$\tau \colon \mathrm{id} \to \mathfrak{M}\mathfrak{P},$$

given by the morphism $T \to \mathfrak{M}(\mathcal{W}(T))$ from Corollary 13.10, where $\mathfrak{M}$ is the functor from Corollary 10.12 that takes an extendable pattern $\mathcal{O}$ to the free Segal $\mathcal{O}$-space monad.

Proof. Suppose we have a morphism of polynomial monads $T \to S$, given by a functor $f\colon \mathcal{I} \to \mathcal{J}$ and a commutative square

$$\begin{array}{ccc} \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) & \xrightarrow{\Phi} & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \\ {\scriptstyle U_S}\downarrow & & \downarrow{\scriptstyle U_T} \\ \mathcal{S}^{\mathcal{J}} & \xrightarrow{f^*} & \mathcal{S}^{\mathcal{I}}. \end{array}$$

By Proposition 11.17, the functor Φ has a left adjoint Ψ which restricts to a functor $\Psi^{\mathrm{op}}\colon \mathcal{W}(T) \to \mathcal{W}(S)$. Lemma 11.18 implies that this functor preserves active and inert morphisms, since the active morphisms are precisely those that are adjoint to generic morphisms by Lemma 12.16, while the inert morphisms are the composites of free morphisms and equivalences. The commutative square from Proposition 11.17(iv) restricts to a commutative square

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{f} & \mathcal{J} \\ \downarrow & & \downarrow \\ \mathcal{W}(T) & \xrightarrow{\Psi^{\mathrm{op}}} & \mathcal{W}(S), \end{array}$$

and so Ψ^{op} also preserves elementary objects. Thus Ψ^{op} is a morphism of algebraic patterns.

It follows from Corollary 13.6 and Corollary 11.19 that $\Phi\colon \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) \to \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ can be identified with the restriction of $(\Psi^{\mathrm{op}})^*$ to Segal objects, thus Ψ^{op} is a Segal morphism by Lemma 4.5.

Since this construction is obviously compatible with composition we obtain a functor

$$\mathfrak{P}\colon \mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}.$$

Using Corollary 13.4 the commutative cube in Corollary 11.19 extends to a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) & & \hookrightarrow & & \mathrm{Fun}(\mathcal{W}(S), \mathcal{S}) & & \\ & \searrow & & & \downarrow & \searrow & \\ \downarrow & & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) & \hookrightarrow & & & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \\ \mathrm{Fun}(\mathcal{W}(S)^{\mathrm{el}}, \mathcal{S}) & & \downarrow & \hookrightarrow & \mathrm{Fun}(\mathcal{W}(S)^{\mathrm{int}}, \mathcal{S}) & & \downarrow \\ & \searrow & & & \downarrow & \searrow & \\ \downarrow & & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) & \hookrightarrow & & & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S}) \\ \mathcal{S}^{\mathcal{J}} & & \downarrow & \hookrightarrow & \mathrm{Fun}(\mathcal{U}(S), \mathcal{S}) & & \downarrow \\ & \searrow & & & & \searrow & \\ & & \mathcal{S}^{\mathcal{I}} & \hookrightarrow & & & \mathrm{Fun}(\mathcal{U}(T), \mathcal{S}), \end{array}$$

where the left side gives the naturality square

$$\begin{array}{ccc} S & \longrightarrow & T_{\mathcal{W}(S)} \\ \downarrow & & \downarrow \\ T & \longrightarrow & T_{\mathcal{W}(T)}. \end{array}$$

Since this construction is again compatible with composition, it gives a natural transformation $\mathrm{id} \to \mathfrak{M}\mathfrak{P}$. □

Variant 13.12. Let us say that a *flagged algebraic pattern* is a pair $(\mathcal{O}, \mathcal{I} \to \mathcal{O}^{\mathrm{el}})$ where $\mathcal{O}$ is an algebraic pattern and $\mathcal{I} \to \mathcal{O}^{\mathrm{el}}$ is an essentially surjective functor of ∞-categories. We write FlAlgPatt for the full subcategory of $\mathrm{AlgPatt} \times_{\mathrm{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ spanned by the flagged algebraic patterns, and $\mathrm{FlAlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$ for the subcategory consisting of flagged algebraic patterns whose underlying patterns are extendable, with morphisms those such that the underlying morphisms of patterns are Segal morphisms. As a variant of the construction of $\mathfrak{P}$ above, we can define a functor

$$\mathfrak{P}'\colon \mathrm{PolyMnd} \to \mathrm{FlAlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$$

that takes a polynomial monad T on $\mathcal{S}^{\mathcal{I}}$ to the flagged algebraic pattern $(\mathcal{W}(T), \mathcal{I} \xrightarrow{e} \mathcal{W}(T)^{\mathrm{el}})$. Note that we can recover the monad T from this flagged pattern, since U_T is equivalent to the composite

$$\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) \xrightarrow{U_{\mathcal{W}(T)}} \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) \xrightarrow{e^*} \mathrm{Fun}(\mathcal{I}, \mathcal{S}).$$

For any flagged extendable pattern $(\mathcal{O}, f \colon \mathcal{I} \to \mathcal{O}^{\mathrm{el}})$ the composite

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{U_{\mathcal{O}}} \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S}) \xrightarrow{f^*} \mathrm{Fun}(\mathcal{I}, \mathcal{S})$$

is a monadic right adjoint (since f^* preserves all limits and colimits and is conservative when f is essentially surjective), but we do not know under what conditions on f the corresponding monad is polynomial. This means that we do not have a satisfactory flagged version of the functor $\mathfrak{M}$ in general. However, if we restrict to patterns $\mathcal{O}$ such that $\mathcal{O}^{\mathrm{el}}$ is an ∞-groupoid, then this construction *does* give a polynomial monad for any essentially surjective morphism f of ∞-groupoids, since in this case the left adjoint $f_!$ preserves weakly contractible limits by [GHK17, Lemma 2.2.10] and the unit and counit for the adjunction $f_! \dashv f^*$ are cartesian transformations by [GHK17, Lemma 2.1.5].

14. Saturation and Canonical Patterns

Suppose $\mathcal{O}$ is an extendable algebraic pattern. Then the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$ is polynomial, and our results in the previous section associate to this another algebraic pattern $\overline{\mathcal{O}} := \mathcal{W}(T_{\mathcal{O}})$ such that there is an equivalence[9]

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \simeq \mathrm{Seg}_{\overline{\mathcal{O}}}(\mathcal{S}).$$

In this section we will explore the relationship between the patterns $\mathcal{O}$ and $\overline{\mathcal{O}}$. We will show that under a mild hypothesis on $\mathcal{O}$ (which can always be enforced by passing to a full subcategory without changing the monad) there is a canonical morphism of patterns $\mathcal{O} \to \overline{\mathcal{O}}$, which gives a natural transformation

$$\mathrm{id} \to \mathfrak{P}\mathfrak{M}.$$

We will also give an explicit necessary and sufficient condition on $\mathcal{O}$ for the map $\mathcal{O} \to \overline{\mathcal{O}}$ to be an equivalence, and discuss some examples where this holds.

Notation 14.1. In the first part of this section we fix an extendable pattern $\mathcal{O}$, and use the notations

$$\mathcal{O}^{\mathrm{el}} \xrightarrow{i} \mathcal{O}^{\mathrm{int}} \xrightarrow{j} \mathcal{O}$$

for the standard inclusions.

We begin by studying the localized Yoneda embedding

$$\mathcal{O}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}, \mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$$

for a pattern $\mathcal{O}$, which will give the canonical map to $\overline{\mathcal{O}}$.

Notation 14.2. Let $\Lambda_{\mathcal{O}}^{(\mathrm{int})} \colon \mathcal{O}^{(\mathrm{int}),\mathrm{op}} \to \mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S})$ denote the composite of the Yoneda embedding $y_{\mathcal{O}}^{(\mathrm{int})} \colon \mathcal{O}^{(\mathrm{int}),\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S})$ with the localization $\mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S})$.

Lemma 14.3. *For $X \in \mathcal{O}$, there is an equivalence*

$$\Lambda_{\mathcal{O}} X \simeq F_{\mathcal{O}} \Lambda_{\mathcal{O}}^{\mathrm{int}} X$$

in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$. This equivalence is natural with respect to inert morphisms, i.e. we have a commutative square

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{op}} & \xrightarrow{\Lambda_{\mathcal{O}}} & \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \\ {\scriptstyle j^{\mathrm{op}}}\uparrow & & \uparrow{\scriptstyle F_{\mathcal{O}}} \\ \mathcal{O}^{\mathrm{int},\mathrm{op}} & \xrightarrow{\Lambda_{\mathcal{O}}^{\mathrm{int}}} & \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}). \end{array}$$

[9] In the next section, we will see that furthermore the patterns $\mathcal{O}$ and $\overline{\mathcal{O}}$ determine the same polynomial monad.

Proof. For $\Phi \in \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$, we have natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}X, \Phi) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}X, \Phi) \\
&\simeq \Phi(X), \\
\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(F_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X, \Phi) &\simeq \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}}X, U_{\mathcal{O}}\Phi) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O}^{\mathrm{int}},\mathcal{S})}(y^{\mathrm{int}}_{\mathcal{O}}X, U_{\mathcal{O}}\Phi) \\
&\simeq U_{\mathcal{O}}\Phi(X) \\
&\simeq \Phi(X).
\end{aligned}$$

The objects $\Lambda_{\mathcal{O}}X$ and $F_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X$ therefore corepresent the same copresheaf on $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ and hence are equivalent. Moreover, this equivalence is by construction natural in $\mathcal{O}^{\mathrm{int}}$. □

Lemma 14.4. *The map*

$$\mathrm{Map}_{\mathcal{O}}(X, Y) \to \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X)$$

given by the functor $\Lambda_{\mathcal{O}}$ fits in a commutative square

$$\begin{array}{ccc}
\mathrm{colim}_{O\to Y \in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X, O) & \longrightarrow & \mathrm{colim}_{O\to Y \in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}}O, \Lambda^{\mathrm{int}}_{\mathcal{O}}X) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{Map}_{\mathcal{O}}(X, Y) & \longrightarrow & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X),
\end{array}$$

where the vertical maps are equivalences and the top horizontal map comes from the functor $\Lambda^{\mathrm{int}}_{\mathcal{O}}$.

Proof. From the commutative square of functors in Lemma 14.3 we get for all $O \in \mathcal{O}$ a commutative square

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X, O) & \longrightarrow & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}}O, \Lambda^{\mathrm{int}}_{\mathcal{O}}X) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{O}}(X, O) & \longrightarrow & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}O, \Lambda_{\mathcal{O}}X),
\end{array}$$

where the right-hand map can be identified with $\Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) \to \mathrm{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(O)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O')$ which is the canonical map to the colimit from the component at id_O. On the other hand, for any active morphism $O \to Y$ we have a natural commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Map}_{\mathcal{O}}(X, O) & \longrightarrow & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}O, \Lambda_{\mathcal{O}}X) & \xrightarrow{\sim} & T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) & \xrightarrow{\sim} & \mathrm{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(O)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{O}}(X, Y) & \longrightarrow & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}X, \Lambda_{\mathcal{O}}Y) & \xrightarrow{\sim} & T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X(Y) & \xrightarrow{\sim} & \mathrm{colim}_{O''\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O''),
\end{array}$$

where the description of $F_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X$ as a left Kan extension implies that the right-hand map is given on the component $\Lambda^{\mathrm{int}}_{\mathcal{O}}X(O')$ for $O' \to O$ by the canonical map $\Lambda^{\mathrm{int}}_{\mathcal{O}}X(O') \to \mathrm{colim}_{O''\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O'')$ for the component at $O' \to O \to Y$. Putting these two diagrams together we therefore obtain natural commutative squares

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X, O) & \longrightarrow & \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{O}}(X, Y) & \longrightarrow & \mathrm{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O'),
\end{array}$$

for every active morphism $\phi\colon O \to Y$, where the right vertical map is the canonical one from the component of the colimit at ϕ. Taking colimits over $\mathrm{Act}_{\mathcal{O}}(Y)$ we therefore get a commutative square

$$\begin{array}{ccc} \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) & \longrightarrow & \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) \\ \downarrow & & \| \\ \mathrm{Map}_{\mathcal{O}}(X,Y) & \longrightarrow & \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O). \end{array}$$

Here the inert–active factorization system on $\mathcal{O}$ implies that the left vertical map is an equivalence, since its fibre at a morphism $\psi\colon X \to Y$ can be identified with the space of inert–active factorizations of ψ, and this completes the proof. □

Remark 14.5. For $Y \in \mathcal{O}$, we have a natural equivalence

$$\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}}Y, \Phi) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O}^{\mathrm{int}},\mathcal{S})}(y^{\mathrm{int}}_{\mathcal{O}}Y, \Phi) \simeq \Phi(Y).$$

In particular,

$$\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}}Y, T_{\mathcal{O}}*) \simeq \mathrm{Act}_{\mathcal{O}}(Y),$$

and so a morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}}Y \to T_{\mathcal{O}}*$ corresponds to an active morphism $X \to Y$ in $\mathcal{O}$.

We will now show that this equivalence identifies active morphisms in $\mathcal{O}$ with generic morphisms in $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$:

Proposition 14.6. *Suppose $\Lambda^{\mathrm{int}}_{\mathcal{O}}Y \xrightarrow{\eta} T_{\mathcal{O}}*$ corresponds to the active morphism $X \overset{\phi}{\rightsquigarrow} Y$ in $\mathrm{Act}_{\mathcal{O}}(Y)$ under the equivalence of Remark 14.5. Then the generic–free factorization of η is*

$$\Lambda^{\mathrm{int}}_{\mathcal{O}}Y \xrightarrow{\hat{\phi}} T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X \to T_{\mathcal{O}}*,$$

where the first morphism is adjoint to $\Lambda_{\mathcal{O}}(\phi)\colon \Lambda_{\mathcal{O}}Y \to \Lambda_{\mathcal{O}}X$.

Proof. We first check that this factorization exists. By Lemma 14.4 the morphism $\hat{\phi}$ adjoint to $\Lambda(\phi)$ corresponds to the point in $T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X(Y) \simeq \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O)$ given by the composite

$$\{\mathrm{id}_X\} \to \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,X) \to \operatorname*{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \to \operatorname*{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O),$$

where the second morphism is the canonical one from the component of the colimit at ϕ. We therefore have a commutative diagram

$$\begin{array}{ccc} & & * \\ & \swarrow & \downarrow \\ \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) & \searrow & \\ \downarrow & & \\ \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) & \longrightarrow & \mathrm{Act}_{\mathcal{O}}(Y), \end{array}$$

where the outer triangle corresponds to the desired factorization

$$\begin{array}{ccc} & \Lambda^{\mathrm{int}}_{\mathcal{O}}Y & \\ {\scriptstyle\hat{\phi}}\swarrow & & \searrow{\scriptstyle\eta} \\ T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X & \longrightarrow & T_{\mathcal{O}}(*). \end{array}$$

Now we must show that $\hat{\phi}$ is generic, so suppose we have a commutative square

$$\begin{array}{ccc} \Lambda^{\mathrm{int}}Y & \xrightarrow{\theta} & T_{\mathcal{O}}\Phi \\ {\scriptstyle\hat{\phi}}\downarrow & & \downarrow \\ T_{\mathcal{O}}\Lambda^{\mathrm{int}}X & \longrightarrow & T_{\mathcal{O}}*, \end{array}$$

where the top horizontal map corresponds to a point p in the fibre $\Phi(X)$ of $T_{\mathcal{O}}\Phi(Y) \simeq \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Phi(O)$ at ϕ. Suppose we have a commutative triangle of the form

$$\begin{array}{ccc} & \Lambda^{\mathrm{int}}_{\mathcal{O}} Y & \\ {}^{\hat{\phi}}\swarrow & & \searrow^{\theta} \\ T_{\mathcal{O}}\Lambda^{\mathrm{int}} X & \xrightarrow{T\psi} & T_{\mathcal{O}}\Phi. \end{array}$$

This amounts to an equivalence between p and the image

$$* \xrightarrow{\mathrm{id}_X} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,X) \to \operatorname*{colim}_{O\in\mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \to \operatorname*{colim}_{O\in\mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}} X(O) \to \operatorname*{colim}_{O\in\mathrm{Act}_{\mathcal{O}}(Y)} \Phi(O).$$

But since the last map arises from $T\psi$, there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,X) & \longrightarrow & \Lambda^{\mathrm{int}}_{\mathcal{O}} X(X) & \longrightarrow & \Phi(X) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{colim}_{O\in\mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) & \longrightarrow & \operatorname{colim}_{O\in\mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}} X(O) & \longrightarrow & \operatorname{colim}_{O\in\mathrm{Act}_{\mathcal{O}}(Y)} \Phi(O), \end{array}$$

which tells us that ψ must be the morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}} X \to \Phi$ obtained by localizing the unique natural transformation $y^{\mathrm{int}}_{\mathcal{O}} X \to \Phi$ that takes id_X to the point p. Thus $\hat{\phi}$ satisfies the universal property of generic morphisms described in Remark 11.2. By the uniqueness of generic–free factorizations, this completes the proof. □

This proposition allows us to identify the objects of $\mathcal{U}(T_{\mathcal{O}})$:

Definition 14.7. We say an object $O \in \mathcal{O}$ is *necessary* if it admits an active morphism $O \to E$ for some $E \in \mathcal{O}^{\mathrm{el}}$, and denote by $\mathcal{O}^{\circ}$ the full subcategory of $\mathcal{O}$ spanned by the necessary objects. We say the pattern $\mathcal{O}$ is *slim* if all objects are necessary, and write $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$ for the full subcategory of $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$ spanned by the slim extendable patterns.

Corollary 14.8. *Let $\mathcal{O}$ be an extendable algebraic pattern, and let $\overline{\mathcal{O}} := \mathcal{W}(T_{\mathcal{O}})$ denote the corresponding canonical pattern. Then:*

(i) The objects of $\mathcal{U}(T_{\mathcal{O}})$ are the objects of $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ of the form $\Lambda^{\mathrm{int}}_{\mathcal{O}} X$ with $X \in \mathcal{O}^{\circ}$. Thus $\Lambda^{\mathrm{int}}_{\mathcal{O}}$ induces an essentially surjective functor $\mathcal{O}^{\circ,\mathrm{int}} \to \mathcal{U}(T_{\mathcal{O}})$.

(ii) The objects of $\overline{\mathcal{O}}$ are the objects of $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ of the form $\Lambda_{\mathcal{O}} X$ with $X \in \mathcal{O}^{\circ}$. Thus $\Lambda_{\mathcal{O}}$ induces an essentially surjective functor $\mathcal{O}^{\circ} \to \overline{\mathcal{O}}$.

(iii) A morphism $\Lambda_{\mathcal{O}} X \to \Lambda_{\mathcal{O}} Y$ is active if and only if it is a composite of an equivalence and the image of an active morphism $X \to Y$ in $\mathcal{O}^{\circ}$. In particular, the functor $\mathcal{O}^{\circ} \to \overline{\mathcal{O}}$ preserves inert and active morphisms.

Proof. By definition, the objects of $\mathcal{U}(T_{\mathcal{O}})$ are the objects Φ of $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ that admit a generic morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}} E \to T_{\mathcal{O}}\Phi$ with $E \in \mathcal{O}^{\mathrm{el}}$. Such a generic morphism is determined by a morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}} E \to T_{\mathcal{O}}*$, and from Proposition 14.6 we see that the generic–free factorizations of such morphisms yield precisely the objects of $\mathcal{O}^{\circ}$. This proves (i), from which (ii) follows using Lemma 14.3. Finally, as active morphisms in $\overline{\mathcal{O}}$ are those morphisms which are adjoint to generic maps by Lemma 12.16, the first part of (iii) follows from the identification of such generic morphisms with active morphisms in $\mathcal{O}$ in Proposition 14.6. This shows that $\Lambda_{\mathcal{O}}$ preserves active morphisms, while the commutative square of Lemma 14.3 implies that it preserves inert morphisms, since free morphisms in $\overline{\mathcal{O}}$ are in particular inert. □

Remark 14.9. If $\mathcal{O}$ is a slim extendable pattern, then Corollary 14.8 says that $\overline{\mathcal{O}}$ has the same objects as $\mathcal{O}$, and the active morphisms are obtained by combining active morphisms from $\mathcal{O}$ with equivalences (which may not all come from $\mathcal{O}$).

Remark 14.10. If O is necessary and $O' \to O$ is an active morphism, then O' is also necessary. This implies that the inert–active factorization system in $\mathcal{O}$ restricts to $\mathcal{O}^\circ$, and that $\mathrm{Act}_{\mathcal{O}}(O) \simeq \mathrm{Act}_{\mathcal{O}^\circ}(O)$ for $O \in \mathcal{O}^\circ$. It follows that $\mathcal{O}^\circ$ is extendable when $\mathcal{O}$ is. In this case we therefore have a commutative diagram

$$\begin{array}{ccc} \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) & \longrightarrow & \mathrm{Seg}_{\mathcal{O}^\circ}(\mathcal{S}) \\ \downarrow & & \downarrow \\ \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) & \longrightarrow & \mathrm{Seg}_{\mathcal{O}^{\circ,\mathrm{int}}}(\mathcal{S}) \\ & \searrow^{\sim} \quad \swarrow^{\sim} & \\ & \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S}), & \end{array}$$

where the vertical maps are monadic right adjoints. The lower horizontal map is an equivalence since the diagonal maps are equivalences. Since the two monads on $\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$ are the same (by definition $\mathrm{Act}_{\mathcal{O}}(E) \simeq \mathrm{Act}_{\mathcal{O}^\circ}(E)$ for $E \in \mathcal{O}^{\mathrm{el}}$), the top horizontal morphism is also an equivalence. Thus the patterns $\mathcal{O}$ and $\mathcal{O}^\circ$ describe the same monad, and so the objects of $\mathcal{O}$ that do not lie in $\mathcal{O}^\circ$ are in this sense *unnecessary*.

Examples 14.11. The examples of patterns discussed in §3 are all slim, with the exception of the pattern $\mathbf{\Delta}_{\Phi}^{\mathrm{op},\natural}$ of Example 3.8. The corresponding slim pattern $\mathbf{\Delta}_{\Phi}^{\mathrm{op},\natural,\circ}$ is the full subcategory spanned by objects $([m], f)$ such that $f(m) \cong *$. Another non-slim example is the extension of the dendroidal category $\mathbf{\Omega}^{\mathrm{op},\natural}$ to a category of forests considered in [HHM16], which has $\mathbf{\Omega}^{\mathrm{op},\natural}$ as its slim subpattern.

Remark 14.12. If T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$ then the algebraic pattern $\mathcal{W}(T)$ is slim. This follows from the fact that objects in $\mathcal{W}(T)$ can be identified with objects in $\mathcal{U}(T)$, i.e. objects X admitting a generic map $I \to TX$ with $I \in \mathcal{I}$. Since $\mathcal{I}$ has the same objects as $\mathcal{W}(T)^{\mathrm{el}}$ and every generic map is adjoint to an active morphism in $\mathcal{W}(T)$, the algebraic pattern $\mathcal{W}(T)$ is indeed slim. We can thus regard $\mathfrak{P}$ as a functor

$$\mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}.$$

Remark 14.13. Suppose $f \colon \mathcal{O} \to \mathcal{P}$ is a Segal morphism between slim extendable patterns. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{op}} & \xrightarrow{f^{\mathrm{op}}} & \mathcal{P}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{O}, \mathcal{S}) & \xrightarrow{f_!} & \mathrm{Fun}(\mathcal{P}, \mathcal{S}) \\ \downarrow & & \downarrow \\ \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) & \longrightarrow & \mathrm{Seg}_{\mathcal{P}}(\mathcal{S}). \end{array}$$

In other words, we have a commutative square

$$\begin{array}{ccc} \mathcal{O}^{(\mathrm{int}),\mathrm{op}} & \xrightarrow{f^{(\mathrm{int}),\mathrm{op}}} & \mathcal{P}^{(\mathrm{int}),\mathrm{op}} \\ {\scriptstyle \Lambda_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \Lambda_{\mathcal{P}}} \\ \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) & \longrightarrow & \mathrm{Seg}_{\mathcal{P}}(\mathcal{S}), \end{array}$$

which restricts to a commutative square

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{f} & \mathcal{P} \\ \downarrow & & \downarrow \\ \overline{\mathcal{O}} & \longrightarrow & \overline{\mathcal{P}}, \end{array}$$

where all the morphisms are Segal morphism of algebraic patterns. Thus we have proved:

Proposition 14.14. *There is a natural transformation* $\sigma\colon \mathrm{id} \to \mathfrak{PM}$ *of functors* $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$.

Our next goal is to identify when the map $\sigma_{\mathcal{O}}$ is an equivalence, which turns out to correspond to the following condition:

Definition 14.15. If $\mathcal{O}$ is a slim extendable pattern, we say that $\mathcal{O}$ is *saturated* if for every object $O \in \mathcal{O}$ the copresheaf

$$\mathrm{Map}_{\mathcal{O}}(O, -)\colon \mathcal{O} \to \mathcal{S}$$

is a Segal $\mathcal{O}$-space. We write $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$ for the full subcategory of $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$ spanned by the saturated patterns.

Proposition 14.16. *The following conditions are equivalent for a slim extendable pattern* $\mathcal{O}$*:*

(1) $\mathcal{O}$ *is saturated.*
(2) For every $X \in \mathcal{O}$*, the canonical functor* $\mathcal{O}^{\mathrm{int},\triangleleft}_{X/} \to \mathcal{O}$ *is a limit diagram.*
(3) The Yoneda embedding $\mathcal{O}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$ *factors through* $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$.
(4) The functor $\Lambda_{\mathcal{O}}\colon \mathcal{O}^{\mathrm{op}} \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ *is fully faithful.*

Proof. The equivalence of (1), (2), and (3) is clear, and it is also clear that (3) implies (4). We prove the remaining implication from (4) to (3) by showing that (4) implies that for every $X \in \mathcal{O}$ there is an equivalence $y_{\mathcal{O}}X \simeq \Lambda_{\mathcal{O}}X$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{S})$. We have

$$\mathrm{Map}_{\mathcal{O}}(X, Y) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, y_{\mathcal{O}}X) \to \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X),$$

where the first map is the Yoneda embedding. Since the composition of the first two morphisms is an equivalence by (4), the second map is an equivalence. The last map is an equivalence because $\Lambda_{\mathcal{O}}X$ is a local object and $y_{\mathcal{O}}Y \to \Lambda_{\mathcal{O}}Y$ is a local equivalence. Hence, we have

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, y_{\mathcal{O}}X) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X)$$

for every object $Y \in \mathcal{O}$, which then implies that $y_{\mathcal{O}}X \simeq \Lambda_{\mathcal{O}}X$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{S})$ by the Yoneda Lemma. □

Lemma 14.17. *Suppose* T *is a polynomial monad. Then the pattern* $\mathcal{W}(T)$ *is saturated.*

Proof. We already know the pattern $\mathcal{W}(T)$ is extendable (by Corollary 13.9) and slim (by Remark 14.12). By definition, $\mathcal{W}(T)^{\mathrm{op}}$ is a full subcategory of $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$, and the Nerve Theorem 11.15 implies that the restricted Yoneda functor $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Fun}(\mathcal{W}(T), \mathcal{S})$ is fully faithful with image $\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$. This implies in particular that the Yoneda embedding of $\mathcal{W}(T)$ takes values in Segal $\mathcal{W}(T)$-spaces, which implies that $\mathcal{W}(T)$ is saturated by Proposition 14.16. □

Lemma 14.17 implies in particular that the pattern $\overline{\mathcal{O}}$ is always saturated, which gives the following:

Corollary 14.18. *The morphism* $\sigma_{\mathcal{O}}\colon \mathcal{O} \to \overline{\mathcal{O}}$ *is an equivalence if and only if* $\mathcal{O}$ *is saturated.* □

Corollary 14.19. *The natural transformation* σ *exhibits the full subcategory* $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$ *as a localization of* $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$.

Proof. Let $L := \mathfrak{PM}$; then the essential image of L is precisely $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$: by Corollary 14.18 the image of L contains all saturated patterns, while all patterns in the image of L are saturated by Lemma 14.17. To see that L and σ exhibit $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$ as a localization, we apply [Lur09, Proposition 5.2.7.4]. It suffices to verify condition (3) of this result, namely that the two morphisms

$$\sigma_{L\mathcal{O}}, L(\sigma_{\mathcal{O}})\colon L\mathcal{O} \to LL\mathcal{O}$$

are both equivalences for all $\mathcal{O}$ in $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$. For $\sigma_{L\mathcal{O}}$ this holds by Corollary 14.18, since $L\mathcal{O}$ is saturated, and for $L(\sigma_{\mathcal{O}})$ it holds since $\sigma_{\mathcal{O}}$ induces an equivalence

$$\sigma^*_{\mathcal{O}}\colon \mathrm{Seg}_{L\mathcal{O}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}),$$

and $L\sigma_{\mathcal{O}}$ is obtained by restricting the inverse of this equivalence. □

The following proposition shows that we can equivalently characterize saturated patterns in terms of their subcategories of inert morphisms:

Proposition 14.20. *The following conditions are equivalent for a slim extendable pattern $\mathcal{O}$:*

(1) $\mathcal{O}$ is saturated.

(2) For every $X \in \mathcal{O}$, the functor

$$\mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X, \text{–})\colon \mathcal{O}^{\mathrm{int}} \to \mathcal{S}$$

is a Segal $\mathcal{O}^{\mathrm{int}}$-space.

(3) For every X in $\mathcal{O}$, the diagram $\mathcal{O}^{\mathrm{el},\triangleleft}_{X/} \to \mathcal{O}^{\mathrm{int}}$ is a limit diagram.

(4) The Yoneda embedding $\mathcal{O}^{\mathrm{int,op}} \to \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S})$ factors through $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$.

(5) The functor $\Lambda^{\mathrm{int}}_{\mathcal{O}}\colon \mathcal{O}^{\mathrm{int,op}} \to \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ is fully faithful.

Proof. The equivalence of conditions (2)–(5) follows exactly as in the proof of Proposition 14.16. It remains to show that these conditions are equivalent to $\mathcal{O}$ being saturated.

Since $\mathcal{O}$ is by assumption extendable, by Proposition 8.8 we have a commutative square

$$\begin{array}{ccc} \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) & \longrightarrow & \mathrm{Fun}(\mathcal{O}, \mathcal{S}) \\ {\scriptstyle F_{\mathcal{O}}}\uparrow & & \uparrow{\scriptstyle j_{\mathcal{O},!}} \\ \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) & \longrightarrow & \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}). \end{array}$$

Omitting notation for the horizontal inclusions, we have equivalences

$$\Lambda_{\mathcal{O}} X \simeq F_{\mathcal{O}} \Lambda^{\mathrm{int}}_{\mathcal{O}} X \simeq j_{\mathcal{O},!} \Lambda^{\mathrm{int}}_{\mathcal{O}} X.$$

If condition (4) holds, then $\Lambda^{\mathrm{int}}_{\mathcal{O}} X$ is the representable presheaf $y^{\mathrm{int}}_{\mathcal{O}} X$, hence

$$j_{\mathcal{O},!} \Lambda^{\mathrm{int}}_{\mathcal{O}} X \simeq j_{\mathcal{O},!} y^{\mathrm{int}}_{\mathcal{O}} X \simeq y_{\mathcal{O}} j_{\mathcal{O}}(X).$$

In other words, $\Lambda_{\mathcal{O}} X$ is precisely the presheaf represented by $X \in \mathcal{O}$, which implies that $\mathcal{O}$ is saturated by Proposition 14.16.

Conversely, suppose $\mathcal{O}$ is saturated. By Proposition 14.16 this means that for every $X \in \mathcal{O}$, the diagram $\mathcal{O}^{\mathrm{el},\triangleleft}_{X/} \to \mathcal{O}$ is a limit diagram. To show that this diagram is then also a limit in the subcategory $\mathcal{O}^{\mathrm{int}}$ (and hence verify condition (3)), it is enough to show that a morphism $\phi\colon Y \to X$ is inert if the composites $Y \to X \rightarrowtail E$ are all inert. Using the inert–active factorization system, we see that it suffices to consider the case where ϕ is active and prove that it is an equivalence. Recall that we have a morphism

$$\mathrm{Act}_{\mathcal{O}}(X) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Act}_{\mathcal{O}}(E),$$

which takes $\phi\colon Y \rightsquigarrow X$ to the active parts of the inert–active decompositions of the composites $Y \rightsquigarrow X \rightarrowtail E$. Since these composites are inert, the image of ϕ is given by id_E for all $E \in \mathcal{O}^{\mathrm{el}}_{X/}$, so that ϕ has the same image as id_X. But since $\mathcal{O}$ is extendable, this map of ∞-groupoids is an equivalence, and hence ϕ is equivalent to id_X in $\mathrm{Act}_{\mathcal{O}}(X)$, which means precisely that ϕ is an equivalence. □

We end this section by looking at some examples of saturated and non-saturated patterns.

Examples 14.21. The patterns $\boldsymbol{\Delta}^{n,\mathrm{op},\natural}$, $\boldsymbol{\Theta}^{\mathrm{op},\natural}_n$, and $\boldsymbol{\Omega}^{\mathrm{op},\natural}$ (described in Examples 3.3, 3.5, and 3.7, respectively) are all saturated. In the case of $\boldsymbol{\Delta}^{\mathrm{op},\natural}$, for example, this amounts to the observation that the object $[n] \in \boldsymbol{\Delta}^{\mathrm{int}}$ is a colimit,

$$[1] \amalg_{[0]} \cdots \amalg_{[0]} [1] \simeq [n],$$

while for $\boldsymbol{\Omega}^{\mathrm{op},\natural}$ the required colimit in $\boldsymbol{\Omega}^{\mathrm{int}}$ amounts to a decomposition of a tree as a colimit of its nodes and edges, and follows from [Koc11, Proposition 1.1.19].

Example 14.22. The pattern $\mathbb{F}_*^\flat$ from Example 3.1 is *not* saturated: The functor $\Lambda^{\mathrm{int}}_{\mathbb{F}_*^\flat}\colon \mathbb{F}_*^{\flat,\mathrm{int},\mathrm{op}} \to \mathcal{S}$ takes $\langle n\rangle$ to a finite set $\mathbf{n}$ with n elements, and an inert morphism $\langle n\rangle \to \langle m\rangle$ to the map $\mathbf{m} \to \mathbf{n}$ that takes $i \in \mathbf{m}$ to its unique preimage $\phi^{-1}(i) \in \mathbf{n}$. Thus inert morphisms correspond bijectively to *injective* morphisms of finite sets, and the functor is not fully faithful. The canonical pattern $\overline{\mathbb{F}}_*^\flat \subseteq \mathrm{Seg}_{\mathbb{F}_*^\flat}(\mathcal{S})^{\mathrm{op}}$ consists of the free commutative monoids on finite sets. By work of Cranch [Cra11] this can be identified with the (2,1)-category $\mathrm{Span}(\mathbb{F})$ whose objects are finite sets and whose morphisms are *spans* of finite sets, with $\mathbb{F}_* \to \overline{\mathbb{F}}_*$ identifying $\mathbb{F}_*$ with the subcategory where the morphisms from I to J are spans $I \leftarrow K \to J$ with the backward map *injective*.

Example 14.23. More generally, for any ∞-operad $\mathcal{O}$ (in the sense of [Lur17]) the canonical pattern $\overline{\mathcal{O}}$ can be identified with the opposite of the ∞-category of finitely generated free $\mathcal{O}$-monoids in $\mathcal{S}$, i.e. the *Lawvere theory* for $\mathcal{O}$-monoids.

Remark 14.24. See [GGN15, Ber20] for more on Lawvere theories in the ∞-categorical context. Note that the monads corresponding to Lawvere theories always preserve sifted colimits, so the (coloured) Lawvere theories that fit into our theory are precisely the monads on $\mathcal{S}^X$ for an ∞-groupoid X that preserve both sifted colimits and weakly contractible limits. These are precisely the *analytic* monads studied in [GHK17], where they are identified with ∞-operads in the sense of dendroidal Segal spaces.

Example 14.25. The pattern $\mathbf{\Gamma}^{\mathrm{op},\natural}$ of Example 3.9 is not saturated. We expect that its saturation is the $(2,1)$-category of graphs implicitly defined by Kock in [Koc16, §3.3].

15. Completion of Polynomial Monads

In this section we will study a class of polynomial monads that is particularly closely related to algebraic patterns, namely the *complete* ones in the following sense:

Definition 15.1. Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. We say that T is *complete* if the functor $\mathcal{I} \to \mathcal{W}(T)^{\mathrm{el}}$ underlying $\tau_T\colon T \to \mathfrak{M}\mathfrak{P}T$ is an equivalence. We write cPolyMnd for the full subcategory of PolyMnd spanned by the complete polynomial monads.

We will see that the polynomial monad corresponding to an extendable pattern is *always* complete, so that the functor $\mathfrak{M}$ takes values in cPolyMnd. Moreover, we will show that the transformation $\tau\colon \mathrm{id} \to \mathfrak{M}\mathfrak{P}$ exhibits cPolyMnd as a localization of PolyMnd, and the functors $\mathfrak{M}$ and $\mathfrak{P}$ restrict to an *equivalence*

$$\mathrm{cPolyMnd} \simeq \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$$

between complete polynomial monads and saturated patterns.

Remark 15.2. The term *complete* is inspired by the equivalence of [GHK17] between dendroidal Segal spaces and *analytic* monads, which are the polynomial monads on presheaves over ∞-groupoids that preserve sifted colimits. Under this equivalence, the complete dendroidal Segal spaces (meaning those whose underlying Segal spaces are complete in the sense of Rezk [Rez01]) are precisely those analytic monads that are complete in our sense.

We begin by giving some alternative descriptions of the complete polynomial monads:

Proposition 15.3. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. The following are equivalent:*

(1) T is complete.
(2) The morphism $\tau_T\colon T \to \mathfrak{M}\mathfrak{P}T$ is an equivalence.
(3) The functor $u\colon \mathcal{U}(T) \to \mathcal{W}(T)^{\mathrm{int}}$ is an equivalence.
(4) The functor $j\colon \mathcal{U}(T) \to \mathcal{W}(T)$ is a subcategory inclusion, i.e. it is faithful and induces an equivalence $\mathcal{U}(T)^{\simeq} \xrightarrow{\sim} \mathcal{W}(T)^{\simeq}$ on underlying ∞-groupoids.
(5) The functor j is faithful and every equivalence is in its image.

Proof. To see that (1) is equivalent to (2), observe that the morphism τ_T in PolyMnd is given by the morphism $e\colon \mathcal{I} \to \mathcal{W}(T)^{\mathrm{el}}$ together with the commutative square

$$\begin{array}{ccc} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) & \xrightarrow{\sim} & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) & \xrightarrow{e^*} & \mathrm{Fun}(\mathcal{I}, \mathcal{S}), \end{array}$$

and so τ_T is an equivalence if and only if e is an equivalence.

It is clear that (3) implies (1), since e is obtained from u by restricting to a full subcategory. Conversely, if e is an equivalence, then the commutative square of Corollary 13.4 gives a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) & \xrightarrow{\sim} & \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \\ \wr\downarrow & & \downarrow u^* \\ \mathrm{Fun}(\mathcal{I}, \mathcal{S}) & \xrightarrow{\sim} & \mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S}), \end{array}$$

where $\mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{U}(T), \mathcal{S})$ of functors right Kan extended from $\mathcal{I}$; the functor

$$u^*\colon \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$$

is therefore an equivalence. Here $\mathcal{W}(T)^{\mathrm{int,op}}$ is a full subcategory of $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$ via the Yoneda embedding by Proposition 14.20, since $\mathcal{W}(T)$ is saturated by Lemma 14.17. Moreover, $\mathcal{U}(T)^{\mathrm{op}}$ is a full subcategory of $\mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$ by Proposition 11.13. The inverse of u^* is given by left Kan extension $u_!$ followed by localization from $\mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S})$ to $\mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$, which restricts to just u on $\mathcal{U}(T)^{\mathrm{op}}$ since $\mathcal{W}(T)^{\mathrm{int,op}}$ is already in $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$. Hence u is the restriction of the equivalence $(u^*)^{-1}$ to a full subcategory, which implies that u is indeed an equivalence.

Since $\mathcal{W}(T)^{\mathrm{int}}$ is by definition a subcategory of $\mathcal{W}(T)$, (3) immediately implies (4). On the other hand, (4) implies (3) since the inert morphisms in $\mathcal{W}(T)$ are precisely those that are composites of morphisms in the image of u and equivalences in $\mathcal{W}(T)$.

Finally, (4) trivially implies (5), while given (5) we know that

$$\mathrm{Map}_{\mathcal{U}(T)}(X, Y) \to \mathrm{Map}_{\mathcal{W}(T)}(jX, jY)$$

is a monomorphism whose image contains the components that correspond to equivalences in $\mathcal{W}(T)$. Since j is conservative by Lemma 12.13, the components that map to these are precisely those that correspond to equivalences in $\mathcal{U}(T)$, so that j restricts to an equivalence $\mathcal{U}(T)^{\simeq} \to \mathcal{W}(T)^{\simeq}$. □

Proposition 15.4. *Suppose $\mathcal{O}$ is a slim extendable pattern. Then $T_{\mathcal{O}}$ is a complete polynomial monad.*

For the proof we need the following observation:

Lemma 15.5. *Suppose $\phi\colon X \to Y$ is an active morphism such that $\Lambda_{\mathcal{O}}\phi$ is an equivalence in* $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$. *Then ϕ is an equivalence in $\mathcal{O}$.*

Proof. Suppose $\alpha\colon \Lambda_{\mathcal{O}}X \to \Lambda_{\mathcal{O}}Y$ is the inverse of $\Lambda_{\mathcal{O}}\phi$. By Proposition 14.6 we can factor α as $\Lambda_{\mathcal{O}}X \xrightarrow{\Lambda_{\mathcal{O}}\psi} \Lambda_{\mathcal{O}}Y' \xrightarrow{\alpha'} \Lambda_{\mathcal{O}}Y$ where α' is free and ψ is an active morphism determined up to equivalence in $\mathcal{O}$ (and both $\Lambda_{\mathcal{O}}\psi$ and α' are equivalences since this is an active–inert factorization). Now the composite $\alpha\Lambda_{\mathcal{O}}\phi$ is the identity, so by Proposition 14.6 the composite $\phi\psi$ lies in the same component of $\mathrm{Act}_{\mathcal{O}}(Y)$ as id_Y, i.e. $\phi\psi$ must be an equivalence. Applying the same argument to ψ, we see that ψ has inverses on both sides in $\mathcal{O}$ and so is an equivalence, hence ϕ is also an equivalence. □

Proof of Proposition 15.4. By Proposition 15.3 the polynomial monad $T_{\mathcal{O}}$ is complete if and only if $j\colon \mathcal{U}(T_{\mathcal{O}}) \to \mathcal{W}(T_{\mathcal{O}})$ is faithful and all equivalences are in its image.

Since $\mathcal{O}$ is slim, the objects of $\mathcal{U}(T_{\mathcal{O}})$ are precisely the objects $\Lambda^{\mathrm{int}}_{\mathcal{O}}X$ for $X \in \mathcal{O}^{\mathrm{int}}$, by Corollary 14.8. To show that j is faithful, we must check that for all $X, Y \in \mathcal{O}^{\mathrm{int}}$, the map

$$\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}}X, \Lambda^{\mathrm{int}}_{\mathcal{O}}Y) \to \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}X, \Lambda_{\mathcal{O}}Y)$$

is a monomorphism. Lemma 14.4 and Remark 14.5 imply that this map can be identified with the map

$$(\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(Y) \to \operatorname*{colim}_{O \in \mathrm{Act}_{\mathcal{O}}(Y)} (\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(O),$$

given by taking $(\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(Y)$ to the component in the colimit corresponding to $\mathrm{id}_Y \in \mathrm{Act}_{\mathcal{O}}(Y)$. This component is of the form $(\mathcal{O}^{\simeq})_{/Y}$ and so is contractible, which means that the colimit decomposes as a disjoint union of $(\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(Y)$ and the colimit over the other components of $\mathrm{Act}_{\mathcal{O}}(Y)$. This means j is indeed faithful.

Now suppose $\alpha\colon \Lambda_{\mathcal{O}}X \to \Lambda_{\mathcal{O}}X'$ is an equivalence in $\mathcal{W}(T_{\mathcal{O}})$. Then by Proposition 14.6 we can factor α as

$$\Lambda_{\mathcal{O}}X \xrightarrow{\Lambda_{\mathcal{O}}\phi} \Lambda_{\mathcal{O}}Y \xrightarrow{j\psi} \Lambda_{\mathcal{O}}X',$$

where ϕ is active and both $\Lambda_{\mathcal{O}}\phi$ and $j\psi$ are equivalences (since this is in particular an active–inert factorization). Then Lemma 15.5 implies that ϕ is an equivalence in $\mathcal{O}$; but then ϕ is also inert, and so the commutative square in Lemma 14.3 implies that $\Lambda_{\mathcal{O}}\phi$ is $j(\Lambda^{\mathrm{int}}_{\mathcal{O}}\phi)$. Thus α is in the image of j, as required. □

Remark 15.6. It follows from Proposition 15.3 that for any slim extendable pattern $\mathcal{O}$, the morphism $\tau_{T_{\mathcal{O}}} : T_{\mathcal{O}} \to T_{\overline{\mathcal{O}}}$ is an equivalence, i.e. the extendable patterns $\mathcal{O}$ and $\overline{\mathcal{O}}$ correspond to the same monad. The saturated pattern $\overline{\mathcal{O}}$ is thus a canonical pattern associated to the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$.

Corollary 15.7. *The natural transformation* $\tau\colon \mathrm{id} \to \mathfrak{MP}$ *exhibits the full subcategory* cPolyMnd *as a localization of* PolyMnd.

Proof. Let $L := \mathfrak{MP}$; then the essential image of L is precisely cPolyMnd: by Proposition 15.3 the image of L contains all complete polynomial monads, while all monads in the image of L are complete by Proposition 15.4.

To see that L and τ exhibit cPolyMnd as a localization, we again apply the criterion of [Lur09, Proposition 5.2.7.4](3). We must thus show that the two morphisms

$$\tau_{LT}, L(\tau_T)\colon LT \to LLT$$

are both equivalences for all T in PolyMnd. For τ_{LT} this holds by Proposition 15.3, since LT is complete, while for $L(\tau_T)$ it holds since $\mathfrak{P}(\tau_T)$ is an equivalence (given by restricting the equivalence $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \xrightarrow{\sim} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ to a full subcategory). □

Theorem 15.8. *The functors* $\mathfrak{M}$ *and* $\mathfrak{P}$ *restrict to give an equivalence*

$$\mathrm{cPolyMnd} \simeq \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}.$$

Proof. We have shown that $\mathfrak{M}\mathcal{O}$ is always complete and $\mathfrak{P}T$ is always saturated, so the functors do restrict to these full subcategories. Moreover, we know that $\sigma_{\mathcal{O}}$ is an equivalence if and only if $\mathcal{O}$ is saturated, and τ_T is an equivalence if and only if T is complete. These natural transformations therefore restrict to natural equivalences on the full subcategories of saturated patterns and complete polynomial monads, and hence exhibit the restrictions of $\mathfrak{P}$ and $\mathfrak{M}$ as inverse equivalences. □

References

[Bar05] Clark Barwick, *(∞, n)-Cat as a closed model category*, ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)–University of Pennsylvania. MR 2706984

[Bar18] ———, *From operator categories to topological operads*, Geom. Topol. **22** (2018), no. 4, 1893–1959.

[Bat98] M. A. Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Adv. Math. **136** (1998), no. 1, 39–103. MR 1623672

[Ber02] Clemens Berger, *A cellular nerve for higher categories*, Adv. Math. **169** (2002), no. 1, 118–175. MR 1916373

[Ber07] ———, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), no. 1, 230–270.

[Ber20] John D. Berman, *Higher Lawvere theories*, J. Pure Appl. Algebra **224** (2020), no. 9, 106362, 17. MR 4083788

[BG19] John Bourke and Richard Garner, *Monads and theories*, Adv. Math. **351** (2019), 1024–1071. MR 3956765

[BMW12] Clemens Berger, Paul-André Melliès, and Mark Weber, *Monads with arities and their associated theories*, Journal of Pure and Applied Algebra **216** (2012), no. 8, 2029 – 2048, Special Issue devoted to the International Conference in Category Theory 'CT2010'.

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973. MR 0420609

[CHH18] Hongyi Chu, Rune Haugseng, and Gijs Heuts, *Two models for the homotopy theory of ∞-operads*, J. Topol. **11** (2018), no. 4, 857–873.

[CM13] Denis-Charles Cisinski and Ieke Moerdijk, *Dendroidal Segal spaces and ∞-operads*, J. Topol. **6** (2013), no. 3, 675–704. MR 3100887

[Cra11] James Cranch, *Algebraic theories, span diagrams and commutative monoids in homotopy theory*, 2011.

[DCH19] Gabriel C. Drummond-Cole and Philip Hackney, *A criterion for existence of right-induced model structures*, Bull. Lond. Math. Soc. **51** (2019), no. 2, 309–326. MR 3937590

[Get09] Ezra Getzler, *Operads revisited*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 675–698. MR 2641184

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153. MR 3450758

[GH15] David Gepner and Rune Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. **279** (2015), 575–716. MR 3345192

[GHK17] David Gepner, Rune Haugseng, and Joachim Kock, *∞-operads as analytic monads*, 2017.

[Hau18] Rune Haugseng, *On the equivalence between Θ_n-spaces and iterated Segal spaces*, Proc. Amer. Math. Soc. **146** (2018), no. 4, 1401–1415. MR 3754328

[Hau20] Rune Haugseng, *On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories*, 2020.

[HHM16] Gijs Heuts, Vladimir Hinich, and Ieke Moerdijk, *On the equivalence between Lurie's model and the dendroidal model for infinity-operads*, Adv. Math. **302** (2016), 869–1043. MR 3545944

[HR18] Philip Hackney and Marcy Robertson, *Lecture notes on infinity-properads*, 2016 MATRIX annals, MATRIX Book Ser., vol. 1, Springer, Cham, 2018, pp. 351–374. MR 3792530

[HRY15] Philip Hackney, Marcy Robertson, and Donald Yau, *Infinity properads and infinity wheeled properads*, Lecture Notes in Mathematics, vol. 2147, Springer, Cham, 2015. MR 3408444

[HRY19] ———, *Higher cyclic operads*, Algebr. Geom. Topol. **19** (2019), no. 2, 863–940. MR 3924179

[HRY20] ———, *A graphical category for higher modular operads*, Adv. Math. **365** (2020), 107044, 61. MR 4064770

[Joy97] André Joyal, *Disks, duality and θ-categories*, 1997, available from `http://ncatlab.org/nlab/files/JoyalThetaCategories.pdf`.

[Koc11] Joachim Kock, *Polynomial functors and trees*, Int. Math. Res. Not. IMRN (2011), no. 3, 609–673. MR 2764874

[Koc16] ———, *Graphs, hypergraphs, and properads*, Collect. Math. **67** (2016), no. 2, 155–190. MR 3484016

[Law68] F. William Lawvere, *Some algebraic problems in the context of functorial semantics of algebraic theories*, Reports of the Midwest Category Seminar. II, Springer, Berlin, 1968, pp. 41–61. MR 0231882

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur17] ———, *Higher algebra*, 2017, available from `http://www.math.ias.edu/~lurie/papers/HA.pdf`.

[May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York, 1972, Lectures Notes in Mathematics, Vol. 271. MR 0420610

[MW07] Ieke Moerdijk and Ittay Weiss, *Dendroidal sets*, Algebr. Geom. Topol. **7** (2007), 1441–1470. MR 2366165

[Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007. MR 1804411

[Rez10] ———, *A Cartesian presentation of weak n-categories*, Geom. Topol. **14** (2010), no. 1, 521–571. MR 2578310

[Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR 0353298

[Sta63] James Dillon Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. **108** (1963), 293–312. MR 0158400

[Web04] Mark Weber, *Generic morphisms, parametric representations and weakly cartesian monads*, Theory and Applications of Categories **13** (2004), no. 14, 191–234.

[Web07] ———, *Familial 2-functors and parametric right adjoints*, Theory and Applications of Categories **18** (2007), no. 22, 665–732.

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ENVELOPES FOR ALGEBRAIC PATTERNS

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Abstract. We generalize Lurie's construction of the symmetric monoidal envelope of an ∞-operad to the setting of algebraic patterns. This envelope becomes fully faithful when sliced over the envelope of the terminal object, and we characterize its essential image. Using this, we prove a comparison result that allows us to compare analogues of ∞-operads over various algebraic patterns. In particular, we show that the G-∞-operads of Nardin–Shah are equivalent to "fibrous patterns" over the $(2,1)$-category $\mathsf{Span}(\mathbb{F}_G)$ of spans of finite G-sets. When G is trivial this means that Lurie's ∞-operads can equivalently be defined over $\mathsf{Span}(\mathbb{F})$ instead of $\mathbb{F}_*$.

Contents

1. Introduction

In Lurie's seminal work on homotopy-coherent algebra [HA], the main objects used to encode algebraic structures are (symmetric) ∞-*operads*, which are defined as a certain type of functor of ∞-categories $\mathcal{O} \to \mathbb{F}_*$, where $\mathbb{F}_*$ is the category of finite pointed sets. However, as illustrated already in [HA], it can sometimes be useful to consider variants of this notion, for instance because they give a combinatorially simpler description of some structure. For example, Lurie also considers *planar* (or non-symmetric) ∞-operads, where the category $\mathbb{F}_*$ is replaced by the simplex category Δ^{op}. As a special case of a general comparison theorem [HA, Theorem 2.3.3.26] using the theory of *approximations* to ∞-operads, Lurie proves that there is an equivalence of ∞-categories

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between planar ∞-operads and ∞-operads over the (symmetric) associative operad Ass, given by pulling back along an explicit map $\Delta^{\mathrm{op}} \to \mathsf{Ass}$.

Our main goal in this paper is to prove a more general version of such comparisons. Before we explain this result in more detail, let us motivate it by (informally) stating the two main new comparisons we will apply it to:

- In the definition of symmetric ∞-operads, we can equivalently replace the category $\mathbb{F}_*$ of finite pointed sets by the (2,1)-category $\mathsf{Span}(\mathbb{F})$ of *spans* of finite sets.
- For G a finite group, the G-equivariant ∞-operads of Nardin and Shah [NS22] can equivalently be described as ∞-operads over the (2,1)-category $\mathsf{Span}(\mathbb{F}_G)$ of spans of finite G-sets.

Fibrous patterns. The general version of our main result is in the setting of *algebraic patterns* in the sense of Chu and Haugseng [CH21], which is a general framework for algebraic structures described by "Segal conditions". More precisely, an algebraic pattern is an ∞-category $\mathcal{O}$ equipped with a factorization system $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$ of "inert" and "active" morphisms and a full subcategory $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{int}}$ of "elementary" objects. This data lets one define *Segal $\mathcal{O}$-objects* in a complete ∞-category $\mathcal{C}$ as functors $F\colon \mathcal{O} \to \mathcal{C}$ such that for any object $O \in \mathcal{O}$ the natural map

$$F(O) \longrightarrow \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E)$$

is an equivalence, where $\mathcal{O}^{\mathrm{el}}_{O/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{O/}$ consists of inert morphisms from O to elementary objects. We can then consider a version of ∞-operads where the category $\mathbb{F}_*$ is replaced by an arbitrary algebraic pattern $\mathcal{O}$; we will refer to them as *fibrous $\mathcal{O}$-patterns*[1]. Such a fibrous $\mathcal{O}$-pattern can be defined as a functor $\pi\colon \mathcal{P} \to \mathcal{O}$ such that:

(1) $\mathcal{P}$ has all π-cocartesian lifts of inert morphisms in $\mathcal{O}$.
(2) For all $O \in \mathcal{O}$, the commutative square of ∞-categories

$$\begin{array}{ccc} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E} \end{array}$$

is cartesian. This square is constructed in Definition 4.1.2 using the factorization system and the cocartesian lifts from (1).[2]

The ∞-category $\mathsf{Fbrs}(\mathcal{O})$ of fibrous $\mathcal{O}$-patterns is then defined as the subcategory of $\mathsf{Cat}_{\infty/\mathcal{O}}$ whose objects are the fibrous $\mathcal{O}$-patterns and whose morphisms are required to preserve cocartesian morphisms over inert maps in $\mathcal{O}$.

Let us mention a few examples of algebraic patterns where the corresponding notion of fibrous pattern has already been studied:

- If we take $\mathbb{F}_*$ with the classes of inert and active maps defined as in [HA] (see Example 3.1.3) and $\langle 1 \rangle := (\{0,1\}, 0)$ as the only elementary object, then a fibrous $\mathbb{F}_*$-pattern is a functor $\pi : \mathcal{P} \to \mathbb{F}_*$ that has cocartesian lifts for inerts and for which

[1] Under the mild technical assumption that $\mathcal{O}$ is *sound*, our definition of fibrous $\mathcal{O}$-patterns agrees with the definition of *weak Segal $\mathcal{O}$-fibrations* studied in [CH21]; see Proposition 4.1.7. However, we prove some results beyond this case, and here the notion of fibrous $\mathcal{O}$-pattern we introduce is better behaved for our purposes.

[2] The bottom horizontal functor is induced by the functors $\alpha_! : \mathcal{O}^{\mathrm{act}}_{/O} \to \mathcal{O}^{\mathrm{act}}_{/E}$ that are defined for an inert map $\alpha : O \rightarrowtail E$ by sending $\omega : X \to O$ to the active part of the factorization $\alpha \circ \omega : X \rightarrowtail \alpha_! X \rightsquigarrow E$. The top horizontal functor is defined similarly, by using the cocartesian lifts for inerts.

the functor

$$\mathcal{P}^{\mathrm{act}} \times_{\mathbb{F}} \mathbb{F}_{/\langle n\rangle} \simeq \mathcal{P} \times_{\mathbb{F}_*} (\mathbb{F}_*)^{\mathrm{act}}_{/\langle n\rangle} \longrightarrow \prod_{\langle n\rangle \rightarrowtail \langle 1\rangle} \mathcal{P} \times_{\mathbb{F}_*} \mathbb{F} \simeq (\mathcal{P}^{\mathrm{act}})^n,$$

is an equivalence. We will show in Proposition 4.1.7 that this is precisely equivalent to $\mathcal{P}$ being a (symmetric) ∞-operad in the sense of Lurie.

- If $O \to \mathbb{F}_*$ is an ∞-operad in the sense of Lurie, then it has a canonical pattern structure for which a fibrous O-pattern $\pi\colon \mathcal{P} \to O$ is simply an ∞-operad over O:
$$\mathsf{Fbrs}(O) \simeq \mathsf{Fbrs}(\mathbb{F}_*)_{/O} = (\mathsf{Opd}_\infty)_{/O}.$$
- Let $\mathbb{F}_*^\natural$ denote the algebraic pattern with underlying category $\mathbb{F}_*$ and the same factorization system as before, but with both $\langle 0\rangle$ and $\langle 1\rangle$ as elementary objects. Then a fibrous $\mathbb{F}_*^\natural$-pattern is a *generalized ∞-operad* in the sense of [HA].
- If we equip Δ^{op} with the usual inert–active factorization system (see Example 3.1.4) and take $[1]$ as the only elementary object, then a fibrous Δ^{op}-pattern is precisely a planar or non-symmetric ∞-operad as in [HA]. If we instead take both $[0]$ and $[1]$ as elementary we get *generalized non-symmetric ∞-operads* as in [GH15].
- For a finite group G, the G-∞-operads of [NS22] are precisely fibrous $\underline{\mathbb{F}}_{G,*}$-patterns for a certain pattern $\underline{\mathbb{F}}_{G,*}$ (see §5.2).

Comparing fibrous patterns. Our first main theorem allows us to compare fibrous patterns over various bases:

Theorem A. *Let $f\colon O \to \mathcal{P}$ be a morphism of algebraic patterns (i.e. a functor that preserves active and inert morphisms and elementary objects). Suppose furthermore that:*

(i) The induced functors $O^{\mathrm{el}}_{O/} \to \mathcal{P}^{\mathrm{el}}_{f(O)/}$ are coinitial for all $O \in O$.
(ii) The pattern $\mathcal{P}$ is sound *in the sense of Definition 3.3.4.*
(iii) The pattern $\mathcal{P}$ is extendable*: for all $P \in \mathcal{P}$ the canonical functor*
$$\mathcal{P}^{\mathrm{act}}_{/P} \longrightarrow \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{P}^{\mathrm{act}}_{/E},$$
is an equivalence.
(iv) The restriction $f^{\mathrm{el}}\colon O^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ of f is an equivalence of ∞-categories,
(v) The functor $(O^{\mathrm{act}}_{/O})^{\simeq} \to (\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$ induced by f is an equivalence for all $O \in O$.

Then pullback along f gives an equivalence
$$f^*\colon \mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(O).$$

Here the condition of *soundness* is a mild but rather technical assumption, which is satisfied in almost all examples of algebraic patterns we are aware of. We can now state the applications of Theorem A that we mentioned above more precisely:

Corollary B. *Let G be a finite group and $\mathsf{Span}(\mathbb{F}_G)$ the $(2,1)$-category of spans of finite G-sets; we regard this as an algebraic pattern where the inert and active maps are the backwards and forwards maps, respectively, and the elementary objects are the orbits G/H for H a subgroup of G. There is a functor $\underline{\mathbb{F}}_{G,*} \to \mathsf{Span}(\mathbb{F}_G)$ such that pullback along it gives an equivalence*

$$\mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G)) \xrightarrow{\simeq} \mathsf{Fbrs}(\underline{\mathbb{F}}_{G*}) = \mathsf{Opd}_{G,\infty}.$$

In the case of the trivial group $G = \{e\}$, Corollary B yields an equivalence

$$\mathsf{Fbrs}(\mathsf{Span}(\mathbb{F})) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*) = \mathsf{Opd}_\infty$$

between fibrous $\mathsf{Span}(\mathbb{F})$-patterns and ∞-operads in the sense of Lurie, given by pulling back along the inclusion of $\mathbb{F}_*$ in $\mathsf{Span}(\mathbb{F})$ as the wide subcategory containing the spans whose backwards map is injective.

Segal envelopes. The crux of our strategy for proving Theorem A is a reduction to a comparison between Segal objects in Cat_∞ for the two patterns. For this purpose we need to develop an analogue of Lurie's *symmetric monoidal envelope* for ∞-operads over a general algebraic pattern.

A symmetric monoidal ∞-category can be viewed both as a commutative monoid in Cat_∞ (i.e. a Segal object for $\mathbb{F}_*$) and as an ∞-operad that is a cocartesian fibration; we thus have a (non-full) subcategory inclusion $\mathsf{CMon}(\mathsf{Cat}_\infty) \to \mathsf{Opd}_\infty$. In [HA, §2.2.4], Lurie shows that this functor has a left adjoint, the symmetric monoidal envelope, which admits a very explicit description as a cocartesian fibration: the envelope of an ∞-operad $\mathcal{O}$ is simply the fiber product $\mathcal{O} \times_{\mathbb{F}_*} \mathrm{Ar}_{\mathrm{act}}(\mathbb{F}_*)$ where $\mathrm{Ar}_{\mathrm{act}}(\mathbb{F}_*)$ is the full subcategory of the arrow category of $\mathbb{F}_*$ on the active morphisms and the fiber product is over the source functor $\mathbb{F}^{\amalg} := \mathrm{Ar}_{\mathrm{act}}(\mathbb{F}_*) \to \mathbb{F}_*$, while the projection to $\mathbb{F}_*$ giving the symmetric monoidal ∞-category is by the target functor. Moreover, it was observed in [HK21] that if we instead regard the envelope as a functor to symmetric monoidal ∞-categories over $(\mathbb{F}, \amalg)$ (that is, finite sets with the disjoint union as symmetric monoidal structure) then it is fully faithful. We want to generalize these results to fibrous $\mathcal{O}$-patterns for a general algebraic pattern $\mathcal{O}$. To simplify exposition we assume here that $\mathcal{O}$ is both sound and extendable. For such $\mathcal{O}$, unstraightening restricts to give a functor $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty) \to \mathsf{Fbrs}(\mathcal{O})$ analogous to the inclusion $\mathsf{CMon}(\mathsf{Cat}_\infty) \to \mathsf{Opd}_\infty$. Our second main result is a description of the left adjoint of this functor.

Theorem C. *Let $\mathcal{O}$ be a sound and extendable pattern. Then:*

(1) The unstraightening functor $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty) \to \mathsf{Fbrs}(\mathcal{O})$ has a left adjoint $\mathsf{Env}_{\mathcal{O}}$ whose value on a fibrous $\mathcal{O}$-pattern $\mathcal{P}$ is given by the functor $O \longmapsto \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}$.

(2) Slicing $\mathsf{Env}_{\mathcal{O}}$ over $\mathcal{A}_{\mathcal{O}} := \mathsf{Env}_{\mathcal{O}}(\mathcal{O})$ yields a fully faithful embedding

$$\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} \colon \mathsf{Fbrs}(\mathcal{O}) \hookrightarrow \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$$

which admits both a left and a right adjoint.

(3) An object $C \to \mathcal{A}_{\mathcal{O}}$ in $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$ lies in the essential image of $\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}$ if and only if it is $\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$-equifibered, i.e. for every active map $O \rightsquigarrow O'$ in $\mathcal{O}$, the square

$$\begin{array}{ccc} C(O) & \xrightarrow{C(\omega)} & C(O') \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{act}}_{/O} & \xrightarrow{\omega_*} & \mathcal{O}^{\mathrm{act}}_{/O'} \end{array}$$

is cartesian.

In §4.2 we actually prove more general (but weaker) versions of this statement that do not require $\mathcal{O}$ to be sound or extendable. The comparison of Theorem A can now be shown by recalling a (simpler) comparison theorem for Segal objects from [Bar22], passing to slices and then showing that the equivalence restricts to the essential image of the envelope.

In §4.3 we spell out Theorem C in several examples. In particular, for $\mathcal{O} = \mathbb{F}_*$, Theorem C recovers a result of [HK21], though with an alternative characterization of the image:[3]

Corollary D. *The left adjoint to the forgetful functor $\mathsf{CMon}(\mathsf{Cat}_\infty) \to \mathsf{Opd}_\infty$ lifts to a fully faithful functor:*

$$\mathsf{Env} \colon \mathsf{Opd}_\infty \hookrightarrow \mathsf{CMon}(\mathsf{Cat}_\infty)_{/(\mathbb{F},\amalg)}$$

[3]See Observation 4.3.2 for a comparison.

This functor has adjoints on both sides. A symmetric monoidal functor $\pi\colon (\mathcal{C},\otimes) \to (\mathbb{F},\amalg)$ *is in the essential image of* Env *if and only if the square*

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{F}\times\mathbb{F} & \xrightarrow{\amalg} & \mathbb{F} \end{array}$$

is a pullback square in Cat_∞.

In §5.2 we also give a similar characterisation of the essential image of the envelope for G-∞-operads, though in that case one has to require additional pullback squares involving the norm maps $\mathrm{Nm}_K^H\colon \mathcal{C}^K \to \mathcal{C}^H$.

Organization. In §2 we prove a key part of Theorem C, which only depends on the factorization system on an algebraic pattern:

Theorem E. *Let* $\mathcal{B}$ *be an* ∞*-category with a factorization system* $(\mathcal{B}_L,\mathcal{B}_R)$.

(1) The forgetful functor $\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}} \to \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ *has a left adjoint, which takes* $\mathcal{E}\to\mathcal{B}$ *to* $\mathcal{E}\times_{\mathcal{B}}\mathrm{Ar}_R(\mathcal{B})$*, where* $\mathrm{Ar}_R(\mathcal{B})$ *is the full subcategory of* $\mathrm{Ar}(\mathcal{B}) := \mathsf{Fun}([1],\mathcal{B})$ *spanned by the morphisms in* $\mathcal{B}_R$*, the fiber product is over evaluation at* $0\in[1]$*, and the projection to* $\mathcal{B}$ *uses evaluation at* 1.

(2) The induced functor $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \to (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathrm{Ar}_R(\mathcal{B})}$ *is fully faithful, and a morphism* $\mathcal{E}\to\mathrm{Ar}_R(\mathcal{B})$ *in* $\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}}$ *lies in the image of* $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ *if and only if it is* equifibered, *meaning that for every* $\varphi\colon a\to b$ *in* $\mathrm{Ar}_R(\mathcal{B})$ *the commutative square*

$$\begin{array}{ccc} \mathcal{E}_a & \xrightarrow{\varphi_!} & \mathcal{E}_b \\ \downarrow & & \downarrow \\ (\mathcal{B}_R)_{/a} & \xrightarrow{\varphi_!} & (\mathcal{B}_R)_{/b} \end{array}$$

is cartesian.

We emphasize that only the second point here is actually new — the first point has already been proved by both Ayala, Mazel-Gee, and Rozenblyum [AMGR17] and Shah [Sha21].

We then review algebraic patterns in §3, where we also introduce the condition of soundness for patterns. In §4 we define fibrous patterns, specialize Theorem E to this context to prove Theorem C, and explore several examples. We are then ready to prove Theorem A in §5, where we also discuss the applications and an $(\infty,2)$-categorical version of Theorem A.

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2. Envelopes for factorization systems

Our goal in this section is to prove Theorem E. We begin in §2.1 by explicitly describing the general procedure of freely adding cocartesian morphisms over a wide subcategory $\mathcal{B}_0$ of $\mathcal{B}$ to a functor $p\colon \mathcal{E}\to\mathcal{B}$, and then in §2.2 we specialize this to the situation where $\mathcal{B}_0$ is the right class of a factorization system and $\mathcal{E}$ already has

p-cocartesian morphisms over the left class. As already mentioned, these results are not new, but we include complete proofs to make the paper more self-contained. In §2.3 we then prove the new part of Theorem E: we observe that for the induced adjunction on slices the left adjoint is fully faithful, and identify its image.

2.1. Adding cocartesian morphisms over a subcategory.

Let $\mathcal{B}$ be an ∞-category equipped with a wide subcategory $\mathcal{B}_0$, and write $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ for the subcategory of $\mathsf{Cat}_{\infty/\mathcal{B}}$ whose objects have all cocartesian lifts of morphisms in $\mathcal{B}_0$ and whose morphisms preserve these. The aim of this subsection is to show that the forgetful functor

$$\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \longrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}$$

admits an (explicitly defined) left adjoint. Before explaining the construction of the left adjoint, let us first fix some notation: We let $\mathrm{Ar}(\mathcal{B}) := \mathsf{Fun}([1], \mathcal{B})$ denote the arrow ∞-category of $\mathcal{B}$, and write $\mathrm{Ar}_0(\mathcal{B})$ for the full subcategory of $\mathrm{Ar}(\mathcal{B})$ spanned by morphisms in $\mathcal{B}_0$. The left adjoint of the forgetful functor above is then given by

$$(\mathcal{E} \longrightarrow \mathcal{B}) \longmapsto (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \longrightarrow \mathcal{B})\,.$$

where the fiber product is over $\mathrm{ev}_0\colon \mathrm{Ar}_0(\mathcal{B}) \to \mathcal{B}$, and the map $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \to \mathcal{B}$ is given by ev_1. We will prove this by showing that for any $\mathcal{E} \in \mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{F} \in \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, restriction yields a natural equivalence:

$$\mathsf{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}), \mathcal{F}) \xrightarrow{\sim} \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}),$$

where the left-hand side consists of functors that preserve cocartesian morphisms over $\mathcal{B}_0$. This result is by no means new, and has already appeared in [AMGR17] and [Sha21], but we include a proof for completeness, as this is the key input needed for our work in this paper.

Notation 2.1.1. Since $\mathcal{B}_0$ is a wide subcategory, the degeneracy map $s_0^*\colon \mathcal{B} \to \mathrm{Ar}(\mathcal{B})$ restricts to a functor $i\colon \mathcal{B} \to \mathrm{Ar}_0(\mathcal{B})$, taking an object of $\mathcal{B}$ to its identity map. We also have evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Ar}_0(\mathcal{B}) \to \mathcal{B}$, and natural transformations $\sigma\colon i \circ \mathrm{ev}_0 \to \mathrm{id}$ and $\tau\colon \mathrm{id} \to i \circ \mathrm{ev}_1$, given for an object $x \xrightarrow{\varphi} y$ by the squares

$$\begin{array}{ccc} x & = & x \\ \| & & \downarrow{\scriptstyle\varphi} \\ x & \xrightarrow{\varphi} & y, \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\varphi} & y \\ \downarrow{\scriptstyle\varphi} & & \| \\ y & = & y, \end{array}$$

respectively. For any functor $p\colon \mathcal{E} \to \mathcal{B}$, the functor i induces a section $i_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})$ of the projection $\mathrm{pr}_{\mathcal{E}}\colon \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \to \mathcal{E}$, and σ induces a natural transformation $\sigma_{\mathcal{E}}\colon i_{\mathcal{E}}\mathrm{pr}_{\mathcal{E}} \to \mathrm{id}$.

Observation 2.1.2. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is cocartesian over $\mathcal{B}_0$. Then $i_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})$ has a left adjoint $\pi_{\mathcal{E}}$: Such an adjoint exists if and only if, given an object $(x, \varphi\colon px \to b)$, there is an initial object in the ∞-category

$$\mathcal{E}_{(x,\varphi)/} := \mathcal{E} \times_{\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})} (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}))_{(x,\varphi)/} \simeq \mathcal{E}_{x/} \times_{\mathcal{B}_{px/}} \mathcal{B}_{b/}$$

with the functor $\mathcal{B}_{b/} \to \mathcal{B}_{px/}$ given by composition with φ. A cocartesian morphism $x \to \varphi_! x$ is precisely an initial object in the right-hand side that maps to the identity in $\mathcal{B}_{b/}$. Thus $\pi_{\mathcal{E}}$ takes $(x, \varphi\colon px \to b)$ to the target $\varphi_! x$ of the cocartesian morphism over φ. Note that we have $\pi_{\mathcal{E}} i_{\mathcal{E}} \simeq \mathrm{id}$, and the unit transformation $\mathrm{id} \to i_{\mathcal{E}}\pi_{\mathcal{E}}$ is given at

(x, φ) by

$$\begin{array}{ccc} x & \qquad & px \xrightarrow{\varphi} b \\ \big\downarrow & & \big\downarrow{\scriptstyle\varphi} \qquad \big\| \\ \varphi_! x, & & b =\!=\!=\!= b. \end{array}$$

Observation 2.1.3. Given $p \colon \mathcal{E} \to \mathcal{B}$, observe that $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})$ is cocartesian over $\mathcal{B}_0$, with cocartesian morphisms given by composition in $\mathrm{Ar}_0(\mathcal{B})$. (For instance, we can write $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})$ as a pullback $(\mathcal{E} \times \mathcal{B}) \times_{(\mathcal{B}\times\mathcal{B})} \mathrm{Ar}_0(\mathcal{B})$ over $\mathcal{B}$, where all three ∞-categories appearing are cocartesian over $\mathcal{B}_0$.)

Proposition 2.1.4. *If $q \colon \mathcal{F} \to \mathcal{B}$ is cocartesian over $\mathcal{B}_0$, composition with $i_{\mathcal{E}}$ gives a functor*

$$\mathsf{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}), \mathcal{F}) \longrightarrow \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}).$$

This is an equivalence, with inverse given by taking $F \colon \mathcal{E} \to \mathcal{F}$ to the composite

$$\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \xrightarrow{F \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})} \mathcal{F} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F}.$$

Proof. Given $G \colon \mathcal{E} \to \mathcal{F}$, the definition of the sections $i_{\mathcal{E}}$ and $i_{\mathcal{F}}$ give

$$(G \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})) \circ i_{\mathcal{E}} \simeq i_{\mathcal{F}} \circ G,$$

and so we have

$$\pi_{\mathcal{F}} \circ (G \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})) \circ i_{\mathcal{E}} \simeq \pi_{\mathcal{F}} \circ i_{\mathcal{F}} \circ G \simeq G.$$

In the other direction, given $F \colon \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \to \mathcal{F}$ that preserves cocartesian morphisms over $\mathcal{B}_0$, we have to show that F is naturally equivalent to $\pi_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}))$. Here we can write $\mathrm{pr}_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}))$ as the composite

$$\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \xrightarrow{\mathrm{pr}_{\mathcal{E}}} \mathcal{E} \xrightarrow{i_{\mathcal{E}}} \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \xrightarrow{F} \mathcal{F},$$

so that $\sigma_{\mathcal{E}}$ induces a natural transformation

$$\alpha \colon \mathrm{pr}_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})) \longrightarrow F.$$

Note that this is given at $(e, \varphi \colon p(e) \to b)$ by the image $F(e, \mathrm{id}_{p(e)}) \to F(e, \varphi)$ of a cocartesian morphism in $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})$, and so is cocartesian in $\mathcal{F}$ since by assumption F preserves cocartesian morphisms over $\mathcal{B}_0$. Projecting to $\mathcal{B}$, we see that $q\alpha$ factors as the projection to $\mathrm{Ar}_0(\mathcal{B})$ followed by the evaluation map $\mathrm{Ar}_0(\mathcal{B}) \times [1] \to \mathcal{B}$. We can therefore define a natural transformation

$$\beta \colon \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \times [1] \longrightarrow \mathcal{F} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})$$

via the commutative diagram

$$\begin{array}{ccc} \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \times [1] & \xrightarrow{\alpha} & \mathcal{F} \\ \big\downarrow & & \\ \mathrm{Ar}_0(\mathcal{B}) \times [1] & \overset{\mathrm{ev}}{\searrow} & \big\downarrow{\scriptstyle q} \\ \big\downarrow{\scriptstyle\tau} & & \\ \mathrm{Ar}_0(\mathcal{B}) & \xrightarrow{s} & \mathcal{B}. \end{array}$$

Here β is a natural transformation $(Fi_{\mathcal{E}} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})) \to i_{\mathcal{F}}F$, and takes $(e, \varphi \colon p(e) \to b)$ to $(F(e, \mathrm{id}_{p(e)}), \varphi) \to (F(e, \varphi), \mathrm{id}_c)$. Composing with $\pi_{\mathcal{F}}$ we get a natural transformation $\pi_{\mathcal{F}}\beta \colon \pi_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})) \to \pi_{\mathcal{F}} i_{\mathcal{F}} F \simeq F$. This is given at (e, φ) by the canonical morphism $\varphi_! F(e, \mathrm{id}) \to F(e, \varphi)$. Since F preserves cocartesian morphisms over $\mathcal{B}_0$, this is an equivalence, and so we have obtained the natural equivalence we required. □

Corollary 2.1.5. *The forgetful functor*

$$\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \longrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}$$

has a left adjoint given by

$$(\mathcal{E} \longrightarrow \mathcal{B}) \quad \mapsto \quad \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) = s^*\mathcal{E} \longrightarrow \mathrm{Ar}_0(\mathcal{B}) \xrightarrow{t} \mathcal{B},$$

and unit given by $i_{\mathcal{E}} \colon \mathcal{E} \to \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})$.

Proof. By Proposition 2.1.4, for $\mathcal{E} \in \mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{F} \in \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0-\text{cocart}}$ the composite

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0-\text{cocart}}}(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}), \mathcal{F}) \longrightarrow \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}), \mathcal{F}) \xrightarrow{i_{\mathcal{E}}^*} \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(\mathcal{E}, \mathcal{F})$$

is an equivalence, hence this natural transformation is indeed the unit of an adjunction. □

Observation 2.1.6. The forgetful functors $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}} \to \mathsf{Cat}_\infty$ detect pullbacks; in particular, the ∞-category $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ has all pullbacks. Indeed, given morphisms $\mathcal{E}_1 \to \mathcal{E}_0 \leftarrow \mathcal{E}_2$ in $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, it is easy to see that a morphism in the fiber product $\mathcal{E}_1 \times_{\mathcal{E}_0} \mathcal{E}_2$ is cocartesian over $\mathcal{B}_0$ if and only if its images in $\mathcal{E}_1$ and $\mathcal{E}_2$ are cocartesian.

Observation 2.1.7. Suppose $\mathcal{A}$ and $\mathcal{B}$ are ∞-categories equipped with wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, and that $f \colon \mathcal{A} \to \mathcal{B}$ is a functor that takes $\mathcal{A}_0$ into $\mathcal{B}_0$. Pullback along f clearly gives a commutative diagram

$$\begin{array}{ccc} \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} & \xrightarrow{f^*} & \mathsf{Cat}_{\infty/\mathcal{A}}^{\mathcal{A}_0\text{-cocart}} \\ \downarrow & & \downarrow \\ \mathsf{Cat}_{\infty/\mathcal{B}} & \xrightarrow{f^*} & \mathsf{Cat}_{\infty/\mathcal{A}}. \end{array}$$

We then have an induced Beck–Chevalley transformation between the left adjoints of the vertical maps, given for $p \colon \mathcal{E} \to \mathcal{B}$ by the natural map

$$(\mathcal{E} \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} \mathrm{Ar}_0(\mathcal{A}) \longrightarrow (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})) \times_{\mathcal{B}} \mathcal{A},$$

which takes $(e \in \mathcal{E}, a \in \mathcal{A}, p(e) \simeq f(a), \varphi \colon a \to a' \in \mathrm{Ar}_0(\mathcal{A}))$ to $(e, f(a), f(\varphi), a)$. Note, however, that this is typically not an equivalence.

2.2. Free fibrations for factorization systems.

In this subsection we specialize our previous results to the case of an ∞-category equipped with a factorization system. We again emphasize that this result already appears in [AMGR17] and [Sha21].

Notation 2.2.1. In this section we fix an ∞-category $\mathcal{B}$ with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$; we write $\mathrm{Ar}_L(\mathcal{B})$ and $\mathrm{Ar}_R(\mathcal{B})$ for the full subcategories of $\mathrm{Ar}(\mathcal{B})$ spanned by the morphisms in $\mathcal{B}_L$ and $\mathcal{B}_R$, respectively. We also abbreviate

$$\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} := \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_L\text{-cocart}}$$

Proposition 2.2.2 ([CH21, Proposition 7.3])**.** *Let* $(q \colon \mathcal{C} \to \mathcal{B}) \in \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$. *Then:*

(1) The functor $q' \colon \mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathcal{B}$ *given by evaluation at the target is a cocartesian fibration.*

(2) *A morphism* $(\alpha, \beta)\colon (c_0, \varphi_0) \to (c_1, \varphi_1)$ *in* $\mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$ *represented by the following diagram*

$$\left(\begin{array}{c} c_0 \\ \downarrow \alpha \\ c_1 \end{array} , \quad \begin{array}{ccc} q(c_0) & \xrightarrow{\varphi_0} & b_0 \\ \downarrow q(\alpha) & & \downarrow \beta \\ q(c_1) & \xrightarrow{\varphi_1} & b_1 \end{array} \right),$$

is a q'*-cocartesian lift of* $\beta\colon b_0 \to b_1$ *if and only if* $q(\alpha)$ *is in* $\mathcal{B}_L$ *and* α *is* q*-cocartesian.*

Proof. We first show that q' is a locally cocartesian fibration. A locally q'-cocartesian morphism over $\beta\colon b_0 \to b_1$ with source $(c_0, \varphi_0\colon q(c_0) \to b_0)$ is an initial object in the ∞-category $(\mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_{(c_0,\varphi_0)/} \times_{\mathcal{B}_{b_0/}} \{\beta\}$. We can identify this ∞-category as the fiber product

$$\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \left(\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/} \right),$$

where $\mathcal{B}^R_{/b_1}$ denotes the full subcategory of $\mathcal{B}_{/b_1}$ spanned by morphisms in $\mathcal{B}_R$.

We first observe that here $\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}$ has an initial object, given by

$$\begin{array}{ccc} qc_0 & \xrightarrow{\lambda} & b' \\ & {\scriptstyle \beta\varphi_0} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & b_1, & \end{array}$$

where (λ, ρ) is the (L, R)-factorization of $\beta\varphi_0$ — this follows from [HTT, Lemma 5.2.8.19].

The projection $\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/} \to \mathcal{B}^R_{/b_1}$ is a left fibration, since it is a base change of the left fibration $(\mathcal{B}_{/b_1})_{\beta\varphi_0/} \to \mathcal{B}_{/b_1}$. The initial object of $\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}$, which maps to ρ in $\mathcal{B}^R_{/b_1}$, therefore gives an equivalence

$$\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/} \simeq (\mathcal{B}^R_{/b_1})_{\rho/}$$

by [Ker, Tag 0199]. We can therefore rewrite our expression for the ∞-category $(\mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_{(c_0,\varphi_0)/} \times_{\mathcal{B}_{b_0/}} \{\beta\}$ as

$$\left(\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/} \right) \times_{\mathcal{B}_{b'/}} (\mathcal{B}^R_{/b_1})_{\rho/}.$$

A q-cocartesian morphism over λ with source c_0, which exists by assumption since λ is in $\mathcal{B}_L$, is precisely an initial object of $\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}$ that maps to the initial object in $\mathcal{B}_{b'/}$. We thus have initial objects in $\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}$ and $(\mathcal{B}^R_{/b_1})_{\rho/}$ that both map to the initial object in $\mathcal{B}_{b'/}$, and these thus give an initial object in the fiber product $(\mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_{(c_0,\varphi_0)/}$. This shows that if $\alpha\colon c_0 \to c_1$ is a q-cocartesian lift of λ, then

$$\left(\begin{array}{c} c_0 \\ \downarrow \alpha \\ c_1 \end{array} , \quad \begin{array}{ccc} q(c_0) & \xrightarrow{\varphi_0} & b_0 \\ \downarrow \lambda & & \downarrow \beta \\ b' & \xrightarrow{\rho} & b_1 \end{array} \right)$$

is a locally q-cocartesian lift of β with source (c_0, φ_0).

We have thus shown that q' is a locally cocartesian fibration, and the locally q'-cocartesian morphisms are precisely those in (2). To see that q' is a cocartesian fibration it then suffices by [HTT, Proposition 2.4.2.8] to check that the locally q'-cocartesian morphisms are closed under composition, which in our case is clear. □

Notation 2.2.3. It follows from Proposition 2.2.2 that the construction $\mathcal{E} \mapsto \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$ restricts to a well-defined functor

$$\mathrm{E}\colon \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \longrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}, \quad (\mathcal{E} \longrightarrow \mathcal{B}) \mapsto (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \longrightarrow \mathcal{B}).$$

Proposition 2.2.4. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be functor admitting cocartesian lifts for all arrows in $\mathcal{B}_L$ and let $q\colon \mathcal{F} \to \mathcal{B}$ be a cocartesian fibration. Then the equivalence of Proposition 2.1.4 restricts to an equivalence*

$$\mathsf{Fun}_{/\mathcal{B}}^{\text{cocart}}(\mathrm{E}(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} \mathsf{Fun}_{/\mathcal{B}}^{L\text{-cocart}}(\mathcal{E}, \mathcal{F}).$$

Proof. We must show that these full subcategories are identified under the equivalence

$$\mathsf{Fun}_{/\mathcal{B}}^{R\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}), \mathcal{F}) \xrightarrow{\sim} \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F})$$

of Proposition 2.1.4. Given a functor $F\colon \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathcal{F}$ that preserves cocartesian morphisms over $\mathcal{B}_R$, we must thus check that F preserves all cocartesian morphisms if and only if $F \circ i_{\mathcal{E}}$ preserves cocartesian morphisms over $\mathcal{B}_L$. We write $p'\colon \mathcal{E} \times_C \mathrm{Ar}_R(\mathcal{B}) \to \mathcal{B}$ for the map induced by ev_1.

First, assume that $F\colon \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathcal{F}$ preserves all cocartesian edges. For a p-cocartesian lift $\alpha\colon c_0 \to c_1$ of an edge $\beta\colon b_0 \to b_1$ in $\mathcal{B}_L$, its image under $i_{\mathcal{E}}$ is the edge

$$\left(\begin{array}{c} c_0 \\ \big\downarrow \alpha \\ c_1 \end{array}, \begin{array}{ccc} b_0 & \xrightarrow{=} & b_0 \\ \big\downarrow \beta & & \big\downarrow \beta \\ b_1 & \xrightarrow{=} & b_1 \end{array} \right)$$

in $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$, which is p'-cocartesian by Proposition 2.2.2. In other words, $i_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$ preserves cocartesian lifts over $\mathcal{B}_L$, and hence so does $F \circ i_{\mathcal{E}}$.

For the converse assume that F preserves cocartesian lifts of edges in $\mathcal{B}_R$ and $F \circ i_{\mathcal{E}}$ preserves cocartesian lifts of edges in $\mathcal{B}_L$. We would like to show that a general p'-cocartesian morphism $(\alpha, \beta)\colon (c_0, \varphi_0) \to (c_1, \varphi_1)$ is sent to a q-cocartesian morphism in $\mathcal{F}$. According to Proposition 2.2.2, the morphism $p(\alpha)$ is in $\mathcal{B}_L$ and α is p-cocartesian. We can fit this morphism into the following diagram by applying the natural transformation $\sigma_{\mathcal{E}}\colon i_{\mathcal{E}}\mathrm{pr}_{\mathcal{E}} \to \mathrm{id}$:

$$\begin{array}{ccc} (c_0, \mathrm{id}) & \xrightarrow{(\mathrm{id}, \varphi_0) = (\sigma_{\mathcal{E}})_{(c_0, \varphi_0)}} & (c_0, \varphi_0) \\ {\scriptstyle (\alpha, q(\alpha))}\big\downarrow & & \big\downarrow{\scriptstyle (\alpha, \beta)} \\ (c_1, \mathrm{id}) & \xrightarrow[(\mathrm{id}, \varphi_1) = (\sigma_{\mathcal{E}})_{(c_1, \varphi_1)}]{} & (c_1, \varphi_1) \end{array}$$

Both horizontal morphisms are cocartesian edges over $\mathcal{B}_R$ (by Proposition 2.2.2) and the left-hand vertical morphism is the image under $i_{\mathcal{E}}$ of a p-cocartesian morphism over $\mathcal{B}_L$. Hence F sends three of the morphisms in the above square to cocartesian edges in $\mathcal{F}$ and it follows by composition and right-cancellation for cocartesian edges that $F(\alpha, \beta)$ is cocartesian too. ◻

Corollary 2.2.5. *The adjunction of Corollary 2.1.5 restricts to an adjunction*

$$\mathrm{E}\colon \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows \mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} : \text{forget}.$$

Observation 2.2.6. Suppose $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ are ∞-categories equipped with factorization systems, and that $f\colon \mathcal{A} \to \mathcal{B}$ is a functor that preserves both classes of maps in these. Pullback along f then gives a commutative diagram

$$\begin{array}{ccc} \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} & \xrightarrow{f^*} & \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{B}} & \xrightarrow{f^*} & \mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{A}}. \end{array}$$

As in Observation 2.1.7, this induces a Beck–Chevalley transformation, but this is typically not an equivalence.

2.3. Full faithfulness on slices.

In this subsection we prove the main new result of this section: We observe that the adjunction of Corollary 2.2.5 induces an adjunction

$$\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \rightleftarrows (\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{B}})_{/\mathrm{Ar}_R(\mathcal{B})}$$

where the left adjoint is fully faithful, and characterize its image as in Theorem E.

To construct this adjunction, we recall the general construction of adjunctions on slices:

Observation 2.3.1. Given an adjunction

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$$

where $\mathcal{C}$ admits pullbacks, we have (by [HTT, Proposition 5.2.5.1]) for any c in $\mathcal{C}$ an induced adjunction

$$L_c\colon \mathcal{C}_{/c} \rightleftarrows \mathcal{D}_{/Lc} : R_c$$

where L_c is simply given by applying L, while R_c is defined at $f\colon d \to Lc$ by the natural pullback square

$$\begin{array}{ccc} R_c d & \longrightarrow & Rd \\ \downarrow & & \downarrow{\scriptstyle Rf} \\ c & \xrightarrow{\eta_c} & RLc \end{array}$$

over the unit map η_c. The unit for the new adjunction is then given at $c' \to c$ by the canonical map $c' \to R_c L_c c'$ obtained by factoring the square

$$\begin{array}{ccc} c' & \xrightarrow{\eta_{c'}} & RLc' \\ \downarrow & & \downarrow \\ c & \xrightarrow{\eta_c} & RLc \end{array}$$

through the pullback, while the counit $L_c R_c d \to d$ is given by the outer square in the diagram

$$\begin{array}{ccccc} LR_c d & \longrightarrow & LRd & \xrightarrow{\epsilon_d} & d \\ \downarrow & & \downarrow & & \downarrow \\ Lc & \xrightarrow{L\eta_c} & LRLc & \xrightarrow{\epsilon_{Lc}} & Lc, \end{array}$$

where ϵ is the counit of the original adjunction.

Proposition 2.3.2. *By applying the construction of Observation 2.3.1 to the adjunction of Corollary 2.2.5 at the terminal object* $(\mathcal{B} \xrightarrow{=} \mathcal{B}) \in \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ *we obtain an adjunction*

$$\mathrm{E}\colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \rightleftarrows \left(\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{B}}\right)_{/\mathrm{Ar}_R(\mathcal{B})} : \mathrm{Q}. \tag{1}$$

The left adjoint in this adjunction is fully faithful.

Proof. Here E sends $\mathcal{E} \to \mathcal{B}$ to the cocartesian fibration $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathcal{B}$, equipped with the canonical projection to $\mathrm{Ar}_R(\mathcal{B}) \to \mathcal{B}$. The right adjoint Q is given by

$$\mathcal{E} \longrightarrow \mathrm{Ar}_R(\mathcal{B}) \quad \longmapsto \quad i^*\mathcal{E} = \mathcal{B} \times_{\mathrm{Ar}_R(\mathcal{B})} \mathcal{E} \longrightarrow \mathcal{B}$$

where the pullback is taken along the inclusion of the identities $i\colon \mathcal{B} \to \mathrm{Ar}_R(\mathcal{B})$. The unit of this adjunction is then the map $\mathcal{E} \to \mathrm{Q}(\mathrm{E}(\mathcal{E}))$ obtained from the commutative square of units for the adjunction $\mathrm{E} \dashv \mathrm{forget}$ (from Corollary 2.2.5) as the canonical map from $\mathcal{E}$ to the pullback. This square of units is the left hand square in the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{i_{\mathcal{E}}} & \mathrm{E}(\mathrm{forget}(\mathcal{E})) & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{i} & \mathrm{Ar}_R(\mathcal{B}) & \xrightarrow{\mathrm{ev}_0} & \mathcal{B}, \end{array}$$

, where the right-hand square is cartesian by construction of $\mathrm{E}(\mathcal{E})$ in Notation 2.2.3. Hence the left-hand square is also cartesian and thus the unit $\mathcal{E} \to \mathrm{Q}(\mathrm{E}(\mathcal{E}))$ is an equivalence, and so E is indeed fully faithful. □

Now that we have the fully faithful envelope functor all that is left to do to prove Theorem E is to characterize its essential image:

Proposition 2.3.3. *A morphism $\mathcal{D} \to \mathrm{Ar}_R(\mathcal{B})$ of cocartesian fibrations over $\mathcal{B}$ is in the essential image of the left adjoint* E *from Proposition 2.3.2 if and only if it is* equifibered, *meaning that for every object $\varphi\colon a \to b$ in $\mathrm{Ar}_R(\mathcal{B})$, the natural square*

$$\begin{array}{ccc} \mathcal{D}_a & \xrightarrow{\varphi_!} & \mathcal{D}_b \\ \downarrow & & \downarrow \\ \mathrm{Ar}_R(\mathcal{B})_a & \xrightarrow{\varphi\circ(-)} & \mathrm{Ar}_R(\mathcal{B})_b \end{array}$$

is cartesian.

Proof. We begin with the "only if" direction for $(\mathcal{E} \to \mathcal{B}) \in \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $(\varphi\colon a \to b) \in \mathrm{Ar}_R(\mathcal{B})$. We need to show that the left square of the following diagram is cartesian:

$$\begin{array}{ccccc} (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_a & \xrightarrow{\varphi_!} & (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_b & \xrightarrow{\mathrm{pr}_{\mathcal{E}}} & \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ar}_R(\mathcal{B})_a & \xrightarrow{\varphi\circ(-)} & \mathrm{Ar}_R(\mathcal{B})_b & \xrightarrow{s} & \mathcal{B}, \end{array}$$

, where the identification of the composite in the top row uses the description of cocartesian morphisms in $(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))$ from Proposition 2.2.2. This follows since the right-hand square and the outer rectangle are both cartesian.

For the "if" direction we must show that the counit $\mathrm{E}(\mathrm{Q}(\mathcal{D})) \to \mathcal{D}$ is an equivalence if $\mathcal{D}$ is equifibered. By Observation 2.3.1 this counit can be factored as the composite of the top horizontal maps in the following diagram:

$$(\star)\qquad \begin{array}{ccccc} \mathrm{E}(\mathrm{Q}(\mathcal{D})) & \longrightarrow & \mathrm{E}(\mathcal{D}) & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ar}_R(\mathcal{B}) & \longrightarrow & \mathrm{E}(\mathrm{Ar}_R(\mathcal{B})) & \longrightarrow & \mathrm{Ar}_R(\mathcal{B}) \end{array}$$

Here the right-hand horizontal maps come from the counit of the adjunction from Corollary 2.2.5. The bottom horizontal composite is an equivalence, so it will suffice to show that the composite rectangle is cartesian. Since the left-hand square is given by E applied to the cartesian square defining Q (as $\mathrm{Ar}_R(\mathcal{B})$ is $\mathrm{E}(\mathcal{B})$), and E preserves weakly contractible limits, it suffices to show that the right-hand square is cartesian.

By assumption, the functor $\mathcal{D} \to \mathcal{B}$ is a cocartesian fibration, and so the projection $\mathrm{E}(\mathcal{D}) \simeq \mathcal{D} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathrm{Ar}_R(\mathcal{B})$ is also a cocartesian fibration, with cocartesian morphisms exactly those that project to cocartesian morphisms in $\mathcal{D}$. Consider now the following square

$$\begin{array}{ccc} \mathrm{E}(\mathcal{D}) & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Ar}_R(\mathcal{B}) & \xrightarrow{t} & \mathcal{B}, \end{array}$$

in which the top map is the counit for the adjunction of Corollary 2.2.5. The top map in the square takes cocartesian morphisms over $\mathrm{Ar}_R(\mathcal{B})$ to π-cocartesian morphisms in $\mathcal{D}$. To see this, note that a cocartesian morphism in $\mathrm{E}(\mathcal{D})$ over $\mathrm{Ar}_R(\mathcal{B})$ is of the form

$$\left(\begin{array}{c} d \\ \downarrow \\ \varphi_! d, \end{array} \qquad \begin{array}{ccc} \pi(d) & \xrightarrow{\alpha} & b \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \beta} \\ a & \xrightarrow{\gamma} & b' \end{array} \right),$$

and this is by construction sent to the canonical map $\alpha_! d \to \gamma_! \varphi_! d$, which is indeed cocartesian over β.

Consequently the top right square of ($\star$) sits as the top face in the following cube

$$\begin{array}{ccccccc} \mathrm{E}(\mathcal{D}) & & \longrightarrow & & \mathcal{D} & & \\ & \searrow & & & & \searrow & \\ \downarrow & & \mathrm{E}(\mathrm{Ar}_R(\mathcal{B})) & \longrightarrow & \downarrow & & \mathrm{Ar}_R(\mathcal{B}) \\ \mathrm{Ar}_R(\mathcal{B}) & & \longrightarrow & & \mathcal{B} & & \downarrow \\ & = & \downarrow & & & = & \\ & & \mathrm{Ar}_R(\mathcal{B}) & & \longrightarrow & & \mathcal{B}, \end{array}$$

in which the vertical maps are cocartesian fibrations and the maps in the top square preserve cocartesian morphisms. Since the bottom square is obviously cartesian, to show that the top square is cartesian it suffices to check that taking fibers over any $\varphi \in \mathrm{Ar}_R(\mathcal{B})$ yields a cartesian square. We thus want to show that the following square is cartesian:

$$\begin{array}{ccc} \mathrm{E}(\mathcal{D})_{\varphi} & \longrightarrow & \mathcal{D}_b \\ \downarrow & & \downarrow \\ \mathrm{E}(\mathrm{Ar}_R(\mathcal{B}))_{\varphi} & \longrightarrow & \mathrm{Ar}_R(\mathcal{B})_b. \end{array}$$

Here there is a canonical equivalence $\mathrm{E}(\mathcal{D})_{\varphi} \simeq (\mathcal{D} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})_{\varphi}) \simeq \mathcal{D}_a$ and similarly $\mathrm{E}(\mathrm{Ar}_R(\mathcal{B}))_{\varphi} \simeq (\mathrm{Ar}_R(\mathcal{B}))_a$. Via these equivalences the horizontal maps are identified with the cocartesian pushforward along φ. The resulting square is then precisely one of the squares that are cartesian by the assumption that $\mathcal{D}$ is equifibered. □

In §4.2 it will be notationally convenient to use a "straightened" version of the adjunction (1); to state this we first introduce some notation:

Notation 2.3.4. Let $\mathcal{B}$ be an ∞-category equipped with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$, and let $\mathcal{R}\colon \mathcal{B} \to \mathsf{Cat}_\infty$ be the straightening of the cocartesian fibration $\mathrm{Ar}_R(\mathcal{B}) \to \mathcal{B}$. We define the functor

$$\mathrm{St}^L_{\mathcal{B}}\colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \longrightarrow \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}},$$

which we think of as a form of "straightening relative to the factorization system", as the composite

$$\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \xrightarrow{\mathrm{E}} \left(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}}\right)_{/\mathrm{Ar}_R(\mathcal{B})} \xrightarrow{\mathrm{St}_{\mathcal{B}}} \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}},$$

sending $(p\colon \mathcal{E} \to \mathcal{B})$ to the straightening of $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathcal{B}$. Dually, we define

$$\mathrm{Un}^L_{\mathcal{B}}\colon \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}} \longrightarrow \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$$

as the composite

$$\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}} \xrightarrow{\mathrm{Un}_{\mathcal{B}}} \left(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}}\right)_{/\mathrm{Ar}_R(\mathcal{B})} \xrightarrow{\mathrm{Q}} \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}.$$

For a functor $F\colon \mathcal{B} \to \mathsf{Cat}$ together with natural transformation $\alpha\colon F \to \mathcal{R}$ we then have that $\mathrm{Un}^L_{\mathcal{B}}(\alpha)$ is the pullback

$$\begin{array}{ccc} \mathrm{Un}^L_{\mathcal{B}}(\alpha) & \longrightarrow & \mathrm{Un}_{\mathcal{B}}(F) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{Un}_{\mathcal{B}}(\alpha)} \\ \mathcal{B} & \longrightarrow & \mathrm{Ar}_R(\mathcal{B}). \end{array}$$

This yields the following reformulation of Theorem E:

Theorem 2.3.5. *The functors $\mathrm{St}^L_{\mathcal{B}}$ and $\mathrm{Un}^L_{\mathcal{B}}$ give an adjunction*

$$\mathrm{St}^L_{\mathcal{B}}\colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \rightleftarrows \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}} : \mathrm{Un}^L_{\mathcal{B}}.$$

The functor $\mathrm{St}^L_{\mathcal{B}}$ is fully faithful and a natural transformation $F \to \mathcal{R}$ is in the essential image of $\mathrm{St}^L_{\mathcal{B}}$ if and only if it is equifibered, *meaning that for every object $a \xrightarrow{\varphi} b$ in $\mathrm{Ar}_R(\mathcal{B})$, the natural square*

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\varphi)} & F(b) \\ \downarrow & & \downarrow \\ \mathcal{R}(a) & \xrightarrow{\mathcal{R}(\varphi)} & \mathcal{R}(b) \end{array}$$

is cartesian. □

A pleasant consequence of Theorem 2.3.5 is that $\mathrm{Un}^L_{\mathcal{B}}$ also has a *left* adjoint and that $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ is presentable. To see this, we use the following observation:

Observation 2.3.6. Let C be a presentable ∞-category, and S a set of morphisms in C. Recall that a morphism $\varphi\colon X \to Y$ in C is *right orthogonal* to S if there exists a unique filler in every commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle \varphi} \\ B & \longrightarrow & Y \end{array}$$

where f is in S. Equivalently, φ is right orthogonal to S if and only if the commutative square

$$\begin{array}{ccc} \mathsf{Map}_C(B,X) & \xrightarrow{f^*} & \mathsf{Map}_C(A,X) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathsf{Map}_C(B,Y) & \xrightarrow{f^*} & \mathsf{Map}_C(A,Y) \end{array}$$

is cartesian for all $f\colon A \to B$ in S. This square is in turn cartesian if and only if for all maps $B \to Y$, the map on fibers

$$\mathsf{Map}_{/Y}(B,X) \longrightarrow \mathsf{Map}_{/Y}(A,X)$$

is an equivalence. Thus the map φ is right orthogonal to S if and only if as an object of $C_{/Y}$ it is local with respect to the set of maps

$$\left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \quad \swarrow & \\ & Y & \end{array} : f \in S \right\}.$$

In particular, the full subcategory of $C_{/Y}$ spanned by the objects that are right orthogonal to S is an accessible localization of $C_{/Y}$, and so is also presentable.

Proposition 2.3.7. *Let $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be a small ∞-category equipped with a factorization system. The functor $\mathrm{Un}^L_{\mathcal{B}}$ has a left adjoint, which exhibits $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ as an accessible localization of $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$. In particular, $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ is a presentable ∞-category.*

Proof. The ∞-category $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$ is clearly presentable, and we know that the functor $\mathrm{St}^L_{\mathcal{B}}$ is fully faithful, with its essential image given by functors equifibered over $\mathcal{R}$. It therefore suffices to show that this is the full subcategory of objects in $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$ that are local with respect to a set of morphisms.

Let S be the collection of morphisms of the form

$$(y(\varphi) \times \mathrm{id})\colon y(b) \times [\epsilon] \longrightarrow y(a) \times [\epsilon]$$

for $\epsilon \in \{0,1\}$ and $(\varphi\colon a \to b) \in \mathrm{Ar}_R(\mathcal{B})$, where $y(a)(-) := \mathsf{Map}_{\mathcal{B}}(a,-)$ is the Yoneda embedding of $\mathcal{B}^{\mathrm{op}}$; this is a set since $\mathcal{B}$ is by assumption a small ∞-category. An object $\gamma\colon F \to \mathcal{R}$ in $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$ is then equifibered if and only if it is right orthogonal to S: The latter means that the commutative squares

$$\begin{array}{ccc} \mathsf{Map}(y(a) \times [\epsilon], F) & \longrightarrow & \mathsf{Map}(y(b) \times [\epsilon], F) \\ \downarrow & & \downarrow \\ \mathsf{Map}(y(a) \times [\epsilon], \mathcal{R}) & \longrightarrow & \mathsf{Map}(y(b) \times [\epsilon], \mathcal{R}) \end{array}$$

are cartesian; by the Yoneda lemma this square can be identified with

$$\begin{array}{ccc} \mathsf{Map}([\epsilon], F(a)) & \longrightarrow & \mathsf{Map}([\epsilon], F(b)) \\ \downarrow & & \downarrow \\ \mathsf{Map}([\epsilon], \mathcal{R}(a)) & \longrightarrow & \mathsf{Map}([\epsilon], \mathcal{R}(b)), \end{array}$$

which is cartesian for $\epsilon = 0, 1$ if and only if the square

$$\begin{array}{ccc} F(a) & \longrightarrow & F(b) \\ \downarrow & & \downarrow \\ \mathcal{R}(a) & \longrightarrow & \mathcal{R}(b) \end{array}$$

is cartesian, since the objects $[0], [1]$ generate Cat_∞ under colimits. The result then follows from Observation 2.3.6. $\square$

Observation 2.3.8. It is easy to see (using the mapping space criterion for cocartesian morphisms) that the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ preserves limits and filtered colimits. Since both ∞-categories are presentable by Proposition 2.3.7, it follows by the adjoint functor theorem that this functor has a left adjoint.

Observation 2.3.9. Let $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be ∞-categories equipped with factorization systems, and let $f\colon \mathcal{A} \to \mathcal{B}$ be a functor that preserves both classes of maps in these.

The functor f then induces a commutative diagram

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{i_{\mathcal{A}}} & \mathrm{Ar}_R(\mathcal{A}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{A} \\
 & & \downarrow{\scriptstyle q} & & \| \\
\Big\downarrow{\scriptstyle f} & & f^*\,\mathrm{Ar}_R(\mathcal{B}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{A} \\
 & & \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
\mathcal{B} & \xrightarrow{i_{\mathcal{B}}} & \mathrm{Ar}_R(\mathcal{B}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{B}.
\end{array}$$

From this we get the following commutative diagram of ∞-categories:
(2)

$$\begin{array}{ccccc}
(\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})} & \xrightarrow{f^*} & (\mathsf{Cat}_{\infty/\mathcal{A}}^{\mathrm{cocart}})_{/f^*\mathrm{Ar}_R(\mathcal{B})} & \xrightarrow{q^*} & (\mathsf{Cat}_{\infty/\mathcal{A}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{A})} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}})_{/\mathrm{Ar}_R(\mathcal{B})} & \xrightarrow{f^*} & (\mathsf{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}})_{/f^*\mathrm{Ar}_R(\mathcal{B})} & \xrightarrow{q^*} & (\mathsf{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}})_{/\mathrm{Ar}_R(\mathcal{A})} \\
\downarrow{\scriptstyle \iota_{\mathcal{B}}^*} & & & & \downarrow{\scriptstyle \iota_{\mathcal{A}}^*} \\
\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} & & \xrightarrow{\qquad f^* \qquad} & & \mathsf{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}}.
\end{array}$$

Let us write $f^{\circledast}$ for the composite in the top row, which takes $\mathcal{E} \to \mathrm{Ar}_R(\mathcal{B})$ to the fiber product $\mathcal{E} \times_{\mathrm{Ar}_R(\mathcal{B})} \mathrm{Ar}_R(\mathcal{A}) \to \mathrm{Ar}_R(\mathcal{A})$. Passing to vertical left adjoints now yields a Beck–Chevalley transformation

$$\mathrm{E}_{\mathcal{A}} f^* \longrightarrow f^{\circledast}\mathrm{E}_{\mathcal{B}};$$

Unwinding the definitions, this is given at $\mathcal{E} \to \mathcal{B}$ in $\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ by the natural map

$$(\mathcal{E} \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} \mathrm{Ar}_R(\mathcal{A}) \longrightarrow (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})) \times_{\mathrm{Ar}_R(\mathcal{B})} \mathrm{Ar}_R(\mathcal{A}).$$

This is an equivalence, so that we also have a commutative square

(3)
$$\begin{array}{ccc}
\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} & \xrightarrow{f^*} & \mathsf{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}} \\
\downarrow{\scriptstyle \mathrm{E}_{\mathcal{B}}} & & \downarrow{\scriptstyle \mathrm{E}_{\mathcal{A}}} \\
(\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})} & \xrightarrow{f^{\circledast}} & (\mathsf{Cat}_{\infty/\mathcal{A}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{A})}.
\end{array}$$

3. Algebraic patterns

In this section we will first review the basic definitions related to algebraic patterns and Segal objects in §3.1, and then look at some examples thereof in §3.2. We then introduce the condition of *soundness* for algebraic patterns in §3.3; this is somewhat technical, but turns out to be the key property needed for some of our results in the next section.

3.1. **Algebraic patterns and Segal objects.** In this subsection we review the definitions of algebraic patterns and Segal objects, and some related basic notions introduced in [CH21]. We also introduce a *relative* version of Segal objects, which will show up later.

Definition 3.1.1. An *algebraic pattern* is an ∞-category $\mathcal{O}$ equipped with a factorization system, whereby every morphism factors (uniquely up to equivalence) as an *inert* morphism followed by an *active* morphism, as well as a collection of *elementary objects*. We write $\mathcal{O}^{\mathrm{int}}$ and $\mathcal{O}^{\mathrm{act}}$ for the subcategories of $\mathcal{O}$ containing only the inert and active morphisms, respectively, and $\mathcal{O}^{\mathrm{el}}$ for the full subcategory of $\mathcal{O}^{\mathrm{int}}$ containing elementary objects and inert morphisms among them. For $X \in \mathcal{O}$, we also write

$$\mathcal{O}^{\mathrm{el}}_{X/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{X/}$$

for the ∞-category of inert maps $X \to E$ with $E \in \mathcal{O}^{\mathrm{el}}$.

Notation 3.1.2. We often indicate inert maps as $X \rightarrowtail Y$ and active maps as $X \rightsquigarrow Y$. These arrows are not meant to indicate any particular intuition about inert or active morphisms.

Example 3.1.3. We write $\mathbb{F}_*$ for a skeleton of the category of pointed finite sets, with objects $\langle n \rangle := (\{0, 1, \ldots, n\}, 0)$, and say a morphism $\varphi \colon \langle n \rangle \to \langle m \rangle$ is *inert* if φ restricts to an isomorphism $\langle n \rangle \setminus \varphi^{-1}(0) \to \langle m \rangle \setminus \{0\}$, and *active* if $\varphi^{-1}(0) = \{0\}$. Then the inert and active morphisms form a factorization system on $\mathbb{F}_*$, and we make this an algebraic pattern[4] by taking $\langle 1 \rangle$ to be the single elementary object.

Example 3.1.4. Another basic example is Δ^{op}, where Δ is the simplex category. Recall that Δ^{op} admits an inert-active factorization system where inert maps are opposite to interval inclusions and active maps are opposite to maps preserving the maximal and minimal elements. To make Δ^{op} an algebraic pattern, we can take the elementary objects to be $[0]$ and $[1]$, in which case we denote the pattern by $\Delta^{\mathrm{op},\natural}$, or alternatively just $[1]$, in which case the pattern is denoted $\Delta^{\mathrm{op},\flat}$.

The main reason for introducing algebraic patterns is that they describe algebraic structures via Segal conditions:

Definition 3.1.5. A functor $F \colon \mathcal{O} \to \mathcal{C}$ is a *Segal $\mathcal{O}$-object* in the ∞-category $\mathcal{C}$ if for every object $X \in \mathcal{O}$ the induced functor

$$(\mathcal{O}^{\mathrm{el}}_{X/})^{\triangleleft} \longrightarrow \mathcal{O} \xrightarrow{F} \mathcal{C}$$

is a limit diagram. If $\mathcal{C}$ has limits for diagrams indexed by $\mathcal{O}^{\mathrm{el}}_{X/}$ for all $X \in \mathcal{O}$, in which case we say that $\mathcal{C}$ is *$\mathcal{O}$-complete*, then this condition is equivalent to the canonical maps

$$F(X) \longrightarrow \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E)$$

being equivalences. We refer to Segal $\mathcal{O}$-objects in the ∞-category $\mathcal{S}$ of spaces as *Segal $\mathcal{O}$-spaces* and Segal $\mathcal{O}$-objects in the ∞-category Cat_∞ of ∞-categories as *Segal $\mathcal{O}$-∞-categories*.

Example 3.1.6. We can identify $(\mathbb{F}_*)^{\mathrm{el}}_{\langle n \rangle/}$ with the set $\{\rho_i : i = 1, \ldots, n\}$, where ρ_i is the inert morphism $\langle n \rangle \to \langle 1 \rangle$ given by

$$\rho_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i. \end{cases}$$

[4]In [CH21] this pattern was denoted $\mathbb{F}_*^\flat$ to distinguish it from the pattern $\mathbb{F}_*^\natural$, where the elementary objects are $\langle 0 \rangle$ and $\langle 1 \rangle$. However, in this paper $\mathbb{F}_* = \mathbb{F}_*^\flat$ is the key example, so we use a simplified notation for it.

A functor $F\colon \mathbb{F}_* \to \mathcal{C}$ is then a Segal $\mathbb{F}_*$-object if for every n the map

$$F(\langle n\rangle) \longrightarrow \prod_{i=1}^{n} F(\langle 1\rangle),$$

induced by the maps ρ_i, is an equivalence. Thus Segal $\mathbb{F}_*$-objects are precisely commutative monoids in the sense of [HA, §2.4.2]. For $\mathcal{C} = \mathcal{S}$, this gives the ∞-categorical analogue of *special* Γ*-spaces* in the sense of Segal [Seg74].

Example 3.1.7. Segal $\Delta^{\mathrm{op},\natural}$-spaces are precisely *Segal spaces* in the sense of [Rez01], while Segal $\Delta^{\mathrm{op},\flat}$-objects in $\mathcal{C}$ are associative monoids (or E_1-algebras).

Later on, we will also need to consider a *relative* version of Segal objects:

Definition 3.1.8. Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-complete ∞-category. A *relative Segal* $\mathcal{O}$*-object* of $\mathcal{C}$ is a morphism $\pi\colon Y \to X$ in $\mathsf{Fun}(\mathcal{O},\mathcal{C})$ such that for every $O \in \mathcal{O}$ the natural commutative square

$$\begin{array}{ccc} Y(O) & \longrightarrow & \lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}} Y(E) \\ {\scriptstyle \pi(O)}\downarrow & & \downarrow{\scriptstyle \lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}} \pi(E)} \\ X(O) & \longrightarrow & \lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}} X(E) \end{array}$$

is cartesian. We denote by $\mathsf{Seg}^{/X}_{\mathcal{O}}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathcal{O},\mathcal{C})_{/X}$ the full subcategory whose objects are the X-relative Segal $\mathcal{O}$-objects.

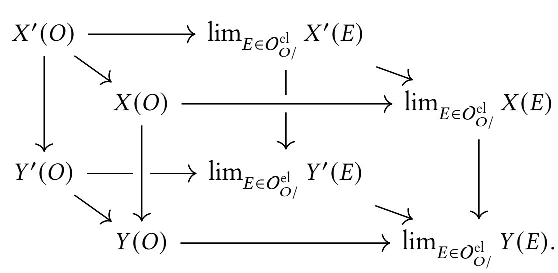
Observation 3.1.9. If $Y \to X$ is a relative Segal $\mathcal{O}$-object of $\mathcal{C}$, then the pasting lemma for cartesian squares implies that a morphism $Z \to Y$ is a relative Segal $\mathcal{O}$-object if and only if the composite $Z \to X$ is one. Moreover, a morphism $X \to *$ to the terminal object is a relative Segal $\mathcal{O}$-object if and only if X is a Segal $\mathcal{O}$-object in $\mathcal{C}$. Combining these two observations, we see that if X is a Segal $\mathcal{O}$-object of $\mathcal{C}$ then an X-relative Segal $\mathcal{O}$-object is just a Segal $\mathcal{O}$-object with a map to X, i.e. we have

$$\mathsf{Seg}^{/X}_{\mathcal{O}}(\mathcal{C}) = \mathsf{Seg}_{\mathcal{O}}(\mathcal{C})_{/X}$$

as full subcategories of $\mathsf{Fun}(\mathcal{C},\mathcal{O})_{/X}$.

Lemma 3.1.10. *Suppose $X \to Y$ is a relative Segal $\mathcal{O}$-object in $\mathcal{C}$. Then for any map $\eta\colon Y' \to Y$, the pullback $X' := X \times_Y Y' \to Y'$ is also a relative Segal $\mathcal{O}$-object. In other words, pullback along η gives a functor $\eta^*\colon \mathsf{Seg}^{/Y}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{Seg}^{/Y'}_{\mathcal{O}}(\mathcal{C})$.*

Proof. For $O \in \mathcal{O}$, consider the commutative cube



Here the left, right, and front faces are all cartesian, hence so is the back face. □

Lemma 3.1.11. *For every presentable ∞-category $\mathcal{C}$ the full subcategory*

$$\mathsf{Seg}^{/X}_{\mathcal{O}}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathcal{O},\mathcal{C})_{/X}$$

is an accessible localization. In particular, it is a presentable ∞-category.

Proof. Consider the following collection of morphisms in $\mathsf{Fun}(\mathcal{O},\mathcal{C})$:

$$\left\{\operatorname{colim}_{E\in\mathcal{O}^{\mathrm{el}}_{X/}} y(E)\otimes C \longrightarrow y(X)\otimes C\right\}_{X\in\mathcal{O},C\in K}$$

where K is a set of compact generators for $\mathcal{C}$, y is the Yoneda embedding for $\mathcal{O}^{\mathrm{op}}$, and $T\otimes C$ for $T\in\mathcal{S}$, $C\in\mathcal{C}$, is the canonical tensoring of $\mathcal{C}$ with $\mathcal{S}$, given by the colimit over T of the constant diagram with value C. A morphism $X\to Y$ in $\mathsf{Fun}(\mathcal{O},\mathcal{C})$ is a relative Segal $\mathcal{O}$-object if and only if it is right orthogonal to this set of morphisms, hence the claim follows from Observation 2.3.6. □

Next, we take a brief look at morphisms between patterns:

Definition 3.1.12. If $\mathcal{O}$ and $\mathcal{P}$ are algebraic patterns, a *morphism of algebraic patterns* is a functor $f\colon\mathcal{O}\to\mathcal{P}$ that preserves inert and active morphisms as well as elementary objects. We say that such a morphism is a *Segal morphism* if for every Segal $\mathcal{P}$-space F and every $X\in\mathcal{O}$ the functor $f^{\mathrm{el}}_{X/}\colon\mathcal{O}^{\mathrm{el}}_{X/}\to\mathcal{P}^{\mathrm{el}}_{f(X)/}$ arising from f induces an equivalence

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} F \xrightarrow{\sim} \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F\circ f;$$

by [CH21, Lemma 4.5] this is equivalent to composition with f giving a functor

$$f^*\colon \mathsf{Seg}_{\mathcal{P}}(\mathcal{C})\longrightarrow \mathsf{Seg}_{\mathcal{O}}(\mathcal{C})$$

for any $\mathcal{O}$-complete ∞-category $\mathcal{C}$. The Segal morphisms that occur in practice are those where the functor $f^{\mathrm{el}}_{X/}$ is coinitial for all $X\in\mathcal{O}$; if this is the case we say that f is a *strong Segal morphism*. In the special case where $f^{\mathrm{el}}_{X/}$ is an *equivalence* for every X, we say that f is an *iso-Segal morphism*.

Example 3.1.13. There is a morphism of algebraic patterns $\mathfrak{c}\colon\Delta^{\mathrm{op},\flat}\to\mathbb{F}_*$, given on objects by $\mathfrak{c}([n])=\langle n\rangle$, and with $\mathfrak{c}(\varphi)\colon\langle n\rangle\to\langle m\rangle$ for a morphism $\varphi\colon[m]\to[n]$ in Δ given by

$$\mathfrak{c}(\varphi)(i)=\begin{cases} j, & \text{if } \varphi(j-1)<i\leq\varphi(j),\\ 0, & \text{otherwise.}\end{cases}$$

It is straightforward to check that this is an iso-Segal morphism.

Notation 3.1.14. We write $\mathsf{AlgPatt}$ for the ∞-category of algebraic patterns together with all morphisms of algebraic patterns.

Observation 3.1.15. Composition with a *strong* Segal morphism $f\colon\mathcal{O}\to\mathcal{P}$ also preserves *relative* Segal objects: If $X\to Y$ is a relative Segal $\mathcal{P}$-object in $\mathcal{C}$, then for $O\in\mathcal{O}$ we have a commutative diagram

$$\begin{array}{ccccc} X(f(O)) & \longrightarrow & \lim_{E\in\mathcal{P}^{\mathrm{el}}_{f(O)/}} X(E) & \xrightarrow{\sim} & \lim_{E'\in\mathcal{O}^{\mathrm{el}}_{O/}} X(f(E')) \\ \downarrow & & \downarrow & & \downarrow \\ Y(f(O)) & \longrightarrow & \lim_{E\in\mathcal{P}^{\mathrm{el}}_{f(O)/}} Y(E) & \xrightarrow{\sim} & \lim_{E'\in\mathcal{O}^{\mathrm{el}}_{O/}} Y(f(E')); \end{array}$$

here both the left and right squares are cartesian, and hence so is the outer composite square. Composition with f thus gives a functor $f^*\colon\mathsf{Seg}^{/Y}_{\mathcal{P}}(\mathcal{C})\to\mathsf{Seg}^{/f^*Y}_{\mathcal{O}}(\mathcal{C})$.

We now recall a simple criterion for a Segal morphism to give an equivalence on Segal objects:

Proposition 3.1.16 ([Bar22, Corollary 2.64])**.** *Suppose $\mathcal{O}$ and $\mathcal{P}$ are algebraic patterns, and $f\colon\mathcal{O}\to\mathcal{P}$ is a strong Segal morphism such that*

(1) $f^{\mathrm{el}}\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ *is an equivalence of* ∞*-categories,*
(2) for every $O \in \mathcal{O}$*, the functor* $(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq} \to (\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$ *is an equivalence of* ∞*-groupoids.*

Then for any complete ∞*-category* $\mathcal{C}$ *the functor* $f^*\colon \mathsf{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathsf{Seg}_{\mathcal{O}}(\mathcal{C})$ *is an equivalence, with inverse given by right Kan extension along* f*.*

Remark 3.1.17. If $(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq}$ is a Segal $\mathcal{O}$-space and $(\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$ is a Segal $\mathcal{P}$-space in Proposition 3.1.16, then it suffices to check condition (2) for elementary objects in $\mathcal{O}$. This holds, for instance, if $\mathcal{O}$ and $\mathcal{P}$ are extendable (see Definition 3.3.16).

3.2. **Examples of algebraic patterns.** We now look at some examples of algebraic patterns. Our focus here will be on examples that will be relevant in the next sections; we refer the reader to [CH21, §3] for many other examples.

Example 3.2.1. We have patterns $\Delta^{n,\mathrm{op},\natural}$ and $\Delta^{n,\mathrm{op},\flat}$ with underlying category $\Delta^{n,\mathrm{op}} := (\Delta^{\mathrm{op}})^{\times n}$, equipped with the factorization system where the inert and active maps are those that are inert or active in Δ^{op} in each component. Here $(\Delta^{n,\mathrm{op},\flat})^{\mathrm{el}} = \{([1],\dots,[1])\}$ while $(\Delta^{n,\mathrm{op},\natural})^{\mathrm{el}}$ consists of all objects whose components are all either $[0]$ or $[1]$. Then Segal $\Delta^{n,\mathrm{op},\natural}$-spaces are n-uple Segal spaces, which model n-fold ∞-categories, while Segal $\Delta^{n,\mathrm{op},\flat}$-objects are $\mathbb{E}_n$-algebras (by the Dunn–Lurie additivity theorem).

Example 3.2.2. Let Θ_n be the inductively defined wreath product $\Delta \wr \Theta_{n-1}$, starting with $\Theta_0 = [0]$; see for example [Ber07, Hau18] for more details. This has a factorization system where the active/inert maps are those whose components in Δ and Θ_{n-1} are both active or inert. There are two interesting pattern structures on Θ_n^{op}: if we define the objects C_i in Θ_n by $C_0 := [0]()$ and $C_i := [1](C_{i-1})$ for $i = 1,\dots,n$, then for $\Theta_n^{\mathrm{op},\flat}$ we take C_n to be the only elementary object, while for $\Theta_n^{\mathrm{op},\natural}$ we take all of $C_0,\dots,C_n$. Then Segal $\Theta_n^{\mathrm{op},\natural}$-spaces are Rezk's model for (∞,n)-categories [Rez10], while Segal $\Theta_n^{\mathrm{op},\flat}$-object are again $\mathbb{E}_n$-algebras (see [Bar18]).

Example 3.2.3. Let $\mathbb{F}_*^{\leq k} \subseteq \mathbb{F}_*$ denote the full subcategory containing pointed finite sets of cardinality $\leq k$ (excluding the basepoint). Consider $\mathbb{F}_*^{\leq k}$ as an algebraic pattern by restricting the inert-active factorization system on $\mathbb{F}_*$ and choosing $\langle 1\rangle$ to be the only elementary object. Segal objects for $\mathbb{F}_*^{\leq k}$ are *arity* k*-restricted commutative monoids* — a variant of commutative monoids in which the homotopy coherence data is only supplied up to arity k. More generally, if $\mathcal{O}$ is an ∞-operad then $\mathcal{O}^{\leq k} := \mathbb{F}_*^{\leq k} \times_{\mathbb{F}_*} \mathcal{O}$ has a natural structure of an algebraic pattern whose Segal objects are arity k-restricted $\mathcal{O}$-monoids. For more details see [Bar22].

The remaining examples we want to discuss are all instances of a general class of algebraic patterns on ∞-categories of spans. For this purpose we briefly recall the construction of such ∞-categories — this is originally due to Barwick [Bar17]; see also [HHLN22] for a more "model-independent" version.

Construction 3.2.4. Let $\mathfrak{X}$ be an ∞-category equipped with a pair of wide subcategories $\mathfrak{X}^b$ and $\mathfrak{X}^f$ (where "b" stands for *backwards* and "f" stands for *forwards*. Following Barwick, we say that the triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ is *adequate* if for every pair of morphisms $\beta\colon x \to y$ in $\mathfrak{X}^b$ and $\varphi\colon y' \to y$ in $\mathfrak{X}^f$, we have:

(1) the pullback $x' := x \times_y y'$ exists in $\mathfrak{X}$,
(2) the projection $x' \to y'$ lies in $\mathfrak{X}^b$.
(3) the projection $x' \to x$ lies in $\mathfrak{X}^f$.

Given an adequate triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ Barwick defines an ∞-category $\mathsf{Span}_{b,f}(\mathfrak{X})$ (denoted $\mathrm{A}^{\mathrm{eff}}(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ in [Bar17]) such that the objects of $\mathsf{Span}_{b,f}(\mathfrak{X})$ are the objects of

$\mathfrak{X}$ and the morphisms from x to y are spans (or correspondences)

$$x \xleftarrow{\beta} w \xrightarrow{\varphi} y$$

where the arrow β lies in $\mathfrak{X}^b$ and the arrow φ lies in $\mathfrak{X}^f$. The assumption that the triple is adequate allows for a composition law defined by taking pullbacks. If $\mathfrak{X}$ is an ∞-category with pullbacks, then we can take $\mathfrak{X}^b = \mathfrak{X}^f = \mathfrak{X}$, in which case we just write $\mathsf{Span}(\mathfrak{X})$ for the corresponding ∞-category of spans.

Observation 3.2.5. By the first part of [HHLN22, Proposition 4.9] the ∞-category $\mathsf{Span}_{b,f}(\mathfrak{X})$ always has a factorization system given by the classes of maps as above with φ or β required to be an equivalence (which we might call the "backwards" and "forwards" maps) and the subcategories of these maps are equivalent to $\mathfrak{X}^{b,\mathrm{op}}$ and $\mathfrak{X}^f$, respectively.

Definition 3.2.6. Given an adequate triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ and a full subcategory $\mathfrak{X}_0 \subseteq \mathfrak{X}$, we denote by $\mathsf{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0)$ the algebraic pattern given by $\mathsf{Span}_{b,f}(\mathfrak{X})$ with the factorization system whose inert and active maps are the backwards and forward maps, respectively, and with the objects of $\mathfrak{X}_0$ as the elementary objects.

Remark 3.2.7. The Segal condition for $\mathsf{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0)$ takes the following form for a functor F:

$$F(x) \simeq \lim_{e \to x \in (\mathfrak{X}^b_{0/x})^{\mathrm{op}}} F(e),$$

where $\mathsf{Span}_{b,f}(\mathfrak{X})^{\mathrm{el}}_{x/} \simeq (\mathfrak{X}^b_{0/x})^{\mathrm{op}}$ with $\mathfrak{X}^b_{0/x} := \mathfrak{X}^b_0 \times_{\mathfrak{X}^b} \mathfrak{X}^b_{/x}$ and $\mathfrak{X}^b_0$ is the full subcategory of $\mathfrak{X}^b$ containing the objects of $\mathfrak{X}_0$.

Example 3.2.8. Let $\mathbb{F}$ denote the category of finite sets. Since this has pullbacks, Construction 3.2.4 produces an ∞-category (in fact a (2,1)-category) $\mathsf{Span}(\mathbb{F})$ whose objects are finite sets, and whose morphisms are spans of the form

$$\mathbf{n} \xleftarrow{\alpha} \mathbf{m} \xrightarrow{\beta} \mathbf{n}'$$

for finite sets $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{n}'$, with composition given by taking pullbacks. We consider $\mathsf{Span}(\mathbb{F}) = \mathsf{Span}(\mathbb{F}; \{1\})$ as an algebraic pattern by taking the backward maps as inerts, forward maps as actives and $1 \in \mathsf{Span}(\mathbb{F})$ as the only elementary object.

Observation 3.2.9. The category $\mathbb{F}_*$ may be thought of as the wide subcategory $\mathsf{Span}_{\mathrm{inj,all}}(\mathbb{F})$ of $\mathsf{Span}(\mathbb{F})$ containing only those morphisms where the backwards map is injective. The inert-active factorization system on $\mathbb{F}_*$ then coincides with the one obtained by restriction from $\mathsf{Span}(\mathbb{F})$, and the inclusion $\mathbb{F}_* \to \mathsf{Span}(\mathbb{F})$ is an iso-Segal morphism.

Example 3.2.10. Let G be a finite group and $\mathbb{F}_G$ the category of finite G-sets. Denote by $\mathrm{Orb}_G \subseteq \mathbb{F}_G$ the collection of G-orbits (i.e. transitive G-sets). Since $\mathbb{F}_G$ has pullbacks we have an ∞-category (really a (2,1)-category) $\mathsf{Span}(\mathbb{F}_G)$. Abusing notation slightly, we also denote the span pattern with the orbits as elementary objects by $\mathsf{Span}(\mathbb{F}_G) := \mathsf{Span}(\mathbb{F}_G; \mathrm{Orb}_G)$. Segal objects for this pattern are precisely G-commutative monoids in the sense of [Nar16]; they also appear in [CMNN20] where they are called semi-Mackey functors. More generally, for any full subcategory $\mathcal{F} \subseteq \mathrm{Orb}_G$ we have a span pattern $\mathsf{Span}(\mathbb{F}_G; \mathcal{F})$ whose Segal objects may be thought of as G-commutative monoids that are Borel-$\mathcal{F}$-complete. Segal objects for $\mathsf{Span}(\mathbb{F}_G;\{G/e\})$ appear implicitly in [CMNN20], where they are called Borel-equivariant.

Example 3.2.11. As a variant of the previous example, we can consider subcategories $\mathbb{F}_G^f$ of $\mathbb{F}_G$ that are closed under base change; if $\mathbb{F}_G^f$ is moreover closed under finite coproducts, this data is equivalent to an *indexing system* in the sense of [BH18]. We can then define the span pattern $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G) := \mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G;\mathrm{Orb}_G)$, whose Segal objects we can think of as G-commutative monoids where only transfers that lie in $\mathbb{F}_G^f$ are allowed. As an illustrative example we may consider the extreme case where all forward maps are isomorphisms, i.e. $\mathbb{F}_G^f := \mathbb{F}_G^{\simeq}$. The corresponding span pattern $\mathsf{Span}_{\mathrm{all},\simeq}(\mathbb{F}_G;\mathrm{Orb}_G)$ has an underlying ∞-category equivalent to $\mathbb{F}_G^{\mathrm{op}}$ with all the maps inert and with $\mathrm{Orb}_G^{\mathrm{op}}$ as the subcategory of elementary objects. Segal objects for this pattern are thus equivalent to presheaves on Orb_G, and by Elemendorf's theorem this ∞-category is equivalent to that of G-spaces.

Example 3.2.12. A space $X \in \mathcal{S}$ is called *m-finite* if it is m-truncated and all of its homotopy groups are finite; we let $\mathcal{S}_m \subseteq \mathcal{S}$ denote the full subcategory of m-finite spaces. Since m-finite spaces are closed under finite limits we may consider the span pattern $\mathsf{Span}(\mathcal{S}_m) := \mathsf{Span}(\mathcal{S}_m;*)$. If we write $\mathcal{S}_m^{n\text{-tr}}$ for the wide subcategory of $\mathcal{S}_m$ whose maps are n-truncated, then $(\mathcal{S}_m, \mathcal{S}_m^{n\text{-tr}}, \mathcal{S}_m)$ is also an adequate triple, and we can likewise consider the pattern

$$\mathsf{Span}_{n\text{-tr},\mathrm{all}}(\mathcal{S}_m) := \mathsf{Span}_{n\text{-tr},\mathrm{all}}(\mathcal{S}_m;*)$$

for any n. For $n = m-1$, the Segal objects for $\mathsf{Span}_{(m-1)\text{-tr},\mathrm{all}}(\mathcal{S}_m)$ are precisely the *m-commutative monoids* of Harpaz [Har20]. It also follows from [Har20, Proposition 5.14] that these are equivalent to Segal objects for $\mathsf{Span}(\mathcal{S}_m)$.

3.3. Sound patterns.

In this subsection we define the notion of a *sound* pattern — a technical condition satisfied in almost all the usual examples. This requires first introducing some notation:

Notation 3.3.1. Fix a morphism $\omega\colon X \to Y$ in an algebraic pattern $\mathcal{O}$. For every elementary object $(\alpha\colon Y \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{Y/}$ we denote the inert-active factorization of $\alpha \circ \omega$ as follows:

$$\begin{array}{ccc} X & \xrightarrow{\omega_\alpha} & \omega_{\alpha!}X \\ {\scriptstyle\omega}\downarrow & & \rightsquigarrow {\scriptstyle(\alpha\circ\omega)^{\mathrm{act}}} \\ Y & \xrightarrow{\alpha} & E \end{array}$$

Factorization defines a functor $\omega_{(-)}\colon \mathcal{O}^{\mathrm{el}}_{Y/} \to \mathcal{O}^{\mathrm{int}}_{X/}$ by sending α to ω_α.

Definition 3.3.2. For $\omega\colon X \rightsquigarrow Y$ we define $\mathcal{O}^{\mathrm{el}}(\omega)$ as the pullback

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{el}}(\omega) & \longrightarrow & \mathrm{Ar}(\mathcal{O}^{\mathrm{int}}_{X/}) \\ \downarrow & & \downarrow{\scriptstyle(s,t)} \\ \mathcal{O}^{\mathrm{el}}_{Y/} \times \mathcal{O}^{\mathrm{el}}_{X/} & \xrightarrow{(\omega_{(-)},\mathrm{id})} & \mathcal{O}^{\mathrm{int}}_{X/} \times \mathcal{O}^{\mathrm{int}}_{X/}. \end{array}$$

An object in $\mathcal{O}^{\mathrm{el}}(\omega)$ can thus be represented by a diagram in $\mathcal{O}$ of the following shape:

$$\begin{array}{ccccc} X & \xrightarrow{\omega_\alpha} & \omega_{\alpha!}X & \rightarrowtail & E' \\ {\scriptstyle\omega}\rightsquigarrow & & \rightsquigarrow & & \\ Y & \xrightarrow{\alpha} & E, & & \end{array}$$

where the arrows labeled by $\rightarrowtail$ and $\rightsquigarrow$ are required to be inert and active, respectively, E and E' are elementary, and ω is fixed. Morphisms in $\mathcal{O}^{\mathrm{el}}(\omega)$ are natural transformations of such diagrams that are constant at $\omega\colon X \rightsquigarrow Y$ and inert at all other objects.

Remark 3.3.3. By construction $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{Y/} \times \mathcal{O}^{\mathrm{el}}_{X/}$ is the bifibration (see [HTT, Definition 2.4.7.2]) corresponding to the functor

$$(\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}} \times \mathcal{O}^{\mathrm{el}}_{X/} \longrightarrow \mathcal{S}, \qquad (\alpha\colon Y \rightarrowtail E, \beta\colon X \rightarrowtail E') \mapsto \mathsf{Map}_{\mathcal{O}^{\mathrm{int}}_{X/}}(\omega_\alpha, \beta).$$

Definition 3.3.4. We say that a pattern $\mathcal{O}$ is *sound* if for every active morphism $\omega\colon X \rightsquigarrow Y$ the functor $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{X/}$ is coinitial.

The point of introducing the condition of soundness is that it allows us to rewrite certain double limits, as described below in Lemma 3.3.7. Before we state this property we look at a first example, namely $\mathbb{F}_*$, where soundness is particularly easy to check; further examples will be given below.

Example 3.3.5. In the pattern $\mathbb{F}_*$ an active morphism $\omega\colon X_+ \rightsquigarrow Y_+$ is simply a map $\omega\colon X \to Y$ in $\mathbb{F}$. The inert undercategory $(\mathbb{F}_*)^{\mathrm{int}}_{Y_+/}$ may be identified with the poset $(\mathrm{Sub}(Y), \supseteq)$ of subsets of Y, by assigning to each $\gamma\colon Y_+ \rightarrowtail Z_+$ the subset $\gamma^{-1}(Z) \subset Y$. The category of elementary objects under Y_+ is given by the one-element subsets, and we may hence identify it with Y itself. For an elementary $\alpha\colon Y_+ \rightarrowtail E$ corresponding to $e \in Y$, the pushforward $\omega_{\alpha!}X_+$ can be identified with $\omega^{-1}(e)_+ \subset X_+$. Hence we have a cartesian square:

$$\begin{array}{ccc} \mathbb{F}^{\mathrm{el}}_*(\omega) & \longrightarrow & \mathrm{Ar}(\mathrm{Sub}(X)) \\ \downarrow & & \downarrow{\scriptstyle (t,s)} \\ X \times Y & \xrightarrow{(\mathrm{id},\omega^{-1})} & \mathrm{Sub}(X) \times \mathrm{Sub}(X). \end{array}$$

and so $\mathbb{F}^{\mathrm{el}}_*(\omega)$ is the poset of pairs $(x,y) \in X \times Y$ such that $\{x\} \subset \omega^{-1}(y)$. In other words, $y = \omega(x)$ and hence the map $\mathbb{F}^{\mathrm{el}}_*(\omega) \to (\mathbb{F}_*)^{\mathrm{el}}_{/X_*} \simeq X$ is an equivalence. In particular it is coinitial and thus $\mathbb{F}_*$ is sound.

Observation 3.3.6. The composite $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{X/} \times \mathcal{O}^{\mathrm{el}}_{Y/} \to \mathcal{O}^{\mathrm{el}}_{Y/}$ is by construction a cartesian fibration. Its straightening is the functor

$$(\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}} \xrightarrow{\omega_{(-)}} (\mathcal{O}^{\mathrm{int}}_{X/})^{\mathrm{op}} \xrightarrow{\mathcal{O}^{\mathrm{el}}_{-/}} \mathsf{Cat}$$

that sends $\alpha\colon Y \rightarrowtail E$ to the ∞-category $\mathcal{O}^{\mathrm{el}}_{\omega_{\alpha!}X/}$ of elementaries under $\omega_{\alpha!}X$. Our definition of $\mathcal{O}^{\mathrm{el}}(\omega)$ therefore matches that given in [CH21, Remark 7.6]. Moreover, a limit over $\mathcal{O}^{\mathrm{el}}(\omega)$ can be rewritten as a double limit, that is for $F\colon \mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{C}$ we have

$$\lim_{\mathcal{O}^{\mathrm{el}}(\omega)} F \simeq \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{Y/}} \lim_{\mathcal{O}^{\mathrm{el}}_{\omega_{\alpha!}X/}} F.$$

If $\mathcal{O}$ is sound, then we can use this to rewrite a limit over $\mathcal{O}^{\mathrm{el}}_{X/}$ as a double limit.

The crucial application of soundness for us will be through the following lemma: this will be used in the proof of Lemma 4.2.4, which is how the assumption of soundness enters our main theorem.

Lemma 3.3.7. *Let $\mathcal{O}$ be a sound pattern and $\mathcal{C}$ a sufficiently complete ∞-category. Consider a natural transformation $(\eta\colon F \Rightarrow G)\colon \mathcal{O}^{\mathrm{int}}_{X/} \to \mathcal{C}$ such that for all $X \rightarrowtail X' \in \mathcal{O}^{\mathrm{int}}_{X/}$ the square*

$$\begin{array}{ccc} F(X') & \longrightarrow & \lim_{X'\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{X'/}} F(E) \\ {\scriptstyle \eta_{X'}}\big\downarrow & & \big\downarrow{\scriptstyle \lim \eta_E} \\ G(X') & \longrightarrow & \lim_{X'\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{X'/}} G(E) \end{array}$$

is cartesian. Then for every active morphism $\omega\colon X \rightsquigarrow Y$ the square

$$\begin{array}{ccc} F(X) & \longrightarrow & \lim_{\alpha\colon Y\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{Y/}} F(\omega_{\alpha!}X) \\ {\scriptstyle \eta_{X}}\big\downarrow & & \big\downarrow{\scriptstyle \lim \eta_E} \\ G(X) & \longrightarrow & \lim_{\alpha\colon Y\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{Y/}} G(\omega_{\alpha!}X) \end{array}$$

is cartesian.

Proof. Consider the commutative cube

$$\begin{array}{ccccc}
F(X) & \longrightarrow & \lim_{\alpha\colon Y\rightarrowtail E'\in \mathcal{O}^{\mathrm{el}}_{Y/}} F(\omega_{\alpha!}X) & & \\
\big\downarrow & \searrow & \big\downarrow & \searrow & \\
 & & \lim_{\beta\colon X\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{X/}} F(E) & \xrightarrow{\sim} & \lim_{(\alpha\colon Y\rightarrowtail E',\gamma\colon \omega_{\alpha!}X\rightarrowtail E)\in \mathcal{O}^{\mathrm{el}}(\omega)} F(E) \\
G(X) & \longrightarrow & \lim_{\alpha\colon Y\rightarrowtail E'\in \mathcal{O}^{\mathrm{el}}_{Y/}} G(\omega_{\alpha!}X) & & \big\downarrow \\
 & \searrow & \big\downarrow & \searrow & \\
 & & \lim_{\beta\colon X\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{X/}} G(E) & \xrightarrow{\sim} & \lim_{(\alpha\colon Y\rightarrowtail E',\gamma\colon \omega_{\alpha!}X\rightarrowtail E)\in \mathcal{O}^{\mathrm{el}}(\omega)} G(E).
\end{array}$$

The front horizontal maps are equivalences because $\mathcal{O}$ is assumed to be sound and hence $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{X/}$ is coinitial. The left square is cartesian by applying the assumption. We would like to show that the back square is cartesian and by pullback pasting it will suffice to show that the right square is cartesian. We may write the limit over $\mathcal{O}^{\mathrm{el}}(\omega)$ as a double limit, by first right Kan extending along the cartesian fibration $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{Y/}$, which is computed by taking limits over the fibers $\mathcal{O}^{\mathrm{el}}_{\omega_{\alpha!}Y/}$, and then taking the limit over $\mathcal{O}^{\mathrm{el}}_{Y/}$. Using this reformulation the right square can be written as a $\mathcal{O}^{\mathrm{el}}_{Y/}$-limit of diagrams of the form

$$\begin{array}{ccc} F(\omega_{\alpha!}X) & \longrightarrow & \lim_{\omega_{\alpha!}X\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{X'/}} F(E) \\ {\scriptstyle \eta_{\omega_{\alpha!}X}}\big\downarrow & & \big\downarrow{\scriptstyle \lim \eta_E} \\ G(\omega_{\alpha!}X) & \longrightarrow & \lim_{\omega_{\alpha!}X\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{X'/}} G(E). \end{array}$$

Each of these diagrams is cartesian by assumption, and hence so is their limit. □

We will now check explicitly that the examples of patterns we discussed above are indeed sound. To do so, the following observation will be useful:

Lemma 3.3.8. *For an algebraic pattern $\mathcal{O}$ the following conditions are equivalent:*

(1) $\mathcal{O}$ is sound.

(2) For every active morphism $\omega\colon X \rightsquigarrow Y$ and $\beta\colon X \rightarrowtail E' \in \mathcal{O}^{\mathrm{el}}_{Y/}$, the ∞-category

$$\mathcal{O}^{\mathrm{el}}_{\beta}(\omega) := \mathcal{O}^{\mathrm{el}}_{Y/} \times_{\mathcal{O}^{\mathrm{int}}_{X/}} (\mathcal{O}^{\mathrm{int}}_{X/})_{/\beta}$$

is weakly contractible.

(3) For every ω and β as in (2) *we have* $\operatorname{colim}_{\alpha\in(\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}}} \mathsf{Map}_{\mathcal{O}^{\mathrm{int}}_{X/}}(\omega_\alpha,\beta) \simeq *$.

Proof. (1 $\Leftrightarrow$ 2) The functor $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{X/}$ is a cocartesian fibration. By the dual of [HTT, Theorem 4.1.3.2.] it is coinitial if and only if its fibers are weakly contractible. Unwinding definitions yields the following description of the straightening:

$$\mathcal{O}^{\mathrm{el}}_{X/} \longrightarrow \mathsf{Cat}, \qquad (\beta\colon X \longrightarrow E') \longmapsto \mathcal{O}^{\mathrm{el}}_{Y/} \times_{\mathcal{O}^{\mathrm{int}}_{X/}} (\mathcal{O}^{\mathrm{int}}_{X/})_{/\beta}.$$

(2 $\Leftrightarrow$ 3) Since $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{Y/}\times\mathcal{O}^{\mathrm{el}}_{X/}$ is a bifibration, passing to the fiber over $\beta \in \mathcal{O}^{\mathrm{el}}_{X/}$ and taking opposites yields a left fibration $q : \mathcal{O}^{\mathrm{el}}_{\beta}(\omega)^{\mathrm{op}} \to (\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}}$. By [HTT, Corollary 3.3.4.6], the ∞-groupoid $|\mathcal{O}^{\mathrm{el}}_{\beta}(\omega)| \simeq |\mathcal{O}^{\mathrm{el}}_{\beta}(\omega)^{\mathrm{op}}|$ can be computed as the colimit of the straightening $\mathrm{St}(q)$, which is given by

$$\mathrm{St}(q)\colon (\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}} \longrightarrow \mathcal{S}, \qquad (\alpha\colon Y \longrightarrow E) \longmapsto \mathsf{Map}_{\mathcal{O}^{\mathrm{int}}_{X/}}(\omega_\alpha,\beta). \qquad \square$$

Observation 3.3.9. Suppose that $\mathcal{O}$ is a pattern such that for all $X \in \mathcal{O}$ the inert under-category $\mathcal{O}^{\mathrm{int}}_{X/}$ is a poset. In this case, spelling out the definition as in Example 3.3.5 we may identify $\mathcal{O}^{\mathrm{el}}_{\beta}(\omega)$ with the following sub-poset of $\mathcal{O}^{\mathrm{el}}_{Y/}$:

$$\mathcal{O}^{\mathrm{el}}_{\beta}(\omega) \simeq \{(\alpha\colon Y \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{Y/} \mid \beta = \gamma\circ\omega_\alpha\}.$$

Example 3.3.10. For the pattern Δ^{op} the inert under-category $(\Delta^{\mathrm{op}})^{\mathrm{int}}_{[n]/}$ is equivalent to the poset of pairs $(a_0 \le a_1) \in [n]$. This is elementary in $\Delta^{\mathrm{op},\flat}$ iff $a_1 - a_0 = 1$ and it is elementary in $\Delta^{\mathrm{op},\natural}$ iff $a_1 - a_0 \le 1$. To check soundness we consider, for a morphism $\omega\colon [m] \to [n]$ in Δ and elementary $(b_0 \le b_1) \in [n]$, the poset

$$(\Delta^{\mathrm{op}})^{\mathrm{el}}_{\beta}(\omega) \simeq \{(a_0 \le a_1) \in (\Delta^{\mathrm{op}})^{\mathrm{el}}_{[m]/} \mid \omega(a_0) \le b_0 \le b_1 \le \omega(a_1)\}.$$

In the case of $\Delta^{\mathrm{op},\flat}$ this poset has a single element, namely that given by $a_0 = \max\{a \in [m] \mid \omega(a) \le b_0\}$ and $a_1 = a_0 + 1$, which satisfies $\omega(a_1) > b_0$ and hence $\omega(a_1) \ge b_1 = b_0 + 1$. For the pattern $\Delta^{\mathrm{op},\natural}$ the poset still has a single element if $b_1 = b_0 + 1$ or if $b_1 = b_0$ with $b_i \notin \omega([m])$. But if $b_1 = b_0 = \omega(a)$ for some $a \in [m]$, then the poset is the category

$$(a-1 \le a) \longrightarrow (a \le a) \longleftarrow (a \le a+1),$$

which is not trivial, but still weakly contractible. This shows that $\Delta^{\mathrm{op},\flat}$ and $\Delta^{\mathrm{op},\natural}$ are both sound.

Example 3.3.11. The pattern $\mathbb{F}_*^\natural$ is sound. The inert under-category $(\mathbb{F}_*^\natural)_{A_+/}$ is the poset of subsets $U \subset A$. Given an active morphism $\omega\colon A_+ \to B_+$ and an elementary $E \subset A$ (i.e. $|E| \le 1$), we need to check that the poset of $E' \subset Y$ with $|E'| \le 1$ and $E \subset \omega^{-1}(E')$ is contractible. If $E = \{a\} \neq \emptyset$, then this poset has exactly one element $E' = \{\omega(a)\}$, and if $E = \emptyset$, then this poset has an initial element $E' = \emptyset$. So the poset is contractible in both cases, which proves that $\mathbb{F}_*^\natural$ is sound.

Lemma 3.3.12. *Products of sound patterns are sound: if $\mathcal{O}_1$ and $\mathcal{O}_2$ are sound patterns, then $\mathcal{O}_1 \times \mathcal{O}_2$ is also a sound pattern.*

Proof. Let $\omega = (\omega_1,\omega_2)\colon (X_1,X_2) \rightsquigarrow (Y_1,Y_2)$ be an active morphism in $\mathcal{O}_1\times\mathcal{O}_2$. The projection $(\mathcal{O}_1\times\mathcal{O}_2)^{\mathrm{el}}(\omega) \longrightarrow (\mathcal{O}_1\times\mathcal{O}_2)^{\mathrm{el}}_{(X_1,X_2)/}$ can be identified with the product of the projections $\mathcal{O}_1^{\mathrm{el}}(\omega_1)\times\mathcal{O}_2^{\mathrm{el}}(\omega_2) \longrightarrow (\mathcal{O}_1)^{\mathrm{el}}_{X_1/}\times(\mathcal{O}_2)^{\mathrm{el}}_{X_2/}$ which, by assumption, is a product of coinitial functors and hence again coinitial. $\square$

Example 3.3.13. Applying Lemma 3.3.12 to Example 3.3.10, we see that the patterns $\Delta^{n,\mathrm{op},\flat}$ and $\Delta^{n,\mathrm{op},\natural}$ are both sound.

Observation 3.3.14. Let $\mathcal{O}$ be a sound algebraic pattern and suppose $\mathcal{P} \hookrightarrow \mathcal{O}$ is a fully faithful inclusion of algebraic patterns. Then $\mathcal{P}^{\mathrm{el}}_{\beta}(\omega) \hookrightarrow \mathcal{O}^{\mathrm{el}}_{\beta}(\omega)$ is fully faithful for all $\omega\colon X \rightsquigarrow Y$ and $\beta\colon Y \rightarrowtail E$. Now assume further that $\mathcal{P}$ satisfies that for any inert morphism $X \rightarrowtail Z$ in $\mathcal{O}$ with $X \in \mathcal{P}$ the object Z is also in $\mathcal{P}$. Then $\mathcal{P}^{\mathrm{el}}_{\beta}(\omega) \simeq \mathcal{O}^{\mathrm{el}}_{\beta}(\omega)$ is an equivalence and hence $\mathcal{P}$ is also sound.

Next, we introduce a further condition for sound patterns; for this we first need some notation:

Notation 3.3.15. By Proposition 2.2.2, evaluation at the target $\mathrm{ev}_1\colon \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \to \mathcal{O}$ is a cocartesian fibration. Its straightening, denoted by $\mathcal{A}_{\mathcal{O}}\colon \mathcal{O} \to \mathsf{Cat}_\infty$, takes $X \in \mathcal{O}$ to the ∞-category $\mathcal{A}_{\mathcal{O}}(X) \simeq \mathcal{O}^{\mathrm{act}}_{/X}$ of active morphisms to X. (Compare with [CH21, Corollary 7.4 and Remark 7.5].)

Definition 3.3.16. We say an algebraic pattern $\mathcal{O}$ is *soundly extendable* if it is sound and in addition the functor $\mathcal{A}_{\mathcal{O}}$ is a Segal $\mathcal{O}$-∞-category, i.e. for every $X \in \mathcal{O}$, the functor

$$\mathcal{O}^{\mathrm{act}}_{/X} \longrightarrow \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathcal{O}^{\mathrm{act}}_{/E}$$

is an equivalence.

Remark 3.3.17. The notion of a soundly extendable pattern is a mild strengthening of the notion of extendable pattern from [CH21, Definition 8.5] (which uses a slightly weaker, but more complicated condition than what we are here calling "soundness"). It was shown in [CH21, Lemma 9.14] that every extendable pattern $\mathcal{O}$ satisfies the condition in Definition 3.3.16, so in particular a sound pattern is extendable if and only if it is soundly extendable. In principle, there could exist extendable patterns that are not sound, but we are not aware of any examples.

Example 3.3.18. The patterns $\mathbb{F}_*$, $\Delta^{\mathrm{op},\natural}$, and $\Delta^{\mathrm{op},\flat}$ are soundly extendable. Their soundness was verified in Example 3.3.5 and Example 3.3.10. For extendability see [CH21, Example 8.13 and 8.14]. The pattern $\Theta_n^{\mathrm{op},\natural}$ is soundly extendable for all n (by [Hau18, Proposition 2.7] and [Hau18, Lemma 3.5]), but note that $\Theta_n^{\mathrm{op},\flat}$ fails to be extendable for $n > 1$. (See [CH21, Example 8.15].)

Example 3.3.19. Let $\mathcal{O} \to \mathbb{F}_*$ be an ∞-operad. Then $\mathcal{O}$ is a soundly extendable pattern. This will follow by Example 4.1.5 and Lemma 4.1.15 in the next section.

Example 3.3.20. The patterns $\mathbb{F}_*^{\le k}$ are sound by Observation 3.3.14, but not soundly extendable. Indeed, $\mathcal{A}_{\mathbb{F}_*^{\le k}}\colon \mathbb{F}_*^{\le k} \to \mathsf{Cat}_\infty$ does not satisfy the Segal condition: for any $n \le k$ the Segal map may be identified with the inclusion

$$(\mathbb{F}^{\times n})^{\le k} \simeq \mathcal{A}_{\mathbb{F}_*^{\le k}}(n) \longrightarrow \mathcal{A}_{\mathbb{F}_*^{\le k}}(1)^{\times n} \simeq (\mathbb{F}^{\le k})^{\times n}$$

where $(\mathbb{F}^{\le k})^{\times n}$ is the category of n-tuples of sets such that each set has size $\le k$, and $(\mathbb{F}^{\times n})^{\le k}$ denotes the full subcategory on those n-tuples of total size $\le k$.

Lemma 3.3.21. *Let $\mathcal{O}$ and $\mathcal{P}$ be soundly extendable patterns such that $\mathcal{O}^{\mathrm{el}}_{O/}$ and $\mathcal{P}^{\mathrm{el}}_{P/}$ are weakly contractible for all $O \in \mathcal{O}$ and $P \in \mathcal{P}$. Then $\mathcal{O} \times \mathcal{P}$ is a soundly extendable pattern.*

Proof. Soundness follows from Lemma 3.3.12. For extendability we have:

$$\begin{aligned} \lim_{(\alpha,\beta)\colon(O,P)\rightarrowtail(E,E')\in(\mathcal{O}\times\mathcal{P})^{\mathrm{el}}_{(O,P)/}} (\mathcal{O}\times\mathcal{P})^{\mathrm{act}}_{/(E,E')} &\simeq \lim_{(\alpha\colon O\rightarrowtail E,\beta\colon P\rightarrowtail E')\in\mathcal{O}^{\mathrm{el}}_{O/}\times\mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E}\times\mathcal{P}^{\mathrm{act}}_{/E'} \\ &\simeq \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E} \times \lim_{\beta\colon P\rightarrowtail E'\in\mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{P}^{\mathrm{act}}_{/E'} \\ &\simeq \mathcal{O}^{\mathrm{act}}_{/O}\times\mathcal{P}^{\mathrm{act}}_{/P} \end{aligned}$$

where in the second line we used that in any ∞-category, products distribute over weakly contractible limits. □

Example 3.3.22. The pattern $\Delta^{n,\mathrm{op},\natural}$ is soundly extendable. Indeed the case $n = 1$ appears in Example 3.3.18, and for $n > 1$ this follows from Lemma 3.3.21 by observing that $(\Delta^{\mathrm{op},\natural})^{\mathrm{el}}_{[k]/}$ is weakly contractible for all k. (Note that this argument fails for $\Delta^{n,\mathrm{op},\flat}$ since $(\Delta^{\mathrm{op},\flat})^{\mathrm{el}}_{[0]/} = \emptyset$, and indeed this pattern is *not* extendable for $n > 1$.)

Proposition 3.3.23. *The pattern* $\mathsf{Span}_{b,f}(\mathfrak{X};\mathfrak{X}_0)$*, as defined in Definition 3.2.6, is*

(1) sound if $\mathfrak{X}^b_{/y} \to \mathfrak{X}_{/y}$ *is fully faithful and the inclusion* $\mathfrak{X}^b_{0/y} \hookrightarrow \mathfrak{X}_{0/y}$ *is cofinal for every* $y \in \mathfrak{X}$.

(2) soundly extendable if and only if it is sound and the functor $\mathfrak{X}^f_{/-}\colon \mathfrak{X}^{b,\mathrm{op}} \to \mathsf{Cat}_\infty$ *(defined on morphisms by pullback) is right Kan extended from* $\mathfrak{X}_0^{b,\mathrm{op}} \subseteq \mathfrak{X}^{b,\mathrm{op}}$.

Proof. (1) By Lemma 3.3.8.(3) the pattern $\mathsf{Span}^{b,f}(\mathfrak{X};\mathfrak{X}_0)$ is sound if and only if for every $\beta\colon e' \to x$ in $\mathfrak{X}^b$ and $\omega\colon x \to y$ in $\mathfrak{X}^f$ the following colimit indexed by $\alpha\colon e \to y \in \mathfrak{X}^b_{0/y}$ is contractible:

$$
\begin{aligned}
&\mathrm{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}^b_{/x}}(\beta\colon e' \to x,\ \omega^*\alpha\colon x\times_y e \to x) \\
&\simeq \mathrm{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}_{/x}}(\beta\colon e' \to x,\ \omega^*\alpha\colon x\times_y e \to x) && (\mathfrak{X}^b_{/x} \subset \mathfrak{X}_{/x} \text{ full}) \\
&\simeq \mathrm{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}_{/y}}(\omega\circ\beta\colon e' \to y,\ \alpha\colon e \to x) && (\omega_! \dashv \omega^*) \\
&\simeq \mathrm{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}_{0/y}}(\omega\circ\beta\colon e' \to y,\ \alpha\colon e \to x) && (\mathfrak{X}_{0/y} \subset \mathfrak{X}_{/y} \text{ full}) \\
&\simeq |\mathfrak{X}^b_{0/y} \times_{\mathfrak{X}_{0/y}} (\mathfrak{X}_{0/y})_{\omega\circ\beta/}|
\end{aligned}
$$

By [HTT, Theorem 4.1.3.1] this category is weakly contractible if $\mathfrak{X}^b_{0/y} \to \mathfrak{X}_{0/y}$ is cofinal, so the claim follows.

(2) Since $\mathsf{Span}_{b,f}(\mathfrak{X};\mathfrak{X}_0)^{\mathrm{act}}_{/-} \simeq \mathfrak{X}^f_{/-}$, this is a consequence of the fact that a functor is Segal if and only if its restriction to the inert category is right Kan extended from the elementaries by [CH21, Lemma 2.9]. □

As an important special case, we have:

Corollary 3.3.24. *If* $\mathfrak{X}^b = \mathfrak{X}$ *then* $\mathsf{Span}_{\mathrm{all},f}(\mathfrak{X};\mathfrak{X}_0)$ *is sound.*

Example 3.3.25. The pattern $\mathsf{Span}(\mathbb{F})$ is soundly extendable.

Example 3.3.26. Let $\mathbb{F}^f_G \subset \mathbb{F}_G$ be closed under base-change and coproduct as in Example 3.2.11. The patterns $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G)$ and $\mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)$ are soundly extendable. The slice $(\mathbb{F}_G)^f_{/A}$ decomposes as a product $\prod_{U\in A/G}(\mathbb{F}_G)^f_{/U}$ since the morphisms of $\mathbb{F}^f_G$ are closed under base-change. This implies that $(\mathbb{F}_G)^f_{/-}$ is a $\mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)$-Segal category since the elementary slice category $\mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)^{\mathrm{el}}_{A/} \simeq (\mathrm{Orb}_G)^{\mathrm{inj}}_{/A}$ is equivalent to the discrete set A/G over which we are taking the product. It also follows that $(\mathbb{F}_G)^f_{/-}$ is a Segal $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G)$-category since $(\mathrm{Orb}_G)^{\mathrm{inj}}_{/A} \simeq \mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)^{\mathrm{el}}_{A/} \hookrightarrow \mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G)^{\mathrm{el}}_{A/}$ is coinitial.

Example 3.3.27. The pattern $\mathsf{Span}(\mathcal{S}_m)$ is soundly extendable. Soundness follows from Corollary 3.3.24. For extendability we need to show that the functor

$$(\mathcal{S}_m)_{/-}\colon \mathcal{S}_m^{\mathrm{op}} \longrightarrow \mathsf{Cat}_\infty$$

is right Kan extended from its value at $* \in \mathcal{S}_m^{\mathrm{op}}$. Since being m-truncated can be checked fiberwise over $Y \in \mathcal{S}_m$, this functor is equivalent to $\mathsf{Fun}(-, \mathcal{S}_m)$ by straightening. This is now right Kan extended because $\mathsf{Fun}(X, \mathcal{S}_m) \simeq \lim_X \mathcal{S}_m$. One can show that $\mathsf{Span}_{(m-1)\text{-tr,all}}(\mathcal{S}_m)$ is also soundly extendable; we will not need this, however.

Finally, we give an example of a pattern that is *not* sound:

Example 3.3.28. We expect that the pattern U^{op} of undirected graphs of Hackney, Robertson, and Yau [RY20] is sound. However, this pattern does not include the nodeless loop S^1. In [Hac21], Hackney gives a simpler description of U^{op} and also defines a variant $\widetilde{\mathrm{U}}^{\mathrm{op}}$ that does include the nodeless loop. We will now show that this is an example of a non-sound pattern $\mathcal{O} = \widetilde{\mathrm{U}}^{\mathrm{op}}$. For the sake of brevity we shall not recall the definition, but rather the following facts:

- The category of elementaries under S^1 is trivial $\mathcal{O}^{\mathrm{el}}_{S^1/} \simeq *$.
- There is an active morphism $\omega\colon S^1 \rightsquigarrow S^1_n$ to the n-vertex loop S^1_n ($n \geq 2$), for which $\mathcal{O}^{\mathrm{el}}_{S^1_n/}$ is the poset of simplices of S^1_n, which is weakly equivalent to S^1.

We can now use the characterisation of soundness from Lemma 3.3.8.(3) in the case of the active morphism $\omega\colon S^1 \rightsquigarrow S^1_n$ described above. Since $\mathcal{O}^{\mathrm{el}}_{S^1/}$ is trivial (and in this case $\omega_{\alpha !}S^1$ is always elementary), the colimit runs over the constant diagram on the point and hence evaluates to the classifying space of $\mathcal{O}^{\mathrm{el}}_{S^1_n/}$, which is not contractible. Note that this could be resolved by introducing a variant of $\widetilde{\mathrm{U}}^{\mathrm{op}}$ where $\mathsf{Map}_{\mathcal{O}}(S^1, S^1) \simeq \mathsf{Map}_{\mathcal{O}}(S^1, e) \simeq O(2)$, in which case $\mathcal{O}^{\mathrm{el}}_{S^1/}$ is equivalent to the ∞-groupoid S^1.

4. Fibrous patterns and Segal envelopes

We begin this section by introducing the notion of *fibrous $\mathcal{O}$-patterns* as a generalization of ∞-operads over an arbitrary base pattern $\mathcal{O}$ in §4.1. We then apply the results of §2 to fibrous patterns in §4.2, where we prove Theorem C. Finally, in §4.3 we give some examples of Segal envelopes.

4.1. Fibrous patterns.

In this subsection we introduce the notion of a *fibrous $\mathcal{O}$-pattern* over a base algebraic pattern $\mathcal{O}$. (We borrow the adjective "fibrous" from [HA, §2.3.3], where it is used for a somewhat related concept.) Fibrous patterns specialize to give, for example, Lurie's ∞-operads and generalized ∞-operads if we take the base pattern to be $\mathbb{F}_*$ or $\mathbb{F}_*^\natural$. The concept is also a variant of the definition of *weak Segal fibrations* given in [CH21]; as we will see in Proposition 4.1.7 the two notions coincide if the base pattern is sound, i.e. for almost all interesting examples of patterns, but the definition of fibrous patterns seems to be simpler and better behaved if we do not assume soundness.

Observation 4.1.1. Let $\mathcal{O}$ be an algebraic pattern. If $\pi\colon \mathcal{P} \to \mathcal{O}$ has cocartesian lifts of inert morphisms, then applying Proposition 2.2.2 to the inert–active factorization system on $\mathcal{O}$ furnishes a cocartesian fibration $\mathcal{P} \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \to \mathcal{O}$ (where this functor is given as $(P, \pi(P) \rightsquigarrow O) \mapsto O$). For a morphism $\omega\colon O_1 \to O_2$ in $\mathcal{O}$ the cocartesian transport functor $\omega_!\colon \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O_1} \to \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O_2}$ is given by

$$(P,\ \varphi\colon \pi(P) \rightsquigarrow O_1) \mapsto (\alpha_! P,\ \beta\colon O' \rightsquigarrow O_2),$$

where

$$\pi(P) \overset{\alpha}{\rightarrowtail} O' \overset{\beta}{\rightsquigarrow} O_2$$

is the inert–active factorization of the composite

$$\pi(P) \overset{\varphi}{\rightsquigarrow} O_1 \xrightarrow{\omega} O_2$$

and $P \to \alpha_! P$ is a cocartesian lift of α.

Definition 4.1.2. Let $\mathcal{O}$ be an algebraic pattern. Then a *fibrous $\mathcal{O}$-pattern* is a functor $\pi\colon \mathcal{P} \to \mathcal{O}$ such that:

(1) $\mathcal{P}$ has all π-cocartesian lifts of inert morphisms in $\mathcal{O}$.

(2) For all $O \in \mathcal{O}$, the commutative square of ∞-categories

$$\begin{array}{ccc} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E} \end{array}$$

is cartesian. Here the horizontal functors are induced by cocartesian transport along the maps $O \rightarrowtail E$ in $\mathcal{O}^{\mathrm{el}}_{O/}$ for the cocartesian fibrations from Observation 4.1.1, applied to π and $\mathrm{id}_{\mathcal{O}}$.

Observation 4.1.3. Condition (2) in Definition 4.1.2 says precisely that the straightening of the projection $\mathcal{P} \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \to \mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$ over $\mathcal{O}$, i.e. the natural transformation $\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{P})\colon \mathrm{St}_{\mathcal{O}}(\mathcal{P} \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O})) \to \mathcal{A}_{\mathcal{O}}$, is a relative Segal $\mathcal{O}$-∞-category.

Remark 4.1.4. For many patterns $\mathcal{O}$, the functor $\mathcal{O}^{\mathrm{act}}_{/-}$ is a Segal $\mathcal{O}$-∞-category; this is the case, for instance, if $\mathcal{O}$ is *extendable* in the sense of [CH21] by [CH21, Lemma 9.14]. In this case, Observation 3.1.9 implies that condition (2) is satisfied if and only if the functor $\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{P})$ is a Segal $\mathcal{O}$-∞-category, i.e. the functor

$$\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \longrightarrow \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E}$$

is an equivalence for all $O \in \mathcal{O}$.

Example 4.1.5. Since $\mathbb{F}_*$ is extendable, a fibrous $\mathbb{F}_*$-pattern is a functor $\pi\colon \mathcal{P} \to \mathbb{F}_*$ such that $\mathcal{P}$ has π-cocartesian lifts for inerts, and for all n the functor

$$\mathcal{P}^{\mathrm{act}} \times_{\mathbb{F}} \mathbb{F}_{/\langle n\rangle} \simeq \mathcal{P} \times_{\mathbb{F}_*} (\mathbb{F}_*)^{\mathrm{act}}_{/\langle n\rangle} \longrightarrow \prod_{\langle n\rangle \rightarrowtail \langle 1\rangle} \mathcal{P} \times_{\mathbb{F}_*} \mathbb{F} \simeq (\mathcal{P}^{\mathrm{act}})^n,$$

is an equivalence. This functor takes an object $P \in \mathcal{P}^{\mathrm{act}}$ over $\langle m\rangle$ in $\mathbb{F}_*$ together with an active map $\omega\colon \langle m\rangle \rightsquigarrow \langle n\rangle$ to the list of objects $(P_1, \ldots, P_n)$ where $P \rightarrowtail P_j$ is the cocartesian lift of the inert map $\omega_j := (\rho_j \circ \omega)^{\mathrm{int}}\colon \langle m\rangle \rightarrowtail \langle m\rangle_j$ where $\rho_j\colon \langle n\rangle \rightarrowtail \langle 1\rangle$ is as in Example 3.1.6. We will see later (Proposition 4.1.7) that this condition is equivalent to $\mathcal{P} \to \mathbb{F}_*$ being an ∞-operad in the sense of Lurie.

We can rewrite the second condition in Definition 4.1.2 to obtain the following equivalent characterization of fibrous patterns:

Proposition 4.1.6. *For any algebraic pattern $\mathcal{O}$, a functor $\pi\colon \mathcal{P} \to \mathcal{O}$ is a fibrous $\mathcal{O}$-pattern if and only if:*

(1) $\mathcal{P}$ has π-cocartesian morphisms over inert morphisms in $\mathcal{O}$.

(2) For every active morphism $\omega\colon O_1 \rightsquigarrow O_2$ in $\mathcal{O}$, and all objects $X_0 \in \mathcal{P}_{O_0}, X_1 \in \mathcal{P}_{O_1}$, the commutative square

$$\begin{array}{ccc} \mathsf{Map}_{\mathcal{P}}(X_0, X_1) & \longrightarrow & \lim_{\alpha\colon O_2 \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O_2/}} \mathsf{Map}_{\mathcal{P}}(X_0, \omega_{\alpha,!}X_1) \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathcal{O}}(O_0, O_1) & \longrightarrow & \lim_{\alpha\colon O_2 \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O_2/}} \mathsf{Map}_{\mathcal{O}}(O_0, \omega_{\alpha,!}O_1) \end{array} \tag{4}$$

is cartesian. Here the horizontal maps are defined using the functor $\omega_{(-)}\colon \mathcal{O}^{\mathrm{el}}_{O_2/} \to \mathcal{O}^{\mathrm{int}}_{O_1/}$ from Notation 3.3.1.

(3) *For every active morphism $\omega\colon O_1 \rightsquigarrow O_2$ in $\mathcal{O}$, the functor*

$$\mathcal{P}^{\simeq}_{O_1} \longrightarrow \lim_{\alpha\colon O_2 \to E \in \mathcal{O}^{\mathrm{el}}_{O_2/}} \mathcal{P}^{\simeq}_{\omega_{\alpha!}O_1},$$

induced by cocartesian transport along the inert morphisms $\omega_\alpha\colon O_1 \rightarrowtail \omega_{\alpha!}O_1$ in $\mathcal{O}^{\mathrm{int}}_{O_1/}$, is an equivalence.

Proof. A square of ∞-categories is cartesian if and only if the underlying square of ∞-groupoids as well as all induced squares of mapping spaces are cartesian. For the square in the definition of a fibrous pattern the underlying square of ∞-groupoids is

$$\begin{array}{ccc} \mathcal{P}^{\simeq} \times_{\mathcal{O}^{\simeq}} (\mathcal{O}_{/O})^{\simeq} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P}^{\simeq} \times_{\mathcal{O}^{\simeq}} (\mathcal{O}_{/E})^{\simeq} \\ \downarrow & & \downarrow \\ (\mathcal{O}_{/O})^{\simeq} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} (\mathcal{O}_{/E})^{\simeq}; \end{array}$$

this is cartesian if and only if the map on fibers over each $\omega\colon O' \rightsquigarrow O$ is an equivalence. This map takes the form

$$\mathcal{P}^{\simeq}_{O'} \longrightarrow \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P}^{\simeq}_{\omega_{\alpha,!}O'}. \tag{5}$$

and is induced by the cocartesian transport along the inert morphisms $\omega_\alpha\colon O' \rightarrowtail \omega_{\alpha!}O'$ as in Notation 3.3.1. This is exactly the map from condition (3), so the square of ∞-groupoids is cartesian if and only if condition (3) holds.

Now consider the square of mapping spaces for two objects $(P, \varphi\colon \pi(P) \rightsquigarrow O)$ and $(P', \varphi'\colon \pi(P') \rightsquigarrow O) \in \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}$:

(6)

$$\begin{array}{ccc} \mathsf{Map}_{\mathcal{P}\times_{\mathcal{O}}\mathcal{O}^{\mathrm{act}}_{/O}}((P,\varphi),(P',\varphi')) & \longrightarrow & \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}_{\mathcal{P}\times_{\mathcal{O}}\mathcal{O}^{\mathrm{act}}_{/E}}((P,\varphi),(\varphi'_{\alpha,!}P',(\alpha\circ\varphi')^{\mathrm{act}})) \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\varphi,\varphi') & \longrightarrow & \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\varphi,(\alpha\circ\varphi')^{\mathrm{act}}). \end{array}$$

A point in $\mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\varphi, \varphi')$ is a (necessarily active) morphism $f\colon \pi(P) \rightsquigarrow \pi(P')$ together with a homotopy $\varphi \simeq \varphi' \circ f$. To compute the fiber of the vertical maps at this point, note that the mapping space in $\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}$ can be computed as:

$$\mathsf{Map}_{\mathcal{P}\times_{\mathcal{O}}\mathcal{O}^{\mathrm{act}}_{/O}}((P,\varphi),(P',\varphi')) \simeq \mathsf{Map}_{\mathcal{P}}(P,P') \times_{\mathsf{Map}_{\mathcal{O}}(\pi(P),\pi(P'))} \mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\varphi,\varphi'),$$

Hence the map on the vertical fibers of the square is given by

$$\mathsf{Map}^{f}_{\mathcal{P}}(P,P') \longrightarrow \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}^{\varphi'_\alpha \circ f}_{\mathcal{P}}(P, \varphi'_{\alpha,!}P'), \tag{7}$$

where the superscripts indicate fibers over maps in $\mathcal{O}$. This agrees with the map on fibers over f of the square in condition (2). Therefore condition (2) implies that the square of mapping spaces is a pullback.

However, we have not shown the converse yet, because we have only considered the fibers in (4) over morphisms $f \in \mathsf{Map}_{\mathcal{O}}(O_0, O_1)$ that are active. Let us now assume that the square of mapping spaces (6) is cartesian. For a general morphism $O_0 \to O_1$ we can find an inert-active factorization $O_0 \overset{j}{\rightarrowtail} Q \overset{g}{\rightsquigarrow} O_1$. Since j is inert we can find a cocartesian lift $\widetilde{j}\colon P_0 \to j_! P_0$ and by virtue of this being cocartesian, pre-composition with $\widetilde{j}$ induces the vertical equivalences in the following diagram:

$$\begin{array}{ccc}
\mathsf{Map}^{g}_{\mathcal{P}}(j_!P, P') & \longrightarrow & \lim_{\alpha\colon O\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}^{\varphi'_\alpha\circ g}_{\mathcal{P}}(j_!P, \varphi'_{\alpha,!}P') \\
{\scriptstyle (-)\circ\widetilde{j}}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle(-)\circ\widetilde{j}} \\
\mathsf{Map}^{g\circ j}_{\mathcal{P}}(P, P') & \longrightarrow & \lim_{\alpha\colon O\rightarrowtail E\in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}^{\varphi'_\alpha\circ g\circ j}_{\mathcal{P}}(P, \varphi'_{\alpha,!}P').
\end{array}$$

Since g is active, the previous argument shows that the top map is an equivalence. Hence the bottom map is an equivalence and as $f = g\circ j$ was arbitrary this shows that condition (2) is implied. □

The conditions in Proposition 4.1.6 are reminiscent of Lurie's definition of an ∞-operad [HA]. Note, however, that in conditions (2) and (3) we need to consider *all* active maps in $\mathcal{O}$, while Lurie's definition of ∞-operads, or the definition of *weak Segal fibrations* in [CH21], only involve the conditions corresponding to identity maps. If the base pattern is *sound*, however, the conditions for all active maps are implied by this special case:

Proposition 4.1.7. *Suppose $\mathcal{O}$ is a sound pattern. Then a functor $\pi\colon \mathcal{P}\to\mathcal{O}$ is a fibrous $\mathcal{O}$-pattern if and only if it is a weak Segal $\mathcal{O}$-fibration in the sense of* [CH21, Definition 9.6]*, i.e. the conditions of Proposition 4.1.6 hold whenever ω is an identity morphism. Concretely:*

(1) $\mathcal{P}$ has all π-cocartesian lifts of inert morphisms in $\mathcal{O}$.

(2) For all $O_0, O_1\in\mathcal{O}$, and all objects $X_0\in\mathcal{P}_{O_0}, X_1\in\mathcal{P}_{O_1}$, the commutative square

$$\begin{array}{ccc}
\mathsf{Map}_{\mathcal{P}}(X_0, X_1) & \longrightarrow & \lim_{\alpha\colon O_1\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O_1/}} \mathsf{Map}_{\mathcal{P}}(X_0, \alpha_! X_1) \\
\downarrow & & \downarrow \\
\mathsf{Map}_{\mathcal{O}}(O_0, O_1) & \longrightarrow & \lim_{\alpha\colon O_1\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O_1/}} \mathsf{Map}_{\mathcal{O}}(O_0, E)
\end{array}$$

is cartesian.

(3) For every $O_1\in\mathcal{O}$, the functor

$$\mathcal{P}^{\simeq}_{O_1} \longrightarrow \lim_{\alpha\colon O_1\longrightarrow E\in\mathcal{O}^{\mathrm{el}}_{O_1/}} \mathcal{P}^{\simeq}_{E},$$

induced by cocartesian transport along $\alpha\colon O_1\rightarrowtail E$ is an equivalence.

Remark 4.1.8. In [CH21] (and [HA]), the analogue of condition (3) says that the functor

$$\mathcal{P}_{O_1} \longrightarrow \lim_{\alpha\colon O_1\longrightarrow E\in\mathcal{O}^{\mathrm{el}}_{O_1/}} \mathcal{P}_{\alpha_! O'_2}$$

is an equivalence, rather than that the underlying map of ∞-groupoids is one. However, it follows from (2) that this functor gives an equivalence on mapping spaces, i.e. it is already fully faithful, and so it is an equivalence if and only if it is an equivalence on underlying ∞-groupoids. In fact, it would suffice in (3) to assume that the map is merely surjective on π_0.

Proof of Proposition 4.1.7. Suppose $\pi\colon\mathcal{P}\to\mathcal{O}$ is a weak Segal fibration. Consider the functor $F\colon \mathcal{O}^{\mathrm{int}}_{O_1/}\to\mathcal{O}^{\mathrm{int}}\to\mathcal{S}$ defined by $F(O_2) := \mathcal{P}^{\simeq}_{O_2}$ and cocartesian transport along inerts. The natural transformation $\eta\colon F\Rightarrow *$ to the terminal functor satisfies the conditions of Lemma 3.3.7. The conclusion of the lemma tells us that (2) holds for all $\omega\colon O_1\rightsquigarrow O_2$.

For property (3), fix $X_0, X_1\in\mathcal{P}$ with $\pi(X_0) = O_0$ and $\pi(X_1) = O_1$. Then cocartesian transport along inerts defines a functor

$$F\colon \mathcal{O}^{\mathrm{int}}_{O_1/}\longrightarrow\mathcal{S},\quad (\varphi\colon O_1\rightarrowtail O_2)\mapsto \mathsf{Map}(X_0, \varphi_! X_1)$$

and this admits a canonical natural transformation to the functor $G(\varphi\colon O_1 \rightarrowtail O_2) := \mathsf{Map}(O_0, O_2)$. Applying lemma 3.3.7 to $\eta\colon F \Rightarrow G$ shows that (3) holds for all $\omega\colon O_1 \rightsquigarrow O_2$. □

Example 4.1.9. Fibrous $\mathbb{F}_*$-patterns are precisely (symmetric) ∞-operads as defined in [HA], while fibrous $\mathbb{F}_*^{\natural}$- patterns are generalized (symmetric) ∞-operads. Similarly, fibrous $\Delta^{\mathrm{op},\flat}$- and $\Delta^{\mathrm{op},\natural}$-patterns are non-symmetric (or planar) ∞-operads and generalized non-symmetric ∞-operads, respectively.

Observation 4.1.10. For a sound pattern $\mathcal{O}$ we can also describe the fibrous $\mathcal{O}$-patterns that are cocartesian fibrations as the unstraightenings of Segal $\mathcal{O}$-∞-categories, i.e. as the *Segal $\mathcal{O}$-fibrations* of [CH21, Definition 9.1]. This is easy to check directly, but it is also a special case of Lemma 4.2.4 (taking $Y = *$), which we will prove below.

Fibrous $\mathcal{O}$-patterns admit a canonical pattern structure, which we now introduce:

Definition 4.1.11. Suppose $\pi\colon \mathcal{P} \to \mathcal{O}$ is a fibrous $\mathcal{O}$-pattern. We say a morphism in $\mathcal{P}$ is *inert* if it is π-cocartesian and lies over an inert morphism in $\mathcal{O}$, and *active* if it just lies over an active morphism in $\mathcal{O}$. The inert and active morphisms then form a factorization system on $\mathcal{P}$ by [HA, Proposition 2.1.2.5], and we give $\mathcal{P}$ an algebraic pattern structure with this factorization system by taking the elementary objects to be all those that lie over elementary objects in $\mathcal{O}$.

Definition 4.1.12. A *morphism of fibrous $\mathcal{O}$-patterns* is a commutative triangle

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\;f\;} & \mathcal{P}' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & \mathcal{O}, & \end{array}$$

where π and π' are fibrous $\mathcal{O}$-patterns and f is a morphism of algebraic patterns. It is immediate from the definition of the pattern structures that for this it suffices to require that f preserves inert morphisms. We write $\mathsf{Fbrs}(\mathcal{O})$ for the full subcategory of $\mathsf{AlgPatt}_{/\mathcal{O}}$ whose objects are the fibrous $\mathcal{O}$-patterns; this is equivalently a full subcategory of $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$.

Lemma 4.1.13. *The inclusion $\mathsf{Fbrs}(\mathcal{O}) \hookrightarrow \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$ preserves limits and κ-filtered colimits where κ is a regular cardinal such that $\mathcal{O}^{\mathrm{el}}_{O/}$ is κ-small for all $O \in \mathcal{O}$. Limits and κ-filtered colimits of $\mathcal{O}$-fibrous patterns can therefore be computed in Cat_∞.*

Proof. By Observation 2.3.8 the forgetful functor $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \to \mathsf{Cat}_{\infty/\mathcal{O}} \to \mathsf{Cat}_\infty$ preserve limits and κ-filtered colimits, and are also conservative. It therefore suffices to observe that the commutative square that is required to be cartesian for an object of $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$ to be a fibrous $\mathcal{O}$-pattern commutes with limits and κ-filtered colimits of ∞-categories. Since a limit or filtered colimit of cartesian squares in Cat_∞ is again cartesian, this implies the result. □

Observation 4.1.14. If $\pi\colon \mathcal{P} \to \mathcal{O}$ is a fibrous $\mathcal{O}$-pattern, then for every object $\overline{X} \in \mathcal{P}$ over X in $\mathcal{O}$, the functor

$$\mathcal{P}^{\mathrm{el}}_{\overline{X}/} \longrightarrow \mathcal{O}^{\mathrm{el}}_{X/}$$

is an equivalence. Indeed, since $\mathcal{P}^{\mathrm{int}} \to \mathcal{O}^{\mathrm{int}}$ is a cocartesian fibration the functor $\mathcal{P}^{\mathrm{int}}_{\overline{X}/} \to \mathcal{O}^{\mathrm{int}}_{X/}$ is an equivalence, and the above functor is obtained by restricting to the full subcategories of elementary objects. In particular, π is an iso-Segal morphism.

More generally, if $f\colon \mathcal{P} \to \mathcal{Q}$ is a morphism of fibrous $\mathcal{O}$-patterns, then f induces an equivalence

$$\mathcal{P}^{\mathrm{el}}_{\overline{X}/} \xrightarrow{\sim} \mathcal{Q}^{\mathrm{el}}_{f(\overline{X})/}$$

for the same reason, so that f is also an iso-Segal morphism.

Lemma 4.1.15. *Suppose $\mathcal{O}$ is a sound pattern and $\pi\colon \mathcal{P} \to \mathcal{O}$ is $\mathcal{O}$-fibrous. Then $\mathcal{P}$ is also a sound pattern. Moreover, if $\mathcal{O}$ is soundly extendable, then so is $\mathcal{P}$.*

Proof. It follows from Observation 4.1.14 that π induces an equivalence

$$\pi\colon \mathcal{P}^{\mathrm{el}}_{\beta}(\omega) = \mathcal{P}^{\mathrm{el}}_{Y/} \times_{\mathcal{P}^{\mathrm{int}}_{X/}} (\mathcal{P}^{\mathrm{int}}_{X/})_{/\beta} \longrightarrow \mathcal{O}^{\mathrm{el}}_{\pi(Y)/} \times_{\mathcal{O}^{\mathrm{int}}_{\pi(X)/}} (\mathcal{O}^{\mathrm{int}}_{\pi(X)/})_{/\pi(\beta)} = \mathcal{O}^{\mathrm{el}}_{\pi(\beta)}(\pi(\omega))$$

for all active $\omega\colon X \rightsquigarrow Y$ and $\beta\colon Y \rightarrowtail E' \in \mathcal{P}^{\mathrm{el}}_{Y/}$. Hence $\mathcal{P}$ is sound by Lemma 3.3.8.(2).

Now assume $\mathcal{O}$ is soundly extendable. Then, by Remark 4.1.4, the functor

$$\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/Y} \longrightarrow \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{Y/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E'}$$

is an equivalence. Since any morphism in $\mathcal{P}$ that is mapped to an active morphism in $\mathcal{O}$ is active by definition and active morphisms satisfy cancellation, we have that $\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/Y} = \mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/Y}$. Consider the case where $Y = \pi(X)$ for $X \in \mathcal{P}$. Since $\mathcal{P} \to \mathcal{O}$ is an equivalence on elementary slices, we can rewrite the limit on the right-hand side as a limit over $E \in \mathcal{P}^{\mathrm{el}}_{X/}$ and set $E' := \pi(E)$:

$$\mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(X)} \xrightarrow{\simeq} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{X/}} \mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(E)}.$$

Now, passing to the over-category of $(X, \mathrm{id}_{\pi(X)})$ we obtain an equivalence:

$$\mathcal{P}^{\mathrm{act}}_{/X} \simeq (\mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(X)})_{/(X,\mathrm{id}_{\pi(X)})} \xrightarrow{\simeq} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{X/}} (\mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(E)})_{/(E,\mathrm{id}_{\pi(E)})} \simeq \lim_{E \in \mathcal{P}^{\mathrm{el}}_{X/}} \mathcal{P}^{\mathrm{act}}_{/E},$$

which shows that $\mathcal{P}$ is soundly extendable. □

Proposition 4.1.16. *Suppose we have a commutative triangle of algebraic patterns*

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{F} & \mathcal{P} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{O}, & \end{array}$$

where $\mathcal{P}$ is $\mathcal{O}$-fibrous. Then $\mathcal{Q}$ is $\mathcal{O}$-fibrous if and only if it is $\mathcal{P}$-fibrous.

Proof. Any inert morphism $\pi\colon P \to P'$ in $\mathcal{P}$ is cocartesian over an inert morphism $\omega\colon O \to O'$ in $\mathcal{O}$; if $\varphi\colon Q \to Q'$ is an inert morphism over ω in $\mathcal{Q}$ such that $F(Q) \simeq P$, then we have $F(\varphi) \simeq \pi$ since F preserves inert morphisms and π is the unique inert morphism over ω with source P. It now follows from [HTT, Proposition 2.4.1.3] that $\varphi\colon Q \to Q'$ is F-cocartesian if and only if it is q-cocartesian. Thus condition (1) in Definition 4.1.2 holds for F if and only if it holds for q.

Assuming this holds, then for $P \in \mathcal{P}_O, P' \in \mathcal{P}_{O'}, Q \in \mathcal{Q}_P, Q' \in \mathcal{Q}_{P'}$ and $\omega\colon O' \to O''$, we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Map}_{\mathcal{Q}}(Q, Q') & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \mathsf{Map}_{\mathcal{Q}}(Q, \omega_{\alpha,!}Q') \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathcal{P}}(P, P') & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \mathsf{Map}_{\mathcal{P}}(P, \omega_{\alpha,!}P') \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathcal{O}}(O, O') & \longrightarrow & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \mathsf{Map}_{\mathcal{O}}(O, \omega_{\alpha,!}O'). \end{array}$$

Here the bottom square is cartesian since $\mathcal{P}$ is $\mathcal{O}$-fibrous, so the top square is cartesian if and only if the outer square is cartesian. But since p is an iso-Segal morphism we can rewrite the top square as

$$\begin{array}{ccc} \mathsf{Map}_{\mathcal{Q}}(Q,Q') & \longrightarrow & \lim_{\alpha\in\mathcal{P}^{\mathrm{el}}_{P'/}} \mathsf{Map}_{\mathcal{Q}}(Q,\omega_{\alpha,!}Q') \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathcal{P}}(P,P') & \longrightarrow & \lim_{\alpha\in\mathcal{P}^{\mathrm{el}}_{P'/}} \mathsf{Map}_{\mathcal{P}}(P,\omega_{\alpha,!}P'), \end{array}$$

and so we have that (2) holds for F if and only if it holds for q. The proof for (3) is similar. □

Corollary 4.1.17. *If $\pi\colon \mathcal{P}\to\mathcal{O}$ exhibits $\mathcal{P}$ as an $\mathcal{O}$-fibrous pattern, then composition with π gives a functor*

$$\pi_!\colon \mathsf{Fbrs}(\mathcal{P}) \longrightarrow \mathsf{Fbrs}(\mathcal{O}),$$

and this induces an equivalence

$$\mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O})_{/\mathcal{P}}.$$

Example 4.1.18. Let $(\pi\colon \mathcal{O}\to\mathbb{F}_*)\in\mathsf{Opd}_\infty$ be an ∞-operad in the sense of Lurie, i.e. a fibrous $\mathbb{F}_*$-pattern. Applying Corollary 4.1.17 we obtain an equivalence:

$$\mathsf{Fbrs}(\mathcal{O}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*)_{/\mathcal{O}} = \mathsf{Opd}_{\infty/\mathcal{O}}$$

so fibrous $\mathcal{O}$-patterns are simply ∞-operads over $\mathcal{O}$.

Lemma 4.1.19. *Suppose $f\colon\mathcal{O}\to\mathcal{P}$ is a strong Segal morphism. Then pullback along f restricts to a functor*

$$f^*\colon \mathsf{Fbrs}(\mathcal{P}) \longrightarrow \mathsf{Fbrs}(\mathcal{O}), \qquad (\pi\colon\mathcal{F}\longrightarrow\mathcal{P}) \mapsto (f^*\pi\colon \mathcal{F}\times_{\mathcal{P}}\mathcal{O}\longrightarrow\mathcal{O}).$$

Proof. Suppose $\pi\colon\mathcal{F}\to\mathcal{P}$ is a $\mathcal{P}$-fibrous pattern. Condition (1) in Definition 4.1.2 for $f^*\mathcal{F}$ follows from the usual description of cocartesian morphisms in a pullback, since f preserves inert morphisms. To prove (2), we observe that $f^*\mathcal{F}\times_{\mathcal{O}}\mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \simeq \mathcal{F}\times_{\mathcal{P}}\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$, so that we have a cartesian square

$$\begin{array}{ccc} f^*\mathcal{F}\times_{\mathcal{O}}\mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) & \longrightarrow & \mathcal{F}\times_{\mathcal{P}}\mathrm{Ar}_{\mathrm{act}}(\mathcal{P})\times_{\mathcal{P}}\mathcal{O} \\ \downarrow & & \downarrow \\ \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) & \longrightarrow & \mathrm{Ar}_{\mathrm{act}}(\mathcal{P})\times_{\mathcal{P}}\mathcal{O} \end{array}$$

of cocartesian fibrations over $\mathcal{O}$. Straightening yields the cartesian square:

$$\begin{array}{ccc} \mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(f^*\mathcal{F}) & \longrightarrow & \mathrm{St}^{\mathrm{int}}_{\mathcal{P}}(\mathcal{F})\circ f \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{O}} & \longrightarrow & \mathcal{A}_{\mathcal{P}}\circ f \end{array}$$

of functors $\mathcal{O}\to\mathsf{Cat}_\infty$. By Observation 4.1.3 the natural transformation $\mathrm{St}^{\mathrm{int}}_{\mathcal{P}}(\mathcal{F})\to\mathcal{A}_{\mathcal{P}}$ is a relative $\mathcal{P}$-Segal ∞-category. This remains true after precomposing with f (by Observation 3.1.15, since f is strong Segal) . Hence the right vertical map in the square is a relative $\mathcal{O}$-Segal ∞-categoryand by Lemma 3.1.10 so is the left vertical arrow. Using Observation 4.1.3 again we see that $f^*\mathcal{F}$ is fibrous. □

Example 4.1.20. The morphism $\mathfrak{c}\colon \Delta^{\mathrm{op},\flat}\to\mathbb{F}_*$ from Example 3.1.13 is iso-Segal and hence Lemma 4.1.19 shows that pulling back along it defines a functor:

$$\mathfrak{c}^*\colon \mathsf{Fbrs}(\mathbb{F}_*) \longrightarrow \mathsf{Fbrs}(\Delta^{\mathrm{op},\flat}).$$

Under the identifications of Example 4.1.9 this is exactly the forgetful functor from (symmetric) ∞-operads to non-symmetric ∞-operads.

Finally, let us note that we can lift the comparison of Proposition 3.1.16 to fibrous patterns:

Proposition 4.1.21. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism that satisfies the conditions of Proposition 3.1.16 and let $\pi\colon \mathcal{Q} \to \mathcal{P}$ be a fibrous pattern. Then $\overline{f}\colon f^*\mathcal{Q} \to \mathcal{Q}$ is also a strong Segal morphism that satisfies the conditions of Proposition 3.1.16 and thus induces an equivalence*

$$\overline{f}^*\colon \mathsf{Seg}_{\mathcal{Q}}(\mathcal{S}) \xrightarrow{\sim} \mathsf{Seg}_{f^*\mathcal{Q}}(\mathcal{S}).$$

Proof. Denote by $\pi'\colon \mathcal{Q}' := f^*\mathcal{Q} \to \mathcal{O}$ the projection map. Since $\mathcal{Q}$ is fibrous and f is strong Segal, it follows from Lemma 4.1.19 that $\mathcal{Q}'$ is also fibrous. By Observation 4.1.14 we have $\mathcal{Q}^{\mathrm{el}}_{Q/} \simeq \mathcal{P}^{\mathrm{el}}_{\pi(Q)/}$ and similarly for $\mathcal{Q}'$ and $\mathcal{O}$. The map $(\mathcal{Q}')^{\mathrm{el}}_{Q/} \to \mathcal{Q}^{\mathrm{el}}_{\overline{f}(Q)/}$ thus identifies with $\mathcal{O}^{\mathrm{el}}_{\pi'(Q)/} \to \mathcal{P}^{\mathrm{el}}_{f(\pi'(Q))/}$ which is coinitial by the assumption that f is strong Segal. We conclude that $\overline{f}$ is strong Segal. We proceed by verifying the conditions. Condition (1) of Proposition 3.1.16 is visibly stable under basechange so it remains to check (2). Observe that for every object $Q \in \mathcal{Q}'$ that lies over $O \in \mathcal{O}$ we have by [HTT, Lemma 5.4.5.4] a pullback square of slice ∞-categories

$$\begin{array}{ccc} \mathcal{Q}'_{/Q} & \longrightarrow & \mathcal{Q}_{/\overline{f}(Q)} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{O}_{/O} & \longrightarrow & \mathcal{P}_{/f(O)}. \end{array}$$

By assumption the bottom map induces an equivalence on the underlying spaces of active maps and since the square is cartesian the same holds for the top map. □

4.2. **Segal envelopes.** In this section we will specialize our results from Section 2 to fibrous $\mathcal{O}$-patterns over an algebraic pattern $\mathcal{O}$. Recall that we have shown that from the inert–active factorization system on $\mathcal{O}$ we obtain an adjunction

$$(-) \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O})\colon \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \rightleftarrows (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}})_{/\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})},$$

where the right adjoint is given by pulling back along the map $\mathcal{O} \to \mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$ given by the degeneracy $[1] \to [0]$. This can equivalently be interpreted as a "straightening–unstraightening" adjunction

$$\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}\colon \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \rightleftarrows \mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}} :\mathrm{Un}^{\mathrm{int}}_{\mathcal{O}}$$

in which the left adjoint is fully faithful with image the $\mathcal{A}_{\mathcal{O}}$-equifibered functors.

We can immediately identify the image of the full subcategory $\mathsf{Fbrs}(\mathcal{O})$ under this fully faithful functor:

Proposition 4.2.1. *For any algebraic pattern $\mathcal{O}$, the fully faithful functor $\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}$ identifies $\mathsf{Fbrs}(\mathcal{O})$ with the full subcategory of $\mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$ spanned by the equifibered maps that are also relative Segal objects. In other words, the functor $\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}$ restricts to a fully faithful functor*

$$\mathsf{Env}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}} := \mathrm{St}^{\mathrm{int}}_{\mathcal{O}}|_{\mathsf{Fbrs}(\mathcal{O})}\colon \mathsf{Fbrs}(\mathcal{O}) \hookrightarrow \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$$

with image the equifibered objects. Moreover, for any strong Segal morphism $f\colon \mathcal{O} \to \mathcal{P}$, we have a commutative square

$$\begin{array}{ccc} \mathsf{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \mathsf{Fbrs}(\mathcal{O}) \\ \downarrow{\scriptstyle \mathsf{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}} & & \downarrow{\scriptstyle \mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}} \\ \mathsf{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) & \xrightarrow{f^\circledast} & \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty) \end{array} \tag{8}$$

where the functor $f^\circledast$ is given by the composite

$$\mathsf{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) \xrightarrow{f^*} \mathsf{Seg}_{\mathcal{O}}^{/f^*\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) \longrightarrow \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$$

of restriction along f and pullback along the natural map $f^*\mathcal{A}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{O}}$ (cf. Observation 3.1.15 and Lemma 3.1.10).

Proof. From Observation 4.1.3 we know that an object $\mathcal{P}$ of $\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ is a fibrous $\mathcal{O}$-pattern if and only if $\mathrm{St}_{\mathcal{O}}^{\text{int}}(\mathcal{P})$ is a relative Segal $\mathcal{O}$-object in Cat_∞. The commutative square (8) likewise follows by restricting the square (3) in Observation 2.3.9 to full subcategories. □

From this observation we can deduce some pleasant properties of the ∞-categories of fibrous patterns:

Corollary 4.2.2. *For any algebraic pattern $\mathcal{O}$, the ∞-category $\mathsf{Fbrs}(\mathcal{O})$ is presentable, and fits in a cartesian square of fully faithful right adjoints*

$$\begin{array}{ccc} \mathsf{Fbrs}(\mathcal{O}) & \xhookrightarrow{\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}} & \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty) \\ \downarrow & & \downarrow \\ \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} & \xhookrightarrow{\mathrm{St}_{\mathcal{O}}^{\text{int}}} & \mathsf{Fun}(\mathcal{O}^{\mathrm{op}}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}. \end{array}$$

Proof. We know from Proposition 4.2.1 that we have the given cartesian square of fully faithful functors; it remains to show that this is a square in Pr^R. For the bottom horizontal and right vertical functor we have shown this in Proposition 2.3.7 and Lemma 3.1.11, respectively. It now follows that the rest of the diagram also lies in Pr^R, since the diagram is cartesian and by [HTT, Theorem 5.5.3.18] Pr^R admits pullbacks and the inclusion $\mathrm{Pr}^R \subset \mathsf{Cat}_\infty$ preserves them. □

Corollary 4.2.3.

(1) For any algebraic pattern $\mathcal{O}$, the following fully faithful inclusions admit left adjoints:

$$\mathsf{Fbrs}(\mathcal{O}) \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{O}}.$$

(2) For any strong Segal morphism $f\colon \mathcal{O} \to \mathcal{P}$, the functor $f^*\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O})$ admits a left adjoint.

Proof. The first claim was shown in Corollary 4.2.2 and Proposition 2.3.7. In particular limits and κ-filtered colimits in $\mathsf{Fbrs}(\mathcal{O})$ for appropriate κ are computed in $\mathsf{Cat}_{\infty/\mathcal{O}}$ (or equivalently in Cat_∞). This implies that $f^*\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{P})$ preserves limits and κ-filtered colimits, since we know pullback along f preserves limits and filtered colimits as a functor $\mathsf{Cat}_{\infty/\mathcal{P}} \to \mathsf{Cat}_{\infty/\mathcal{O}}$. Hence the claim follows from the adjoint functor theorem. □

Note that in Proposition 4.2.1 we only showed that the left adjoint $\mathrm{St}_{\mathcal{O}}^{\text{int}}$ restricts to a functor from fibrous patterns to relative Segal objects — in general the right adjoint

$\mathrm{Un}^{\mathrm{int}}_{\mathcal{O}}$ does not necessarily take relative Segal $\mathcal{O}$-∞-categories over $\mathcal{A}_{\mathcal{O}}$ to fibrous $\mathcal{O}$-patterns. However, this is the case if $\mathcal{O}$ is sound; to see this, we first need a technical lemma:

Lemma 4.2.4. *Let $\mathcal{O}$ be a sound algebraic pattern and let $\gamma\colon X \to Y$ be a morphism in $\mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)$, with $\Gamma\colon \mathcal{X} \to \mathcal{Y}$ denoting its unstraightening. Then the following are equivalent:*

(1) $\gamma\colon X \to Y$ is a relative Segal object.
(2) $\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\Gamma)\colon \mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{X}) \to \mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{Y})$ is a relative Segal object, i.e. the commutative square

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \\ \downarrow & & \downarrow \\ \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \end{array}$$

is cartesian for all $O \in \mathcal{O}$.

Proof. For $O \in \mathcal{O}$, we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} & & \longrightarrow & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \\ & \searrow & & & & \searrow \\ & & \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} & \longrightarrow & & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \\ & \swarrow & & & \searrow & \swarrow \\ \mathcal{O}^{\mathrm{act}}_{/O} & & \longrightarrow & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}. \end{array}$$

Here all four functors to the bottom row are cocartesian fibrations, and the morphisms in the top square preserve cocartesian morphisms. We therefore see that condition (2), which asks for the top square to be cartesian, is equivalent to all squares of fibers over $\omega\colon O' \rightsquigarrow O$ in $\mathcal{O}^{\mathrm{act}}_{/O}$ being cartesian. The relevant square of fibers is

$$\begin{array}{ccc} X(O') & \longrightarrow & \lim_{(\alpha\colon O \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{O/}} X(\omega_{\alpha !} O') \\ \downarrow & & \downarrow \\ Y(O') & \longrightarrow & \lim_{(\alpha\colon O \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{O/}} Y(\omega_{\alpha !} O'). \end{array}$$

Considering the special case $\omega = \mathrm{id}_O$ we see that (2) implies (1), while to see that the converse holds when $\mathcal{O}$ is sound we apply Lemma 3.3.7 with $F = X$ and $G = Y$. □

Proposition 4.2.5. *If the pattern $\mathcal{O}$ is sound, then the adjunction of Notation 2.3.4 restricts to an adjunction*

$$\mathsf{Env}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}\colon \mathsf{Fbrs}(\mathcal{O}) \rightleftarrows \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty) : \mathrm{Un}^{\mathrm{int}}_{\mathcal{O}}.$$

Moreover, if $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism between sound patterns, then in addition to the square (8) we also have a commutative square

$$\begin{array}{ccc} \mathsf{Seg}^{/\mathcal{A}_{\mathcal{P}}}_{\mathcal{P}}(\mathsf{Cat}_\infty) & \xrightarrow{f^{\circledast}} & \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty) \\ {\scriptstyle \mathrm{Un}^{\mathrm{int}}_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Un}^{\mathrm{int}}_{\mathcal{O}}} \\ \mathsf{Fbrs}(\mathcal{P}) & \xrightarrow{f^{*}} & \mathsf{Fbrs}(\mathcal{O}). \end{array} \tag{9}$$

Proof. We need to show that $\mathrm{Un}^{\mathrm{int}}_{O}\colon \mathsf{Fun}(O^{\mathrm{op}}, \mathsf{Cat}_\infty)_{/\mathcal{A}_O} \to \mathsf{Cat}^{\text{int-cocart}}_{\infty/O}$ sends $\mathcal{A}_O$-relative Segal objects to fibrous O-patterns. Since we know an object of $\mathsf{Cat}^{\text{int-cocart}}_{\infty/O}$ is fibrous if and only if its image under $\mathrm{St}^{\mathrm{int}}_O$ is a relative Segal object, it suffices to show that $\mathrm{St}^{\mathrm{int}}_O \circ \mathrm{Un}^{\mathrm{int}}_O$ preserves relative Segal objects.

Let $X \to \mathcal{A}_O$ be a relative Segal object; then $\mathrm{St}^{\mathrm{int}}_O(\mathrm{Un}^{\mathrm{int}}_O(X))$ fits into a cartesian square

$$\begin{array}{ccc} \mathrm{St}^{\mathrm{int}}_O(\mathrm{Un}^{\mathrm{int}}_O(X)) & \longrightarrow & \mathrm{St}^{\mathrm{int}}_O(\mathrm{Un}_O(X)) \\ \downarrow & & \downarrow \\ \mathcal{A}_O & \longrightarrow & \mathrm{St}^{\mathrm{int}}_O(\mathrm{Un}_O(\mathcal{A}_O)) \end{array}$$

obtained from applying $\mathrm{St}^{\mathrm{int}}_O$ to the cartesian square defining $\mathrm{Un}^{\mathrm{int}}_O(X)$. Since relative Segal objects are stable under base change by 3.1.10, it suffices to show the right vertical map is a relative Segal object, which follows from Lemma 4.2.4. The commutative square (9) follows by restricting the square (2) in Observation 2.3.9 to full subcategories. □

For soundly extendable patterns O we can furthermore think of this adjunction as being induced by one between fibrous patterns and Segal O-∞-categories:

Theorem 4.2.6. *Let O be a soundly extendable pattern. Then there is an adjunction*

$$\mathsf{Env}_O\colon \mathsf{Fbrs}(O) \rightleftarrows \mathsf{Seg}_O(\mathsf{Cat}_\infty),$$

where $\mathsf{Env}_O(\mathcal{P})(X) := \mathcal{P} \times_O O^{\mathrm{act}}_{/X}$ and the right adjoint is given by unstraightening. This induces an adjunction

$$\mathsf{Env}^{/\mathcal{A}_O}_O\colon \mathsf{Fbrs}(O) \rightleftarrows \mathsf{Seg}_O(\mathsf{Cat}_\infty)_{/\mathcal{A}_O}$$

where the left adjoint is fully faithful and the image consists of the Segal O-∞-categories that are equifibered over $\mathcal{A}_O$.

Proof. It remains to show that the adjunction

$$(-) \times_O \mathrm{Ar}_{\mathrm{act}}(O)\colon \mathsf{Cat}^{\text{int-cocart}}_{\infty/O} \rightleftarrows \mathsf{Cat}^{\mathrm{cocart}}_{\infty/O} \simeq \mathsf{Fun}(O, \mathsf{Cat}_\infty)$$

from Corollary 2.2.5 restricts to an adjunction between $\mathsf{Fbrs}(O)$ and $\mathsf{Seg}_O(\mathsf{Cat}_\infty)$. Since $\mathcal{A}_O$ is a Segal O-∞-category, we have by Observation 3.1.9 and Proposition 4.2.1 that the left adjoint takes fibrous patterns to Segal O-∞-categories. On the other hand, the right adjoint takes the latter to fibrous patterns by Observation 4.1.10. □

Remark 4.2.7. Note that in the context of Theorem 4.2.6 the right adjoint of Env_O is faithful and replete. It induces an equivalence between $\mathsf{Seg}_O(\mathsf{Cat}_\infty)$ and the subcategory of $\mathsf{Fbrs}(O)$ whose objects are cocartesian fibrous patterns and whose morphisms preserve all cocartesian edges.

Remark 4.2.8. If $f\colon O \to \mathcal{P}$ is a strong Segal morphism between soundly extendable patterns, then pullback/restriction along f gives a commutative square

$$\begin{array}{ccc} \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty) & \xrightarrow{f^*} & \mathsf{Seg}_O(\mathsf{Cat}_\infty) \\ \downarrow & & \downarrow \\ \mathsf{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \mathsf{Fbrs}(O). \end{array}$$

Note, however, that the corresponding Beck–Chevalley transformation is usually not invertible, so we have to slice over $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{O}}$ to get a commutative square of envelopes

$$\begin{array}{ccc} \mathsf{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \mathsf{Fbrs}(\mathcal{O}) \\ {\scriptstyle \mathsf{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}}\downarrow & & \downarrow{\scriptstyle \mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}} \\ \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}} & \xrightarrow{f^\circledast} & \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}} \end{array} \tag{10}$$

as a special case of (8).

4.3. Examples of Segal envelopes.

Example 4.3.1. For the soundly extendable pattern $\mathbb{F}_*$, we know that fibrous patterns are exactly ∞-operads, while Segal $\mathbb{F}_*$-∞-categories are symmetric monoidal ∞-categories; here $\mathcal{A}_{\mathbb{F}_*}$ is the symmetric monoidal category $\mathbb{F}^{\amalg}$ of finite sets under disjoint union. Hence Theorem 4.2.6 yields an adjunction

$$\mathsf{Env}_{\mathbb{F}_*}^{/\mathbb{F}^{\amalg}}\colon \mathsf{Opd}_\infty = \mathsf{Fbrs}(\mathbb{F}_*) \rightleftarrows \mathsf{Seg}_{\mathbb{F}_*}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathbb{F}_*}} = \mathsf{CMon}(\mathsf{Cat}_\infty)_{/(\mathbb{F},\amalg)}.$$

The left adjoint is fully faithful and a symmetric monoidal functor $\pi\colon (\mathcal{C},\otimes) \to (\mathbb{F},\amalg)$ lies in the essential image if and only if it is equifibered. This means that the following square is cartesian for all maps $\omega\colon X \to Y$ in $\mathbb{F}$:

$$\begin{array}{ccc} \mathcal{C}^X & \xrightarrow{\omega_\otimes} & \mathcal{C}^Y \\ {\scriptstyle \pi^X}\downarrow & & \downarrow{\scriptstyle \pi^Y} \\ \mathbb{F}^X & \xrightarrow{\omega_\amalg} & \mathbb{F}^Y. \end{array}$$

Here the horizontal functors tensor over fibers of ω. In fact, it follows by taking products and pasting pullback diagrams[5] that it suffices to check the case of $\omega\colon \{1,2\} \to \{1\}$.

Observation 4.3.2. The essential image of the sliced envelope functor $\mathsf{Env}_{\mathbb{F}_*}^{/\mathbb{F}^{\amalg}}\colon \mathsf{Opd}_\infty \hookrightarrow \mathsf{CMon}(\mathsf{Cat}_\infty)_{/(\mathbb{F},\amalg)}$ was first described in [HK21], but the characterization there looks at first glance quite different from ours. Let us therefore compare these two descriptions:

For a symmetric monoidal functor $\pi\colon \mathcal{C} \to \mathbb{F}$, let us write $\mathcal{C}_{(1)} \subset \mathcal{C}$ for the full subcategory of those $x \in \mathcal{C}$ with $|\pi(x)| = 1$. Then the characterization of [HK21] is that the essential image consists of those π that satisfy the following pair of conditions:

(1) Every object $x \in \mathcal{C}$ is equivalent to $x_1 \otimes \cdots \otimes x_n$ for some $x_i \in \mathcal{C}_{(1)}$.
(2) For every $n, m \geq 0$ and any two tuples $x_1, \ldots, x_m \in \mathcal{C}_{(1)}$ and $y_1, \ldots, y_n \in \mathcal{C}_{(1)}$, the canonical map

$$\coprod_{\varphi\colon m \to n} \prod_{i=1}^{n} \mathsf{Map}_{\mathcal{C}}\left(\bigotimes_{j\in\varphi^{-1}(i)} x_j, y_i\right) \longrightarrow \mathsf{Map}(\otimes_{j=1}^m x_j, \otimes_{i=1}^n y_i)$$

is an equivalence.

These conditions must be equivalent to our equifiberedness condition since they describe the same full subcategory. To check this more explicitly, we consider the functor

$$D_n\colon \mathcal{C}^n \longrightarrow \mathbb{F}^n \times_{\mathbb{F}} \mathcal{C},$$

which is an equivalence for all n if and only if $p\colon \mathcal{C} \to \mathbb{F}$ is equifibered. The functor D_n is essentially surjective if and only if for any $x \in \mathcal{C}$ and a decomposition $\pi(x) = A_1 \amalg \cdots \amalg A_n$ there is a decomposition $x = x_1 \otimes \cdots \otimes x_n$ such that $\pi(x_i) \simeq A_i$ compatibly with the decomposition. By choosing the trivial decomposition with $|A_i| = 1$ this recovers

[5] See Lemma 5.2.16 for an elaboration of this argument.

condition (1). Conversely, given condition (1) we can decompose x as $\otimes_{a\in\pi(x)} y_a$ and then find the desired x_i as $x_i = \otimes_{a\in A_i} y_a$.

To see that the full faithfulness of the D_n's corresponds to condition (2), we first observe that in the presence of condition (1) we can replace condition (2) with the following:

(2′) For every $n \geq 0$ and any two tuples $z_1, \ldots, z_n \in C$ and $y_1, \ldots, y_n \in C$, the canonical map

$$\prod_{i=1}^{n} \mathsf{Map}_C(z_i, y_i) \longrightarrow \coprod_{\varphi_i\colon \pi(z_i)\to\pi(y_i)} \mathsf{Map}^{\coprod_{i=1}^{n}\varphi_i}(\otimes_{i=1}^{n} z_i, \otimes_{i=1}^{m} y_i)$$

is an equivalence. Here we write $\mathsf{Map}_C^{\varphi}(a, b)$ for the fiber of $\mathsf{Map}_C(a, b)$ over some $\varphi\colon \pi(a) \to \pi(b)$.

To relate this to condition (2), first decompose y_i using condition (1) and use 2-out-of-3 to reduce to the case where $|\pi(y_i)| = 1$. Then write $z_i = \otimes_{j\in\varphi^{-1}(i)} x_j$ and argue as in [HK21, Remark 2.4.8].

Now we can observe that D_n is fully faithful if and only if condition (2′) holds: indeed, the mapping space in $\mathbb{F}^n \times_{\mathbb{F}} C$ can be described as

$$\begin{aligned}
&\mathsf{Map}_{\mathbb{F}^n\times_{\mathbb{F}} C}((x, \pi(x) = A_1 \amalg \cdots \amalg A_n), (y, \pi(y) := B_1 \amalg \cdots \amalg B_n)) \\
&\qquad \simeq \mathsf{Map}_{\mathbb{F}^n}((A_i), (B_i)) \times_{\mathsf{Map}_{\mathbb{F}}(\pi(x),\pi(y))} \mathsf{Map}_C(x, y) \\
&\qquad \simeq \coprod_{(\varphi_i\colon A_i \longrightarrow B_i)} \mathsf{Map}^{\amalg \varphi_i}(x, y).
\end{aligned}$$

Applying this to the images of $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ under $C^n \to \mathbb{F}^n \times_{\mathbb{F}} C$ yields the desired form.

It is interesting to note that while in condition (2) we need to quantify over all $n, m \geq 0$, in condition (2′) it suffices to consider only the case $n = 2$ as all other cases can be obtained inductively. This works because the objects z_i and y_i in condition (2′) are themselves allowed to be composite.

Example 4.3.3. For the soundly extendable pattern $\Delta^{\mathrm{op},\flat}$ fibrous patterns are non-symmetric ∞-operads, while Segal $\Delta^{\mathrm{op},\flat}$-$\infty$-categories are monoidal ∞-categories. We therefore denote $\mathsf{Opd}_\infty^{\mathrm{ns}} := \mathsf{Fbrs}(\Delta^{\mathrm{op},\flat})$ and $\mathsf{Mon}(\mathsf{Cat}_\infty) := \mathsf{Seg}_{\Delta^{\mathrm{op},\flat}}(\mathsf{Cat}_\infty)$. The Segal $\Delta^{\mathrm{op},\flat}$-category $\mathcal{A}_{\Delta^{\mathrm{op},\flat}}$ is equivalent to the category Δ_+ of finite (possibly empty) linearly ordered sets, with the monoidal structure given by concatenation. The envelope functor $\mathsf{Env}_{\Delta^{\mathrm{op},\flat}}^{/\Delta_+}$ can then be interpreted as a fully faithful embedding:

$$\mathsf{Env}_{\Delta^{\mathrm{op},\flat}}^{/\Delta_+} : \mathsf{Opd}_\infty^{\mathrm{ns}} \hookrightarrow \mathsf{Mon}(\mathsf{Cat}_\infty)_{/\Delta_+}$$

Similarly to Example 4.3.1 we can describe the essential image as those monoidal functors $\pi : \mathcal{V} \to \Delta_+$ for which the following natural square is cartesian:

$$\begin{array}{ccc}
\mathcal{V}\times\mathcal{V} & \xrightarrow{\otimes} & \mathcal{V} \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
\Delta_+\times\Delta_+ & \xrightarrow{\otimes} & \Delta_+.
\end{array}$$

Example 4.3.4. For the soundly extendable pattern $\Delta^{\mathrm{op},\natural}$, fibrous patterns are generalized non-symmetric ∞-operads as defined in [GH15], while Segal $\Delta^{\mathrm{op},\natural}$-$\infty$-categories are category objects in Cat_∞, i.e. double ∞-categories. We thus write $\mathsf{Opd}_\infty^{\mathrm{gen,ns}} := \mathsf{Fbrs}(\Delta^{\mathrm{op},\natural})$ and $\mathsf{DblCat}_\infty := \mathsf{Seg}_{\Delta^{\mathrm{op},\natural}}(\mathsf{Cat}_\infty)$. We may regard $(\infty,2)$-categories (in the form of complete 2-fold Segal spaces) as those double ∞-categories $\mathcal{X}_\bullet$ such that $\mathcal{X}_0$ is an ∞-groupoid and which satisfy a completeness condition. In particular, the Segal

$\Delta^{\mathrm{op},\natural}$-$\infty$-category $\mathcal{A}_{\Delta^{\mathrm{op},\natural}} \simeq \mathcal{A}_{\Delta^{\mathrm{op},\flat}}$ may be thought of as the one-object $(\infty,2)$-category $\mathfrak{B}\Delta_+$ where the endomorphisms of the single object are Δ_+, with the monoidal structure corresponding to composition. The envelope functor $\mathsf{Env}_{\Delta^{\mathrm{op},\natural}}^{/\mathfrak{B}\Delta_+}$ can then be interpreted as giving fully faithful embedding:

$$\mathsf{Env}_{\Delta^{\mathrm{op},\natural}}^{/\mathfrak{B}\Delta_+} : \mathsf{Opd}_\infty^{\mathrm{gen,ns}} \hookrightarrow \mathsf{DblCat}_{\infty/\mathfrak{B}\Delta_+}$$

The essential image is characterized by a pullback square analogous to the one from Example 4.3.3. Note that the morphisms in $\mathsf{Opd}_\infty^{\mathrm{gen,ns}}$ among the cocartesian fibrations that correspond to $(\infty,2)$-categories are precisely *lax functors* as defined for instance in [GR17], so we obtain a description of these in terms of $\mathsf{DblCat}_{\infty/\mathfrak{B}\Delta_+}$. (More generally, we can also consider the envelope for $\Delta^{n,\mathrm{op},\natural}$, which was briefly discussed in [Hau17].)

Example 4.3.5. Let $\mathcal{O} \to \mathbb{F}_*$ be an ∞-operad. Fibrous $\mathcal{O}$-patterns are, by Example 4.1.18, exactly ∞-operads over $\mathcal{O}$, while Segal $\mathcal{O}$-∞-categories are precisely $\mathcal{O}$-monoidal ∞-categories which we denote by $\mathsf{Mon}_\mathcal{O}(\mathsf{Cat}_\infty) := \mathsf{Seg}_\mathcal{O}(\mathsf{Cat}_\infty)$. By Example 3.3.19, $\mathcal{O}$ is soundly extendable and our construction recovers the $\mathcal{O}$-monoidal envelope of [HA, §2.2.4]. In particular, we see that this gives a fully faithful embedding

$$\mathsf{Env}_\mathcal{O}^{/\mathcal{A}_\mathcal{O}} : \mathsf{Opd}_{\infty/\mathcal{O}} \longrightarrow \mathsf{Mon}_\mathcal{O}(\mathsf{Cat}_\infty)_{/\mathcal{A}_\mathcal{O}}.$$

In the case $\mathcal{O} = \mathbb{E}_n$, the ∞-category $\mathcal{A}_{\mathbb{E}_n}$ admits an alternative description as the $\mathbb{E}_n$-monoidal ∞-category of embedded n-disks in $\mathbb{R}^n$.

5. The comparison theorem

In §5.1 we use the Segal envelopes to prove the comparison result, Theorem A. We then discuss the application of this to equivariant ∞-operads, Corollary B, in §5.2. Finally, we explain how to upgrade the envelope and comparison equivalences to equivalences of $(\infty,2)$-categories in §5.3.

5.1. Comparing fibrous patterns.

In this subsection we will use Segal envelopes to obtain a criterion for a morphism of patterns $f : \mathcal{O} \to \mathcal{P}$ to induce via pullback an equivalence

$$f^* : \mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O})$$

between the corresponding ∞-categories of fibrous patterns. We specialize this to recover some comparison results from [HA] without using the technical results on approximations to ∞-operads from [HA, §2.3.3]. As new applications, we show that (symmetric) ∞-operads can also be described as fibrous patterns over $\mathsf{Span}(\mathbb{F})$, and that fibrous patterns over $\mathsf{Span}(\mathcal{S}_m)$ and $\mathsf{Span}_{(m-1)\text{-tr,all}}(\mathcal{S}_m)$ are equivalent.

Theorem 5.1.1. *Suppose $\mathcal{O}$ is a pattern, $\mathcal{P}$ is a soundly extendable pattern, and $f : \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism such that the following conditions hold:*

(i) $f^{\mathrm{el}} : \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ is an equivalence of ∞-categories,
(ii) $(\mathcal{O}^{\mathrm{act}}_{/X})^\simeq \to (\mathcal{P}^{\mathrm{act}}_{/f(X)})^\simeq$ is an equivalence for all $X \in \mathcal{O}$.

Then pullback along f gives an equivalence

$$f^* : \mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O}).$$

Remark 5.1.2. If we also assume that $\mathcal{A}_\mathcal{O}^\simeq = (\mathcal{O}^{\mathrm{act}}_{/-})^\simeq$ is an $\mathcal{O}$-Segal space, for example if $\mathcal{O}$ is soundly extendable, then it suffices to check condition (ii) when X is elementary.

Example 5.1.3. Let $\mathcal{P}$ be a soundly extendable pattern, and define $\mathcal{O} \subset \mathcal{P}$ as the full subpattern on the "necessary objects" in the sense of [CH21, Definition 14.7]. This means that $\mathcal{O}$ contains those $X \in \mathcal{P}$ for which there exists an active morphism $X \rightsquigarrow E$

with E elementary. Then Theorem 5.1.1 applies to the full inclusion $\mathcal{O} \subset \mathcal{P}$ and hence restriction yields an equivalence $\mathsf{Fbrs}(\mathcal{P}) \simeq \mathsf{Fbrs}(\mathcal{O})$.

First we show that condition (ii) can always be strengthened as follows.

Lemma 5.1.4. *In the situation of Theorem 5.1.1 the induced natural transformation*

$$\alpha\colon \mathcal{A}_{\mathcal{O}} \longrightarrow f^*\mathcal{A}_{\mathcal{P}}$$

of functors $\mathcal{O} \to \mathsf{Cat}_\infty$ is an equivalence. In particular $\mathcal{A}_{\mathcal{O}}$ is $\mathcal{O}$-Segal.

Proof. By assumption, the functor $\mathcal{A}_{\mathcal{O}}(O) \to \mathcal{A}_{\mathcal{P}}(f(O))$ is an equivalence on underlying ∞-groupoids, so it remains to show that it is fully faithful. To see this, observe that given active maps $\varphi\colon O \rightsquigarrow X$ and $\varphi'\colon O' \rightsquigarrow X$, the mapping space $\mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/X}}(\varphi', \varphi)$ is the fiber at φ' of the map $(\varphi \circ -) : \mathcal{A}^{\simeq}_{\mathcal{O}}(O) \to \mathcal{A}^{\simeq}_{\mathcal{O}}(X)$. This map fits into a square

$$\begin{array}{ccc} \mathcal{A}^{\simeq}_{\mathcal{O}}(O) & \xrightarrow{\sim} & \mathcal{A}^{\simeq}_{\mathcal{P}}(fO) \\ \downarrow & & \downarrow \\ \mathcal{A}^{\simeq}_{\mathcal{O}}(X) & \xrightarrow{\sim} & \mathcal{A}^{\simeq}_{\mathcal{P}}(fX) \end{array}$$

where the horizontal maps are equivalences. Then we also have equivalences on fibers, which gives the desired full faithfulness. Finally we note that $\mathcal{A}_{\mathcal{O}} \simeq f^*\mathcal{A}_{\mathcal{P}}$ implies that $\mathcal{A}_{\mathcal{O}}$ is Segal since $\mathcal{A}_{\mathcal{P}}$ was assumed to be Segal and f^* preserves Segal objects. □

The following lemma tells us that for sound patterns it suffices to check $\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$-equifiberedness on active morphisms that end in elementary objects.

Lemma 5.1.5. *Let $\mathcal{O}$ be a sound pattern and let $(\eta\colon F \Rightarrow G)$ be a relative Segal object over $\mathcal{O}$ in a sufficiently complete ∞-category $\mathcal{C}$. Suppose that the naturality squares*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\omega)} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(\omega)} & G(Y) \end{array}$$

are cartesian for active morphisms $\omega\colon X \rightsquigarrow Y$ where Y is elementary. Then they are also cartesian for arbitrary Y, i.e. η is $\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$-equifibered.

Proof. For an arbitrary active morphism $\omega\colon X \rightsquigarrow Y$ consider the commutative cube

$$\begin{array}{ccccccc} & & \lim_{\alpha\colon Y \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{Y/}} F(\omega_{\alpha!}X) & & \xrightarrow{\lim F(\omega_\alpha)} & & \lim_{\alpha\colon Y \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{Y/}} F(E) \\ & \nearrow & & & & \nearrow & \\ F(X) & & \xrightarrow{F(\omega)} & & F(Y) & & \downarrow{\scriptstyle \lim \eta_E} \\ & & \downarrow{\scriptstyle \lim \eta_{\omega_{\alpha!}X}} & & & & \\ \downarrow{\scriptstyle \eta_X} & & \lim_{\alpha\colon Y \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{Y/}} G(\omega_{\alpha!}X) & & \xrightarrow{\lim G(\omega_\alpha)} & & \lim_{\alpha\colon Y \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{Y/}} G(E) \\ & \nearrow & & & \downarrow{\scriptstyle \eta_Y} & \nearrow & \\ G(X) & & \xrightarrow{G(\omega)} & & G(Y). & & \end{array}$$

The back square is cartesian as it is a limit over squares that we have assumed to be cartesian. (Note that $\omega_\alpha\colon \omega_{\alpha!}X \rightsquigarrow E$ is an active morphism with elementary target.) The right face is cartesian because η is a relative Segal object, and so is the left face from this and Lemma 3.3.7. Therefore the front face is cartesian by the pullback pasting lemma. □

Proof of Theorem 5.1.1. It follows from Proposition 3.1.16 that the functor

$$f^*\colon \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty) \longrightarrow \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$$

is an equivalence. From Lemma 5.1.4 we have $\mathcal{A}_{\mathcal{O}} \simeq f^*\mathcal{A}_{\mathcal{P}}$ and that $\mathcal{A}_{\mathcal{O}}$ is Segal. Hence the induced functor

$$f^{\circledast}\colon \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}} \longrightarrow \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$$

is also an equivalence. This means in the commutative square

$$\begin{array}{ccc} \mathsf{Fbrs}(\mathcal{P}) & \xrightarrow{\quad f^* \quad} & \mathsf{Fbrs}(\mathcal{O}) \\ \big\downarrow{\scriptstyle \mathsf{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}} & & \big\downarrow{\scriptstyle \mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}} \\ \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}} & \xrightarrow{\ f^{\circledast}\ } & \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}} \end{array}$$

from Proposition 4.2.1, the bottom horizontal functor $f^{\circledast}$ is an equivalence, while the vertical functors are fully faithful. It follows that the top horizontal functor f^* is also fully faithful. To prove that it is also essentially surjective, it suffices to show that an object of $\mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}}$ is in the image of $\mathsf{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}$ if its image under the equivalence $f^{\circledast}$ is in the image of $\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}$.

Suppose we are given some $(\eta\colon F \Rightarrow \mathcal{A}_{\mathcal{P}}) \in \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}}$ such that $f^{\circledast}F \Rightarrow \mathcal{A}_{\mathcal{O}}$ is equifibered. Equivalently, $\eta_{\circ f}\colon (F \circ f) \Rightarrow (\mathcal{A}_{\mathcal{P}} \circ f)$ is equifibered. By Lemma 5.1.5 it suffices to check that the naturality squares are cartesian for active morphisms $\omega\colon X \rightsquigarrow E \in \mathcal{P}$ ending in an elementary. Since $f\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ is an equivalence, we may write $E \simeq f(E')$ for $E' \in \mathcal{O}$. Moreover, since $f\colon \mathcal{O}^{\mathrm{act}}_{/E'} \to \mathcal{P}^{\mathrm{act}}_{/f(E')}$ is an equivalence, we can find $\rho\colon Y \rightsquigarrow E' \in \mathcal{O}$ such that $f(\rho) \simeq \omega$ as objects of $\mathrm{Ar}_{\mathrm{act}}(\mathcal{P})$. Now it follows that the naturality square of η at ω is cartesian since we assumed that the naturality square of $\eta_{\circ f}$ at ρ is equifibered. This shows that η is $\mathrm{Ar}_{\mathrm{act}}(\mathcal{P})$-equifibered, and hence that $f^*\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O})$ is essentially surjective. □

Example 5.1.6. Let Ass be the (symmetric) associative ∞-operad as defined in [HA, Definition 4.1.1.1.], and let $\mathrm{Cut}\colon \Delta^{\mathrm{op}} \to \mathsf{Ass}$ denote the functor defined in [HA, Construction 4.1.2.9.]. Then pullback along Cut gives an equivalence

$$\mathsf{Fbrs}(\Delta^{\mathrm{op},\flat}) \xleftarrow{\sim} \mathsf{Fbrs}(\mathsf{Ass}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*)_{/\mathsf{Ass}}$$

between non-symmetric ∞-operads and symmetric ∞-operads over Ass, where the second equivalence is that of Corollary 4.1.17. In other words, non-symmetric ∞-operads are equivalent to symmetric ∞-operads over the associative ∞-operad.

The equivalence of Example 5.1.6 is also proved by Lurie as [HA, Theorem 4.1.3.14], which is a special case of [HA, Theorem 2.3.3.26]. This more general statement can also be proved by our methods; to see this, we first need to recall some definitions:

Definition 5.1.7. Let $\pi\colon \mathcal{O} \to \mathbb{F}_*$ be an ∞-operad. We say a functor $f\colon \mathcal{C} \to \mathcal{O}$ is an *approximation* if the following conditions hold:

(1) For $C \in \mathcal{C}$ over $\langle n\rangle$ in $\mathbb{F}_*$, there exists for $i = 1, \dots, n$ a locally cocartesian morphism $\rho_i^C\colon C \to C_i$ in $\mathcal{C}$ over $\rho_i\colon \langle n\rangle \to \langle 1\rangle$. Moreover, the image of ρ_i^C in $\mathcal{O}$ is inert.
(2) $\mathcal{C}$ has all f-cartesian lifts of active morphisms in $\mathcal{O}$.

Following [Hin20], we say that f is a *strong approximation* if we additionally have:

(3) The functor $\mathcal{C}_{\langle 1\rangle} \to \mathcal{O}_{\langle 1\rangle}$ is an equivalence.

Remark 5.1.8. Suppose $\mathcal{O}$ is an ∞-operad and $f\colon \mathcal{C} \to \mathcal{O}$ is an approximation. We say a morphism in $\mathcal{C}$ is *inert* if its image in $\mathcal{O}$ is inert, and *active* if it is f-cartesian and its image in $\mathcal{O}$ is active. Then the inert and active morphisms in $\mathcal{C}$ give a factorization system. We think of $\mathcal{C}$ as an algebraic pattern using this factorization system, with the elementary objects being those that map to $\langle 1\rangle$ in $\mathbb{F}_*$; then f is a morphism of algebraic patterns.

Proposition 5.1.9. *Suppose $\mathcal{O}$ is an ∞-operad and $f\colon \mathcal{C} \to \mathcal{O}$ is a strong approximation. Then*

(i) $\mathcal{C}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{el}}$ *is an equivalence.*
(ii) $\mathcal{C}^{\mathrm{el}}_{C/} \to \mathcal{O}^{\mathrm{el}}_{f(C)/}$ *is an equivalence for all $C \in \mathcal{C}$, i.e. f is an iso-Segal morphism.*
(iii) $\mathcal{C}^{\mathrm{act}}_{/C} \to \mathcal{O}^{\mathrm{act}}_{/f(C)}$ *is an equivalence for all $C \in \mathcal{C}$.*

Proof. For (i), observe that from the equivalence $\mathcal{C}_{\langle 1\rangle} \xrightarrow{\sim} \mathcal{O}_{\langle 1\rangle}$ it follows that a morphism in $\mathcal{C}$ over $\langle 1\rangle$ is inert if and only if it is an equivalence (since the equivalences are precisely the inert morphisms in $\mathcal{O}_{\langle 1\rangle}$). Hence $\mathcal{C}^{\mathrm{el}} = \mathcal{C}^{\simeq}_{\langle 1\rangle}$, so the functor $\mathcal{C}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{el}}$ is just the underlying morphism of ∞-groupoids of the functor between fibers over $\langle 1\rangle$ that is an equivalence by assumption.

To show (ii), we first observe that $\mathcal{C}^{\mathrm{el}}_{C/}$ is an ∞-groupoid, since morphisms are given by inert maps over $\langle 1\rangle$ and these are invertible. Moreover, if C lies over $\langle n\rangle$ then the fiber of $\mathcal{C}^{\mathrm{el}}_{C/}$ over ρ_i is contractible, since there by assumption exists a locally cocartesian morphism over ρ_i — this is then initial in the ∞-category $(\mathcal{C}_{C/})_{\rho_i}$ and so in particular has no automorphisms.

We thus have a commutative triangle

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{el}}_{C/} & \longrightarrow & \mathcal{O}^{\mathrm{el}}_{f(C)/} \\ & {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle\sim} & \\ & (\mathbb{F}^{\mathrm{el}}_*)_{\langle n\rangle} & \end{array}$$

where both maps to $(\mathbb{F}^{\mathrm{el}}_*)_{\langle n\rangle}$ are equivalences, hence so is the top horizontal map.

To prove (iii), observe that by assumption $\mathcal{C}^{\mathrm{act}} \to \mathcal{O}^{\mathrm{act}}$ is the underlying right fibration of the cartesian fibration $\mathcal{C} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}} \to \mathcal{O}^{\mathrm{act}}$. This gives the required equivalence of slices by [Ker, Tag 00TE]. □

Corollary 5.1.10. *Suppose $f\colon \mathcal{C} \to \mathcal{O}$ is a strong approximation to an ∞-operad $\mathcal{O}$.*

(1) If $\mathcal{X}$ is an ∞-category with finite products, then restriction along f gives an equivalence

$$f^*\colon \mathsf{Seg}_{\mathcal{O}}(\mathcal{X}) \xrightarrow{\sim} \mathsf{Seg}_{\mathcal{C}}(\mathcal{X}).$$

(2) Pullback along f gives an equivalence

$$f^*\colon \mathsf{Fbrs}(\mathcal{O}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{C}).$$

Proof. Combine Proposition 5.1.9 with Proposition 3.1.16 and Theorem 5.1.1. □

Remark 5.1.11. Lurie's proof of [HA, Theorem 2.3.3.26] uses envelopes for approximations to ∞-operads, just as our proof of Theorem 5.1.1, and we do not claim that our proof is different in any essential way.

We end this section with a couple of examples that do not follow from Corollary 5.1.10 or [HA, Theorem 2.3.3.26]. These involve patterns defined using spans, so we start with a general observation about comparisons of these:

Observation 5.1.12. Consider two adequate triples $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ and $(\mathfrak{Y}, \mathfrak{Y}^b, \mathfrak{Y}^f)$ and a functor $F\colon \mathfrak{X} \to \mathfrak{Y}$ that preserves the two subcategories and also preserves pullbacks of backwards maps along forwards maps. Suppose further that we have full subcategories $\mathfrak{X}_0 \subset \mathfrak{X}$ and $\mathfrak{Y}_0 \subset \mathfrak{Y}$ such that $F(\mathfrak{X}_0) \subset \mathfrak{Y}_0$. Then F induces a morphism of patterns:

$$F\colon \mathsf{Span}_{b,f}(\mathfrak{X};\mathfrak{X}_0) \longrightarrow \mathsf{Span}_{b,f}(\mathfrak{Y};\mathfrak{Y}_0).$$

We may apply Theorem 5.1.1 to this if the following conditions hold:

(1) $\mathsf{Span}_{b,f}(\mathfrak{Y};\mathfrak{Y}_0)$ is soundly extendable. (See Proposition 3.3.23.)
(2) For all $x \in \mathfrak{X}$, the map $\mathfrak{X}_0^b \times_{\mathfrak{X}^b} \mathfrak{X}^b_{/x} \to \mathfrak{Y}_0^b \times_{\mathfrak{Y}^b} \mathfrak{Y}^b_{/F(x)}$ is cofinal.
(3) $F\colon \mathfrak{X}_0^b \to \mathfrak{Y}_0^b$ is an equivalence of ∞-categories.
(4) $F\colon \mathfrak{X}^f_{/x} \to \mathfrak{Y}^f_{/F(x)}$ induces an equivalence on maximal subgroupoids for all $x \in \mathfrak{X}$.

Note that point (2) ensures that F is a strong Segal morphism since $\mathsf{Span}_{b,f}(\mathfrak{X};\mathfrak{X}_0)^{\mathrm{int}} \simeq (\mathfrak{X}^b)^{\mathrm{op}}$ with the elemetaries being $(\mathfrak{X}_0^b)^{\mathrm{op}}$.

Corollary 5.1.13. *Pullback along the inclusion* $\mathfrak{i}\colon \mathbb{F}_* \simeq \mathsf{Span}_{\mathrm{inj},\mathrm{all}}(\mathbb{F}) \to \mathsf{Span}(\mathbb{F})$ *gives an equivalence*

$$\mathfrak{i}^*\colon \mathsf{Fbrs}(\mathsf{Span}(\mathbb{F})) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*) \simeq \mathsf{Opd}_\infty.$$

Proof. We check the conditions of Theorem 5.1.1 in the form stated in Observation 5.1.12:

(1) The pattern is soundly extendable by Example 3.3.25.
(2) For $A \in \mathbb{F}$ the relevant functor is the restriction of $\mathbb{F}^{\mathrm{inj}}_{/A} \to \mathbb{F}_{/A}$ to elementaries. But every map out of a one-point set is injective, so this is an equivalence.
(3) Similarly, the functor on backwards morphisms $\mathbb{F}^{\mathrm{inj}} \to \mathbb{F}$ restricts to an equivalence on elementaries.
(4) Both categories have the same forward morphisms. □

More generally, we have:

Corollary 5.1.14. *Pullback along the inclusion* $\mathfrak{i}_m\colon \mathsf{Span}_{(m-1)\text{-tr},\mathrm{all}}(\mathcal{S}_m) \to \mathsf{Span}(\mathcal{S}_m)$ *induces an equivalence*

$$\mathfrak{i}_m^*\colon \mathsf{Fbrs}(\mathsf{Span}(\mathcal{S}_m)) \longrightarrow \mathsf{Fbrs}(\mathsf{Span}_{(m-1)\text{-tr},\mathrm{all}}(\mathcal{S}_m)).$$

Proof. We can apply Theorem 5.1.1: The target pattern $\mathsf{Span}(\mathcal{S}_m)$) is soundly extendable by Example 3.3.27 and in this example we also note that $\mathfrak{i}_m$ is an iso-Segal morphism. Condition (i) of Theorem 5.1.1 holds because in both cases the elementary ∞-category is the terminal ∞-category. Condition (ii) holds because both span ∞-categories have the same forward morphisms. □

5.2. **G-equivariant ∞-operads.** In this section we apply the theory of fibrous patterns and envelopes in the setting of G-equivariant ∞-operads developed in [NS22]. While their paper works in the generality of T-parametrized ∞-operads, we will restrict to the special case of the orbit category $T = \mathrm{Orb}_G$ for simplicity. Our main result is that the G-∞-operads of [NS22] are equivalent to fibrous $\mathsf{Span}(\mathbb{F}_G)$-patterns; we will also show that the sliced envelope for G-∞-operads is fully faithful and characterize the image, giving a third description of these objects.

First, we recall some constructions in equivariant higher algebra, which were pioneered in [Bar17] and further developed in [Nar16] and [NS22]. Fix a finite group G throughout.

Definition 5.2.1. Let $\mathbb{F}_G$ be the category of finite G-sets, $\mathbb{F}_{G,*}$ the category of finite pointed G-sets, and $\mathrm{Orb}_G \subset \mathbb{F}_G$ the full subcategory of G-orbits.

Definition 5.2.2. A *G-∞-category* is a functor $\mathrm{Orb}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty$ and a *G-symmetric monoidal ∞-category* is a $\mathsf{Span}(\mathbb{F}_G)$-Segal object in Cat_∞. We write

$$\mathsf{Cat}_{G,\infty} := \mathsf{Fun}(\mathrm{Orb}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \qquad \text{and} \qquad \mathsf{Cat}^{\otimes}_{G,\infty} := \mathsf{Seg}_{\mathsf{Span}(\mathbb{F}_G)}(\mathsf{Cat}_\infty)$$

and define the forgetful functor $\mathsf{Cat}^{\otimes}_{G,\infty} \to \mathsf{Cat}_{G,\infty}$ by restricting to the elementaries $\mathrm{Orb}_G^{\mathrm{op}} \to \mathsf{Span}(\mathbb{F}_G)$.

Notation 5.2.3. For a G-∞-category $\mathcal{C}\colon \mathrm{Orb}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty$ we denote its value at G/H by $\mathcal{C}^H$ an refer to it as the H-fixed point category of $\mathcal{C}$. There are restriction maps $\mathcal{C}^H \to \mathcal{C}^K$ for $K \subset H \subset G$. Given a G-symmetric monoidal ∞-category $\mathcal{D}\colon \mathsf{Span}(\mathbb{F}_G) \to \mathsf{Cat}_\infty$ we further have tensor products $\otimes\colon \mathcal{D}^H \times \mathcal{D}^H \to \mathcal{D}^H$ and so-called norm maps $\mathrm{Nm}_K^H\colon \mathcal{D}^K \to \mathcal{D}^H$ for all $K \subset H \subset G$ coming from the span $(G/K \xleftarrow{=} G/K \to G/H)$.

Example 5.2.4. Since $\mathsf{Span}(\mathbb{F}_G)$ is an extendable pattern (Example 3.3.26) $\mathcal{A}_{\mathsf{Span}(\mathbb{F}_G)}$ is a Segal object in Cat_∞. We denote this G-symmetric monoidal ∞-category by

$$\underline{\mathbb{F}}_G := \mathcal{A}_{\mathsf{Span}(\mathbb{F}_G)}(-) = \mathsf{Span}(\mathbb{F}_G)^{\mathrm{act}}_{/-}\colon \mathsf{Span}(\mathbb{F}_G) \longrightarrow \mathsf{Cat}_\infty.$$

The H-fixed point category is the category of finite H-sets:

$$(\underline{\mathbb{F}}_G)^H = \mathsf{Span}(\mathbb{F}_G)^{\mathrm{act}}_{/(G/H)} \simeq (\mathbb{F}_G)_{/(G/H)} \simeq \mathbb{F}_H.$$

The restriction maps are given by restriction, the tensor product by disjoint union, and the norm maps are $(- \times H)_{/K}\colon \mathbb{F}_K \to \mathbb{F}_H$. In summary, $\underline{\mathbb{F}}_G$ is $\mathbb{F}_G$ with its natural structure as a G-symmetric monoidal ∞-category.

Below we will see that fibrous $\mathsf{Span}(\mathbb{F}_G)$-patterns model G-∞-operads. We now explain how $\mathcal{N}_\infty$-operads fit into this framework:

Example 5.2.5. Let $\mathbb{F}_G^f \subset \mathbb{F}_G$ be a wide subcategory closed under base-change and disjoint union. Then the inclusion functor $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G) \to \mathsf{Span}(\mathbb{F}_G)$ defines a fibrous $\mathsf{Span}(\mathbb{F}_G)$-pattern. To see that it has cocartesian lifts for inerts, note that any functor of the form $\mathsf{Span}_{b,f}(\mathcal{C}) \to \mathsf{Span}_{b,\mathrm{all}}(\mathcal{C})$ has cocartesian lifts for backwards maps. For the second condition we need to show that

$$(\mathbb{F}_G^f)_{/A} \longrightarrow \lim_{U \in (\mathrm{Orb}_G)_{/A}} (\mathbb{F}_G^f)_{/U}$$

is an equivalence. The limit may be rewritten as a product over the set of orbits of A and then the equivalence follows because $\mathbb{F}_G^f$ is closed under base-change and disjoint union.

Categories $\mathbb{F}_G^f$ that in addition to the above also contain all fold maps $\nabla : G/H \amalg G/H \to G/H$, are in bijection with the indexing systems of [BH18], see [NS22, Remark 2.4.12]. Under the equivalence $\mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G)) \simeq \mathsf{Opd}_{G,\infty}$ proved below the fibrous $\mathsf{Span}(\mathbb{F}_G)$-patters described above are the "commutative G-∞-operads" from [NS22, Definition 2.4.10], which correspond to the $\mathcal{N}_\infty$-operads of [BH18] by [NS22, Remark 2.4.12].

We now quickly recall the necessary notation from [NS22] to state their definition of G-∞-operads, but we refer the reader there for details.

Definition 5.2.6. Define $\underline{\mathbb{F}}_G^v \subset \mathrm{Ar}(\mathbb{F}_G)$ as the full subcategory of those morphisms $(f\colon U \to V)$ where V is an orbit: $\underline{\mathbb{F}}_G^v := \mathrm{Ar}(\mathbb{F}_G) \times_{\mathbb{F}_G} \mathrm{Orb}_G$. We say that a morphism $f \to g$ given by

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{k} & Y \end{array}$$

- lies in $(\underline{\mathbb{F}}_G^v)^{\mathrm{si}}$ if $U \to X \times_Y V$ is a injective,
- lies in $(\underline{\mathbb{F}}_G^v)^{\mathrm{tdeg}}$ if $k\colon V \to Y$ is an equivalence.

Definition 5.2.7. Define $\underline{\mathbb{F}}_{G,*}$ as the algebraic pattern

$$\underline{\mathbb{F}}_{G,*} := \mathsf{Span}_{\mathrm{si,tdeg}}(\underline{\mathbb{F}}_G^v; \mathrm{Orb}_G),$$

where the elementary objects are those in the essential image of the identity inclusion $\mathrm{Orb}_G \to \mathrm{Ar}(\mathrm{Orb}_G) \subset \underline{\mathbb{F}}_G^v$.

Remark 5.2.8. The functor $\mathrm{ev}_1 : \underline{\mathbb{F}}_G^v \to \mathrm{Orb}_G$ induces a cocartesian fibration

$$\underline{\mathbb{F}}_{G,*} = \mathsf{Span}_{\mathrm{si,tdeg}}(\underline{\mathbb{F}}_G^v) \xrightarrow{\mathrm{ev}_1} \mathsf{Span}_{\mathrm{all,iso}}(\mathrm{Orb}_G) \simeq \mathrm{Orb}_G^{\mathrm{op}}.$$

Straightening this yields a G-∞-category whose H-fixed point category is $(\underline{\mathbb{F}}_{G,*})^H \simeq \mathbb{F}_{H,*}$, similarly to Example 5.2.4.

Observation 5.2.9. For $(U \to V) \in \underline{\mathbb{F}}_{G,*}$ the category of elementaries under $(U \to V)$ is equivalent to the opposite of the category of orbits over U (as in Remark 3.2.7):

$$(\underline{\mathbb{F}}_{G,*})^{\mathrm{el}}_{(U\longrightarrow V)/} \simeq (\mathrm{Orb}_G \times_{(\underline{\mathbb{F}}_G^v)} (\underline{\mathbb{F}}_G^{v,si})_{/(U\longrightarrow V)})^{\mathrm{op}} \simeq (\mathrm{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G)_{/U})^{\mathrm{op}}.$$

Here we used that any morphism $(Q \xrightarrow{=} Q) \to (U \to V)$ (where Q is an orbit) is automatically in $(\underline{\mathbb{F}}_G^v)^{\mathrm{si}}$ since $Q \to Q \times_V U$ is injective. Now consider the full subcategory on those $(Q \to U)$ that are injective. This subcategory is equivalent to the discrete set of orbits U/G and moreover the inclusion of the subcategory is a left adjoint:

$$U/G \hookrightarrow (\mathrm{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G)_{/U})^{\mathrm{op}} \simeq (\underline{\mathbb{F}}_{G,*})^{\mathrm{el}}_{(U\longrightarrow V)/},$$

with right adjoint given by sending $(f : Q \to U)$ to $(f(Q) \hookrightarrow U)$. In particular, the inclusion of U/G is a coinitial functor. This means that for any kind of (weak) Segal condition over $\underline{\mathbb{F}}_{G,*}$ the limit involved can be rewritten as a product indexed by the finite set U/G.

Corollary 5.2.10. *The pattern $\underline{\mathbb{F}}_{G,*}$ is sound.*[6]

Proof. We check the conditions of Proposition 3.3.23. First we show that the backwards maps satisfy cancellation. Consider two morphisms in $\underline{\mathbb{F}}_G^v$:

$$\begin{array}{ccccc} A & \xrightarrow{a} & U & \xrightarrow{h} & X \\ \big\downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow{b} & V & \xrightarrow{k} & Y \end{array}$$

such that $A \to B \times_Y X$ is injective. We can write this map as a composite $A \to B \times_V U \to B \times_Y X$, the first map of which then has to be injective. In other words $(a, b) : e \to f$ is in $\underline{\mathbb{F}}_G^{v,\mathrm{si}}$ as claimed.

We also need to show that the inclusion $\mathfrak{X}^b_{0/y} \hookrightarrow \mathfrak{X}_{0/y}$ is cofinal. In the case at hand this inclusion is $\mathrm{Orb}_G \times_{\underline{\mathbb{F}}_G^v} (\underline{\mathbb{F}}_G^{v,\mathrm{si}})_{/(U\to V)} \to \mathrm{Orb}_G \times_{\underline{\mathbb{F}}_G^v} (\underline{\mathbb{F}}_G^v)_{/(U\to V)}$, which is an equivalence by the argument from Observation 5.2.9. □

Definition 5.2.11 ([NS22]). A G-∞-*operad* is a weak Segal fibration over $\underline{\mathbb{F}}_{G,*}$ in the sense of [CH21, Definition 9.6], see also Proposition 4.1.7. Let $\mathsf{Opd}_{G,\infty}$ denote the full subcategory of $\mathsf{Cat}^{\mathrm{int\text{-}cocart}}_{\infty/\underline{\mathbb{F}}_{G,*}}$ on the G-∞-operads.

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[6] In fact this pattern is soundly extendable. This follows because the functor $\underline{\mathbb{F}}_{G,*} \to \mathsf{Span}(\mathbb{F}_G)$ discussed in Proposition 5.2.14 is iso-Segal and induces an equivalence on forward maps. However, the extendability of $\underline{\mathbb{F}}_{G,*}$ will not be needed here.

Observation 5.2.12. This agrees with the definition of [NS22]. First we note that given $p\colon \mathcal{P} \to \underline{\mathbb{F}}_{G,*}$ with cocartesian lifts for inerts, the composite $\mathrm{ev}_1 \circ p\colon \mathcal{P} \to \mathrm{Orb}_G^{\mathrm{op}}$ exhibits $\mathcal{P}$ as a cocartesian fibration over $\mathrm{Orb}_G^{\mathrm{op}}$, i.e. an Orb_G-∞-category, and p as an Orb_G-functor. This holds because the inert morphisms in $\underline{\mathbb{F}}_{G,*}$ contain all the cocartesian lifts of $\mathrm{ev}_1\colon \underline{\mathbb{F}}_{G,*} \to \mathrm{Orb}_G^{\mathrm{op}}$. We hence have an identification:

$$\mathsf{Cat}_{\infty/\underline{\mathbb{F}}_{G,*}}^{\text{int-cocart}} = (\mathsf{Cat}_{G,\infty})_{/\underline{\mathbb{F}}_{G,*}}^{\text{int-cocart}}.$$

It remains to see that their conditions (2) and (3) exactly amount to the weak Segal conditions (2) and (3) in [CH21, Definition 9.6]. Indeed, this follows by inspection using Observation 5.2.9 and [CH21, Remark 9.7].

Corollary 5.2.13. *We have* $\mathsf{Opd}_{G,\infty} = \mathsf{Fbrs}(\underline{\mathbb{F}}_{G,*})$.

Proof. The pattern $\underline{\mathbb{F}}_{G,*}$ is sound by Corollary 5.2.10 and hence weak Segal fibrations and fibrous patterns are the same by Proposition 4.1.7. □

Proposition 5.2.14. *Restriction along the morphism of patterns* $\underline{\mathbb{F}}_{G,*} \xrightarrow{s} \mathsf{Span}(\mathbb{F}_G)$ *induced by the functor* $\underline{\mathbb{F}}_G^{v} \to \mathbb{F}_G$ *given by evaluation at* 0 *yields an equivalence*

$$s^*\colon \mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G)) \xrightarrow{\simeq} \mathsf{Fbrs}(\underline{\mathbb{F}}_{G,*}) = \mathsf{Opd}_{G,\infty}.$$

Proof. We need to show that the morphism of patterns

$$s\colon \underline{\mathbb{F}}_{G,*} = \mathsf{Span}_{\mathrm{si,tdeg}}(\underline{\mathbb{F}}_G^{v}; \mathrm{Orb}_G) \longrightarrow \mathsf{Span}(\mathbb{F}_G; \mathrm{Orb}_G)$$

satisfies the conditions of Theorem 5.1.1. Since this comes from a morphism of adequate triples, we can use the formulation in Observation 5.1.12. We check each of the conditions there in turn:

(1) It was checked in Example 3.3.25 that $\mathsf{Span}(\mathbb{F}_G)$ is soundly extendable.
(2) We need to show that

$$(\mathrm{Orb}_G \times_{(\underline{\mathbb{F}}_G^{v})} (\underline{\mathbb{F}}_G^{v,\mathrm{si}})_{/(U \longrightarrow V)})^{\mathrm{op}} \longrightarrow (\mathrm{Orb}_G \times_{\mathbb{F}_G} (\underline{\mathbb{F}}_G)_{/U})^{\mathrm{op}}$$

is cofinal. But we have already noted in Observation 5.2.9 that it is an equivalence.
(3) This holds since the functor induces the identity on Orb_G.
(4) For all $U \in \mathbb{F}_G$ the functor

$$(\underline{\mathbb{F}}_G^{v,\mathrm{tdeg}})_{/(U \longrightarrow V)} \longrightarrow (\mathbb{F}_G)_{/U}$$

is an equivalence by inspection of the definition of $(\underline{\mathbb{F}}_G^{v})^{\mathrm{tdeg}}$. □

As a consequence we obtain a fully faithful envelope into the ∞-category of G-symmetric monoidal ∞-categories over $\underline{\mathbb{F}}_G$ and a characterization of the image.

Corollary 5.2.15. *There is an adjunction*

$$\mathsf{Env}_G\colon \mathsf{Opd}_{G,\infty} \rightleftarrows \mathsf{Cat}_{G,\infty}^{\otimes} :\mathrm{forget}$$

where the left adjoint may be lifted to a fully faithful functor

$$\mathsf{Env}_G\colon \mathsf{Opd}_{G,\infty} \hookrightarrow (\mathsf{Cat}_{G,\infty}^{\otimes})_{/\underline{\mathbb{F}}_G}.$$

This functor has both adjoints and its essential image consists of those G-symmetric monoidal functors $p\colon \mathcal{C} \to \underline{\mathbb{F}}_G$ that are $\mathrm{Ar}_{\mathrm{act}}(\mathsf{Span}(\mathbb{F}_G))$-equifibered.

Proof. Using that $\mathsf{Opd}_{G,\infty} \simeq \mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G))$ by Proposition 5.2.14, this is an instance of Theorem 4.2.6. Note that the envelope of the terminal G-∞-operad is $\mathsf{Env}_{\mathsf{Span}(\mathbb{F}_G)}(*) = \mathcal{A}_{\mathsf{Span}(\mathbb{F}_G)} = \underline{\mathbb{F}}_G$ by Example 5.2.4. □

We elaborate further on the characterization of the image:

Lemma 5.2.16. *A G-symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ is $\mathrm{Ar}_{\mathrm{act}}(\mathsf{Span}(\mathbb{F}_G))$-equifibered if and only if*

$$\begin{array}{ccc} \mathcal{C}^H \times \mathcal{C}^H & \xrightarrow{\otimes} & \mathcal{C}^H \\ \downarrow & & \downarrow \\ \mathcal{D}^H \times \mathcal{D}^H & \xrightarrow{\otimes} & \mathcal{D}^H \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \mathcal{C}^K & \xrightarrow{\mathrm{Nm}_K^H} & \mathcal{C}^H \\ \downarrow & & \downarrow \\ \mathcal{D}^K & \xrightarrow{\mathrm{Nm}_K^H} & \mathcal{D}^H \end{array}$$

are pullback squares of ∞-categories for all subgroups $K \subset H \subset G$.

Proof. F induces a natural transformation of functors $\mathbb{F}_G \to \mathsf{Cat}$, defined by restricting to forwards maps in $\mathsf{Span}(\mathbb{F}_G)$. Let $\mathcal{K} \subset \mathbb{F}_G$ denote the maximal subcategory such that the restriction of F to $\mathcal{K}$ is a cartesian natural transformation. Then F is $\mathrm{Ar}_{\mathrm{act}}(\mathsf{Span}(\mathbb{F}_G))$-equifibered if and only if $\mathcal{K} = \mathbb{F}_G$. Note that $\mathcal{K}$ is closed under composition and right-cancellation, since pullback squares are, and contains all equivalences. Moreover, $\mathcal{K}$ is closed under disjoint union since both functors $\mathcal{C}, \mathcal{D}\colon \mathbb{F}_G \to \mathsf{Cat}$ send disjoint unions to products. Using this one can see that to show $\mathcal{K} = \mathbb{F}_G$, it suffices to check that $\mathcal{K}$ contains the morphisms

$$\nabla\colon G/H \amalg G/H \longrightarrow G/H, \qquad \text{and} \qquad G/K \longrightarrow G/H$$

for all subgroups $K \subset H \subset G$. This is exactly the condition stated in the lemma. □

Remark 5.2.17. One might hope that G-∞-operads are also equivalent to fibrous $\mathbb{F}_{G,*}$-patterns, in analogy with what we showed in Corollary 5.1.13 for $G = \{e\}$, but this is false for non-trivial groups. Note that the orbit functor $(-)_G\colon \mathbb{F}_{G,*} \to \mathbb{F}_*$ exhibits $\mathbb{F}_{G,*}$ as a fibrous $\mathbb{F}_*$-pattern, i.e. an ∞-operad in the sense of Lurie. Therefore there is an equivalence $\mathsf{Fbrs}(\mathbb{F}_{G,*}) \simeq (\mathsf{Opd}_\infty)_{/\mathbb{F}_{G,*}}$. We refer to this as the ∞-category of *naive G-∞-operads*. There is an inclusion of patterns $\mathbb{F}_{G,*} \to \mathsf{Span}(\mathbb{F}_G)$ similar to the one used in Corollary 5.1.13, and this is a strong Segal morphism by an argument as in Observation 5.2.9. Therefore there is a restriction functor:

$$\mathsf{Opd}_{G,\infty} \simeq \mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G)) \longrightarrow \mathsf{Fbrs}(\mathbb{F}_{G,*}) \simeq (\mathsf{Opd}_\infty)_{/\mathbb{F}_{G,*}}$$

which forgets from (genuine) G-∞-operads to naive G-∞-operads. However, we cannot apply the comparison theorem 5.1.1 since $(\mathrm{Orb}_G^{\mathrm{op}})^{\simeq} \simeq \mathbb{F}_{G,*}^{\mathrm{el}} \to \mathsf{Span}(\mathbb{F}_G)^{\mathrm{el}} \simeq \mathrm{Orb}_G^{\mathrm{op}}$ is not an equivalence.

5.3. **Upgrading to $(\infty, 2)$-categories.** In this subsection we will upgrade our main results from ∞-categories to $(\infty,2)$-categories: we will see that the comparison equivalence of Theorem 5.1.1 is an equivalence of $(\infty,2)$-categories and the fully faithful envelope functor of Proposition 4.2.1 is a fully faithful functor of $(\infty,2)$-categories. More precisely, we will show that these functors are compatible with natural Cat_∞-module structures on the ∞-categories involved. It then follows from results of Hinich [Hin20] and Heine [Hei20] that these ∞-categories can be upgraded to $(\infty,2)$-categories and the functors to functors of $(\infty,2)$-categories. We will not comment further on this, however, as our primary interest is in showing that our equivalences are compatible with the natural ∞-categories of maps, which is an immediate consequence of compatibility with the Cat_∞-module structures. We begin by defining such module structures on the ∞-categories and functors we studied in §2:

Construction 5.3.1. Let $\mathcal{B}$ be an ∞-category equipped with a wide subcategory $\mathcal{B}_0$. The forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}} \to \mathsf{Cat}_\infty$ has a right adjoint, taking $\mathcal{C} \in \mathsf{Cat}_\infty$ to the projection $\mathcal{C} \times \mathcal{B} \to \mathcal{B}$; this factors through the subcategory $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ and thus gives symmetric monoidal functors

$$\mathsf{Cat}_\infty \longrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \longrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}$$

with respect to the cartesian products. It follows that both $\mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ are Cat_∞-modules, with the tensoring in both cases simply given by cartesian product, i.e.

$$(C, \mathcal{E} \longrightarrow \mathcal{B}) \quad \mapsto \quad \mathcal{E} \times C \longrightarrow \mathcal{B},$$

and that the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ is a Cat_∞-module functor. Moreover, both Cat_∞-module structures are adjoint to an enrichment in Cat_∞, given respectively by $\mathsf{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(-,-)$ and $\mathsf{Fun}_{/\mathcal{B}}(-,-)$. Similarly, if $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, then the ∞-categories $\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ are Cat_∞-modules, with the tensoring given by the cartesian product, and the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ is a Cat_∞-module functor; it is easy to see that this Cat_∞-module structure on $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ corresponds under the equivalence with $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)$ to that given by taking products with constant functors.

Proposition 5.3.2.

(i) *For any ∞-category $\mathcal{B}$, the tensoring of $\mathsf{Cat}_{\infty/\mathcal{B}}$ over Cat_∞ from Construction 5.3.1 is adjoint to a cotensoring, with the cotensor of $C \in \mathsf{Cat}_\infty$ and $\mathcal{E} \to \mathcal{B}$ given by the pullback*

$$\mathcal{E}^{C}_{/\mathcal{B}} := \mathsf{Fun}(C, \mathcal{E}) \times_{\mathsf{Fun}(C,\mathcal{B})} \mathcal{B}$$

along the constant diagram functor $\mathcal{B} \to \mathsf{Fun}(C, \mathcal{B})$.

(ii) *If $\mathcal{B}_0$ is a wide subcategory of $\mathcal{B}$, then $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ is also cotensored over Cat_∞, with the cotensor of $C \in \mathsf{Cat}_\infty$ and $\mathcal{E} \to \mathcal{B}$ again given by $\mathcal{E}^{C}_{/\mathcal{B}}$. In particular, the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ preserves the cotensoring.*

Proof. Part (i) follows from the natural equivalences

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(C \times \mathcal{F}, \mathcal{E}) \simeq \left\{ \begin{array}{ccc} C \times \mathcal{F} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ C \times \mathcal{B} & \xrightarrow{\text{proj}} & \mathcal{B} \end{array} \right\} \simeq \left\{ \begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathsf{Fun}(C, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{\text{const}} & \mathsf{Fun}(C, \mathcal{B}) \end{array} \right\} \simeq \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(\mathcal{F}, \mathcal{E}^{C}_{/\mathcal{B}}).$$

To prove (ii), we observe that if $\mathcal{E} \to \mathcal{B}$ is in $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, then so is $\mathcal{E}^{C}_{/\mathcal{B}}$ by [HTT, Proposition 3.1.2.3], and a morphism $[1] \to \mathcal{E}^{C}_{/\mathcal{B}}$ is cocartesian if and only if the corresponding map $[1] \times C \to \mathcal{E}$ has cocartesian components at every $c \in C$. Thus a morphism $\mathcal{F} \to \mathcal{E}^{C}_{/\mathcal{B}}$ over $\mathcal{B}$ preserves cocartesian morphisms over $\mathcal{B}_0$ if and only if the corresponding map $\mathcal{F} \times C \to \mathcal{E}$ preserves cocartesian morphisms over $\mathcal{B}_0$, so that the previous equivalence of mapping spaces restricts on subspaces to an equivalence

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(C \times \mathcal{F}, \mathcal{E}) \simeq \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(\mathcal{F}, \mathcal{E}^{C}_{/\mathcal{B}}),$$

as required. ◻

Observation 5.3.3. If $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, then the ∞-categories $\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ are similarly cotensored over Cat_∞, with the same cotensors as in Proposition 5.3.2, and the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ preserves the cotensoring.

Proposition 5.3.4.

(i) *Let $\mathcal{B}$ be an ∞-category with a wide subcategory $\mathcal{B}_0$. Then the left adjoint*

$$(-) \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \colon \mathsf{Cat}_{\infty/\mathcal{B}} \longrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$$

of the forgetful functor from Corollary 2.1.5 is a Cat_∞-module functor, with the adjunction being an adjunction of Cat_∞-modules.

(ii) If $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, then the left adjoint

$$(-) \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \colon \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \longrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$$

of the forgetful functor from Corollary 2.2.5 is a Cat_∞-module functor, with the adjunction being an adjunction of Cat_∞-modules.

Proof. The forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ is a Cat_∞-module functor by Construction 5.3.1. By [HHLN21, Theorem 3.4.7], the left adjoint then has a canonical oplax Cat_∞-module structure, given for $\mathcal{C} \in \mathsf{Cat}_\infty$ and $\mathcal{E} \to \mathcal{B}$ in $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ by the natural map

$$(\mathcal{C} \times \mathcal{B}) \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \longrightarrow \mathcal{C} \times (\mathcal{B} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}));$$

this is clearly an equivalence, so the adjunction of Corollary 2.1.5 lifts to an adjunction of Cat_∞-modules. This proves (i), and the proof of (ii) is the same. □

Remark 5.3.5. The Cat_∞-module structures on $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ and $\mathsf{Cat}_{\infty/\mathcal{B}}$ are adjoint to enrichments in Cat_∞, given respectively by $\mathsf{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(-,-)$ and $\mathsf{Fun}_{/\mathcal{B}}(-,-)$; the equivalence of Proposition 2.1.4 is then precisely that induced by the Cat_∞-module adjunction from Proposition 5.3.4. Similarly, if $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, then the equivalence of Proposition 2.2.4 is also induced by the Cat_∞-module adjunction above.

Lemma 5.3.6.

(i) For any functor of ∞-categories $f \colon \mathcal{A} \to \mathcal{B}$ the functor $f^* \colon \mathsf{Cat}_{\infty/\mathcal{B}} \to \mathsf{Cat}_{\infty/\mathcal{A}}$ given by pullback along f is a Cat_∞-module functor and also preserves the cotensoring with Cat_∞.

(ii) Suppose $\mathcal{A}$ and $\mathcal{B}$ are ∞-categories equipped with wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, and that $f \colon \mathcal{A} \to \mathcal{B}$ is a functor that takes $\mathcal{A}_0$ into $\mathcal{B}_0$. Then the functor $f^* \colon \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{A}}^{\mathcal{A}_0\text{-cocart}}$ given by pullback along f is a Cat_∞-module functor and also preserves the cotensoring with Cat_∞.

Proof. We prove (i); the proof of (ii) is the same. The functor f^* fits in a commutative triangle

$$\begin{array}{ccc} & \mathsf{Cat} & \\ \swarrow & & \searrow \\ \mathsf{Cat}_{\infty/\mathcal{B}} & \xrightarrow{\;f^*\;} & \mathsf{Cat}_{\infty/\mathcal{A}} \end{array}$$

where all three functors preserve finite products, and so are symmetric monoidal with respect to the cartesian products. Hence $f^* \colon \mathsf{Cat}_{\infty/\mathcal{B}} \to \mathsf{Cat}_{\infty/\mathcal{A}}$ is a Cat_∞-module functor. To see that f^* also preserves the cotensoring, observe that for $\mathcal{E} \to \mathcal{B}$ in $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ or $\mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{C} \in \mathsf{Cat}_\infty$ we have a natural commutative cube

$$\begin{array}{ccccc} (f^*\mathcal{E})^{\mathcal{C}}_{/\mathcal{A}} & \longrightarrow & \mathsf{Fun}(\mathcal{C}, f^*\mathcal{E}) & & \\ & \searrow & & \searrow & \\ \downarrow & & \mathcal{E}^{\mathcal{C}}_{/\mathcal{B}} & \longrightarrow & \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \\ \mathcal{A} & \longrightarrow & \mathsf{Fun}(\mathcal{C}, \mathcal{A}) & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & \mathcal{B} & \longrightarrow & \mathsf{Fun}(\mathcal{C}, \mathcal{B}) \end{array}$$

where the front, back and right faces are cartesian. The left vertical square is therefore also cartesian, giving an equivalence

$$(f^*\mathcal{E})^{\mathcal{C}}_{/\mathcal{A}} \xrightarrow{\sim} f^*(\mathcal{E}^{\mathcal{C}}_{/\mathcal{B}}),$$

as required. □

Observation 5.3.7. For $f\colon \mathcal{A} \to \mathcal{B}$ a functor that preserves wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, we have a commutative diagram

$$\begin{tikzcd} \mathsf{Cat}_\infty \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \arrow[r, "f^*"] \arrow[d] & \mathsf{Cat}_{\infty/\mathcal{A}}^{\mathcal{A}_0\text{-cocart}} \arrow[d] \\ & \mathsf{Cat}_{\infty/\mathcal{B}} \arrow[r, "f^*"] & \mathsf{Cat}_{\infty/\mathcal{A}}, \end{tikzcd}$$

of symmetric monoidal functors (with the cartesian monoidal structures). It follows that the commutative square on the bottom right (as in Observation 2.1.7) is a square of Cat_∞-modules. Similarly, if f is compatible with factorization systems $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$, then the commutative square

$$\begin{tikzcd} \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}} \arrow[r, "f^*"] \arrow[d] & \mathsf{Cat}_{\infty/\mathcal{A}}^{\mathrm{cocart}} \arrow[d] \\ \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \arrow[r, "f^*"] & \mathsf{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}}, \end{tikzcd}$$

is a square of Cat_∞-modules. It follows that for both squares the Beck–Chevalley map is a natural transformation of Cat_∞-modules.

Proposition 5.3.8. *Let $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be a factorization system. Then there is a natural Cat_∞-module structure on the ∞-category $(\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})}$, with the tensoring given by cartesian products, and the adjunction*

$$\mathrm{E}\colon \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows (\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})} :\mathrm{Q}$$

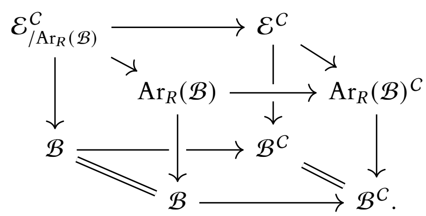
is compatible with the Cat_∞-module structures. Moreover, $(\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})}$ is also cotensored over Cat_∞, with the cotensor of $C \in \mathsf{Cat}_\infty$ and $\mathcal{E} \to \mathrm{Ar}_R(\mathcal{B})$ in $(\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})}$ being $\mathcal{E}^C_{/\mathrm{Ar}_R(\mathcal{B})}$.

Proof. The forgetful functor $(\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})} \to \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}}$ has a right adjoint, which takes a cocartesian fibration $\mathcal{E} \to \mathcal{B}$ to the projection $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathrm{Ar}_R(\mathcal{B})$. We thus have a commutative diagram

$$\begin{tikzcd} & \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}} \arrow[dr, "\text{forget}"] \arrow[dd, "(-)\times_{\mathcal{B}}\mathrm{Ar}_R(\mathcal{B})"] & \\ \mathsf{Cat}_\infty \arrow[ur, "(-)\times\mathcal{B}"] \arrow[dr, "(-)\times\mathrm{Ar}_R(\mathcal{B})"'] & & \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \\ & (\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})} \arrow[ur, "\mathrm{Q}"] & \end{tikzcd}$$

of right adjoints, which are then symmetric monoidal functors with respect to cartesian products. This in particular shows that $(\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})}$ is a Cat_∞-module, with the tensoring given by taking cartesian products, and the functor Q is compatible with the Cat_∞-module structures. As in Construction 5.3.1, it follows that the left adjoint E is an oplax Cat_∞-module functor, and that the oplax structure maps are equivalences; thus we have a Cat_∞-module adjunction.

To identify the cotensor, we first observe that $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathrm{Ar}_R(\mathcal{B})}$ can be described as a subcategory of $\mathsf{Cat}_{\infty/\mathrm{Ar}_R(\mathcal{B})}$; the Cat_∞-module structures on both are clearly compatible, and the latter has a cotensoring given by $(C, \mathcal{E}) \mapsto \mathcal{E}^C_{/\mathrm{Ar}_R(\mathcal{B})}$ by Proposition 5.3.2. It thus suffices to show that $\mathcal{E}^C_{/\mathrm{Ar}_R(\mathcal{B})}$ is an object of $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathrm{Ar}_R(\mathcal{B})}$, i.e. that the composite to $\mathcal{B}$ is a cocartesian fibration, that the morphism to $\mathrm{Ar}_R(\mathcal{B})$ preserves cocartesian morphisms over $\mathcal{B}$, and that a morphism $\mathcal{F} \to \mathcal{E}^C_{/\mathrm{Ar}_R(\mathcal{B})}$ preserves cocartesian morphisms over $\mathcal{B}$ if and only if the adjoint map $\mathcal{F} \times C \to \mathcal{E}$ does so. To see this, consider the commutative cube

(11) 

Here the top and bottom squares are cartesian, the vertical maps are cocartesian fibrations, and both maps to $\mathrm{Ar}_R(\mathcal{B})^C$ preserve cocartesian morphisms. It follows that $\mathcal{E}^C_{/\mathrm{Ar}_R(\mathcal{B})} \to \mathcal{B}$ is a cocartesian fibration, and a morphism here is cocartesian if and only if its images in $\mathrm{Ar}_R(\mathcal{B})$ and $\mathcal{E}^C$ are both cocartesian. Combining this with the description of cocartesian morphisms in $\mathcal{E}^C$ from [HTT, Proposition 3.1.2.1] gives the required description of cocartesian morphisms in $\mathcal{E}^C_{/\mathrm{Ar}_R(\mathcal{B})}$. □

Observation 5.3.9. Let us write $\mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathrm{Ar}_R(\mathcal{B})}(-,-)$ for

the enrichment adjoint to the Cat_∞-module structure on $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathrm{Ar}_R(\mathcal{B})}$; this satisfies

$$\mathsf{Map}_{\mathsf{Cat}_\infty}(C, \mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathrm{Ar}_R(\mathcal{B})}(-,-)) \simeq \mathsf{Map}_{(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathrm{Ar}_R(\mathcal{B})}}(C \times -, -);$$

identifying the right-hand side as a fiber product we see that for $\alpha \colon \mathcal{E} \to \mathrm{Ar}_R(\mathcal{B})$, $\beta \colon \mathcal{F} \to \mathrm{Ar}_R(\mathcal{B})$ we have a natural cartesian square

$$\begin{array}{ccc} \mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathrm{Ar}_R(\mathcal{B})}((\mathcal{E},\alpha),(\mathcal{F},\beta)) & \longrightarrow & \mathsf{Fun}^{\mathrm{cocart}}_{/\mathcal{B}}(\mathcal{E},\mathcal{F}) \\ \downarrow & & \downarrow \\ \{\alpha\} & \longrightarrow & \mathsf{Fun}^{\mathrm{cocart}}_{/\mathcal{B}}(\mathcal{E},\mathrm{Ar}_R(\mathcal{B})). \end{array}$$

Since the functor E is fully faithful and compatible with the Cat_∞-module structures we conclude that it gives a natural equivalence

$$\mathsf{Fun}^{L\text{-cocart}}_{/\mathcal{B}}(-,-) \xrightarrow{\sim} \mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathrm{Ar}_R(\mathcal{B})}(\mathrm{E}(-),\mathrm{E}(-)).$$

Observation 5.3.10. Suppose $f \colon \mathcal{A} \to \mathcal{B}$ is a functor compatible with specified factorization systems. Passing to vertical left adjoints in the commutative square Observation 2.3.9 yields a Beck–Chevalley transformation

$$\mathrm{E}_{\mathcal{A}} f^* \longrightarrow f^{\circledast} \mathrm{E}_{\mathcal{B}};$$

Unwinding the definitions, this is given at $\mathcal{E} \to \mathcal{B}$ in $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ by the natural map

$$(\mathcal{E} \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} \mathrm{Ar}_R(\mathcal{A}) \longrightarrow (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})) \times_{\mathrm{Ar}_R(\mathcal{B})} \mathrm{Ar}_R(\mathcal{A}),$$

which is an equivalence. The functors and transformations here are also compatible with the Cat_∞-module structures, by the same argument as in Observation 5.3.7 , so for

$\mathcal{E}, \mathcal{F} \to \mathrm{Ar}_R(\mathcal{B})$ we have a natural commutative square in which the vertical maps are equivalences:

$$\begin{array}{ccc} \mathsf{Fun}^{L\text{-cocart}}_{/\mathcal{B}}(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathsf{Fun}^{L\text{-cocart}}_{/\mathcal{A}}(f^*\mathcal{E}, f^*\mathcal{E}) \\ \downarrow\wr & & \downarrow\wr \\ \mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathrm{Ar}_R(\mathcal{B})}(\mathrm{E}_{\mathcal{B}}\mathcal{E}, \mathrm{E}_{\mathcal{B}}\mathcal{F}) & \longrightarrow & \mathsf{Fun}^{\mathcal{A}\text{-cocart}}_{/\mathrm{Ar}_R(\mathcal{A})}(\mathrm{E}_{\mathcal{A}} f^*\mathcal{E}, \mathrm{E}_{\mathcal{A}} f^*\mathcal{F}). \end{array} \tag{12}$$

After these preliminaries we are finally ready to consider fibrous patterns and their envelopes. First, we want to show that the ∞-categories $\mathsf{Fbrs}(\mathcal{O})$ and $\mathsf{Seg}^{/\mathcal{A}_\mathcal{O}}_\mathcal{O}(\mathsf{Cat}_\infty)$ have Cat_∞-module structures inherited from those we have already considered. This is slightly complicated by the fact that $\mathsf{Fbrs}(\mathcal{O})$ may not be closed under tensors in $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$, and similarly for the relative Segal objects. (For example, for $\mathcal{O} \in \mathsf{Fbrs}(\mathbb{F}_*)$ and $\mathcal{C} \in \mathsf{Cat}_\infty$, the ∞-category $\mathcal{C} \times \mathcal{O}$ is not an object of $\mathsf{Fbrs}(\mathbb{F}_*)$ since its fiber over $\langle 0 \rangle$ is $\mathcal{C}$, not $*$; on the other hand, $\mathsf{Fbrs}(\mathbb{F}_*^\natural)$ *is* closed under tensoring with Cat_∞.) Luckily, cotensors are better behaved:

Proposition 5.3.11. *Let $\mathcal{O}$ be an algebraic pattern.*

(i) For $\mathcal{P} \in \mathsf{Fbrs}(\mathcal{O})$ and $\mathcal{C} \in \mathsf{Cat}$, the cotensor $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}$ in $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$ is again fibrous.

(ii) For $X \in \mathsf{Seg}^{/\mathcal{A}_\mathcal{O}}_\mathcal{O}(\mathsf{Cat}_\infty)$ corresponding to $\mathcal{X} \in (\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{O}})_{/\mathrm{Ar}_{\text{act}}(\mathcal{O})}$ and $\mathcal{C} \in \mathsf{Cat}$, the cotensor $\mathcal{X}^{\mathcal{C}}_{/\mathrm{Ar}_{\text{act}}(\mathcal{O})}$ in $(\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{O}})_{/\mathrm{Ar}_{\text{act}}(\mathcal{O})}$ again straightens to a relative Segal object.

Proof. To prove (i), first observe that we can identify $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}} \times_\mathcal{O} \mathcal{O}^{\text{act}}_{/O}$ as the fiber product $\mathsf{Fun}(\mathcal{C}, \mathcal{P} \times_\mathcal{O} \mathcal{O}^{\text{act}}_{/O}) \times_{\mathsf{Fun}(\mathcal{C}, \mathcal{O}^{\text{act}}_{/O})} \mathcal{O}^{\text{act}}_{/O}$, so that we have a commutative cube

$$\begin{array}{ccccc} \mathcal{P}^{\mathcal{C}}_{/\mathcal{O}} \times_\mathcal{O} \mathcal{O}^{\text{act}}_{/O} & & \longrightarrow & & \mathsf{Fun}(\mathcal{C}, \mathcal{P} \times_\mathcal{O} \mathcal{O}^{\text{act}}_{/O}) \\ & \searrow & & & & \searrow \\ & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{P}^{\mathcal{C}}_{/\mathcal{O}} \times_\mathcal{O} \mathcal{O}^{\text{act}}_{/E} & \longrightarrow & & \mathsf{Fun}(\mathcal{C}, \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{P} \times_\mathcal{O} \mathcal{O}^{\text{act}}_{/E}) \\ \downarrow & & \downarrow & & \downarrow & \downarrow \\ \mathcal{O}^{\text{act}}_{/O} & & \longrightarrow & & \mathsf{Fun}(\mathcal{C}, \mathcal{O}^{\text{act}}_{/O}) \\ & \searrow & & & & \searrow \\ & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{O}^{\text{act}}_{/E} & \longrightarrow & & \mathsf{Fun}(\mathcal{C}, \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{O}^{\text{act}}_{/E}). \end{array}$$

where the front and back faces are cartesian. Here the right vertical face is also cartesian since $\mathcal{P}$ is $\mathcal{O}$-fibrous. It then follows that the left vertical face is also cartesian, i.e. $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}$ is also $\mathcal{O}$-fibrous.

For (ii), we extract the following commutative diagram from the cube (11) that describes $\mathcal{X}^{\mathcal{C}}_{/\mathrm{Ar}_{\text{act}}(\mathcal{O})}$:

$$\begin{array}{ccccc} (\mathcal{X}^{\mathcal{C}}_{/\mathrm{Ar}_{\text{act}}(\mathcal{O})})_O & & \longrightarrow & & (\mathcal{X}^{\mathcal{C}})_O \\ & \searrow & & & & \searrow \\ & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} (\mathcal{X}^{\mathcal{C}}_{/\mathrm{Ar}_{\text{act}}(\mathcal{O})})_E & \longrightarrow & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} (\mathcal{X}^{\mathcal{C}})_E \\ \downarrow & & \downarrow & & \downarrow & \downarrow \\ \mathrm{Ar}_{\text{act}}(\mathcal{O})_O & & \longrightarrow & & (\mathrm{Ar}_{\text{act}}(\mathcal{O})^{\mathcal{C}})_O \\ & \searrow & & & & \searrow \\ & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathrm{Ar}_{\text{act}}(\mathcal{O})_E & \longrightarrow & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} (\mathrm{Ar}_{\text{act}}(\mathcal{O})^{\mathcal{C}})_E. \end{array}$$

(Here we have also used O for the constant functor $\mathcal{C} \to \mathcal{O}$ with this value.) The front and back vertical faces in this cube are cartesian by the definition of $\mathcal{X}^{\mathcal{C}}_{/\mathrm{Ar}_{\text{act}}(\mathcal{O})}$, while the right vertical face is cartesian since $\mathcal{X}$ by assumption straightens to a relative Segal object (and we can identify $(\mathcal{X}_{\mathcal{O}}^{\mathcal{C}}$ as $\mathsf{Fun}(\mathcal{C}, \mathcal{X}_{\mathcal{O}})$ etc.). Hence the left vertical face is also cartesian, and this is precisely the relative Segal condition for $\mathcal{X}^{\mathcal{C}}_{/\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})}$. □

Corollary 5.3.12. *Let $\mathcal{O}$ be an algebraic pattern.*

(i) The localization $L_{\mathrm{fbrs}}\colon \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \to \mathsf{Fbrs}(\mathcal{O})$ is a localization of Cat_∞-modules.
(ii) The localization $L_{\mathrm{rseg}}\colon (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}})_{/\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})} \to \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ is a localization of Cat_∞-modules.

Proof. We prove the first claim; the proof of the second is the same — in particular, both follow from [HA, Proposition 2.2.1.9]. In order to apply this to L_{fbrs}, we must verify the required hypothesis, which amounts to checking that for $\mathcal{C} \in \mathsf{Cat}_\infty$ and $\mathcal{E} \in \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$, the canonical map $\mathcal{C} \times \mathcal{E} \to \mathcal{C} \times L_{\mathrm{fbrs}}(\mathcal{E})$ is taken to an equivalence by L_{fbrs}. Equivalently, we must show that for $\mathcal{P} \in \mathsf{Fbrs}(\mathcal{O})$, the induced map

$$\mathsf{Map}_{\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}}(\mathcal{C} \times L_{\mathrm{fbrs}}(\mathcal{E}), \mathcal{P}) \longrightarrow \mathsf{Map}_{\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}}(\mathcal{C} \times \mathcal{E}, \mathcal{P})$$

is an equivalence. Using the cotensoring, this is the same as the map

$$\mathsf{Map}_{\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}}(L_{\mathrm{fbrs}}(\mathcal{E}), \mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}) \longrightarrow \mathsf{Map}_{\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}}(\mathcal{E}, \mathcal{P}^{\mathcal{C}}_{/\mathcal{O}})$$

given by composition with the localization map $\mathcal{E} \to L_{\mathrm{fbrs}}(\mathcal{E})$. This map is indeed an equivalence, since $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}$ is fibrous by Proposition 5.3.11. □

Corollary 5.3.13. *Let $\mathcal{O}$ be a sound pattern. Then we have a commutative square*

$$\begin{array}{ccc}
\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} & \xrightarrow{L_{\mathrm{fbrs}}} & \mathsf{Fbrs}(\mathcal{O}) \\
\downarrow{\scriptstyle \mathrm{E}} & & \downarrow{\scriptstyle \mathsf{Env}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}} \\
(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}})_{/\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})} & \xrightarrow{L_{\mathrm{rseg}}} & \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)
\end{array}$$

of Cat_∞-module functors. Moreover, the adjunction

$$\mathsf{Env}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}\colon \mathsf{Fbrs}(\mathcal{O}) \rightleftarrows \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty) : \mathrm{Un}^{\mathrm{int}}_{\mathcal{O}}$$

of Proposition 4.2.5 is an adjunction of Cat_∞-modules, with the right adjoint being a lax *Cat_∞-module functor.*

Proof. Let us use the universal property of $\mathsf{Fbrs}(\mathcal{O})$ as a Cat_∞-module localization to verify that the composite

$$\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \xrightarrow{\mathrm{E}} (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}})_{/\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})} \xrightarrow{L_{\mathrm{rseg}}} \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$$

factors through L_{fbrs}, as a functor of Cat_∞-modules. Thus we need to verify that if a morphism $\mathcal{E} \to \mathcal{F}$ in $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$ is taken to an equivalence by L_{fbrs}, then $\mathrm{E}\mathcal{E} \to \mathrm{E}\mathcal{F}$ is taken to an equivalence by L_{rseg}. The latter condition is equivalent to the induced morphism

$$\mathsf{Map}(\mathrm{E}\mathcal{F}, \mathcal{X}) \longrightarrow \mathsf{Map}(\mathrm{E}\mathcal{E}, \mathcal{X})$$

being an equivalence provided $\mathcal{X}$ is the unstraightening of an object in $\mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$. By adjunction this holds if and only if the map

$$\mathsf{Map}(\mathcal{F}, \mathrm{Q}\mathcal{X}) \longrightarrow \mathsf{Map}(\mathcal{E}, \mathrm{Q}\mathcal{X})$$

is an equivalence for all such $\mathcal{X}$, but since $\mathcal{O}$ is sound the object $\mathrm{Q}\mathcal{X}$ is fibrous, and hence this is indeed an equivalence as by assumption $\mathcal{E} \to \mathcal{F}$ is taken to an equivalence by L_{fbrs}. It follows that the right adjoint inherits a lax Cat_∞-module structure. □

Remark 5.3.14. For any pattern $\mathcal{O}$ the Segal envelope

$$\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} \colon \mathsf{Fbrs}(\mathcal{O}) \longrightarrow \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$$

is a *lax* Cat_∞-module functor, since it can be defined by restricting $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ to these full subcategories, the inclusions of which are lax Cat_∞-module functors. This suffices to upgrade the envelope to a functor of $(\infty,2)$-categories, and we can see that it is fully faithful since it is obtained by restricting the functor $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}} \colon \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \to \mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$, which is a fully faithful functor of $(\infty,2)$-categories by Observation 5.3.9.

Proposition 5.3.15. *Let $\mathcal{O}$ and $\mathcal{P}$ be algebraic patterns and $f \colon \mathcal{O} \to \mathcal{P}$ a strong Segal morphism.*

(i) The functor $f^* \colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O})$ is a lax Cat_∞-module functor and its left adjoint $f_!$ is a Cat_∞-module functor.

(ii) The functor $f^\circledast \colon \mathsf{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) \to \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$ is a lax Cat_∞-module functor and its left adjoint $f_!$ is a Cat_∞-module functor.

Proof. To prove (i), we observe that f^* is obtained by restricting $f^* \colon \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{P}}^{\text{int-cocart}}$, which is a Cat_∞-module functor by Observation 5.3.10, to full subcategories; it is therefore a lax Cat_∞-module functor. The left adjoint $f_!$ is then automatically an oplax Cat_∞-module functor, and the oplax structure map is an equivalence if and only if the right adjoint f^* preserves Cat_∞-cotensors, which we know from Lemma 5.3.6 and Proposition 5.3.11. The proof of (ii) is the same. □

Remark 5.3.16. It follows that for $\mathcal{Q} \in \mathsf{Fbrs}(\mathcal{O})$ and $\mathcal{R} \in \mathsf{Fbrs}(\mathcal{P})$ we have a natural equivalence

$$\mathsf{Fun}_{/\mathcal{O}}^{\text{int-cocart}}(f_!\mathcal{Q}, \mathcal{R}) \simeq \mathsf{Fun}_{/\mathcal{P}}^{\text{int-cocart}}(\mathcal{Q}, f^*\mathcal{R}).$$

Corollary 5.3.17. *Let $f \colon \mathcal{O} \to \mathcal{P}$ be a strong Segal morphism between soundly extendable patterns that satisfies the hypotheses of Theorem 5.1.1. Then pullback along f gives an equivalence*

$$f^* \colon \mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O})$$

of Cat_∞-modules. In particular, for any $\mathcal{Q}, \mathcal{Q}'$ in $\mathsf{Fbrs}(\mathcal{P})$, the induced functor

$$\mathsf{Fun}_{/\mathcal{O}}^{\text{int-cocart}}(\mathcal{Q}, \mathcal{Q}') \longrightarrow \mathsf{Fun}_{/\mathcal{P}}^{\text{int-cocart}}(f^*\mathcal{Q}, f^*\mathcal{Q}')$$

is an equivalence. □

References

[AMGR17] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum, *Factorization homology of enriched ∞-categories* (2017), available at arXiv:1710.06414.

[Bar22] Shaul Barkan, *Arity Approximation of ∞-Operads* (2022), available at arXiv:2207.07200.

[Bar17] Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Adv. Math. **304** (2017), 646–727, available at arXiv:1404.0108.

[Bar18] ______, *From operator categories to higher operads*, Geom. Topol. **22** (2018), no. 4, 1893–1959, available at arXiv:1302.5756.

[Ber07] Clemens Berger, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), no. 1, 230–270.

[BH18] Andrew J. Blumberg and Michael A. Hill, *Incomplete Tambara functors*, Algebr. Geom. Topol. **18** (2018), no. 2, 723–766.

[CMNN20] Dustin Clausen, Akhil Mathew, Niko Nauman, and Justin Noel, *Descent and vanishing in chromatic algebraic K-theory via group actions* (2020), available at arXiv:2011.08233.

[CH21] Hongyi Chu and Rune Haugseng, *Homotopy-coherent algebra via Segal conditions*, Advances in Mathematics **385** (2021), 107733, available at arXiv:1907.03977.

[GR17] Dennis Gaitsgory and Nick Rozenblyum, *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, Mathematical Surveys and Monographs, vol. 221, American Mathematical Society, Providence, RI, 2017. Available from http://www.math.harvard.edu/~gaitsgde/GL/.

[GH15] David Gepner and Rune Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. **279** (2015), 575–716, available at arXiv:1312.3178.

[Hac21] Philip Hackney, *Categories of graphs for operadic structures* (2021), available at arXiv:2109.06231.

[RY20] Marcy Robertson and Donald Yau, *A graphical category for higher modular operads*, Adv. Math. **365** (2020), 107044, 61.

[Har20] Yonatan Harpaz, *Ambidexterity and the universality of finite spans*, Proc. Lond. Math. Soc. (3) **121** (2020), no. 5, 1121–1170, available at arXiv:1703.09764.

[Hau17] Rune Haugseng, *The higher Morita category of E_n-algebras*, Geom. Topol. **21** (2017), 1631–1730, available at arXiv:1412.8459.

[Hau18] ———, *On the equivalence between Θ_n-spaces and iterated Segal spaces*, Proc. Amer. Math. Soc. **146** (2018), no. 4, 1401–1415, available at arXiv:1604.08480.

[HHLN21] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten, *Lax monoidal adjunctions, two-variable fibrations and the calculus of mates* (2021), available at arXiv:2011.08808.

[HHLN22] ———, *Two-variable fibrations, factorisation systems and ∞-categories of spans* (2022), available at arXiv:arXiv:2011.11042.

[HK21] Rune Haugseng and Joachim Kock, *∞-operads as symmetric monoidal ∞-categories* (2021), available at arXiv:2106.12975.

[Hei20] Hadrian Heine, *An equivalence between enriched ∞-categories and ∞-categories with weak action* (2020), available at arXiv:2009.02428.

[Hin20] Vladimir Hinich, *Yoneda lemma for enriched ∞-categories*, Adv. Math. **367** (2020), 107129, available at arXiv:1805.07635.

[HTT] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available from http://math.ias.edu/~lurie/.

[HA] ———, *Higher Algebra*, 2017. Available at http://math.ias.edu/~lurie/.

[Ker] ———, *Kerodon*. Available at http://kerodon.net.

[Nar16] Denis Nardin, *Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞-category* (2016), available at arXiv:1608.07704.

[NS22] Denis Nardin and Jay Shah, *Parametrized and equivariant higher algebra* (2022), available at arXiv:2203.00072.

[Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007.

[Rez10] ———, *A Cartesian presentation of weak n-categories*, Geom. Topol. **14** (2010), no. 1, 521–571.

[Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Sha21] Jay Shah, *Parametrized higher category theory II: Universal constructions* (2021), available at arXiv:2109.11954.

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Part VI

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Genuine equivariant factorization homology

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Abstract

We construct a genuine G-equivariant extension of factorization homology for G a finite group, assigning a genuine G-spectrum to a manifold with G-action. We show that G-factorization homology is compatible with Hill-Hopkins-Ravenel norms and satisfies equivariant $\otimes$-excision. Following Ayala-Francis we prove an axiomatic characterization of genuine G-factorization homology. Applications include a description of real topological Hochschild homology and relative topological Hochschild homology of C_n-rings using genuine G-factorization homology.

Contents

1 Introduction

Factorization homology, introduced by Lurie under the name topological chiral homology ([Lur09b], [Lur]), is an invariant of an $\mathbb{E}_n$-algebra and a framed n-dimensional manifold. The factorization homology of a framed n-dimensional manifold M with coefficients in an $\mathbb{E}_n$-ring spectrum A is a spectrum denoted $\int_M A$. If M admits an action of a finite group G then $\int_M A$ admits an G-action by functoriality. However, this action is defined only up to coherent homotopy, as $\int_M A$ is defined by an ∞-categorical colimit. A fundamental observation of equivariant homotopy theory is that such a "naive" action does not determine the homotopy type of the fixed points. In particular the action of G on $\int_M A$ does not define a *genuine* G-spectrum structure on $\int_M A$.

The first goal of this paper is to construct and study such a genuine equivariant extension of factorization homology for a fixed finite group G. We draw on two points of view in order to explain the expected properties of genuine equivariant factorization homology.

Factorization homology as a tensor product First, according to [Lur09b, rem. 4.1.19] one can intuitively think of $\int_M A$ as a continuous tensor product $\otimes_{x\in M} A$ indexed by the points of M. One should have this intuition in mind when considering the behavior of factorization homology with respect to disjoint unions[1], namely

$$\int_{M_1 \sqcup M_2} A \simeq \int_{M_1} A \otimes \int_{M_2} A. \tag{1}$$

In order to generalize this behavior to genuine G-spectra we now recall the interaction of the smash product with the group action. If X is a genuine H-spectrum for $H < G$ a subgroup then the smash product $\otimes_{|G/H|} X$ of G/H copies of X has a naive G-action, induced by the combining the action of H on X with the action of G on the indexing set G/H. Hill-Hopkins-Ravenel [HHR16] extended this naive G-action to a genuine G-spectrum, $N_H^G(X)$, the Hill-Hopkins-Ravenel norm of X. More generally they define smash products indexed by finite G-sets as the smash product of Hill-Hopkins-Ravenel norms (see [HHR16, app. A.3]). Let U be a finite G-set, given by a coproduct of orbits $U = \coprod_{i\in I} G/H_i$ with stabilizers $H_i < G$. The U-indexed smash product of a family $X_\bullet = \{X_i\}_{i\in I}$, where each X_i is a genuine H_i-spectrum, is the genuine G-spectrum given by the smash product of the norms $\otimes_U X_\bullet = \otimes_{i\in I} N_{H_i}^G(X_i)$. The indexed smash product interacts with smash products and norms as follows.

[1]We distinguish between disjoint unions and coproducts since disjoint union is not the categorical coproduct in the category of n-dimensional manifolds and open embeddings which we consider below.

- The indexed smash product takes disjoint unions to smash products: if U', U'' are of finite G-sets then the indexed smash product along $U' \coprod U''$ is equivalent to smash product of the indexed products, $\otimes_{U' \coprod U''} X_\bullet \simeq (\otimes_{U'} X_\bullet) \otimes (\otimes_{U''} X_\bullet)$.
- The indexed smash product takes topological inductions to norms: given a subgroup $H < G$ and a finite H-set U, denote the quotient $G \times_H U$ by $\coprod_{G/H} U$. The left action of G on the first coordinate makes $\coprod_{G/H} U$ a G-set which we call the *topological induction* of U from H to G. The norm of an indexed product is given by an indexed product along the topological induction, $\otimes_{\coprod_{G/H} U} X_\bullet \simeq N_H^G(\otimes_U X_\bullet)$.

Note that stating the second property required us to consider tensor products indexed by finite H-sets for all $H < G$.

Interpreting genuine equivariant factorization homology as a tensor product indexed by a G-manifold M, one would expect a similar behavior. To state it, we consider the genuine factorization homology of H-manifolds for all subgroups $H < G$. Namely, for any subgroup $H < G$ and H-manifold M we expect genuine equivariant factorization homology to assign a genuine H-spectrum $\int_M A \in \mathbf{Sp}^H$, which interacts with smash products and norms as follows.

- Genuine equivariant factorization homology takes disjoint unions to smash products: if M', M'' are n-dimensional G-manifolds then the genuine equivariant factorization homology along $M' \sqcup M''$ is equivalent to the smash products of the genuine equivariant factorization homologies along M' and M'',
$$\int_{M' \sqcup M''} A \simeq (\int_{M'} A) \otimes (\int_{M''} A),$$
as genuine G-spectra.
- Genuine equivariant factorization homology takes topological inductions to norms: given a subgroup $H < G$ and an n-dimensional H-manifold M, denote the topological induction $G \times_H M$ by $\sqcup_{G/H} M$, with left G-action induced by acting on the first coordinate. The norm of genuine equivariant factorization homology along M is equivalent to genuine equivalent factorization homology along the topological induction $\sqcup_{G/H} M$,
$$\int_{\sqcup_{G/H} M} A \simeq N_H^G(\int_M A), \tag{2}$$
as genuine G-spectra.

Factorization homology as a homology theory A second point of view on factorization homology is given by Ayala-Francis [AF15], where factorization homology is considered as a homology theory of n-dimensional manifolds. Ayala-Francis start from the observation that factorization homology is functorial with respect to open embeddings of framed n-dimensional manifolds. Let $\mathbf{Mfld}_n^{fr}$ be the ∞-category of framed n-dimensional manifolds and framed open embeddings, and let $\mathcal{C}$ be a cocomplete symmetric monoidal ∞-category. Fixing an $\mathbb{E}_n$-algebra A in $\mathcal{C}$, Ayala-Francis consider factorization homology $M \mapsto \int_M A$ as a functor of ∞-categories $\int_{\_} A \colon \mathbf{Mfld}_n^{fr} \to \mathcal{C}$. Factorization homology extends to a symmetric monoidal functor $\int_{\_} A \colon \mathbf{Mfld}_n^{fr,\sqcup} \to \mathcal{C}^{\otimes}$ with respect to disjoint union of manifolds (a functorial version of eq. (1)) under mild conditions[2] on $\mathcal{C}$.

[2]Namely that the tensor product in $\mathcal{C}$ distributes over sifted colimits.

Taking the view that excision is the characterizing property of a homology theory, Ayala-Francis define a homology theory for manifolds as a symmetric monoidal functor $\mathbf{Mfld}_n^{fr,\sqcup} \to \mathcal{C}^{\otimes}$ satisfying $\otimes$-excision, and show that $\int_{\_} A$ is indeed such a homology theory for manifolds.

Furthermore, they show that the Eilenberg-Steenrod characterization of generalized homology theories admits the following generalization. Let $\mathcal{H}(\mathbf{Mfld}_n^{fr}, \mathcal{C}) \subseteq \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n^{fr}, \mathcal{C})$ be the full subcategory of symmetric monoidal functors satisfying $\otimes$-excision.

Theorem 1.0.1 (Ayala-Francis). *There is an equivalence of ∞-categories*

$$\int : Alg_{\mathbb{E}_n}(\mathcal{C}) \xrightarrow{\sim} \mathcal{H}(\mathbf{Mfld}_n^{fr}, \mathcal{C}), \quad A \mapsto (\int_{\_} A \colon \mathbf{Mfld}_n^{fr} \to \mathcal{C})$$

sending an $\mathbb{E}_n$-algebra A to factorization homology with coefficients in A.

In fact, this theorem holds in greater generality, replacing framed manifolds with B-framed manifolds and $\mathbb{E}_n$-algebras with B-framed n-disk-algebras. The second goal of this paper is to provide such an axiomatic characterization of genuine equivariant factorization homology (see theorem 6.0.2).

Framed G-manifolds We now describe V-framed G-manifolds, which serve as the geometric inputs of genuine G-factorization homology theories. The notion of V-framed G-manifolds has already been studied by [Wee18], though our construction differs from his.

Fix a finite group G and $n \in \mathbb{N}$. In what follows a *G-manifold* is an n-dimensional smooth manifold M with a smooth action of G. We organize G-manifolds and G-equivariant smooth open embeddings using a topological category $\mathbf{Mfld}^G$, which we consider as an ∞-category by taking its coherent nerve.

Recall that a framing of M is trivialization of its tangent bundle, i.e an isomorphism of tangent bundles $TM \cong M \times \mathbb{R}^n$. In order to define a framing of G-manifolds we consider TM as G-vector bundle, with G-action induced from the smooth action of G on M by taking differentials. Fix a real n-dimensional G-representation V. A *V-framing* of M as an isomorphism of G-vector bundles $TM \cong M \times V$ over M. The ∞-category of $\mathbf{Mfld}^G$ can be enhanced to an ∞-category $\mathbf{Mfld}^{G,V-fr}$ of V-framed G-manifolds.

In fact, we consider genuine equivariant factorization homology theories of G-manifolds with more general tangential structures (see definition 4.1.2). These tangential structures include unframed G-manifolds, equivariant orientations in the sense of [CMW01] and manifolds with a free G-action (see section 3.3).

We plan to compare this notion of an equivariant tangential structure with the one introduced by [Wee18, sec. 2.2] in future work.

Equivariant factorization homology as a single functor of ∞-categories Viewing factorization homology as a homology theory suggests a natural generalization to G-manifolds. Namely, define a G-factorization homology theory as a symmetric monoidal functor

$$\mathbf{Mfld}^{G,V-fr} \to \mathcal{C}$$

satisfying $\otimes$-excision. This is essentially the approach taken by Weelinck in [Wee18], which leads to a natural generalization of the axiomatic characterization of factorization homology discussed above. In particular, taking $\mathcal{C} = \mathbf{Sp}^G$ to be the ∞-category of genuine G-spectra produces invariants of G-manifolds valued in genuine G-spectra.

However, this is not the approach we take in this paper, for two reasons. First, we are looking for an *extension* of factorization homology to genuine G-spectra. If M is a G-manifold and $F\colon \mathbf{Mfld}^{G,V-fr} \to \mathbf{Sp}^G$ is a G-factorization homology theory in the sense of [Wee18] then the underlying spectrum of $F(M) \in \mathbf{Sp}^G$ need not agree with the factorization homology of M. Second, using a single functor $\mathbf{Mfld}^{G,V-fr} \to \mathcal{C}$ to encode a G-factorization homology theory prevents us from expressing its expected compatibility with norms described in eq. (2).

Our emphasis on the compatibly of equivariant factorization homology with norms implies that our notion an equivariant disk algebra, serving as a coefficient system for equivariant factorization homology, is different from the one introduced in [Wee18]. For a specific example, compare [Wee18, ex. 1.3] with the description of $\mathbb{E}_\sigma$-algebras in section 7.1. A detailed comparison of these two notions will appear in future work.

Parametrized ∞-categories. In order to express both the functoriality of genuine equivariant factorization homology with respect to equivariant embeddings and the compatibilities of eq. (2) we view genuine factorization homology as a collection of symmetric monoidal functors

$$\forall H < G: \quad \int_{-} A\colon \mathbf{Mfld}^{H,V-fr} \to \mathbf{Sp}^H$$

from the ∞-category of V-framed H-manifolds[3] to the category of genuine H-spectra, coherently compatible with restrictions and topological inductions. [4]

To make this coherent compatibilities precise we use the theory of parametrized ∞-categories, developed by Barwick-Dotto-Glasman-Nardin-Shah in [BDG$^+$16b, Sha18, Nar17, BDG$^+$, Nar16]. Informally, a G-∞-category is a diagram of ∞-categories $\mathcal{O}_G^{op} \to \mathcal{C}\mathrm{at}_\infty$ indexed contravariantly by the orbits of G. A G-symmetric monoidal structure encodes a symmetric monoidal structure on each of the ∞-categories in the diagram together with norm functors and all their expected compatibilities. In section 2 we review parametrized ∞-category theory in more detail.

In particular, we use the G-∞-category $\underline{\mathbf{Sp}}^G$ of genuine G-spectra constructed in [Nar17]. As a G-∞-category $\underline{\mathbf{Sp}}^G$ encodes the ∞-categories $\mathbf{Sp}^H$ for all subgroups $H < G$ and the restriction functors relating them. The G-symmetric monoidal structure on $\underline{\mathbf{Sp}}^G$ encodes smash products and Hill-Hopkins-Ravenel norms. Nardin gives an axiomatic characterization of this G-symmetric monoidal, see [Nar17, cor. 3.28]. This characterization allows us to work with the Hill-Hopkins-Ravenel norms at a formal level, avoiding the original point set definition of [HHR16].

$\mathbb{E}_V$-algebras and V-framed disks. Genuine equivariant factorization homology is an invariant of a geometric input, a V-framed G-manifold (described above), and of an algebraic input, an $\mathbb{E}_V$-algebra. We now briefly describe this algebraic structure.

Conceptually, factorization homology is constructed by gluing local data, given by a coefficient system. Such a coefficient system is an algebraic structure indexed by the local geometry of manifolds: an n-disk algebra in the case of factorization homology of n-dimensional manifolds and an $\mathbb{E}_n$-algebra in the case of factorization homology of framed n-dimensional manifolds.

Similarly, the structure of an $\mathbb{E}_V$-algebra is determined by the local structure of V-framed G-manifolds. Let M be a V-framed G-manifold and $x \in M$ a point with stabilizer $H < G$, then H acts linearly on the tangent space T_xM, and the H-representation T_xM is isomorphic to V (with the action restricted to $H < G$). [5] It follows that $x \in M$ has an H-equivariant

[3] Here we consider V as an H-representation by restricting the G-action to $H < G$.

[4] In particular, $\int_{-} A$ defines a natural transformation between two functors from $\mathcal{O}_G^{op}$ to symmetric monoidal ∞-categories. However, such natural transformation does not capture the compatibility of norms with topological inductions.

[5] To see this, pull the V-framing $TM \cong M \times V$ along $\{x\} \to M$.

neighborhood isomorphic to an open disk in V. [6] Therefore the orbit of x (considered as a G-submanifold of codimension 0) has a G-tubular neighborhood isomorphic to the topological induction $\coprod_{G/H} V = G \times_H V$.

Let $\mathcal{D}_V$ be the G-operad of little V-disks, and $\mathbb{E}_V$ its genuine operadic nerve (see [Bon19]). We define $\mathbb{E}_V$-algebras in $\underline{\mathbf{Sp}}^G$ as maps of G-∞-operads

$$\mathbb{E}_V \to \underline{\mathbf{Sp}}^G.$$

Informally, an $\mathbb{E}_V$-algebra A in $\underline{\mathbf{Sp}}^G$ assigns to V a genuine G-spectrum A (the "underlying G-spectrum" of A). The algebraic structure on A is indexed by H-equivariant embeddings[7] for $H < G$. To an H-embedding $V \sqcup V \hookrightarrow V$ the algebra A assigns a map of genuine H-spectra $A \otimes A \to A$ (a "multiplication map"), and to a H-embedding $\sqcup_{H/K} V \hookrightarrow V$ the algebra A assigns a map $N_K^H(A) \to A$ (a "multiplicative norm map") from the Hill-Hopkins-Ravenel norm of A. All of these maps are coherently compatible with smash products, restrictions of the group action and with each other. We use G-∞-category theory, and specifically G-symmetric monoidal structures, to handle these coherent compatibilities.

The G-∞-operad $\mathbb{E}_V$ is closely related to $\underline{\mathbf{Mfld}}^{G,V-fr}$, as we now explain. Let $\underline{\mathbf{Disk}}^{G,V-fr}$ be the full G-∞-subcategory of $\underline{\mathbf{Mfld}}^{G,V-fr}$ generated from the G-manifold V by restricting the group action, disjoint unions and topological induction (see section 3 for details). By construction, the G-symmetric monoidal structure of $\underline{\mathbf{Mfld}}^{G,V-fr}$ induces a G-symmetric monoidal structure on $\underline{\mathbf{Disk}}^{G,V-fr}$. In section 3.9 we show that $\underline{\mathbf{Disk}}^{G,V-fr}$ is equivalent to the G-symmetric monoidal envelope of $\mathbb{E}_V$. In particular, an $\mathbb{E}_V$-algebra in $\underline{\mathbf{Sp}}^G$ corresponds to an essentially unique G-symmetric monoidal functor

$$\underline{\mathbf{Disk}}^{G,V-fr} \to \underline{\mathbf{Sp}}^G.$$

We call such functors *V-framed G-disk algebras* in $\underline{\mathbf{Sp}}^G$.

Genuine equivariant factorization homology We encode the functors $\int_{\_} A \colon \mathbf{Mfld}^{H,V-fr} \to \mathbf{Sp}^H$ as a single G-symmetric monoidal G-functor $\underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Sp}}^G$ from the G-∞-category of V-framed G-manifolds to the G-∞-category of genuine G-spectra. Given an $\mathbb{E}_V$-algebra A in $\underline{\mathbf{Sp}}^G$, let $A \colon \underline{\mathbf{Disk}}^{G,V-fr} \to \underline{\mathbf{Sp}}^G$ denote the corresponding V-framed G-disk algebra. We construct genuine G-factorization homology

$$\int_{\_} A \colon \underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Sp}}^G$$

as the G-left Kan extension of A along the inclusion $\underline{\mathbf{Disk}}^{G,V-fr} \hookrightarrow \underline{\mathbf{Mfld}}^{G,V-fr}$. By work of Shah [Sha18] the genuine G-spectrum $\int_M A$ has an explicit description as a G-colimit indexed by "little disks in M", see definition 4.1.2 and proposition 4.1.4. This construction is indeed a homology theory of G-manifolds, as it extends to a G-symmetric monoidal functor satisfying G-$\otimes$-excision.

Genuine equivariant factorization homology satisfies the following extension of the Ayala-Francis axiomatic characterization.

[6] Choose a G-equivariant Riemannian metric on M use the fact that the exponential map $T_x M \dashrightarrow M$ is H-invariant.

[7] compatible with the G-framing

Theorem 1.0.2. *Let $\mathcal{H}(\underline{\mathbf{Mfld}}^{G,V-fr},\underline{\mathbf{Sp}}^G) \subset \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Mfld}}^{G,V-fr},\underline{\mathbf{Sp}}^G)$ be the full subcategory of the ∞-category of G-symmetric monoidal G-functors $\underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Sp}}^G$ which satisfy G-$\otimes$-excision and respect sequential unions. Then there is an equivalence of ∞-categories*

$$\int : Alg_{\mathbb{E}_V}(\underline{\mathbf{Sp}}^G) \xrightarrow{\sim} \mathcal{H}(\underline{\mathbf{Mfld}}^{G,V-fr},\underline{\mathbf{Sp}}^G), \quad A \mapsto (\int_{-} A \colon \underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Sp}}^G) \tag{3}$$

sending an $\mathbb{E}_V$-algebra A to G-equivariant factorization homology with coefficients in A.

The above result holds in greater generality. First, V-framed G-manifolds can be replaced with G-manifolds with more general equivariant tangential structures[8]. Second, the G-∞-category of genuine G-spectra can be replaced with any presentable G-symmetric monoidal G-∞-category $\underline{\mathcal{C}}$[9]. The general statement is given in theorem 6.0.2, which is the main result of this paper.

Applications As an application of theorem 6.0.2, we describe two variants of topological Hochschild homology using genuine G-factorization homology.

In section 7.1 we show that the real topological Hochschild homology spectrum of Hesselholt-Madsen [HM13] is equivalent to genuine C_2-factorization homology over S^1.

Proposition 1.0.3 (proposition 7.1.1)**.** *For A an $\mathbb{E}_\sigma$-algebra in $\underline{\mathbf{Sp}}^{C_2}$ there is an equivalence of genuine C_2-spectra*

$$\int_{S^1} A \simeq A \otimes_{N_e^{C_2} A} A.$$

where C_2 acts on S^1 by reflection.

By a theorem of ([DMPR17]) it follows that for A a flat ring spectrum with anti-involution there is as an equivalence of genuine C_2-spectra

$$\int_{S^1} A \simeq THR(A),$$

where $THR(A)$ is the real topological Hochschild homology of A, see remark 7.1.2.

In section 7.2 we show that the "twisted" topological Hochschild homology of a genuine C_n-ring spectrum of [ABG$^+$14, sec. 8] is equivalent to the geometric fixed points of C_n-factorization homology over S^1.

Proposition 1.0.4 (proposition 7.2.2)**.** *Let A be an $\mathbb{E}_1$-ring spectrum in $\mathbf{Sp}_{C_n}$, and $C_n \curvearrowright S^1$ be the standard action. Then there exists an equivalence of spectra*

$$\left(\int_{S^1} A\right)^{\Phi C_n} \simeq THH(A; A^\tau).$$

In particular, $THH(A; A^\tau)$ admits a natural circle action.

This circle action is equivalent to the circle action on the nerve of the "twisted cyclic bar construction" of [ABG$^+$14, sec. 8], which gives an alternative description of the relative norm of [ABG$^+$14, def. 8.2].

[8]This requires replacing V-framed G-disk algebras with more general G-disk algebras.

[9]Conjecturally, this condition can be weakened to distributivity of the tensor product over parametrized sifted colimits

Construction of V-framed G-disk algebras. Above we gave a rough description of a V-framed G-disk algebra as encoding multiplication maps, multiplicative norm maps and their coherent compatibilities. Unwinding these compatibilities implied by definition 3.6.11 is usually a non-trivial task (especially when $\dim V \geq 2$), and so it is inadvisable to construct a V-framed G-disk algebra by specifying multiplication maps, multiplicative norm maps and associated coherence data. It is therefore desirable to have some general mechanisms for constructing V-framed G-disk algebras.

For example, one would expect to be able to construct a V-framed G-disk algebra from an algebra over the G-operad $\mathcal{D}_V$. Such a construction would provide many examples of coefficients for genuine equivalent factorization homology of V-framed G-manifolds. More generally, it would be reassuring to have a "rectification" result showing that classical algebras over $\mathcal{D}_V$ form a model for the ∞-category of V-framed G-disk algebras, in the style of [PS14, thm. 7.10].

We leave such constructions for future work.

What about compact Lie groups? It is natural to want to extend genuine equivariant factorization homology from finite groups to compact Lie groups. There are two different points in which one encounters complications.

First, we prove theorem 6.0.2 (the axiomatic characterization of genuine G-factorization homology) inductively using equivariant handle bundle decompositions (see [Was69]). We produce these decompositions using equivariant Morse theory, which is more complicated over a compact Lie group. Choosing an invariant Morse function gives rise to a handle bundle decomposition, where each handle bundle is an equivariant disk bundle over a critical orbit. However, for a compact Lie group of positive dimension these handle bundles can be non trivial, since critical orbits are submanifolds of possibly positive dimension.

Second, and more fundamental, is the lack of good G-∞-category theory for a compact Lie group G. The source of the problem is the lack of multiplicative norms for subgroups $H < G$ of non-finite index. In order to understand the significance of this fact for genuine G-factorization homology, consider $M = \mathbb{C}$ the complex plane with the standard action of the circle group $S^1 = \mathbb{C}^\times$. The unit circle is an S^1-orbit in $\mathbb{C}$, with S^1-tubular neighborhood given by the open annulus. The embedding of the open annulus in $\mathbb{C}$ should induce a "multiplication norm map" $\otimes_{S^1} A \to A$ of genuine S^1-spectra, where the tensor product is indexed over the free orbit S^1. However, we do not have a good definition for the domain of this map as a genuine S^1-spectrum.

Organization We start by reviewing some parts of parametrized ∞-category theory in section 2. We hope this short exposition will assist the reader unfamiliar with the theory of G-∞-categories.

In section 3 we construct the G-∞-categories of G-manifolds and G-disks with equivariant tangential structures, and their G-symmetric monoidal structure which encodes disjoint unions and topological induction. These constructions provide a bridge between the geometry of G-manifolds and parametrized ∞-category theory, and enables the construction of genuine G-factorization homology in section 4.2.

Our definition of equivariant tangential structures in section 3.3 uses an equivariant version of the tangent classifier of [AF15] which may be of independent interest, see section 3.2. While we focus on framed G-manifolds, our definition is flexible enough to consider more general tangential structures such as equivariant orientations, as well as allowing us to restrict our attention to manifolds with a free G-action.

We finish section 3 by studying some aspects of these constructions. In section 3.8 we study the relation between embedding spaces of G-disks and G-configuration spaces. In section 3.9 use the work of [Bon19] to show that the G-∞-operad encoding V-framed G-disk algebras is closely related to the G-operad of little disks in a representation V.

The technical results and constructions of section 3 provide a solid foundation for the use of abstract theory of parametrized ∞-categories in the following sections.

In section 4 we define framed G-disk algebras and construct G-factorization homology, first as a G-functor (by G-left Kan extension, see section 4.1) and then as a G-symmetric monoidal G-functor (section 4.2).

In section 5 we study the properties of G-factorization homology. In section 5.1 we define G-collar decompositions and construct an "inverse image" functor. We use these in section 5.2, where we define G-$\otimes$-excision for a general G-symmetric monoidal functor $\underline{\mathbf{Mfld}}^G \to \underline{\mathcal{C}}$, and show that G-factorization homology satisfies G-tensor excision. In section 5.3 we show that G-factorization homology respects sequential unions.

In section 6 we prove our main result, giving an axiomatic characterization of G-factorization homology using equivariant Morse theory.

In section 7 we describe real topological Hochschild homology using G-factorization homology (section 7.1), and the relative norm of a genuine C_n-ring spectrum as the geometric fixed points of G-factorization homology (section 7.2).

In appendix A we show how to model ∞-slice categories in the framework of topological categories. For the convenience of the reader we recall the definition of G-symmetric monoidal categories in appendix B. We collect some Some general statements about mapping spaces in over categories in appendix C.

Notation. In this work we use the quasi-categories as a model ∞-categories (with the exception of remark 2.1.4). We assume the reader is familiar with the theory of ∞-categories, as developed in [Lur09a] and [Lur]. Explicitly, an ∞-category is a simplicial set $\mathcal{C}$ satisfying the left lifting property with respect to inner horns: for every $0 < i < n$, any map $\Lambda^n_i \to \mathcal{C}$ admits an extension to $\Delta^n \to \mathcal{C}$.

All of the manifolds we consider are smooth and n-dimensional for a fixed $n \in \mathbb{N}$. We fix a finite group G, and only consider manifolds with actions of subgroups $H < G$.

We frequently construct ∞-categories from topological categories by taking their coherent nerve (which is called the topological nerve in [Lur09a, def. 1.1.5.5]). We emphasize that the coherent nerve of a topological category $\mathcal{C}$ is a two step construction. First, taking the singular nerve of each mapping space, produces a simplicial category $\mathbf{Sing}(\mathcal{C})$. Second, applying the simplicial nerve functor of [Lur09a, def. 1.1.5.5] to $\mathbf{Sing}(\mathcal{C})$ produces an ∞-category. We denote the resulting ∞-category by $\mathbf{N}(\mathcal{C})$.

We denote parametrized ∞-categories with an underline, for example $\underline{\mathcal{C}}$. In general, if $\underline{\mathcal{C}}$ is parametrized over an ∞-category S we refer to $\underline{\mathcal{C}}$ as an S-∞-category. We say that $\underline{\mathcal{C}}$ is a *G-category* (see definition 2.1.3) if it is parametrized over $\mathcal{O}_G^{op}$, where $\mathcal{O}_G$ is the orbit category of G. No other notion of G-categories is used; a G-category is by definition an $\mathcal{O}_G^{op}$-∞-category.

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2 Background on parametrized ∞-category theory

In this section we review parametrized ∞-category theory of Barwick, Dotto, Glassman, Nardin and Shah, developed in [BDG$^+$16b, Sha18, Nar16, Nar17, BDG$^+$]. We recall the notions of G-∞-category theory employed below and fix our notation. We restrict our discussion to the case of G-∞-categories, though nothing substantial would change when working over an arbitrary indexing category.

This section contains no original results, all the results of this section are entirely due to Barwick, Dotto, Glassman, Nardin and Shah.

2.1 From Elmendorf-McClure's theorem to G-categories

A good starting point to a discussion of G-categories is the Elmendorf-McClure theorem, which recasts the equivariant homotopy theory of G-spaces as a presheaf category. Throughout we fix a finite group G.

Definition 2.1.1. *The orbit category $\mathcal{O}_G$ is the full subcategory of G-sets supported by transitive G-sets.*

Note that every orbit in $\mathcal{O}_G$ is isomorphic to a quotient of G by some subgroup $H < G$. This isomorphism depends only on the choice of a base point of the orbit, with H the stabilizer of the chosen basepoint. We denote the objects of $\mathcal{O}_G$ either by O by G/H. Despite the suggestive notation, we try to refrain from a choice of basepoint when possible.

Define the ∞-category of G-spaces $\mathbf{Top}^G$ as the coherent nerve of the topological category of G-CW spaces and G-maps.

Theorem 2.1.2 (Elmendorf-McClure, [Elm83])**.** *There is an equivalence of ∞-categories*

$$\mathbf{Top}^G \xrightarrow{\sim} \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S}),$$

sending a G-space X to its diagram of fixed points, $G/H \mapsto X^H$.

Using straightening/unstraightening ([Lur09a, thm. 2.2.1.2]) we get a third description of a G-space X as the left fibration over $\mathcal{O}_G^{op}$ classifying the diagram of fixed points of X.

Definition 2.1.3. *A G-∞-category is a coCartesian fibration $\underline{\mathcal{C}} \twoheadrightarrow \mathcal{O}_G^{op}$.*

For sake of readability we refer to G-∞-categories simply as G-categories. Other notions of G-categories present in the literature are not present in this paper.

A G-category $\underline{\mathcal{C}}$ is classified by a diagram of ∞-categories $\underline{\mathcal{C}}_\bullet \colon \mathcal{O}_G^{op} \to \mathcal{C}\mathrm{at}_\infty$ sending $G/H \in \mathcal{O}_G^{op}$ to the fiber $\underline{\mathcal{C}}_{[G/H]}$ of $\underline{\mathcal{C}} \twoheadrightarrow \mathcal{O}_G^{op}$ over G/H. We systematically use the subscript-square bracket notation $\mathcal{C}_{[G/H]}$ for the fiber ∞-category in order to avoid confusion with other subscript notations. As above, straightening/unstraightening ([Lur09a, sec. 3.2]) ensures that this is an equivalent description of the G-∞-category $\underline{\mathcal{C}}$.

Remark 2.1.4. Describing $\mathcal{C}\mathrm{at}_\infty$ as a complete Segal object in the ∞-category of spaces, we can use the Elmendorf-McClure theorem to get a third equivalent description of a G-category as a complete Segal object in $\mathbf{Top}^G$. This follows from following the Segal conditions and completeness conditions along the equivalences

$$\mathrm{Fun}(\Delta^{op}, \mathbf{Top}^G) \simeq \mathrm{Fun}(\Delta^{op}, \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Fun}(\Delta^{op}, \mathcal{S})),$$

where the first equivalence is induced by the Elmendorf-McClure theorem.

In particular, categories internal to G-spaces (and to G-sets) are examples of G-categories. Note that [GM17] defines G-categories as categories internal to G-spaces, making them examples of G-(∞-)categories in the sense of [BDG$^+$16b], used here.

While these equivalent descriptions of a G-category are good to have in mind, we stick to the definition of a G-category as a coCartesian fibration for its explicit nature.

When we need more general parametrized ∞-categories we use the following definition (and the notation of [Sha18]).

Definition 2.1.5. *Let S be an ∞-category. An S-∞-category is a coCartesian fibration $\underline{\mathcal{C}} \twoheadrightarrow S$. We denote the fiber of $\underline{\mathcal{C}}$ over $s \in S$ by $\underline{\mathcal{C}}_{[s]}$.*

We refer to S-∞-categories as S-categories.

Remark 2.1.6. Most results recalled in this section hold for general S-categories. One notable exception is the description of $\underline{\mathbf{Sp}}^S$, the S-stabilization of the S-category of S-spaces, using spectral Mackey functors. Another exception is the uniqueness of S-symmetric monoidal structure on $\underline{\mathbf{Sp}}^S$. However, these results hold under mild conditions on S. [10]

Handling H-categories as G-categories Occasionally we have to consider H-categories for some subgroup $H < G$. When doing so we use the slice category $\underline{G/H} := (\mathcal{O}_G^{op})_{(G/H)/}$, the opposite of the category of G-orbits over G/H. The category $\underline{G/H}$ is equivalent to the category of H-orbits. Moreover, the forgetful functor $\underline{G/H} \to \mathcal{O}_G^{op}$ is left fibration classified by the representable functor $\mathrm{Map}(-, G/H) \colon \mathcal{O}_G^{op} \to \mathcal{S}$. In particular a $\underline{G/H}$-category $\underline{\mathcal{C}} \twoheadrightarrow \underline{G/H}$ is a G-category by postcomposition with the forgetful functor, $\underline{G/H} \twoheadrightarrow \underline{\mathcal{O}_G^{op}}$. Note that this construction also avoid a choice of basepoint to G/H. When referring to the fibers of $\underline{\mathcal{C}} \twoheadrightarrow \underline{G/H}$ we adopt the notation $\underline{\mathcal{C}}_{[\varphi]}$ for the fiber over $\varphi \colon G/K \to G/H$ as an object in the slice category $\underline{G/H}$.

The G-category $\underline{G/H}$ has a second role for us, since a G-functor $\underline{G/H} \to \underline{\mathcal{C}}$ corresponds to an object in the fiber of $\underline{\mathcal{C}} \twoheadrightarrow \mathcal{O}_G^{op}$ over G/H, as we now explain. Under straightening/unstraightening the left fibration $\underline{G/H} \twoheadrightarrow \mathcal{O}_G^{op}$ corresponds to the representable functor of the orbit G/H, given by $\mathrm{Hom}_{\mathcal{O}_G}(-, G/H) \colon \mathcal{O}_G^{op} \to \mathcal{S}$, and therefore by the Yoneda lemma ([Lur09a, lem. 5.1.5.2]) corresponds to an object of $\underline{\mathcal{C}}_{[G/H]}$ [11]. We denote the G-functor corresponding to $x \in \underline{\mathcal{C}}_{[G/H]}$ by $\sigma_x : \underline{G/H} \to \underline{\mathcal{C}}$. A more explicit construction is given by choosing a section of the trivial fibration $\underline{\mathbf{Arr}}^{\mathrm{coCart}}_{x\to}(\underline{\mathcal{C}}) \overset{\sim}{\twoheadrightarrow} \underline{G/H}$ of [Sha18, not. 2.28] and composing with $ev_1 \colon \underline{\mathbf{Arr}}^{\mathrm{coCart}}_{x\to}(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$.

The G-category of G-spaces The ∞-categories $\mathrm{Fun}(\underline{G/H}, \mathcal{S})$ assemble as the fibers of a G-category $\underline{\mathbf{Top}}^G$, the G-category of G-spaces ([BDG$^+$16b, ex. 7.5]). By the Elmendorf-McClure theorem the fiber over G/H is equivalent to $\underline{\mathbf{Top}}^G_{[G/H]} \cong \mathrm{Fun}(\underline{G/H}, \mathcal{S}) \simeq \mathrm{Fun}(\mathcal{O}_H^{op}, \mathcal{S}) \simeq \mathbf{Top}^H$, the ∞-category of H-spaces. By an H-space we always mean an H-CW space. The G-category of G-spaces is characterized by the following universal property (see [BDG$^+$16b, thm. 7.8]).

For any G-category $\underline{\mathcal{C}}$ we have an equivalence of ∞-categories $\mathrm{Fun}_G(\underline{\mathcal{C}}, \underline{\mathbf{Top}}^G) \simeq \mathrm{Fun}(\underline{\mathcal{C}}, \mathcal{S})$, i.e $\underline{\mathbf{Top}}^G$ is the cofree G-category co-generated by the ∞-category of spaces.

Taking our cue from the Elmendorf-McClure theorem, we think of a G-category as capturing the notion of a G-action on an ∞-category. With this intuition in mind one may think of $\underline{\mathbf{Top}}^G$

[10] Specifically, they hold for S an atomic orbital ∞-category. See [BDG$^+$16a] for examples and [BDG$^+$16b], [Sha18], [Nar17] for the general theory.

[11] To make this argument precise we need to replace $\underline{\mathcal{C}}$ with a presheaf of spaces. To achieve that we straighten $\underline{\mathcal{C}}^{\simeq} \subseteq \underline{\mathcal{C}}$, the maximal G-subgroupoid of $\underline{\mathcal{C}}$, given as a left fibration by the full maximal sub-simplicial set supported on the coCartesian edges of $\underline{\mathcal{C}}$.

as follows. Imagine that the ∞-category of spaces admits a non-trivial G-action, whose H-fixed points is the ∞-category of H-spaces for all $H < G$. Think of $\underline{\mathbf{Top}}^G$ as capturing this imagined G-action.

Remark 2.1.7. In section 3.2 we use the following explicit model for $\underline{\mathbf{Top}}^G$. Construct an auxiliary *topological* category $\mathcal{O}_G$-**Top** as follows. An object of $\mathcal{O}_G$-**Top** is G-map $X \to O$ where the domain X is a G-CW complex and codomain $O \in \mathcal{O}_G$ is a G-orbit. We refer to an object of $\mathcal{O}_G$-**Top** as $\mathcal{O}_G$-space, though it should rightfully be called a "G-space over an orbit". A map of $\mathcal{O}_G$-spaces is given by a (strictly) commuting squares of G-spaces

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ O_1 & \longrightarrow & O_2. \end{array} \tag{4}$$

The mapping spaces of $\mathcal{O}_G$-**Top** are given by

$$\mathrm{Map}_{\mathcal{O}_G\text{-}\mathbf{Top}}(X_1 \to O_1, X_2 \to O_2) = \mathrm{Map}_G(X_1, X_2) \times_{\mathrm{Map}_G(X_1, O_2)} \mathrm{Map}_G(O_1, O_2),$$

where $\mathrm{Map}_G(X, Y)$ is the space of G-maps $X \to Y$ with the compact-open topology.

We think of an $\mathcal{O}_G$-space $X \to G/H$ as representing the H-space given by the fiber $X|_H$ of $X \to G/H$ over the coset H. On the other hand, given an H-space X_0 we can use topological induction to construct a $\mathcal{O}_G$-space $G \times_H X_0$ whose fiber over H is X_0. Note that the $\mathcal{O}_G$-space $X \to O$ *does not* represent the G-space X (in fact, choosing an isomorphism $O \cong G/H$ for some $H < G$ exhibits the G-space X as the topological induction of the H-space represented by $X \to G/H$).

Applying topological nerve construction of [Lur09a, def. 1.1.5.5] produces an ∞-category $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Top})$. The forgetful $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Top}) \to \mathcal{O}_G$, $(X \to O) \mapsto O$ is a Cartesian fibration, and a commuting square (4) describes a coCartesian edge in $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Top})$ if it is a pullback square. To see this use [Lur09a, prop. 2.4.1.1 (2)] as in the proof of proposition 3.1.14. The dual coCartesian fibration $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Top})^\wedge \to \mathcal{O}_G^{op}$, described in [BGN14], is a G-category equivalent to $\underline{\mathbf{Top}}^G$. We can explicitly describe an object of $\underline{\mathbf{Top}}^G_{[G/H]}$ in this model as a G-map $X \to G/H$, which we interpret as the H-space $X|_{eH}$ given by the fiber over the coset eH. A map in $\underline{\mathbf{Top}}^G$ is given by a (strictly) commutative diagram of G-spaces

$$\begin{array}{ccccc} X_1 & \longleftarrow & X' & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ O_1 & \longleftarrow & O_2 & \xrightarrow{=} & O_2 \end{array}$$

in which the left square is a pullback square. It is a coCartesian edge if and only of the G-map $X' \to Y$ is a G-homotopy equivalence over O_2 (see [BGN14]). Equivalently, if $O_2 = G/H$ then the above edge is coCartesian precisely when the map of fibers $X'|_{eH} \to Y|_{eH}$ is an H-homotopy equivalence.

By definition maps in fiber $\underline{\mathbf{Top}}^G_{[O]}$ are commutative diagrams as above, with row given by $O \xleftarrow{=} O \xrightarrow{=} O$. Unwinding the definitions we see that $\underline{\mathbf{Top}}^G_{[O]}$ is equivalent to $\mathbf{N}(\mathbf{Top}^G_{/O})$, the coherent nerve of the topological category of G-CW-spaces over O. If $O = G/H$ then restriction to the fiber over eH defines an equivalence of topological categories $\mathbf{Top}^G_{/G/H} \xrightarrow{\sim} \mathbf{Top}^H$ to the topological category of H-CW-spaces.

Finally, we note that $\mathbf{N}(\mathbf{Top}^G_{/O}) \simeq \mathbf{N}(\mathbf{Top}^G)_{/O}$ are equivalent ∞-categories . We use the Moore over-category of appendix A to see this. By corollary A.0.5 we have $\mathbf{N}(\mathbf{Top}^G)_{/O} \simeq \mathbf{N}\left((\mathbf{Top}^G)^{Moore}_{/O}\right)$. However, since the orbit O is a discrete G-space we see that for every $X \in \mathbf{Top}^G$ the only Moore paths in $\mathrm{Map}_{\mathbf{Top}^G}(X,O)$ are constant, so $\mathbf{Top}^G_{/O} \to (\mathbf{Top}^G)^{Moore}_{/O}$ is an equivalence of topological categories. Therefore the fiber $\underline{\mathbf{Top}}^G_{[O]}$ is equivalent to the slice category $\mathbf{N}(\mathbf{Top}^G)_{/O}$. The mapping spaces of $\underline{\mathbf{Top}}^G_{[O]} \simeq \mathbf{N}(\mathbf{Top}^G_{/O})$ will be denoted by $\mathrm{Map}^G_O(X,Y)$.

The G-category of G-spectra A more interesting example is given by $\underline{\mathbf{Sp}}^G$, the G-category of G-spectra, with fiber over G/H is equivalent to $\underline{\mathbf{Sp}}^G_{[G/H]} \simeq \mathbf{Sp}^H$, the ∞-category genuine orthogonal H-spectra (see [Nar17, thm. 2.40], with origins in [GM11]). For a construction of $\underline{\mathbf{Sp}}^G$ as the G-stabilization of $\underline{\mathbf{Top}}^G$ see [Nar17, def. 2.35 and thm. 2.36].

2.2 Constructing G-categories

We frequently use the following constructions of G-categories.

Construction 2.2.1. Given two S-categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$, the fiber product $\underline{\mathcal{C}} \times_S \underline{\mathcal{D}}$ is an S-category, the fiberwise product of $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$. If $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ are G-categories, we denote the fiberwise product $\underline{\mathcal{C}} \times_{\mathcal{O}_G^{op}} \underline{\mathcal{D}}$ by $\underline{\mathcal{C}} \underline{\times} \underline{\mathcal{D}}$. In particular, we use the fiberwise product to restrict a G-category $\underline{\mathcal{C}} \twoheadrightarrow \mathcal{O}_G^{op}$ to a $\underline{G/H}$-category $\underline{\mathcal{C}} \underline{\times} \underline{G/H} \twoheadrightarrow \underline{G/H}$ ("forgetting the G-action on $\mathcal{C}$ to get an H-action").

Construction 2.2.2. Given a G-category $\underline{\mathcal{C}}$ define the *fiberwise arrow category* $\underline{\mathbf{Arr}}_G(\underline{\mathcal{C}})$ as the fiber product $\mathcal{O}_G^{op} \times_{\mathrm{Fun}(\Delta^1, \mathcal{O}_G^{op})} \mathrm{Fun}(\Delta^1, \underline{\mathcal{C}})$ (see [Sha18, not. 4.29]). Note that $\underline{\mathbf{Arr}}_G(\underline{\mathcal{C}})$ is equivalent to the functor G-category $\underline{\mathrm{Fun}}_G(\mathcal{O}_G^{op} \times \Delta^1, \underline{\mathcal{C}})$, where the G-category $\mathcal{O}_G^{op} \times \Delta^1$ is the constant G-category on Δ^1. More generally, for any S-category $\underline{\mathcal{C}} \twoheadrightarrow S$ define the fiberwise arrow S-category $\underline{\mathbf{Arr}}_S(\underline{\mathcal{C}})$ as the fiber product $S \times_{\mathrm{Fun}(\Delta^1, S)} \mathrm{Fun}(\Delta^1, \underline{\mathcal{C}})$.

Construction 2.2.3. Let $\underline{\mathcal{C}}$ be a G-category and $x \in \underline{\mathcal{C}}_{[G/H]}$ an object over G/H, corresponding to the G-functor $\sigma_x \colon \underline{G/H} \to \underline{\mathcal{C}}$. Following [Sha18, not. 4.29], we define the *parametrized slice-category* $\underline{\mathcal{C}}_{/\underline{x}} \twoheadrightarrow \underline{G/H}$ by pulling back the coCartsian fibration $ev_1 \colon \underline{\mathbf{Arr}}_G(\underline{\mathcal{C}}) \twoheadrightarrow \underline{\mathcal{C}}$ along σ_x, i.e. $\underline{\mathcal{C}}_{/\underline{x}} := \underline{\mathbf{Arr}}_G(\underline{\mathcal{C}}) \times_{\underline{\mathcal{C}}} \underline{G/H}$. We will also consider $\underline{\mathcal{C}}_{/\underline{x}} \twoheadrightarrow \underline{G/H}$ as a $\underline{G/H}$-category.

Note that the fiber of $\mathcal{C}_{/\underline{x}} \twoheadrightarrow \underline{G/H}$ over $\varphi \colon G/K \to \underline{G/H}$ is equivalent to the ∞-over-category $(\mathcal{C}_{[G/K]})_{/\varphi^* x}$, where $\varphi^* x \in \mathcal{C}_{[G/K]}$ is determined by choosing a coCartesian lift $x \to \varphi^* x$ of φ.

Construction 2.2.4. For $\underline{\mathcal{C}} \twoheadrightarrow S$ an S-category, the fiberwise cone S-category of $\underline{\mathcal{C}}$ is defined as the parametrized join $\underline{\mathcal{C}} \star_S S$ (see [Sha18, not. 4.2] or appendix B).

Parametrized functors and parametrized functor categories

Definition 2.2.5. *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be S-categories, i.e. coCartesian fibrations $\underline{\mathcal{C}} \twoheadrightarrow S, \underline{\mathcal{D}} \twoheadrightarrow S$. An S-functor is a functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ over S which preserves coCartesian edges. Let* $\mathrm{Fun}_S(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \mathrm{Fun}_{/S}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ *be the full subcategory of functors $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ over S which preserve coCartesian edges. When $S = \mathcal{O}_G^{op}$ we refer to a $\mathcal{O}_G^{op}$-functor as a G-functor, and denote the ∞-category of G-functors by* $\mathrm{Fun}_G(\underline{\mathcal{C}}, \underline{\mathcal{D}})$.

Remark 2.2.6. An S-functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ encodes the data of a coherent natural transformation $\underline{\mathcal{C}}_\bullet \Rightarrow \underline{\mathcal{D}}_\bullet$ between the S-diagrams $\underline{\mathcal{C}}_\bullet, \underline{\mathcal{D}}_\bullet \colon S \to \mathcal{C}\mathrm{at}_\infty$ classified by the coCartesian fibrations $\underline{\mathcal{C}} \twoheadrightarrow S$ and $\underline{\mathcal{D}} \twoheadrightarrow S$.

Remark 2.2.7. Since the left fibration $\underline{G/H} \to \mathcal{O}_G^{op}$ is corepresentable by construction, we have $\underline{\mathcal{C}}_{[G/H]} \simeq \mathrm{Fun}_G(\underline{G/H}, \underline{\mathcal{C}})$.

The ∞-category of G-categories admits an internal hom, a G-category denoted $\underline{\mathrm{Fun}}_G(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ see [BDG$^+$16b, thm. 9.7] and [BDG$^+$16b, def. 9.2] for an explicit construction. The fiber of $\underline{\mathrm{Fun}}_G(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \twoheadrightarrow \mathcal{O}_G^{op}$ over G/H admits the following description. Forget the G-action on $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ to an H-action by taking the fiber products $\underline{\mathcal{C}} \underline{\times} \underline{G/H}$, $\underline{\mathcal{D}} \underline{\times} \underline{G/H}$. The fiber $\underline{\mathrm{Fun}}_G(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{[G/H]}$ is equivalent to the ∞-category $\mathrm{Fun}_{\underline{G/H}}(\underline{\mathcal{C}} \underline{\times} \underline{G/H}, \underline{\mathcal{D}} \underline{\times} \underline{G/H})$ of $\underline{G/H}$-functors $\underline{\mathcal{C}} \underline{\times} \underline{G/H} \to \underline{\mathcal{D}} \underline{\times} \underline{G/H}$, (which we think of as modeling "H-equivariant functors from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$").

More generally, for any two S-categories $\underline{\mathcal{C}} \twoheadrightarrow S, \underline{\mathcal{D}} \twoheadrightarrow S$ there is an S-category of functors $\underline{\mathrm{Fun}}_S(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ with fibers $\underline{\mathrm{Fun}}_S(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{[s]} \simeq \mathrm{Fun}_{\underline{s}}(\underline{\mathcal{C}} \times_S \underline{s}, \underline{\mathcal{D}} \times_S \underline{s})$ where $\underline{s} = S_{s/}$. The S-category of functors possesses the universal property of internal hom, from [BDG$^+$16b, thm. 9.7].

Theorem 2.2.8 (Barwick-Dotto-Glasman-Nardin-Shah)**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{\mathcal{E}}$ be S-categories. Then there are natural equivalences*

$$\underline{\mathrm{Fun}}_S(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}_S(\underline{\mathcal{D}}, \underline{\mathcal{E}})) \xrightarrow{\sim} \underline{\mathrm{Fun}}_S(\underline{\mathcal{C}} \times_S \underline{\mathcal{D}}, \underline{\mathcal{E}}), \quad \mathrm{Fun}_S(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}_S(\underline{\mathcal{D}}, \underline{\mathcal{E}})) \xrightarrow{\sim} \mathrm{Fun}_S(\underline{\mathcal{C}} \times_S \underline{\mathcal{D}}, \underline{\mathcal{E}}).$$

Note that if $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{\mathcal{E}}$ are G-categories, then the second equivalence follows from the first by restricting to the fiber over the orbit $[G/G]$, the terminal object of $\mathcal{O}_G$.

2.3 Parametrized adjoints, colimits, left Kan extensions

We follow [Nar17], defining parametrized colimits and parametrized left Kan extensions using parametrized adjoints.

Parametrized adjoints Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be S-categories. An S-adjunction ([Sha18, def. 8.1]) is a relative adjunction $L \colon \underline{\mathcal{C}} \leftrightarrows \underline{\mathcal{D}} : R$ over S ([Lur, def. 7.3.2]) where both L and R are S-functors. In particular, for each $s \in S$ we have an adjunction $L_{[s]} \colon \underline{\mathcal{C}}_{[s]} \leftrightarrows \underline{\mathcal{D}}_{[s]} : R_{[s]}$ between the fibers over s. When $S = \mathcal{O}_G^{op}$ we will refer to an $\mathcal{O}_G^{op}$-adjunction as a G-adjunction.

Parametrized colimits Let $p \colon \underline{I} \to \underline{\mathcal{C}}$ be an S-functor, which we think of as an S-diagram in $\underline{\mathcal{C}}$. The S-colimit of p is an S-object of $\underline{\mathcal{C}}$, i.e a coCartesian section $S - \underrightarrow{colim}(p) \colon S \to \underline{\mathcal{C}}$ of the structure fibration $\underline{\mathcal{C}} \twoheadrightarrow S$. For a general definition of $\underrightarrow{colim}(p)$ as the S-initial S-cone under p see [Sha18, def. 5.2]. We define $\underline{I}$-shaped S-colimits as the S-left adjoint to the "constant $\underline{I}$-diagram" S-functor, following [Nar16, def. 2.1]. This definition is justified by [Sha18, 10.4], since we only take S-colimits in S-cocomplete S-categories.

Explicitly, precomposition with the coCartesian fibration $\underline{I} \twoheadrightarrow S$ induces an S-functor $\Delta_I : \underline{\mathcal{C}} \simeq \underline{\mathrm{Fun}}_S(\underline{S}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}_S(\underline{I}, \underline{\mathcal{C}})$, where $\underline{S}$ is the terminal S-category (given by $id \colon S \to S$). If $\Delta_{\underline{I}}$ admits an S-left adjoint we say that $\underline{\mathcal{C}}$ admits $\underline{I}$-indexed S-colimits, and denote the S-left adjoint by $S - \underrightarrow{colim} \colon \underline{\mathrm{Fun}}_S(\underline{I}, \underline{\mathcal{C}}) \to \underline{\mathcal{C}}$. Note that for every index $s \in S$ we have an adjunction of ∞-categories

$$S - \underrightarrow{colim} \colon \mathrm{Fun}_{\underline{s}}(\underline{I} \times_S \underline{s}, \underline{\mathcal{C}} \times_S \underline{s}) \leftrightarrows \mathrm{Fun}_{\underline{s}}(\underline{S} \times_S \underline{s}, \underline{\mathcal{C}} \times_S \underline{s}) \simeq \underline{\mathcal{C}}_{[s]} : \Delta_{\underline{I}}.$$

Particularly, we will use the following type of $\underline{G/H}$-colimit.

Example 2.3.1. Let $\underline{\mathcal{C}}$ be a G-category, $\underline{I} \twoheadrightarrow \underline{G/H}$ a $\underline{G/H}$-category and $p \colon \underline{I} \to \underline{\mathcal{C}}$ a G-functor. Since $\underline{G/H} \twoheadrightarrow \mathcal{O}_G^{op}$ is a left fibration we have $\mathrm{Fun}_G(\underline{I}, \underline{\mathcal{C}}) \simeq \mathrm{Fun}_{\underline{G/H}}(\underline{I}, \underline{\mathcal{C}} \underline{\times} \underline{G/H})$, under which

p corresponds to a $\underline{G/H}$-functor $p\colon \underline{I} \to \underline{\mathcal{C}}\underline{\times}\underline{G/H}$, or in other words $p \in \underline{\mathrm{Fun}}_{G/H}(\underline{I}, \underline{\mathcal{C}}\underline{\times}\underline{G/H})$. Then $\underline{G/H} - \underrightarrow{colim}(p) \in \underline{\mathcal{C}}_{[G/H]}$ is given by applying the left adjoint of

$$\underline{G/H} - \underrightarrow{colim}\colon \underline{\mathrm{Fun}}_{G/H}(\underline{I}, \underline{\mathcal{C}}\underline{\times}\underline{G/H}) \leftrightarrows \underline{\mathrm{Fun}}_{G/H}(\underline{G/H}, \underline{\mathcal{C}}\underline{\times}\underline{G/H}) \simeq \underline{\mathcal{C}}_{[G/H]} : \Delta_{\underline{I}}.$$

We say that an S-category $\underline{\mathcal{C}}$ is S-cocomplete if for every $s \in S$ the $\underline{s}$-category $\underline{\mathcal{C}}\underline{\times}\underline{s}$ admits $\underline{I}$-indexed $\underline{s}$-colimits for any $\underline{s}$-category $\underline{I}$.

Parametrized left Kan extensions We follow [Nar17, def. 2.12] and define S-left Kan extension using the give a global characterization as a left adjoint. For a general definition of pointwise parametrized left Kan extensions see [Sha18, def. 10.1], which satisfies the global characterization by [Sha18, 10.4]. We only use the pointwise definition in the proof of proposition 4.2.4, a G-categorical statement independent from the rest of the paper.

Let $\iota\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ be an S-functor and $\underline{\mathcal{C}}$ an S-category. Restriction along ι induces an S-functor $\iota^*\colon \underline{\mathrm{Fun}}_S(\underline{\mathcal{M}}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}_S(\underline{\mathcal{D}}, \underline{\mathcal{C}})$. The S-left Kan extension along ι is the S-left adjoint to ι^* and denoted by $\phi_!$.

We will use the following propositions from [Sha18].

Proposition 2.3.2. *[Sha18, thm. 10.3] Let $A\colon \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ and $\iota\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ be S-categories, and suppose that for every $x \in \underline{\mathcal{M}}$ over $s \in S$ the $\underline{s}$-colimit*

$$\underline{s} - \underrightarrow{colim}\left(\underline{\mathcal{D}}_{/\underline{x}} \to \underline{\mathcal{D}} \times_S \underline{s} \xrightarrow{A\times_S \underline{s}} \underline{\mathcal{C}} \times_S \underline{s}\right)$$

exists. Then the S-left Kan extension of A along ι exists (and is essentially unique), and acts on $x \in \underline{\mathcal{D}}$ by sending it to the $\underline{s}$-colimit above, considered as an object in the fiber $\underline{\mathcal{C}}_{[s]}$.

Proposition 2.3.3. *[Sha18, cor. 10.6] Let $\underline{\mathcal{C}}$ be a S-cocomplete S-category and $\iota\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ a fully faithful S-functor (i.e fiberwise fully faithful, see [BDG$^+$16b, def. 1.6]). Then the S-left Kan extension $\iota_!\colon \underline{\mathcal{M}} \to \underline{\mathcal{C}}$ exists and is S-fully faithful.*

When $S = \mathcal{O}_G^{op}$ we refer to S-left Kan extensions as G-left Kan extensions, which we use to define G-factorization homology as a G-functor (see proposition 4.1.4).

Parametrized Yoneda embedding Another useful tool available to us is the parametrized Yoneda embedding of [BDG$^+$16b, sec. 10], which we use in the construction of the G-tangent classifier (see construction 3.2.8). Let $\underline{\mathcal{C}}$ be a G-category, and $\underline{\mathcal{C}}^{vop}$ the fiberwise opposite G-category (with fibers $(\underline{\mathcal{C}}^{vop})_{[G/H]} \cong (\underline{\mathcal{C}}_{[G/H]})^{op}$, see [BDG$^+$16b, def. 3.1]). According to [BDG$^+$16b, def. 10.2] there exists a G-functor $j\colon \mathcal{C} \to \underline{\mathrm{Fun}}_G(\mathcal{C}^{vop}, \underline{\mathbf{Top}}^G)$, the parametrized Yoneda embedding, which can be informally described as follows. The G-functor j takes $x \in \underline{\mathcal{C}}_{[G/H]}$ the $\underline{G/H}$-functor $\underline{\mathrm{Map}}(-,x)\colon \underline{\mathcal{C}}^{vop}\underline{\times}\underline{G/H} \to \underline{\mathbf{Top}}^G\underline{\times}\underline{G/H}$ sending an object $y \in ((\underline{\mathcal{C}}^{vop})\underline{\times}\underline{G/H})_{[\varphi]} \cong (\underline{\mathcal{C}}_{[G/K]})^{op}$ in the fiber over $\varphi\colon G/K \to G/H$ to the mapping space $\mathrm{Map}(y, \varphi^* x)$ of the ∞-category $\underline{\mathcal{C}}_{[G/K]}$.

2.4 G-symmetric monoidal structures

The notion of a G-symmetric monoidal structure plays a central role in our presentation of G-factorization homology. In this subsection we give some intuition for G-symmetric monoidal structure, hopefully making it more approachable. This subsection is expository in nature, the formal definition of a G-symmetric monoidal G-category can be found in [Nar17, sec. 3.1], or in appendix B.

Informally, the data of a G-symmetric monoidal structure on a G-category $\underline{\mathcal{C}}$ is given by collection of symmetric monoidal structures on the fibers $\underline{\mathcal{C}}_{[G/H]}$, together with symmetric monoidal functors $\underline{\mathcal{C}}_{[G/K]} \to \underline{\mathcal{C}}_{[G/H]}$, called norm functors, for each map of orbits $G/K \to G/H$. We have the following examples in mind.

- The coCartesian G-symmetric monoidal structure on $\underline{\mathbf{Top}}^G$, which is given by disjoint unions in $\underline{\mathbf{Top}}^G_{[G/H]} \simeq \mathbf{Top}^H$ and norm functors

$$\forall K < H < G : \quad \coprod_{H/K} : \mathbf{Top}^K \to \mathbf{Top}^H, \quad \coprod_{H/K} X = H \times_K X,$$

where $H \times_K X$ is the quotient of $G \times X$ by the diagonal action of K.

- The Cartesian G-symmetric monoidal structure on $\underline{\mathbf{Top}}^G$, which is given by products of H-spaces and norm functors

$$\forall K < H < G : \quad \prod_{H/K} : \mathbf{Top}^K \to \mathbf{Top}^H, \quad \prod_{H/K} X = \mathrm{Map}_K(H, X),$$

where $\mathrm{Map}_K(H, X)$ is the space of K-equivariant maps $H \to X$ with K acting on H by multiplication from the right.

- The G-category $\underline{\mathbf{Sp}}^G$ of G-spectra has a G-symmetric monoidal which is given by smash products in $\underline{\mathbf{Sp}}^G_{[G/H]} \simeq \mathbf{Sp}^H$ and the Hill-Hopkins-Ravenel norm functors, informally given by taking $X \in Sp^H$ to the smash product of $|G/H|$ copies of X with induced G-action. Nardin gave a universal property characterizing this G-symmetric monoidal structure by proving that $\underline{\mathbf{Sp}}^G$ admits an essentially unique G-symmetric monoidal structure for which the sphere spectrum is the unit, see [Nar17, cor. 3.28].

The data of a G-symmetric monoidal structure, along with its coherent compatibility, is encoded by a single coCartesian fibration over the indexing category $\underline{\mathbf{Fin}}^G_*$, satisfying certain Segal conditions. In what follows, we try to explain how this technical description is related to the intuition presented above.

We regard the symmetric monoidal structure on each fiber and the norm functors on equal footing. To that end, consider the G-symmetric monoidal structure as acting on a U-family of objects, where we index our family be a finite G-set. The members of a U-family $x_\bullet$ in a G-category $\underline{\mathcal{C}}$ correspond to the orbits of U, with $x_W \in \underline{\mathcal{C}}_{[W]}$ for each orbit $W \in \mathrm{Orbit}(U)$. Given a G-map $I \colon U \to G/H$, we can use the G-symmetric monoidal structure to construct an element $\otimes_I x_\bullet \in \underline{\mathcal{C}}_{[G/H]}$. Using the operations $\otimes_I$ we can encapsulate the data G-symmetric monoidal structure on $\underline{\mathcal{C}}$.

The various operations $\otimes_I$ are subject to certain compatibility conditions, which hold upto coherent homotopy. In order to encapsulate the compatibility of $\otimes_I$ for various I it is convenient to extend $\otimes_I$ from $I \colon U \to G/H$ to general G-maps of finite G-sets $\varphi \colon U \to V$. The generalized operation $\otimes_\varphi$ takes a U-family to a V-family by acting on the fibers of φ, i.e

$$\forall W' \in \mathrm{Orbit}(V) : \quad (\otimes_\varphi x_\bullet)_{W'} = \otimes_{\varphi^{-1}(W')} \left(x_\bullet|_{\varphi^{-1}(W')} \right) \in \underline{\mathcal{C}}_{[W']}.$$

Note that we also need to keep track of restrictions taking a U-family $x_\bullet$ to a U'-family $x_\bullet|_{U'}$ for each inclusion of G-sets $U' \hookrightarrow U$.

All these operations are encoded by a coCartesian fibration over $\underline{\mathbf{Fin}}^G_*$, the G-category of finite pointed G-sets (see appendix B), which we think of as our indexing category. Note that the fiber of $\underline{\mathbf{Fin}}_*^G$ over G/H is the given by the category of spans of finite G-sets $U \hookleftarrow U' \to V$ over G/H, where the wrong way map $U \hookleftarrow U'$ is an inclusion. Restriction to the fiber over eH defines an equivalence $(\underline{\mathbf{Fin}}_*^G)_{[G/H]} \xrightarrow{\sim} \mathbf{Fin}_*^H$ to the category of finite pointed H-sets, described here as partly defined H-maps given by spans of finite H-sets $\tilde{U} \hookleftarrow \tilde{U}' \to \tilde{V}$, where the wrong way map is an inclusion of finite H-sets.

We end this subsection by briefly sketching how to extract the tensor products and norms from a coCartesian fibration $p\colon \underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ describing a G-symmetric monoidal structure on a G-category $\underline{\mathcal{C}}$.

First we describe the tensor product of two objects $x_1, x_2 \in \underline{\mathcal{C}}_{[G/H]}$. The ∞-category $\underline{\mathcal{C}}_{[G/H]}$ is given as the fiber of p over $G/H \xrightarrow{=} G/H$. Let $U = G/H \coprod G/H$ and $I \in \underline{\mathbf{Fin}}_*^G$ given by the fold map $I\colon U \to G/H$. By the Segal conditions we have an equivalence $\underline{\mathcal{C}}_I^{\otimes} \xrightarrow{\sim} \underline{\mathcal{C}}_{[G/H]} \times \underline{\mathcal{C}}_{[G/H]}$ from the fiber of p over I. Through this equivalence we identify the ordered pair $(x_1, x_2) \in \underline{\mathcal{C}}_{[G/H]} \times \underline{\mathcal{C}}_{[G/H]}$ with an object $x_\bullet \in \underline{\mathcal{C}}_I^{\otimes}$ (a U-family). Choose a p-coCartesian lift $x_\bullet \to y$ of the span $U \xleftarrow{=} U \xrightarrow{I} G/H$ over G/H. The tensor product $x_1 \otimes x_2$ is given by $y \in \underline{\mathcal{C}}_{[G/H]}$.

Next we describe the norm of an object $x \in \underline{\mathcal{C}}_{[G/K]}$ along $\varphi\colon G/K \to G/H$. As before, the ∞-category $\underline{\mathcal{C}}_{[G/K]}$ is the fiber of p over $G/K \xrightarrow{=} G/K$. Consider the map $\varphi\colon G/K \to G/H$ as an object of $\underline{\mathbf{Fin}}_*^G$. By the Segal conditions we have an equivalence $\underline{\mathcal{C}}_{\varphi}^{\otimes} \xrightarrow{\sim} \underline{\mathcal{C}}_{[G/K]}$ from the fiber of p over φ. Through this equivalence we identify $x \in \underline{\mathcal{C}}_{[G/K]} \times \underline{\mathcal{C}}_{[G/H]}$ with an object $x_\bullet \in \underline{\mathcal{C}}_{\varphi}^{\otimes}$ (a φ-family). Choose a p-coCartesian lift $x_\bullet \to y$ of the span $G/K \xleftarrow{=} G/K \xrightarrow{\varphi} G/H$ over G/H. The norm $\otimes_\varphi x$ is given by $y \in \underline{\mathcal{C}}_{[G/H]}$.

3 G-manifolds and G-disks

Genuine G-factorization homology will be constructed in section 4 using parametrized ∞-category theory. Our goal in this section is to construct and study the G-∞-categories needed there. Most of this section is devoted to the construction of these G-∞-categories and their G-symmetric monoidal structures. These constructions may be of independent interest, as they provide a bridge between geometry of manifolds with a finite group action and the theory of parametrized ∞-categories.

In section 3.1 we construct $\underline{\mathbf{Mfld}}^G$, the G-category of G-manifolds. The construction is inspired by the model of $\underline{\mathbf{Top}}^G$ described in remark 2.1.7. We then turn to study its relation to G-vector bundles, and construct an equivariant version of the tangent classifier functor of [AF15]. This G-tangent classifier is used in section 3.3 to construct framed variants of $\underline{\mathbf{Mfld}}^G$.

Next, we turn our attention to G-disjoint unions. In section 3.4 we define a G-symmetric monoidal structure on $\underline{\mathbf{Mfld}}^G$ encoding disjoint unions and topological inductions. The construction is quite explicit, and relies on the unfurling construction Barwick, introduced in [Bar14]. In section 3.4 we lift G-disjoint unions to a G-symmetric monoidal structures on the framed variants of $\underline{\mathbf{Mfld}}^G$. Our main tool will be the G-coCartesian structures constructed in [BDG+].

The G-symmetric monoidal structure of G-disjoint unions will be used in section 4 when defining factorization homology in two ways. First, the expected interaction of genuine G-factorization homology with disjoint unions and topological inductions is expressed by being a G-symmetric monoidal functor from $\underline{\mathbf{Mfld}}^G$. Second, the definition of G-disk algebras relies on the definition of the G-symmetric monoidal G-∞-category of G-disks, defined in section 3.6.

Next, we turn to study our constructions. In section 3.7 we show that G-disks are exactly the G-manifolds generated from linear representations of subgroups $H < G$ by taking disjoint unions and topological inductions. In section 3.8 we compare equivariant embeddings of G-disks with equivariant configurations spaces. The results of this comparison will be used in section 5.2

to show that genuine G-factorization homology satisfies $\otimes$-excision. In section 3.9 we define the G-∞-operad $\mathbb{E}_V$ of little V-disks, and use the results of section 3.8 to relate $\mathbb{E}_V$ to V-framed G-disks.

3.1 The G-category of G-manifolds

The goal of this subsection is to give an explicit model for the G-∞-category $\underline{\mathbf{Mfld}}^G$ of n-dimensional G-manifolds.

Before going into the details of the construction, let us first recall the construction of the ∞-category $\mathbf{Mfld}^G$ of G-manifolds, achieved by a standard procedure. Let M_1, M_2 be smooth n-dimensional manifolds equipped with a smooth action of a finite group G. The set $Emb^G(M_1, M_2)$ of smooth G-equivariant open embeddings $M_1 \hookrightarrow M_2$ comes with a natural topology, making the category $\mathbf{Mfld}^G$ of n-dimensional G-manifolds into a topological category. We consider $\mathbf{Mfld}^G$ as an ∞-category by taking its coherent nerve ([Lur09a, def. 1.1.5.5]).

We can extend the construction of $\mathbf{Mfld}^G$ to construct the G-∞-category $\underline{\mathbf{Mfld}}^G$ as follows. Consider the ∞-categories $\mathbf{Mfld}^H$ of n-dimensional H-manifolds and H-embeddings for all subgroups $H < G$. The ∞-categories $\mathbf{Mfld}^H$ form a diagram of ∞-categories, by related by two types of functors:

1. First, if M is a G-manifold and $H < G$ we can consider M as an H-manifold, which defines a functor of topological categories $\mathbf{Mfld}^G \to \mathbf{Mfld}^H$. Similarly we have $\mathbf{Mfld}^H \to \mathbf{Mfld}^K$ for $K < H < G$.

2. Second, suppose $K, H < G$ are conjugate subgroups, i.e $H = gKg^{-1}$ for some $g \in G$, and M is an H-manifold. We can consider M as a K-manifold by twisting the H-action by conjugation, defining an isomorphism of topological categories $conj_K^H \colon \mathbf{Mfld}^H \to \mathbf{Mfld}^K$.

A standard verification shows that the topological categories $\mathbf{Mfld}^H$ define a diagram of topological categories indexed by subgroups $H < G$, with functors indexed contravariantly by G-maps $G/K \to G/H$. Note that this indexing category is equivalent to the orbit category $\mathcal{O}_G$ (see definition 2.1.1). Composing with the topological nerve we get a diagram of ∞-categories

$$\mathbf{Mfld}^\bullet \colon \mathcal{O}_G^{op} \to \mathcal{C}\mathrm{at}_\infty, \quad G/H \mapsto \mathbf{N}(\mathbf{Mfld}^H),$$

which we can unstraighten to a coCartesian fibration $UnSt(\mathbf{Mfld}^\bullet) \twoheadrightarrow \mathcal{O}_G^{op}$ (see [Lur09a, sec. 3.2]). The casual reader can use $UnSt(\mathbf{Mfld}^\bullet)$ as the definition of the G-category of G-manifolds, and skip the rest of this subsection.

The construction of $UnSt(\mathbf{Mfld}^\bullet)$ is unsatisfying to us in two respects. First, it depends on an implicit choice of an inverse to the inclusion of the full subcategory $\{G/H\}_{H<G} \subset \mathcal{O}_G$ into the category of G-orbits (which is equivalent to choosing a basepoint for every transitive G-set). Second, manipulating $UnSt(\mathbf{Mfld}^\bullet)$ as a simplicial set is inconvenient, as unstraightening is a right adjoint functor. Instead of working with $UnSt(\mathbf{Mfld}^\bullet)$ we construct an equivalent G-∞-category $\underline{\mathbf{Mfld}}^G$ (definition 3.1.16) which admits a more accessible description as a simplicial set. This is the main construction of this subsection.

Let us briefly describe our strategy for constructing $\underline{\mathbf{Mfld}}^G$, inspired by the model of $\underline{\mathbf{Top}}^G$ described in remark 2.1.7. First we construct a topological category $\mathcal{O}_G$-$\mathbf{Mfld}$ equipped with functor to the orbit category, and show that the topological nerve defines a Cartesian fibration $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Mfld}) \to \mathcal{O}_G$ of simplicial sets, which classifies a diagram of ∞-categories equivalent to $\mathbf{N}(\mathbf{Mfld}^\bullet)$. We then define $\underline{\mathbf{Mfld}}^G$ (definition 3.1.16) as the coCartesian fibration dual to $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Mfld}) \to \mathcal{O}_G$, which classifies the same diagram $\mathbf{N}(\mathbf{Mfld}^\bullet)$. The dual coCartesian fibration admits an explicit construction span categories (see [BGN14]) which we use to describe the objects and morphisms of $\underline{\mathbf{Mfld}}^G$ and the coCartesian morphisms of $\underline{\mathbf{Mfld}}^G \twoheadrightarrow \mathcal{O}_G^{op}$.

Remark 3.1.1. In this section we denote objects of the orbit category by $O \in \mathcal{O}_G^{op}$, as opposed to G/H elsewhere. This is merely for notational convenience.

$\mathcal{O}_G$-manifolds and their spaces of smooth equivariant embeddings. We start by defining $\mathcal{O}_G$-manifolds and spaces of smooth equivariant embeddings which will serve as objects and mapping spaces of the topological category $\mathcal{O}_G$-**Mfld**, see definition 3.1.8.

Definition 3.1.2. *An $\mathcal{O}_G$-manifold $M \to O$ is a smooth n-dimensional manifold M with an action of G on M by smooth maps, together with a G-map $M \to O$ from the underlying G-space of the manifold M to a G-orbit $O \in \mathcal{O}_G$.*

We *always* think of an $\mathcal{O}_G$-manifold $M \to G/H$ as encoding a smooth n-dimensional manifold with an action of H, given by the fiber $M|_H$ of the G-map $M \to G/H$ over the coset H. Note that a choice of a basepoint $o \in O$ induces an isomorphism $G/H \xrightarrow{\cong} O$, $gH \mapsto g \cdot o$, where $H < G$ is the stabilizer of o. We therefore think of an $\mathcal{O}_G$-manifold $M \to O$ as encoding the smooth action of $H = Stab(o)$ on the fiber $M|_H$.

Notation 3.1.3. Suppose M, N are smooth n-dimensional manifolds. Denote by $C^\infty(M, N)$ the space of *smooth* maps $M \to N$ with the *compact-open* topology.

Definition 3.1.4. *Let $M_1 \to O_1, M_2 \to O_2$ be $\mathcal{O}_G$-manifolds. For $\varphi \colon O_1 \to O_2$ a map in $\mathcal{O}_G$, define $Emb_\varphi^{\mathcal{O}_G}(M_1, M_2) \subset C^\infty(M_1, M_2)$ as the subspace of smooth maps $f \colon M_1 \to M_2$ such that*

1. *f is a G-map*
2. *f is over φ, i.e*

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{\varphi} & O_2 \end{array} \tag{5}$$

   is a commutative square of G-spaces.

3. *the induced map $M_1 \to O_1 \times_{O_2} M_2$ is an embedding.*

Define the topological space $Emb^{\mathcal{O}_G}(M_1, M_2)$ as the coproduct

$$Emb^{\mathcal{O}_G}(M_1, M_2) := \coprod_\varphi Emb_\varphi^{\mathcal{O}_G}(M_1, M_2), \tag{6}$$

where the coproduct is indexed by the set $\mathrm{Hom}_{\mathcal{O}_G}(O_1, O_2)$.

Notation 3.1.5. When the orbit map φ is an identity $G/H \xrightarrow{=} G/H$ we use the notation $Emb^G_{G/H}(M_1, M_2)$ for the space $Emb_\varphi^{\mathcal{O}_G}(M_1, M_2)$ of G-equivariant embeddings $M_1 \to M_2$ over $G/H \xrightarrow{=} G/H$. Restriction to the fiber over H defines a homeomorphism from $Emb^G_{G/H}(M_1, M_2)$ with the space of H-equivariant embeddings $M_1|_H \to M_2|_H$ between the fibers over H.

Definition 3.1.6. *Let $M_1 \to O_1, M_2 \to O_2$ be $\mathcal{O}_G$-manifolds. A G-*isotopy over *$\varphi \colon O_1 \to O_2$ is a path in $Emb_\varphi^{\mathcal{O}_G}(M_1, M_2)$. When $M_1 \to O, M_2 \to O$ are over the same orbit we call a path in $Emb_O^{\mathcal{O}_G}(M_1, M_2)$ a G-isotopy over O.*

Note that a G-isotopy over G/H is equivalent to an H-equivariant isotopy between two H-equivariant embeddings $M_1|_H \to M_2|_H$.

The topological category of $\mathcal{O}_G$-manifolds. We now turn to the definition of the topological category of $\mathcal{O}_G$-manifolds. Note that the pullback of smooth embeddings of n-dimensional manifolds is a smooth embedding, therefore we have

Lemma 3.1.7. *Let $M_1 \to O_1, M_2 \to O_2, M_3 \to O_3$ be $\mathcal{O}_G$-manifolds. The composition of smooth functions defines a continuous map*

$$Emb^{\mathcal{O}_G}(M_2, M_3) \times Emb^{\mathcal{O}_G}(M_1, M_2) \to Emb^{\mathcal{O}_G}(M_1, M_3), \quad (g, f) \mapsto g \circ f.$$

Definition 3.1.8. *The* category of $\mathcal{O}_G$-manifolds $\mathcal{O}_G$-**Mfld** *is the topological category whose objects are a $\mathcal{O}_G$-manifolds. The morphism space from $M_1 \to O_1$ to $M_2 \to O_2$ is given by* $\mathrm{Map}_{\mathcal{O}_G\text{-}\mathbf{Mfld}}(M_1, M_2) := Emb^{\mathcal{O}_G}(M_1, M_2)$.

*Define a forgetful functor $q\colon \mathcal{O}_G$-**Mfld** $\to \mathcal{O}_G$ by sending $M \to O$ to the orbit O, and the subspace $Emb^{\mathcal{O}_G}_{\varphi}(M_1, M_2) \subset Emb^{\mathcal{O}_G}(M_1, M_2)$ to $\varphi \in \mathrm{Hom}_{\mathcal{O}_G}(O_1, O_2)$.*

By [Lur09a, ex. 1.1.5.12] the topological nerve $\mathbf{N}(\mathcal{O}_G$-**Mfld**$)$ *is an ∞-category, and by [Lur09a, ex. 1.1.5.8] the topological nerve of $\mathcal{O}_G$ can be identified with its ordinary nerve, which we identify with $\mathcal{O}_G$ by standard abuse of notation.*

Applying the topological nerve functor of [Lur09a, 1.1.5.5] to q produces a functor of ∞-categories $\mathbf{N}(q)\colon \mathbf{N}(\mathcal{O}_G$-**Mfld**$) \to \mathcal{O}_G$.

In particular, an object of the ∞-category $\mathbf{N}(\mathcal{O}_G$-**Mfld**$)$ is an $\mathcal{O}_G$-manifold $M \to O$, a map is given by a commutative square eq. (5) satisfying the conditions of definition 3.1.4, and by [Lur09a, thm. 1.1.5.13] the mapping spaces of $\mathbf{N}(\mathcal{O}_G$-**Mfld**$)$ are weakly equivalent to the mapping spaces of $\mathcal{O}_G$-**Mfld**.

Remark 3.1.9. The fiber of $\mathcal{O}_G$-**Mfld** $\to \mathcal{O}_G$ over an orbit G/H is the topological nerve of the topological category whose objects are $\mathcal{O}_G$-manifolds $M \to G/H$ and morphism spaces are $Emb^G_{G/H}(M_1, M_2)$. This topological category is equivalent to the category $\mathbf{Mfld}^H$ of H-manifolds and H-equivariant embeddings by restriction to the fibers over H.

Remark 3.1.10. We caution the reader not to pass to ∞-categories prematurely. One can construct the topological category $\mathcal{O}_G$-**Mfld** as a subcategory of the topological arrow category $\mathbf{Mfld}^G \downarrow \mathcal{O}_G$. However, the ∞-category $\mathbf{N}(\mathcal{O}_G$-**Mfld**$)$ is *not* a subcategory of the topological nerve $\mathbf{N}(\mathbf{Mfld}^G \downarrow \mathcal{O}_G)$ in the sense of [Lur09a, sec. 1.2.11]. To see this note that a subcategory of $\mathbf{N}(\mathbf{Mfld}^G \downarrow \mathcal{O}_G)$ is specified by a subcategory of its homotopy category $ho\mathbf{N}(\mathbf{Mfld}^G \downarrow \mathcal{O}_G)$, and therefore given by a choosing connected components of each mapping space of $\mathbf{Mfld}^G \downarrow \mathcal{O}_G$. On the other hand condition (3) of definition 3.1.8 is not preserved by G-homotopy equivalence, so the subspace

$$Emb^G_{G/H}(M_1 \to O_1, M_2 \to O_2) \subset \mathrm{Map}_{\mathbf{Mfld}^G \downarrow \mathcal{O}_G}(M_1 \to O_1, M_2 \to O_2)$$

is not given by a set of connected components. The same phenomenon exists in the non-equivariant setting.

Equivalences of $\mathcal{O}_G$-manifolds. Unwinding the definition of equivalence in a nerve of a topological category, we see that a map $f\colon M_1 \to M_2$ in $\mathcal{O}_G$-**Mfld** is an equivalence in $\mathbf{N}(\mathcal{O}_G$-**Mfld**$)$ if it has a G-isotopy inverse: a map $g\colon M_2 \to M_1$ in $\mathcal{O}_G$-**Mfld**, together with a G-isotopy over $id_{q(M_1)}$ from $g \circ f$ to id_{M_1} and a G-isotopy over $id_{q(M_2)}$ from $f \circ g$ to id_{M_2}.

Definition 3.1.11. *We say that a map $f\colon M_1 \to M_2$ of $\mathcal{O}_G$-manifolds is a* G-isotopy equivalence *if it is an equivalence in the ∞-category* $\mathbf{N}(\mathcal{O}_G$-**Mfld**$)$.

Note that an equivalence f always lies over an isomorphism of orbits $q(f)\colon O_1 \to O_2$. Using the homeomorphism between the mapping space $Emb^G_{G/H}(M_1, M_2)$ over an orbit G/H and the space of H-equivariant embeddings $M_1|_H \to M_2|_H$ we see that a map $f\colon M_1 \to M_2$ over an orbit G/H is an equivalence in $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Mfld})$ if and only if its restriction to the fibers $f|_H\colon M_1|_H \to M_2|_H$ is invertible upto H-isotopy. In particular, f need not induce an equivariant diffeomorphism. Nonetheless, its existence is enough to ensure that there exists an equivariant diffeomorphism between underlying manifolds. We learned the following argument from an answer of Ian Agol on MathOverflow [ha], which we reproduce here (with addition of a G-action).

Proposition 3.1.12. *Let $M_1 \to G/H$ and $M_2 \to G/H$ be two $\mathcal{O}_G$-manifolds over G/H. If $f \in Emb^G_{G/H}(M_1, M_2)$ and $g \in Emb^G_{G/H}(M_2, M_1)$ are G-isotopy inverses over G/H then there exists a G-equivariant diffeomorphism $M_1 \cong M_2$ over G/H.*

Proof. We prove the statement by reduction. Since $Emb^G_{G/H}(M_1, M_2)$ is homeomorphic to the space of H-invariant embeddings between $M_1|_H \to M_2|_H$ it is enough to consider the case $G = H$.

Suppose M, N are n-dimensional manifolds with smooth actions of G, and we are given G-equivariant embeddings $f\colon M \to N$, $g\colon N \to M$. Consider the direct limit

$$X = \underrightarrow{colim}(M \xrightarrow{f} N \xrightarrow{g} M \xrightarrow{f} N \xrightarrow{g} \cdots),$$

given by the explicit model $M \times \mathbb{N} \sqcup N \times \mathbb{N}/\sim$ with equivalence relation generated by $(m, k) \simeq (f(m), k)$ and $(n, k) \simeq (g(n), k + 1)$. Then X is a smooth manifold with an action of G, as a sequential union of nested open submanifolds.

Since X is G-diffeomorphic to $Y = \underrightarrow{colim}(N \xrightarrow{g} M \xrightarrow{f} N \xrightarrow{g} \cdots)$ (removing the first term of the sequence does not change the colimit), it is enough to show that X is G-diffeomorphic to M.

Note that X is G-diffeomorphic to $\underrightarrow{colim}(M \xrightarrow{F_1} M \xrightarrow{F_1} M \xrightarrow{F_1} \cdots)$ for $F_1 = g \circ f$, and F_1 is G-isotopic to id_M. Let $F_t\colon M \to M, t \in [0, 1]$ be the G-isotopy from $F_0 = id_M$ to $F_1 = g \circ f$, and define $X_t = \underrightarrow{colim}(M \xrightarrow{F_t} M \xrightarrow{F_t} \cdots)$, so that $X_1 = X$ and $X_0 = M$.

Choose a sequence of compact G-submanifolds with boundary $K_1 \subset K_2 \subset K_3 \subset \cdots M$ such that $M = \cup_i K_i$ and $F(K_i \times [0, 1]) \subset int(K_{i+1})$. Such a sequence can be chosen inductively using a G-invariant Morse function on M (which exists by [Was69, cor. 4.10]). Define $Y_t = \underrightarrow{colim}(K_1 \xrightarrow{F_t} K_2 \xrightarrow{F_t} K_3 \xrightarrow{F_t} \cdots)$ using the restrictions of the F_t to the subsets K_i. We claim that $Y_t = X_t$, using the standard model for direct limits. Write $X_t = M \times \mathbb{N}/(x, i) \sim (F_t(x), i+1)$, and note that $Y_t \subseteq X_t$ as the points (x, i) with $x \in K_i$. We claim that each point $x \in X_t$ is in Y_t. Represent x by $(x, i) \in M \times \mathbb{N}$, then since $M = \cup K_i$ we have $x \in K_j$ for some $j \in \mathbb{N}$. If $j \leq i$ then $K_j \subset K_i$, so $x \in K_i$, hence (x, i) represents an point in Y_t. Otherwise $(x, i) \sim (F_t^{j-i}(x), j)$ in represents the same point in X_t, and since $F_t^{j-i}(K_i) \subset K_j$ we get $F_t^{j-i}(x) \in K_j$, so $(F_t^{j-i}(x), j)$ represents an element of Y_t.

We showed that $Y_t = X_t$, so it is enough to prove that $Y_0 \cong M$ is G-diffeomorphic to $Y_1 \cong X$. By definition we have $Y_0 = \underrightarrow{colim}(K_1 \hookrightarrow K_2 \hookrightarrow K_3 \hookrightarrow \cdots)$ and $Y_1 = \underrightarrow{colim}(K_1 \xrightarrow{F_1} K_2 \xrightarrow{F_1} K_3 \xrightarrow{F_1} \cdots)$, hence it is enough to construct compatible G-diffeomorphims $\phi_i\colon K_i \hookrightarrow K_i$, i.e satisfying $\phi_{i+1}|_{K_i} = F_1 \circ \phi_i$.

We now inductively construct G-equivariant maps $G^i\colon K_i \times [0, 1] \to K_i$ such that $G_0 = Id_{K_i}$, $\forall t \in [0, 1] : G_t\colon K_i \to K_i$ is a diffeomorphism and $\forall x \in K_i, t \in [0, 1] :\ F_t \circ G^i_t(x) = G^{i+1}_t(x)$, i.e the diagram

$$\begin{array}{ccc} K_i \times [0,1] & \hookrightarrow & K_{i+1} \times [0,1] \\ \downarrow{\scriptstyle G^i \times Id} & & \downarrow{\scriptstyle G^{i+1}} \\ K_i \times [0,1] & \xrightarrow{F|_{K_i \times [0,1]}} & K_{i+1} \end{array}$$

commutes. [12]

We start with setting $G^1_t = Id_{K_1}$. Assume that a G^i has been constructed. Consider the isotopy $K_i \times [0,1] \xrightarrow{G^i \times Id} K_i \times [0,1] \xrightarrow{F|_{K_i \times [0,1]}} K_{i+1}$. Since $K_i \subset K_{i+1}$ is a compact submanifold and $F(K_i) \subset Int(K_{i+1})$ the conditions of the isotopy extension theorem [Hir12, ch. 8 thm. 1.3] are satisfied. Therefore there exists a diffeotopy $\tilde{G}^{i+1} \colon K_{i+1} \times [0,1] \to K_{i+1}$ which extends the isotopy $K_i \times [0,1] \xrightarrow{G^i \times Id} K_i \times [0,1] \xrightarrow{F|_{K_i \times [0,1]}} K_{i+1}$ and satisfies $\tilde{G}^{i+1}_0 = Id_{K_{i+1}}$, but might not be G-equivariant. Since K_{i+1} is compact we can apply [Bre72, thm 3.1], and get a G-equivariant diffeotopy $G^{i+1} \colon K_{i+1} \times [0,1] \to K_{i+1}$ with $G^{i+1}_0 = \tilde{G}^{i+1}_0 = Id_{K_{i+1}}$ and which agrees with $\tilde{G}^{i+1}$ on the subset $\left\{ x \in K_{i+1} \,|\, \forall g \in G, t \in [0,1] : \tilde{G}^{i+1}_t(gx) = g\tilde{G}^{i+1}_t(x) \right\}$. In particular, for $x \in K_i$ we have $\tilde{G}^{i+1}_t(x) = F_t G^i_t(x)$, so the G-equivariant diffeotopy G^{i+1} agrees with $\tilde{G}^{i+1}$ on $K_i \times [0,1]$.

Setting $\phi^i = G^i_1$ gives the compatible G-diffeomorphisms proving that $Y_0 \cong M$ is indeed G-diffeomorphic to $Y_1 \cong X$. ☐

Cartesian edges in $\mathcal{O}_G$-Mfld. We now identify the Cartesian edges of the forgetful functor $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Mfld}) \to \mathcal{O}_G$, as well as the coCartesian edges over isomorphisms. We start with

Lemma 3.1.13. *The forgetful functor* $\mathbf{N}(q) \colon \mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Mfld}) \to \mathcal{O}_G$ *is an inner fibration.*

Proof. For every pair $M_1 \to O_1, M_2 \to O_2$ of $\mathcal{O}_G$-manifolds, q induces a Kan fibration

$$\mathrm{Map}_{\mathbf{Sing}(\mathcal{O}_G\text{-}\mathbf{Mfld})}(M_1, M_2) \to \mathrm{Map}_{\mathbf{Sing}(\mathcal{O}_G)}(O_1, O_2) = \mathrm{Hom}_{\mathcal{O}_G}(O_1, O_2),$$

because its a map from a Kan simplicial complex and to discrete simplicial set. Therefore by [Lur09a, prop. 2.4.1.10(1)] the functor $\mathbf{N}(q)$ is an inner fibration. ☐

Note that a map $M \to O$ from an n-dimensional manifold to a finite set is always a submersion, so its pullback along any map of finite sets is an n-dimensional manifold.

Proposition 3.1.14. *Suppose that* $\varphi \colon O_1 \to O_2$ *be a map of orbits, and* $M \to O_2$ *a* $\mathcal{O}_G$*-manifold. Then the pullback square of topological G-spaces*

$$\begin{array}{ccc} O_1 \times_{O_2} M & \xrightarrow{f} & M \\ \downarrow & \ulcorner & \downarrow \\ O_1 & \longrightarrow & O_2 \end{array}$$

defines a $\mathbf{N}(q)$*-Cartesian morphism* f *in* $\mathcal{O}_G$-**Mfld**. *In particular,* $\mathbf{N}(q)$ *is a Cartesian fibration.*

[12] The map G^i is an equivariant *diffeotopy* in terminology of [Hir12] and an *equivariant isotopy starting from the identity* in the terminology of [Bre72].

Proof. Checking that f satisfies the conditions of definition 3.1.8 is immediate.

By [Lur09a, prop. 2.4.1.1 (2)] the morphism f is $\mathbf{N}(q)$-Cartesian if and only if, for every $\mathcal{O}_G$-manifold $T \to O$, the square of spaces

$$\begin{array}{ccc} Map_{\mathbf{Sing(Mfld)}}(T, O_1 \times_{O_2} M) & \xrightarrow{f_*} & Map_{\mathbf{Sing}(\mathcal{O}_G\text{-}\mathbf{Mfld})}(T, M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_G}(O, O_1) & \xrightarrow{q(f)_*} & \mathrm{Hom}_{\mathcal{O}_G}(O, O2) \end{array}$$

is a homotopy pullback square. Since the vertical maps are Kan fibrations, this square is a homotopy pullback if and only if the horizontal map

$$\begin{array}{ccc} \mathrm{Map}(T, O_1 \times_{O_2} M) & \longrightarrow & \mathrm{Hom}(O, O_1) \times_{\mathrm{Hom}(O,O_2)} \mathrm{Map}(T, M) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}(O, O_1) & \end{array}$$

is a homotopy equivalence, or equivalently, if f_* induces an equivalence between the fiber over every $\tau \in \mathrm{Hom}(O, O_1)$.

Let $\tau : O \to O_1$. Then f_* induces a map of fibers over τ

$$Emb_\tau^{\mathcal{O}_G}(T, O_1 \times_{O_2} M) \to \{\tau\} \times_{\mathrm{Hom}(O,O_2)} \mathrm{Map}(T, M) = Emb_{q(f)\circ\tau}^{\mathcal{O}_G}(T, M),$$

$$\left(\begin{array}{ccc} T & \xrightarrow{g} & O_1 \times_{O_2} M \\ \downarrow & & \downarrow \\ O & \xrightarrow{\tau} & O_1 \end{array}\right) \mapsto \left(\begin{array}{ccccc} T & \xrightarrow{g} & O_1 \times_{O_2} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \ulcorner & & \downarrow \\ O & \xrightarrow{\tau} & O_1 & \xrightarrow{q(f)} & O_2 \end{array}\right)$$

This continuous map is a bijection by the universal property of the pullback. We leave to it to the reader to verify it is an open map using the definition of the compact-open topology. □

This gives the following complete description of the Cartesian edges in $\mathcal{O}_G$-**Mfld**.

Corollary 3.1.15. *A morphism* (5) *is* $\mathbf{N}(q)$*-Cartesian if and only if it is equivalent to a pullback, i.e. the morphism*

$$\begin{array}{ccc} M_1 & \longrightarrow & O_1 \times_{O_2} M_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{=} & O_1 \end{array} \tag{7}$$

is a G-isotopy equivalence.

Proof. Factor the morphism (5) as the composition of (7) and a pullback square. Combining proposition 3.1.14 and [Lur09a, prop. 2.4.1.7] we see that the morphism (5) is $\mathbf{N}(q)$-Cartesian if and only if the map above is $\mathbf{N}(q)$-Cartesian. Since the morphism (7) lies over an equivalence it is $\mathbf{N}(q)$-Cartesian if and only if it is an equivalence, by [Lur09a, prop. 2.4.1.5]. □

Construction of the G-category of G-manifolds. The construction of the G-category $\underline{\mathbf{Mfld}}^G$ now follows easily from the description of the Cartesian fibration $\mathbf{N}(q)$ and the explicit construction of [BGN14].

Definition 3.1.16. *Let $p\colon \underline{\mathbf{Mfld}}^G \to \mathcal{O}_G^{op}$ be the dual of the Cartesian fibration $\mathcal{O}_G$-$\mathbf{Mfld} \to \mathcal{O}_G$ in the sense of [BGN14, def. 3.5]. Explicitly, $\underline{\mathbf{Mfld}}^G$ is the pullback of the effective Burnside category*

$$A^{eff}(\mathcal{O}_G\text{-}\mathbf{Mfld},\, \mathcal{O}_G\text{-}\mathbf{Mfld} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong},\, \text{q-Cart}(\mathcal{O}_G\text{-}\mathbf{Mfld}))$$

along the equivalence $\mathcal{O}_G^{op} \xrightarrow{\sim} A^{eff}(\mathcal{O}_G, \mathcal{O}_G^{\cong}, \mathcal{O}_G)$, where $\mathcal{O}_G^{\cong}$ is the maximal subgroupoid of $\mathcal{O}_G$ and q-Cart$(\mathcal{O}_G$-$\mathbf{Mfld}) \subset \mathcal{O}_G$-$\mathbf{Mfld}$ *is the subcategory spanned by all objects and morphisms which are q-Cartesian.*

By [BGN14, prop. 3.4] the map $p\colon \underline{\mathbf{Mfld}}^G \to \mathcal{O}_G^{op}$ is a coCartesian fibration, and we have an explicit description of the objects and morphisms of $\underline{\mathbf{Mfld}}^G$. The objects of the total ∞-category $\underline{\mathbf{Mfld}}^G$ are $\mathcal{O}_G$-manifolds $M \to O$. A morphism in $\underline{\mathbf{Mfld}}^G$ from $M_1 \to O_1$ to $M_2 \to O_2$ is a diagram of the form

$$\begin{array}{ccccc} M_1 & \longleftarrow & M & \longrightarrow & M_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \longleftarrow & O_2 & \xrightarrow{=} & O_2 \end{array} \tag{8}$$

where the left square is a coCartesian edge in $\mathcal{O}_G$-$\mathbf{Mfld}$ (in other words, it is equivalent to a pullback square, see corollary 3.1.15). This arrow is p-coCartesian exactly when the right square is a G-isotopy equivalence.

Without loss of generality we will represent a morphism in $\underline{\mathbf{Mfld}}^G$ by a span (8) where the left square is a pullback square.

Remark 3.1.17. Let $H < G$ be a subgroup. Topological induction defines a functor

$$G \times_H (-)\colon \mathbf{Mfld}^H \to \underline{\mathbf{Mfld}}^G_{[G/H]}, \quad G \times_H M = ((G \times M)/G \to (G \times pt)/H = G/H)$$

where we quotient by the H-action $h \cdot (g, x) = (gh^{-1}, gx)$. Topological induction if a functor of topological categories, and in fact an equivalence of topological categories $\mathbf{Mfld}^H \xrightarrow{\sim} \underline{\mathbf{Mfld}}^G_{[G/H]}$, with inverse $(M \to G/H) \mapsto M|_{eH}$ given by restriction to the fiber over eH.

Informally, the coCartesian fibration $\underline{\mathbf{Mfld}}^G \to \mathcal{O}_G^{op}$ classifies the functor $\mathcal{O}_G^{op} \to \mathcal{C}\mathrm{at}_\infty$ sending G/H to $\mathbf{Mfld}^H$.

Notation 3.1.18. We will refer to $\underline{\mathbf{Mfld}}^G$ as *the G-category of G-manifolds*, to stress its conceptual role and not its technical construction. We urge the reader to regards the objects of $\underline{\mathbf{Mfld}}^G$ not as $\mathcal{O}_G$-manifolds (which they are), but as a technical means of encoding manifolds with an action of a subgroup of G. This naming convention is also compatible with [BDG$^+$16b, ex. 7.5], where $\underline{\mathbf{Top}}_T$ is referred to as the T-∞-category of T-spaces.

By construction, we have a simple description of the fiberwise opposite[13] category $(\underline{\mathbf{Mfld}}^G)^{vop}$, introduced in [BDG$^+$16b, sec. 3]. It is helpful to keep this description in mind when we use the parametrized Yoneda embedding to construct the equivariant tangent classifier in construction 3.2.8.

[13] The superscript "vop" stands for taking "vertical opposites".

Proposition 3.1.19. *Applying the opposite ∞-category functor $(-)^{op}$ to the Cartesian fibration $\mathcal{O}_G$-$\mathbf{Mfld} \to \mathcal{O}_G$ produces a G-category $(\mathcal{O}_G\text{-}\mathbf{Mfld})^{op} \twoheadrightarrow \mathcal{O}_G^{op}$ equivalent to $(\underline{\mathbf{Mfld}}^G)^{vop} \twoheadrightarrow \mathcal{O}_G^{op}$.*

Proof. By [BDG$^+$16b, def. 3.1] the opposite G-category $(\underline{\mathbf{Mfld}}^G)^{vop} \twoheadrightarrow \mathcal{O}_G^{op}$ is given by taking the opposite of the dual Cartesian fibration $(\underline{\mathbf{Mfld}}^G)_\wedge \twoheadrightarrow \mathcal{O}_G$. The result follows, since taking the dual coCartesian fibration is homotopy inverse to taking the dual Cartesian fibration (see [BGN14, thm. 1.7]). □

3.2 Representations, G-vector bundles and the G-tangent classifier

In this subsection we study the relation between G-vector bundles, H-representations of subgroups $H < G$ and the G-category of G-manifolds, $\underline{\mathbf{Mfld}}^G$, constructed in section 3.1. We do this by identifying H-representations with G-vector bundles over G/H, which in turn span a full G-subcategory $\underline{\mathbf{Rep}}_n^G \subset \underline{\mathbf{Mfld}}^G$. An equivariant version of "smooth Kister's theorem" implies that $\underline{\mathbf{Rep}}_n^G$ is in fact a G-∞-groupoid, which can be identified with the G-space classifying n-dimensional G-vector bundles, $BO_n(G)$. We use $\underline{\mathbf{Rep}}_n^G$ to construct an equivariant version of the tangent classifier of [AF15, sec 2.1] (see construction 3.2.8), which will be used in section 3.3 to define equivariant tangential structures on G-manifolds. It is worth noting that parametrized ∞-category theory is essential for construction 3.2.8, which relies on the identification of the *G-space* $BO_n(G)$ with a full G-subcategory of $\underline{\mathbf{Mfld}}^G$.

We start by recalling the standard definition of G-vector bundles.

Definition 3.2.1 (see [Bre72, sect. VI.2], [tD87, ch. I, def. 9.1])**.** *Let X be a G-space. A* G-vector bundle *over X is a (real) vector bundle $p\colon E \to X$ together with a G-action on E by bundle maps (i.e linear action on each fiber) such that p is a G-map. We say $p\colon E \to X$ is* smooth *if E, X are (smooth) G-manifolds and p is a smooth map. Let $G - \mathbf{Vect}_{/X}$ denote the category of G-vector bundles over X.*

Note that G-vector bundles are stable under pullback along G-maps, and that a G-vector bundle over a point is the same as a G-representation. It is useful to keep in mind the correspondence between representations of subgroups $H < G$ and G-vector bundles over the orbit G/H:

Proposition 3.2.2. *[tD87, special case of prop. I.9.2] Let $H < G$ be a subgroup. Restriction to the fiber over $[eH]$ gives an equivalence $G - \mathbf{Vect}_{/(G/H)} \xrightarrow{\sim} H - \mathbf{Vect}_{/pt} \cong \mathbf{Rep}^H$ from the category of G-vector bundles over the orbit G/H to the category of H-representations. An inverse is given by sending a representation of H on $\mathbb{R}^n$ to its topological induction $G \times_H \mathbb{R}^n$.*

The subject of this subsection is the following G-subcategory.

Definition 3.2.3. *Let $\underline{\mathbf{Rep}}_n^G \subset \underline{\mathbf{Mfld}}^G$ be the full G-subcategory spanned by G-vector bundles $(E \to G/H)$, i.e $\mathcal{O}_G$-manifolds $E \to G/H$ such that E can be endowed with a structure of a G-vector bundle over G/H.*

Remark 3.2.4. We will use G-vector bundles as a model for "G-disks". Specifically, an embedding of a G-disk in an $\mathcal{O}_G$-manifold $M \in \underline{\mathbf{Mfld}}^G$ is just a map in $\underline{\mathbf{Mfld}}^G$ with target is M and domain in $\underline{\mathbf{Rep}}_n^G$. Genuine G-factorization homology is defined as a parametrized colimit over finite disjoint unions of G-disks in M (see definition 4.1.2). In section 3.6 we organize these disjoint unions into a G-∞-category $\underline{\mathbf{Disk}}^G$.

In order to see the close relation of $\underline{\mathbf{Rep}}_n^G$ with representation theory we use the following equivariant version of the "smooth Kister-Mazur" theorem (see [Kup]).

Proposition 3.2.5. *Let V be a finite dimensional real representation of $H < G$. Let $\mathrm{Aut}_{\mathbf{Rep}^H}(V)$ be the automorphism group of V as an H-representation, i.e linear H-equivariant isomorphisms. Let $Emb_0^H(V,V)$ denote the subspace of smooth H-equivariant embedding fixing the origin, and $\mathrm{Aut}_0^H(V) \subset Emb_0^H(V,V)$ the subspace of H-equivariant diffeomorphisms. Then the inclusions*

$$\mathrm{Aut}_{\mathbf{Rep}^H}(V) \hookrightarrow \mathrm{Aut}_0^H(V) \hookrightarrow Emb_0^H(V,V)$$

are homotopy equivalences.

Proof. The proof of [Kup, thm. 2.4] applies verbatim when restricting to subspaces of H-equivariant maps after checking that the formulas for $G_s^{(1)}, G_s^{(2)}$ produce H-equivariant homotopies. □

The central role played by $\underline{\mathbf{Rep}}_n^G$ in what follows stems from the following characterization.

Proposition 3.2.6. *The G-category $\underline{\mathbf{Rep}}_n^G$ is a G-∞-groupoid, with fibers $(\underline{\mathbf{Rep}}_n^H)_{[G/H]}$ equivalent to the topological groupoid $\mathbf{Rep}_n^H$ of n-dimensional real representations of H and (linear, H-equivalent) isomorphisms, where the mapping space $\mathrm{Iso}_{\mathbf{Rep}^H}(V_0, V_1)$ is endowed with the compact-open topology.*

Proof. In order to show that $\underline{\mathbf{Rep}}_n^G$ is a G-∞-groupoid we have to prove that the coCartesian fibration $\underline{\mathbf{Rep}}_n^G \twoheadrightarrow \mathcal{O}_G^{op}$ is a left fibration. By [Lur09a, prop. 2.4.2.4] it is enough to show that the fibers $(\underline{\mathbf{Rep}}_n^G)_{[G/H]}$ are ∞-groupoids. The equivalence $\underline{\mathbf{Mfld}}^G_{[G/H]} \cong \mathbf{Mfld}^H$ of remark 3.1.17 takes a G-vector bundle $E \to G/H$ to an H-vector bundle $E|_{eH} \to pt$, i.e. an n-dimensional real H-representation $V = (H \curvearrowright \mathbb{R}^n)$, so we have to show that for every $V_0, V_1 \in \mathbf{Rep}_n^H$ the inclusion $\mathrm{Aut}^H(V_0, V_1) \subset Emb^H(V_0, V_1)$ is a weak equivalence.

Let $Emb_0^H(V_0, V_1) \subset Emb^H(V_0, V_1)$ denote the subspace of origin fixing maps. Clearly the inclusion $Emb_0^H(V_0, V_1) \hookrightarrow Emb^H(V_0, V_1)$ is a homotopy equivalence. By proposition 3.2.5 the inclusion $\mathrm{Iso}_{\mathbf{Rep}^H}(V_0, V_1) \hookrightarrow Emb_0^H(V_0, V_1)$ is a weak equivalence, so $\mathrm{Iso}_{\mathbf{Rep}^H}(V_0, V_1) \xhookrightarrow{\sim} Emb^H(V_0, V_1)$ is a weak equivalence.

In other words, the functor $\mathbf{Rep}_n^H \to (\underline{\mathbf{Rep}}_n^G)_{[G/H]}$ is fully faithful. Since by definition it is essentially surjective it is an equivalence of ∞-categories. In particular $(\underline{\mathbf{Rep}}_n^G)_{[G/H]}$ is equivalent to the (coherent nerve of) the topological groupoid $\mathbf{Rep}_n^H$, hence an ∞-groupoid. □

By construction of the classifying space of G-vector bundles (see [LR78, Wan80]) we have the following statement.

Corollary 3.2.7. *The G-∞-groupoid $\underline{\mathbf{Rep}}_n^G$ corresponds to $BO_n(G) \in \mathbf{Top}^G$, the classifying G-space of rank n real G-vector bundles.*

We can now construct an equivariant version of the tangent classifier of Ayala-Francis (see [AF15, sec. 2.1]).

Construction 3.2.8 (G-tangent classifier)**.** Let $j\colon \underline{\mathbf{Mfld}}^G \to \underline{\mathrm{Fun}}_G((\underline{\mathbf{Mfld}}^G)^{vop}, \underline{\mathrm{Top}}^G)$ be the parametrized Yoneda embedding G-functor of [BDG+16b] (see proposition 3.1.19 for a description of the fiberwise opposite $(\underline{\mathbf{Mfld}}^G)^{vop}$). Define a *$G$-tangent classifier* by the composition of G-functors

$$\tau\colon \underline{\mathbf{Mfld}}^G \xrightarrow{j} \underline{\mathrm{Fun}}_G((\underline{\mathbf{Mfld}}^G)^{vop}, \underline{\mathbf{Top}}^G) \to \underline{\mathrm{Fun}}_G((\underline{\mathbf{Rep}}_n^G)^{vop}, \underline{\mathbf{Top}}^G) \simeq \underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}$$

where the last equivalence is given by parametrized straightening/unstraightening.

In order to show that the G-tangent classifier sends a G-manifold M to the G-map classifying its tangent bundle we will use the following description of the G-slice category $\underline{\mathbf{Top}}^G_{/\underline{B}}$.

Remark 3.2.9. A G-space B defines a G-object $\underline{B}\colon \mathcal{O}_G^{op} \to \underline{\mathbf{Top}}^G$ (i.e. a coCartesian section, see [BDG+16b, def. 7.1]). Using the explicit model of $\underline{\mathbf{Top}}^G$ given in remark 2.1.7 we can describe $\underline{B}$ as

$$\underline{B}\colon \mathcal{O}_G^{op} \to \underline{\mathbf{Top}}^G,\ [G/H] \mapsto (B \times G/H \to G/H).$$

By [AF15, lem] and remark 2.1.7 it follows that the fibers of the parametrized slice category $\underline{\mathbf{Top}}^G_{/\underline{B}}$ are given by

$$\left(\underline{\mathbf{Top}}^G_{/\underline{B}}\right)_{[G/H]} \simeq \left(\underline{\mathbf{Top}}^G_{[G/H]}\right)_{/\underline{B}(G/H)} \simeq \left(\mathbf{Top}^G_{/G/H}\right)_{/(B\times G/H \to G/H)} \xrightarrow{\sim} \mathbf{Top}^G_{/B\times G/H}.$$

In particular an object of $\left(\underline{\mathbf{Top}}^G_{/\underline{B}}\right)_{[G/H]}$ is given by a G-space over $B \times G/H$, which we consider as an object $(Y \to G/H) \in \mathbf{Top}^G_{/G/H} \simeq \underline{\mathbf{Top}}^G_{[G/H]}$, together with a G-map $f\colon Y \to B$. We write $\bar{f}\colon Y \to B \times G/H$ for the G-map corresponding to the pair $(Y \to G/H, Y \xrightarrow{f} B)$.

The mapping spaces of the slice category $\left(\underline{\mathbf{Top}}^G_{[G/H]}\right)_{/\underline{B}(G/H)} \simeq \mathbf{Top}^G_{/B\times G/H}$ will be denoted by $\mathrm{Map}^G_{/\underline{B}(G/H)}(X,Y)$. An explicit description of these mapping spaces is given by the Moore over category, see appendix A.

Proposition 3.2.10. *Let $(M \to G/H)$ be an $\mathcal{O}_G$-manifold, and consider the tangent bundle $TM \to M$ as a G-vector bundle. Then $\tau_M \in \left(\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}\right)_{[G/H]}$ is given by $(M \to G/H) \in \underline{\mathbf{Top}}^G_{[G/H]}$ together with the G-map $\tau_M\colon M \to BO_n(G)$ classifying the tangent bundle of M.*

Proof. Recall that an $\mathcal{O}_G$-manifold $M \to G/H$ has an open cover by G-embeddings $E_\alpha \hookrightarrow M$ over G/H, where the patches $(E_\alpha \to G/H)$ are G-vector bundles. The mapping space $\mathrm{Map}^G(M, BO_n(G))$ is the homotopy limit of $\mathrm{Map}^G(E_\alpha, BO_n(G))$, so by the functionality of τ in M we are reduced to verifying the statement for $E \to G/H$ a G-vector bundle.

By construction the restriction of τ to $\underline{\mathbf{Rep}}^G_n$ is given by straightening the functor associated to the Yoneda embedding $\underline{\mathbf{Rep}}^G_n \hookrightarrow \underline{\mathrm{Fun}}_G((\underline{\mathbf{Rep}}^G_n)^{vop}, \underline{\mathbf{Top}}^G)$. Recalling the construction of the parametrized Yoneda embedding ([BDG+16b, sec. 10]) we see that $\tau|_{\underline{\mathbf{Rep}}^G_n}$ is associated to the left fibration of the parametrized twisted arrow category $\widetilde{\mathscr{O}}(\underline{\mathbf{Rep}}^G_n/\mathcal{O}_G^{op}) \twoheadrightarrow (\underline{\mathbf{Rep}}^G_n)^{vop} \underline{\times} \underline{\mathbf{Rep}}^G_n$, end τ_E is associated to its pullback

$$\begin{array}{ccc} P_E & \longrightarrow & \widetilde{\mathscr{O}}(\underline{\mathbf{Rep}}^G_n/\mathcal{O}_G^{op}) \\ \downarrow & \ulcorner & \downarrow \\ (\underline{\mathbf{Rep}}^G_n)^{vop}\underline{\times}\underline{G/H} & \xrightarrow{id \underline{\times} \underline{E}} & (\underline{\mathbf{Rep}}^G_n)^{vop}\underline{\times}\underline{\mathbf{Rep}}^G_n. \end{array}$$

By proposition 3.2.6 this is a pullback square of G-∞-groupoids, and using corollary 3.2.7 we can identify it a homotopy pullback of G-spaces given by the top square of the following diagram

$$\begin{array}{ccccc}
G/H & \xrightarrow{\simeq} & \mathcal{P}_e & \longrightarrow & \mathrm{Map}(\Delta^1, BO_n(G)) \\
 & & \downarrow \ulcorner & & \downarrow \\
 & & BO_n(G) \times G/H & \xrightarrow{id \times e} & BO_n(G) \times BO_n(G) \\
 & & \downarrow{\scriptstyle proj} \ulcorner & & \downarrow{\scriptstyle proj} \\
 & \overset{=}{\searrow} & G/H & \xrightarrow{e} & BO_n(G).
\end{array}$$

Since the bottom square (given by projections to the second coordinate) is a homotopy pullback square it follows that the outer rectangle is a homotopy limit diagram. Observe that the composition of the right vertical maps is an equivalence, and therefore the composition of the left vertical maps is an equivalence as well. It follows that τ_E is equivalent to the G-map $(e, id)\colon G/H \to BO_n(G) \times G/H)$, where $e\colon G/H \to BO_n(G)$ classifies the G-vector bundle $E \to G/H$.

On the other hand the tangent bundle TE is given by fiber product $TE \cong E \times_{G/H} E$ and therefore classified by the composition of the bottom maps in

$$\begin{array}{ccccc}
TE & \longrightarrow & E & & \\
\downarrow & \ulcorner & \downarrow & & \\
E & \xrightarrow{\simeq} & G/H & \xrightarrow{e} & BO_n(G),
\end{array}$$

which is clearly equivalent to $E \xrightarrow{\simeq} G/H \xrightarrow{(e,id)} BO_n(G) \times G/H \xrightarrow{proj} BO_n(G)$. □

3.3 The G-category of f-framed G-manifolds

We now turn to the definition of the G-∞-category of G-manifolds with additional tangential structure. Our main interest is in the tangential structure defining V-framed G-manifolds, for V a G-representation. However, the definition of equivariant framing on G-manifolds supports other interesting tangential structures, including equivariant orientations in the sense of [CMW01], and free G-manifolds (an example not a priori associated with tangential structures).

The specific type of G-tangential structure, such as equivariant framing or equivariant orientation, is specified by a G-space B and a G-map $f\colon B \to BO_n(G)$, in the following manner. An f-framing on a G-manifold M is given by a G-map $M \to B$ such that the composition $M \to B \xrightarrow{f} BO_n(G)$ classifies the tangent bundle of M. Similarly, if $H < G$ is a subgroup and M is an H-manifold, we say that M is f-framed its tangent bundle is classified by the H-map $M \to B \xrightarrow{f} BO_n(G)$.

The ∞-categories of f-framed H-manifold for $H < G$ can be arranged into an $\mathcal{O}_G^{op}$-diagram, encoded by a G-∞-category $\underline{\mathbf{Mfld}}^{G,f-fr}$. We start by giving a precise definition of $\underline{\mathbf{Mfld}}^{G,f-fr}$ and the G-functor $\underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathbf{Mfld}}^{G}$ that forgets the tangential structure.

Definition 3.3.1. *Let $B \in \mathbf{Top}^G$ be a G-space and $f\colon B \to BO_n(G)$ be a G-map. Define the*

G-categories of f-framed G-manifolds as the pullback

$$\begin{array}{ccc} \underline{\mathbf{Mfld}}^{G,f-fr} & \longrightarrow & \underline{\mathbf{Top}}^G_{/\underline{B}} \\ \big\downarrow & \ulcorner & \big\downarrow f_* \\ \underline{\mathbf{Mfld}}^G & \xrightarrow{\tau} & \underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}. \end{array}$$

Remark 3.3.2. Unwinding the definition, an object of $(\underline{\mathbf{Mfld}}^{G,f-fr})_{[G/H]}$ is given by $(M \to G/H) \in \underline{\mathbf{Mfld}}^G_{[G/H]}$, a G-map $f_M \colon M$ and a G-homotopy between $f \circ f_M$ exhibiting

$$\begin{array}{ccc} & & B \\ & \overset{f_M}{\nearrow} & \big\downarrow f \\ M & \xrightarrow{\tau_M} & BO_n(G) \end{array}$$

as homotopy coherent diagram of G-spaces.

The mapping spaces of $\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ are given by homotopy pullbacks

$$\begin{array}{ccc} Emb^{G,f-fr}_{G/H}(M,N) & \xrightarrow{\qquad\quad} & \mathrm{Map}^G_{/B\times G/H}(M \xrightarrow{\bar{f_M}} B \times G/H, N \xrightarrow{\bar{f_N}} B \times G/H) \\ \big\downarrow & \ulcorner & \big\downarrow (f\times G/H)_* \\ Emb^G_{G/H}(M,N) & \xrightarrow{\tau} & \mathrm{Map}^G_{/BO_n(G)\times G/H}(M \xrightarrow{\bar{\tau_M}} BO_n(G) \times G/H, N \xrightarrow{\bar{\tau_N}} BO_n(G) \times G/H). \end{array}$$

We finish this subsection with some examples of equivariant tangential structures on G-manifolds. We are primarily interested in equivariantly framed G-manifolds, which is our first example.

Example 3.3.3 (V-framed G-manifolds)**.** Let $B = pt$. A G-map $f \colon pt \to BO_n(G)$ factors through the space of G-fixed points $(BO_n(G))^G = \coprod_V B\,\mathrm{Aut}_{\mathbf{Rep}^G_n}(V)$, so choosing f is equivalent to choosing a connected component, i.e a real n-dimensional G-representation V. A V-framing of an H-manifold M is therefore a homotopy lift

$$\begin{array}{ccc} & & pt \\ & \nearrow & \big\downarrow V \\ M & \xrightarrow{\tau_M} & \mathrm{BAut}(V), \end{array}$$

which under proposition 3.2.10 and restriction to fibers over the coset eH is equivalent to a choice of trivialization $TM \cong M \times V$ as an H-vector bundle.

Example 3.3.4 (G-manifolds with no tangential structure)**.** Apply definition 3.3.1 for the G-space $B = BO_n(G)$ and $id \colon BO_n(G) \to BO_n(G)$ constructs $\underline{\mathbf{Mfld}}^{G,id-fr} \cong \underline{\mathbf{Mfld}}^G$.

Example 3.3.5 (G-orientated G-manifolds)**.** Orientations of G-vector bundles were studied by Costenoble, May and Waner in [CMW01][14], and used in [CW] to prove equivariant versions of Poincaré duality.

[14]see [CMW01, def. 2.8] for a precise definition

Let us recall the relevant results from [CMW01]. First, there exists a universal oriented G-n-plane bundle, given by a G-map $EO_n(G,S) \to BO_n(G,S)$, see [CMW01, thm. 22.4]. Second, there is a G-map $f\colon BO_n(G,S) \to BO_n(G)$ representing the forgetful functor from oriented n-plane bundles to G-n-plane bundles. Therefore an orientation on a G-vector bundle is given by a G-homotopy lift of its classifying map along the G-map f.

Applying definition 3.3.1 to $B = BO_n(G,S)$ and $f\colon BO_n(G,S) \to BO_n(G)$ we get a G-∞-category $\underline{\mathbf{Mfld}}^{G,or}$ of oriented G-manifolds.

Remark 3.3.6. The notion of an oriented G-manifold seems not to agree with the notion of oriented global orbifold (see, for example, [ALR07, p. 34]).

Finally, we can use equivariant tangential structures to restrict the class of G-manifolds we consider, an idea introduced in [AFT17b, rem. 1.1.9].

Example 3.3.7. Applying definition 3.3.1 with $B = BO_n(G) \times EG$ and a G-map given by the projection $pr\colon BO_n(G) \times EG \to BO_n(G)$ produces a G-∞-category $\underline{\mathbf{Mfld}}^{G,pr-fr}$. In this example the forgetful G-functor $\underline{\mathbf{Mfld}}^{G,pr-fr} \to \underline{\mathbf{Mfld}}^{G}$ is fully faithful, and exhibits $\underline{\mathbf{Mfld}}^{G,pr-fr}$ as the full G-subcategory of $\underline{\mathbf{Mfld}}^{G}$ spanned by $\mathcal{O}_G$-manifolds $M \to O$ where M is a free G-manifold. We now give a quick sketch the argument.

We consider a manifold M with an action of G, describing an object $(M \to G/G) \in \underline{\mathbf{Mfld}}^{G}_{[G/G]}$ (the argument for an $\mathcal{O}_G$-manifold $M \to G/H$ is similar). A homotopy lift of $\tau_M\colon M \to BO_n(G)$ along the projection the same as a G-map $M \to EG$. A G-map $M \to EG$ exists if and only if the action of G on M is free, in which case the space of G-maps $\mathrm{Map}_G(M, EG)$ is contractible. This is easily seen by using the Elmendorf-McClure theorem; the presheaves that represents M and EG send

$$M, EG\colon \mathcal{O}_G^{op} \to \mathcal{S}, \quad M\colon G/H \mapsto M^H,\, EG\colon G/H \mapsto \begin{cases} pt, & H = e, \\ \emptyset & H \neq e, \end{cases}$$

and a map $M^H \to \emptyset$ exists if and only if M^H is empty. It follows that a map of $\mathcal{O}_G$-presheaves $M \to EG$ exists if and only if the action of G on M is free. Finally, if G acts freely on M then $\mathrm{Map}_{\mathrm{Fun}(\mathcal{O}_G^{op},\mathcal{S})}(M, EG) \simeq \mathrm{Map}(M, pt) \simeq pt$.

3.4 G-disjoint union of G-manifolds

The goal of this section is to endow the G-∞-category $\underline{\mathbf{Mfld}}^{G}$ with a G-symmetric monoidal structure associated to disjoint unions.

Recall that n-dimensional manifolds with G-action and G-equivariant embedding can be organized into a topological category $\mathbf{Mfld}^G$. Despite the fact that $\mathbf{Mfld}^G$ does not have coproducts[15], we can still endow $\mathbf{Mfld}^G$ with a symmetric monoidal structure by taking disjoint unions. Therefore the ∞-category $\mathbf{N}(\mathbf{Mfld}^G)$ admits a symmetric monoidal structure $\mathbf{N}^{\otimes}(\mathbf{Mfld}^G) \twoheadrightarrow \mathbf{Fin}_*$, given by applying the *operadic nerve* construction of [Lur, def. 2.1.1.23].

Similarly, disjoint unions endow the ∞-category $\mathbf{N}(\mathbf{Mfld}^H)$ with a symmetric monoidal structure, making the restriction and conjugation functors symmetric monoidal. We can therefore enhance $\mathbf{Mfld}^\bullet$ from a diagram of ∞-categories to a diagram of symmetric monoidal ∞-categories $\mathbf{N}^{\otimes}(\mathbf{Mfld}^\bullet)$. However, this construction does not encode the operation of topological induction and its coherent compatibility with the symmetric monoidal structure and restriction and conjugation of the action. The main point of this subsection is that all of the structure we are interested in can be encoded as a G-symmetric monoidal structure on the G-category of G-manifolds (see

[15] Note that $Emb^G(M_1 \sqcup M_2, M) \not\simeq Emb^G(M_1, M) \times Emb^G(M_2, M)$.

definition 3.1.16). It would be preferable to define this G-symmetric monoidal structure by an appropriate variant of the operadic nerve construction, however we are unaware of such construction. We therefore define the G-symmetric monoidal structure by explicitly constructing a coCartesian fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ (see definition 3.4.19).

Our construction can be briefly described as follows. The category $\underline{\mathbf{Fin}}_*^G$ is constructed as a category of spans in the category of finite G-sets over an orbit, $\mathcal{O}_G$-$\mathbf{Fin}$, (see lemma 3.4.2), so it is natural to construct $\underline{\mathbf{Mfld}}^{G,\sqcup}$ as category of spans of an auxiliary ∞-category $\mathcal{O}_G$-$\mathbf{Fin}$-$\mathbf{Mfld}$, defined over $\mathcal{O}_G$-$\mathbf{Fin}$. In definition 3.4.5 we construct $\mathcal{O}_G$-$\mathbf{Fin}$-$\mathbf{Mfld}$ as a topological category over $\mathcal{O}_G$-$\mathbf{Fin}$. We want to apply Barwick's unfurling construction, see [Bar14], to the functor $\mathbf{N}(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}) \to \mathcal{O}_G\text{-}\mathbf{Fin}$, in order to produce a coCartesian fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ between the respected ∞-categories of spans. There is a simple criterion, described in [Bar14], that ensures that the unfurled functor is a coCartesian fibration:

1. Egressive arrows in $\mathcal{O}_G$-$\mathbf{Fin}$, serving as the "wrong way arrows" in the span category $\underline{\mathbf{Fin}}_*^G$, have Cartesian lifts (verified in lemma 3.4.9).

2. Ingressive arrows in $\mathcal{O}_G$-$\mathbf{Fin}$, serving as the "right way arrows" in the span category $\underline{\mathbf{Fin}}_*^G$, have coCartesian lifts (verified in lemma 3.4.13).

3. The pullback squares appearing in the definition of composition in the span category $\underline{\mathbf{Fin}}_*^G$ satisfies a "Beck-Chevalley condition" (verified in proposition 3.4.15).

The resulting "unfurled" ∞-category $\underline{\mathbf{Mfld}}^{G,\sqcup}$ (see definition 3.4.19) admits an explicit description as an ∞-category of spans. In particular we have a description of the objects, morphisms and coCartesian morphisms of $\underline{\mathbf{Mfld}}^{G,\sqcup}$. Using the explicit description of $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ we show that it satisfies the G-Segal conditions and that its underlying G-∞-category is $\underline{\mathbf{Mfld}}^G$ (proposition 3.4.21).

Construction of the auxiliary category $\mathcal{O}_G$-Fin-Mfld

In this subsection we define a topological category $\mathcal{O}_G$-$\mathbf{Fin}$-$\mathbf{Mfld}$ with a functor to the category $\mathcal{O}_G$-$\mathbf{Fin}$ of finite G-sets over orbits. The topological category $\mathcal{O}_G$-$\mathbf{Fin}$-$\mathbf{Mfld}$ serves as input to the unfurling construction ([Bar14, sec. 11]), producing a coCartesian fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ that defines the G-symmetric monoidal structure of G-disjoint union on $\underline{\mathbf{Mfld}}^G$ (see definition 3.4.19).

We start with a definition of the category $\mathcal{O}_G$-$\mathbf{Fin}$, which serves as the base category of the unfurling construction.

Definition 3.4.1. *The category $\mathcal{O}_G$-$\mathbf{Fin}$ is the pullback $\mathcal{O}_G\text{-}\mathbf{Fin} := \mathrm{Fun}(\Delta^1, Fin^G) \times_{\mathrm{Fun}(\{1\}, Fin^G)} \mathcal{O}_G$. The category $\mathcal{O}_G$-$\mathbf{Fin}$ is a full subcategory of the arrow category $\mathrm{Fun}(\Delta^1, Fin^G)$, whose objects are arrows $U \to O$ in Fin^G such that $O \in \mathcal{O}_G$. A morphism in $\mathcal{O}_G$-$\mathbf{Fin}$ is a* summand-inclusion *([Nar16, def. 4.12]) if it factors as an inclusion over orbit-identity followed by a pullback square*

$$\begin{array}{ccccc} U_1 & \hookrightarrow & \varphi^* U_2 & \longrightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \xrightarrow{=} & O_1 & \xrightarrow{\varphi} & O_2 \end{array} \quad . \tag{9}$$

Note that we the inclusion of G-sets $U_1 \hookrightarrow \varphi^* U_2$ exhibits $\varphi^* U_2$ as the coproduct of the G-sets U_1 and $U' = \varphi^* U_2 \setminus U_1$. We can therefore identify $\varphi^* U_2 \cong U_1 \coprod U'$.

Let $\mathcal{O}_G$-$\mathbf{Fin}^\dagger \subset \mathcal{O}_G$-$\mathbf{Fin}$ be the subcategory consisting of all objects while morphisms are summand-inclusions.

It is straightforward to see that the G-category $\underline{\mathbf{Fin}}^G_*$ of [Nar16, def. 4.12] can be defined by the following unfurling construction.

Lemma 3.4.2. *The triple $(\mathcal{O}_G\text{-}\mathbf{Fin}, \mathcal{O}_G\text{-}\mathbf{Fin} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}, \mathcal{O}_G\text{-}\mathbf{Fin}^{\dagger})$ is an adequate triple in the sense of [Bar14, def. 5.2], and its effective Burnside category fits into a pullback square*

$$\begin{array}{ccc} \underline{\mathbf{Fin}}^G_* & \hookrightarrow & A^{eff}(\mathcal{O}_G\text{-}\mathbf{Fin}, \mathcal{O}_G\text{-}\mathbf{Fin} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}, \mathcal{O}_G\text{-}\mathbf{Fin}^{\dagger}) \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{O}_G & \hookrightarrow & A^{eff}(\mathcal{O}_G, \mathcal{O}_G^{\cong}, \mathcal{O}_G) \end{array} \tag{10}$$

We now define a topological category of "parametrized $\mathcal{O}_G$-manifolds" over $\mathcal{O}_G$-$\mathbf{Fin}$.

Definition 3.4.3. *An $\mathcal{O}_G$-$\mathbf{Fin}$-*manifold *$M \to U \to O$ is*

1. *a smooth n-dimensional manifold M with an action of G on M by smooth maps,*
2. *together with a G-map $M \to U$ from the underlying G-space of the manifold M to a G-finite set $U \in Fin^G$,*
3. *and an arrow $U \to O$ in $\mathbf{Fin}^G$ such that $O \in \mathcal{O}_G$.*

An morphism of $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds is given by a commuting square of G-spaces

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow & & \downarrow \\ U_1 & \xrightarrow{\overline{\varphi}} & U_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{\varphi} & O_2, \end{array}$$

such that the induced map $M_1 \to O_1 \times_{O_2} M_2$ is an embedding.

Definition 3.4.4. *Let $M_1 \to U_1 \to O_1$, $M_2 \to U_2 \to O_2$ be $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds and $\varphi\colon I_1 \to I_2$ a morphism in $\mathcal{O}_G$-$\mathbf{Fin}$ given by*

$$\begin{array}{ccc} U_1 & \xrightarrow{\overline{\varphi}} & U_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{\varphi} & O_2 \end{array}$$

. Define $Emb_{\varphi}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(M_1, M_2) \subset C^{\infty}(M_1, M_2)$ as the subspace of smooth maps $f: M_1 \to M_2$ such that $(f, \overline{\varphi}, \varphi)$ is a morphism of $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds from $M_1 \to U_1 \to O_1$ to $M_2 \to U_2 \to O_2$.

Definition 3.4.5. *The* Category of $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds $\mathcal{O}_G$-$\mathbf{Fin}$-$\mathbf{Mfld}$ *is the topological category whose objects are $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds. The morphism space from M_1 to M_2 is given by*

$$\mathrm{Map}_{\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}}(M_1, M_2) := \coprod_{\varphi} Emb_{\varphi}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(M_1, M_2),$$

where the coproduct is indexed by $\mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_1 \to O_1, U_2 \to O_2)$.

Define a forgetful functor $p\colon \mathcal{O}_G$-$\mathbf{Fin}$-$\mathbf{Mfld} \to \mathcal{O}_G$-$\mathbf{Fin}$ by sending $M \to U \to O$ to $U \to O$, and the subspace $Emb_{\varphi}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(M_1, M_2) \subset Map(M_1, M_2)$ to $\varphi \in$ $\mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_1 \to O_1, U_2 \to O_2)$.

From here on we will abuse notation, writing $\mathcal{O}_G$-**Fin**-**Mfld** for both the topological category $\mathcal{O}_G$-**Fin**-**Mfld**, its incarnation as a fibrant simplicial category **Sing**($\mathcal{O}_G$-**Fin**-**Mfld**) and its incarnation as an ∞-category **N**($\mathcal{O}_G$-**Fin**-**Mfld**), distinguishing between these incarnations by context.

Remark 3.4.6. Note that an equivalence $f\colon M \to N$ in $\mathcal{O}_G$-**Fin**-**Mfld** is always an embedding of smooth manifolds, since it lies over an isomorphism of orbits. Moreover, it is G-isotopic to an identity-of-manifolds over the isomorphism $p(f)$. On the other hand, if f is G-isotopic to an identity-of-manifolds over an isomorphism of finite G-sets then f is an equivalence in $\mathcal{O}_G$-**Fin**-**Mfld**, so we have a complete characterization of equivalences in $\mathcal{O}_G$-**Fin**-**Mfld**.

Some Cartesian and coCartesian edges of $\mathcal{O}_G$-Fin-Mfld $\to$ $\mathcal{O}_G$-Fin

We characterize p-Cartesian edges of $\mathcal{O}_G$-**Fin**-**Mfld** over summand-inclusions and p-coCartesian edges over isomorphisms of orbits. We summarize the results of this subsection as follows.

Proposition 3.4.7. *A morphism f of $\mathcal{O}_G$-**Fin**-**Mfld** over $\mathcal{O}_G$-**Fin**† is p-Cartesian if and only if it is equivalent to a pullback over a summand-inclusion. A morphism g of $\mathcal{O}_G$-**Fin**-**Mfld** over $\mathcal{O}_G$-**Fin** $\times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}$ is p-coCartesian if and only if it is G-isotopic to an identity-of-manifolds over an orbit-isomorphism.*

The characterization of p-Cartesian edges is given in corollary 3.4.12, and the characterization of p-coCartesian edges is given in corollary 3.4.14.

Remark 3.4.8. By [Lur09a, prop. 2.4.1.10(1)] the map $\mathcal{O}_G$-**Fin**-**Mfld** $\to$ $\mathcal{O}_G$-**Fin** is an inner fibration.

Lemma 3.4.9. *Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_1 \to O_1, U_2 \to O_2)$ be a morphism in $\mathcal{O}_G$-**Fin** given by a pullback square* $\begin{array}{ccc} U_1 & \longrightarrow & U_2 \\ \downarrow & \ulcorner & \downarrow \\ O_1 & \longrightarrow & O_2 \end{array}$*, and $N \to U_2 \to O_2$ a $\mathcal{O}_G$-**Fin**-manifold over its target. Then the pullback*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & \ulcorner & \downarrow \\ U_1 & \longrightarrow & U_2 \\ \downarrow & \ulcorner & \downarrow \\ O_1 & \longrightarrow & O_2 \end{array}$$

*defines a p-Cartesian morphism f in $\mathcal{O}_G$-**Fin**-**Mfld** lifting φ.*

Proof. According to [Lur09a, prop. 2.4.1.10(2)] we have to show that for every $\mathcal{O}_G$-**Fin**-manifold $T \to U \to O$ the commutative square

$$\begin{array}{ccc} \mathrm{Map}(T, M) & \xrightarrow{f_*} & \mathrm{Map}(T, N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(p(T), p(M)) & \xrightarrow{p(f)_*} & \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(p(T), p(N)) \end{array}$$

is a homotopy pullback. Since the vertical maps are Kan fibrations, this square is a homotopy pullback if and only if f_* induces an equivalence between the fibers over every vertex of the base $\mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(p(T), p(M))$.

Let $\tau \in \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(p(T), p(M))$. The functor f_* induces a map of the fibers over τ

$$(f_*)|_\tau \colon Emb_\tau^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, M) \to \{\tau\} \times_{\mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(p(T),p(M))} \mathrm{Map}(T, N).$$

Unwinding the definition of the mapping space in $\mathcal{O}_G$-**Fin-Mfld**, we have

$$\begin{aligned}\{\tau\} \times_{\mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(p(T),p(M))} \mathrm{Map}(T, N) &= \{\tau\} \times_{\mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(p(T),p(M))} \left(\coprod_{\varphi} Emb_\varphi^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, N)\right)\\ &= Emb_{p(f)\circ\tau}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, N),\end{aligned}$$

where the last equality holds since pullback along a fixed map preserve coproducts.

Suppose that the $\mathcal{O}_G$-**Fin**-manifold T is given by $T \to U \to O$ and $\tau \colon p(T) \to p(N)$ is given by the square $\begin{array}{ccc} U & \longrightarrow & U_1 \\ \downarrow & & \downarrow \\ O & \longrightarrow & O_1 \end{array}$. Then the map $(f_*)|_\tau \colon Emb_\tau^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, M) \to Emb_{p(f)\circ\tau}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, N)$ sends $h \colon T \to M$ to $f \circ h$:

$$(f_*)|_\tau \colon h = \left(\begin{array}{ccc} T & \xrightarrow{h} & M \\ \downarrow & & \downarrow \\ U & \longrightarrow & U_1 \\ \downarrow & & \downarrow \\ O & \longrightarrow & O_1 \end{array}\right) \mapsto \left(\begin{array}{ccccc} T & \xrightarrow{h} & M & \xrightarrow{f} & N \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ U & \longrightarrow & U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ O & \longrightarrow & O_1 & \longrightarrow & O_2 \end{array}\right).$$

The universal property of the pullback $M = N \times_{U_2} U_1$ shows that $(f_*)|_\tau$ is a continuous bijection: injectivity follows from uniqueness of maps to the pullback. Surjectivity: suppose $g \in Emb_{p(f)\circ\tau}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, N)$, by existence of a map to the pullback we have a candidate map $h \colon T \to M$ over τ such that $g = f \circ g$. We have to show that $h \in Emb_\tau^{\mathcal{O}_G\text{-}\mathbf{Fin}}$. Clearly h is a smooth G-map, so we only have to verify condition (3) of definition 3.4.4: h induces an embedding $T \to O \times_{O_1} M$. To see that observe that g induces an embedding $T \hookrightarrow O \times_{O_2} N$ which factors as the map induced by h followed by the isomorphism $O \times_{O_1} M = O \times_{O_1} (O_1 \times_{O_2} N) \cong O \times_{O_2} N$.

We leave it as an exercise to the reader to verify that $(f_*)|_\tau$ is an open map, and therefore a homeomorphism. ☐

Note that every G-map $M \to U_1 \coprod U_2$ from a manifold with G-action to a coproduct of G-sets factors as coproduct of G-maps $M = M_1 \coprod M_2 \to U_1 \coprod U_2$.

Lemma 3.4.10. *Let φ be an inclusion of finite G-sets over id_O in $\mathcal{O}_G$-**Fin**, given by the diagram* $\begin{array}{ccc} U_1 & \hookrightarrow & U_1 \coprod U_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{=} & O_1 \end{array}$*, and $M_1 \coprod M_2 \to U_1 \coprod U_2 \to O_2$ a $\mathcal{O}_G$-**Fin**-manifold over its target. Then* *the pullback*

$$\begin{array}{ccc} M_1 & \overset{i}{\hookrightarrow} & M_1 \coprod M_2 \\ \downarrow & \ulcorner & \downarrow \\ U_1 & \hookrightarrow & U_1 \coprod U_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{=} & O_1 \end{array}$$

*defines a p-Cartesian morphism i in $\mathcal{O}_G$-**Fin-Mfld** lifting φ.*

Proof. As in lemma 3.4.9, we have to show that for every $\mathcal{O}_G$-**Fin**-manifold $T \to U \to O$ and every $\tau\colon p(T) \to p(M_1)$ the map i_* induces equivalence of the fibers

$$Emb_{\tau}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, M_1) \to Emb_{p(i)\circ\tau}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(T, M_1 \coprod M_2).$$

As above, we use the universal property of the pullback to show this map is a bijection, and leave it to the reader to verify it is an open map.

The only part which is different is the verification of condition (3) of definition 3.4.4: g induces an embedding $T \hookrightarrow O \times_{O_1} (M_1 \coprod M_2)$, which factors as the composition of the map induced by h, an inclusion and an isomorphism

$$T \to O \times_{O_1} M_1 \hookrightarrow O \times_{O_1} M_1 \coprod O \times_{O_1} M_2 \cong O \times_{O_1} (M_1 \coprod M_2).$$

Since the composition is an embedding, the map $T \to O \times_{O_1} M_1$ induced by h is an embedding. □

Together, the lemmas above show the existence of p-Cartesian lifts over summand-inclusions and characterizes them.

Corollary 3.4.11. *Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_1 \to O_1, U_2 \to O_2)$ be a morphism in $\mathcal{O}_G$-**Fin**† and $N \to U_2 \to O_2$ an $\mathcal{O}_G$-**Fin**-manifold over its target. Then the pullback*

$$\begin{array}{ccc} M & \overset{f}{\longrightarrow} & N \\ \downarrow & \ulcorner & \downarrow \\ U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow \\ O_1 & \longrightarrow & O_2 \end{array}$$

*defines a p-Cartesian morphism f in $\mathcal{O}_G$-**Fin-Mfld** lifting φ.*

Proof. Factor the summand-inclusion φ as in (9), apply lemma 3.4.9 and lemma 3.4.10. □

By [Lur09a, prop. 2.4.1.7 and 2.4.1.5], we have

Corollary 3.4.12. *A morphism f of $\mathcal{O}_G$-**Fin-Mfld** over $\mathcal{O}_G$-$\mathbf{Fin}^{\dagger}$ is p-Cartesian if and only if it is equivalent to a pullback over a summand-inclusion, i.e the left map in the factorization*

$$f = \left(\begin{array}{ccccc} M_1 & \hookrightarrow & M_2 \times_{U_2} U_1 & \longrightarrow & M_2 \\ \downarrow & & \downarrow \ulcorner & & \downarrow \\ U_1 & \xrightarrow{=} & U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \xrightarrow{=} & O_1 & \longrightarrow & O_2 \end{array} \right)$$

*is an equivalence in $\mathcal{O}_G$-**Fin-Mfld** (a G-isotopy equivalence over U_1).*

Next, we construct p-coCartsian lifts over isomorphism of orbits.

Lemma 3.4.13. *Let* $\varphi = \left(\begin{array}{ccc} U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{\cong} & O_2 \end{array} \right)$ *be a morphism of $\mathcal{O}_G$-$\mathbf{Fin} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}$ and $M \to U_1 \to O_1$ an $\mathcal{O}_G$-**Fin**-manifold. Then* $f = \left(\begin{array}{ccc} M & \xrightarrow{=} & M \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow \\ O_1 & \xrightarrow{\cong} & O_2 \end{array} \right)$ *is a p-coCartesian lift of φ.*

Proof. By the dual version of [Lur09a, prop. 2.4.1.10(2)] we have to show that for every $\mathcal{O}_G$-**Fin**-manifold $T \to U \to O$ the square

$$\begin{array}{ccc} Map(M \to U_2 \to O_1, T \to U \to O) & \xrightarrow{f^*} & Map(M \to U_1 \to O_1, T \to U \to O) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_2 \to O_1, U \to O) & \xrightarrow{p(f)^*} & \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_1 \to O_1, U \to O) \end{array}$$

is a homotopy pullback square. Since the vertical maps are Kan fibrations, this square is a homotopy pullback if and only if f^* induces an equivalence between the fibers. Next, note that the map f^* is induced by composition with id_M, and the fibers over $\tau \in \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_2 \to O_1, U \to O)$ and $\tau \circ p(f) \in \mathrm{Hom}_{\mathcal{O}_G\text{-}\mathbf{Fin}}(U_1 \to O_1, U \to O)$ are both subspaces of the space of smooth maps $C^{\infty}(M, T)$:

$$Emb_{\tau}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(M, T) \subset C^{\infty}(M, T), \quad Emb_{\tau \circ p(f)}^{\mathcal{O}_G\text{-}\mathbf{Fin}}(M, T) \subset C^{\infty}(M, T).$$

We finish the proof by observing that these subspaces are equal: conditions (1),(3) of definition 3.4.4 coincide, while the equivalence of condition (2) follows from the commutativity of the square $\begin{array}{ccc} M & \xrightarrow{=} & M \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & U_2 \end{array}$, the top square of f. ∎

We therefore have a characterisation of p-coCartesian edges over orbit isomorphisms.

Corollary 3.4.14. *A morphism f of $\mathcal{O}_G$-**Fin-Mfld** over an orbit-isomorphism is p-Cartesian if and only if it is equivalent to an identity-of-manifolds, i.e. the right map in the factorization*

$$f = \begin{pmatrix} M_1 & \xrightarrow{=} & M_1 & \hookrightarrow & M_2 \\ \downarrow & & \downarrow & & \downarrow \\ U_1 & \longrightarrow & U_2 & \xrightarrow{=} & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \xrightarrow{\cong} & O_2 & \xrightarrow{=} & O_2 \end{pmatrix}$$

*is an equivalence in $\mathcal{O}_G$-**Fin-Mfld** (a G-isotopy equivalence over U_2).*

Construction of the G-symmetric monoidal category $\underline{\mathbf{Mfld}}^{G,\sqcup}$

We now turn to the goal of this subsection, the construction of a G-symmetric monoidal structure on the G-category of G-manifolds. In definition 3.4.19 we use the unfurling construction of [Bar14, sect. 11] to define a coCartesian fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$, and in proposition 3.4.21 we verify the Segal conditions, showing that it defines a G-symmetric monoidal structure on $\underline{\mathbf{Mfld}}^G$.

We first make sure that the conditions for applying Barwick's unfurling construction hold. Since Cartesian lifts of egressive morphisms and coCartesian lifts of ingressive morphisms were constructed in proposition 3.4.7 it remains to verify the Beck-Chevalley conditions.

Proposition 3.4.15. *The inner fibration $\mathcal{O}_G$-**Fin-Mfld** $\to \mathcal{O}_G$-**Fin** is adequate over the triple $(\mathcal{O}_G\text{-}\mathbf{Fin}, \mathcal{O}_G\text{-}\mathbf{Fin} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}, \mathcal{O}_G\text{-}\mathbf{Fin}^\dagger)$ ([Bar14, def. 10.3]).*

Proof. Conditions [Bar14, cond. (10.3.1),(10.3.2)] follow from proposition 3.4.7. To verify condition [Bar14, cond. (10.3.3)] construct the natural map $i_! \circ q^*(\tilde{N}) \to q'^* \circ j_!(\tilde{N})$ by choosing appropriate p-Cartesian and p-coCartesian lifts, and show that map is the universal map between two models of the same pullback, hence a diffeomorphism over an identity map.

Let $\begin{array}{ccc} s & \overset{i}{\rightarrowtail} & s' \\ {\scriptstyle q}\downarrow & \ulcorner & \downarrow{\scriptstyle q'} \\ t & \underset{j}{\rightarrowtail} & t' \end{array}$ be an ambigressive pullback square in $\mathcal{O}_G$-**Fin**, whose objects and morphisms are given by

$$s = \begin{pmatrix} \tilde{U} \\ \downarrow \\ \tilde{O}_1 \end{pmatrix}, \quad s' = \begin{pmatrix} U \\ \downarrow \\ O_1 \end{pmatrix}, \quad t = \begin{pmatrix} \tilde{V} \\ \downarrow \\ \tilde{O}_2 \end{pmatrix}, \quad t' = \begin{pmatrix} V \\ \downarrow \\ O_2 \end{pmatrix}, \quad i = \begin{pmatrix} \tilde{U} & \longrightarrow & U \\ \downarrow & & \downarrow \\ \tilde{O}_1 & \xrightarrow{\cong} & O_1 \end{pmatrix},$$

$$j = \begin{pmatrix} \tilde{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ \tilde{O}_2 & \xrightarrow{\cong} & O_2 \end{pmatrix}, \quad q = \begin{pmatrix} \tilde{U} & \longrightarrow & \tilde{V} \\ \downarrow & & \downarrow \\ \tilde{O}_1 & \longrightarrow & \tilde{O}_2 \end{pmatrix}, \quad q' = \begin{pmatrix} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ O_1 & \longrightarrow & O_2 \end{pmatrix}$$

And $\tilde{N} = (\tilde{N} \to \tilde{V} \to \tilde{O}_2)$ an object in the fiber of p over t. We compute $i_! \circ q^*(\tilde{N})$, $q'^* \circ j_!(\tilde{N})$ and the map $i_! \circ q^*(\tilde{N}) \to q'^* \circ j_!(\tilde{N})$ (natural in $\tilde{N}$) by choosing appropriate p-Cartesian and p-coCartesian lifts.

Let $\tilde{M} := \tilde{N} \times_{\tilde{N}} \tilde{U}$. Since q is a summand-inclusion by corollary 3.4.11 the map

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & \tilde{N} \\ \downarrow & \ulcorner & \downarrow \\ \tilde{U} & \longrightarrow & \tilde{V} \\ \downarrow & & \downarrow \\ \tilde{O}_1 & \longrightarrow & \tilde{O}_2 \end{array}$$

is p-Cartesian over q , so $q^*(\tilde{N}) := (\tilde{M} \to \tilde{U} \to \tilde{O}_1)$.

Since i is over an isomorphism of orbits, by lemma 3.4.13 the map

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{=} & \tilde{M} \\ \downarrow & & \downarrow \\ \tilde{U} & \longrightarrow & U \\ \downarrow & & \downarrow \\ \tilde{O}_1 & \xrightarrow{\cong} & O_1 \end{array}$$

is p-coCartesian over i, so $i_! \circ q^*(\tilde{N}) := (\tilde{M} \to U \to O_1)$.

Since j is over an isomorphism of orbits, by lemma 3.4.13 the map

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{=} & \tilde{N} \\ \downarrow & & \downarrow \\ \tilde{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ \tilde{O}_2 & \xrightarrow{\cong} & O_2 \end{array}$$

is p-coCartesian over j, so $j_!(\tilde{N}) := (\tilde{N} \to V \to O_2)$.

Let $M := \tilde{N} \times_V U$. Since q' is a summand-inclusion by corollary 3.4.11 the map

$$\begin{array}{ccc} M & \longrightarrow & \tilde{N} \\ \downarrow & \ulcorner & \downarrow \\ U & \longrightarrow & V \\ \downarrow & & \downarrow \\ O_1 & \longrightarrow & O_2 \end{array}$$

is p-Cartesian over q', so $q'^* \circ j_!(\tilde{N}) := (M \to U \to O_1)$.

Next, we choose a map $\xi \colon q^*(\tilde{N}) \to q'^* \circ j_!(\tilde{N})$ over i by composing the lifts of q and j above and using the universal property of the pullback M

$$
\begin{array}{ccccc}
\tilde{M} & \longrightarrow & \tilde{N} & \xrightarrow{=} & \tilde{N} \\
\downarrow & \ulcorner & \downarrow & & \downarrow \\
\tilde{U} & \longrightarrow & \tilde{V} & \longrightarrow & V \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{O}_1 & \longrightarrow & \tilde{O}_2 & \xrightarrow{\cong} & O_2
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccccc}
\tilde{M} & \overset{\exists ! \xi}{\dashrightarrow} & M & \longrightarrow & \tilde{N} \\
\downarrow & & \downarrow & \ulcorner & \downarrow \\
\tilde{U} & \longrightarrow & U & \longrightarrow & V \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{O}_1 & \xrightarrow{\cong} & O_1 & \longrightarrow & O_2
\end{array}
.
$$

The map ξ induces the natural map $\overline{\xi}\colon i_! \circ q^*(\tilde{N}) \to q'^* \circ j_!(\tilde{N})$ over $id_{s'}$ by

$$
\begin{array}{ccccc}
\tilde{M} & \xrightarrow{=} & \tilde{M} & \overset{\exists ! \xi}{\dashrightarrow} & M \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{U} & \longrightarrow & U & \xrightarrow{=} & U \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{O}_1 & \xrightarrow{\cong} & O_1 & \xrightarrow{=} & O_1
\end{array}
.
$$

In order to verify [Bar14, cond. (10.3.3)] we have to show that $\overline{\xi}$ is an equivalence in the fiber over s'. We show that ξ is a diffeomorphism. Consider the diagram

$$
\begin{array}{ccc}
\tilde{M} & \longrightarrow & \tilde{N} \\
\downarrow & \ulcorner & \downarrow \\
\tilde{U} & \longrightarrow & \tilde{V} \\
\downarrow & \ulcorner & \downarrow \\
U & \longrightarrow & V
\end{array}
$$

the top square is a pullback square by definition of $\tilde{M}$, and the bottom square is a pullback square by assumption. Therefore the outer rectangle is a pullback square. By the universal property of $M = \tilde{N} \times_U V$ the induced map ξ is a diffeomorphism, as claimed.

This ends the proof of proposition 3.4.15. □

We can now define $\underline{\mathbf{Mfld}}^{G,\sqcup}$ by applying the unfurling construction to $\mathcal{O}_G$-**Fin**-**Mfld** $\to$ $\mathcal{O}_G$-**Fin**.

Definition 3.4.16. *Define a subcategory* $(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})^\dagger \subset \mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}$ *with the same objects as* $\mathcal{O}_G$-**Fin**-**Mfld***, and with morphisms the* p*-Cartesian edges over summand-inclusions (i.e over edges over* $\mathcal{O}_G\text{-}\mathbf{Fin}^\dagger$*). Define a subcategory* $(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_\dagger \subset \mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}$ *by*

$$(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_\dagger := \mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld} \times_{\mathcal{O}_G\text{-}\mathbf{Fin}} (\mathcal{O}_G\text{-}\mathbf{Fin} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}) \cong \mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}.$$

Construction 3.4.17. By lemma 3.4.2, proposition 3.4.15 and [Bar14, prop. 11.2] the triple $(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}, (\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_\dagger, (\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})^\dagger)$ is adequate. This condition ensures we can form the ∞-category of spans $A^{eff}(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}, (\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_\dagger, (\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})^\dagger)$. Applying the effective Burnside construction to $p\colon \mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld} \to \mathcal{O}_G\text{-}\mathbf{Fin}$ we get a functor

$$\begin{array}{c} A^{eff}(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}, (\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_\dagger, (\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})^\dagger) \\ \Big\downarrow \Upsilon(p) \\ A^{eff}(\mathcal{O}_G\text{-}\mathbf{Fin}, \mathcal{O}_G\text{-}\mathbf{Fin} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}, \mathcal{O}_G\text{-}\mathbf{Fin}^\dagger), \end{array}$$

called the unfurling of p in [Bar14, def. 11.3].

Lemma 3.4.18. *The functor $\Upsilon(p)$ is a coCartesian fibration.*

Proof. The functor $\Upsilon(p)$ is an inner fibration by [Bar14, lem. 11.4], and a coCartesian fibration by [Bar14, lem. 11.5] and proposition 3.4.7. $\square$

Definition 3.4.19. *Define a coCartesian fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ by pulling $\Upsilon(p)$ along the inclusion $\underline{\mathbf{Fin}}_*^G \hookrightarrow A^{eff}(\mathcal{O}_G\text{-}\mathbf{Fin}, \mathcal{O}_G\text{-}\mathbf{Fin} \times_{\mathcal{O}_G} \mathcal{O}_G^{\cong}, \mathcal{O}_G\text{-}\mathbf{Fin}^\dagger)$ of* (10).

Remark 3.4.20. Unwinding the definition of the effective Burnside category, we see that the objects of $\underline{\mathbf{Mfld}}^{G,\sqcup}$ are $\mathcal{O}_G$-**Fin**-manifolds, and a morphism $f\colon M_1 \to M_2$ is represented by a span

$$f = \begin{pmatrix} M_1 & \leftarrow & M & \rightarrow & M_2 \\ \downarrow & & \downarrow & & \downarrow \\ U_1 & \leftarrow & U & \rightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \leftarrow & O_2 & \xrightarrow{=} & O_2 \end{pmatrix},$$

where the 'backwards arrow' is equivalent to a pullback over a summand-inclusion. The morphism f is coCartsian exactly when the 'forward arrow' is equivalent to an identity-of-manifolds (see proposition 3.4.7 and [Bar14, lem. 11.5]).

Proposition 3.4.21. *The coCartesian fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ of definition 3.4.19 is G-symmetric monoidal category whose underlying G-category is isomorphic to the G-category $\underline{\mathbf{Mfld}}^G$ of definition 3.1.16. We call this G-symmetric monoidal structure G-*disjoint union *of G-manifolds.*

Proof. By definition B.0.4 the underlying G-category of $\underline{\mathbf{Mfld}}^{G,\sqcup}$ has objects $\mathcal{O}_G$-**Fin**-manifolds of the form $(M \to O \xrightarrow{=} O)$ and maps represented by spans of the form

$$\begin{pmatrix} M_1 & \leftarrow & M & \rightarrow & M_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \leftarrow & O_2 & \xrightarrow{=} & O_2 \\ \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\ O_1 & \leftarrow & O_2 & \xrightarrow{=} & O_2 \end{pmatrix}$$

with left square equivalent to a pullback. This G-category is isomorphic to $\underline{\mathbf{Mfld}}^G$ by the forgetful functor $(M \to O \xrightarrow{=} O) \mapsto (M \to O)$.

By lemma B.0.10 it is enough to show that for every $I = (U \to O) \in \underline{\mathbf{Fin}}_*^G$ the induced functor $\prod \rho_*^W : \underline{\mathbf{Mfld}}_I^{G,\sqcup} \to \prod_{W \in \mathrm{Orbit}(U)} \underline{\mathbf{Mfld}}_{[W]}^G$ is an equivalence of ∞-categories, where ρ_*^W is induced by the fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ and the inert edge

$$\rho^W = \left(\begin{array}{ccccc} U & \hookleftarrow & W & \xrightarrow{=} & W \\ \downarrow & & \downarrow{=} & & \downarrow{=} \\ O & \leftarrow & W & \xrightarrow{=} & W. \end{array} \right), \quad \rho^W \in \underline{\mathbf{Fin}}_*^G.$$

Let $(M \to U \to O) \in \underline{\mathbf{Mfld}}_I^G$ be an $\mathcal{O}_G$-$\mathbf{Fin}$-manifold. The decomposition $U = \coprod_{W \in \mathrm{Orbit}(U)} W$ into orbits induces a decomposition of M into a disjoint union $M = \sqcup_{W \in \mathrm{Orbit}(U)} M_W$. The action of ρ_*^W on $(M \to U \to O)$ is specified by a choice of coCartesian lift over ρ^W. By the above description of coCartesian edges we see that

$$\left(\begin{array}{ccccc} M & \hookleftarrow & M_W & \xrightarrow{=} & M_W \\ \downarrow & & \downarrow & & \downarrow \\ U & \hookleftarrow & W & \xrightarrow{=} & W \\ \downarrow & & \downarrow{=} & & \downarrow{=} \\ O & \leftarrow & W & \xrightarrow{=} & W. \end{array} \right)$$

is such a coCartesian edge, therefore the functor $\prod \rho_*^W$ is given by

$$\prod \rho_*^W : \underline{\mathbf{Mfld}}_I^{G,\sqcup} \to \prod_{W \in \mathrm{Orbit}(U)} \underline{\mathbf{Mfld}}_{[W]}^G,$$

$$\prod \rho_*^W : \begin{pmatrix} M \\ \downarrow \\ U \\ \downarrow \\ O \end{pmatrix} = \begin{pmatrix} \bigsqcup_{W \in \mathrm{Orbit}(U)} M_W \\ \downarrow \\ \coprod_{W \in \mathrm{Orbit}(U)} W \\ \downarrow \\ O \end{pmatrix} \mapsto \begin{pmatrix} M_W \\ \downarrow \\ W \\ \downarrow{=} \\ W \end{pmatrix}_{W \in \mathrm{Orbit}(U)}$$

which is an equivalence by inspection. □

3.5 G-disjoint union of f-framed G-manifolds

In this subsection we lift G-disjoint union of G-manifolds to a G-symmetric monoidal structure on $\underline{\mathbf{Mfld}}^{G,f-fr}$. Recall that $\underline{\mathbf{Mfld}}^{G,f-fr}$ was defined as the pullback of G-∞-categories (see definition 3.3.1). We will show that the G-symmetric monoidal structure of $\underline{\mathbf{Mfld}}^G$ lifts to $\underline{\mathbf{Mfld}}^{G,f-fr}$ by exhibiting the pullback square of definition 3.3.1 as underlying a pullback square of G-symmetric monoidal G-∞-categories and G-symmetric monoidal functors.

In addition to G-disjoint unions of G-manifolds we will use the G-coCartesian structure, constructed in [BDG+] and given by G-coproducts. In general the G-coCartesian structure on a G-category $\underline{\mathcal{C}}$ is given by a G-∞-operad $\underline{\mathcal{C}}^{\amalg}$. However, we will only use this construction for $\underline{\mathcal{C}}$ with finite G-coproducts, in which case $\underline{\mathcal{C}}^{\amalg}$ is a G-symmetric monoidal G-∞-category.

We show show that the G-functors in the pulback square of definition 3.3.1 extend to G-symmetric monoidal functors in two steps. By a formal argument these G-functors extend to *lax*

G-symmetric monoidal functors. It then remains to verify that these lax G-symmetric monoidal functors are in fact G-symmetric monoidal.

The following claim allows us to extend G-functors to $\underline{\mathcal{C}}$ from certain G-∞-operads to lax G-symmetric monoidal functors.

Lemma 3.5.1. *Let $\underline{\mathcal{C}}$ be a G-category and $\underline{\mathcal{O}}^{\otimes}$ a unital G-∞-operad. Restriction to the underlying G-category induces an equivalence*

$$Alg_G(\underline{\mathcal{O}}, \underline{\mathcal{C}}) \to \mathrm{Fun}_G(\underline{\mathcal{O}}, \underline{\mathcal{C}})$$

between the ∞-category of morphisms of G-∞-operads from $\underline{\mathcal{O}}^{\otimes}$ to $\underline{\mathcal{C}}^{\amalg}$ and the ∞-category of G-functors between the underlying G-categories.

Let $B \in \mathbf{Top}^G$ be a G-space and $f \colon B \to BO_n(G)$ be a G-map. Endow the parametrized slice G-categories $\underline{\mathbf{Top}}^G_{/\underline{B}}$, $\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}$ with the G-coCartesian G-symmetric monoidal structure. By lemma 3.5.1 the G-functors

$$f_* \colon \underline{\mathbf{Top}}^G_{/\underline{B}} \to \underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}, \quad \tau \colon \underline{\mathbf{Mfld}}^G \to \underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}$$

admit an essentially unique lift to lax G-symmetric monoidal functors

$$f_* \colon \underline{\mathbf{Top}}^G_{/\underline{B}} \to (\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}})^{\amalg}, \quad \tau \colon \underline{\mathbf{Mfld}}^{G,\sqcup} \to (\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}})^{\amalg}$$

The following description of the G-coCartesian structure $(\underline{\mathbf{Top}}^G_{/\underline{B}})^{\amalg}$ is useful when verifying that the lax G-symmetric monoidal functors τ, f_* constructed above are in fact G-symmetric monoidal.

Remark 3.5.2. Let $I = (U \to G/H) \in \underline{\mathbf{Fin}}^G_*$. Then a U-family $x_\bullet \colon \underline{U} \to \underline{\mathbf{Top}}^G$ can be described by a G-map $X \to U$. Moreover, under this description the parametrized coproduct $\coprod_I x_\bullet \colon \underline{G/H} \to \underline{\mathbf{Top}}^G$ is given by the G-map $X \to U \to G/H$.

To see this first construct the left fibration associated to $x_\bullet$, and then notice it is a map of G-∞-groupoids and therefore can identified with a map of G-spaces $X \to U$. One should think of the family $X \to U$ as assigning to each $W \in \mathrm{Orbit}(U)$ the G-map $(X|_W \to W) \in \underline{\mathbf{Top}}^G_{[W]}$, where we use the explicit model of remark 2.1.7. In order to see that $\coprod_I x_\bullet$ is given by $(X \to U \to G/H) \in \underline{\mathbf{Top}}^G_{[G/H]}$ recall that $\coprod_I$ is given by G-left Kan extension along $\underline{U} \to \underline{G/H}$, which by [Sha18, prop. 10.9] is given by (unparametrized) left Kan extension along $\underline{U} \to \underline{G/H}$. Applying straightening/unstraightening, we see that $\coprod_I$ is let adjoint to pulling back along $\underline{U} \to \underline{G/H}$, and therefore given by post-composition with $\underline{U} \to \underline{G/H}$.

Let B be a G-space. Combining remark 3.2.9 with the description of U-families in $\underline{\mathbf{Top}}^G$ above, we get the following description of G-coproducts in $\underline{\mathbf{Top}}^G_{/\underline{B}}$. A U-family $x_\bullet \colon \underline{U} \to \underline{\mathbf{Top}}^G_{/\underline{B}}$ is given by a G-map $X \to U$ together with a collection of G-maps $\{X|_W \to B\}$ indexed by $W \in \mathrm{Orbit}(U)$. Equivalently, $x_\bullet \colon \underline{U} \to \underline{\mathbf{Top}}^G_{/\underline{B}}$ is given by a pair of G-maps $(X \to U,\, X \to B)$. The G-coproduct $\coprod_I x_\bullet \in \left(\underline{\mathbf{Top}}^G_{/\underline{B}}\right)_{[G/H]}$ is given by $(X \to U \to G/H) \in \underline{\mathbf{Top}}^G_{[G/H]}$ together with the G-map $X \to B$.

Lemma 3.5.3. *The functor $\tau \colon \underline{\mathbf{Mfld}}^{G,\sqcup} \to (\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}})^{\amalg}$ is a G-symmetric monoidal functor.*

Proof. By proposition 3.2.10 and the Segal conditions we have a concrete description of τ. Namely, if $I = (U \to G/H) \in \underline{\mathbf{Fin}}_*^G$ and $(M \to U \to G/H) \in \underline{\mathbf{Mfld}}_I^{G,\sqcup}$ is a $\mathcal{O}_G$-**Fin**-manifold then $\tau(M \to U \to G/H) \in (\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}})^{\amalg}$ is given by $(M \to U \to G/H) \in \underline{\mathbf{Top}}^G_I$ together with the G-map $M \to BO_n(G)$ classifying $TM \to M$. Therefore the G-coproduct $\coprod_I \tau(M \to U \to G/H)$ is given by $(M \to U \to G/H) \in \underline{\mathbf{Top}}^G_{[G/H]}$ together with the G-map $M \to BO_n(G)$ classifying $TM \to M$.

On the other hand, by remark 3.4.20 the G-disjoint union $\sqcup_I M \in \underline{\mathbf{Mfld}}^G_{[G/H]}$ is the $\mathcal{O}_G$-manifold given by the composition $M \to U \to G/H$, therefore $\tau(\sqcup_I M)$ is given by the $\mathcal{O}_G$-manifold $(M \to U \to G/H) \in \underline{\mathbf{Mfld}}^G_{[G/H]}$ together with the G-map $M \to BO_n(G)$ classifying $TM \to M$. $\square$

Proposition 3.5.4. *The G-functor $f_* \colon \underline{\mathbf{Top}}^G_{/\underline{B}} \to \underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}$ extends to a G-symmetric monoidal functor $f_* \colon (\underline{\mathbf{Top}}^G_{/\underline{B}})^{\amalg} \to (\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}})^{\amalg}$.*

Proof. This is an immediate consequence of the description of G-coproducts in $\underline{\mathbf{Top}}^G_{/\underline{B}}$ and $\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}$: for $I = (U \to G/H)$ the diagram

$$\begin{array}{ccc} \left(\underline{\mathbf{Top}}^G_{/\underline{B}}\right)^{\amalg}_I & \xrightarrow{f_*} & \left(\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}\right)^{\amalg}_I \\ \downarrow{\scriptstyle \sqcup_I} & & \downarrow{\scriptstyle \amalg_I} \\ \left(\underline{\mathbf{Top}}^G_{/\underline{B}}\right)_{[G/H]} & \xrightarrow{f_*} & \left(\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}}\right)_{[G/H]} \end{array}$$

is commutativity, since remark 3.5.2 implies it is given by

$$\begin{array}{ccc} (X \to U, X \to B) & \stackrel{f_*}{\longmapsto} & (X \to U, X \to B \xrightarrow{f} BO_n(G)) \\ \downarrow{\scriptstyle \sqcup_I} & & \downarrow{\scriptstyle \amalg_I} \\ (X \to U \to G/H, X \to B) & \stackrel{f_*}{\longmapsto} & (X \to U \to G/H, X \to B \xrightarrow{f} BO_n(G)). \end{array}$$

$\square$

It follows that given a G-map $f \colon B \to BO_n(G)$ over G/H we can endow $\underline{\mathbf{Mfld}}^{G,f-fr}$ with a G-symmetric monoidal structure.

Corollary 3.5.5. *The G-symmetric monoidal structure of G-disjoint union on $\underline{\mathbf{Mfld}}^G$ lifts to a G-symmetric monoidal structure on $\underline{\mathbf{Mfld}}^{G,f-fr}$, given by the pullback*

$$\begin{array}{ccc} \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup} & \longrightarrow & (\underline{\mathbf{Top}}^G_{/\underline{B}})^{\amalg} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle f_*} \\ \underline{\mathbf{Mfld}}^{G,\sqcup} & \xrightarrow{\tau} & (\underline{\mathbf{Top}}^G_{/\underline{BO_n(G)}})^{\amalg}. \end{array}$$

Proof. The ∞-category $\mathcal{C}\mathrm{at}_\infty^{G,\otimes}$ of G-symmetric monoidal categories admits limits, and the forgetful G-functor $\mathcal{C}\mathrm{at}_\infty^{G,\otimes} \to \mathcal{C}\mathrm{at}_\infty^{G}$ sending a G-symmetric monoidal category $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ to its underlying G-category $\underline{\mathcal{C}} = \underline{\mathcal{C}}^{\otimes} \times_{\underline{\mathbf{Fin}}_*^G} \mathcal{O}_G^{op}$ preserves limits. ☐

Remark 3.5.6. Informally, we can describe an object of $\underline{\mathbf{Mfld}}^{G,f-fr,\sqcup}$ over $(U \to G/H) \in \underline{\mathbf{Fin}}_*^G$ as an $\mathcal{O}_G$-**Fin**-manifold $(M \to U \to G/H)$ together an f-framing $f_M \colon M \to B \times G/H$.

Definition 3.5.7. *Let* $\underline{\mathbf{Rep}}_n^{G,f-fr,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup}$ *be the full G-subcategory of* $\underline{\mathbf{Mfld}}^{G,f-fr,\sqcup}$ *given by the pullback*

$$\begin{array}{ccc} \underline{\mathbf{Rep}}_n^{G,f-fr,\sqcup} & \hookrightarrow & \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup} \\ \downarrow & \ulcorner & \downarrow \\ \underline{\mathbf{Rep}}_n^{G,\sqcup} & \hookrightarrow & \underline{\mathbf{Mfld}}^{G,\sqcup}. \end{array}$$

It follows that $\underline{\mathbf{Rep}}_n^{G,f-fr,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup}$ *is the full subcategory of f-framed* $\mathcal{O}_G$-**Fin**-*manifolds* $(E \to U \to G/H)$ *where* $E \to U$ *is a G-vector bundle. Note that* $\underline{\mathbf{Rep}}_n^{G,f-fr,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ *is a G-*∞*-operad.*

3.6 The G-category of G-disks and the definition of G-disk algebras

Our next goal is to define the G-symmetric monoidal G-∞-category of G-disks $\underline{\mathbf{Disk}}^{G,\sqcup}$, and its framed variants $\underline{\mathbf{Disk}}^{G,f-fr,\sqcup}$. These G-∞-categories are the point of contact between equivariant algebra and equivariant geometry.

On the one hand, we use $\underline{\mathbf{Disk}}^{G,\sqcup}$ to define G-disk algebras, which serve as coefficients for genuine equivariant factorization homology. In a nutshell, the algebraic structure of a G-disk algebra is indexed by equivariant embeddings of G-disks.

On the other hand, G-disks capture the local geometry of G-manifolds: G-disks are designed to be the G-tubular neighbourhoods of a configuration of orbits in a G-manifold. We will therefore define G-disks as a full G-subcategory $\underline{\mathbf{Disk}}^G \subset \underline{\mathbf{Mfld}}^G$ of the G-∞-category of G-manifolds.

After defining $\underline{\mathbf{Disk}}^G$ we show that G-disjoint unions endow it with G-symmetric monoidal structure (see definition 3.6.5 and corollary 3.6.8). Finally, we construct $\underline{\mathbf{Disk}}^{G,f-fr}$, a framed version of the G-∞-category of G-disks (see definition 3.6.9) and define f-framed G-disk algebras (see definition 3.6.11).

Definition 3.6.1 (G-disks)**.** *A G-disk is a G-vector bundle* $E \to O$ *rank n, where* $O \in \mathcal{O}_G$ *is an orbit. Clearly a G-disk is an* $\mathcal{O}_G$*-manifold.*

Let $\underline{\mathbf{Disk}}^G \subset \underline{\mathbf{Mfld}}^G$, $\mathbf{Disk}^G \subset \mathbf{Mfld}^G$ *be the full subcategories spanned by* $\mathcal{O}_G$*-manifolds equivalent to a composition* $E \to U \to O$ *of G-vector bundle* $E \to U$ *of rank n over a finite G-set* $(U \to O) \in \underline{\mathbf{Fin}}^G$.

Remark 3.6.2. The G-subcategory $\underline{\mathbf{Disk}}^G \subset \underline{\mathbf{Mfld}}^G$ is the full G-subcategory generated from G-disks by finite G-disjoint unions. We think of $(E \to U \to O) \in \underline{\mathbf{Mfld}}^G$ as a G-disjoint union of G-disks: the decomposition $U = \sqcup_{W \in \mathrm{Orbit}(U)} W$ into orbits decomposes E into a disjoint union of G-vector bundles $E_W \to W$, and each composition $E_W \to W \to O$ exhibits $E_W \to O$ as the topological induction of $E_W \to W$ along $W \to O$.

In fact, $\underline{\mathbf{Disk}}^G$ is the free G-category generated from H-representations for $H < G$, considered as G-vector bundles over G/H, by disjoint unions and topological induction (see lemma 3.7.2 below).

We first verify that $\underline{\mathbf{Disk}}^G$ is a G-∞-category.

Proposition 3.6.3. *The subcategory $\underline{\mathbf{Disk}}^G \subset \underline{\mathbf{Mfld}}^G$ is a G-subcategory stable under equivalences.*

Proof. By [BDG⁺16b, lem. 4.5] it is enough to show that for any coCartesian edge $x \to y$ in $\underline{\mathbf{Mfld}}^G$ if $x \in \underline{\mathbf{Disk}}^G$ then $y \in \underline{\mathbf{Disk}}^G$. Recall that an edge

$$\begin{pmatrix} M_1 & \longleftarrow & M & \longrightarrow & M_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \xleftarrow{\ \varphi\ } & O_2 & \xrightarrow{\ =\ } & O_2 \end{pmatrix}$$

in $\underline{\mathbf{Mfld}}^G$ is coCartesian if and only if the left square is equivalent to a pullback square and the right square is a G-isotopy equivalence. Let $(M_1 \to O_1) \in \underline{\mathbf{Disk}}^G$, then by definition it is equivalent to $E \to U \to O_1$ for U a finite G-set and $E \to U$ a G-vector bundle. Pulling back along φ shows that $M \to O_2$ is equivalent to $\varphi^* E \to \varphi^* U \to O$, a G-vector bundle over a finite G-set. Since $M_2 \to O_2$ is equivalent to $M \to O_2$ it follows that $(M_2 \to O) \in \underline{\mathbf{Disk}}^G$. ☐

Remark 3.6.4. The coCartesian fibration $\underline{\mathbf{Disk}}^G \twoheadrightarrow \mathcal{O}_G^{op}$ is dual to the Cartesian fibration $\mathbf{Disk}^G \to \mathcal{O}_G$.

G-disjoint union of G-disks We now show (corollary 3.6.8) that G-disjoint union of G-manifolds (see proposition 3.4.21) induces a G-symmetric monoidal structure on $\underline{\mathbf{Disk}}^G$.

Definition 3.6.5. *Define $\underline{\mathbf{Disk}}^{G,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,\sqcup}$ to be the full subcategory spanned by the $\mathcal{O}_G$-**Fin**-manifolds $M \to U \to O$ equivalent to $E \to U' \to U \to O$ where $E \to U'$ is a G-vector bundle over a finite G-set U'.*

Remark 3.6.6. Note that if $M \to U \to O$ is equivalent to $E \to U' \to U \to O$ where $E \to U'$ is a G-vector bundle over a finite G-set U', then $U' = \pi_0(E) \cong \pi_0(M)$ is the set of connected components of M with the induced action.

Lemma 3.6.7. *The subcategory $\underline{\mathbf{Disk}}^{G,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,\sqcup}$ is a G-subcategory stable under equivalences.*

Proof. The proof follows from the characterization of coCartesian edges of $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ as spans of $\mathcal{O}_G$-**Fin**-manifolds where the 'backwards arrow' is equivalent to a pullback over a summand-inclusion and the 'forwards arrow' is equivalent to an identity-of-manifolds, following the outline of proposition 3.6.3. ☐

Corollary 3.6.8. *The operation of G-disjoint union on $\underline{\mathbf{Mfld}}^G$ induces a G-symmetric monoidal structure on the G-subcategory $\underline{\mathbf{Disk}}^G$.*

Proof. By lemma 3.6.7 it is enough to show that to show that the underlying G-category of $\underline{\mathbf{Disk}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ is equivalent to $\underline{\mathbf{Disk}}^G$. Indeed, pulling back along the G-functor

$$\sigma_{<G/G>} \colon \mathcal{O}_G^{op} \to \underline{\mathbf{Fin}}_*^G, \quad O \mapsto (O \xrightarrow{=} O)$$

we see that the underlying category $\underline{\mathbf{Disk}}^{G,\sqcup}_{<G/G>}$ has objects $\mathcal{O}_G$-**Fin**-manifolds equivalent to $(E \to U' \to O \xrightarrow{=} O)$ for $E \to U'$ a G-vector bundle over a G-finite set. Therefore the full G-subcategory $\underline{\mathbf{Disk}}^{G,\sqcup}_{<G/G>} \subset \underline{\mathbf{Mfld}}^{G,\sqcup}_{<G/G>}$ corresponds to $\underline{\mathbf{Disk}}^G \subset \underline{\mathbf{Mfld}}^G$ under the identification

$$\underline{\mathbf{Mfld}}^{G,\sqcup}_{<G/G>} \simeq \underline{\mathbf{Mfld}}^G, \quad (M \to O \xrightarrow{=} O) \mapsto (M \to O).$$ ☐

Framed G-disks We now define f-framed G-disks by restricting the underlying $\mathcal{O}_G$-manifolds of f-framed $\mathcal{O}_G$-manifolds to G-disks.

Definition 3.6.9. *Let $B \in \mathbf{Top}^G$ be a G-space and $f\colon B \to BO_n(G)$ be a G-map. Define the G-categories of f-framed G-disks as the pullback on the left.*

$$\begin{array}{ccc} \underline{\mathbf{Disk}}^{G,f-fr} & \hookrightarrow & \underline{\mathbf{Mfld}}^{G,f-fr} \\ \downarrow & \ulcorner & \downarrow \\ \underline{\mathbf{Disk}}^{G} & \hookrightarrow & \underline{\mathbf{Mfld}}^{G}, \end{array} \qquad \begin{array}{ccc} \underline{\mathbf{Disk}}^{G,f-fr,\sqcup} & \hookrightarrow & \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup} \\ \downarrow & \ulcorner & \downarrow \\ \underline{\mathbf{Disk}}^{G,\sqcup} & \hookrightarrow & \underline{\mathbf{Mfld}}^{G,\sqcup}. \end{array}$$

The G-symmetric monoidal structure of G-disjoint union on $\underline{\mathbf{Disk}}^G$ lifts to a G-symmetric monoidal structure on $\underline{\mathbf{Disk}}^{G,f-fr}$, given by the right pullback above.

G-disk algebras We define G-disk algebras using G-symmetric monoidal functors.

Notation 3.6.10. Let $p\colon \underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$, $q\colon \mathcal{D}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ be two G-symmetric monoidal categories. A G-symmetric monoidal functor from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$ is a functor of ∞-categories $f: \underline{\mathcal{C}}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\underline{\mathbf{Fin}}_*^G$ that takes p-coCartesian edges to q-coCartesian edges. Denote the ∞-category of G-symmetric monoidal functors from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$ by $\mathrm{Fun}_G^{\otimes}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) := \mathrm{Fun}_{\underline{\mathbf{Fin}}_*^G}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes})$.

Definition 3.6.11. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ be a G-symmetric monoidal category. A* G-disk algebra *with values in $\underline{\mathcal{C}}$ is a G-symmetric monoidal functor $A\colon \underline{\mathbf{Disk}}^{G,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ (see definition 3.6.5). Denote the ∞-category of G-disk algebras in $\underline{\mathcal{C}}$ by $\mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^G, \underline{\mathcal{C}})$.*

Let $f\colon B \to BO_n(G)$ a G-map, as in definition 3.3.1. An f-framed G-disk algebra *with values in $\underline{\mathcal{C}}$ is a G-symmetric monoidal functor $A\colon \underline{\mathbf{Disk}}^{G,f-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ (see corollary 3.5.5). Denote the ∞-category of G-disk algebras in $\underline{\mathcal{C}}$ by $\mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}})$.*

We will use G-disk algebras as coefficients in the definition of G-factorization homology in section 4.

Example 3.6.12. Let $V : pt \to BO_n(G)$ be the G-map corresponding to a real n-dimensional G-representation V (see example 3.3.3), and $\underline{\mathbf{Disk}}^{G,V-fr,\sqcup}$ be the G-symmetric monoidal category of V-framed G-disks. A V-framed G-disk algebra is a G-symmetric monoidal functor $\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$. In corollary 3.9.9 we will see that V-framed G-disk algebras are equivalent to $\mathbb{E}_V$-algebras.

3.7 G-disks as a G-symmetric monoidal envelope

There is a close relationship between $\underline{\mathbf{Disk}}^G$ and the G-∞-category $\underline{\mathbf{Rep}}_n^G$ of definition 3.2.3. To state it we first define a G-∞-operad $\underline{\mathbf{Rep}}_n^{G,\sqcup}$ whose underlying G-∞-category is $\underline{\mathbf{Rep}}_n^G$ (see definition 3.5.7), and then show that $\underline{\mathbf{Disk}}^G$ is the G-symmetric monoidal envelope of $\underline{\mathbf{Rep}}_n^{G,\sqcup}$.

See [BDG+] for the construction and universal property of the G-symmetric monoidal envelope.

Definition 3.7.1. *Let $\underline{\mathbf{Rep}}_n^{G,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,\sqcup}$ be the full G-subcategory on the objects of $\underline{\mathbf{Rep}}_n^G$ (using the Segal conditions on the fibers of $\underline{\mathbf{Mfld}}^{G,\sqcup}$). Note that $\underline{\mathbf{Rep}}_n^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ is a G-∞-operad. Equivalently, $\underline{\mathbf{Rep}}_n^{G,\sqcup}$ is the full subcategory on $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds $E \to U \to O$ where $E \to U$ is a G-vector bundle.*

Lemma 3.7.2. *The G-symmetric monoidal G-category of G-disks,* $\underline{\mathbf{Disk}}^{G,\sqcup}$*, is equivalent to* $Env_G(\underline{\mathbf{Rep}}_n^{G,\sqcup})$*, the G-symmetric monoidal envelope of* $\underline{\mathbf{Rep}}_n^{G,\sqcup}$*.*

Proof. Recall that $Env_G(\underline{\mathbf{Rep}}_n^{G,\sqcup})$ is given by the fiber product $\underline{\mathbf{Rep}}_n^{G,\sqcup} \times_{\underline{\mathbf{Fin}}_*^G} \underline{\mathbf{Arr}}_G^{act}(\underline{\mathbf{Fin}}_*^G)$, where $\underline{\mathbf{Arr}}_G^{act}(\underline{\mathbf{Fin}}_*^G) \subset \underline{\mathbf{Arr}}_G(\underline{\mathbf{Fin}}_*^G)$ is the full subcategory of fiberwise active arrows. Unwinding the definition, we identify the objects of $Env_G(\underline{\mathbf{Rep}}_n^{G,\sqcup})$ with

$$\begin{array}{ccccc} E & & & & \\ \downarrow & & & & \\ U_1 & \xleftarrow{=} & U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ O & \xleftarrow{=} & O & \xrightarrow{=} & O, \end{array}$$

where $E \to U_1$ is a G-vector bundle.

The inclusion $\underline{\mathbf{Rep}}_n^{G,\sqcup} \hookrightarrow \underline{\mathbf{Mfld}}^{G,\sqcup}$ is a morphism of G-∞-operads, so by the universal property of the enveloping G-symmetric monoidal G-category induces a G-symmetric monoidal G-functor $Env_G(\underline{\mathbf{Rep}}_n^{G,\sqcup}) \to \underline{\mathbf{Mfld}}^{G,\sqcup}$, taking an object

$$\begin{array}{ccccc} E & & & & \\ \downarrow & & & & \\ U_1 & \xleftarrow{=} & U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ O & \xleftarrow{=} & O & \xrightarrow{=} & O \end{array}$$

to the $\mathcal{O}_G$-**Fin**-manifold $E \to U_1 \to U_2 \to O$. Therefore the essential image of $Env_G(\underline{\mathbf{Rep}}_n^{G,\sqcup}) \to \underline{\mathbf{Mfld}}^{G,\sqcup}$ is $\underline{\mathbf{Disk}}^{G,\sqcup}$.

We have to show that the G-functor $Env_G(\underline{\mathbf{Rep}}_n^{G,\sqcup}) \to \underline{\mathbf{Disk}}^{G,\sqcup}$ is a fully faithful (i.e. that it is fiberwise fully faithful). However, the mapping spaces of $Env_G(\underline{\mathbf{Rep}}_n^{G,\sqcup}) = \underline{\mathbf{Rep}}_n^{G,\sqcup} \times_{\underline{\mathbf{Fin}}_*^G} \underline{\mathbf{Arr}}_G^{act}(\underline{\mathbf{Fin}}_*^G)$ are given by homotopy pullbacks of the mapping spaces of $\underline{\mathbf{Rep}}_n^{G,\sqcup}$ and $\underline{\mathbf{Arr}}_G^{act}(\underline{\mathbf{Fin}}_*^G)$ over $\underline{\mathbf{Fin}}_*^G$. This follows from the definition of the mapping spaces of $\underline{\mathbf{Mfld}}^{G,\sqcup}$ after decomposing the mapping spaces of $\underline{\mathbf{Disk}}^{G,\sqcup}$ using the Segal conditions. ☐

It follows that G-disk algebras (see definition 3.6.11) are equivalent to algebras over the G-∞-operad $\underline{\mathbf{Rep}}_n^{G,\sqcup}$.

Corollary 3.7.3. *Let* $\underline{\mathcal{C}}^{\otimes}$ *be a G-symmetric monoidal category. The* ∞*-category* $\mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^G, \underline{\mathcal{C}})$ *of G-symmetric monoidal functors* $A\colon \underline{\mathbf{Disk}}^{G,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ *is equivalent to the* ∞*-category* $Alg_G(\underline{\mathbf{Rep}}^G, \underline{\mathcal{C}})$ *of morphisms of G-*∞*-operads* $\underline{\mathbf{Rep}}_n^{G,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$*, i.e algebras of the G-*∞*-operad* $\underline{\mathbf{Rep}}_n^{G,\sqcup}$ *in* $\underline{\mathcal{C}}$*.*

A similar result holds for f-framed G-disks, for B a G-space and $f\colon B \to BO_n(G)$ a G-map as in definition 3.6.9.

Proposition 3.7.4. *The G-symmetric monoidal category* $\underline{\mathbf{Disk}}^{G,f-fr,\sqcup}$ *is equivalent to the G-symmetric monoidal envelope of* $\underline{\mathbf{Rep}}^{G,f-fr,\sqcup}$*.*

3.8 Embedding spaces of G-disks and equivariant configuration spaces

We compare the mapping spaces of f-framed $\mathcal{O}_G$-manifolds with equivariant configuration spaces.

Notation 3.8.1. Let $(M \to G/H) \in \underline{\mathbf{Mfld}}^G$ be an $\mathcal{O}_G$-manifold over G/H and $(U \to G/H) \in \underline{\mathbf{Fin}}^G$ a finite G-set over G/H. Denote by $\mathbf{Conf}^G_{G/H}(U; M) \subset \mathrm{Map}^G_{G/H}(U, M)$ the space of injective G-equivariant functions $U \to M$ over G/H with compact-open topology.

Remark 3.8.2. The space $\mathbf{Conf}^G_{G/H}(U; M)$ can be identified with the space of configurations of disjoint orbits in the H-manifold $M|_{eH}$ (the fiber of $M \to G/H$ over the base coset eH), where the orbits of the configurations are indexed by the orbits of $U|_{eH}$, with stabilizers specified by $Stab(W)$, $W \in \mathrm{Orbit}(U|_{eH})$.

In order to compare equivariant embedding spaces of G-disks in M with equivariant configuration spaces we first study the equivariant embedding space of a single G-disk.

Definition 3.8.3. *Let $E \to U$ be a G-vector bundle over a finite G-set, and choose a G-equivariant metric on E. For $t > 0$ define $B_t(E) \subset E$, $B_t(E) = \{v \in E \,|\, \|v\| < t\}$, so $B_t(E) \to U$ is the "open ball of radius t" subbundle. Define $Germ(E, M) = \underrightarrow{colim}_n\, Emb^G_{G/H}(B_{\frac{1}{2^n}}(E), M)$.*

Lemma 3.8.4. *For $s < t$ the restriction map $Emb^G_{G/H}(B_t(E), M) \to Emb^G_{G/H}(B_s(E), M)$ is a homotopy equivalence.*

Proof. By radial dilation we see that the inclusion $B_s(E) \hookrightarrow B_t(E)$ is G-isotopic over G/H to a G-equivariant homeomorphism. □

Corollary 3.8.5. *The restriction map $Emb^G_{G/H}(E, M) \to Germ(E, M)$ is a homotopy equivalence.*

Let $(E \to U \to G/H) \in \underline{\mathbf{Disk}}^G$ be a finite G-disjoint union of G-disks, i.e. $E \to U$ a G-vector bundle, $U = \pi_0 E$, and $(M \to G/H) \in \underline{\mathbf{Mfld}}^G$ an $\mathcal{O}_G$-manifold. Precomposition with the zero section inclusion $U \to E$ defines a fibration

$$c\colon Emb^G_{G/H}(E, M) \twoheadrightarrow \mathbf{Conf}^G_{G/H}(U; M), \tag{11}$$

which we think of as sending a configuration of G-disks in M to the configuration of points which are in the centers these G-disks.

Similarly, for $t > 0$ we have fibrations $Emb^G_{G/H}(B_t(E), M) \twoheadrightarrow \mathbf{Conf}^G_{G/H}(U; M)$, whose colimit forms a fibration $c\colon Germ(E, M) \twoheadrightarrow \mathbf{Conf}^G_{G/H}(U; M)$.

The following corollary is used in the proof of the axiomatic properties of G-factorization homology (see the proofs of lemma 5.2.7 and lemma 5.3.4).

Corollary 3.8.6. *Let $(E \to U \to G/H) \in \underline{\mathbf{Disk}}^G$ be a finite G-disjoint union of G-disks, i.e. $E \to U$ a G-vector bundle, $U = \pi_0 E$. Let $(M \to G/H) \in \underline{\mathbf{Mfld}}^G$ be an $\mathcal{O}_G$-manifold and $N \subset M$ an open G-submanifold. Then*

$$\begin{array}{ccc} Emb^G_{G/H}(E, N) & \longrightarrow & Emb^G_{G/H}(E, M) \\ \big\downarrow c & & \big\downarrow c \\ \mathbf{Conf}^G_{G/H}(U; N) & \longrightarrow & \mathbf{Conf}^G_{G/H}(U; M) \end{array}$$

is a homotopy Cartesian square of spaces, where the vertical maps are given by (11).

Proof. By corollary 3.8.5 the left horizontal maps in the diagram

$$\begin{array}{ccccc} Emb^G_{G/H}(E,N) & \xrightarrow{\sim} & Germ(E,N) & \xrightarrow{c} & \mathbf{Conf}^G_{G/H}(U;N) \\ \downarrow & & \downarrow & & \downarrow \\ Emb^G_{G/H}(E,M) & \xrightarrow{\sim} & Germ(E,M) & \xrightarrow{c} & \mathbf{Conf}^G_{G/H}(U;M) \end{array} \tag{12}$$

are homotopy equivalences, so we have to show the right square is a homotopy pullback square. Since the right horizontal arrows are fibrations, it is enough to show that the right square is a pullback square.

Let $x_\bullet \in \mathbf{Conf}^G_{G/H}(U;N)$, given by an injective G-map $x_\bullet \colon U \to N$. For $t \in \mathbb{R}$ denote the fiber of $Emb^G_{G/H}(B_t(E),N) \twoheadrightarrow Conf_U(N)$ by $Emb^G_{G/H}(B_t(E),N)_{x_\bullet}$. We have a map of fibrations

$$\begin{array}{ccccc} Emb^G_{G/H}(B_t(E),N)_{x_\bullet} & \longrightarrow & Emb^G_{G/H}(B_t(E),N) & \twoheadrightarrow & \mathbf{Conf}^G_{G/H}(U;N) \\ \downarrow & & \downarrow & & \downarrow \\ Emb^G_{G/H}(B_t(E),M)_{x_\bullet} & \longrightarrow & Emb^G_{G/H}(B_t(E),M) & \twoheadrightarrow & \mathbf{Conf}^G_{G/H}(U;M). \end{array}$$

For small enough $t > 0$ the left vertical arrow is an isomorphism, hence the right square is a pullback square. Since filtered colimits commute with pullbacks in **Top**, we see that the right square of diagram (12) is indeed a pullback square. □

Our goal for the rest of this subsection is to study the framed version of the map (11), and show that its V-framed variant is an equivalence (example 3.8.10). This fact will be used in section 3.9 to compare the G-∞-operad $\underline{\mathbf{Rep}}_n^{G,V-fr,\sqcup}$ (definition 3.5.7) with the classical G-operad of little disks in V.

We begin by showing that the decomposition of the configuration of G-disks E into orbits of G-disks induces a decomposition on its space of G-embeddings into M.

Proposition 3.8.7. *Let $(M \to G/H) \in \underline{\mathbf{Mfld}}^G_{[G/H]}$ be an $\mathcal{O}_G$-manifold over G/H and $(E \to U \to G/H) \in \underline{\mathbf{Disk}}^G_{[G/H]}$. For $W \in \mathrm{Orbit}(U)$ let $E_W \in \underline{\mathbf{Disk}}^G_{[G/H]}$ denote $(E|_W \to W \to G/H)$, the restriction of the vector bundle $E \to U$ to the orbit $W \subseteq U$. Then the commutative square of spaces*

$$\begin{array}{ccc} Emb^G_{G/H}(E,M) & \longrightarrow & \prod_W Emb^G_{G/H}(E_W,M) \\ \downarrow c & \ulcorner & \downarrow c \\ \mathbf{Conf}^G_{G/H}(U;M) & \longrightarrow & \prod_W \mathbf{Conf}^G_{G/H}(W;M) \end{array}$$

is a homotopy pullback square, where the products are indexed by $W \in \mathrm{Orbit}(U)$, and the vertical maps are given by (11).

Proof. By corollary 3.8.5 the left horizontal maps in the diagram

$$\begin{array}{ccccc} Emb^G_{G/H}(E,M) & \xrightarrow{\sim} & Germ(E,M) & \xrightarrow{c} & \mathbf{Conf}^G_{G/H}(U;M) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_W Emb^G_{G/H}(E_W,M) & \xrightarrow{\sim} & \prod_W Germ(E_W,M) & \xrightarrow{c} & \prod_W \mathbf{Conf}^G_{G/H}(W;M) \end{array}$$

are equivalences, so it is enough to show that right square is a homotopy pullback square. The right horizontal maps are Kan fibrations, therefore it is enough to show this square is a pullback square. This is clear, since the induced map on the fibers of the horizontal maps is a homeomorphism. □

Proposition 3.8.8. *Let $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ be an f-framed $\mathcal{O}_G$-manifold over G/H, given by a $\mathcal{O}_G$-manifold $M \to G/H$ together with an f-framing $f_M \colon M \to B$ lifting $\tau_M \colon M \to BO_n(G)$. Let $E \in \underline{\mathbf{Disk}}^{G,f-fr}_{[G/H]}$, given by $(E \to U \to G/H) \in \underline{\mathbf{Disk}}^{G}_{[G/H]}$ and f-framing $f_E \colon E \to B$. For $W \in \mathrm{Orbit}(U)$ denote $E_W \in \underline{\mathbf{Disk}}^{G,f-fr}_{[G/H]}$ denote the restricted G-vector bundle $(E|_W \to W \to G/H)$, with the restricted framing $f_W \colon E|_W \subset E \xrightarrow{f_E} B \times G/H$.*

Then the commutative square of spaces

$$\begin{array}{ccc}
Emb^{G,f-fr}_{G/H}(E,M) & \longrightarrow & \prod_W Emb^{G,f-fr}_{G/H}(E_W,M) \\
\big\downarrow c \ \ulcorner & & \big\downarrow c \\
\mathbf{Conf}^G_{G/H}(U;M) & \longrightarrow & \prod_W \mathbf{Conf}^G_{G/H}(W;M)
\end{array}$$

is a homotopy pullback square, where the products are indexed by $W \in \mathrm{Orbit}(U)$, and the vertical maps are given by precomposition with the zero section.

Proof. Recall the notation of remark 3.2.9,

$$\begin{aligned}
&\underline{B}(G/H),\ \underline{BO_n(G)}(G/H) \in \mathbf{Top}^G_{/G/H},\\
&\underline{B}(G/H) = (B \times G/H \to G/H), \quad \underline{BO_n(G)}(G/H) = (BO_n(G) \times G/H \to G/H).
\end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccccccc}
Emb^{G,f-fr}_{G/H}(E,M) & & \longrightarrow & & \mathrm{Map}^G_{/\underline{B}(G/H)}(E,M) & & \\
 & \searrow & & & & \searrow & \\
\big\downarrow & & \prod_W Emb^{G,f-fr}_{G/H}(E_W,M) & \longrightarrow & \big\downarrow & & \prod_W \mathrm{Map}^G_{/\underline{B}(G/H)}(E_W,M) \\
Emb^G_{G/H}(E,M) & & \big\downarrow & \longrightarrow & \mathrm{Map}^G_{/\underline{BO_n(G)}(G/H)}(E,M) & & \big\downarrow \\
 & \searrow & & & & \searrow & \\
 & & \prod_W Emb^G_{G/H}(E_W,M) & \longrightarrow & & & \prod_W \mathrm{Map}^G_{/\underline{BO_n(G)}(G/H)}(E_W,M),
\end{array}$$

where

$$\begin{aligned}
\mathrm{Map}^G_{/\underline{B}(G/H)}(E,M) &= \mathrm{Map}^G_{/\underline{B}(G/H)}(E \xrightarrow{\overline{f_E}} B \times G/H, M \xrightarrow{\overline{f_M}} B \times G/H),\\
\mathrm{Map}^G_{/\underline{BO_n(G)}(G/H)}(E,M) &= \mathrm{Map}^G_{/\underline{BO_n(G)}(G/H)}(E \xrightarrow{\overline{f_E}} BO_n(G) \times G/H, M \xrightarrow{\overline{f_M}} BO_n(G) \times G/H).
\end{aligned}$$

The forward and backward faces of the cube are homotopy pullback squares by definition (see remark 3.3.2). The diagonal morphisms on the right are equivalences, since

$$\left(E \xrightarrow{\overline{f_E}} B \times G/H\right) = \coprod_W \left(E_W \xrightarrow{\overline{f_W}} B \times G/H\right),$$
$$\left(E \xrightarrow{\overline{\tau_E}} BO_n(G) \times G/H\right) = \coprod_W \left(E_W \xrightarrow{\overline{\tau_W}} BO_n(G) \times G/H\right),$$

and in particular the right face is a homotopy pullback square. By [Lur09a, lem. 4.4.2.1] the left face is a homotopy pullback square.

Note that the left face is above diagram the same as the top square of the diagram

$$\begin{array}{ccc} Emb^{G,f-fr}_{G/H}(E,M) & \longrightarrow & \prod_W Emb^{G,f-fr}_{G/H}(E_W,M) \\ \downarrow & \ulcorner & \downarrow \\ Emb^{G}_{G/H}(E,M) & \longrightarrow & \prod_W Emb^{G}_{G/H}(E_W,M) \\ \downarrow c & \ulcorner & \downarrow c \\ \mathbf{Conf}^G_{G/H}(U;M) & \longrightarrow & \prod_W \mathbf{Conf}^G_{G/H}(W;M) \end{array}$$

By proposition 3.8.7 the bottom square is a homotopy pullback square, hence by [Lur09a, lem. 4.4.2.1] so is the outer rectangle. ☐

The endomorphism space of a single framed G-disk We identify the endomorphism space of a single framed G-disk as a loop space. Let $E \xrightarrow{\pi} G/H$ be a G-vector bundle. Note that as an object of $\underline{\mathbf{Top}}^G_{[G/H]}$ it is equivalent to the terminal object $(G/H \xrightarrow{=} G/H)$, so the G-tangent classifier $\tau_E \colon E \to BO_n(G)$ is given by a choice of connected component of $(BO_n(G))^H \simeq \coprod_V \mathrm{BAut}_{\mathbf{Rep}^H}(V)$, i.e an H-representation V of dimension n. In particular, we have an isomorphism $E \cong V \times_H G$ of G-vector bundles over G/H.

An f-framing on E is given by a G-map $e\colon E \to B$ lifting $V \colon E \to BO_n(G)$ up to G-homotopy. Using the equivalence $\mathrm{Map}^G(E,B) \simeq \mathrm{Map}^G(G/H,B) \simeq \mathrm{Map}^H(pt,B) \simeq B^H$ we can consider e as a point in B^H.

Proposition 3.8.9. *Let $E \to G/H$ be a G-vector bundle with f-framing $e\colon E \to B$. Then the endomorphism space of $(E \to G/H) \in \mathbf{Mfld}^{G,f-fr}_{[G/H]}$ is weakly equivalent to the loop space of B^H with base point e, $Emb^{G,f-fr}_{G/H}(E,E) \simeq \Omega_e B^H$.*

Proof. The endomorphism space of E is given by the homotopy pullback

$$\begin{array}{ccc} Emb^{G,f-fr}_{G/H}(E,E) & \longrightarrow & \mathrm{Map}^G_{/\underline{B}(G/H)}(E \xrightarrow{\overline{e}} B \times G/H, E \xrightarrow{\overline{e}} B \times G/H) \\ \downarrow & \ulcorner & \downarrow (f\times G/H)_* \\ Emb^{G}_{G/H}(E,E) & \xrightarrow{\tau} & \mathrm{Map}^G_{/\underline{BO_n(G)}(G/H)}(E \xrightarrow{\overline{V}} BO_n(G) \times G/H, E \xrightarrow{\overline{V}} BO_n(G) \times G/H). \end{array} \tag{13}$$

We prove our claim by identifying the mapping spaces on the right column with loop spaces and showing that the horizontal maps are equivalences.

Since $\mathrm{Map}^G_{/\underline{B}(G/H)}(E \xrightarrow{\overline{e}} B \times G/H, E \xrightarrow{\overline{e}} B \times G/H)$ is a mapping space in the slice category $\left(\underline{\mathbf{Top}}^G_{[G/H]}\right)_{/\underline{B}(G/H)}$ it is equivalent to the homotopy pullback

$$\begin{array}{ccc} \mathrm{Map}^G_{/\underline{B}(G/H)}(E \xrightarrow{\overline{e}} B \times G/H, E \xrightarrow{\overline{e}} B \times G/H) & \longrightarrow & \mathrm{Map}^G_{G/H}(E \to G/H, E \to G/H) \\ \downarrow & \ulcorner & \downarrow \overline{e}_* \\ * & \xrightarrow{\overline{e}} & \mathrm{Map}^G_{G/H}(E \to G/H, B \times G/H \to G/H). \end{array}$$

Since $(E \to G/H) \in \underline{\mathbf{Top}}^G_{[G/H]}$ is terminal we have

$$\begin{aligned} \mathrm{Map}^G_{G/H}(E \to G/H, E \to G/H) &\simeq \mathrm{Map}^G_{G/H}(G/H \xrightarrow{=} G/H, G/H \xrightarrow{=} G/H) = *, \\ \mathrm{Map}^G_{G/H}(E \to G/H, B \times G/H \to G/H) &\simeq \mathrm{Map}^G_{G/H}(G/H \xrightarrow{=} G/H, B \times G/H \to G/H) \\ &\simeq \mathrm{Map}^G(G/H, B) \cong B^H, \end{aligned}$$

hence $\mathrm{Map}^G_{/\underline{B}(G/H)}(E \xrightarrow{\overline{e}} B \times G/H, E \xrightarrow{\overline{e}} B \times G/H) \simeq \Omega_e B^H$.

Replacing B with $BO_n(G)$, the same calculation shows

$$\begin{aligned} \mathrm{Map}^G_{/\underline{BO_n(G)}(G/H)}(E \xrightarrow{\overline{V}} BO_n(G) \times G/H, E \xrightarrow{\overline{V}} BO_n(G) \times G/H) &\simeq \Omega_V \left(\coprod_{\rho\colon\, H \curvearrowright \mathbb{R}^n} \mathrm{BAut}_{\mathbf{Rep}^H}(\rho) \right) \\ &= \Omega\, \mathrm{BAut}_{\mathbf{Rep}^H}(V). \end{aligned}$$

Identify $Emb^G_{G/H}(E, E) \cong Emb^G_{G/H}(V \times G/H, V \times G/H) \cong Emb^H(V, V)$ in the homotopy pullback square (13) we get a homotopy pullback square

$$\begin{array}{ccccccc} & & & & Emb^{G,f-fr}_{G/H}(E,E) & \longrightarrow & \Omega_e B^H \\ & & & & \downarrow & \ulcorner & \downarrow \\ \mathrm{Aut}_{\mathbf{Rep}^H}(V) & \hookrightarrow & Emb^H_0(V,V) & \hookrightarrow & Emb^H(V,V) & \xrightarrow{\tau} & \Omega\, \mathrm{BAut}_{\mathbf{Rep}^H}(V), \end{array}$$

where $Emb^H_0(V, V)$ is the subspace of H-equivariant self embeddings $V \hookrightarrow V$ that fix the origin.

By proposition 3.2.5 the bottom left map is a weak equivalence, and the middle bottom arrow is clearly a homotopy equivalence. Since the composition of the bottom maps is the known equivalence $\mathrm{Aut}_{\mathbf{Rep}^H}(V) \to \Omega\, \mathrm{BAut}_{\mathbf{Rep}^H}(V)$, we conclude that τ is a weak equivalence, and therefore the top map of the homotopy pullback square is a weak equivalence as well. □

Finally, we return to the V-framed variant of the map (11).

Example 3.8.10. Consider V-framed manifolds for V be a real n-dimensional G-representation (example 3.3.3), and let $E \in \underline{\mathbf{Disk}}^{G,V-fr}_{[G/H]}$ be given by $(E \to U \to G/H) \in \underline{\mathbf{Disk}}^G_{[G/H]}$, with V-framing inducing a trivialization $E \cong U \times V$ of the G-vector bundle $E \to U$. Consider the mapping space from E to $(V \times G/H \to G/H) \in \underline{\mathbf{Disk}}^{G,V-fr}_{[G/H]}$. For every orbit $W \in \mathrm{Orbit}(U)$ we have $E_W \cong W \times V$ as G-vector bundles over W. Therefore the homotopy fiber of $c\colon Emb^{G,V-fr}_{G/H}(E_W, V \times G/H) \to \mathbf{Conf}^G_{G/H}(W; V \times G/H)$ is equivalent to the loop space of a point (see proposition 3.8.9), hence contractible. It follows that the map

$\prod_W Emb^{G,V-fr}_{G/H}(E_W, V \times G/H) \to \prod_W \mathbf{Conf}^G_{G/H}(W; V \times G/H)$ is an equivalence, since its homotopy fibers are contractible. By proposition 3.8.8 precomposition with the zero section of $E \to U$ induces a homotopy equivalence

$$c\colon Emb^{G,V-fr}_{G/H}(E, V \times G/H) \xrightarrow{\sim} \mathbf{Conf}^G_{G/H}(U; V \times G/H). \tag{14}$$

More generally, for any V-framed $\mathcal{O}_G$-manifold $M \in \underline{\mathbf{Mfld}}^{G,V-fr}_{[G/H]}$ we have

$$c\colon Emb^{G,V-fr}_{G/H}(E, M) \xrightarrow{\sim} \mathbf{Conf}^G_{G/H}(U; M), \tag{15}$$

since by proposition 3.8.8 the homotopy fibers are contractible.

We will use example 3.8.10 in section 3.9.

3.9 Comparison of the equivariant little disks G-operad and the G-∞-operad of V-framed representations

Let V be a real n-dimensional representation of G, and $\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$ the G-∞-operad of definition 3.5.7. In this subsection we define the G-∞-operad $\underline{\mathbb{E}_V}$ of little G-disks (definition 3.9.5) using the genuine operadic nerve construction of Bonventre, and show that it is equivalent to $\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$ (proposition 3.9.8), hence $\mathbb{E}_V$-algebras are equivalent to V-framed G-disk algebras.

We first review the relevant details of Bonventre's construction. This construction is best understood in the light of [BP17, thm. III] which gives a (right) Quillen equivalence

$$i_*\colon sOp^G \to sOp_G$$

between the G-graph model structure on simplicial G-operads (where weak equivalences is detected on graph-subgroup fixed points) and the projective model structure on genuine G-operads.

Construction 3.9.1 (The genuine equivariant category of operators, see [Bon19, def. 4.1]). Let $\mathcal{P} \in sOp_G$ be a genuine G-operad. Define a simplicial category $\mathcal{P}^\otimes$ as follows. The objects of $\mathcal{P}^\otimes$ are objects of $\underline{\mathbf{Fin}}^G_*$, i.e. G-maps $U \to G/H$ from a finite G-set to a G-orbit. The simplicial space of maps $\mathcal{P}^\otimes(U_1 \to G/H, U_2 \to G/K)$ is given by

$$\mathrm{Map}_{\mathcal{P}^\otimes}\left(\begin{smallmatrix} U_1 \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} U_2 \\ \downarrow \\ G/K \end{smallmatrix}\right) = \coprod_{\varphi} \prod_{W \in \mathrm{Orbit}(U_2)} \mathcal{P}\left(\begin{smallmatrix} f^{-1}(W) \\ \downarrow \\ W \end{smallmatrix}\right),$$

where the coproduct is indexed by $\varphi \in \mathrm{Map}_{\underline{\mathbf{Fin}}^G_*}\left(\begin{smallmatrix} U_1 \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} U_2 \\ \downarrow \\ G/K \end{smallmatrix}\right)$. Composition in $\mathcal{P}^\otimes$ is defined using coproducts of the composition maps of the genuine G-operad $\mathcal{P}$.

Theorem 3.9.2 ([Bon19, thm. 4.10]). *Let $\mathcal{P} \in sOp_G$ be a genuine G-operad, and $N^\otimes(\mathcal{P})$ the coherent nerve of the $\mathcal{P}^\otimes$. If $\mathcal{P} \in sOp_G$ is locally fibrant, then $N^\otimes(\mathcal{P})$ is a G-∞-operad.*

We call $N^\otimes(\mathcal{O})$ as the *genuine operadic nerve* of $\mathcal{O}$.

Corollary 3.9.3 ([Bon19, cor 6.3]). *Let $\mathcal{O} \in sOp^G$ be a graph-fibrant simplicial G-operad with a single color. Then $i_*\mathcal{O} \in sOp_G$ is locally fibrant, and thus there exists a G-∞-operad $N^\otimes(\mathcal{O})$ associated to $\mathcal{O}$.*

In particular, the genuine coherent nerve construction associates a G-∞-operad to the equivariant little disk operad.

Example 3.9.4 ([Bon19, ex. 6.5])**.** Let V be a real orthogonal n-dimensional G-representation, and $D(V)$ the open unit disk of V. For $H < G$ and U a finite H-set let $Emb^{Aff,H}(U \times D(V), D(V))$ denote the space of H-equivariant affine embeddings $U \times D(V) \hookrightarrow D(V)$. Let $\mathcal{D}_V$ be the *little V-disks operad* (see e.g [GM17, def. 1.1] or [BH15, def. 3.11(ii)]). Applying the functor **Sing** to the spaces $(\mathcal{D}_V)_n$ we get a locally fibrant simplicial G-operad, hence an associated G-∞-operad $N^{\otimes}(\mathcal{D}_V)$.

The mapping spaces of $N^{\otimes}(\mathcal{D}_V)$ are given by

$$\mathrm{Map}_{N^{\otimes}(\mathcal{D}_V)}\left(\begin{smallmatrix} U_1 \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} U_2 \\ \downarrow \\ G/K \end{smallmatrix} \right) = \coprod_{\varphi} \prod_{Gx \in \mathrm{Orbit}(U_2)} Emb^{Aff,Stab(x)}(f^{-1}(x) \times D(V), D(V)).$$

Definition 3.9.5. *Fix a real orthogonal G-representation V, and let $\mathcal{D}_V$ denote the G-operad of little V-disks. Let $\mathbb{E}_V^{\otimes}$ denote the genuine operadic nerve $N^{\otimes}(\mathcal{D}_V)$ of [Bon19, ex. 6.5].*

Before defining $\mathbb{E}_V$-algebras we recall the definition of a G-∞-operad map.

Notation 3.9.6. Let $\mathbb{P}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$, $\mathbb{Q}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ be G-∞-operads (see [Nar17, def. 3.1]). A map of G-∞-operads from $\mathbb{P}^{\otimes}$ to $\mathbb{Q}^{\otimes}$ is a map of simplicial sets $f \colon \mathbb{P}^{\otimes} \to \mathbb{Q}^{\otimes}$ such that

1. The diagram

$$\begin{array}{ccc} \mathbb{P}^{\otimes} & \xrightarrow{\quad f \quad} & \mathbb{Q}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \underline{\mathbf{Fin}}_*^G & \end{array}$$

commutes.

2. The functor f carries coCartesian edges over inert morphisms to coCartesian edges.

Definition 3.9.7. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ be a G-symmetric monoidal category. An* $\mathbb{E}_V$-algebra *in $\underline{\mathcal{C}}$ is a map of G-∞-operads $A \colon \mathbb{E}_V \to \underline{\mathcal{C}}^{\otimes}$. Let $Alg_{\mathbb{E}_V}(\underline{\mathcal{C}}) \subseteq \mathrm{Fun}_{/\underline{\mathbf{Fin}}_*^G}(\mathbb{E}_V^{\otimes}, \underline{\mathcal{C}}^{\otimes})$ denote the full subcategory spanned by $\mathbb{E}_V$-algebras.*

Comparison with $\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$. We can now easily compare the G-∞-operads $\mathbb{E}_V$ of definition 3.9.5 and $\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$ of definition 3.5.7.

We start with some observations. Fix a V-framed G-diffeomorphism $D(V) \cong V$. Let $H < G$ and U' a finite H-set and $U = G \times_H U'$ its topological induction. Then the topological induction of $U' \times D(V)$ from H to G is $U \times D(V)$. Note that the induced map $U \times D(V) \to G/H$ is G-vector bundle equivalent to $U \times V \to U$ by our chosen diffeomorphism, and hence V-framed. Let $Emb_{G/H}^{Aff,G}(U \times D(V), G/H \times D(V))$ denote the space of affine G-embeddings over G/H. Note that restriction to the fiber over eH defines a homeomorphism

$$Emb^{Aff,H}(U' \times D(V), D(V)) \cong Emb_{G/H}^{Aff,G}(U \times D(V), G/H \times D(V)).$$

On the other hand, affine G-embeddings $U \times D(V) \hookrightarrow G/H \times D(V)$ over G/H are clearly V-framed (using the chosen G-diffeomorphism $D(V) \cong V$. Therefore we have a map

$$Emb_{G/H}^{Aff,G}\left(U \times D(V), G/H \times D(V)\right) \hookrightarrow Emb_{G/H}^{G,V-fr}\left(U \times D(V), G/H \times D(V)\right).$$

Construct a functor $F\colon \mathbb{E}_V \to \underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$ over $\underline{\mathbf{Fin}}^G_*$ as follows. For every finite G-set U define $F(U \to G/H) = (U \times D(V) \to U \to G/H)$. Define F on mapping spaces by the embeddings

$$\begin{aligned}
\mathrm{Map}_{\mathbb{E}_V}\left(\begin{smallmatrix} U_1 \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} U_2 \\ \downarrow \\ G/K \end{smallmatrix}\right) &= \coprod_{\varphi} \prod_{Gx \in \mathrm{Orbit}(U_2)} Emb^{Aff,Stab(x)}(f^{-1}(x) \times D(V), D(V)) \\
&\hookrightarrow \coprod_{\varphi} \prod_{W \in \mathrm{Orbit}(U_2)} Emb^{G,V-fr}_W(f^{-1}(W) \times D(V), G/H \times D(V)) \\
&= \mathrm{Map}_{\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}}(U_1 \times D(V), U_2 \times D(V)).
\end{aligned}$$

Proposition 3.9.8. *The functor $F\colon \mathbb{E}_V \to \underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$ is an equivalence of G-∞-operads.*

Proof. By construction F is a functor over $\underline{\mathbf{Fin}}^G_*$, therefore it is enough to show that F is an equivalence of ∞-categories. Clearly F is essentially surjective, since any V-framed G-vector bundle over a finite G-set U is equivariant to $U \times V \cong U \times D(V)$. We therefore have to show that F is fully faithful.

By the Segal conditions it is enough to show that F induces an equivalence of spaces

$$F\colon\ \mathrm{Map}_{\mathbb{E}_V}\left(\begin{smallmatrix} U \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} G/H \\ \downarrow \\ G/H \end{smallmatrix}\right) \to \mathrm{Map}_{\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}}\left(\begin{smallmatrix} U\times D(V) \\ \downarrow \\ U \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} G/H\times D(V) \\ \downarrow \\ G/H \\ \downarrow \\ G/H \end{smallmatrix}\right)$$

on the mapping spaces over $\varphi \in \mathrm{Map}_{\underline{\mathbf{Fin}}^G_*}\left(\begin{smallmatrix} U \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} G/H \\ \downarrow \\ G/H \end{smallmatrix}\right)$. By example 3.9.4 we have

$$\mathrm{Map}_{\mathbb{E}_V}\left(\begin{smallmatrix} U \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} G/H \\ \downarrow \\ G/H \end{smallmatrix}\right) \simeq \prod_{W \in \mathrm{Orbit}(G/H)} Emb^{Aff,Stab(W)}(f^{-1}(W) \times D(V), D(V)),$$

and since $\underline{\mathbf{Rep}}^{G,V-fr,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,V-fr,\sqcup}$ is a full G-subcategory we have

$$\begin{aligned}
\mathrm{Map}_{\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}}\left(\begin{smallmatrix} U\times D(V) \\ \downarrow \\ U \\ \downarrow \\ G/H \end{smallmatrix}, \begin{smallmatrix} G/H\times D(V) \\ \downarrow \\ G/H \\ \downarrow \\ G/H \end{smallmatrix}\right) &= Emb^{G,V-fr}_{G/H}(U \times D(V), G/H \times D(V)) \\
&\cong Emb^{G,V-fr}_{G/H}(U \times V, G/H \times V).
\end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccc}
\prod_{W \in \mathrm{Orbit}(U)} Emb^{Aff,Stab(W)}(f^{-1}(W) \times D(V), D(V)) & \xrightarrow{F} & Emb^{G,V-fr}_{G/H}(U \times V, G/H \times V) \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\
\prod_{W \in \mathrm{Orbit}(U)} \mathbf{Inj}^{Stab(W)}(f^{-1}(W), V) & \longrightarrow & \mathbf{Conf}^G_{G/H}(U; V \times G/H)
\end{array}$$

where the vertical map is given by taking the centers of disks, and the right vertical map is given by precomposition with the zero section. We wish to prove that the top horizontal map is an equivalence. The left vertical map is known to be an equivalence (see [BH15, prop. 4.19] and [GM17, lem 1.2]). The right vertical map is an equivalence by example 3.8.10, and the bottom horizontal map is a homeomorphism by inspection. □

We immediately see that $\mathbb{E}_V$-algebras are equivalent to V-framed G-disk algebras.

Corollary 3.9.9. *There is an equivalence of* ∞*-categories* $Alg_{\mathbb{E}_V}(\underline{\mathcal{C}}) \simeq \mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Disk}}^{G,V-fr}, \underline{\mathcal{C}})$.

Proof. Precomposition with the equivalence $F : \mathbb{E}_V \xrightarrow{\sim} \underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$ of proposition 3.9.8 induces an equivalence

$$Alg_{\mathbb{E}_V}(\underline{\mathcal{C}}) \xrightarrow{\sim} Alg_{\underline{\mathbf{Rep}}^{G,V-fr}}(\underline{\mathcal{C}}).$$

By proposition 3.7.4 the G-symmetric monoidal envelope of $\underline{\mathbf{Rep}}^{G,V-fr,\sqcup}$ is equivalent to $\underline{\mathbf{Disk}}^{G,V-fr,\sqcup}$, so by its universal property we have

$$Alg_{\underline{\mathbf{Rep}}^{G,V-fr}}(\underline{\mathcal{C}}) \simeq \mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Disk}}^{G,V-fr}, \underline{\mathcal{C}}).$$

□

4 Genuine G-factorization homology

In this section we use the G-categories $\underline{\mathbf{Mfld}}^{G,f-fr}$ and $\underline{\mathbf{Disk}}^{G,f-fr}$ to define genuine equivariant factorization homology. We define G-factorization homology, first as a parametrized colimit (definition 4.1.2), then as a G-functor (proposition 4.1.4) and finally as a G-symmetric monoidal functor (definition 4.2.3).

4.1 The definition of G-disk algebras and G-factorization homology as a G-functor

In this subsection we define equivariant factorization homology (see proposition 4.1.4). This is an smooth equivariant version of the factorization homology of [AF15] and of topological chiral homology of [Lur, 7.5.2].

In order to define genuine G-factorization homology using parametrized ∞-colimits we first recall the definition of a parametrized over-category from [Sha18]. The parametrized over-category plays the role of an indexing category in the G-colimit defining factorization homology below (see definition 4.1.2), and more generally in the G-colimit formula for G-left Kan extensions (see [Sha18, thm. 10.3]).

Let $\underline{\mathcal{C}}$ be a G-category and $x \in \underline{\mathcal{C}}_{[G/H]}$ an object over G/H, classified by the G-functor $\sigma_x \colon \underline{G/H} \to \underline{\mathcal{C}}$. Define the parametrized over-category $\underline{\mathcal{C}}_{/\underline{x}} \twoheadrightarrow \underline{G/H}$ (see [Sha18, not. 4.29]) as the fiber product $\underline{\mathbf{Arr}}_G(\underline{\mathcal{C}}) \times_{\underline{\mathcal{C}}} \underline{G/H}$, considered as a $\underline{G/H}$-category by pulling back the coCartesian fibration $ev_1 \colon \underline{\mathbf{Arr}}_G(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ along $\sigma_x \colon \underline{G/H} \to \underline{\mathcal{C}}$. Note that the fiber of $\underline{\mathcal{C}}_{/\underline{x}} \twoheadrightarrow \underline{G/H}$ over $\varphi \colon G/K \to G/H$ is equivalent to the ∞-over-category $(\underline{\mathcal{C}}_{[G/K]})_{/\varphi^* x}$, where $\varphi^* x \in \underline{\mathcal{C}}_{[G/K]}$ is determined by choosing a coCartesian lift $x \to \varphi^* x$ of φ.

If $\underline{\mathcal{C}}' \subseteq \underline{\mathcal{C}}$ is a full G-subcategory we abuse notation and write $\underline{\mathcal{C}}'_{/\underline{x}}$ for the restricted G-over-category, given by the fiber product $\underline{\mathcal{C}}' \times_{\underline{\mathcal{C}}} \underline{\mathcal{C}}_{/\underline{x}}$.

We now return to the definition of genuine G-factorization homology.

Let $A \in \mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}})$ be an f-framed G-disk algebra with values in $\underline{\mathcal{C}}$, and $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ an f-framed $\mathcal{O}_G$-manifold. In the following definition we use the parametrized over-category $\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}$ associated to $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ and $\underline{\mathbf{Disk}}^{G,f-fr} \subset \underline{\mathbf{Mfld}}^{G,f-fr}$.

Remark 4.1.1. Note that $\underline{\mathbf{Disk}}^{G}_{/\underline{M}} \to \underline{G/H}$ is the coCartesian fibration dual to the Cartesian fibration $(\mathbf{Disk}^G)_{/M} \to (\mathcal{O}_G)_{/[G/H]}$ (see [Lur09a, prop 2.4.3.1], compare [Sha18, prop. 4.31]), and therefore can be modeled by the topological Moore over category (see appendix A).

Construct a G-functor over $\underline{G/H}$ by composing

$$\begin{array}{ccccc} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} & \longrightarrow & \underline{\mathbf{Disk}}^{G,f-fr}\underline{\times}\underline{G/H} & \xrightarrow{A\underline{\times}id} & \underline{\mathcal{C}}\underline{\times}\underline{G/H} \\ & \searrow & \downarrow & \swarrow & \\ & & \underline{G/H}. & & \end{array} \tag{16}$$

Consider the functor (16) as an $\underline{G/H}$-diagram in the $\underline{G/H}$-category $\underline{\mathcal{C}}\underline{\times}\underline{G/H}$. Note that the $\underline{G/H}$-colimit of the above diagram is a coCartesian section of $\underline{\mathcal{C}}\underline{\times}\underline{G/H} \twoheadrightarrow \underline{G/H}$, or equivalently a G-functor $\underline{G/H} \to \underline{\mathcal{C}}$ and that a G-functor $\underline{G/H} \to \underline{\mathcal{C}}$ represents an object of $\underline{\mathcal{C}}$ over $[G/H]$.

Definition 4.1.2. *Let $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ be an f-framed $\mathcal{O}_G$-manifold, and A an f-framed G-disk algebra. Define the G-*factorization homology *of M with coefficients in A by the parametrized colimit*

$$\int_M A \in \underline{\mathcal{C}}, \quad \int_M A := \underline{G/H} - \underrightarrow{colim}\left(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr}\underline{\times}\underline{G/H} \xrightarrow{A\underline{\times}id} \underline{\mathcal{C}}\underline{\times}\underline{G/H}\right). \tag{17}$$

In what follows, assume that $\underline{\mathcal{C}}$ is a G-cocomplete G-category (i.e $\underline{\mathcal{C}}$ has all $\underline{G/H}$-colimits for every $H < G$, see [Sha18, def. 5.12]), so that all the parametrized colimits of proposition 4.1.4 exist. Next we show that the assignment $M \mapsto \int_M A$ extends to a G-functor $\int_{\_} A \colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$, and that the G-functors $\int_{\_} A$ are in turn functorial in A (proposition 4.1.4).

Construction 4.1.3. Let $\iota\colon \underline{\mathbf{Disk}}^{G,f-fr} \hookrightarrow \underline{\mathbf{Mfld}}^{G,f-fr}$ denote the inclusion of the full G-subcategory of finite G-disjoint unions of G-disks and $\underline{\mathcal{C}}$ be a cocomplete G-symmetric monoidal category. The inclusion G-functor ι induces a restriction G-functor $\iota^*\colon \underline{\mathrm{Fun}}_G(\underline{\mathbf{Mfld}}^{G,f-fr},\underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}_G(\underline{\mathbf{Disk}}^{G,f-fr},\underline{\mathcal{C}})$. By [Sha18, cor. 10.6] (proposition 2.3.3) the restriction G-functor has a fully faithful left G-adjoint

$$\iota_!\colon \underline{\mathrm{Fun}}_G(\underline{\mathbf{Disk}}^{G,f-fr},\underline{\mathcal{C}}) \leftrightarrows \underline{\mathrm{Fun}}_G(\underline{\mathbf{Mfld}}^{G,f-fr},\underline{\mathcal{C}}) :\iota^*.$$

In particular, define $\iota_!$ to be the fully faithful left adjoint of

$$\iota_!\colon \mathrm{Fun}_G(\underline{\mathbf{Disk}}^{G,f-fr},\underline{\mathcal{C}}) \leftrightarrows \mathrm{Fun}_G(\underline{\mathbf{Mfld}}^{G,f-fr},\underline{\mathcal{C}}) :\iota^*, \tag{18}$$

the adjunction of ∞-categories between the fibers over the terminal orbit $[G/G]$

Proposition 4.1.4. *Let $\underline{\mathcal{C}}$ be a cocomplete G-symmetric monoidal category. Then the functor*

$$\mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Disk}}^{G,f-fr},\underline{\mathcal{C}}) \to \mathrm{Fun}_G(\underline{\mathbf{Disk}}^{G,f-fr},\underline{\mathcal{C}}) \xrightarrow{\iota_!} \mathrm{Fun}_G(\underline{\mathbf{Mfld}}^{G,f-fr},\underline{\mathcal{C}}),$$
$$(A\colon \underline{\mathbf{Disk}}^{G,f-fr}D \to \underline{\mathcal{C}}^{\otimes}) \mapsto (A\colon \underline{\mathbf{Disk}}^{G,f-fr} \to \underline{\mathcal{C}}) \mapsto (\iota_! A\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}})$$

sends a G-disk algebra A to a G-functor

$$\iota_! A\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}, \quad , M \mapsto (\iota_! A)(M) = \int_M A.$$

Proof. By [Sha18, thm. 10.4] for every G-disk algebra $A \in \mathrm{Fun}_G(\underline{\mathbf{Mfld}}^{G,f-fr},\underline{\mathcal{C}})$ the left G-adjoint $\iota_!(A)\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}})$ is given by left G-Kan extension of A along ι. By [Sha18, thm 10.3] applying the $\iota_!(A)$ to $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ is given by the $\underline{G/H}$-colimit

$$(\iota_! A)(M) = \underline{G/H} - \underrightarrow{colim}\left(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr}\underline{\times}\underline{H/G} \xrightarrow{A\underline{\times}id} \underline{\mathcal{C}}\underline{\times}\underline{G/H}\right) = \int_M A.$$

□

4.2 Extensding G-factorization homology to a G-symmetric monoidal functor

In this subsection we prove (proposition 4.2.2) that G-factorization homology with values in a presentable G-symmetric monoidal category extends to a G-symmetric monoidal functor (see definition 4.2.3).

Definition 4.2.1. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^{G}_{*}$ be a G-symmetric monoidal category. We say $\underline{\mathcal{C}}^{\otimes}$ is a* presentable G-symmetric monoidal category *if the underlying G-category is presentable and for every active map $\alpha \colon I \to J$ in $\underline{\mathbf{Fin}}^{G}_{*}$ the G-functor $\otimes_{\alpha} \colon \underline{\mathcal{C}}^{\otimes}_{<I>} \to \underline{\mathcal{C}}^{\otimes}_{<J>}$ is distributive ([Nar17, sec. 3.3]).*

Proposition 4.2.2. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^{G}_{*}$ be a presentable G-symmetric monoidal category.*

Then the adjunction eq. (18) *lifts to an adjunction*

$$\begin{array}{ccc} (\iota^{\otimes})_{!} \colon \mathrm{Fun}^{\otimes}_{G}(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}}) & \rightleftarrows & \mathrm{Fun}^{\otimes}_{G}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}}) : (\iota^{\otimes})^{*} \\ \downarrow & & \downarrow \\ \iota_{!} \colon \mathrm{Fun}_{G}(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}}) & \rightleftarrows & \mathrm{Fun}_{G}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}}) : \iota^{*} \end{array} \tag{19}$$

where $\iota^{\otimes} \colon \underline{\mathbf{Disk}}^{G,f-fr,\sqcup} \to \underline{\mathbf{Mfld}}^{G,f-fr.\sqcup}$ is the inclusion of the subcategory of f-framed indexed disks (see corollary 3.5.5).

Note that since $\iota_{!}$ is fully faithful the Segal conditions imply that $(\iota^{\otimes})_{!} \colon \mathrm{Fun}^{\otimes}_{G}(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}}) \to \mathrm{Fun}^{\otimes}_{G}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}})$ is fully faithful.

Definition 4.2.3. *For $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^{G}_{*}$, $A \colon \underline{\mathbf{Disk}}^{G,f-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ as in proposition 4.2.2, denote the G-symmetric monoidal functor $(\iota^{\otimes})_{!}$ by*

$$\begin{array}{ccc} \int_{\_} A \colon \underline{\mathbf{Mfld}}^{G,f-fr.\sqcup} & \longrightarrow & \underline{\mathcal{C}}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \underline{\mathbf{Fin}}^{G}_{*}. & \end{array}$$

Commutativity of the diagram (19) *shows that $\int_{\_} A$ extends the G-functor $\iota_{!}A \colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$ of eq.* (18) *which sends an $\mathcal{O}_G$-manifold $(M \to G/H)$ to its G-factorization homology $\iota_{!}A(M) = \int_{M} A$ (proposition 4.1.4) to a G-symmetric monoidal functor. We call the G-symmetric monoidal functor $\int_{\_} A \colon \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ the* G-factorization homology functor with coefficients in A.

In the remainder of this subsection we prove proposition 4.2.2. The proof has two parts, the first is a general G-categorical lemma, lemma 4.2.5, giving conditions ensuring that a G-left Kan extension lifts to a G-symmetric monoidal functor, and the second is a verification of these conditions.

We start with by recalling the notion of a G-lax monoidal functor and stating a useful proposition from [BDG+]. Let $\underline{\mathcal{D}}^{\otimes}, \underline{\mathcal{C}}^{\otimes}$ be G-symmetric monoidal categories. Recall that a lax G-symmetric monoidal G-functor F from $\underline{\mathcal{D}}$ to $\underline{\mathcal{C}}$ is a functor $F \colon \underline{\mathcal{D}}^{\otimes} \to \underline{\mathcal{C}}^{\otimes}$ over $\underline{\mathbf{Fin}}^{G}_{*}$ which preserves inert edges (i.e. coCartesian edges over inert morphisms). Let $Alg(\underline{\mathcal{D}}, \underline{\mathcal{C}}) \subset \mathrm{Fun}_{/\underline{\mathbf{Fin}}^{G}_{*}}(\underline{\mathcal{D}}^{\otimes}, \underline{\mathcal{C}}^{\otimes})$ be the full subcategory of functors over $\underline{\mathbf{Fin}}^{G}_{*}$ which are lax G-symmetric monoidal.

Proposition 4.2.4. *Let $\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be a presentable G-symmetric monoidal category, let $\underline{\mathcal{M}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be a small G-symmetric monoidal category and $\iota^\otimes \colon \underline{\mathcal{D}}^\otimes \hookrightarrow \underline{\mathcal{M}}^\otimes$ an inclusion of a full G-symmetric monoidal subcategory. Denote by $\iota\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ the induced G-functor on the underlying categories.*

Then the restriction along $\iota^\otimes$ has a left adjoint $(\iota^\otimes)_!\colon Alg(\underline{\mathcal{D}},\underline{\mathcal{C}}) \to Alg(\underline{\mathcal{M}},\underline{\mathcal{C}})$. Moreover, the adjunction $(\iota^\otimes)_!\colon Alg(\underline{\mathcal{D}},\underline{\mathcal{C}}) \leftrightarrows Alg(\underline{\mathcal{M}},\underline{\mathcal{C}}) :(\iota^\otimes)^*$ restricts to the adjunction $\iota_!\colon \mathrm{Fun}(\underline{\mathcal{D}},\underline{\mathcal{C}}) \leftrightarrows \mathrm{Fun}(\underline{\mathcal{M}},\underline{\mathcal{C}}) :\iota^*$, where $\iota_!\colon \mathrm{Fun}(\underline{\mathcal{D}},\underline{\mathcal{C}}) \to \mathrm{Fun}(\underline{\mathcal{M}},\underline{\mathcal{C}})$ is left adjoint to the restriction along ι.

In particular we have a commuting square of ∞-categories

$$\begin{array}{ccc} Alg(\underline{\mathcal{D}},\underline{\mathcal{C}}) & \xrightarrow{(\iota^\otimes)_!} & Alg(\underline{\mathcal{M}},\underline{\mathcal{C}}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_G(\underline{\mathcal{D}},\underline{\mathcal{C}}) & \xrightarrow{\iota_!} & \mathrm{Fun}_G(\underline{\mathcal{M}},\underline{\mathcal{C}}). \end{array}$$

We will prove proposition 4.2.2 by applying the following G-categorical lemma (a G-categorical version of [AFT17a, lem. 2.16]).

Lemma 4.2.5. *Let $\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be a presentable G-symmetric monoidal category, let $\underline{\mathcal{D}}^\otimes, \underline{\mathcal{M}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be small G-symmetric monoidal categories and $\iota^\otimes\colon \underline{\mathcal{D}}^\otimes \hookrightarrow \underline{\mathcal{M}}^\otimes$ be an inclusion of a full G-symmetric monoidal subcategory. Denote by $\iota\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ the induced G-functor on the underlying categories.*

If for every active morphism $\psi\colon I \to J$ in a fiber $(\underline{\mathbf{Fin}}^G_*)_{[G/H]}$ and every coCartesian lift $x \to y$ of ψ to $\underline{\mathcal{M}}^\otimes$ the $\underline{G/H}$-functor $\otimes_\psi\colon (\underline{\mathcal{D}}^\otimes_{<I>})_{/\underline{x}} \to (\underline{\mathcal{D}}^\otimes_{<J>})_{/\underline{y}}$ is $\underline{G/H}$-cofinal then the diagram

$$\begin{array}{ccc} \mathrm{Fun}^\otimes_G(\underline{\mathcal{D}},\underline{\mathcal{C}}) & \xrightarrow{(\iota^\otimes)_!} & \mathrm{Fun}^\otimes_G(\underline{\mathcal{M}},\underline{\mathcal{C}}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_G(\underline{\mathcal{D}},\underline{\mathcal{C}}) & \xrightarrow{\iota_!} & \mathrm{Fun}_G(\underline{\mathcal{M}},\underline{\mathcal{C}}) \end{array}$$

commutes, where $(\iota^\otimes)_!$ and $\iota_!$ the left adjoins to the restrictions along $\iota^\otimes$ and ι, respectively.

Proof. Applying proposition 4.2.4 we have:

$$\begin{array}{ccc} (\iota^\otimes)_!\colon Alg(\underline{\mathcal{D}},\underline{\mathcal{C}}) & \leftrightarrows & Alg(\underline{\mathcal{M}},\underline{\mathcal{C}}) :(\iota^\otimes)^* \\ \downarrow & & \downarrow \\ \iota_!\colon \mathrm{Fun}_G(\underline{\mathcal{D}},\underline{\mathcal{C}}) & \leftrightarrows & \mathrm{Fun}_G(\underline{\mathcal{M}},\underline{\mathcal{C}}) :\iota^* \end{array}$$

We need to show that the adjunction $(\iota^\otimes)_!\colon Alg(\underline{\mathcal{D}},\underline{\mathcal{C}}) \leftrightarrows Alg(\underline{\mathcal{M}},\underline{\mathcal{C}}) :(\iota^\otimes)^*$ restricts to an adjunction between the full subcategories

$$\mathrm{Fun}^\otimes_G(\underline{\mathcal{D}},\underline{\mathcal{C}}) \subset Alg(\underline{\mathcal{D}},\underline{\mathcal{C}}), \quad \mathrm{Fun}^\otimes_G(\underline{\mathcal{M}},\underline{\mathcal{C}}) \subset Alg(\underline{\mathcal{M}},\underline{\mathcal{C}}).$$

Clearly precomposition with the G-symmetric monoidal functor $\iota^\otimes\colon \underline{\mathcal{D}}^\otimes \to \underline{\mathcal{M}}^\otimes$ takes G-symmetric monoidal functors to G-symmetric monoidal functors, so the right adjoint restricts to a functor

$$(\iota^\otimes)^*\colon \mathrm{Fun}^\otimes_G(\underline{\mathcal{M}},\underline{\mathcal{C}}) \to \mathrm{Fun}^\otimes_G(\underline{\mathcal{D}},\underline{\mathcal{C}}).$$

Let $F^\otimes \colon \underline{\mathcal{D}}^\otimes \to \underline{\mathcal{C}}^\otimes$ be a G-symmetric monoidal functor, with $F \colon C \to D$ the induced G-functor on the underlying categories. Applying the left adjoint $(\iota^\otimes)_!$ to $F^\otimes$ we get a lax G-symmetric monoidal functor $(\iota^\otimes)_! F^\otimes \colon \underline{\mathcal{M}}^\otimes \to \underline{\mathcal{C}}^\otimes$, in other words $(\iota^\otimes)_! F^\otimes$ preserves coCartesian edges over inert morphisms. We have to show that $(\iota^\otimes)_! F^\otimes$ preserves all coCartesian edges. Using the inert-fiberwise active factorization system on $M^\otimes$ (which exists on any G-∞-operad, see [BDG$^+$]), we are reduced to showing that $(\iota^\otimes)_! F^\otimes$ preserves fiberwise active coCartesian edges. By the Segal conditions it is enough to show $(\iota^\otimes)_! F^\otimes$ preserves arrows over maps $I \to J$ in $\underline{\mathbf{Fin}}^G_*$ with $J = (G/H \xrightarrow{=} G/H)$.

Before showing that $(\iota^\otimes)_! F^\otimes$ preserve these coCartesian edges, let us first recall how the functor $(\iota^\otimes)_! F^\otimes$ acts on morphisms.

By definition $\iota_! \colon \underline{\mathcal{M}} \to \underline{\mathcal{C}}$ is a left G-Kan extension. Using the construction of [Sha18, def. 10.1] we have a G-functor

$$(\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}})) \star_{\underline{\mathcal{M}}} \underline{\mathcal{M}} \to \underline{\mathcal{C}}$$

which is an $\underline{\mathcal{M}}$-parametrized G-colimit diagram, where

$$\underline{\mathbf{Arr}}_G(\underline{\mathcal{M}}) = \mathcal{O}_G^{op} \times_{\mathrm{Fun}(\underline{\Delta}^1, \mathcal{O}_G^{op})} \mathrm{Fun}(\underline{\Delta}^1, \underline{\mathcal{M}}) \simeq \underline{\mathrm{Fun}}_G(\mathcal{O}_G^{op} \times \underline{\Delta}^1, \underline{\mathcal{M}})$$

is the fiberwise arrow category (see [Sha18, not. 4.29]). Note that by definition the restriction to the first coordinate $\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}}) \to \big(\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}})\big) \star_{\underline{\mathcal{M}}} \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ factors as $\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}}) \xrightarrow{\pi_{\underline{\mathcal{D}}}} \underline{\mathcal{D}} \xrightarrow{F} \underline{\mathcal{C}}$. and the restriction to the second coordinate is the left G-Kan extension functor $\iota_! F$, i.e $\iota_! F \colon \underline{\mathcal{M}} \to (\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}})) \star_{\underline{\mathcal{M}}} \underline{\mathcal{M}} \to \underline{\mathcal{C}}$.

Let $x \in \underline{\mathcal{M}}^\otimes$ be an object over $I = (U \to G/H)$ and $\psi \colon I \to J$ be an active morphism in the fiber $(\underline{\mathbf{Fin}}^G_*)_{[G/H]}$ with target $J = (G/H \xrightarrow{=} G/H)$, given by the span

$$\psi = \begin{pmatrix} U & \xleftarrow{=} & U & \xrightarrow{f} & G/H \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} \\ G/H & \xleftarrow{=} & G/H & \xrightarrow{=} & G/H \end{pmatrix}.$$

Denote the G-functor classified by x by $x_\bullet \colon \underline{U} \to \underline{\mathcal{M}}$ (see remark B.0.8). Pulling back the coCartesian fibration $(\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}})) \star_{\underline{\mathcal{M}}} \underline{\mathcal{M}} \twoheadrightarrow \underline{\mathcal{M}}$ along $x_\bullet$ we get a $\underline{U}$-parametrized G-colimit diagram $(\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}}) \times_{\underline{\mathcal{M}}} \underline{U}) \star_{\underline{U}} \underline{U} \to (\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}})) \star_{\underline{\mathcal{M}}} \underline{\mathcal{M}} \to \underline{\mathcal{C}}$ (implicitly using [Sha18, lem. 4.4]), and therefore a $\underline{U}$-colimit diagram

$$\overline{p} \colon (\underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}}) \times_{\underline{\mathcal{M}}} \underline{U}) \star_{\underline{U}} \underline{U} \to \underline{\mathcal{C} \times U}.$$

Denote the $\underline{U}$-category indexing the colimit diagram above by $\underline{\mathcal{D}}_{/\underline{x_\bullet}} := \underline{\mathcal{D}} \times_{\underline{\mathcal{M}}} \underline{\mathbf{Arr}}_G(\underline{\mathcal{M}}) \times_{\underline{\mathcal{M}}} \underline{U}$. Note that by definition the restriction of $\overline{p}$ to $\underline{\mathcal{D}}_{/\underline{x_\bullet}}$ factors as the $\underline{U}$-functor

$$\underline{\mathcal{D}}_{/\underline{x_\bullet}} \to \underline{\mathcal{D} \times U} \xrightarrow{\underline{F \times U}} \underline{\mathcal{C} \times U}$$

and the restriction to $\underline{U}$ is the $\underline{U}$-functor

$$\iota_! F(x_\bullet) \colon \underline{U} \xrightarrow{\underline{x_\bullet \times U}} \underline{\mathcal{M} \times U} \xrightarrow{\underline{\iota_! F \times U}} \underline{\mathcal{C} \times U}.$$

Since $\underline{\mathcal{C}}^\otimes$ is a presentable G-symmetric monoidal category the tensor product functor

$$\otimes_\psi \colon \prod_I \underline{\mathcal{C} \times U} \to \underline{\mathcal{C} \times G/H}$$

of definition B.0.11 is a distributive $\underline{G/H}$-functor (see [Nar17, def. 3.15]). Therefore the $\underline{U}$-colimit diagram $\overline{p}$ induces a $\underline{G/H}$-colimit diagram

$$\left(\prod_I \underline{\mathcal{D}}_{/\underline{x_\bullet}}\right) \star_{\underline{G/H}} \underline{G/H} \to \prod_I \left(\underline{\mathcal{D}}_{/\underline{x_\bullet}} \star_{\underline{U}} \underline{U}\right) \xrightarrow[\psi]{\prod \overline{p}} \prod_I (\underline{\mathcal{C}}\underline{\times}\underline{U}) \xrightarrow{\otimes_\psi} \underline{\mathcal{C}}\underline{\times}\underline{G/H} \tag{20}$$

exhibiting the $\underline{G/H}$-object

$$\otimes_\psi \left(\prod_I \iota_! F(x_\bullet)\right) : \underline{G/H} \xrightarrow{\simeq} \prod_I \underline{U} \xrightarrow{\prod_I \iota_! F(x_\bullet)} \prod_I \underline{\mathcal{C}}\underline{\times}\underline{U} \xrightarrow{\otimes_\psi} \underline{\mathcal{C}}\underline{\times}\underline{G/H}$$

as the $\underline{G/H}$-colimit of

$$p\colon \prod_I \underline{\mathcal{D}}_{/\underline{x_\bullet}} \to \prod_I \underline{\mathcal{D}}\underline{\times}\underline{U} \to \prod_I \underline{\mathcal{C}}\underline{\times}\underline{U} \xrightarrow{\otimes_\psi} \underline{\mathcal{C}}\underline{\times}\underline{G/H}.$$

First, note that we can express the $\underline{G/H}$-colimit $\otimes_\psi \left(\prod_I \iota_! F(x_\bullet)\right)$ of (20) in simpler terms. Since $(\iota^\otimes)_! F^\otimes \colon D^\otimes \to \underline{\mathcal{C}}^\otimes$ is a lax G-symmetric monoidal functor we have

$$\begin{array}{ccccc}
\underline{G/H} & \xrightarrow{x} & \underline{\mathcal{M}}^\otimes_{<I>} & \xrightarrow{((\iota^\otimes)_! F^\otimes)_{<I>}} & \underline{\mathcal{C}}^\otimes_{<I>} \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\prod_I \underline{U} & \xrightarrow{\prod_I x_\bullet \underline{\times} \underline{U}} & \prod_I \underline{\mathcal{M}}\underline{\times}\underline{U} & \xrightarrow{\prod_I \iota_! F \underline{\times} \underline{U}} & \prod_I C\underline{\times}\underline{U},
\end{array}$$

with the top composite labelled $(\iota^\otimes)_! F^\otimes(x)$ and the bottom composite labelled $\prod_I \iota_! F(x_\bullet)$, therefore $\otimes_\psi \left(\prod_I \iota_! F(x_\bullet)\right) \simeq \otimes_\psi \left((\iota^\otimes)_! F^\otimes(x)\right)$.

On the other hand, we can also express the $\underline{G/H}$-diagram p in simpler terms. To see this observe the commutative diagram

$$\begin{array}{ccccc}
(\underline{\mathcal{D}}^\otimes_{<I>})_{/\underline{x}} & \longrightarrow & \underline{\mathcal{D}}^\otimes_{<I>} & \xrightarrow{F^\otimes_{<I>}} & \underline{\mathcal{C}}^\otimes_{<I>} \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\prod_I \underline{\mathcal{D}}_{/\underline{x_\bullet}} & \longrightarrow & \prod_I \underline{\mathcal{D}}\underline{\times}\underline{U} & \xrightarrow{\prod_I F \underline{\times} \underline{U}} & \prod_I \underline{\mathcal{C}}\underline{\times}\underline{U} \\
\downarrow \otimes_\psi & & \downarrow \otimes_\psi & & \downarrow \otimes_\psi \\
\underline{\mathcal{D}}_{/\underline{\otimes_\psi x}} \cong \left(\underline{\mathcal{D}}\underline{\times}\underline{G/H}\right)_{/\underline{\otimes_\psi x}} & \longrightarrow & \underline{\mathcal{D}}\underline{\times}\underline{G/H} & \xrightarrow{F \underline{\times} \underline{G/H}} & \underline{\mathcal{C}}\underline{\times}\underline{G/H}
\end{array} \tag{21}$$

where the left vertical column is induced by taking the $\underline{G/H}$-limit of the rows of the following diagram of $\underline{G/H}$-categories:

$$\begin{array}{ccccccccc}
\underline{\mathcal{D}}^{\otimes}_{<I>} & \longrightarrow & \underline{\mathcal{M}}^{\otimes}_{<I>} & \longleftarrow & \underline{\mathbf{Arr}}_{\underline{G/H}}(\underline{\mathcal{M}}^{\otimes}_{<I>}) & \longrightarrow & \underline{\mathcal{M}}^{\otimes}_{<I>} & \xleftarrow{x} & \underline{G/H} \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\prod_I \underline{\mathcal{D}} \underline{\times} \underline{U} & \longrightarrow & \prod_I \underline{\mathcal{M}} \underline{\times} \underline{U} & \longleftarrow & \underline{\mathbf{Arr}}_{\underline{G/H}}(\prod_I \underline{\mathcal{M}} \underline{\times} \underline{U}) & \longrightarrow & \prod_I \underline{\mathcal{M}} \underline{\times} \underline{U} & \xleftarrow{\prod_I x_\bullet} & \prod_I \underline{U} \\
\downarrow \otimes_\psi & & \downarrow \otimes_\psi & & \downarrow \otimes_\psi & & \downarrow \otimes_\psi & & \downarrow \simeq \\
\underline{\mathcal{D}} \underline{\times} \underline{G/H} & \longrightarrow & \underline{\mathcal{M}} \underline{\times} \underline{G/H} & \longleftarrow & \underline{\mathbf{Arr}}_{\underline{G/H}}(\underline{\mathcal{M}} \underline{\times} \underline{G/H}) & \longrightarrow & \underline{\mathcal{M}} \underline{\times} \underline{G/H} & \xleftarrow{\otimes_\psi x} & \underline{G/H}.
\end{array}$$

Note that p is the composition of the middle row of diagram (21) followed by the lower right vertical $\underline{G/H}$-functor $\otimes_\psi$. Therefore p is equivalent to the composition of the left vertical column of diagram (21) followed by the bottom row:

$$(\underline{\mathcal{D}}^{\otimes}_{<I>})_{/\underline{x}} \xrightarrow{\otimes_\psi} \underline{\mathcal{D}}_{/\underline{\otimes_\psi x}} \to \underline{\mathcal{D}} \underline{\times} \underline{G/H} \xrightarrow{F \underline{\times} \underline{G/H}} \underline{\mathcal{C}} \underline{\times} \underline{G/H}.$$

Finally, by the assumption of the lemma the $\underline{G/H}$-functor $\otimes_\psi \colon (\underline{\mathcal{D}}^{\otimes}_{<I>})_{/\underline{x}} \to \left(\underline{\mathcal{D}} \underline{\times} \underline{G/H}\right)_{/\underline{\otimes_\psi x}}$ is $\underline{G/H}$-cofinal, therefore

$$\begin{aligned}
\otimes_\psi \left((\iota^{\otimes})_! F^{\otimes}(x)\right) &\simeq \underline{G/H} - \underrightarrow{colim} \left((\underline{\mathcal{D}}^{\otimes}_{<I>})_{/\underline{x}} \xrightarrow{\otimes_\psi} \underline{\mathcal{D}}_{/\underline{\otimes_\psi x}} \to \underline{\mathcal{D}} \underline{\times} \underline{G/H} \xrightarrow{F \underline{\times} \underline{G/H}} \underline{\mathcal{C}} \underline{\times} \underline{G/H} \right) \\
&\xrightarrow{\sim} \underline{G/H} - \underrightarrow{colim} \left(\underline{\mathcal{D}}_{/\underline{\otimes_\psi x}} \to \underline{\mathcal{D}} \underline{\times} \underline{G/H} \xrightarrow{F \underline{\times} \underline{G/H}} \underline{\mathcal{C}} \underline{\times} \underline{G/H} \right) \simeq \iota_! F(\otimes_\psi x),
\end{aligned}$$

so we have a coCartesian edge $e \colon (\iota^{\otimes})_! F^{\otimes}(x) \to \iota_! F(\otimes_\psi x)$ in $\underline{\mathcal{C}}^{\otimes}$ over ψ.

We can now show that $(\iota^{\otimes})_! F^{\otimes} \colon \underline{\mathcal{M}}^{\otimes} \to \underline{\mathcal{C}}^{\otimes}$ preserves coCartesian edges over $\psi \colon I \to J$ as above. Let $e' \colon x \to y$ be a coCartesian edge in $\underline{\mathcal{M}}^{\otimes}$ over ψ. By definition of $\otimes_\psi$ this coCartesian edge factors as $x \to \otimes_\psi x \xrightarrow{\sim} y$ over $I \xrightarrow{\psi} J \xrightarrow{=} J$ (See [Lur09a, rem. 2.4.1.4 and prop. 2.4.1.5]). Applying $(\iota^{\otimes})_! F^{\otimes}$ we get $(\iota^{\otimes})_! F^{\otimes}(e') \colon (\iota^{\otimes})_! F^{\otimes}(x) \to (\iota^{\otimes})_! F^{\otimes}(y) = \iota_! F(y)$, and we need to show $(\iota^{\otimes})_! F^{\otimes}(e')$ is a coCartesian lift of ψ. However, we already have a coCartesian lift of ψ, the edge e we constructed above. Therefore $(\iota^{\otimes})_! F^{\otimes}(e')$ factors through e as $(\iota^{\otimes})_! F^{\otimes}(x) \xrightarrow{e} \iota_! F(\otimes_\psi x) \to \iota_! F(y)$. Note that the morphism $\iota_! F(\otimes_\psi x) \to \iota_! F(y)$ is induced from $\otimes_\psi x \xrightarrow{\sim} y$, and therefore an equivalence. Hence $(\iota^{\otimes})_! F^{\otimes}(e')$ is coCartesian as a composition of a coCartesian edge and an equivalence.

This ends the proof of lemma 4.2.5. □

We can now prove proposition 4.2.2 by verifying the cofinality conditions of lemma 4.2.5. In fact, we prove the cofinality of the maps in lemma 4.2.5 by showing that they are equivalences.

We rely on the following result to reduce our calculations to the non-framed case $B = BO_n(G)$.

Proposition 4.2.6. *Let $f \colon B \to BO_n(G)$ be a G-map as in definition 3.3.1, and $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ an f-framed $\mathcal{O}_G$-manifold over G/H. Then the $\underline{G/H}$-functor*

$$(\underline{\mathbf{Mfld}}^{G,f-fr})_{/\underline{M}} \to \underline{\mathbf{Mfld}}^{G}_{/\underline{M}}$$

is an equivalence of $\underline{G/H}$-categories.

In particular, every G-submanifold $N \subseteq M$ has an essentially unique lift to $N \in \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}}$ (informally, M induces an f-framing of N).

Proof. We show that for every $\varphi \in \underline{G/H}$, $\varphi \colon G/K \to G/H$ the induced functor on the fibers over φ, $\left((\underline{\mathbf{Mfld}}^{G,f-fr})_{/\underline{M}}\right)_{[\varphi]} \to \left(\underline{\mathbf{Mfld}}^{G}_{/\underline{M}}\right)_{[\varphi]}$, is an equivalence. By construction the fibers of the parametrized over category are equivalent to the over categories

$$\left((\underline{\mathbf{Mfld}}^{G,f-fr})_{/\underline{M}}\right)_{[\varphi]} \simeq \left(\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/K]}\right)_{/\varphi^* M}, \quad \left((\underline{\mathbf{Mfld}}^{G})_{/\underline{M}}\right)_{[\varphi]} \simeq \left(\underline{\mathbf{Mfld}}^{G}_{[G/K]}\right)_{/\varphi^* M}.$$

By definition 3.3.1 the fiber $\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/K]}$ is given by the pullback of ∞-categories

$$\begin{array}{ccc}
\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/K]} & \longrightarrow & (\underline{\mathbf{Top}}^{G}_{[G/K]})_{/\underline{B}(G/K)} \\
\downarrow & \ulcorner & \downarrow \\
\underline{\mathbf{Mfld}}^{G}_{[G/K]} & \longrightarrow & (\underline{\mathbf{Top}}^{G}_{[G/K]})_{/\underline{BO_n(G)}(G/K)},
\end{array}$$

where

$$\underline{B}(G/K) = (B \times G/K \to G/K), \quad \underline{BO_n(G)}(G/K) = (BO_n(G) \times G/K \to G/K),$$

see remark 2.1.7. We can simplify the pullback square above using the equivalences of remark 3.2.9:

$$(\underline{\mathbf{Top}}^{G}_{[G/K]})_{/\underline{B}(G/K)} \xrightarrow{\sim} \mathbf{Top}^{G}_{/B \times G/K}, \quad (\underline{\mathbf{Top}}^{G}_{[G/K]})_{/\underline{BO_n(G)}(G/K)} \xrightarrow{\sim} \mathbf{Top}^{G}_{/BO_n(G) \times G/K}.$$

We can now express the slice category $(\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/K]})_{/\varphi^* M}$ as a pullback of slice categories

$$\begin{array}{ccc}
(\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/K]})_{/\varphi^* M} & \longrightarrow & \left(\mathbf{Top}^{G}_{/B \times G/K}\right)_{/(\varphi^* M \to B \times G/K)} \\
\downarrow & \ulcorner & \downarrow \\
(\underline{\mathbf{Mfld}}^{G}_{[G/K]})_{/\varphi^* M} & \longrightarrow & \left(\mathbf{Top}^{G}_{/BO_n(G) \times G/K}\right)_{/(\varphi^* M \to BO_n(G) \times G/K)} \\
 & & \downarrow \\
 & & \mathbf{Top}^{G}_{/\varphi^* M}.
\end{array}$$

By [AF15, lem. 2.5] both the bottom right vertical arrow and the composition of the right vertical arrows are equivalences of ∞-categories. By the two-out-of-three property we see that the top vertical arrow is an equivalence of ∞-categories, and therefore the left vertical arrow is also an equivalence, as claimed. □

Proof of proposition 4.2.2. By the Segal conditions and proposition 4.2.6 the $\underline{G/H}$-functors

$$(\underline{\mathbf{Disk}}^{G,f-fr,\sqcup}_{<I_i>})_{/\underline{M_i}} \to (\underline{\mathbf{Disk}}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}}, \quad i = 1, 2$$

are equivalences of $\underline{G/H}$-categories, therefore it is enough to prove the non-framed case.

Let $\psi\colon I \to J$ be an active morphism in the fiber $(\underline{\mathbf{Fin}}^G_*)_{[G/H]}$. Without loss of generality, ψ is represented by the span

$$\psi = \left(\begin{array}{ccccc} U_1 & \xleftarrow{=} & U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ G/H & \xleftarrow{=} & G/H & \xrightarrow{=} & G/H \end{array} \right).$$

By remark 3.4.20 a coCartesian lift $f\colon M_1 \to M_2$ of ψ is represented by a span

$$f = \left(\begin{array}{ccccc} M_1 & \xleftarrow{=} & M_1 & \xrightarrow{\sim} & M_2 \\ \downarrow & & \downarrow & & \downarrow \\ U_1 & \xleftarrow{=} & U_1 & \longrightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ G/H & \xleftarrow{=} & G/H & \xrightarrow{=} & G/H \end{array} \right).$$

By lemma 4.2.5 it is enough to show that the $\underline{G/H}$-functor $(\underline{\mathbf{Disk}}^{G,\sqcup}_{<I>})_{/\underline{M_1}} \to (\underline{\mathbf{Disk}}^{G,\sqcup}_{<J>})_{/\underline{M_2}}$ is $\underline{G/H}$-cofinal. We prove that it is in fact an equivalence, by showing that it is induces fiberwise equivalences.

Consider the induced functor $\left((\underline{\mathbf{Disk}}^{G,\sqcup}_{<I>})_{/\underline{M_1}}\right)_{[\varphi]} \to \left((\underline{\mathbf{Disk}}^{G,\sqcup}_{<J>})_{/\underline{M_2}}\right)_{[\varphi]}$ between the fibers over $\varphi \in \underline{G/H}$, $\varphi\colon G/K \to G/H$.

We now inspect each fiber. By definition, we have

$$(\underline{\mathbf{Disk}}^{G,\sqcup}_{<I>})_{/\underline{M_i}} := \underline{\mathbf{Disk}}^{G,\sqcup}_{<I>} \times_{\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I>}} (\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I>})_{/\underline{M_i}}, \quad i = 1, 2,$$

and therefore the fibers over φ are given by

$$\left((\underline{\mathbf{Disk}}^{G,\sqcup}_{<I>})_{/\underline{M_i}}\right)_{[\varphi]} = \left(\underline{\mathbf{Disk}}^{G,\sqcup}_{<I>}\right)_{[\varphi]} \times_{\left(\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I>}\right)_{[\varphi]}} \left((\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I>})_{/\underline{M_i}}\right)_{[\varphi]}, \quad i = 1, 2.$$

By the definition of parametrized slice category [Sha18, not. 4.29] we have

$$\left((\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I>})_{/\underline{M_i}}\right)_{[\varphi]} \cong \left((\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I>})_{[\varphi]}\right)_{/\varphi^* M_i}, \quad i = 1, 2,$$

where $\varphi^* M_i$, $i = 1, 2$ is the pullback of $M_i \to U_i \to G/H$ along $\varphi\colon G/K \to G/H$.

Next, note that $(\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I_i>})_{[\varphi]} \cong (\underline{\mathbf{Mfld}}^{G,\sqcup})_{\varphi^* I_i}$ is the fiber of $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ over $\varphi^* I_i = (U_i \times_{G/H} G/K \to G/H) \in \underline{\mathbf{Fin}}^G_*$.

However, using the definition of the coCartesian fibration $\underline{\mathbf{Mfld}}^{G,\sqcup} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ (definition 3.4.19) and the definition of the unfurling construction (see [Bar14, prop. 11.6] and the description of the fibers following it) we see that $(\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I_i>})_{[\varphi]}$ is equivalent to $(\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_{\varphi^* I_i}$, (the coherent nerve of) the full topological subcategory of $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds, $\mathcal{O}_G$-$\mathbf{Fin}$-$\mathbf{Mfld}$ (definition 3.4.5) spanned by $\mathcal{O}_G$-$\mathbf{Fin}$-manifolds over $\varphi^* I_i$. It follows that the ∞-category $\left((\underline{\mathbf{Mfld}}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}}\right)_{[\varphi]}$ is equivalent to the slice category $((\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_{\varphi^* I_i})_{/\varphi^* M_i}$, modeled by the coherent nerve of the Moore over category $((\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_{\varphi^* I_i})^{\text{Moore}}_{/\varphi^* M_i}$. Therefore, the fiber $\left((\underline{\mathbf{Disk}}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}}\right)_{[\varphi]}$ is equivalent to the full subcategory of $((\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_{\varphi^* I_i})^{\mathrm{Moore}}_{/\varphi^* M_i}$ spanned by objects represented by morphisms $(E \to U' \to U_i \times_{G/H} G/K \to G/H) \to (\varphi^* M_i \to U_i \times_{G/H} G/K \to G/H)$ over $\phi^* I_i$, where $E \to U'$ is a G-vector bundle. Recall that $U' = \pi_0(E)$ (remark 3.6.6). Unwinding the definition of morphisms in $\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld}$ over $\varphi^* I_i = U_i \times_{G/H} G/K$, we see that such morphisms are represented by commutative diagrams

$$\begin{array}{ccccc}
E & \xleftarrow{\sim} & E' & \hookrightarrow & M_i \\
\downarrow & & \downarrow & & \\
\pi_0(E) & \xleftarrow{=} & \pi_0(E) & & \downarrow \\
\downarrow & & \downarrow & & \\
\varphi^* I_i & \xleftarrow{=} & \varphi^* I_i & \xrightarrow{=} & \varphi^* I_i \\
\downarrow & & \downarrow & & \downarrow \\
G/K & \xleftarrow{=} & G/K & \xrightarrow{=} & G/K,
\end{array}$$

or equivalently, by a G-equivariant embedding $E \hookrightarrow M_i$ over $\varphi^* I_i$.

With this concrete description of the fibers at hand, the induced functor between the fibers $\left((\underline{\mathbf{Disk}}^{G,\sqcup}_{<I_1>})_{/\underline{M_1}}\right)_{[\varphi]} \to \left((\underline{\mathbf{Disk}}^{G,\sqcup}_{<I_2>})_{/\underline{M_2}}\right)_{[\varphi]}$ is given by composition with

$$\varphi^* f = \left(\begin{array}{ccccc}
\varphi^* M_1 & \xleftarrow{=} & \varphi^* M_1 & \xrightarrow{\sim} & \varphi^* M_2 \\
\downarrow & & \downarrow & & \downarrow \\
\varphi^* I_1 & \xleftarrow{=} & \varphi^* I_1 & \longrightarrow & \varphi^* I_2 \\
\downarrow & & \downarrow & & \downarrow \\
G/K & \xleftarrow{=} & G/K & \xrightarrow{=} & G/K
\end{array}\right).$$

By inspection the induced functor

$$(\varphi^* f) \circ - \colon ((\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_{\varphi^* I_1})^{\mathrm{Moore}}_{/\varphi^* M_1} \to ((\mathcal{O}_G\text{-}\mathbf{Fin}\text{-}\mathbf{Mfld})_{\varphi^* I_2})^{\mathrm{Moore}}_{/\varphi^* M_2}$$

$$\left(\begin{array}{ccc}
E & \hookrightarrow & \varphi^* M_1 \\
\downarrow & & \downarrow \\
\pi_0(E) & & \downarrow \\
 & \searrow & \downarrow \\
 & & \varphi^* I_1 \\
 & & \downarrow \\
 & & G/K
\end{array}\right) \mapsto \left(\begin{array}{ccccccc}
E & \hookrightarrow & \varphi^* M_1 & \xleftarrow{=} & \varphi^* M_1 & \xrightarrow{\sim} & \varphi^* M_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_0(E) & & \downarrow & & \downarrow & & \downarrow \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & \varphi^* I_1 & \xleftarrow{=} & \varphi^* I_1 & \longrightarrow & \varphi^* I_2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & G/K & \xleftarrow{=} & G/K & \xrightarrow{=} & G/K
\end{array}\right)$$

is an equivalence of topological categories. □

5 Properties of G-factorization homology

In this subsection we prove two properties of G-factorization homology: it satisfies G-$\otimes$-excision (proposition 5.2.3) and respects G-sequential colimits (proposition 5.3.3).

5.1 Collar decomposition of G-manifolds

We define G-collar decompositions of G-manifolds and construct inverse image functors (construction 5.1.4). In the next subsection we use these constructions to define G-$\otimes$-excision and prove that G-factorization homology satisfies G-$\otimes$-excision (proposition 5.2.3).

We begin with an equivariant version of collar-gluing, see [AF15, def. 3.13]. The same definition is given in [Wee18, def. 4.20].

Definition 5.1.1. *Let $M \in \mathbf{Mfld}^G$ be an n-dimensional G-manifold. A G-*collar decomposition *of M is a smooth G-invariant function $f\colon M \to [-1,1]$ to the closed interval for which the restriction $f|_{(-1,1)}\colon M|_{(-1,1)} \to (-1,1)$ is a manifold fiber bundle, with a choice of trivialization $M|_{(-1,1)} \cong M_0 \times (-1,1)$. Here $M_{(-1,1)} = f^{-1}(-1,1)$, $M_0 = f^{-1}(0)$. For such a decomposition, denote $M_+ := f^{-1}(-1,1]$, $M_- := f^{-1}[-1,1)$.*

A G-collar decomposition of an f-framed $\mathcal{O}_G$-manifold $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ is a G-collar decomposition of the underlying G-manifold M.

Remark 5.1.2. Note that $M|_{(-1,1)}$ is a tubular neighborhood of the codimension one G-submanifold M_0, and that M_0 splits M into two G-manifolds, i.e. there exists a continuous G-invariant function $M \setminus M_0 \to [-1,1] \setminus \{0\} \to \{-1,1\}$ to the set with two elements. On the other hand, a G-submanifold $M_0 \subset M$ of codimension one that splits M into two G-manifolds has an equivariant tubular neighbourhood $T \subset M$ equivalent to the total space $\nu(M_0)$ of the normal bundle of M_0 (compatible with the $M_0 \subset M$ and the zero section $M_0 \to \nu(M)$). By assumption, the normal bundle of M_0 is a trivial vector bundle of rank 1 with trivial G action. A choice of G-diffeomorphisms $T \cong \nu(M) \cong M_0 \times \mathbb{R} \cong M_0 \times (-1,1)$ (compatible with M_0) determines a G-collar decomposition of M.

Remark 5.1.3. A G-collar decomposition $f\colon M \to [-1,1]$ defines a decomposition of M into a union of open G-submanifolds $M = M_- \cup M_+$ with a chosen isomorphism $M_- \cap M_+ \cong M_0 \times (-1,1)$. The purpose of the above definition is to specify these decompositions among all decompositions $M = U \cup V$ of M as a union of two open G-submanifolds. We will see that G-equivariant homology is compatible with G-collar decompositions (definition 5.2.2 and proposition 5.2.3). This should be compared with Bredon homology, which is compatible with all decompositions $M = U \cup V$ into two equivariant open subsets (the equivariant Mayer-Vietoris property).

Next we construct an "inverse image" functor $f^{-1}\colon \mathbf{Mfld}^{\partial,or}_{/[-1,1]} \to (\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]})_{/M}$ from the ∞-category of 1-dimensional oriented manifolds with boundary over the interval $[-1,1]$ (see [AF15]). By proposition 4.2.6 we have an equivalence of ∞-categories

$$(\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]})_{/M} \xrightarrow{\sim} (\underline{\mathbf{Mfld}}^{G}_{[G/H]})_{/(M\to G/H)}, \tag{22}$$

so it is enough to construct $f^{-1}\colon \mathbf{Mfld}^{\partial,or}_{/[-1,1]} \to (\underline{\mathbf{Mfld}}^{G}_{[G/H]})_{/(M\to G/H)}$ for $(M \to G/H) \in \underline{\mathbf{Mfld}}^{G}_{[G/H]}$ the underlying $\mathcal{O}_G$-manifold of $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$.

Note that both the domain and the codomain of the functor f^{-1} can be described using coherent nerve of the Moore over categories (see appendix A), since the ∞-categories

$$\mathbf{Mfld}^{bnd,or}, \quad \underline{\mathbf{Mfld}}^{G}_{[G/H]} \cong \mathbf{Mfld}^{G}_{[G/H]}$$

are coherent nerves of topological categories. We construct the inverse image functor as the coherent nerve of a functor of topological categories between the Moore over categories.

Construction 5.1.4. Let $(M \to G/H)$ be an $\mathcal{O}_G$-manifold with a collar decomposition $f\colon M \to [-1,1]$. Define a topological functor $(\mathbf{Mfld}^{\partial,or})^{\text{Moore}}_{/[-1,1]} \to (\underline{\mathbf{Mfld}}^{G}_{[G/H]})^{\text{Moore}}_{/(M \to G/H)}$ between the Moore over categories:

1. Send an object of $(\mathbf{Mfld}^{\partial,or})^{\text{Moore}}_{/[-1,1]}$ given by oriented embedding $\varphi\colon V \hookrightarrow [-1,1]$ to its inverse image, $f^{-1}(\varphi)\colon f^{-1}V \hookrightarrow M$ given by the pullback of φ along f. Since the function f is G-invariant the embedding $f^{-1}(\varphi)$ is G-equivariant. The composition

$$f^{-1}V \xhookrightarrow{f^{-1}(\varphi)} M \longrightarrow G/H$$

makes $f^{-1}V$ a G-manifold over G/H, hence $f^{-1}(\varphi)$ is a point in the topological space $Emb^{G}_{G/H}(f^{-1}V, M)$, i.e an object of the Moore over category $(\underline{\mathbf{Mfld}}^{G}_{[G/H]})^{\text{Moore}}_{/(M \to G/H)}$.

2. Let $\varphi\colon V \hookrightarrow [-1,1]$ and $\varphi'\colon V' \hookrightarrow [-1,1]$ be two objects of the Moore over category $(\mathbf{Mfld}^{\partial,or})^{\text{Moore}}_{/[-1,1]}$. Let $(h,(r,\gamma))$ be a point in $\mathrm{Map}_{(\mathbf{Mfld}^{\partial,or})^{\text{Moore}}_{/[-1,1]}}(\varphi,\varphi')$, where $h\colon V \hookrightarrow V'$ is an oriented embedding and $(r,\gamma) \in [0,\infty) \times \left(Emb^{\partial,or}(V,[-1,1])\right)^{[0,\infty)}$ is a Moore path from φ to $\varphi' \circ h$. Define a continuous function

$$f^{-1}\colon \mathrm{Map}_{(\mathbf{Mfld}^{\partial,or})^{\text{Moore}}_{/[-1,1]}}(\varphi,\varphi') \to \mathrm{Map}_{(\underline{\mathbf{Mfld}}^{G}_{[G/H]})^{\text{Moore}}_{/(M\to G/H)}}\left(f^{-1}(\operatorname{Im}\varphi) \subset M, f^{-1}(\operatorname{Im}\varphi') \subset M\right),$$

$$f^{-1}(h,(r,\gamma)) := (f^{-1}(h),(r,\alpha)), \qquad \begin{array}{ccccc} f^{-1}V & \xhookrightarrow{f^{-1}(h)} & f^{-1} & \xhookrightarrow{f^{-1}(\varphi')} & M \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow f \\ V & \xhookrightarrow{\varphi} & V & \xhookrightarrow{h} & [-1,1] \end{array}$$

where $f^{-1}(h)$ is given by the pullback

$$\begin{array}{ccccc} f^{-1}V & \xhookrightarrow{f^{-1}(h)} & f^{-1} & \xhookrightarrow{f^{-1}(\varphi')} & M \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow f \\ V & \xhookrightarrow{\varphi} & V & \xhookrightarrow{h} & [-1,1] \end{array}$$

and $\alpha\colon [0,\infty) \to Emb^{G}(f^{-1}(V), M)$ is the Moore path of length r defined as follows. If $x \in M|_{(-1,1)} \cong M_0 \times (-1,1)$ corresponds to $(y,s) \in M_0 \times (-1,1)$ define

$$\alpha_t(x) = (y, \gamma_t \circ \varphi^{-1}(s)) \in M_0 \times (-1,1) \cong M|_{(-1,1)},$$

otherwise (i.e. $f(x) = \pm 1$) define $\alpha_t(x) = x$. Verification that (r,α) is a smooth G-equivariant isotopy depending continuously on γ is left to the reader.

Clearly f^{-1} preserve disjoint unions.

Remark 5.1.5. More generally, one can try to define an inverse image functor along a general smooth invariant map $M \to N$ to a oriented manifold with boundary N. However, not every map f will do. First, in order to define the isotopy lift α assume that the restrictions of f to $f^{-1}(N \setminus \partial N)$ and $f^{-1}(\partial N)$ are smooth fiber bundles, and use G-equivariant parallel transport between the fibers. The connections on the fiber bundles need to be compatible in order for α to be continuous and smooth. However, such parallel transport defines functions which are only continuous in the C^1-topology on $Emb^G(f^{-1}V, M)$, since they depend on the time derivative of the isotopy γ. Nevertheless, if the connections chosen are flat then parallel transport depends only on the end points, and therefore defines a continuous function relative to the compact-open topology. All these conditions can be can be captured together by assuming that $f\colon M \to N$ is a G-invariant flat complete Riemannian submersion. This condition implies that the restrictions to $N \setminus \partial N$ and ∂N are flat fiber bundles, with compatibly chosen flat G-equivariant Ehresmann connections (i.e. a constructible fiber bundle relative to the boundary stratification).

5.2 G-$\otimes$-excision

We define an equivariant version of $\otimes$-excision of [AF15, def. 3.15] (see definition 5.2.2), and prove it is satisfied by G-factorization homology (proposition 5.2.3).

Given a G-symmetric monoidal functor $F\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$ and a G-collar decomposition of an f-framed $\mathcal{O}_G$-manifold $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ we construct a comparison map $F(M_-)\otimes_{F(M_0\times(-1,1))} F(M_+) \to F(M)$ in $\underline{\mathcal{C}}_{[G/H]}$. This construction depends on the "inverse image" functor of construction 5.1.4.

Construction 5.2.1. Let $F\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$ be a G-symmetric monoidal functor. Let $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ with underlying $\mathcal{O}_G$-manifold $(M \to G/H) \in \underline{\mathbf{Mfld}}^{G}_{[G/H]}$, and $f\colon M \to [-1,1]$ a G-collar decomposition. Consider the $\mathbf{Disk}^{\partial,or}_{/[-1,1]}$-shaped diagram in $\underline{\mathcal{C}}_{[G/H]}$ given by the functor

$$\mathbf{Disk}^{\partial,or}_{/[-1,1]} \to \mathbf{Mfld}^{\partial}_{/[-1,1]} \xrightarrow{f^{-1}} (\underline{\mathbf{Mfld}}^{G}_{[G/H]})_{/(M\to G/H)} \simeq (\underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]})_{/M} \xrightarrow{F} (\underline{\mathcal{C}}_{[G/H]})_{/F(M)} \tag{23}$$

where the first functor is the embedding of disks in manifolds followed by the functor forgetting orientation (see [AF15, def. 2.18]), the second functor is the inverse image functor defined in construction 5.1.4, followed by the equivalence of eq. (22), and the third functor is induced by the action of F on the over categories. By [AF15, lem. 3.11] there is a cofinal map $\Delta^{op} \to \mathbf{Disk}^{\partial,or}$, therefore the colimit of eq. (23) in $\underline{\mathcal{C}}_{[G/H]})_{/F(M)}$ is given by

$$\begin{pmatrix} \underrightarrow{colim}\,(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} F(M_-)\otimes F(M_0\times(-1,1))\otimes F(M_+) \rightrightarrows F(M_-)\otimes F(M_+)) \\ \downarrow \\ F(M) \end{pmatrix} \in (\underline{\mathcal{C}}_{[G/H]})_{/F(M)},$$

known as the two sided bar construction. Assume that $\underline{\mathcal{C}}_{[G/H]}$ admits sifted colimits and that the tensor product functor of $\underline{\mathcal{C}}_{[G/H]}$ preserves sifted colimits separately in each variable (i.e the coCartesian fibration $\underline{\mathcal{C}}^{\otimes}_{[G/H]} \to \mathbf{Fin}_*$ is compatible with sifted colimits in the sense of [Lur, def. 3.1.1.18]). Then the relative tensor product $F(M_-)\otimes_{F(M_0\times(-1,1))} F(M_+)$ can be identified with the colimit of this two sided bar construction (see [Lur, thm. 4.4.2.8]). Hence we identify the colimit of the diagram eq. (23) with

$$\begin{pmatrix} F(M_-)\otimes_{F(M_0\times(-1,1))} F(M_+) \\ \downarrow \\ F(M) \end{pmatrix} \in (\underline{\mathcal{C}}_{[G/H]})_{/F(M)}. \tag{24}$$

Definition 5.2.2. *A G-symmetric monoidal functor $F\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$ satisfies G-$\otimes$-excision if for every $M \in \underline{\mathbf{Mfld}}^{G,f-fr}$ with underlying $\mathcal{O}_G$-manifold $(M \to G/H)$ together with a G-collar decomposition $f\colon M \to [-1,1]$ the morphism (24) is an equivalence in $\underline{\mathcal{C}}_{[G/H]}$.*

The main result of this subsection is

Proposition 5.2.3. *Let $A\colon \underline{\mathbf{Disk}}^{G,f-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ be an f-framed G-disk algebra. Then the G-factorization homology functor $\int A\colon \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ of definition 4.2.3 satisfies G-$\otimes$-excision.*

Remark 5.2.4. We view the proof of proposition 5.2.3 as an instance of "equivariant push-forward". We conjecture that the pushforward paradigm of [AF15, sec. 3.4] and [AFT17a, sec. 2.5] has an equivariant generalization to a smooth constructible G-fiber bundle between equivariantly-framed $\mathcal{O}_G$-manifolds with boundary. However, the definition of equivariantly framed $\mathcal{O}_G$-manifolds with boundary is beyond the scope of this work.

Instead, we are able to prove proposition 5.2.3 without these definitions because the action of G on the oriented manifold $[-1,1]$ is trivial.

We could have followed a slightly more general approach, considering a G-constructible bundle $M \to N$ where N has boundary and trivial G-action. To do this, note that since G acts trivially on N, any G-embedding $V \hookrightarrow N$ must have a trivial G-action as well, so the slice category of G-disks over N is a constant G-diagram. This allows us to harness the definition of (nonequivariant) framing given in [AF15] to construct a replacement for the expected "G-slice category of f-framed G-embeddings $V \hookrightarrow N$" needed to preform pushforward. We chose not to prove this generalization since we do not currently need it, and we believe it would further obfuscate the proof.

In order to prove proposition 5.2.3 we need the following auxiliary construction.

Construction 5.2.5. Let $M \to G/H$ be an $\mathcal{O}_G$-manifold and $f\colon M \to [-1,1]$ be a G-collar decomposition of M. Define a $\underline{G/H}$-category $\underline{\mathbf{X}}_f \to \underline{G/H}$ and $\underline{G/H}$-functors

$$ev_0\colon \underline{\mathbf{X}}_f \to \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}, \quad ev_1\colon \underline{\mathbf{X}}_f \to \underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}$$

by the taking the limit of the following diagram of $\underline{G/H}$-categories.

$$\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}} \leftarrow \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{G/H} \times \Delta^1, \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}}) \to \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}} \xleftarrow{f^{-1}} \underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}$$

where f^{-1} is the inverse image functor of construction 5.1.4.

Remark 5.2.6. Using proposition 4.2.6 and unwinding the definitions shows that the fiber of $\underline{\mathbf{X}}_f \to \underline{G/H}$ over $\varphi\colon G/K \to G/H$ is given by the limit of

$$(\mathbf{Disk}^G_{[G/K]})_{/\varphi^* M} \to (\mathbf{Mfld}^G_{[G/K]})_{/\varphi^* M} \leftarrow \mathrm{Fun}(\Delta^1, (\mathbf{Disk}^G_{[G/K]})_{/\varphi^* M}) \to (\mathbf{Mfld}^G_{[G/K]})_{/\varphi^* M} \xleftarrow{f^{-1}} \mathbf{Disk}^{\partial,or}_{/[-1,1]},$$

where the ∞-over categories $(\mathbf{Disk}^G_{[G/H]})_{/\varphi^*M}$, $(\mathbf{Mfld}^G_{[G/H]})_{/\varphi^*M}$ can be modeled as the coherent nerve of the Moore over category (see appendix A). Explicitly, an object of $(\underline{\mathbf{X}}_f)_{[\varphi]}$ is given by $(g\colon V \hookrightarrow [-1,1], h\colon E \hookrightarrow \varphi^*M, h'\colon E \hookrightarrow f^{-1}V, \gamma)$ where

- V is a finite disjoint union of 1-dimensional oriented disks with boundary, i.e oriented open intervals equivalent to $\mathbb{R}$ and oriented half open intervals equivalent to $[0,1)$ or $(0,1]$,
- g is an orientation preserving embedding of V into the closed interval $[-1,1]$,
- $E \to U \to G/K$ is a finite G-disjoint union of G-disks (i.e $E \to U$ a G-vector bundle over a finite G-set),
- h is a G-equivariant embedding over G/K of E into the pullback of $M \to G/H$ along φ,
- h' is a G-equivariant embedding over G/K of E into the preimage $f^{-1}V$
- γ is a Moore path in $Emb^G_{[G/K]}(E, \varphi^*M)$ from h to $E \xrightarrow{h'} f^{-1}V \xrightarrow{f^{-1}(g)} \varphi^*M$.

The functor ev_0 sends the object $(g\colon V \hookrightarrow [-1,1], h\colon E \hookrightarrow \varphi^*M, h'\colon E \hookrightarrow f^{-1}V, \gamma)$ described above to $(h\colon E \hookrightarrow \varphi^*M) \in (\mathbf{Mfld}^G_{[G/K]})_{/\varphi^*M}$, while the functor ev_1 sends it to $(g\colon V \hookrightarrow [-1,1]) \in \mathbf{Disk}^{\partial,or}_{/[-1,1]}$.

By [Lur09a, prop. 2.4.7.12] it follows that for every $\varphi \in \underline{G/H}$ the functor

$$(ev_0)_{[\varphi]}\colon (\underline{\mathbf{X}}_f)_{[\varphi]} \to (\mathbf{Disk}^G_{[G/H]})_{/\varphi^*M}$$

is a Cartesian fibration (and therefore that ev_1 is a $\underline{G/H}$-Cartesian fibration, see [Sha18, def. 7.1]).

The following lemma is the main ingredient in the proof of proposition 5.2.3.

Lemma 5.2.7. *The $\underline{G/H}$-functor $ev_0\colon \underline{\mathbf{X}}_f \to \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}$ is $\underline{G/H}$-cofinal.*

The following proof is an adaptation of [Lur, thm. 5.5.3.6], [AF15, lem. 3.21] and [AFT17a, lem. 2.27] to the equivariant setting.

Proof of lemma 5.2.7. By proposition 4.2.6 we have to prove that $ev_0\colon \underline{\mathbf{X}}_f \to \underline{\mathbf{Disk}}^G_{/\underline{M}}$ is $\underline{G/H}$-cofinal. By [Sha18, thm. 6.7, def. 6.8] the $\underline{G/H}$ functor ev_0 is $\underline{G/H}$-cofinal if and only if for each $(\varphi\colon G/K \to G/H) \in \underline{G/H}$ the functor $(ev_0)_{[\varphi]}\colon (\underline{\mathbf{X}}_f)_{[\varphi]} \to (\underline{\mathbf{Disk}}^G_{/\underline{M}})_{[\varphi]}$ is cofinal.

By replacing $f\colon M \to [-1,1]$ with $\varphi^*M \to M \xrightarrow{f} [-1,1]$ we reduce to $\varphi = (G/H \xrightarrow{=} G/H) \in \underline{G/H}$: it is enough to prove that $(ev_0)_{[G/H]}$ is cofinal.

By remark 5.2.6 the functor $(ev_0)_{[G/H]}$ is a Cartesian fibration, therefore by [Lur09a, prop. 4.1.3.2] it is enough to show that for each $(E \hookrightarrow M) \in (\mathbf{Disk}^G_{[G/H]})_{/M}$ the fiber $(ev_0)^{-1}(E \hookrightarrow M)$ is weakly contractible.

Note that the category $(ev_0)^{-1}(E \hookrightarrow M)$ has a functor to $\mathbf{Disk}^{\partial,or}_{/[-1,1]}$ by construction:

$$\begin{array}{ccc}
(ev_0)^{-1}(E \hookrightarrow M) & \xrightarrow{\qquad\ulcorner\qquad} & \mathbf{Disk}^{\partial,or}_{/[-1,1]} \cong \{E \hookrightarrow M\} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]} \\
\downarrow & & \downarrow \\
(\underline{\mathbf{X}}_f)_{[G/H]} & \xrightarrow[\ulcorner]{((ev_0)_{[G/H]},(ev_1)_{[G/H]})} & (\mathbf{Disk}^G_{[G/H]})_{/M} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]} \\
\downarrow & & \downarrow{\scriptstyle (\iota, f^{-1})} \\
Fun(\Delta^1, (\mathbf{Mfld}^G_{[G/H]})_{/M}) & \longrightarrow & (\mathbf{Mfld}^G_{[G/H]})_{/M} \times (\mathbf{Mfld}^G_{[G/H]})_{/M}
\end{array}$$

The top horizontal functor $(ev_0)^{-1}(E \hookrightarrow M) \to \mathbf{Disk}^{\partial,or}_{/[-1,1]}$ is pullback of a left fibration, since it can be written as

$$\begin{array}{ccc}
(ev_0)^{-1}(E \hookrightarrow M) & \xrightarrow{\qquad\qquad} & \mathbf{Disk}^{\partial,or}_{/[-1,1]} \\
\downarrow & \ulcorner & \downarrow{\scriptstyle f^{-1}} \\
\Big((\mathbf{Mfld}^G_{[G/H]})_{/M}\Big)_{(E\hookrightarrow M)/} & \xrightarrow{\qquad\qquad} & (\mathbf{Mfld}^G_{[G/H]})_{/M} \\
\downarrow & \ulcorner & \downarrow{\scriptstyle (\{E\hookrightarrow M\},id)} \\
Fun(\Delta^1, (\mathbf{Mfld}^G_{[G/H]})_{/M}) & \longrightarrow & (\mathbf{Mfld}^G_{[G/H]})_{/M} \times (\mathbf{Mfld}^G_{[G/H]})_{/M},
\end{array}$$

where the middle horizontal arrow is a left fibration by [Lur09a, cor. 2.1.2.2].

The left fibration $(ev_0)^{-1}(E \hookrightarrow M) \to \mathbf{Disk}^{\partial,or}_{/[-1,1]}$ classifies the functor

$$(\mathbf{Mfld}^G_{[G/H]})_{/M} \to \mathcal{S}, \quad (V \hookrightarrow [-1,1]) \mapsto \mathrm{Map}_{(\mathbf{Mfld}^G_{[G/H]})_{/M}}(E \hookrightarrow M, f^{-1}V \hookrightarrow M),$$

and by [Lur09a, 3.3.4.5] we have to show that the colimit

$$\underrightarrow{colim}_{(V\hookrightarrow[-1,1])\in\mathbf{Disk}^{\partial,or}_{[-1,1]}} \mathrm{Map}_{(\mathbf{Mfld}^G_{[G/H]})_{/M}}(E \hookrightarrow M, f^{-1}V \hookrightarrow M)$$

is weakly contractible.

Let $\mathbf{Disk}^{\partial,or}([-1,1])$ denote the ordinary category with the same objects as $\mathbf{Disk}^{\partial,or}_{/[-1,1]}$ and sets of morphisms given by forgetting the topology of the mapping spaces of $\mathbf{Disk}^{\partial,or}_{/[-1,1]}$ (see [AF15, def. 2.8]). Note that the category $\mathbf{Disk}^{\partial,or}([-1,1])$ is equivalent to the partial ordered set of open subsets $V \subsetneq [-1,1]$ for V a finite disjoint union of intervals in $[-1,1]$, possibly containing the edge points $-1,1$, after excluding the whole interval $[-1,1] \subseteq [-1,1]$.

By [AF15, prop. 2.19] the functor $\mathbf{Disk}^{\partial,or}([-1,1]) \to \mathbf{Disk}^{\partial,or}_{/[-1,1]}$ is cofinal, hence it is enough to show that the homotopy colimit

$$\underrightarrow{hocolim}_{(V\subsetneq[-1,1])\in\mathbf{Disk}^{\partial,or}([-1,1])} \mathrm{Map}_{(\mathbf{Mfld}^G_{[G/H]})_{/M}}(E \hookrightarrow M, f^{-1}V \hookrightarrow M)$$

is contractible.

Using observation 1 we see that the space $\mathrm{Map}_{(\mathbf{Mfld}^G_{[G/H]})_{/M}}(E \hookrightarrow M, f^{-1}V \hookrightarrow M)$ is the homotopy fiber of $Emb^G_{[G/H]}(E, f^{-1}V) \to Emb^G_{[G/H]}(E, M)$, hence by [Lur09a] it is enough to show that the map

$$\underrightarrow{hocolim}_{(V\subsetneq[-1,1])\in\mathbf{Disk}^{\partial,or}([-1,1])} Emb^G_{[G/H]}(E, f^{-1}V) \to Emb^G_{[G/H]}(E, M)$$

is an equivalence.

By [Lur09a, thm. 6.1.0.6] colimits in $\mathcal{S}$ are universal, therefore by corollary 3.8.6 it is enough to prove that the map

$$\underrightarrow{hocolim}_{(V\subsetneq[-1,1])\in\mathbf{Disk}^{\partial,or}([-1,1])} \underline{\mathbf{Conf}}^G_{G/H}(U; f^{-1}V) \to \underline{\mathbf{Conf}}^G_{G/H}(U; M)$$

is an equivalence. Since $\left\{\underline{\mathbf{Conf}}^G_{G/H}(U; f^{-1}V)\right\}_{(V\subsetneq[-1,1])\in\mathbf{Disk}^{\partial,or}([-1,1])}$ is a complete open cover of $\underline{\mathbf{Conf}}^G_{G/H}(U; M)$ it follows from [DI04, cor. 1.6] that the above map is an equivalence. $\square$

We will also need a simple cofinality lemma. Assume we have a coCartesian fibration $p\colon \underline{\mathcal{D}} \twoheadrightarrow \underline{\mathcal{C}}$ between S-categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ (i.e. an S-coCartesian fibration, see [Sha18, rem. 7.3]), and an S-object $x\colon S \to \underline{\mathcal{C}}$. Let $p^{-1}(x) := S \times_{\underline{\mathcal{C}}} \underline{\mathcal{D}}$ be the pullback of p along x. Since $p^{-1}(x) \twoheadrightarrow S$ is a coCartesian fibration we can considered $p^{-1}(x)$ as an S-category, which we denote by $\underline{p^{-1}(x)}$.

Lemma 5.2.8. *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be S-categories and $p\colon \underline{\mathcal{D}} \twoheadrightarrow \underline{\mathcal{C}}$ be a coCartesian fibration, and $x\colon S \to \underline{\mathcal{C}}$ an S-object of $\underline{\mathcal{C}}$. Then the S-functor $\underline{p^{-1}(x)} \to \underline{\mathcal{D}}_{/\underline{x}}$ is S-cofinal.*

Proof. By [Sha18, thm. 6.7] we have to show that for each $s \in S$ the functor $\underline{p^{-1}(x)}_{[s]} \to (\underline{\mathcal{D}}_{\underline{x}})_{[s]}$ between the fibers of $\underline{p^{-1}(x)} \to \underline{\mathcal{D}}_{/\underline{x}}$ over s is cofinal. Since $p_{[s]}\colon \underline{\mathcal{D}}_{[s]} \twoheadrightarrow \underline{\mathcal{C}}_{[s]}$ is a coCartesian fibration it follows that $(p_{[s]}^{-1}(x(s)) \to (\underline{\mathcal{D}}_{[s]})_{/x(s)}$ is cofinal. The result now follows from the equivalence $p_{[s]}^{-1} \cong (p_{[s]})^{-1}(x(s))$. ☐

With lemma 5.2.7 at hand we turn to the proof of proposition 5.2.3. The proof follows the outline of the proof of [AF15, prop. 3.23] ("pushforward").

Proof of proposition 5.2.3. By eq. (17) and lemma 5.2.7 we have

$$\begin{aligned}\int_M A &:= \underline{G/H} - \underrightarrow{colim}(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} \xrightarrow{A \underline{\times} G/H} \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \\ &= \underline{G/H} - \underrightarrow{colim}(\underline{\mathbf{X}}_f \xrightarrow{ev_0} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} \xrightarrow{A \underline{\times} G/H} \underline{\mathcal{C}} \underline{\times} \underline{G/H}).\end{aligned}$$

Using the characterization of parametrized Kan extensions as parametrized left adjoints (see [Sha18, thm. 10.4], and also [Nar16, def. 2.10 and def. 2.12]) we can express the above $\underline{G/H}$-colimit as a left $\underline{G/H}$-Kan extension of $L\colon \underline{\mathbf{X}}_f \to \underline{\mathcal{C}} \underline{\times} \underline{G/H}$ along the structure map $\underline{\mathbf{X}}_f \to \overline{\underline{G/H}}$, where L is the $\overline{\underline{G/H}}$-functor given by the composition

$$L\colon \underline{\mathbf{X}}_f \xrightarrow{ev_0} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} \xrightarrow{A \underline{\times} G/H} \underline{\mathcal{C}} \underline{\times} \underline{G/H}. \tag{25}$$

Equivalently the $\underline{G/H}$-colimit over $\underline{\mathbf{X}}_f$ is given by the left $\underline{G/H}$-adjoint to restriction along the structure map $\underline{\mathbf{X}}_f \to \underline{G/H}$,

$$\underline{G/H} - \underrightarrow{colim}\colon \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{\mathbf{X}}_f, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \leftrightarrows \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{G/H}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \simeq \underline{\mathcal{C}} \underline{\times} \underline{G/H}.$$

By construction $ev_1\colon \underline{\mathbf{X}}_f \to \underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}$ is a $\underline{G/H}$ category, therefore the structure map $\underline{\mathbf{X}}_f \to \underline{G/H}$ factors as $\underline{\mathbf{X}}_f \xrightarrow{ev_1} \underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]} \to \underline{G/H}$. We can now extend L along $\underline{\mathbf{X}}_f \to \underline{G/H}$ in two steps, again using [Sha18, thm. 10.4], as the composition of left $\underline{G/H}$-adjoints

$$(ev_1)_!\colon \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{\mathbf{X}}_f, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \longleftrightarrow \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) : (ev_1)^*,$$

$$\underline{G/H} - \underrightarrow{colim}\colon \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \longleftrightarrow \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{G/H}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \simeq \underline{\mathcal{C}} \underline{\times} \underline{G/H},$$

where $(ev_1)_!$ is the left $\underline{G/H}$-Kan extension of (25) along ev_1. In particular restricting to fibers over $(G/H \xleftarrow{=} G/H) \in \underline{G/H}$ we get composition of unparametrized left adjoints (see [Lur, prop. 7.3.2.6] and [Sha18, def. 8.1]):

$$(ev_1)_!\colon \mathrm{Fun}_{\underline{G/H}}(\underline{\mathbf{X}}_f, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \longleftrightarrow \mathrm{Fun}_{\underline{G/H}}(\underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) : (ev_1)^*,$$

$$\underline{G/H} - \underrightarrow{colim}\colon \mathrm{Fun}_{\underline{G/H}}(\underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \longleftrightarrow \mathrm{Fun}_{\underline{G/H}}(\underline{G/H}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \simeq \underline{\mathcal{C}}_{[G/H]}.$$

Applying both left adjoints to the $\underline{G/H}$-functor $L\colon \underline{\mathbf{X}}_f \to \underline{\mathcal{C}} \underline{\times} \underline{G/H}$ of (25) produces the G-factorization homology $\int_M A = \underline{G/H} - \underrightarrow{colim}(L\colon \underline{\mathbf{X}}_f \to \underline{\mathcal{C}} \underline{\times} \underline{G/H})$. Let $L' := (ev_1)_!(L) \in \mathrm{Fun}_{\underline{G/H}}(\underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}} \underline{\times} \underline{G/H})$ be the left $\underline{G/H}$-Kan extension of L along ev_1. Then the $\underline{G/H}$-colimit of L' is

$$\underline{G/H} - \underrightarrow{colim}(L') = \underline{G/H} - \underrightarrow{colim}\,((ev_1)_!(L)) \simeq \underline{G/H} - \underrightarrow{colim}(L) = \int_M A.$$

Next, note that the $\underline{G/H}$-colimit over the constant diagram $\underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}$ is equivalent to an unparametrized colimit over $\mathbf{Disk}^{\partial,or}_{/[-1,1]}$. To see this, use the equivalence

$$\mathrm{Fun}_{\underline{G/H}}(\underline{G/H} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}} \underline{\times} \underline{G/H}) \xrightarrow{\sim} \mathrm{Fun}(\mathbf{Disk}^{\partial,or}_{/[-1,1]}, C_{[G/H]}),$$
$$L' \mapsto L'|_{\left\{G/H \xleftarrow{=} G/H\right\} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}}$$

and the global definition of a colimit as a left adjoint

$$\underrightarrow{colim}\colon \mathrm{Fun}(\mathbf{Disk}^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}}_{[G/H]}) \leftrightarrows \underline{\mathcal{C}}_{[G/H]}.$$

Therefore, we have

$$\int_M A \simeq \underline{G/H} - \underrightarrow{colim}(L') \simeq \underrightarrow{colim}\left(L'|_{\left\{G/H \xleftarrow{=} G/H\right\} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]}}\right),$$

which we write as

$$\int_M A \simeq \underrightarrow{colim}_{(V \hookrightarrow [-1,1]) \in \mathbf{Disk}^{\partial,or}_{/[-1,1]}} L'(G/H \xrightarrow{=} G/H, V \hookrightarrow [-1,1]).$$

Out next goal is to calculate $L'(y)$ for $y = (G/H \xrightarrow{=} G/H, V \hookrightarrow [-1,1])$ where $(V \hookrightarrow [-1,1]) \in \mathbf{Disk}^{\partial,or}_{/[-1,1]}$ is an oriented embedding. We claim that $L'(y) \simeq \int_{f^{-1}V} A \in \underline{\mathcal{C}}_{[G/H]}$ is the G-factorization homology of $f^{-1}(V) \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$. After asserting our claim we use the cofinal map $\Delta^{op} \to \mathbf{Disk}^{\partial,or}_{/[-1,1]}$ of [AF15, lem. 3.11] to deduce $\int_M A$ is equivalent to the colimit of the simplicial diagram $\Delta^{op} \to \left\{G/H \xleftarrow{=} G/H\right\} \times \mathbf{Disk}^{\partial,or}_{/[-1,1]} \xrightarrow{L'} \underline{\mathcal{C}}_{[G/H]}$. Since the functor $L'(V \hookrightarrow [-1,1]) \simeq \int_{f^{-1}V} A$ takes disjoint unions over $[-1,1]$ to tensor product in $\underline{\mathcal{C}}_{[G/H]}$ (see proposition 4.2.2), and using the equivalence of oriented open and half open intervals over $[-1,1]$ (as objects of $\mathbf{Disk}^{\partial,or}_{/[-1,1]}$) we see that $\int_M A \in \underline{\mathcal{C}}_{[G/H]}$ is equivalent to the realization of the two sided bar construction

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \int_{M_-} A \otimes \int_{M_0 \times (-1,1)} A \otimes \int_{M_-} A \rightrightarrows \int_{M_-} A \otimes \int_{M_-} A,$$
$$[n] \mapsto \left(\int_{M_-} A\right) \otimes \left(\int_{M_0 \times (-1,1)} A\right)^{\otimes n} \otimes \left(\int_{M_-} A\right).$$

By [Lur, 4.4.2.8-11] we see that G-factorization homology of M is equivalent to the relative tensor product $\int_M A \simeq (\int_{M_-} A) \otimes_{(\int_{M_0 \times (-1,1)} A)} (\int_{M_-} A)$. Therefore it is enough to prove our claim that $L'(y) \simeq \int_{f^{-1}V} A \in \underline{\mathcal{C}}_{[G/H]}$.

Since L' is the left $\underline{G/H}$-Kan extension of $L\colon \underline{\mathbf{X}}_f \to \underline{\mathcal{C}}\underline{\times}\underline{G/H}$ along ev_1, it is given by the following $\underline{G/H}$-colimit

$$\begin{aligned} L'(y) &= \underline{G/H} - \underrightarrow{colim}\left((\underline{\mathbf{X}}_f)_{/\underline{y}} \to \underline{\mathbf{X}}_f \xrightarrow{L} \underline{\mathcal{C}}\underline{\times}\underline{G/H}\right) \\ &= \underline{G/H} - \underrightarrow{colim}\left((\underline{\mathbf{X}}_f)_{/\underline{y}} \to \underline{\mathbf{X}}_f \xrightarrow{ev_0} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr}\underline{\times}\underline{G/H} \xrightarrow{A\underline{\times}\underline{G/H}} \underline{\mathcal{C}}\underline{\times}\underline{G/H}\right) \end{aligned}$$

Next we replace the $\underline{G/H}$-category $\underline{\mathbf{X}}_f$ indexing the above colimit by a $\underline{G/H}$-category which is more closely related to G-disks in $f^{-1}V$. Note that $ev_1\colon \underline{\mathbf{X}}_f \to \underline{G/H}\times\mathbf{Disk}^{\partial,or}$ is a coCartesian fibration, and let $\underline{(ev_1)^{-1}(y)}$ denote the pullback of $\underline{\mathbf{X}}_f$ along the $\underline{G/H}$-functor

$$\underline{G/H} \to \underline{G/H}\times\mathbf{Disk}^{\partial,or}, \quad (G/K \to G/H) \mapsto (G/K \to G/H, V \hookrightarrow [-1,1])$$

corresponding to $y = (G/H \xrightarrow{=} G/H, V \hookrightarrow [-1,1]) \in \underline{G/H}\times\mathbf{Disk}^{\partial,or}$. By lemma 5.2.8 the $\underline{G/H}$-functor $\underline{(ev_1)^{-1}(y)} \to (\underline{\mathbf{X}}_f)_{/\underline{y}}$ is $\underline{G/H}$-cofinal, hence $\underline{L'(y)}$ is the $\underline{G/H}$-colimit of the $\underline{G/H}$-diagram

$$\underline{(ev_1)^{-1}(y)} \to (\underline{\mathbf{X}}_f)_{/\underline{y}} \to \underline{\mathbf{X}}_f \xrightarrow{ev_0} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr}\underline{\times}\underline{G/H} \xrightarrow{A\underline{\times}\underline{G/H}} \underline{\mathcal{C}}\underline{\times}\underline{G/H}.$$

Since $ev_1\colon \underline{\mathbf{X}}_f \to \underline{G/H}\times\mathbf{Disk}^{\partial,or}$ factors through $(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}})\times_{\underline{G/H}}\left(\underline{G/H}\times\mathbf{Disk}^{\partial,or}_{/[-1,1]}\right)$ we can express $\underline{(ev_1)^{-1}(y)}$ as the iterative pullback

$$\begin{array}{ccc} \underline{(ev_1)^{-1}(y)} & \longrightarrow & \underline{\mathbf{X}}_f \\ \downarrow & & \downarrow{\scriptstyle (ev_0,ev_1)} \\ \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}\times_{\underline{G/H}}\underline{G/H} & \xrightarrow{id\underline{\times}y} & (\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}})\times_{\underline{G/H}}\left(\underline{G/H}\times\mathbf{Disk}^{\partial,or}_{/[-1,1]}\right) \\ \downarrow & & \downarrow \\ \underline{G/H}^{-}y & \longrightarrow & \underline{G/H}\times\mathbf{Disk}^{\partial,or}_{/[-1,1]}. \end{array}$$

On the other hand we can express $\underline{\mathbf{X}}_f$ is the pullback

$$\begin{array}{ccc} \underline{\mathbf{X}}_f & \longrightarrow & \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{G/H}\times\Delta^1, \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}}) \\ \downarrow{\scriptstyle (ev_0,ev_1)} & & \downarrow \\ (\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}})\times_{\underline{G/H}}\left(\underline{G/H}\times\mathbf{Disk}^{\partial,or}_{/[-1,1]}\right) & \xrightarrow{\iota\times f^{-1}} & \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}}\times_{\underline{G/H}}\underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}}. \end{array}$$

Notice that the composition

$$\begin{array}{c} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}\times_{\underline{G/H}}\underline{G/H} \\ \downarrow{\scriptstyle id\underline{\times}y} \\ (\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}})\times_{\underline{G/H}}\left(\underline{G/H}\times\mathbf{Disk}^{\partial,or}_{/[-1,1]}\right) \\ \downarrow{\scriptstyle \iota\times f^{-1}} \\ \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}}\times_{\underline{G/H}}\underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}} \end{array}$$

is equivalent to

$$\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \times_{G/H} \underline{G/H} \xrightarrow{(\iota, f^{-1}(y))} \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}} \underline{\times} \underline{\mathbf{Mfld}}^{G,f-fr}_{/\underline{M}},$$

and therefore that

$$\underline{(ev_1)^{-1}(y)} \cong \left(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}\right)_{/\underline{(f^{-1}V \hookrightarrow M)}} \simeq \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{f^{-1}V}}$$

(compare [AF15, lem. 2.1]). Finally, since the diagram

$$\begin{array}{ccccccc}
\underline{(ev_1)^{-1}(y)} & \longrightarrow & (\underline{\mathbf{X}}_f)_{/\underline{y}} & \longrightarrow & \underline{\mathbf{X}}_f & & \\
\downarrow \simeq & & & & \downarrow & & \\
\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{f^{-1}V}} & \longrightarrow & \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} & \longrightarrow & \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} & \xrightarrow{A \underline{\times} G/H} & \underline{\mathcal{C}} \underline{\times} \underline{G/H}
\end{array}$$

commutes, we get

$$L'(y) \simeq \underline{G/H} - \underrightarrow{colim} \left(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{f^{-1}V}} \to \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} \xrightarrow{A \underline{\times} G/H} \underline{\mathcal{C}} \underline{\times} \underline{G/H} \right).$$

Therefore, by the definition of left $\underline{G/H}$-Kan extension we see that indeed $L'(y) \simeq \int_{f^{-1}V} A$. □

5.3 G-sequential unions

Definition 5.3.1. *Let M be a G-manifold. A G-*sequential union *of M is a sequence of open G-submanifolds $M_1 \subset M_2 \subset \cdots \subset M$ with $M = \cup_{i=1}^{\infty} M_i$. A G-sequential union of an f-framed $\mathcal{O}_G$-manifold $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ is a G-sequential union of its underlying G-manifold.*

If $F\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$ is a G-symmetric monoidal functor and $M = \cup_{i=1}^{\infty} M_i$ is a G-sequential union of $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$, then we have a comparison morphism $\underrightarrow{colim}\, F(M_i) \to F(M)$ in $\underline{\mathcal{C}}_{[G/H]}$.

Definition 5.3.2. *We say that G-symmetric monoidal functor $F\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$* respects G-sequential unions *if for every G-sequential union $M = \cup_{i=1}^{\infty} M_i$ the comparison morphism*

$$\underrightarrow{colim}\, F(M_i) \to F(M)$$

is an equivalence in $\underline{\mathcal{C}}_{[G/H]}$.

Proposition 5.3.3. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be a G-symmetric monoidal G-category and A be an f-framed G-disk algebra with values in $\underline{\mathcal{C}}$. Then G-factorization homology $\int_{\_} A\colon \underline{\mathbf{Mfld}}^{G,f-fr} \to \underline{\mathcal{C}}$ of definition 4.2.3 respects G-sequential unions.*

The proof of proposition 5.3.3 relies on the following lemma.

Lemma 5.3.4. *Let $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ be an f-framed $\mathcal{O}_G$-manifold over G/H, and $M = \cup_{i=1}^{\infty} M_i$ a G-sequential union of M. Then the $\underline{G/H}$-functor $\underrightarrow{colim}\, \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M_i}} \xrightarrow{\sim} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}$ is an equivalence of $\underline{G/H}$-categories.*

Proof. By proposition 4.2.6 it is enough to prove that the $\underline{G/H}$-functor $\underrightarrow{colim}\,\underline{\mathbf{Disk}}^G_{/\underline{M_i}} \to \underline{\mathbf{Disk}}^G_{/\underline{M}}$ is a fiberwise equivalence. Without loss of generality we show that the functor between the fiber over $(G/H \overset{=}{\longleftarrow} G/H) \in \underline{G/H}$ is an equivalence. Since colimits of G-categories are computed fiberwise, we have to show that $\underrightarrow{colim}_i(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i} \to (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$ is an equivalence of ∞-categories.

In order to show that this functor is fully faithful we first show that $Emb^G_{G/H}(E, M)$ is equivalent to the homotopy colimit $\underrightarrow{hocolim}_i\, Emb^G_{G/H}(E, M_i)$. Let $(E \to U \to G/H) \in \underline{\mathbf{Disk}}^G$ be a finite G-disjoint union of G-disks, i.e. $E \to U$ a G-vector bundle, $U = \pi_0 E$. By corollary 3.8.6 the square

$$\begin{array}{ccc} Emb^G_{G/H}(E, M_i) & \longrightarrow & Emb^G_{G/H}(E, M) \\ \downarrow & & \downarrow \\ \underline{\mathbf{Conf}}^G_{G/H}(U; M_i) & \longrightarrow & \underline{\mathbf{Conf}}^G_{G/H}(U; M) \end{array}$$

is a homotopy pullback square for each $i \in \mathbb{N}$. Since filtered homotopy colimits preserves homotopy pullbacks, the square

$$\begin{array}{ccc} \underrightarrow{hocolim}_i\, Emb^G_{G/H}(E, M_i) & \longrightarrow & Emb^G_{G/H}(E, M) \\ \downarrow & & \downarrow \\ \underrightarrow{hocolim}_i\, \underline{\mathbf{Conf}}^G_{G/H}(U; M_i) & \longrightarrow & \underline{\mathbf{Conf}}^G_{G/H}(U; M) \end{array}$$

is also a homotopy pullback square. However, $\left\{\underline{\mathbf{Conf}}^G_{G/H}(U; M_i)\right\}_i \in \mathbb{N}$ is a complete open cover of $\underline{\mathbf{Conf}}^G_{G/H}(U; M)$, so by [DI04, cor. 1.6] the bottom map is a weak equivalence. Therefore the map $\underrightarrow{hocolim}_i\, Emb^G_{G/H}(E, M_i) \xrightarrow{\sim} Emb^G_{G/H}(E, M)$ is a weak equivalence.

We now show that $\underrightarrow{colim}_i(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i} \to (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$ is fully faithful.

Let $(E' \to U' \to G/H)$, $(E'' \to U'' \to G/H) \in \underline{\mathbf{Disk}}^G_{[G/H]}$ and $f' \colon E' \hookrightarrow M_{i'}$ $f'' \colon E'' \hookrightarrow M_{i''})$ be two G-embeddings over G/H, representing two objects in $\underrightarrow{colim}_i(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i}$. For i greater then i' and i'' they represent objects of the same slice category

$$(f'_i \colon E' \xrightarrow{f'} M_{i'} \subseteq M_i),\ (f''_i \colon E'' \xrightarrow{f''} M_{i''} \subseteq M_i) \in (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i},$$

with mapping space $\mathrm{Map}_{(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i}}(f'_i \colon E' \hookrightarrow M_i, f''_i \colon E'' \hookrightarrow M_i)$ given by the homotopy fiber of $(f''_i)_* \colon Emb^G_{G/H}(E', E'') \to Emb^G_{G/H}(E', M_i)$ over $f'_i \in Emb^G_{G/H}(E', M_i)$. Homotopy fibers are preserved by filtered homotopy colimits, so the homotopy fiber of the map

$$Emb^G_{G/H}(E', E'') \to \underrightarrow{hocolim}_i\, Emb^G_{G/H}(E', M_i) \simeq Emb^G_{G/H}(E', M)$$

induced by post composition with $f'' \colon E'' \hookrightarrow M_i \subset M$ over $f' \colon E' \hookrightarrow M_i \subseteq M$ is equivalent to $\underrightarrow{hocolim}_i\, \mathrm{Map}_{(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i}}(f'_i \colon E' \hookrightarrow M_i, f''_i \colon E'' \hookrightarrow M_i)$. On the other hand, this homotopy fiber is equivalent to the mapping space of the slice category $(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$, hence

$$\underrightarrow{colim}_i\, \mathrm{Map}_{(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i}}(f'_i \colon E' \hookrightarrow M_i, f''_i \colon E'' \hookrightarrow M_i)$$

is homotopy equivalent to

$$\mathrm{Map}_{(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}}(f'\colon E' \hookrightarrow M, f''\colon E'' \hookrightarrow M),$$

so the functor $\underrightarrow{colim}_i (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i} \to (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$ is fully faithful.

It remains to show that $\underrightarrow{colim}_i (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i} \to (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$ is essentially surjective. Let $(E \to U \to G/H) \in \underline{\mathbf{Disk}}^G_{[G/H]}$, $(f\colon E \hookrightarrow M) \in (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$ for $E \to U$ a G-vector bundle. Choose $t > 0$ small enough so that the restriction of f to the open ball of radius t bundle, $B_t(E) \hookrightarrow E \xrightarrow{f} M$, factors through some $M_i \subseteq M$. By radial dilation we see that the inclusion $(B_t(E) \to G/H) \to (E \to G/H)$ is an equivalence in $\underline{\mathbf{Disk}}^G_{[G/H]}$. Postcomposition with $f\colon E \hookrightarrow M$ induces an equivalence $(f\colon E \hookrightarrow M) \simeq (B_t(E) \hookrightarrow E \xrightarrow{f} M)$ of objects in the slice category $(\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$. On the other hand, since $(B_t(E) \hookrightarrow E \xrightarrow{f} M)$ factors through M_i this object is clearly in the image of the functor $\underrightarrow{colim}_i (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M_i} \to (\underline{\mathbf{Disk}}^G_{[G/H]})_{/M}$, showing the functor is indeed essentially surjective. □

We now show that G-factorization homotopy respects sequential colimits.

Proof of proposition 5.3.3. Let $M \in \underline{\mathbf{Mfld}}^{G,f-fr}_{[G/H]}$ be an f-framed $\mathcal{O}_G$-manifold and $M = \cup_{i=1}^\infty M_i$ a G-sequential union of M. The assembly map $\underrightarrow{colim}_i \int_{M_i} A \to \int_M A$ factors as a sequence of equivalences

$$\begin{aligned}
\underrightarrow{colim}_i \int_{M_i} A &= \underrightarrow{colim}_i \left(\underline{G/H} - \underrightarrow{colim} \left(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M_i}} \to \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} \xrightarrow{A \underline{\times} id} \underline{\mathcal{C}} \underline{\times} \underline{G/H} \right) \right) \\
&\simeq \underline{G/H} - \underrightarrow{colim} \left(\underrightarrow{colim}_i \left(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M_i}} \to \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} \xrightarrow{A \underline{\times} id} \underline{\mathcal{C}} \underline{\times} \underline{G/H} \right) \right) \\
&\xrightarrow{\sim} \underline{G/H} - \underrightarrow{colim} \left(\underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}} \to \underline{\mathbf{Disk}}^{G,f-fr} \underline{\times} \underline{G/H} \xrightarrow{A \underline{\times} id} \underline{\mathcal{C}} \underline{\times} \underline{G/H} \right) = \int_M A,
\end{aligned}$$

where the second equivalence is induced by the equivalence $\underrightarrow{colim}\, \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M_i}} \xrightarrow{\sim} \underline{\mathbf{Disk}}^{G,f-fr}_{/\underline{M}}$ of lemma 5.3.4. □

6 Axiomatic characterization of G-factorization homology theories

In this subsection we give an axiomatic characterization of G-factorization homology theories with values in a presentable G-symmetric monoidal G-category (definition 4.2.1), as G-symmetric monoidal functors that satisfy G-$\otimes$-excision (definition 5.2.2) and respects G-sequential unions (definition 5.3.2).

Definition 6.0.1. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be a G-symmetric monoidal category and $B \to BO_n(G)$ a G-map, as in definition 3.3.1. An* equivariant homology theory of G-manifolds *is a G-symmetric monoidal functor $F\colon \underline{\mathbf{Mfld}}^{G,f-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ which satisfies G-$\otimes$-excision and respects G-sequential unions. We denote the full subcategory of equivariant homology theories by $\mathcal{H}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}}) \subset$ $\mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}})$.*

The main result in this subsection is the following characterization of G-factorization homology.

Theorem 6.0.2. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ be a presentable G-symmetric monoidal category. Then the full subcategory $\mathcal{H}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}}) \subset \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}})$ is spanned by objects for which the counit map of the adjunction*

$$(\iota^{\otimes})_! \colon \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}}) \rightleftarrows \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}}) : (\iota^{\otimes})^*$$

of (19) *is an equivalence. In particular, the adjunction restricts to an equivalence*

$$(\iota^{\otimes})_! \colon \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}}) \xrightarrow{\sim} \mathcal{H}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}}), \quad A \mapsto \int_{-} A$$

sending an f-framed G-disk algebra A to G-factorization homology with coefficients in A.

Proof. Let A be a G-disk algebra. By proposition 5.2.3 and proposition 5.3.3 the functor

$$(\iota^{\otimes})_! \colon \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^{G,f-fr}, \underline{\mathcal{C}}) \to \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}})$$

factors though the full G-subcategory $\mathcal{H}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}}) \subset \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}})$.

On the other hand, let $F \in \mathcal{H}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}})$ be an equivariant homology theory of G-manifolds. Denote by $A \colon \underline{\mathbf{Disk}}^{G,f-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ the restriction of F along $\iota^{\otimes}$. We have to show that the counit $\int_{-} A \to F$ is an equivalence. Since F, $\int_{-} A$ are G-symmetric monoidal functors it is enough to show that for every f-framed $\mathcal{O}_G$-manifold $M \in \underline{\mathbf{Mfld}}^{G,f-fr}$ the counit map $\int_M A \to F(M)$ is an equivalence in $\underline{\mathcal{C}}$. We proceed by induction.

For $k = 0, 1, \dots, n$ let $\mathcal{F}_{\leq k} \subseteq \underline{\mathbf{Mfld}}^{G,f-fr}$ be the full G-subcategory of f-framed $\mathcal{O}_G$-manifolds whose underlying $\mathcal{O}_G$-manifold is of the form $(M \times_{G/H} D \to G/H)$ where $G/H \in \mathcal{O}_G$ is a G-orbit, $M \to G/H$ is a *k-dimensional* $\mathcal{O}_G$-manifold and $(D \to G/H)$ is a finite G-disjoint union of $(n-k)$-dimensional G-disks, i.e. equivalent to $D \to U \to G/H$ where U is a finite G-set and $D \to U$ is a G-vector bundle of rank $n-k$ (and therefore $U = \pi_0(D)$).

We now prove that the counit map is an equivalence on objects of $\mathcal{F}_{\leq k}$ by induction on k.

For $k = 0$ the underlying $\mathcal{O}_G$-manifold of $M \in \mathcal{F}_{\leq 0}$ is simply a finite G-disjoint union of G-disks, $(D \to G/H) \in \underline{\mathbf{Disk}}^G$, therefore $M \in \underline{\mathbf{Disk}}^{G,f-fr}$ and $\int_M A \simeq A(M) = F(M)$, since $\iota_!$ is fully faithful A is the restriction of F along ι.

For $k \geq 1$, let $N \in \mathcal{F}_{\leq k}$ with underlying $\mathcal{O}_G$-manifold $(M \times_{G/H} D \to G/H)$. We show that the counit map $\int_N A \to F(N)$ is an equivalence using equivariant Morse theory. In what follows we only consider G-submanifolds of $M \times_{G/H} D \to G/H$, which by proposition 4.2.6 have an essentially unique f-framing induced from the inclusion into N. Therefore we omit the identification of such G-submanifolds with their f-framed lift to $\underline{\mathbf{Mfld}}^{G,f-fr}$.

Choose a G-equivariant Morse function $f \colon M \to \mathbb{R}$ with $f^{-1}(-\infty, r]$ a compact G-submanifold for every $r \in \mathbb{R}$ (see [Was69, thm. 4.10]). Choose an increasing sequence of regular values $r_0 < r_1 < r_2 < \cdots$ such that $f^{-1}(-\infty, r_0) = \emptyset$, the interval (r_i, r_{i+1}) contains a single critical value and $r_i \to \infty$.

Let $M_i := f^{-1}((-\infty, r_i))$, then $M = \bigcup_{i=0}^{\infty} M_i$ and therefore $M \times_{G/H} D \cong \left(\bigcup_{i=0}^{\infty} M_i\right) \times_{G/H} D \cong \bigcup_{i=0}^{\infty} \left(M_i \times_{G/H} D\right)$ is a G-sequential union of $(M \times_{G/H} D \to G/H)$ (definition 5.3.1). Since both $F \in \mathcal{H}(\underline{\mathbf{Mfld}}^{G,f-fr}, \underline{\mathcal{C}})$ and $\int_{-} A$ respect G-sequential unions (definition 5.3.2 and proposition 5.3.3) we have $F(M \times_{G/H} D) \simeq \underrightarrow{colim}\, F(M_i \times_{G/H} D)$, $\int_{M \times_{G/H} D} A \simeq \underrightarrow{colim} \int_{M_i \times_{G/H} D} A$. Therefore it is enough to prove that the counit map $\int_{M_i \times_{G/H} D} A \to F(M_i \times_{G/H} D)$ is an equivalence, which we prove by induction on i.

Let $\overline{M_i} := f^{-1}(-\infty, r_i]$. Since $\overline{M_i}$ is compact $\overline{M_{i+1}} \setminus M_i$ has only a finite number of critical orbits, $x_j : W_j \hookrightarrow M$, $j = 1, \ldots, s$. Note that the tangent bundle $T_{x_j} M \to W_j$ over the critical orbit x_j is a G-vector bundle which decomposes as a direct sum of two G-bundles $T_{x_j} \cong P_j \oplus E_j$ on which the Hessian is negative definite (called the index E_j) and positive definite (called the co-index P_j).

By [Was69, thm. 4.6] $\overline{M_{i+1}}$ is equivariantly diffeomorphic to $\overline{M_i}$ with s handle-bundles $N_1, \ldots, N_s$ disjointly attached, where the handle-bundle $N_j := \overline{\mathbb{D}}(P_j) \times_{W_j} \overline{\mathbb{D}}(E_j)$ is the fiberwise product of the closed unit disk bundles $\overline{\mathbb{D}}(P_j) \to W_j$, $\overline{\mathbb{D}}(E_j) \to W_j$, attached to M_i along $\overline{\mathbb{D}}(P_j) \times_{W_j} \mathbb{S}(E_j)$ where $\mathbb{S}(E_j) \to W_j$ is the unit sphere bundle of the negative definite G-subbundle (the index).

Since the handle-bundles are attached disjointly and $F, \int_{\_} A$ are G-symmetric monoidal we can reduce to the case of a single handle-bundle by attaching one handle-bundle at a time. Therefore we assume that there is a single critical orbit $x : W \hookrightarrow M$ in $\overline{M_{i+1}} \setminus M_i$ with $T_x M \cong P \oplus E$, and $\overline{M_{i+1}} \cong \overline{M_i} \bigcup_{\overline{\mathbb{D}}(P) \times_W \mathbb{S}(E)} \left(\overline{\mathbb{D}}(P) \times_W \overline{\mathbb{D}}(E)\right)$.

Let $\mathbb{A}(E) \to W$ denote the unit annulus bundle of E, i.e. the open unit disk bundle minus the zero section. Note that $\mathbb{A}(E)$ is a G-tubular neighbourhood of $\mathbb{S}(E)$, therefore $\overline{M_{i+1}} \cong \overline{M_i} \bigcup_{\overline{\mathbb{D}}(P) \times_W \mathbb{A}(E)} \left(\overline{\mathbb{D}}(P) \times_W \mathbb{A}(E)\right)$ a union of k-dimensional G-manifolds with boundary along a k-dimensional manifold with boundary.

Discarding boundary points we see that the M_{i+1} is equivariantly diffeomorphic to the union of M_i with the G-manifold $\mathbb{D}(P) \times_W \mathbb{D}(E)$ along the G-manifold $\mathbb{D}(P) \times_W \mathbb{A}(E)$. After taking fibered product with the fibration map $D \to G/H$ we have

$$M_{i+1} \times_{G/H} D \cong (M_i \times_{G/H} D) \bigcup_{\left((\mathbb{D}(P) \times_W \mathbb{A}(E)) \times_{G/H} D\right)} \left((\mathbb{D}(P) \times_W \mathbb{D}(E)) \times_{G/H} D\right). \tag{26}$$

This decomposition has the following properties:

1. The decomposition of eq. (26) is in fact a G-collar decomposition. Intuitively, the codimension one G-submanifold $(\mathbb{D}(P) \times_W \mathbb{S}(E)) \times_{G/H} D$ splits the handle bundle of eq. (26) to two G-submanifolds, M_i and the handle bundle. Explicitly, construct a G-collar decomposition by defining a G-invariant smooth G-invariant function $M_{i+1} \to [-1, 1]$ for which the restriction to the open interval $(-1, 1)$ is a manifold bundle as follows. Compose the G-diffeomorphism of eq. (26) with the restriction of the Morse function $f : M \to \mathbb{R}$ to the handle-bundle of eq. (26), followed by a smooth function $\Psi : \mathbb{R} \to [-1, 1]$ such that

   (a) it sends the closed interval $(-\infty, a + \epsilon]$ to -1 for some small $\epsilon > 0$.

   (b) it sends $[c - \epsilon, \infty)$ to 1 for c the unique critical value of f in the interval $[a, b]$.

   (c) it has a positive derivative in the open interval $(a + \epsilon, c - \epsilon)$.

   Note that the fibers of $M_{i+1} \to [-1, 1]$ over $(-1, 1)$ are $\left((\mathbb{D}(P) \times_W \mathbb{S}(E, r)) \times_{G/H} D\right)$ where $\mathbb{S}(E, r)$ is the radius-r-sphere bundle, for various radii r.

2. The induced handle-bundle $\left((\mathbb{D}(P) \times_W \mathbb{D}(E)) \times_{G/H} D \to G/H\right) \in \underline{\mathbf{Disk}}^G$ is a finite G-disjoint union of G-disks, since the open unit disk bundle of a G-vector bundle is equivalent to the entire vector bundle.

We now distinguish between two cases, according to the rank of the bundle $E \to W$.

1. If the critical orbit x has zero index, i.e. the Hessian is positive definite on $T_x M$, then $E \to W$ is a rank zero G-vector bundle, and its unit annulus $\mathbb{A}(E) = \emptyset$ is empty. In this

case the G-collar decomposition of eq. (26) is a disjoint union

$$M_{i+1} \times_{G/H} D \cong (M_i \times_{G/H} D) \sqcup \big((\mathbb{D}(P) \times_W \mathbb{D}(E)) \times_{G/H} D\big).$$

Since F, $\int_{\_} A$ are G-symmetric monoidal functors we have

$$\int_{M_{i+1}} A \simeq \left(\int_{M_i} A\right) \otimes \left(\int_{\left((\mathbb{D}(P)\times_W \mathbb{D}(E))\times_{G/H} D\right)} A\right),$$
$$F(M_{i+1}) \simeq F(M_i) \otimes F\left((\mathbb{D}(P) \times_W \mathbb{D}(E)) \times_{G/H} D\right)$$

where $\left((\mathbb{D}(P) \times_W \mathbb{D}(E)) \times_{G/H} D\right) \simeq \left((P \times_W E) \times_{G/H} D\right) \in \underline{\mathbf{Disk}}^G$ is a finite G-disjoint union of G-disks. Therefore $\int_{M_{i+1}} A \xrightarrow{\sim} F(M_{i+1})$ by induction on i.

2. Otherwise the critical orbit x has positive index, i.e. $\mathrm{rank}(E) > 0$. In this case, $\mathbb{A}(E) \cong \mathbb{S}(E) \times (-1, 1)$ where G acts trivially on the open interval $(-1, 1)$, since the Morse function f is G-invariant. It follows that

$$(\mathbb{D}(P) \times_W \mathbb{A}(E)) \times_{G/H} D \cong \mathbb{A}(E) \times_W (P \times_{G/H} D) \cong \mathbb{S}(E) \times_W ((-1, 1) \times P \times_{G/H} D),$$

hence $(\mathbb{S}(E) \to W \to G/H)$ is a G-manifold of dimension

$$\dim \mathbb{S}(E) = \mathrm{rank}(E) - 1 \le \dim M - 1 = k - 1,$$

so we have $(\mathbb{D}(P) \times_W \mathbb{A}(E)) \times_{G/H} D \in \mathcal{F}_{k-1}$. It follows by induction on k that the counit map $\int_{(\mathbb{D}(P)\times_W \mathbb{A}(E))\times_{G/H} D} A \xrightarrow{\sim} F((\mathbb{D}(P) \times_W \mathbb{A}(E)) \times_{G/H} D)$ is an equivalence.

The G-functor $\int_{\_} A$ satisfies G-$\otimes$-excision by proposition 5.2.3 and F satisfies G-$\otimes$-excision by assumption, therefore applying $F, \int_{\_} A$ to the G-collar decomposition of eq. (26) we get

$$F(M_i \times_{G/H} D) \otimes_{F\left((\mathbb{D}(P)\times_W \mathbb{A}(E))\times_{G/H} D\right)} F\left((\mathbb{D}(P) \times_W \mathbb{D}(E)) \times_{G/H} D\right) \xrightarrow{\sim} F(M_{i+1}),$$
$$\left(\int_{(M_i \times_{G/H} D)} A\right) \otimes_{\left(\int_{(\mathbb{D}(P)\times_W \mathbb{A}(E))\times_{G/H} D} A\right)} \left(\int_{(\mathbb{D}(P)\times_W \mathbb{D}(E))\times_{G/H} D}\right) A \xrightarrow{\sim} \int_{M_{i+1}} A$$

and by induction on i the map $\int_{M_{i+1}} A \xrightarrow{\sim} F(M_{i+1})$ is an equivalence. □

7 Equivariant versions of Hochschild homology

As an application of the G-$\otimes$-excision property (proposition 5.2.3) we describe two variants of topological Hochschild homology using G-factorization homology.

7.1 Real topological Hochschild homology as G-factorization homology

Let C_2 denote the cyclic group of order two and let σ be its one dimensional sign representation.

The structure of an $\mathbb{E}_\sigma$-algebra in $\underline{\mathbf{Sp}}^{C_2}$. Let us first describe the algebraic structure of an $\mathbb{E}_\sigma$-algebra A in $\underline{\mathbf{Sp}}^{C_2}$. We will use this description in the proof of proposition 7.1.1.

- By corollary 3.9.9 we have an equivalence

$$Alg_{\mathbb{E}_\sigma}(\underline{\mathbf{Sp}}^{C_2}) \simeq \mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Disk}}^{C_2,\sigma-fr}, \underline{\mathbf{Sp}}^{C_2}),$$

so A corresponds to a C_2-symmetric monoidal functor $A\colon \underline{\mathbf{Disk}}^{C_2,\sigma-fr} \to \underline{\mathbf{Sp}}^{C_2}$. In particular the G-symmetric monoidal functor A restricts to symmetric monoidal functors

$$A_{[C_2/C_2]}\colon \underline{\mathbf{Disk}}^{C_2,\sigma-fr}_{[C_2/C_2]} \to \mathbf{Sp}_{C_2}, \quad A_{[C_2/C_2]}\colon \underline{\mathbf{Disk}}^{C_2,\sigma-fr}_{[C_2/e]} \to \mathbf{Sp}. \tag{27}$$

- By abuse of notation, we use A to denote the "underlying" genuine C_2-spectrum,

$$A_{[C_2/C_2]}(\mathbb{R}^\sigma) \in \mathbf{Sp}^{C_2},$$

where $\mathbb{R}^\sigma \in \underline{\mathbf{Disk}}^{C_2,\sigma-fr}$ is the one dimensional sign representation of C_2, considered as a σ-framed C_2-manifold.

- Unwinding the definitions we see that $\underline{\mathbf{Disk}}^{C_2,\sigma-fr}_{[C_2/e]}$ is equivalent to the ∞-category $\mathbf{Disk}^{fr}_1$ of [AF15, rem 2.10]. Since $A\colon \underline{\mathbf{Disk}}^{C_2,\sigma} \to \underline{\mathbf{Sp}}^{C_2}$ is a G-functor it is compatible with the forgetful functors

$$Res^{C_2}_e\colon \underline{\mathbf{Disk}}^{C_2,\sigma}_{[C_2/C_2]} \to \underline{\mathbf{Disk}}^{C_2,\sigma}_{[C_2/e]} \simeq \mathbf{Disk}^{fr}_1, \quad Res^{C_2}_e\colon \mathbf{Sp}_{C_2} \to \mathbf{Sp},$$

therefore $A_{[C_2/e]}(\mathbb{R}^1) = A_{[C_2/e]}(Res^{C_2}_e\mathbb{R}^\sigma) \simeq Res^{C_2}_e A_{[C_2/C_2]}(\mathbb{R}^\sigma) = Res^{C_2}_e A$.

- Observe that $Res^{C_2}_e A$ is endowed with a structure of an $\mathbb{E}_1$-sing spectrum. To see this, recall that $\mathbb{R}^1 \in \mathbf{Disk}^{fr}_1$ is an $\mathbb{E}_1$-algebra in $\mathbf{Disk}^{fr}_1$, which induces an equivalence between the symmetric monoidal envelope of $\mathbb{E}_1$ and $\mathbf{Disk}^{fr}_1$ (see [AFT17a, prop. 2.12]).

- Let

$$\sqcup_{C_2}\mathbb{R}^1 \in \underline{\mathbf{Disk}}^{C_2,\sigma-fr}_{[C_2/C_2]}, \quad \sqcup_{C_2}\mathbb{R}^1 = C_2 \times \mathbb{R}^1$$

denote the topological induction of $\mathbb{R}^1 \in \mathbf{Disk}^{fr}_1$. The compatibility of the G-symmetric monoidal functor $A\colon \underline{\mathbf{Disk}}^{C_2,\sigma-fr} \to \underline{\mathbf{Sp}}^{C_2}$ with with topological induction and the Hopkins-Hill-Ravenel norm,

$$\sqcup_{C_2}\colon \mathbf{Disk}^{fr}_1 \simeq \underline{\mathbf{Disk}}^{C_2,\sigma}_{[C_2/C_2]} \to \underline{\mathbf{Disk}}^{C_2,\sigma}_{[C_2/e]}, \quad N^{C_2}_e\colon \mathbf{Sp} \to \mathbf{Sp}_{C_2},$$

implies that $A_{[C_2/C_2]}(\sqcup_{C_2}\mathbb{R}^1) \simeq N^{C_2}_e A$.

- Note that $N^{C_2}_e A$ is an $\mathbb{E}_1$-algebra in $\mathbf{Sp}_{C_2}$, since $N^{C_2}_e\colon \mathbf{Sp} \to \mathbf{Sp}_{C_2}$ is a symmetric monoidal functor and $Res^{C_2}_e A$ is an $\mathbb{E}_1$-ring spectrum.

- The "underlying" C_2-spectrum A has the structure of a module over $N^{C_2}A$. To see this structure, note that an equivariant oriented embeddings

$$\left(\sqcup_{C_2}\mathbb{R}^1\right) \sqcup \mathbb{R}^\sigma \hookrightarrow \mathbb{R}^\sigma$$

induces a map

$$N^{C_2}_e A \otimes A \to A.$$

Proposition 7.1.1. *For A an $\mathbb{E}_\sigma$-algebra in $\underline{\mathbf{Sp}}^{C_2}$ there is an equivalence of genuine C_2-spectra*

$$\int_{S^1} A \simeq A \otimes_{N^{C_2}_e A} A.$$

where C_2 acts on S^1 by reflection.

Proof. Consider the C_2-collar gluing $S_1 = \mathbb{R}^\sigma \cup_{\sqcup_{C_2}\mathbb{R}^1} \mathbb{R}^\sigma$ into two hemispheres, where each hemisphere is reflected onto itself by the action of C_2. Note that the intersection $\sqcup_{C_2}\mathbb{R}^1$ consists of two segments interchanged by the action of C_2. Applying proposition 5.2.3 we get an equivalence of genuine $\mathcal{C}_2$-spectra

$$\int_{S^1} A \simeq \left(\int_{\mathbb{R}^\sigma} A\right) \otimes_{\left(\int_{\sqcup_{C_2}\mathbb{R}^1} A\right)} \left(\int_{\mathbb{R}^\sigma} A\right) \simeq A \otimes_{N_e^{C_2} A} A.$$

□

Remark 7.1.2. The tensor product $A \otimes_{N_e^{C_2} A} A$ appearing in proposition 7.1.1 is equivalent to the derived smash product $A \wedge^{\mathbf{L}}_{N_e^{C_2} A} A$ of left and right $N_e^{C_2}$-modules. Dotto, Moi, Patchkoria and Reeh ([DMPR17]) show that for A a flat ring spectrum with anti-involution there is a stable equivalence of genuine C_2-spectra

$$THR(A) \simeq A \wedge^{L}_{N_e^{C_2} A} A,$$

where $THR(A)$ is the Bökstedt model for real topological Hochschild homology.

By [DMPR17, def. 2.1] we can interpret a ring spectrum with anti-involution as an algebra over an operad Ass^σ in C_2-sets. Direct inspection shows Ass^σ is equivalent to G-operad $\mathcal{D}_\sigma$ of the little σ-disks [16], whose genuine operadic nerve is $\mathbb{E}_\sigma$. Regarding a flat ring spectrum with anti-involution A as an $\mathbb{E}_\sigma$-algebra in $\underline{\mathbf{Sp}}^{C_2}$, we can reinterpret proposition 7.1.1 as an equivalence

$$\int_{S^1} A \simeq THR(A)$$

of genuine C_2-spectra.

7.2 Twisted Topological Hochschild Homology of genuine C_n-ring spectra

We start with a general lemma relating trivially framed G-disk algebras to $\mathbb{E}_n$-algebras. Let G be a finite group acting trivially on $\mathbb{R}^n$, and $\underline{\mathbf{Mfld}}^{G,\mathbb{R}^n-fr}$ the G-category of trivially framed G-manifolds.

Lemma 7.2.1. *Let $\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ be a G-symmetric monoidal ∞-category. The ∞-category $\mathrm{Fun}_G^\otimes(\underline{\mathbf{Disk}}^{G,\mathbb{R}^n-fr}, \underline{\mathcal{C}})$ of trivially framed G-disk algebras in $\mathcal{C}$ is equivalent to the ∞-category $Alg_{\mathbb{E}_n}(\underline{\mathcal{C}}_{[G/G]})$ of $\mathbb{E}_n$-algebras in the fiber $\underline{\mathcal{C}}_{[G/G]}$.*

The structure of a trivially framed C_n-disk algebra Let C_n the cyclic group of order n and $\underline{\mathcal{C}} = \underline{\mathbf{Sp}}^{C_n}$, the C_n-∞-category of genuine C_n-spectra. We will use the following an explicit description of the trivially framed C_n-disk algebra corresponding to A. The C_n-functor $A\colon \underline{\mathbf{Disk}}^{C_n,\mathbb{R}^n-fr,\sqcup} \to \underline{\mathbf{Sp}}^{C_n}$ sends

$$\forall H < C_n: \quad A_{[C_n/H]}\colon \sqcup_{C_n/H} \mathbb{R}^1 \mapsto N_H^{C_n}(A) \in \mathbf{Sp}_H,$$

where $N_H^{C_n}(A)$ denotes the Hill-Hopkins-Ravenel norm applied to the restriction of the genuine C_n-spectrum $A \in \mathbf{Sp}_{C_n}$ to $\mathbf{Sp}_H$. In particular, $A\colon \underline{\mathbf{Disk}}^{C_n,\mathbb{R}^n-fr,\sqcup} \to \underline{\mathbf{Sp}}^{C_n}$ sends $\mathbb{R}^n$ with trivial C_n-action to $A \in \mathbf{Sp}_{C_n}$ and the topological induction $\sqcup_{C_n}\mathbb{R}^1 = C_n \times \mathbb{R}^1 \in$ to $N_e^{C_n}(A) \in \mathbf{Sp}_{C_n}$.

[16] This also follows from a direct analysis of the mapping spaces of $\underline{\mathbf{Rep}}^{C_2,\sigma-fr,\sqcup}$, which are homotopically discrete.

We will need some notation for our next statement. Let A be an $\mathbb{E}_n$-ring spectrum in $\mathbf{Sp}_{C_n}$. Define an $A - A^{op}$-bimodule structure on $A \in \mathbf{Sp}_{C_n}$ with "twisted" left multiplication, given by first acting on the scalar by the generator $\tau \in C_n$:

$$A \otimes A^\tau \otimes A \to A^\tau, \quad x \otimes a \otimes y \mapsto \tau x \cdot a \cdot y.$$

We denote this "twisted" $A - A$-bimodule by A^τ. Let $THH(A; A^\tau)$ denote the topological Hochschild homology of A with coefficients in A^τ.

Proposition 7.2.2. *Let A be an $\mathbb{E}_1$-ring spectrum in $\mathbf{Sp}_{C_n}$, and $C_n \curvearrowright S^1$ be the standard action. Then there exists an equivalence of spectra*

$$\left(\int_{S^1} A\right)^{\Phi C_n} \simeq THH(A; A^\tau).$$

In particular, $THH(A; A^\tau)$ admits a natural circle action.

Proof. Consider S^1 as the n-fold covering space $p\colon S^1 \to S^1$, with the standard C_n-action given by deck transformations. Let $S^1 = U \cup_{U\cap V} V$ be the standard collar decomposition of the base S^1 by hemispheres. Construct a C_n-collar decomposition $S^1 = p^{-1}(U) \cup_{p^{-1}(U\cap V)} p^{-1}(V)$ of the covering space by taking preimages. Observe that the pieces of this C_n-collar decomposition are given by topological induction,

$$\begin{aligned} &p^{-1}(U) = \sqcup_{C_n} U \cong \sqcup_{C_n} \mathbb{R}^1, \quad p^{-1}(V) = \sqcup_{C_n} V \cong \sqcup_{C_n} \mathbb{R}^1, \\ &p^{-1}(U \cap V) = \sqcup_{C_n} (U \cap V) \cong \sqcup_{C_n} (\mathbb{R}^1 \sqcup \mathbb{R}^1) = (\sqcup_{C_n} \mathbb{R}^1) \sqcup (\sqcup_{C_n} \mathbb{R}^1). \end{aligned}$$

Therefore by C_n-$\otimes$-excision

$$\begin{aligned} \int_{S^1} A &\simeq \left(\int_{p^{-1}(U)} A\right) \bigotimes_{\int_{p^{-1}(U\cap V)} A} \left(\int_{p^{-1}(V)} A\right) \simeq \left(\int_{\sqcup_{C_n} \mathbb{R}^1} A\right) \bigotimes_{\int_{(\sqcup_{C_n} \mathbb{R}^1) \sqcup (\sqcup_{C_n} \mathbb{R}^1)} A} \left(\int_{\sqcup_{C_n} \mathbb{R}^1} A\right) \\ &\simeq (N_e^{C_n} A) \bigotimes_{(N_e^{C_n} A) \otimes (N_e^{C_n} A)^{op}} (N_e^{C_n} A)^\tau. \end{aligned}$$

Let us pause and explain the superscript decorations in the last term. The $\left(\int_{p^{-1}(U\cap V)} A\right)$-module structure of $\int_{p^{-1}(U)} A$ is induced by the inclusion $p^{-1}(U \cap V) \hookrightarrow p^{-1}(U)$. When we identify $p^{-1}(U\cap V) \cong (\sqcup_{C_n} \mathbb{R}^1) \sqcup (\sqcup_{C_n} \mathbb{R}^1)$ the module structure on $\int_{p^{-1}(U)} A$ is naturally identified with an $(N_e^{C_n} A) - (N_e^{C_n} A)$-bimodule structure, or equivalently a right $(N_e^{C_n} A) \otimes (N_e^{C_n} A)^{op}$-module structure. Similarly, $\int_{p^{-1}(V)} A$ is naturally a left $N_e^{C_n}(A) - N_e^{C_n}(A)^{op}$-module. However the left module structure is induced by an embedding $\sqcup_{C_n} \mathbb{R}^1 \hookrightarrow p^{-1} V$ which defers from the standard embedding (the topological induction of $\mathbb{R}^1 \hookrightarrow V$) by a deck transformation. Therefore the left multiplication is "twisted", i.e. given by first acting on the scalar by the generator $\tau \in C_n$. In order to remember this twist in the module structure of the right hand side we add the superscript τ.

Next we take geometric fixed points of $\int_{S^1} A$. Since the geometric fixed points functor $(-)^{\Phi C_n}\colon \mathbf{Sp}_{C_n} \to \mathbf{Sp}$ is symmetric monoidal and preserve homotopy colimits

$$\left(\int_{S^1} A\right)^{\Phi C_n} \simeq (N_e^{C_n} A)^{\Phi C_n} \bigotimes_{(N_e^{C_n} A)^{\Phi C_n} \otimes ((N_e^{C_n} A)^{op})^{\Phi C_n}} ((N_e^{C_n} A)^\tau)^{\Phi C_n} \simeq A \bigotimes_{A \otimes A^{op}} A^\tau.$$

The right hand side is equivalent to the topological Hochschild homology $THH(A; A^\tau)$ of $A \in \mathbf{Sp}$ with coefficients in the $A-A$-bimodule A^τ.

Finally, we describe the natural circle action on $\left(\int_{S^1} A\right)^{\Phi C_n}$. Note that the automorphism space of $S^1 \in \underline{\mathbf{Mfld}}^{C_n,1-fr}_{[C_n/C_n]}$ acts on S^1, so by functoriality it induces a natural action on $\left(\int_{S^1} A\right)^{\Phi C_n}$. The endomorphism space of $S^1 \in \underline{\mathbf{Mfld}}^{C_n,1-fr}_{[C_n/C_n]}$ is the space of C_n-equivariant oriented embeddings $Emb^{C_n}(S^1, S^1)$. In particular the endomorphism space $S^1 \in \underline{\mathbf{Mfld}}^{C_n,1-fr}_{[C_n/C_n]}$ includes rotations of S^1, therefore the circle group acts on $\int_{S^1} A$ by rotations, and by functoriality on $\left(\int_{S^1} A\right)^{\Phi C_n}$. □

Remark 7.2.3. The inclusion of the circle group into $Emb^{C_n}(S^1, S^1)$ is in fact a deformation retract.

Remark 7.2.4. This theorem can be seen as an instance of a more general principle: factorization homology with local coefficients on a manifold M can be constructed as the fixed points of G-factorization homology on a cover of M.

Relation to the relative norm construction The spectrum $THH(A; A^\tau)$ and its circle action have been used to define the relative norm in [ABG+14]. In order to give a precise statement we recall the notation of [ABG+14].

Fix U a complete universe of the circle group (in the sense of orthogonal spectra), and define a complete C_n-universe $\tilde{U} = \iota^*_{C_n} U$. Let R be an associative ring orthogonal C_n-spectrum indexed on the universe $\tilde{U}$. Let $I^{\mathbb{R}^\infty}_{\tilde{U}}, I^U_{\mathbb{R}^\infty}$ denote the "change of universe" functors. The relative norm $N^{S^1}_{C_n} R$ of [ABG+14, def. 8.2] is the genuine S^1-spectra defined as

$$I^U_{\mathbb{R}^\infty} \left| N^{cyc,C_n}_\wedge (I^{\mathbb{R}^\infty}_{\tilde{U}} R) \right|,$$

where $N^{cyc,C_n}_\wedge(-)$ is the "twisted cyclic bar construction" of [ABG+14, def. 8.1].

Note that the geometric realization $|N^{cyc,C_n}_\wedge (I^{\mathbb{R}^\infty}_{\tilde{U}} R)|$ is equivalent to $THH(R; R^\sigma)$, computed using the standard bar resolution. By proposition 7.2.2 there exists an equivalence of spectra

$$\left(\int_{S^1} R \right)^{\Phi C_n} \simeq \left| N^{cyc,C_n}_\wedge \left(I^{\mathbb{R}^\infty}_{\tilde{U}} R \right) \right|,$$

where one the left hand side we consider R as an $\mathbb{E}_1$-algebra in $\mathbf{Sp}_{C_n}$.

Moreover, by inspection the above equivalence respects the circle action, hence after applying the change of universe functor $I^U_{\mathbb{R}^\infty}$ we get an equivalence of genuine S^1-spectra

$$N^{S^1}_{C_n} R \simeq I^U_{\mathbb{R}^\infty} \left(\left(\int_{S^1} R \right)^{\Phi C_n} \right).$$

Appendix A The Moore over category

Let $\mathcal{C}$ be a topological category and $x \in \mathcal{C}$ an object. Denote by $N(\mathcal{C}) \in \mathcal{C}\mathrm{at}_\infty$ the coherent nerve of $\mathcal{C}$, and by $N(\mathcal{C})_{/x} \in \mathcal{C}\mathrm{at}_\infty$ the over category. Note that $N(\mathcal{C})_{/x}$ is not equivalent to the coherent nerve of $\mathcal{C}_{/x}$, the topological over category: both have the same objects, but a point in $\mathrm{Map}_{\mathcal{C}_{/x}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x)$ is an given by a map $h \in \mathrm{Map}_{\mathcal{C}}(y_0, y_1)$ satisfying $f_0 = f_1 \circ h$, while a point in $\mathrm{Map}_{N(\mathcal{C})_{/x}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x)$ is given by a map $h \in \mathrm{Map}_{\mathcal{C}}(y_0, y_1)$ together with a *path* in $\mathrm{Map}_{\mathcal{C}}(y_0, x)$ from f_0 to $f_1 \circ h$. Nevertheless, it is useful to have a topological category whose coherent nerve is equivalent to $N(\mathcal{C})_{/x}$. Of course, this could be achieved by applying homotopy coherent realization to $N(\mathcal{C})_{/x}$, but unwinding the construction one sees that an explicit description of topological category involves a lot of simplicial combinatorics. In what follows we construct a topological category $\mathcal{C}_{/x}^{\mathrm{Moore}}$ whose coherent nerve is equivalent to $N(\mathcal{C})_{/x}$, which avoids simplicial combinatorics.

An obvious candidate for the mapping space $\mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x)$ is the space of maps $h\colon y_0 \to y_1$ in $\mathcal{C}$ together with a path from f_0 to $f_1 \circ f$ in $\mathrm{Map}_{\mathcal{C}}(y_0, x)$, formally given by the fiber product $\mathrm{Map}_{\mathcal{C}}(y_0, y_1) \times_{\mathrm{Map}_{\mathcal{C}}(y_0,x)} P(\mathrm{Map}_{\mathcal{C}}(y_0, x))$. However, one runs into trouble when trying to define composition functions which are strictly associative, since the composition action uses concatenation of paths. The problem of defining strictly associative concatenation of paths has a classical solution, namely replacing the space of paths with the homotopy equivalent space of Moore paths. Defining the mapping space $\mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x)$ using Moore paths leads to a simple construction of a topological category $\mathcal{C}_{/x}^{\mathrm{Moore}}$, the Moore over category (definition A.0.1), whose coherent nerve is equivalent to $N(\mathcal{C})_{/x}$ (corollary A.0.5).

We first recall the definition of the Moore path space and concatenation of Moore paths. Let X be a topological space. The Moore path space of X is the subspace

$$M(X) \subset [0,\infty) \times X^{[0,\infty)}, \quad M(X) = \left\{(r,\gamma) |\text{ the restriction } \gamma|_{[r,\infty)} \text{ is a constant function}\right\},$$

where $X^{[0,\infty)}$ is the space of functions $[0,\infty) \to X$ endowed with the compact-open topology. The "starting point" and "finishing point" fibrations $\alpha, \omega\colon M(X) \twoheadrightarrow X$ are the given by $\alpha(r,\gamma) = \gamma(0)$, $\omega(r,\gamma) = \gamma(r)$. Moreover, the "ends points" map $(\alpha,\omega)\colon M(X) \twoheadrightarrow X \times X$ is also a Serre fibration. Concatenation of Moore paths is defined by

$$*\colon M(X) \times_X M(X) \to M(X), \quad (r_0,\gamma_0) * (r_1,\gamma_1) = \left(r_0 + r_1, t \mapsto \begin{cases} \gamma_0(t) & t \le r_0 \\ \gamma_1(t - r_0) & t \ge r_0 \end{cases} \right).$$

It is straightforward to verify that concatenation of paths is associative, i.e.

$$((r_0,\gamma_0) * (r_1,\gamma_1)) * (r_2,\gamma_2) = (r_0,\gamma_0) * ((r_1,\gamma_1) * (r_2,\gamma_2)).$$

For $x \in X$ a point, the "constant instant Moore path" $(0, t \mapsto x) \in M(X)$ is a neutral element for concatenation.

With the definition of Moore paths at hand, we can define the Moore path category.

Definition A.0.1. *Let $\mathcal{C}$ be a topological category and $x \in \mathcal{C}$ an object. Define a topological category $\mathcal{C}_{/x}^{\mathrm{Moore}}$ with objects arrows $f\colon y \to x$, i.e pairs (y, f) where $y \in \mathcal{C}$, $f \in \mathrm{Map}_{\mathcal{C}}(y, x)$, and morphism spaces $\mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x)$ given by the fiber products*

$$\begin{aligned} &\{f_0\} \times_{\mathrm{Map}_{\mathcal{C}}(y_0,x)} M(\mathrm{Map}_{\mathcal{C}}(y_0,x)) \times_{\mathrm{Map}_{\mathcal{C}}(y_0,x),(f_1 \circ_{\mathcal{C}} (-))} \mathrm{Map}_{\mathcal{C}}(y_0,y_1) \\ &\qquad = \{((r,\gamma), h) \,|\, \gamma(0) = f_0,\, \gamma(r) = f_1 \circ_{\mathcal{C}} h\}. \end{aligned}$$

Define composition in $\mathcal{C}_{/x}^{\mathrm{Moore}}$ by

$$\begin{aligned} \circ\colon &\mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x) \times \mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y_0 \xrightarrow{f_1} x, y_1 \xrightarrow{f_2} x) \to \mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_2} x) \\ &(((r,\gamma),h),((r',\gamma'),h')) \mapsto ((r,\gamma) * (r',\gamma' \circ_{\mathcal{C}} h), h' \circ_{\mathcal{C}} h) \end{aligned}$$

and identity of $f\colon y \to x$ by $((0, t \mapsto f), id_y) \in \mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y \xrightarrow{f} x, y \xrightarrow{f} x)$, using the constant instant Moore path at f. We call $\mathcal{C}_{/x}^{\mathrm{Moore}}$ the Moore over category *of $\mathcal{C}$ over x.*

Observation 1. The mapping space $\mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}$ is the homotopy fiber of

$$f_1 \circ (-)\colon \mathrm{Map}_{\mathcal{C}}(y_0, y_1) \to \mathrm{Map}_{\mathcal{C}}(y_0, x).$$

Remark A.0.2. If the mapping spaces $\mathrm{Map}_{\mathcal{C}}(y, x)$ of $\mathcal{C}$ has a smooth structure one can replace the Moore spaces of continuous Moore paths by spaces of piecewise smooth Moore paths, without changing the ∞-category represented by $N(\mathcal{C}_{/x}^{\mathrm{Moore}})$.

Lemma A.0.3. *The coherent nerve of the Moore category $\mathcal{C}_{/x}^{\mathrm{Moore}}$ has a terminal object $(x \xrightarrow{=} x) \in \mathcal{C}_{/x}^{\mathrm{Moore}}$.*

Proof. For every object $(y \xrightarrow{f} x) \in \mathcal{C}_{/x}^{\mathrm{Moore}}$ the mapping space $\mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(y \xrightarrow{f} x, x \xrightarrow{=} x)$ is the space of Moore paths in $\mathrm{Map}_{\mathcal{C}}(y, x)$ starting at f, a contractible space. □

Define a functor of topological categories $U\colon \mathcal{C}_{/x}^{\mathrm{Moore}} \to \mathcal{C}$ sending $U\colon (y \xrightarrow{f} x) \mapsto y$ and

$$U\colon \mathrm{Map}_{\mathcal{C}_{/x}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x) \to \mathrm{Map}_{\mathcal{C}}(y_0, y_1), \quad U\colon ((r, \gamma), h) \mapsto h$$

on mapping spaces by projection.

Lemma A.0.4. *The induced map of coherent nerves $N(U)\colon N(\mathcal{C}_{/x}^{\mathrm{Moore}}) \to N(\mathcal{C})$ is a right fibration.*

Proof. First we observe that $N(U)$ is an inner fibration. For each pair of objects $(y_0 \xrightarrow{f_0} x), (y_1 \xrightarrow{f_1} x) \in \mathcal{C}_{/x}^{\mathrm{Moore}}$ the map $U\colon \mathrm{Map}_{\mathcal{C}_{/x}}(y_0 \xrightarrow{f_0} x, y_1 \xrightarrow{f_1} x) \to \mathrm{Map}_{\mathcal{C}}(y_0, y_1)$ is a pullback of the "end points" fibration (α, ω) along $\{f_0\} \times (f_1 \circ_{\mathcal{C}} (-))$, and therefore a fibration. By [Lur09a, prop. 2.4.1.10 (1)] it follows that $N(U)$ is an inner fibration.

By [Lur09a, prop. 2.4.2.4] we need to show that every morphism $((r, \gamma), h)\colon (y_0 \xrightarrow{f_0} x) \to (y_1 \xrightarrow{f_1} x)$ in $\mathcal{C}_{/x}^{\mathrm{Moore}}$ is U-Cartesian. By [Lur09a, prop. 2.4.1.10 (2)] we have to show that for every $(y \xrightarrow{f} x)$ in $\mathcal{C}_{/c}^{\mathrm{Moore}}$ the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(f, f_0) & \xrightarrow{((r,\gamma),h)\circ -} & \mathrm{Map}_{\mathcal{C}_{/x}^{\mathrm{Moore}}}(f, f_1) \\ \downarrow U & & \downarrow U \\ \mathrm{Map}_{\mathcal{C}}(y, y_0) & \xrightarrow{h \circ_{\mathcal{C}} -} & \mathrm{Map}_{\mathcal{C}}(y, y_1) \end{array}$$

is homotopy Cartesian. We show that the induced map between the fibers is a homotopy equivalence. For every point $(h'\colon y \to y_0) \in \mathrm{Map}_{\mathcal{C}}(y, y_0)$, the fiber over h' is the space of Moore paths in $\mathrm{Map}_{\mathcal{C}}(y, x)$ starting at f and ending at $f_0 \circ h$, the fiber over $h \circ h'$ is the space of Moore paths in $\mathrm{Map}_{\mathcal{C}}(y, x)$ starting at f and ending at $f_1 \circ h' \circ h$, and the map between the fibers is given by concatenation with the Moore path $(r, \gamma \circ h')$ starting at $f_0 \circ h'$ and ending at $f_1 \circ h \circ h'$, a homotopy equivalence. □

Corollary A.0.5. *Let $\mathcal{C}$ be a topological category and $x \in \mathcal{C}$ an object. The coherent nerve $N(\mathcal{C}_{/x}^{\mathrm{Moore}})$ is equivalent to the ∞-over category $N(\mathcal{C})_{/x}$.*

Proof. The right fibration $N(U)\colon N(\mathcal{C}_{/x}^{\mathrm{Moore}}) \to N(\mathcal{C})$ takes the terminal object $(x \xrightarrow{=} x) \in N(\mathcal{C}_{/x}^{\mathrm{Moore}})$ to $x \in \mathcal{C}$. By [Lur09a, prop. 4.4.4.5] the right fibrations $N(U)\colon N(\mathcal{C}_{/x}^{\mathrm{Moore}}) \to N(\mathcal{C})$ and $N(\mathcal{C})_{/x} \twoheadrightarrow N(\mathcal{C})$ are equivalent fibrant objects of the contravariant model structure on $sSet_{/N(\mathcal{C})}$ (both right fibrations classify the representable functor $\mathrm{Map}(-, x)\colon \mathcal{C}^{op} \to \mathcal{S}$), and claim the follows. □

Appendix B The definition of a G-Symmetric Monoidal category

This appendix contains no original results or definitions. The notion of G-symmetric monoidal ∞-category, developed by Barwick, Dotto, Glasman, Nardin and Shah, is central to our treatment of G-factorization homology. For the convenience of the reader we include the definition here (see definition B.0.7), which is equivalent to the definition given in [Nar17].

Parametrized join First we recall the parametrized version of the join construction.

Definition B.0.1. *Let S be an ∞-category. Restricting along $S\times\partial\Delta^1 \to S\times\Delta^1$ defines a functor $sSet_{/S\times\Delta^1} \to sSet_{/S\times\partial\Delta^1} \cong sSet_{/S\times\{0\}} \times sSet_{/S\times\{1\}}$ which carries coCartesian fibrations over $S \times \Delta^1$ to coCartesian fibrations over $S \times \partial\Delta^1 = S \times \{0\} \coprod S \times \{1\}$. This functor has a right adjoint which is called the S-parametrized join and denoted by*

$$sSet_{/S\times\Delta^1} \leftrightarrows sSet_{/S\times\{0\}} \times sSet_{/S\times\{1\}} : \star_S.$$

By [Sha18, prop. 4.3], if $\underline{\mathcal{C}} \twoheadrightarrow S, \underline{\mathcal{D}} \twoheadrightarrow S$ are coCartesian fibrations (i.e S-categories), then $\underline{\mathcal{C}} \star_S \underline{\mathcal{D}} \twoheadrightarrow S$ is a coCartesian fibration.

It follows from [Sha18, thm. 4.16] that the parametrized join carries coCartesian fibrations over $S \times \partial\Delta^1$ to inner fibrations over $S \times \Delta^1$ with coCartesian lifts over $S \times \partial\Delta^1$.

The parametrized join $X \star_S Y \to S \times \Delta^1$ of $X \to S$, $Y \to S$ can be informally described as follows (see [Sha18, lem. 4.4]): its restriction to $S \times \{0\}$ is $X \to S$, its restriction to $S \times \{1\}$ is $Y \to S$, and for each $s \in S$ its restriction to $\{s\} \times \Delta^1$ is the join $X_{[s]} \star Y_{[s]}$, where $X_{[s]}, Y_{[s]}$ are the fibers of $X \to S$, $Y \to S$ over $s \in S$.

Fact: for the case $Y = S$ one gets a coCartesian fibration $X \star_S S \twoheadrightarrow S$.

Finite pointed G-sets We denote by $\underline{\mathbf{Fin}}^G_*$ the G-category of finite pointed G-sets of [Nar16, def. 4.12]. An object $I \in \underline{\mathbf{Fin}}^G_*$ over the orbit G/H is a G-equivariant map $I = (U \to G/H)$ from a finite G-set U. A morphism in $\underline{\mathbf{Fin}}^G_*$ over $\varphi\colon G/K \to G/H$ is a span of the form

$$\begin{array}{ccccc} U & \longleftarrow & U' & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ G/H & \overset{\psi}{\longleftarrow} & G/K & \overset{=}{\longrightarrow} & G/K \end{array}$$

where the left square is a summand inclusion, i.e it induces an inclusion of U' into the pullback $\psi^*U = G/K \times_{G/H} U$. The span above is a coCartesian edge if the left square is Cartesian and the map $U' \to V$ is an isomorphism of finite G-sets ([Nar16, lem. 4.9, def. 4.12]). We call the span above *inert* if $U' \to V$ is an isomorphism.

Notation B.0.2. Let $G/K \in \mathcal{O}_G^{op}$ be an orbit. Denote by $I_+(G/K) = (G/K \xrightarrow{=} G/K) \in \underline{\mathbf{Fin}}^G_*$ the finite pointed set given by the identity map of G/K.

Definition B.0.3. *Let $I \in \underline{\mathbf{Fin}}^G_*$, $I = (U \to G/H)$ be a finite pointed G-set over G/H. Recall that the left fibration*

$$\underline{G/H} = (\mathcal{O}_G^{op})_{[G/H]/} \to \mathcal{O}_G^{op}, \quad (G/H \leftarrow G/K) \mapsto G/K$$

classifies the representable functor $\mathrm{Hom}(-, G/H)\colon \mathcal{O}_G^{op} \to Set$ *(see [BDG+16b, ex. 2.4]). By Yoneda's lemma the set $(\underline{\mathbf{Fin}}^G_*)_{[G/H]}$ of finite G-sets over G/H is in bijection with the set of* *natural transformations* $Nat\Big(\mathrm{Hom}(-,G/H),(\underline{\mathbf{Fin}}^G_*)_{(-)}\Big)$*, which in turn is in bijection with the set of G-functors* $\underline{G/H} \to \underline{\mathbf{Fin}}^G_*$*. Define* $\sigma_{<I>} \colon \underline{G/H} \to \underline{\mathbf{Fin}}^G_*$ *as the G-functor corresponding to I under the bijection above. Explicitly,* $\sigma_{<I>}$ *acts on objects by* $\sigma_{<I>} \colon (G/H \xleftarrow{\varphi} G/K) \mapsto (\varphi^* U \to G/K)$ *.*

The underlying G-categories of the G-diagram classified by $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ By straightening/unstraightening for G-categories ([BDG+16b, prop. 8.3]) the coCartesian fibration $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ corresponds to a G-functor $\underline{\mathbf{Fin}}^G_* \to \underline{\mathcal{C}\mathrm{at}}_{\infty,G}$, which we can interpret as a $\underline{\mathbf{Fin}}^G_*$-shaped G-diagram in $\underline{\mathcal{C}\mathrm{at}}_{\infty,G}$. The functor $\underline{\mathbf{Fin}}^G_* \to \underline{\mathcal{C}\mathrm{at}}_{\infty,G}$ assigns to each $I \in (\underline{\mathbf{Fin}}^G_*)_{[G/H]}$ an object of $(\underline{\mathcal{C}\mathrm{at}}_{\infty,G})_{[G/H]} = \mathrm{Fun}(\underline{G/H}, \mathcal{C}\mathrm{at}_\infty) \simeq (\mathcal{C}\mathrm{at}_\infty)^{coCart}_{/\underline{G/H}}$ (see [BDG+16b, ex. 7.5]), i.e a coCartesian fibration over $\underline{G/H}$ [17], which can be constructed as follows.

Definition B.0.4. *Let* $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ *be a coCartesian fibration, and* $I \in \underline{\mathbf{Fin}}^G_*$ *a G-set over G/H. Define a coCartesian fibration* $\underline{\mathcal{C}}^{\otimes}_{<I>} \twoheadrightarrow \underline{G/H}$ *by pulling back* $\underline{\mathcal{C}}^{\otimes}$ *along* $\sigma_{<I>}$*,*

$$\begin{array}{ccc} \underline{\mathcal{C}}^{\otimes}_{<I>} & \longrightarrow & \underline{\mathcal{C}}^{\otimes} \\ \downarrow & \ulcorner & \downarrow \\ \underline{G/H} & \xrightarrow{\sigma_{<I>}} & \underline{\mathbf{Fin}}^G_*. \end{array}$$

In particular, for $I_+(G/G) = (G/G \xrightarrow{=} G/G)$*, the terminal object of* $\underline{\mathbf{Fin}}^G_*$*, denote by* $\underline{\mathcal{C}} := \underline{\mathcal{C}}^{\otimes}_{<I_+(G/G)>}$ *the* underlying G-category of $\underline{\mathcal{C}}^{\otimes}$.

An inert diagram in $\underline{\mathbf{Fin}}^G_*$ Let $I = (U \to G/H)$ be a finite pointed G-set over G/H, as before. Applying the parametrized join construction for $S = \underline{G/H}$ and the left fibrations $\underline{G/H} \xrightarrow{=} \underline{G/H}$, $\underline{U} \twoheadrightarrow \underline{G/H}$ [18] we get a coCartesian fibration $\underline{U} \star_{\underline{G/H}} \underline{G/H} \twoheadrightarrow \underline{G/H} \times \Delta^1$, which we can consider as a G-category by composing with the coCartesian fibration $\underline{G/H} \times \Delta^1 \twoheadrightarrow \underline{G/H}$ and the left fibration $\underline{G/H} \twoheadrightarrow \mathcal{O}_G^{op}$.

For each $I \in \underline{\mathbf{Fin}}^G_*$ we construct a G-functor $\Phi_{<I>} \colon \underline{U} \star_{\underline{G/H}} \underline{G/H} \to \underline{\mathbf{Fin}}^G_*$ (a G-diagram in $\underline{\mathbf{Fin}}^G_*$):

Definition B.0.5. *Let* $I = (U \to G/H)$ *be a finite pointed G-set over G/H. We define a G-functor*

$$\begin{array}{ccccc} \Phi_{<I>} \colon \underline{U} \star_{\underline{G/H}} \underline{G/H} & & \longrightarrow & & \underline{\mathbf{Fin}}^G_* \\ \downarrow & & & & \downarrow \\ \underline{G/H} \times \Delta^1 & \twoheadrightarrow & \underline{G/H} & \twoheadrightarrow & \mathcal{O}_G^{op} \end{array}$$

by specifying its restrictions to $\underline{U} \twoheadrightarrow \underline{G/H} \times \{0\}$ *and* $\underline{G/H} \twoheadrightarrow \underline{G/H} \times \{1\}$*, together with its action on morphisms over* $(id, 0 \to 1) \in \underline{G/H} \times \Delta^1$*:*

---

[17] We think of a coCartesian fibration over $\underline{G/H}$ as representing an H-category, since the category $\underline{G/H} = (\mathcal{O}_G^{op})_{/[G/H]}$ is equivalent to $\mathcal{O}_H^{op}$.

[18] Since $\mathbf{Fin}^G$ is a category, a map of finite G-sets $U \to V$ induces a G-functor $\underline{U} \to \underline{V}$. By comparison, a map $f \colon x \to y$ in an ∞-category C induces a span $\underline{x} = C_{/x} \xleftarrow{\sim} C_{/f} \to C_{/y} = \underline{y}$ where both arrows are left fibrations and the left arrow is an equivalence of ∞-categories.

1. *The G-functor* $\underline{U} \to \underline{\mathbf{Fin}}_*^G$ *is the composition* $\underline{U} \twoheadrightarrow \underline{G/H} \xrightarrow{\sigma_{<I>}} \underline{\mathbf{Fin}}_*^G$ *, where the first map is the left fibration induced by* $U \to G/H$*.*

2. *The G-functor* $\underline{G/H} \to \underline{\mathbf{Fin}}_*^G$ *is the composition* $\underline{G/H} \twoheadrightarrow \underline{G/G} \xrightarrow{\sigma_{<I+(G/G)>}} \underline{\mathbf{Fin}}_*^G$ *, where the first map is the structure map* $\underline{G/H} \twoheadrightarrow \mathcal{O}_G^{op} = \underline{G/G}$ *and the second map is the G-functor corresponding to* $I_+(G/G)$ *(in fact, the composition is just* $\sigma_{<I_+(G/H)>}$*).*

3. *Let* $(G/H \xleftarrow{\psi} G/K) \in \underline{G/H}$*, then the fiber of* $\underline{U} \star_{\underline{G/H}} \underline{G/H} \to \underline{G/H} \times \Delta^1$ *over* $(\{\psi\}, 0 \to 1)$ *is* $\left(\underline{U} \star_{\underline{G/H}} \underline{G/H}\right)_\psi = \underline{U}_\psi \star \{\psi\}$*, a co-cone diagram on the finite set of maps* $\varphi \colon G/H \to U$ *such that*
$$\begin{array}{ccc} G/K & \xrightarrow{\varphi} & U \\ & \searrow^{\psi} & \downarrow \\ & & G/H \end{array}$$
commutes. Therefore, morphisms of $\underline{U} \star_{\underline{G/H}} \underline{G/H} \twoheadrightarrow \underline{G/H} \times \Delta^1$ *over* $(id_\psi, 0 \to 1) \in \underline{G/H} \times \Delta^1$ *are in bijection to* $\varphi \colon U \to G/H$ *making the above diagram commute. Let* $\bar{\varphi} \colon G/K \to \psi^* U$ *be the unique map given by*
$$\begin{array}{ccccc} G/K & & & & \\ & \overset{\exists ! \bar{\varphi}}{\searrow} & & \overset{\varphi}{\searrow} & \\ & & \psi^* U & \longrightarrow & U \\ & {\scriptstyle =} \searrow & \downarrow & \ulcorner & \downarrow \\ & & G/K & \xrightarrow{\psi} & G/H \end{array}$$
The functor $\Phi_{<I>}$ *sends the morphism over* $(id_\psi, 0 \to 1)$ *corresponding to* $\varphi \colon U \to G/H$ *to the span of finite pointed G-sets*
$$\begin{array}{ccccc} \psi^* U & \xleftarrow{\bar{\varphi}} & G/K & \xrightarrow{=} & G/K \\ \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ G/K & \xleftarrow{=} & G/K & \xrightarrow{=} & G/K. \end{array}$$
Using the fact that $\mathcal{O}_G$ *is atomic (i.e orbits have no non-trivial retracts) one can check that the left square is a summand inclusion.*

Steps 1 and 2 define $\Phi_{<I>}$ *on every morphism over* $\underline{G/H} \times (0 \to 1)$*, since every such morphism uniquely decomposes as a morphism in* $\underline{U}$ *followed by a morphism over* $(\{\psi\}, 0 \to 1)$ *for some* $\psi \in \underline{G/H}$*. Verifying that* $\Phi_{<I>}$ *is well defined is a straightforward calculation, using the fact that every morphism of* $\underline{U} \star_{\underline{G/H}} \underline{G/H}$ *can be uniquely decomposed as a morphism over* $(\{\psi'\}, 0 \to 1)$ *followed by a morphism in* $\underline{G/H}$*.*

Construction of Segal maps and definition of a G-symmetric monoidal ∞-category For any coCartesian fibration over $\underline{\mathbf{Fin}}_*^G$ we construct 'Segal maps':

Definition B.0.6. *Let* $\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ *be a coCartesian fibration and* $I = (U \to G/H)$ *a finite pointed G-set over* G/H*. Construct a G-functor over* $\underline{G/H}$ *by the following steps:*

1. *Pulling* $\underline{\mathcal{C}}^\otimes$ *along* $\Phi_{<I>}$ *produces a coCartesian fibration* $(\Phi_{<I>})^*\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{U}\star_{\underline{G/H}}\underline{G/H}$, *which we can consider as a coCartesian fibration over* $\underline{G/H}\times\Delta^1$ *by the composition*

$$(\Phi_{<I>})^*\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{U}\star_{\underline{G/H}}\underline{G/H} \twoheadrightarrow \underline{G/H}\times\Delta^1. \qquad (28)$$

2. *The restriction of the coCartesian fibration* (28) *to* $\underline{G/H}\times\{0\}$ *is given by*

$$\underline{U}\times_{\underline{G/H}}\underline{\mathcal{C}}^\otimes_{<I>} \twoheadrightarrow \underline{U} \twoheadrightarrow \underline{G/H}\times\{0\},$$

as it is the pullback of $\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ *along* $\underline{U} \twoheadrightarrow \underline{G/H} \xrightarrow{\sigma_{<I>}} \underline{\mathbf{Fin}}^G_*$.

3. *The restriction of the coCartesian fibration* (28) *to* $\underline{G/H}\times\{1\}$ *is given by*

$$\underline{G/H}\underline{\times}\underline{\mathcal{C}} \twoheadrightarrow \underline{G/H} \xrightarrow{=} \underline{G/H}\times\{1\},$$

as it is the pullback of $\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ *along* $\underline{G/H} \twoheadrightarrow \underline{G/G} \xrightarrow{\sigma_{<I+(G/G)>}} \underline{\mathbf{Fin}}^G_*$.

4. *Therefore, the coCartesian fibration* (28) *classifies a G-functor over* $\underline{G/H}$

$$\begin{array}{ccc} \underline{U}\times_{\underline{G/H}}\underline{\mathcal{C}}^\otimes_{<I>} & \longrightarrow & \underline{G/H}\underline{\times}\underline{\mathcal{C}} \\ & \searrow \quad \swarrow & \\ & \underline{G/H}, & \end{array}$$

which by [BDG+16b, thm. 9.7] is equivalent to a G-functor over $\underline{G/H}$

$$\begin{array}{ccc} \phi_{<I>}\colon \underline{\mathcal{C}}^\otimes_{<I>} & \longrightarrow & \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{U},\underline{G/H}\underline{\times}\underline{\mathcal{C}}) \\ & \searrow \quad \swarrow & \\ & \underline{G/H}. & \end{array} \qquad (29)$$

We call (29) *the Segal map of* I.

We can now give the definition of a G-symmetric monoidal G-category.

Definition B.0.7. *A* G-symmetric monoidal G-category *is a coCartesian fibration* $\underline{\mathcal{C}}^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ *such that for every finite pointed G-set* $I = (U \to G/H)$ *the Segal map* $\phi_{<I>}$ *of eq.* (29) *is an equivalence of* $\underline{G/H}$*-categories.*

Remark B.0.8. Let $\underline{\mathcal{C}}^\otimes \to \underline{\mathbf{Fin}}^G_*$ be a G-symmetric monoidal G-category. The Segal conditions imply that an object $x \in \underline{\mathcal{C}}^\otimes$ over $I = (U \to G/H) \in \underline{\mathbf{Fin}}^G_*$ classifies a G-functor $x_\bullet\colon \underline{U} \to \underline{\mathcal{C}}$. To see this, first note that by Yoneda's lemma x defines a $\underline{G/H}$ object $\sigma_x\colon \underline{G/H} \to \underline{\mathcal{C}}^\otimes$. Since $x \in \underline{\mathcal{C}}^\otimes$ is over $I \in \underline{\mathbf{Fin}}^G_*$ the composition $\underline{G/H} \xrightarrow{\sigma_x} \underline{\mathcal{C}}^\otimes \to \underline{\mathbf{Fin}}^G_*$ is equivalent to $\sigma_{<I>}\colon \underline{G/H} \to \underline{\mathbf{Fin}}^G_*$, so σ_x factors as $\sigma_x\colon \underline{G/H} \to \underline{\mathcal{C}}^\otimes_{<I>} \to \underline{\mathcal{C}}^\otimes$. Therefore we can regard σ_x as a G/H-object of $\underline{\mathcal{C}}^\otimes_{<I>}$. Using the Segal conditions we identify σ_x with a $\underline{G/H}$-object of $\underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{U},\underline{\mathcal{C}}\underline{\times}\underline{G/H})$. Finally we use the equivalence

$$\mathrm{Fun}_{\underline{G/H}}(\underline{G/H},\underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{U},\underline{\mathcal{C}}\underline{\times}\underline{G/H}) \simeq \mathrm{Fun}_{\underline{G/H}}(\underline{U},\underline{\mathcal{C}}\underline{\times}\underline{G/H}) \simeq \mathrm{Fun}_G(\underline{U},\underline{\mathcal{C}}).$$

to identify $\sigma_x\colon \underline{G/H} \to \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{U},\underline{\mathcal{C}}\underline{\times}\underline{G/H})$ with a G-functor $x_\bullet\colon \underline{U} \to \underline{\mathcal{C}}$.

Remark B.0.9. The codomain of the above Segal map is equivalent to a parametrized product: The "internal hom" $\underline{G/H}$-functor $\underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{U}, -) \colon \mathcal{C}\mathrm{at}_\infty^{\underline{G/H}} \to \mathcal{C}\mathrm{at}_\infty^{\underline{G/H}}$ is right adjoint to the composition

$$\begin{array}{ccccc} \mathcal{C}\mathrm{at}_\infty^{\underline{G/H}} & \longrightarrow & \mathcal{C}\mathrm{at}_\infty^{\underline{U}} & \longrightarrow & \mathcal{C}\mathrm{at}_\infty^{\underline{G/H}}, \\ (\underline{\mathcal{D}} \twoheadrightarrow \underline{G/H}) & \longmapsto & (\underline{\mathcal{D}} \times_{\underline{G/H}} \underline{U} \twoheadrightarrow \underline{U}) & \longmapsto & (X \times_{\underline{G/H}} \underline{U} \twoheadrightarrow \underline{U} \to \underline{G/H}). \end{array}$$

Therefore, it decomposes as the composition of the right adjoints:

Under this equivalence, the Segal map of $I = (U \to G/H)$ is given by

$$\begin{array}{ccc} \phi_{<I>} \colon \underline{\mathcal{C}}^{\otimes}_{<I>} & \longrightarrow & \prod_I \underline{\mathcal{C}} \times \underline{U} \\ & \searrow \quad \swarrow & \\ & \underline{G/H}. & \end{array} \tag{30}$$

In particular, we can identify an object $x \in \underline{\mathcal{C}}^{\otimes}$ over I with a $\underline{G/H}$-object of $\prod_I \underline{\mathcal{C}} \times \underline{U}$ as follows. Since $\underline{\mathbf{Fin}}_*^G \twoheadrightarrow \mathcal{O}_G^{op}$, $I \mapsto [G/H]$ the object x belongs to the fiber $\underline{\mathcal{C}}^{\otimes}_{[G/H]}$, and by Yoneda's lemma is classified by a G-functor $\sigma_x \colon \underline{G/H} \to \underline{\mathcal{C}}^{\otimes}$. Since $x \in \underline{\mathcal{C}}^{\otimes}$ is over $I \in \underline{\mathbf{Fin}}_*^G$, the G-functor $\underline{G/H} \to \underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ classifies $I \in \underline{\mathbf{Fin}}_*^G$, and is therefore equivalent to $\sigma_{<I>}$. Therefore it induces a $\underline{G/H}$-functor $\underline{G/H} \to \underline{\mathcal{C}}^{\otimes}_{<I>}$. Post-composing with the Segal map of eq. (30) we get our desired $\underline{G/H}$-object $\underline{G/H} \to \underline{\mathcal{C}}^{\otimes}_{<I>} \xrightarrow{\sim} \prod_I \underline{\mathcal{C}} \times \underline{U}$, which by abuse of notation we also denote by $\sigma_x \colon \underline{G/H} \to \prod_I \underline{\mathcal{C}} \times \underline{U}$.

Unpacking the construction of the Segal maps (29) in definition B.0.7 gives the following fiberwise characterization of G-symmetric monoidal categories, which is easier to verify.

Lemma B.0.10. *A coCartesian fibration $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ is a G-symmetric monoidal category (definition B.0.7) if and only if for each finite pointed G-set $J = (V \to G/K) \in \underline{\mathbf{Fin}}_*^G$ the functor*

$$\underline{\mathcal{C}}^{\otimes}_J \to \prod_{W \in \mathrm{Orbit}(\psi^* U)} \underline{\mathcal{C}}_{[W]}$$

is an equivalence of ∞-categories, where $\underline{\mathcal{C}}^{\otimes}_J$ is the fiber of $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ over $J = (V \to G/K)$ and the above functor is the product of $\underline{\mathcal{C}}^{\otimes}_J \to \underline{\mathcal{C}}_{[W]}$ associated to the $\underline{\mathbf{Fin}}_*^G$ edges

$$\forall W \in \mathrm{Orbit}(V) : \qquad \begin{array}{ccccc} V & \longleftarrow & W & \xrightarrow{=} & W \\ \downarrow & & \downarrow{=} & & \downarrow{=} \\ G/K & \longleftarrow & W & \xrightarrow{=} & W. \end{array}$$

Proof. The Segal condition of G-symmetric monoidal G-categories states that the Segal map $\phi_{<I>}$ is a parametrized equivalence, i.e for each $(G/H \xleftarrow{\psi} G/K) \in \underline{G/H}$, the Segal map $\phi_{<I>}$ induces an equivalence between the fibers

$$(\underline{\mathcal{C}}^{\otimes}_{<I>})_{[\psi]} \to \underline{\mathrm{Fun}}_{\underline{G/H}}(\underline{U}, \underline{G/H} \times \underline{\mathcal{C}})_{[\psi]}.$$

The fiber of $\underline{\mathcal{C}}^{\otimes}_{<I>}$ over ψ is the fiber of $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ over the finite pointed G-set $J := (\psi^* U \to G/K)$. The fiber of $\underline{\mathrm{Fun}}_{G/H}(\underline{U}, \underline{G/H} \underline{\times} \underline{\mathcal{C}})$ over ψ is the ∞-category of G-functors $\mathrm{Fun}_{\mathcal{O}_G^{op}}(\underline{\psi^* U}, \underline{\mathcal{C}})$. Decomposing the finite G-set $\psi^* U = \coprod_{W \in \mathrm{Orbit}(\psi^* U)} W$ into orbits we have

$$\mathrm{Fun}_{\mathcal{O}_G^{op}}(\underline{\psi^* U}, \underline{\mathcal{C}}) \cong \mathrm{Fun}_{\mathcal{O}_G^{op}}(\coprod \underline{W}, \underline{\mathcal{C}}) \simeq \prod_W \mathrm{Fun}_{\mathcal{O}_G^{op}}(\underline{W}, \underline{\mathcal{C}}) \simeq \prod_{W \in \mathrm{Orbit}(\psi^* U)} \mathcal{C}_{[W]}.$$

Since both sides depend only on $J = (\psi^* U \to G/K) \in \underline{\mathbf{Fin}}^G_*$ the result follows. □

We end this appendix with the definition of parametrized tensor product functors in a G-symmetric monoidal category.

Definition B.0.11. *Let $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be a G-symmetric monoidal category. Let $I = (U \to G/H)$, $J = (V \to G/H) \in \underline{\mathbf{Fin}}^G_*$ be two object over the orbit G/H, and $f \colon I \to J$ a morphism in $(\underline{\mathbf{Fin}}^G_*)_{[G/H]}$, given by*

$$\begin{array}{ccccc} U & \xleftarrow{=} & U & \xrightarrow{f} & V \\ \downarrow & & \downarrow & & \downarrow \\ G/H & \xleftarrow{=} & G/H & \xrightarrow{=} & G/H. \end{array}$$

The morphism f corresponds to a functor $\Delta^1 \to (\underline{\mathbf{Fin}}^G_*)_{[G/H]}$, or equivalently to a G-functor $\sigma_{<f>} \colon \underline{G/H} \times \Delta^1 \to \underline{\mathbf{Fin}}^G_*$, which restricts to $\sigma_{<I>}$ over $\underline{G/H} \times \{0\}$ and to $\sigma_{<J>}$ over $\underline{G/H} \times \{1\}$.

Pulling back $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ along $\sigma_{<f>}$ we get a coCartesian fibration $\underline{\mathcal{C}}^{\otimes}_{<f>} \twoheadrightarrow \underline{G/H} \times \Delta^1$ which restricts to $\underline{\mathcal{C}}^{\otimes}_{<I>}$ over $\underline{G/H} \times \{0\}$ and to $\underline{\mathcal{C}}^{\otimes}_{<J>}$ over $\underline{G/H} \times \{1\}$. Therefore this coCartesian fibration classifies a $\underline{G/H}$-functor

$$\begin{array}{ccc} \otimes_f \colon \underline{\mathcal{C}}^{\otimes}_{<I>} & \longrightarrow & \underline{\mathcal{C}}^{\otimes}_{<J>} \\ & \searrow \quad \swarrow & \\ & \underline{G/H} & \end{array}$$

which we refer to as the tensor product over f. Composing with the Segal maps of eq. (30), *we can rewrite the tensor product over f as*

$$\begin{array}{ccc} \otimes_f \colon \prod_I \underline{\mathcal{C}} \underline{\times} \underline{U} & \longrightarrow & \prod_J \underline{\mathcal{C}} \underline{\times} \underline{V} \\ & \searrow \quad \swarrow & \\ & \underline{G/H}. & \end{array}$$

Appendix C Mapping spaces in over-categories

We prove some simple properties of mapping spaces in over categories.

Lemma C.0.1. *Consider the over category $\mathcal{C}_{/b}$ for b an object in an ∞-category $\mathcal{C}$. Let $x \to b, y_1 \to b, y_2 \to b$ be objects in $\mathcal{C}_{/b}$, and a morphism φ in $\mathcal{C}_{/b}$ from $y_1 \to b$ to $y_2 \to b$. Then*

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_1 \to b) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x, y_1) \\ \downarrow \varphi_* & & \downarrow \varphi_* \\ \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_2 \to b) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x, y_2) \end{array}$$

is a homotopy pullback square.

Proof. The mapping space $\mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y \to b)$ is the homotopy fiber of the postcomposition map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, b)$. Therefore the lower square and outer rectangle in the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_1 \to b) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x, y_1) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_2 \to b) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x, y_2) \\ \downarrow & & \downarrow{\scriptstyle (y_2 \to b)_*} \\ * & \xrightarrow{x \to b} & \mathrm{Map}_{/\mathcal{C}}(x, b) \end{array}$$

are homotopy pullback diagram. It follows that the top square is a homotopy pullback square. □

Lemma C.0.2. *Let $f\colon b \to b'$ be a morphism in an ∞-category $\mathcal{C}$, and consider the postcomposition functor $f_*\colon \mathcal{C}_{/b} \to \mathcal{C}_{/b'}$. Let $x \to b, y_1 \to b, y_2 \to b$ be objects in $\mathcal{C}_{/b}$, and a morphism φ in $\mathcal{C}_{/b}$ from $y_1 \to b$ to $y_2 \to b$. Then*

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_1 \to b) & \xrightarrow{f_*} & \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_1 \to b \xrightarrow{f} b') \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_2 \to b) & \xrightarrow{f_*} & \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_2 \to b \xrightarrow{f} b') \end{array}$$

is a homotopy pullback square.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_1 \to b) & \xrightarrow{f_*} & \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_1 \to b \xrightarrow{f} b') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x, y_1) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_2 \to b) & \xrightarrow{f_*} & \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_2 \to b \xrightarrow{f} b') & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(x, y_2). \end{array}$$

By lemma C.0.1 the right square and the outer rectangle are homotopy pullback squares, hence the left square is a homotopy pullback square. □

Next, let $f\colon b \to b'$ be a morphism in an ∞-category $\mathcal{C}$ as before, and $T\colon \mathcal{M} \to \mathcal{C}_{/b'}$ a functor of ∞-categories. Define an ∞-category $\mathcal{M}_T$ as the pullback

$$\begin{array}{ccc} \mathcal{M}_T & \longrightarrow & \mathcal{C}_{/b} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f_*} \\ \mathcal{M} & \xrightarrow{T} & \mathcal{C}_{/b'}. \end{array}$$

Lemma C.0.3. *Let $\mathcal{C}, f: b \to b', T: \mathcal{M} \to \mathcal{C}_{/b'}$ and $\mathcal{M}_T$ be as above. Let X, Y_1, Y_2 be objects in $\mathcal{M}_T$ and $\Phi: Y_1 \to Y_2$ be morphism in $\mathcal{M}_T$. Then*

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{M}_T}(X, Y_1) & \xrightarrow{\Phi_*} & \mathrm{Map}_{\mathcal{M}_T}(X, Y_2) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{M}}(u(X), u(Y_1)) & \xrightarrow{u(\Phi)_*} & \mathrm{Map}_{\mathcal{M}}(u(X), u(Y_2))
\end{array}$$

is a homotopy pullback square.

Proof. Denote the images of $X, Y_1, Y_2 \in \mathcal{M}_F$ in $\mathcal{C}_{/b}$ by $x \to b$, $x \to y_1, x \to y_2$, and the image of Φ by φ. Using the equivalences $u(X) \simeq (x \to b \xrightarrow{f} b'), u(Y_1) \simeq (y_1 \to b \xrightarrow{f} b'), u(Y_2) \simeq (y_2 \to b \xrightarrow{f} b')$ in $\mathcal{C}_{/b'}$ we can identify the mapping spaces

$$\mathrm{Map}_{\mathcal{C}_{/b'}}(Tu(X), Tu(Y_i)) \simeq \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_i \xrightarrow{f} b'), \quad i = 1, 2.$$

Under these identifications we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{M}_T}(X, Y_1) & \longrightarrow & \mathrm{Map}_{\mathcal{M}}(u(X), u(Y_1)) \\
\downarrow & & \downarrow T \\
\mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_1 \to b) & \xrightarrow{f_*} & \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_1 \to b \xrightarrow{f} b') \\
\downarrow \varphi_* & & \downarrow \varphi_* \\
\mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_2 \to b) & \xrightarrow{f_*} & \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_2 \xrightarrow{f} b').
\end{array}$$

The top square is a homotopy pullback square by the definition of $\mathcal{M}_T$ as a pullback, and the bottom square is a homotopy pullback square by lemma C.0.2. Therefore the outer rectangle is a homotopy pullback square. On the other hand, this is also the outer rectangle in the diagram

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathcal{M}_T}(X, Y_1) & \xrightarrow{\Phi_*} & \mathrm{Map}_{\mathcal{M}_T}(X, Y_2) & \longrightarrow & \mathrm{Map}_{\mathcal{C}_{/b}}(x \to b, y_2 \to b) \\
\downarrow & & \downarrow & & \downarrow f_* \\
\mathrm{Map}_{\mathcal{M}}(u(X), u(Y_1)) & \xrightarrow{u(\Phi)_*} & \mathrm{Map}_{\mathcal{M}}(u(X), u(Y_2)) & \xrightarrow{T} & \mathrm{Map}_{\mathcal{C}_{/b'}}(x \to b \xrightarrow{f} b', y_2 \xrightarrow{f} b').
\end{array}$$

By definition of $\mathcal{M}_T$ is a pullback the right square is a homotopy pullback square, hence the left square is a homotopy pullback square, as claimed. ☐

References

[ABG+14] Vigleik Angeltveit, Andrew Blumberg, Teena Gerhardt, Michael Hill, Tyler Lawson, and Michael Mandell. Topological cyclic homology via the norm. *arXiv preprint arXiv:1401.5001*, 2014.

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *Journal of Topology*, 8(4):1045–1084, 2015.

[AFT17a] David Ayala, John Francis, and Hiro Lee Tanaka. Factorization homology of stratified spaces. *Selecta Mathematica*, 23(1):293–362, 2017.

[AFT17b] David Ayala, John Francis, and Hiro Lee Tanaka. Local structures on stratified spaces. *Advances in Mathematics*, 307:903–1028, 2017.

[ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and stringy topology*, volume 171. Cambridge University Press, 2007.

[Bar14] Clark Barwick. Spectral mackey functors and equivariant algebraic k-theory (i). *arXiv preprint arXiv:1404.0108*, 2014.

[BDG$^+$] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. Parametrized and equivariant higher algebra. To appear.

[BDG$^+$16a] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. Parametrized higher category theory and higher algebra: A general introduction. *arXiv preprint arXiv:1608.03654*, 2016.

[BDG$^+$16b] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. Parametrized higher category theory and higher algebra: Exposé I–elements of parametrized higher category theory. *arXiv preprint arXiv:1608.03657*, 2016.

[BGN14] Clark Barwick, Saul Glasman, and Denis Nardin. Dualizing cartesian and cocartesian fibrations. *arXiv preprint arXiv:1409.2165*, 2014.

[BH15] Andrew J Blumberg and Michael A Hill. Operadic multiplications in equivariant spectra, norms, and transfers. *Advances in Mathematics*, 285:658–708, 2015.

[Bon19] Peter Bonventre. The genuine operadic nerve. *arXiv preprint arXiv:1904.01465*, 2019.

[BP17] Peter Bonventre and Luis A Pereira. Genuine equivariant operads. *arXiv preprint arXiv:1707.02226*, 2017.

[Bre72] Glen E Bredon. *Introduction to compact transformation groups*, volume 46. Academic press, 1972.

[CMW01] SR Costenoble, J Peter May, and Stefan Waner. Equivariant orientation theory. *Homology, Homotopy and Applications*, 3(2):265–339, 2001.

[CW] SR Costenoble and S Waner. Equivariant ordinary homology and cohomology preprint, 2003.

[DI04] Daniel Dugger and Daniel C Isaksen. Topological hypercovers and 1-realizations. *Mathematische Zeitschrift*, 246(4):667–689, 2004.

[DMPR17] Emanuele Dotto, Kristian Moi, Irakli Patchkoria, and Sune Precht Reeh. Real topological hochschild homology. *arXiv preprint arXiv:1711.10226*, 2017.

[Elm83] Anthony D Elmendorf. Systems of fixed point sets. *Transactions of the American Mathematical Society*, 277(1):275–284, 1983.

[GM11] Bertrand Guillou and JP May. Models of g-spectra as presheaves of spectra. *arXiv preprint arXiv:1110.3571*, 2011.

[GM17] Bertrand Guillou and Peter May. Equivariant iterated loop space theory and permutative g–categories. *Algebraic & geometric topology*, 17(6):3259–3339, 2017.

[ha] Ian Agol (https://mathoverflow.net/users/1345/ian agol). isotopy inverse embeddings vs. diffeomorphisms. MathOverflow. URL:https://mathoverflow.net/q/100660 (version: 2013-06-26).

[HHR16] Michael A Hill, Michael J Hopkins, and Douglas C Ravenel. On the nonexistence of elements of kervaire invariant one. *Annals of Mathematics*, 184(1):1–262, 2016.

[Hir12] Morris W Hirsch. *Differential topology*, volume 33. Springer Science & Business Media, 2012.

[HM13] Lars Hesselholt and Ib Madsen. Real algebraic k-theory. *to appear*, 2013.

[Kup] Alexander Kupers. Three lectures on topological manifolds. http://www.math.harvard.edu/ kupers/notes/toplectures.pdf.

[LR78] R Lashof and M Rothenberg. G-smoothing theory. In *Proc. Symp. Pure Math*, volume 32, pages 211–266, 1978.

[Lur] Jacob Lurie. Higher algebra. http://www.math.harvard.edu/ lurie/.

[Lur09a] Jacob Lurie. *Higher topos theory*, volume 170. Princeton University Press, 2009.

[Lur09b] Jacob Lurie. On the classification of topological field theories. *Current developments in mathematics*, 2008:129–280, 2009.

[Nar16] Denis Nardin. Parametrized higher category theory and higher algebra: Exposé IV–stability with respect to an orbital ∞-category. *arXiv preprint arXiv:1608.07704*, 2016.

[Nar17] Denis Nardin. *Stability and distributivity over orbital ∞-categories.* PhD thesis, Massachusetts Institute of Technology, 2017.

[PS14] Dmitri Pavlov and Jakob Scholbach. Admissibility and rectification of colored symmetric operads. *arXiv preprint arXiv:1410.5675*, 2014.

[Sha18] Jay Shah. Parametrized higher category theory and higher algebra: Exposé II–indexed homotopy limits and colimits. *arXiv preprint arXiv:1809.05892*, 2018.

[tD87] Tammo tom Dieck. *Transformation groups*, volume 8. Walter de Gruyter, 1987.

[Wan80] Stefan Waner. Equivariant classifying spaces and fibrations. *Transactions of the American Mathematical Society*, 258(2):385–405, 1980.

[Was69] Arthur G Wasserman. Equivariant differential topology. *Topology*, 8(2):127–150, 1969.

[Wee18] TAN Weelinck. Equivariant factorization homology of global quotient orbifolds. *arXiv preprint arXiv:1810.12021*, 2018.

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Parametrised Presentability over Orbital Categories

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In this paper, we develop the notion of presentability in the parametrised homotopy theory framework of [BDG+16a] over orbital categories. We formulate and prove a characterisation of parametrised presentable categories in terms of its associated straightening. From this we deduce a parametrised adjoint functor theorem from the unparametrised version, prove various localisation results, and we record the interactions of the notion of presentability here with multiplicative matters. Such a theory is of interest for example in equivariant homotopy theory, and we will apply it in [Hil22b] to construct the category of parametrised noncommutative motives for equivariant algebraic K-theory.

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1 Introduction

Parametrised homotopy theory is the study of higher categories fibred over a base ∞-category. This is a generalisation of the usual theory of higher categories, which can be viewed as the parametrised homotopy theory over a point. The advantage of this approach is that many structures can be cleanly encoded by the morphisms in the base ∞-category. For example, in the algebro-geometric world, various forms of pushforwards exist for various classes of scheme morphisms (see [BH21] for more details). Another example, which is the main motivation of this work for our applications in [Hil22b], is that of genuine equivariant homotopy theory for a finite group G - here the base ∞-category would be $\mathcal{O}_G^{\mathrm{op}}$, the opposite of the G-orbit category. In this case, for subgroups $H \leq K \leq G$, important and fundamental constructions such as *indexed coproducts*, *indexed products*, and *indexed tensors*

$$\coprod_{K/H} \qquad \prod_{K/H} \qquad \bigotimes_{K/H}$$

can be encoded by the morphisms in $\mathcal{O}_G^{\mathrm{op}}$. One framework in which to study this is the series of papers following [BDG+16a] and the results in this paper should be viewed as a continuation of the vision from the aforementioned series - we refer the reader to them for more motivations and examples.

For an ∞-category to admit all small colimits and limits is a very desirable property as it means that many constructions can be done in it. However, this property entails that it has to be large enough and we might lose control of it due to size issues. Fortunately, there is a fix to this problem in the form of the very well-behaved class of *presentable* ∞-categories: these are cocomplete ∞-categories that are "essentially generated" by a small subcategory. One of the most important features of presentable ∞-categories is the adjoint functor theorem which says that one can test whether or not a functor between presentables is right or left adjoint by checking that it preserves limits or colimits respectively. The ∞-categorical theory of presentability was developed by Lurie in [Lur09, Chapter 5], generalising the classical 1-categorical notion of *locally presentable categories*.

The goal of this paper is to translate the above-mentioned theory of presentable ∞-categories to the parametrised setting and to understand the relationship between the notion of parametrised presentability and its unparametrised analogue in [Lur09]. As a signpost to the expert reader, we will always assume throughout this paper that the base category is orbital in the sense of [Nar17]. Besides that, we will adopt the convention in said paper of defining a $\mathcal{T}$-category, for a fixed based ∞-category $\mathcal{T}$, to be a cocartesian fibration over the *opposite*, $\mathcal{T}^{\mathrm{op}}$. This convention is geared towards equivariant homotopy theory as introduced in the motivation above where $T = \mathcal{O}_G$. Note that by the straightening-unstraightening equivalence of [Lur09], a $\mathcal{T}$-category can equivalently be thought of as a functor $\mathcal{T}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. The first main result we obtain is then the following characterisation of parametrised presentable ∞-categories.

Theorem A (Straightening characterisation of parametrised presentables, full version in Theorem 6.1.2)**.** *Let $\mathcal{C}$ be a $\mathcal{T}$-category. Then it is $\mathcal{T}$-presentable if and only if the associated straightening $C : \mathcal{T}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ factors through the non-full subcategory $\mathrm{Pr}^{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty$ of presentable categories and left adjoint functors, and morevoer these functors themselves have left adjoints satisfying certain Beck-Chevalley conditions (3.1.8).*

In the full version, we also give a complete parametrised analogue of the characterisations of presentable ∞-categories due to Lurie and Simpson (cf. [Lur09, Thm. 5.5.1.1]), which in particular shows that the notion defined in this paper is equivalent to the one defined in [Nar17, §1.4]. While it is generally expected that the theory of ∞-cosmoi in [RV22] should absorb the statement and proof of the Lurie-Simpson-style characterisations of presentability, the value of the theorem above is in clarifying the relationship between the notion of presentability in the parametrised sense and in the unparametrised sense. Indeed, the description in Theorem A is a genuinely parametrised statement that is not seen in the unparametrised realm where $T = *$. One consequence of this is that we can easily deduce the parametrised adjoint functor theorem *from* the unparametrised version instead of repeating the same arguments in the parametrised setting.

Theorem B (Parametrised adjoint functor theorem, Theorem 6.2.1)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a $\mathcal{T}$-functor between $\mathcal{T}$-presentable categories. Then:*

1. *If F strongly preserves $\mathcal{T}$-colimits, then F admits a $\mathcal{T}$-right adjoint.*
2. *If F strongly preserves $\mathcal{T}$-limits and is $\mathcal{T}$-accessible, then F admits a $\mathcal{T}$-left adjoint.*

Another application of Theorem A is the construction of *presentable Dwyer-Kan localisations*, Theorem 6.3.7. This is deduced essentially by performing fibrewise localisations, which are in turn furnished by [Lur09]. It is an extremely important construction, much like in the unparametrised world, and we will for example use it in [Hil22b] to understand the parametrised enhancement of the noncommutative motives of [BGT13]. Other highlights include the *localisation-cocompletions* construction in Theorem 6.4.2, the idempotent-complete-presentables correspondence Theorem 6.5.4, as well as studying the various interactions between presentability with multiplicative matters and functor categories in §6.7 where, among other things, we prove a formula for the tensor product of parametrised presentable categories that was claimed in [Nar17] without proof.

We now comment on the methods and philosophy of this article. The approach taken here is an axiomatic one and is slightly different in flavour from the series of papers in [BDG+16a] in that we freely pass between the viewpoint of parametrised ∞-categories as cocartesian fibrations and as ∞-category-valued functors via the straightening-unstraightening equivalence of Lurie. This allows us to work model-independently, ie. without thinking of our ∞-categories as simplicial sets. The point is that, as far as presentability and adjunctions are concerned, the foundations laid in [BDG+16b; BGN14; Sha23; Sha22; Nar17] are sufficient for us to make model-independent formulations and proofs via universal properties. Indeed, a recurring trick in this paper is to say that relevant universal properties guarantee the existence of certain functors, and then we can just check that certain diagrams of ∞-categories commute by virtue of the essential uniqueness of left/right adjoints.

Outline of paper. In Sections 2 to 4 we collect all the background materials, together with references, that will be needed for the rest of the paper. We hope that this establishes notational consistency and makes the paper as self-contained as possible. We have denoted by "recollections" those subsections which contain mostly only statements that have appeared in the literature. We recommend the reader to skim this section on first reading and refer to it as necessary. In §5 we introduce the notions of $\mathcal{T}$-compactness and $\mathcal{T}$-idempotent-completeness.

We then come to the heart of the paper in §6 where we state and prove various basic results about parametrised presentable ∞-categories as enumerated above.

Conventions and assumptions. This paper is written in the language of ∞-categories and so from now on we will drop the adjective ∞- and mean ∞-categories when we say categories. Moreover, throughout the paper the base category $\mathcal{T}$ will be assumed to be *orbital* (cf. Definition 2.1.11) unless stated otherwise. We will also use the notation Cat for the category of small categories and $\widehat{\mathrm{Cat}}_\infty$ for the category of large categories.

Related work. Since the appearance of this article, Louis Martini and Sebastian Wolf have also independently produced many similar results in [MW22] using a different formalism of working internal to an ∞–topos. An important point of departure of this work from [MW22] is the the following: because this article is geared towards the applications we have in mind in [Hil22b], our setup crucially interacts well with the notion of parametrised symmetric monoidal structures (a.k.a. multiplicative norms), which to the best of our knowledge, is not yet formulated in their formalism at the present moment. Moreover, this work has also been incorporated as Chapters 1 and 2 of the author's PhD thesis [Hil22a].

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2 Preliminaries: general base categories

2.1 Recollections: basic objects and constructions

Recollections 2.1.1. For a category $\mathcal{T}$, there is Lurie's straightening-unstraightening equivalence $\mathrm{coCart}(\mathcal{T}^{\mathrm{op}}) \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ (cf. for example [HW21, Thm. I.23]). The category of *$\mathcal{T}$–categories* is then defined simply as $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ and we also write this as $\mathrm{Cat}_{\mathcal{T}}$. We will always denote a $\mathcal{T}$–category with an underline $\underline{\mathcal{C}}$. Under the equivalence above, the datum of a $\mathcal{T}$–category is equivalent to the datum of a cocartesian fibration $p\colon \mathrm{Total}(\underline{\mathcal{C}}) \to \mathcal{T}^{\mathrm{op}}$, and a $\mathcal{T}$–functor is defined just to be a morphism of $\mathcal{T}$–categories $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$, is then equivalently a map of cocartesian fibrations $\mathrm{Total}(\underline{\mathcal{C}}) \to \mathrm{Total}(\underline{\mathcal{D}})$ over $\mathcal{T}^{\mathrm{op}}$. For an object $V \in \mathcal{T}$, we will write $\mathcal{C}_V$ or $\underline{\mathcal{C}}_V$ for the fibre of $\mathrm{Total}(\underline{\mathcal{C}}) \to \mathcal{T}^{\mathrm{op}}$ over V.

Remark 2.1.2. The product $\underline{\mathcal{C}} \times \underline{\mathcal{D}}$ in $\mathrm{Cat}_{\mathcal{T}}$ of two $\mathcal{T}$–categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ is given as the pullback $\mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{\mathcal{D}})$ in the cocartesian fibrations perspective. We will always denote with $\times$ when we are viewing things as $\mathcal{T}$–categories and we reserve $\times_{\mathcal{T}^{\mathrm{op}}}$ for when we are viewing things as total categories. In this way, there will be no confusion as to whether or not $\times_{\mathcal{T}^{\mathrm{op}}}$ denotes a pullback in $\mathrm{Cat}_{\mathcal{T}}$: this will never be the case.

Notation 2.1.3. Since $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ is naturally even a 2–category, for $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$, we have the *category* of $\mathcal{T}$–functors from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$: this we write as $\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$. Unstraightening, we obtain $\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \mathrm{Fun}^{\mathrm{cocart}}(\mathrm{Total}(\underline{\mathcal{C}}), \mathrm{Total}(\underline{\mathcal{D}})) \times_{\mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}), \mathcal{T}^{\mathrm{op}})} \{p\}$ where $\mathrm{Fun}^{\mathrm{cocart}}$ is the full subcategory of functors preserving $\mathcal{T}^{\mathrm{op}}$–cocartesian morphisms.

Example 2.1.4. We now give some basic examples of $\mathcal{T}$–categories to set notation.

- (Fibrewise $\mathcal{T}$–categories) Let $K \in \mathrm{Cat}$. Write $\underline{\mathrm{const}}_{\mathcal{T}}(K) \in \mathrm{Cat}_{\mathcal{T}}$ for the constant K–valued diagram. In other words, $\mathrm{Total}(\underline{\mathrm{const}}_{\mathcal{T}}(K)) \simeq K \times \mathcal{T}^{\mathrm{op}}$.
- We write $\underline{*} \coloneqq \underline{\mathrm{const}}_{\mathcal{T}}(*)$. This is clearly a final object in $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$.
- (Corepresentable $\mathcal{T}$–categories) Let $V \in \mathcal{T}$. Then we can consider the left (and so cocartesian) fibration associated to the functor $\mathrm{Map}_{\mathcal{T}} \colon \mathcal{T}^{\mathrm{op}} \to \mathcal{S}$ and denote this $\mathcal{T}$–category by $\underline{V}$. Note that $\mathrm{Total}(\underline{V}) \simeq (\mathcal{T}_{/V})^{\mathrm{op}}$. By corepresentability of $\underline{V}$, we have $\mathrm{Fun}_{\mathcal{T}}(\underline{V}, \underline{\mathcal{C}}) \simeq \mathcal{C}_V$. To wit, for $K \in \mathrm{Cat}$, by Construction 2.1.13, we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cat}}\left(K, \mathrm{Fun}_{\mathcal{T}}(\underline{V}, \underline{\mathcal{C}})\right) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}\left(\underline{V}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}(K), \underline{\mathcal{C}})\right)\\ &\simeq \mathrm{Map}_{\mathrm{Cat}}(K, \mathcal{C}_V)\end{aligned}$$

Definition 2.1.5. The category of *$\mathcal{T}$–objects of $\underline{\mathcal{C}}$* is defined to be $\mathrm{Fun}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}})$.

Remark 2.1.6. If $\mathcal{T}^{\mathrm{op}}$ has an initial object $T \in \mathcal{T}^{\mathrm{op}}$, then this means that the category of $\mathcal{T}$–objects in $\underline{\mathcal{C}}$ is just $\mathcal{C}_T$.

Construction 2.1.7 (Parametrised opposites)**.** For a $\mathcal{T}$–category $\underline{\mathcal{C}}$, its *$\mathcal{T}$–opposite* $\underline{\mathcal{C}}^{\underline{\mathrm{op}}}$ is defined to be the image under the functor obtained by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to $(-)^{\mathrm{op}} \colon \mathrm{Cat} \to \mathrm{Cat}$. In the unstraightened view, this is given by taking fibrewise opposites in the total category. In [BDG+16b] this was called vertical opposites $(-)^{\mathrm{vop}}$ to invoke just such an impression.

Observation 2.1.8. Let $\underline{V}$ be a corepresentable $\mathcal{T}$–category. Then $\underline{V}^{\underline{\mathrm{op}}} \simeq \underline{V}$ since the functor $(-)^{\mathrm{op}} \colon \mathrm{Cat} \to \mathrm{Cat}$ restricts to the identity on $\mathcal{S}$.

Construction 2.1.9. The cone and cocone are functors $(-)^{\triangleleft}, (-)^{\triangleright} \colon \mathrm{Cat} \to \mathrm{Cat}$ which add a (co)cone point to a category. Applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to this functor yields the *$\mathcal{T}$–cone* and *–cocone* functors $(-)^{\underline{\triangleleft}}$ and $(-)^{\underline{\triangleright}}$ respectively. We refer to [Sha23] for more on this.

Definition 2.1.10. A $\mathcal{T}$–functor is *$\mathcal{T}$-fully faithful* (resp. *$\mathcal{T}$-equivalence*) if it is so fibrewise. There is the expected characterisation of $\mathcal{T}$–fully faithfulness in terms of $\mathcal{T}$–mapping spaces, see Remark 3.3.2.

Definition 2.1.11. We say that the category $\mathcal{T}$ is *orbital* if the finite coproduct cocompletion $\mathrm{Fin}_{\mathcal{T}}$ admits finite pullbacks. Here, by finite coproduct cocompletion, we mean the full subcategory of the presheaf category $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by finite coproduct of representables. We say that it is *atomic* if every retraction is an equivalence.

Notation 2.1.12 (Basechange)**.** As in [Nar17], we will write $\underline{\mathcal{C}}_{\underline{V}} \coloneqq \underline{\mathcal{C}} \times \underline{V} = \mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V})$ for the basechanged parametrised category, which is now viewed as a $\mathcal{T}_{/V}$–category. The $(-)_{\underline{V}}$ is a useful reminder that we have basechanged to V, and so for example we will often use the notation $\mathrm{Fun}_{\underline{V}}$ to mean $\mathrm{Fun}_{\mathcal{T}_{/V}}$ and *not* $\mathrm{Fun}_{\mathrm{Total}(\underline{V})} \simeq \mathrm{Fun}_{(\mathcal{T}_{/V})^{\mathrm{op}}}$.

Construction 2.1.13 (Internal $\mathcal{T}$–functor category, [BDG+16b, §9])**.** For $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$, there is a $\mathcal{T}$–category $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ such that

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{E}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}})$$

This is because $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ is presentable and the endofunctor $- \times \underline{\mathcal{C}}$ has a right adjoint since it preserves colimits. In particular, by a Yoneda argument we get $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) \simeq \underline{\mathcal{D}}$. Moreover, plugging in $\underline{\mathcal{E}} = \underline{*}$ we see that $\mathcal{T}$–objects of the internal $\mathcal{T}$–functor object are just $\mathcal{T}$–functors. Furthermore, the $\mathcal{T}$–functor categories basechange well in that

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{\underline{V}} \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}})$$

so the fibre over $V \in \mathcal{T}^{\mathrm{op}}$ is given by $\mathrm{Fun}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}})$. To wit, for any $\mathcal{T}_{/V}$–category $\underline{\mathcal{E}}$,

$$\begin{aligned}
\mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{\underline{V}}) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \\
&\simeq \mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}}_{\underline{V}}) \\
&\simeq \mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}} \times_{\underline{V}} \mathcal{C}_{\underline{V}}, \mathcal{D}_{\underline{V}}) \\
&\simeq \mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\underline{V}}(\mathcal{C}_{\underline{V}}, \mathcal{D}_{\underline{V}}))
\end{aligned}$$

Notation 2.1.14 (Parametrised cotensors)**.** Let I be a small unparametrised category. Then the adjunction $- \times I : \mathrm{Cat} \rightleftarrows \mathrm{Cat} : \mathrm{Fun}(I, -)$ induces the adjunction

$$(- \times I)_* : \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) \rightleftarrows \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) : \mathrm{Fun}(I, -)_*$$

Under the identification $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{Cat}_{\mathcal{T}}$ where $\mathrm{Cat}_{\mathcal{T}}$ is the category of $\mathcal{T}$-categories, it is clear that $(- \times I)_*$ corresponds to the $\mathcal{T}$-functor $\underline{\mathrm{const}}_{\mathcal{T}}(I) \times -$, whose right adjoint we know is $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(I), -)$. Therefore $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(I), -)$ implements the *fibrewise functor construction*. We will introduce the notation $\mathrm{fun}(I, -)$ for $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(I), -)$. This satisfies the following properties whose proofs are immediate.

1. $\underline{\mathrm{Cat}}_{\mathcal{T}}$ is cotensored over Cat in the sense that for any $\mathcal{T}$-categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ we have

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \mathrm{fun}(I, \underline{\mathcal{D}})) \simeq \mathrm{fun}(I, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$$

2. $\mathrm{fun}(I, -)$ preserves $\mathcal{T}$-adjunctions.

Observation 2.1.15*.* There is a natural equivalence of $\mathcal{T}$–categories

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})^{\underline{\mathrm{op}}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}^{\underline{\mathrm{op}}})$$

This is because $(-)^{\underline{\mathrm{op}}} : \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ is an involution, and so for any $\underline{\mathcal{E}} \in \mathrm{Cat}_{\mathcal{T}}$,

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})^{\underline{\mathrm{op}}}) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}}) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}} \times \underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}^{\underline{\mathrm{op}}}) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}^{\underline{\mathrm{op}}}))
\end{aligned}$$

Construction 2.1.16 (Cofree parametrisation, [Nar17, Def. 1.10])**.** Let $\mathcal{D}$ be a category. There is a $\mathcal{T}$–category $\underline{\mathrm{Cofree}}(\mathcal{D}) \colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}$ classified by $V \mapsto \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{D})$. This has the following universal property: if $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$, then there is a natural equivalence

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Cofree}}(\mathcal{D})) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}), \mathcal{D})$$

of ordinary ∞-categories. This construction is of foundational importance and it allows us to define the following two fundamental $\mathcal{T}$–categories.

Notation 2.1.17. We will write $\underline{\mathrm{Cat}}_{\mathcal{T}} \coloneqq \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat})$ for the *$\mathcal{T}$–category of $\mathcal{T}$–categories*; we write $\underline{\mathcal{S}}_{\mathcal{T}} \coloneqq \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathcal{S})$ for the *$\mathcal{T}$–category of $\mathcal{T}$–spaces.*

Theorem 2.1.18 (Parametrised straightening-unstraightening, [BDG+16b, Prop. 8.3])**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. Then there are equivalences*

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{coCart}(\mathrm{Total}(\underline{\mathcal{C}})) \qquad \mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Left}(\mathrm{Total}(\underline{\mathcal{C}}))$$

Proof. This is an immediate consequence of the usual straightening-unstraightening and the universal property of $\mathcal{T}$–categories of $\mathcal{T}$–objects above. For example,

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}), \mathrm{Cat}) \simeq \mathrm{coCart}(\mathrm{Total}(\underline{\mathcal{C}}))$$

and similarly for spaces. ∎

2.2 Parametrised adjunctions

$\mathcal{T}$–adjunctions as introduced in [Sha23] is based on the *relative adjunctions* of [Lur17].

Definition 2.2.1 ([Lur17, Def. 7.3.2.2])**.** Suppose we have diagrams of categories

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{G} & \mathcal{D} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{E} & \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{E} & \end{array}$$

Then we say that:

- For the first diagram, G admits a *left adjoint* F *relative to* $\mathcal{E}$ if G admits a left adjoint F such that for every $C \in \mathcal{C}$, q sends the unit $\eta : C \to GFC$ to an equivalence in $\mathcal{E}$ (equivalently, if $q\eta : q \Rightarrow p \circ F$ exhibits a commutation $p \circ F \simeq q$ by [Lur17, Prop. 7.3.2.1]).
- For the second diagram, F admits a *right adjoint* G *relative to* $\mathcal{E}$ if F admits a right adjoint G such that for every $D \in \mathcal{D}$, p maps the counit $\varepsilon : FGD \to D$ to an equivalence in $\mathcal{E}$ (equivalently if $p\varepsilon : q \circ G \Rightarrow p$ exhibits $q \circ G \simeq p$ by [Lur17, Prop. 7.3.2.1]).

Observe that when $\mathcal{E} \simeq *$, this specialises to the usual notion of adjunctions.

Remark 2.2.2. These two definitions are compatible. To see this, assume the first condition for example, ie. that G has a left adjoint F relative to $\mathcal{E}$. We need to see that F then admits a right adjoint G relative to $\mathcal{E}$ in the sense of the second condition, ie. that p sends the counit $\varepsilon : FGD \to D$ to an equivalence in $\mathcal{E}$. For this just consider the commutative diagram

$$\begin{array}{ccccc} qG & \xrightarrow[\simeq]{q\eta_G} & qGFG & \xrightarrow{\simeq} & pFG \\ & \searrow\!\!\!\searrow & \downarrow{\scriptstyle qG\varepsilon} & & \downarrow{\scriptstyle p\varepsilon} \\ & & qG & \xrightarrow{\simeq} & p \end{array}$$

where the triangle is by the adjunction, and the square is by the natural equivalence $qG \simeq p$.

Definition 2.2.3. Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then a $\mathcal{T}$*-adjunction* $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ is defined to be a relative adjunction such that F, G are $\mathcal{T}$–functors. A $\mathcal{T}$*-Bousfield localisation* is a $\mathcal{T}$–adjunction where the $\mathcal{T}$–right adjoint is $\mathcal{T}$–fully faithful.

Proposition 2.2.4 (Stability of relative adjunctions under pullbacks, [Lur17, Prop. 7.3.2.5])**.** *Suppose we have a relative adjunction*

$$\begin{array}{ccc} \mathcal{C} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathcal{D} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{E} & \end{array}$$

Then for any functor $\mathcal{E}' \to \mathcal{E}$ the diagram of pullbacks

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{E}} \mathcal{E}' & \underset{G}{\overset{F}{\rightleftarrows}} & \mathcal{D} \times_{\mathcal{E}} \mathcal{E}' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{E}' & \end{array}$$

is again a relative adjunction.

We now have the following criteria to obtain relative adjunctions - these are just modified from Lurie's more general assumptions.

Proposition 2.2.5 (Criteria for relative adjunctions, [Lur17, Prop. 7.3.2.6])**.** *Suppose $p : \mathcal{C} \to \mathcal{E}$, $q : \mathcal{D} \to \mathcal{E}$ are cocartesian fibrations. If we have a map of cocartesian fibrations F*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{E} & \end{array}$$

Then:

(1) F admits a right adjoint G relative to $\mathcal{E}$ if and only if for each $E \in \mathcal{E}$ the map of fibres $F_E : \mathcal{C}_E \to \mathcal{D}_E$ admits a right adjoint G_E. The right adjoint need no longer be a map of cocartesian fibrations.

(2) F admits a left adjoint L relative to $\mathcal{E}$ if and only if for each $E \in \mathcal{E}$ the map of fibres $F_E : \mathcal{C}_E \to \mathcal{D}_E$ admits a left adjoint L_E and the canonical comparison maps (constructed in [Lur17, Prop. 7.3.2.11])

$$Lf^* \to LFf^*L \xrightarrow{\varepsilon} f^*L$$

constructed from the fibrewise adjunction are equivalences - here f^* is the pushforward given by the cocartesian lift along some $f : E' \to E$ in $\mathcal{E}$. The relative left adjoint, if it exists, must necessarily be a map of cocartesian fibrations.

Proof. We prove each in turn. To see (1), suppose F has an $\mathcal{E}$-right adjoint G. Then for each $e \in \mathcal{E}$ the inclusion $\{e\} \hookrightarrow \mathcal{E}$ induces a pullback relative adjunction over the point $\{e\}$ by Proposition 2.2.4, and so we get the statement on fibres. Conversely, suppose we have fibrewise right adjoints. To construct an $\mathcal{E}$-right adjoint G, since adjunctions can be constructed objectwise by the unparametrised version of Proposition 3.3.9 below, we need to show that for each $e \in \mathcal{E}$ and $d \in \mathcal{D}_e$, there is a $Gd \in \mathcal{C}_e$ and a map $\varepsilon : FGd \to d$ such that:

(a) For every $c \in \mathcal{C}$ the following composition is an equivalence

$$\mathrm{Map}_{\mathcal{C}}(c, Gd) \xrightarrow{F} \mathrm{Map}_{\mathcal{D}}(Fc, FGd) \xrightarrow{\varepsilon} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$

(b) The morphism $p\varepsilon : pFGd \to pd$ is an equivalence in $\mathcal{E}$.

We can just define $Gd \coloneqq G_e(d) \in \mathcal{C}_e$ given by the fibrewise right adjoint and let $\varepsilon : FGd \to d$ be the fibrewise counit. Since these are fibrewise, point (b) is automatic. To see point (a), let $c \in \mathcal{C}_{e'}$ for some $e' \in \mathcal{E}$. Since the mapping space in the total category of cocartesian fibrations are just disjoint unions over the components lying under $\mathrm{Map}_{\mathcal{C}}(c, Gd)$, we can work over some $f \in \mathrm{Map}_{\mathcal{E}}(e', e)$. Consider

$$\begin{array}{ccc} \mathrm{Map}^f_{\mathcal{C}}(c, Gd) & \longrightarrow & \mathrm{Map}_{\mathcal{D}_e}(f^*Fc, d) \simeq \mathrm{Map}^f_{\mathcal{D}}(Fc, d) \\ \downarrow\simeq & \nearrow\simeq & \uparrow\varepsilon \\ \mathrm{Map}_{\mathcal{C}_e}(f^*c, Gd) & \xrightarrow{F} & \mathrm{Map}_{\mathcal{D}_e}(f^*Fc, FGd) \end{array}$$

where we have used also that F was a map of cocartesian fibrations so that $f^*F \simeq Ff^*$ and that the diagonal map is an equivalence since we had a fibrewise adjunction $F_e \dashv G_e$ by hypothesis. This completes the proof of part (1).

For case (2), to see the cocartesianness of a relative left adjoint L, note

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(Lf^*d, c) \simeq \mathrm{Map}_{\mathcal{D}}(f^*d, Fc) &\simeq \mathrm{Map}^f_{\mathcal{D}}(d, Fc) \\ &\simeq \mathrm{Map}^f_{\mathcal{C}}(Ld, c) \simeq \mathrm{Map}_{\mathcal{C}}(f^*Ld, c) \end{aligned}$$

The proof for right adjoints in (1) go through in this case but now we use

$$\begin{array}{ccc} \mathrm{Map}^f_{\mathcal{C}}(Ld,c) & \longrightarrow & \mathrm{Map}_{\mathcal{D}_E}(f^*d,Fc) \simeq \mathrm{Map}^f_{\mathcal{D}}(d,Fc) \\ \downarrow\simeq & & \simeq\uparrow \\ \mathrm{Map}_{\mathcal{C}_E}(f^*Ld,c) & \longrightarrow & \mathrm{Map}_{\mathcal{C}_E}(Lf^*d,c) \end{array}$$

and so the technical condition in the statement says that there is a canonical map inducing the bottom map in the square which must necessarily be an equivalence. □

Remark 2.2.6. One might object to the notation we have adopted for the pushforward being f^* instead of $f_!$. This convention is standard in the framework of [BDG+16a] because the latter notation is reserved for the *left adjoint* of f^* (the so-called $\mathcal{T}$–coproducts) that will be recalled later.

Corollary 2.2.7 (Fibrewise criteria for $\mathcal{T}$–adjunctions)**.** *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor. Then it admits a $\mathcal{T}$–right adjoint if and only if it has fibrewise right adjoints G_V for all $V \in \mathcal{T}$ and*

$$\begin{array}{ccc} \mathcal{C}_W & \xleftarrow{G_W} & \mathcal{D}_W \\ f^*\uparrow & & \uparrow f^* \\ \mathcal{C}_V & \xleftarrow[G_V]{} & \mathcal{D}_V \end{array}$$

commutes for all $f : W \to V$ in $\mathcal{T}$. Similarly for left $\mathcal{T}$–adjoints.

Proof. The commuting square ensures that the relative right adjoint is a $\mathcal{T}$–functor. □

Proposition 2.2.8 (Criteria for $\mathcal{T}$–Bousfield localisations, "[Lur09, Prop. 5.2.7.4]")**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and $L : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ a $\mathcal{T}$–functor equipped with a fibrewise natural transformation $\eta : \mathrm{id} \Rightarrow L$. Let $j : L\underline{\mathcal{C}} \subseteq \underline{\mathcal{C}}$ be the inclusion of the $\mathcal{T}$–full subcategory spanned by the image of L. Suppose the transformations $L\eta, \eta_L : L \Longrightarrow L \circ L$ are equivalences. Then the pair (L, j) constitutes a $\mathcal{T}$–Bousfield localisation with unit η.*

Proof. We want to apply Corollary 2.2.7. Since we are already provided with the fact that L was a $\mathcal{T}$–functor, all that is left to show is that it is fibrewise left adjoint to the inclusion $L\underline{\mathcal{C}} \subseteq \underline{\mathcal{C}}$. But this is guaranteed by [Lur09, Prop. 5.2.7.4], and so we are done. □

Finally, we show that parametrised adjunctions have the expected internal characterisation in terms of the parametrised mapping spaces recalled in Construction 3.3.1.

Lemma 2.2.9 (Mapping space characterisation of $\mathcal{T}$–adjunctions)**.** *Let $F : \underline{\mathcal{C}} \leftrightarrows \underline{\mathcal{D}} : G$ be a pair of $\mathcal{T}$–functors. Then there is a $\mathcal{T}$–adjunction $F \dashv G$ if and only if we have a natural equivalence*

$$\underline{\mathrm{Map}_{\mathcal{D}}}(F-,-) \simeq \underline{\mathrm{Map}_{\mathcal{C}}}(-,G-) : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{D}} \longrightarrow \underline{\mathcal{S}}_{\mathcal{T}}$$

Proof. The if direction is clear: since F and G were already $\mathcal{T}$–functors, by Corollary 2.2.7 the only thing left to do is to show fibrewise adjunction, and this is easily implied by the equivalence which supplies the unit and counits. For the only if direction, by definition of a relative adjunction, we have a fibrewise natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ (ie. a morphism in $\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{C}})$) and so we obtain a natural comparison

$$\underline{\mathrm{Map}_{\mathcal{D}}}(F-,-) \xrightarrow{G} \underline{\mathrm{Map}_{\mathcal{C}}}(GF-,G-) \xrightarrow{\eta^*} \underline{\mathrm{Map}_{\mathcal{C}}}(-,G-)$$

Since equivalences between $\mathcal{T}$–functors are checked fibrewise, let $c \in \mathcal{C}_V, d \in \mathcal{D}_V$. Then

$$\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F-,-): \quad (c,d) \mapsto \quad \Big((W \xrightarrow{f} V) \mapsto (\mathrm{Map}_{\mathcal{D}_V}(Fc,d) \to \mathrm{Map}_{\mathcal{D}_W}(f^*Fc, f^*d)\Big) \in \underline{\mathcal{S}}_{\underline{V}}$$
$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-,G-): \quad (c,d) \mapsto \quad \Big((W \xrightarrow{f} V) \mapsto (\mathrm{Map}_{\mathcal{C}_V}(c,Gd) \to \mathrm{Map}_{\mathcal{C}_W}(f^*c, f^*Gd)\Big) \in \underline{\mathcal{S}}_{\underline{V}}$$

Since F, G were $\mathcal{T}$–functors, we have $Ff^* \simeq f^*F$ and $Gf^* \simeq f^*G$, and so the natural comparison coming from the relative adjunction unit given above exhibits a pointwise equivalence between $\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F-,-)$ and $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-,G-)$ by Corollary 2.2.7. □

3 Preliminaries: orbital base categories

Some of the notions here still make sense for general $\mathcal{T}$, but we want orbitality in order to make formulations involving Beck-Chevalley conditions. Hence, from now on, we assume that $\mathcal{T}$ is orbital.

3.1 Recollections: colimits and Kan extensions

Definition 3.1.1. Let $\underline{K} \in \mathrm{Cat}_{\mathcal{T}}$ and $q\colon \underline{K} \to \underline{*}$ be the unique map. Then precomposition induces the $\mathcal{T}$–functor $q^* : \underline{\mathcal{D}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathcal{D}})$. The $\mathcal{T}$–left adjoint $q_!$, if it exists, is called the $\underline{K}$–indexed *$\mathcal{T}$–colimit*, and similarly for *T-limits* q_*.

Example 3.1.2. Here are some special and important classes of these:

- A $\mathcal{T}$–(co)limit indexed by $\underline{\mathrm{const}}_{\mathcal{T}}(K)$ for some ordinary ∞–category K is called a *fibrewise $\mathcal{T}$-(co)limit.*
- A $\mathcal{T}$–(co)limit indexed by a corepresentable $\mathcal{T}$–category $\underline{V}$ (cf. Example 2.1.4) of some $V \in \mathcal{T}$ is called the *T-(co)product.*

Definition 3.1.3 ([Sha23, Def. 11.2])**.** Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor.

- We say that it *preserves $\mathcal{T}$-colimits* if for all $\mathcal{T}$–colimit diagrams $\overline{d} : K^{\trianglerighteq} \to \underline{\mathcal{C}}$, the postcomposed diagram $F \circ \overline{d} : K^{\trianglerighteq} \to \underline{\mathcal{D}}$ is a $\mathcal{T}$–colimit. Similarly for $\mathcal{T}$–limits.
- We say that F *strongly preserves $\mathcal{T}$-colimits* if for all $V \in \mathcal{T}$, $F_{\underline{V}} : \underline{\mathcal{C}}_{\underline{V}} \to \underline{\mathcal{D}}_{\underline{V}}$ preserves $\mathcal{T}_{/V}$–colimits. Similarly for $\mathcal{T}$–limits.

Warning 3.1.4 ([Sha23, Rmk. 5.14]). Note that being $\mathcal{T}$–cocomplete is much stronger than just admitting all $\mathcal{T}$–colimits. This is because admitting all $\mathcal{T}$–colimits just means that any $\mathcal{T}_{/V}$–diagram $\underline{K}_{\underline{V}} \to \underline{\mathcal{C}}_{\underline{V}}$ pulled back from a $\mathcal{T}$–diagram $\underline{K} \to \underline{\mathcal{C}}$ admits a $\mathcal{T}_{/V}$–colimit. However not every $\mathcal{T}_{/V}$–diagram is pulled back as such. We will elaborate on the distinction of these definitions in the next subsection. In this document, we will never consider preservations, but only strong preservations.

Definition 3.1.5 ([Sha23, Def. 5.13])**.** Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. Then we say $\underline{\mathcal{C}}$ is *$\mathcal{T}$–(co)complete* if for all $V \in \mathcal{T}$ and $\mathcal{T}_{/V}$–diagram $p : \underline{K} \to \underline{\mathcal{C}}_{\underline{V}}$ with $\underline{K}$ small, p admits a $\mathcal{T}_{/V}$–(co)limit.

Terminology 3.1.6. When we want to specify particular kinds of parametrised (co)limits that a $\mathcal{T}$–category admits, it is convenient to use the following terminology: for $\mathcal{K} = \{\mathcal{K}_V\}_{V \in \mathcal{T}}$ some collection of diagrams varying over $V \in \mathcal{T}$, we say that $\underline{\mathcal{C}}$ *strongly admits* $\mathcal{K}$–(co)limits if for all $V \in \mathcal{T}$, $\underline{\mathcal{C}}_{\underline{V}}$ admits $\underline{K}$–colimits for all $\underline{K} \in \mathcal{K}_V$. Examples include:

- $\underline{\mathcal{C}}$ strongly admits all $\mathcal{T}$–(co)limits means that it is $\mathcal{T}$–(co)cocomplete,
- Let κ be a regular cardinal. We say that $\underline{\mathcal{C}}$ strongly admits κ–small $\mathcal{T}$–(co)products to mean that it has $\mathcal{T}$–(co)limits for any diagram indexed over $\coprod_{a \in A} \underline{V}_a$ where A is a κ–small set. Hence, strongly admitting finite $\mathcal{T}$–(co)products means admitting finite fibrewise (co)products and (co)limits for all corepresentable diagrams $\underline{V}$.

Lemma 3.1.7 (Decomposition of indexed coproducts). *Let $R_a, V \in \mathcal{T}$ and $\coprod_a f_a : \coprod_a R_a \to V$ be a map where the coproduct is not necessarily finite. Suppose $\underline{\mathcal{C}}$ strongly admits finite $\mathcal{T}$–coproducts and arbitrary fibrewise coproducts. Then $\underline{\mathcal{C}}$ admits $\coprod_a f_a$-coproducts and this is computed by composing fibrewise $\mathcal{T}$–coproduct $\coprod_a$ with the individual indexed $\mathcal{T}$–coproducts.*

Proof. We will in fact show that we have $\mathcal{T}_{/V}$–adjunctions

$$\underline{\mathrm{Fun}}_{\underline{V}}(\textstyle\coprod_a \underline{R_a}, \underline{\mathcal{C}}_{\underline{V}}) = \prod_a \underline{\mathrm{Fun}}_{\underline{V}}(\underline{R_a}, \underline{\mathcal{C}}_{\underline{V}}) \underset{\prod_a (f_a)^*}{\overset{\prod_a (f_a)_!}{\rightleftarrows}} \prod_a \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}}_{\underline{V}}) \underset{\Delta}{\overset{\coprod_a}{\rightleftarrows}} \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}}_{\underline{V}})$$

That these $\mathcal{T}_{/V}$–adjunctions exist is by our hypotheses, and all that is left to do is check that $\prod_a (f_a)^* \circ \Delta \simeq (\coprod_a f_a)^*$. But this is also clear since we have the commuting diagram

$$\begin{array}{ccc} \coprod_a \underline{R_a} & \xrightarrow{\coprod_a f_a} & \underline{V} \\ {\scriptstyle f_a}\downarrow & \nearrow_{\coprod_a \mathrm{id}_V} & \\ \coprod_a \underline{V} & & \end{array}$$

Applying $(-)^*$ to this triangle completes the proof. □

Terminology 3.1.8 (Beck-Chevalley conditions). Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ that admits finite fibrewise coproducts (resp. products) and such that for each $f : W \to V$ in $\mathcal{T}$, $f^* : \mathcal{C}_V \to \mathcal{C}_W$ admits a left adjoint $f_!$ (resp. right adjoint f_*). We say that $\underline{\mathcal{C}}$ satisfies the *left Beck-Chevalley condition* (resp. *right Beck-Chevalley condition*) if for every pair of morphisms $f : W \to V$ and $g : Y \to V$ in $\mathcal{T}$: in the pullback (whose orbital decomposition exists by orbitality of $\mathcal{T}$)

$$\begin{array}{ccc} \coprod_a R_a = Y \times_V W & \xrightarrow{\coprod_a f_a} & Y \\ {\scriptstyle \coprod_a g_a}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ W & \xrightarrow[f]{} & V \end{array}$$

the canonical basechange transformation

$$\coprod_a g_{a!} f_a^* \Longrightarrow f^* g_! \quad \Big(\text{resp. } f^* g_* \Longrightarrow \prod_a g_{a*} f_a^*\Big)$$

is an equivalence.

Here is an omnibus of results due to Jay Shah.

Theorem 3.1.9 ([Sha23, 5.5-5.12 and §12]). *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. Then:*

(1) (Fibrewise criterion) $\underline{\mathcal{C}}$ strongly admits $\mathcal{T}$–colimits indexed by $\underline{\mathrm{const}}_{\mathcal{T}}(K)$ if and only if for every $V \in \mathcal{T}$ the fibre $\mathcal{C}_V$ has all colimits indexed by K and for every morphism $f : W \to V$ in $\mathcal{T}$ the cocartesian lift $f^* : \mathcal{C}_V \to \mathcal{C}_W$ preserves colimits indexed by K. A cocone diagram $\overline{p} : \underline{\mathrm{const}}_{\mathcal{T}}(K)^{\underline{\triangleright}} \to \underline{\mathcal{C}}$ is a $\mathcal{T}$–colimit if and only if it is so fibrewise.

(1) (T-coproducts criteria) $\underline{\mathcal{C}}$ strongly admits finite $\mathcal{T}$–coproducts if and only if we have:

(a) For every $W \in \mathcal{T}$ the fibre $\mathcal{C}_W$ has all finite coproducts and for every $f : W \to V$ in $\mathcal{T}$ the map $f^* : \mathcal{C}_V \to \mathcal{C}_W$ preserves finite coproducts,

(b) $\mathcal{C}$ satisfies the left Beck-Chevalley condition (cf. Terminology 3.1.8).

(3) (Decomposition principle) $\underline{\mathcal{C}}$ is $\mathcal{T}$–cocomplete if and only if it has all fibrewise colimits and strongly admits finite $\mathcal{T}$–coproducts.

Similar statements hold for $\mathcal{T}$–limits, and the right adjoint to f^* will be denoted f_*.

Theorem 3.1.10 (Omnibus $\mathcal{T}$–adjunctions, [Sha23, §8])**.** *Let $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ be a $\mathcal{T}$–adjunction and $\underline{I}$ be a $\mathcal{T}$–category. Then:*

(1) We get adjunctions $F_* \colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{D}}) : G_*$, $G^* \colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{I}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{I}) : F^*$. By Corollary 2.2.7 this implies ordinary adjunctions when we replace $\underline{\mathrm{Fun}}_{\mathcal{T}}$ by $\mathrm{Fun}_{\mathcal{T}}$.

(2) F strongly preserves $\mathcal{T}$–colimits and G strongly preserves $\mathcal{T}$–limits.

Proof. In [Sha23, Cor. 8.9], part (2) was stated only as ordinary preservation, not strong preservation. But then strong preservation was implicit since relative adjunctions are stable under pullbacks by Proposition 2.2.4, and the statement in [Sha23] also holds after pulling back to $- \times \underline{V}$ for all $V \in \mathcal{T}$. ☐

Proposition 3.1.11 ($\mathcal{T}$–cocompleteness of Bousfield local subcategories)**.** *If $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : j$ is a $\mathcal{T}$–Bousfield localisation where $\underline{\mathcal{C}}$ is $\mathcal{T}$–cocomplete, then $\underline{\mathcal{D}}$ is too and $\mathcal{T}$–colimits in $\underline{\mathcal{D}}$ is computed as L applied to the $\mathcal{T}$–colimit computed in $\underline{\mathcal{C}}$.*

Proof. This is an immediate consequence of Lemma 2.2.9. ☐

Proposition 3.1.12 ($\mathcal{T}$–(co)limits of functor categories is pointwise)**.** *Let $\underline{K}, \underline{I}, \underline{\mathcal{C}}$ be $\mathcal{T}$–categories. Suppose $\underline{\mathcal{C}}$ strongly admits $\underline{K}$–indexed diagrams. Then so does $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}})$ and the parametrised (co)limits are inherited from that of $\underline{\mathcal{C}}$.*

Proof. This is a direct consequence of the adjunction $(\underline{\mathrm{colim}}_{\underline{K}})_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}})) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathcal{C}})) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}}) : \mathrm{const}$ for $\mathcal{T}$–colimits. The other case is similar. ☐

Definition 3.1.13 ($\mathcal{T}$–Kan extensions)**.** Let $j : \underline{I} \to \underline{K}$ be a $\mathcal{T}$–functor. If $j^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{D}})$ has a $\mathcal{T}$–left adjoint, then we denote it by $j_!$ and call it the *T-left Kan extension*. Similarly for $\mathcal{T}$–right Kan extensions.

Proposition 3.1.14 (Fully faithful $\mathcal{T}$–Kan extensions, [Sha23, Prop. 10.6])**.** *Let $i : \underline{\mathcal{C}} \hookrightarrow \underline{\mathcal{D}}$ be a $\mathcal{T}$–fully faithful functor and $F : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ be another $\mathcal{T}$–functor. If the $\mathcal{T}$–left Kan extension $i_! F$ exists, then the adjunction unit $F \Rightarrow i^* i_! F \colon \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ is an equivalence.*

Theorem 3.1.15 (Omnibus $\mathcal{T}$–Kan extensions, [Sha23, Thm. 10.5])**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ be $\mathcal{T}$–cocomplete. Then for every $\mathcal{T}$–functor of small $\mathcal{T}$–categories $f : \underline{I} \to \underline{K}$, the $\mathcal{T}$–left Kan extension $f_! : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathcal{C}})$ exists.*

3.2 Strong preservation of $\mathcal{T}$-colimits

We now explain in more detail the notion of strong preservation. In particular, the reader may find Proposition 3.2.2 to be a convenient alternative description, and we will have many uses of it in the coming sections.

Observation 3.2.1 (Strong preservations vs preservations). Here are some comments for the distinction. Proposition 3.2.2 will then characterise strong preservations more concretely.

(1) Recall Warning 3.1.4 that admitting $\mathcal{T}$–colimits is weaker than being $\mathcal{T}$–cocomplete. In the proof of the Lurie-Simpson characterisation Theorem 6.1.2, we will see that we really need $\mathcal{T}$–cocompleteness via Proposition 3.2.2.

(2) However, $\underline{\mathcal{C}}$ admitting $\mathcal{T}$–colimits indexed by $p : \underline{K} \to \mathcal{T}^{\mathrm{op}}$ does imply $\underline{\mathcal{C}}_{\underline{V}}$ admits $\mathcal{T}_{/V}$–colimits indexed by $\underline{K}_{\underline{V}}$. This is because the adjunction $p_! \colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}}) : p^*$ pulls back to the $p_! \colon \underline{\mathrm{Fun}}_{\underline{V}}(\underline{K}_{\underline{V}}, \underline{\mathcal{C}}_{\underline{V}}) \rightleftarrows \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}}_{\underline{V}}) : p^*$ adjunction by Proposition 2.2.4. We have also used that functor $\mathcal{T}$–categories basechange well by Construction 2.1.13.

(3) Strongly preserving fibrewise $\mathcal{T}$–(co)limits is equivalent to preserving these (co)limits on each fibre since by Theorem 3.1.9 fibrewise (co)limits are constructed fibrewise.

The following result was also recorded in the recent [Sha22, Thm. 8.6].

Proposition 3.2.2 (Characterisation of strong preservations)**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$–cocomplete categories and $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ a $\mathcal{T}$–functor. Then F strongly preserves $\mathcal{T}$–colimits if and only if it preserves colimits in each fibre and for all $f : W \to V$ in $\mathcal{T}$, the following square commutes (and similarly for $\mathcal{T}$–limits)*

$$\begin{array}{ccc} \mathcal{C}_W & \xrightarrow{f_!} & \mathcal{C}_V \\ {\scriptstyle F_W}\downarrow & & \downarrow{\scriptstyle F_V} \\ \mathcal{D}_W & \xrightarrow{f_!} & \mathcal{D}_V \end{array}$$

Proof. To see the only if direction, that F preserves colimits in each fibre is clear since F in particular preserves fibrewise $\mathcal{T}$–colimits. Now for $f : W \to V$, we basechange to $\underline{V}$. Since F strongly preserves $\mathcal{T}$–colimits, we get commutative squares

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V}) & \xrightarrow{f_!} & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}} \times \underline{V}) \\ {\scriptstyle (F_{\underline{V}})_*}\downarrow & & \downarrow{\scriptstyle (F_{\underline{V}})_*} \\ \underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{D}} \times \underline{V}) & \xrightarrow{f_!} & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{D}} \times \underline{V}) \end{array}$$

Taking global sections by using that $\mathrm{Fun}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V}) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{W}, \underline{\mathcal{C}}) \simeq \mathcal{C}_W$ from Example 2.1.4, we get the desired square.

For the if direction, we know by Theorem 3.1.9 that all $\mathcal{T}$–colimits can be decomposed as fibrewise $\mathcal{T}$–colimits and indexed $\mathcal{T}$–coproducts, and so if we show strong preservation of these we would be done. By Observation 3.2.1 (3) strong preservation of fibrewise $\mathcal{T}$–colimits is the same as preserving colimits in each fibre, so this case is covered. Since arbitrary indexed $\mathcal{T}$–coproducts are just compositions of orbital $\mathcal{T}$–coproducts and arbitrary fibrewise coproducts by Lemma 3.1.7, we need only show for orbital $\mathcal{T}$–coproducts, so let $f : W \to V$ be a morphism in $\mathcal{T}$. We need to show that the canonical comparison in

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V}) & \xrightarrow{f_!} & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}} \times \underline{V}) \\ {\scriptstyle (F_{\underline{V}})_*}\downarrow & & \downarrow{\scriptstyle (F_{\underline{V}})_*} \\ \underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{D}} \times \underline{V}) & \xrightarrow{f_!} & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{D}} \times \underline{V}) \end{array}$$

is a natural equivalence. Since equivalences is by definition a fibrewise notion, we can check this on each fibre. So let $\varphi : Y \to V$ be in $\mathcal{T}$, and consider the pullback

$$\begin{array}{ccc} \coprod_a R_a & \xrightarrow{\coprod f_a} & Y \\ {\scriptstyle \coprod \varphi_a}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi} \\ W & \xrightarrow{f} & V \end{array}$$

by orbitality of $\mathcal{T}$. We need to show that

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V})_Y & \xrightarrow{f_!} & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}} \times \underline{V})_Y \\ {\scriptstyle (F_{\underline{V}})_*}\downarrow & & \downarrow{\scriptstyle (F_{\underline{V}})_*} \\ \underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{D}} \times \underline{V})_Y & \xrightarrow{f_!} & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{D}} \times \underline{V})_Y \end{array}$$

commutes. But then by the universal property of the internal functor $\mathcal{T}$–categories from Construction 2.1.13, this is the same as

$$\begin{array}{ccc} \mathrm{Fun}_{\underline{Y}}(\coprod_a \underline{R_a}, \underline{\mathcal{C}}\times\underline{Y}) \simeq \mathrm{Fun}_{\underline{Y}}(\underline{W}\times_{\underline{V}}\underline{Y}, \underline{\mathcal{C}}\times\underline{Y}) & \xrightarrow{f_!} & \mathrm{Fun}_{\underline{Y}}(\underline{Y}, \underline{\mathcal{C}}\times\underline{Y}) \\ \downarrow{\scriptstyle (F_{\underline{Y}})_*} & & \downarrow{\scriptstyle (F_{\underline{Y}})_*} \\ \mathrm{Fun}_{\underline{Y}}(\coprod_a \underline{R_a}, \underline{\mathcal{D}}\times\underline{Y}) \simeq \mathrm{Fun}_{\underline{Y}}(\underline{W}\times_{\underline{V}}\underline{Y}, \underline{\mathcal{D}}\times\underline{Y}) & \xrightarrow{f_!} & \mathrm{Fun}_{\underline{Y}}(\underline{Y}, \underline{\mathcal{D}}\times\underline{Y}) \end{array}$$

and this is in turn

$$\begin{array}{ccc} \prod_a \mathcal{C}_{R_a} & \xrightarrow{\coprod (f_a)_!} & \mathcal{C}_Y \\ \downarrow{\scriptstyle \prod F_a} & & \downarrow{\scriptstyle F_V} \\ \prod_a \mathcal{D}_{R_a} & \xrightarrow[\coprod (f_a)_!]{} & \mathcal{D}_Y \end{array}$$

which commutes by hypothesis together with that F commutes with fibrewise $\mathcal{T}$–colimits (and so in particular finite fibrewise coproducts). This finishes the proof of the result. ☐

3.3 Recollections: mapping spaces and Yoneda

Construction 3.3.1 (Parametrised mapping spaces and Yoneda, [BDG+16b, Def. 10.2])**.** Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category. Then the $\mathcal{T}$–twisted arrow construction gives us a left $\mathcal{T}$–fibration

$$(s,t) : \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}}) \longrightarrow \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}$$

$\mathcal{T}$–straightening this via Theorem 2.1.18 we get a $\mathcal{T}$–functor

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}} : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{S}}_{\mathcal{T}}$$

By [BGN14, §5] we know that $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-,-) : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ is given on fibre over V by the map $\mathcal{C}_V^{\mathrm{op}} \times \mathcal{C}_V \to \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{S})$

$$(c,c') \mapsto \Big((W \xrightarrow{f} V) \mapsto (\mathrm{Map}_{\mathcal{C}_V}(c,c') \to \mathrm{Map}_{\mathcal{C}_W}(f^*c, f^*c')\Big)$$

Moreover, by currying we obtain the $\mathcal{T}$–Yoneda embedding

$$j : \underline{\mathcal{C}} \longrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) = \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$$

which on level $V \in \mathcal{T}$ is given by

$$\begin{aligned} j_V : \mathcal{C}_V \hookrightarrow \mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}) &\hookrightarrow \mathrm{Fun}_{\underline{V}}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{V}, \underline{\mathcal{S}}_{\underline{V}}) \\ &\simeq \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}), \mathcal{S}) \end{aligned}$$

Remark 3.3.2. By the explicit fibrewise description of the parametrised mapping spaces above, we see immediately that a $\mathcal{T}$–functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is $\mathcal{T}$–fully faithful if and only if it induces equivalences on $\underline{\mathrm{Map}}(-,-)$.

Lemma 3.3.3 ($\mathcal{T}$–Yoneda Lemma, [BDG+16b, Prop. 10.3])**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category and let $X \in \mathcal{C}_V$ for some $V \in \mathcal{T}$. Then for any $\mathcal{T}_{/V}$–functor $F : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{V} \longrightarrow \underline{\mathcal{S}}_{\underline{V}}$, we have an equivalence of $\mathcal{T}_{/V}$–spaces*

$$F(X) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})}\big(j_V(X),\ F\big)$$

In particular, the $\mathcal{T}$–Yoneda embedding $j : \underline{\mathcal{C}} \longrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is $\mathcal{T}$–fully faithful.

Proof. First of all note that the V-fibre Yoneda map above factors as

$$\begin{array}{ccc} \mathcal{C}_V & \xrightarrow{\quad j_V \quad} & \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}), \mathcal{S}) \\ \big\downarrow & \nearrow_{\widetilde{j}_V} & \\ (\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}))^{\mathrm{op}} & & \end{array}$$

This already gives that j_V is fully faithful, and so by definition of parametrised fully faithfulness, the $\mathcal{T}$–yoneda functor $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is $\mathcal{T}$–fully faithful. On the other hand, by the universal property of the $\mathcal{T}$–category of $\mathcal{T}$–objects from Construction 2.1.16, we can regard F as an ordinary functor $F : \mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}) \to \mathcal{S}$. And so by ordinary Yoneda we get

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}_{\underline{V}}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{V}, \underline{\mathcal{S}}_{\underline{V}})}\big(j_V(X),\ F\big) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}), \mathcal{S})}\big(j_V(X),\ F\big) \\ &\simeq F(X) \in \mathcal{S} \end{aligned}$$

as required. □

Theorem 3.3.4 (Continuity of $\mathcal{T}$–Yoneda, [Sha23, Cor. 11.10])**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. The $\mathcal{T}$–yoneda embedding $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ strongly preserves and detects $\mathcal{T}$–limits.*

Corollary 3.3.5. *Let $f : V \to W$ be a map in $\mathcal{T}$. Let $B \in \mathcal{C}_V$, $X \in \mathcal{C}_W$, and $f_! \dashv f^* \dashv f_*$. Then*

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{W}}}(f_!B, X) \simeq f_*\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(B, f^*X) \in \underline{\mathcal{S}}_{\underline{W}}$$

Proof. Applying Theorem 3.3.4 on $\underline{\mathcal{C}}^{\mathrm{op}}$, we see that

$$\underline{\mathcal{C}}^{\mathrm{op}} \hookrightarrow \underline{\mathrm{Fun}}(\underline{\mathcal{C}}, \underline{\mathcal{S}}) \quad :: \quad A \mapsto \underline{\mathrm{Map}}_{\underline{\mathcal{C}}^{\mathrm{op}}}(-, A) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(A, -)$$

strongly preserves $\mathcal{T}$–limits. Hence, since f_* in $\underline{\mathcal{C}}^{\mathrm{op}}$ is given by $f_!$ in $\underline{\mathcal{C}}$, we see that

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{W}}}(f_!B, -) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{W}}{}^{\mathrm{op}}}(-, f_!B) \simeq f_*\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}{}^{\mathrm{op}}}(-, B) \simeq f_*\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(B, -)$$

as required. □

Theorem 3.3.6 ($\mathcal{T}$–Yoneda density, [Sha23, Lem. 11.1])**.** *Let $j : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the $\mathcal{T}$–yoneda embedding. Then $\mathrm{id}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})} \simeq j_!j$, that is, everything in the $\mathcal{T}$–presheaf is a $\mathcal{T}$–colimit of representables.*

Theorem 3.3.7 (Universal property of $\mathcal{T}$–presheaves, [Sha23, Thm. 11.5])**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$ and suppose $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete. Then the precompositions $j^* : \mathrm{Fun}^L_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \longrightarrow \mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ and $j^* : \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are equivalences with the inverse given by left Kan extensions. Here $\mathrm{Fun}^L_{\mathcal{T}}$ means those functors which strongly preserve $\mathcal{T}$–colimits (cf. Notation 3.3.10).*

We learnt of the following useful procedure from Fabian Hebestreit.

Definition 3.3.8 (Adjoint objects)**.** Let $R : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ be a $\mathcal{T}$–functor. Let $x \in \underline{\mathcal{C}}$ and $y \in \underline{\mathcal{D}}$ and $\eta : x \to R(y)$. We say that η *witnesses y as a left adjoint object to x under R* if

$$\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(y, -) \xrightarrow{R} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(Ry, R-) \xrightarrow{\eta^*} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(x, R-)$$

is an equivalence of $\mathcal{T}$–functors $\underline{\mathcal{D}} \to \underline{\mathcal{S}}_{\mathcal{T}}$.

The following observation, due to Lurie, is quite surprising for ∞-categories: adjunctions can be constructed objectwise, ie. to check that we have an adjunction, it is enough to construct a left adjoint object for each object.

Proposition 3.3.9 (Pointwise construction of adjunctions)**.** *$R : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ admits a left adjoint $L : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ if and only if all objects in $\underline{\mathcal{C}}$ admits a left adjoint object. More generally, writing $\underline{\mathcal{C}}_R$ for the full subcategory of objects admitting left adjoint objects, we obtain a $\mathcal{T}$–functor $L : \underline{\mathcal{C}}_R \to \underline{\mathcal{D}}$ that is $\mathcal{T}$–left adjoint to the restriction of $R : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ to the subcategory of $\underline{\mathcal{D}}$ landing in $\underline{\mathcal{C}}_R$.*

Proof. The trick is to use the $\mathcal{T}$–Yoneda lemma to help us assemble the various left adjoint objects into a coherent $\mathcal{T}$–functor. We consider $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-, R-) : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{D}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ as a $\mathcal{T}$–functor $H : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}})$. Hence by definition of $\underline{\mathcal{C}}_R$, the bottom left composition lands in the essential image of the Yoneda embedding and so we obtain a lift L^{op} in the commuting square

$$\begin{array}{ccc} \underline{\mathcal{C}}_R^{\mathrm{op}} & \xrightarrow{L^{\mathrm{op}}} & \underline{\mathcal{D}}^{\mathrm{op}} \\ \big\downarrow & & \big\downarrow y \\ \underline{\mathcal{C}}^{\mathrm{op}} & \xrightarrow{H} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}}) \end{array}$$

To see that when $\underline{\mathcal{C}}_R = \underline{\mathcal{C}}$, we get a $\mathcal{T}$–left adjoint, note that by construction $y \circ L^{\mathrm{op}} \simeq H$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}}, \mathrm{Fun}(\underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}}))$, and hence $\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(L-, -) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-, R-)$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}})$. By the characterisation of $\mathcal{T}$–adjunctions from Lemma 2.2.9, we are done. ☐

Notation 3.3.10. We write $\underline{\mathrm{RFun}}_{\mathcal{T}}$ (resp. $\underline{\mathrm{LFun}}_{\mathcal{T}}$) for the $\mathcal{T}$–full subcategories in $\underline{\mathrm{Fun}}_{\mathcal{T}}$ of $\mathcal{T}$–right adjoint functors (resp. $\mathcal{T}$–left adjoint functors). This is distinguished from the notations $\underline{\mathrm{Fun}}^R_{\mathcal{T}}$ (resp. $\underline{\mathrm{Fun}}^L_{\mathcal{T}}$) by which we mean the $\mathcal{T}$–full subcategories in $\underline{\mathrm{Fun}}_{\mathcal{T}}$ of strongly $\mathcal{T}$–limit-preserving functors (resp. strongly $\mathcal{T}$–colimit-preserving functors).

Proposition 3.3.11 ("[Lur09, Prop. 5.2.6.2]")**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then there is a canonical equivalence $\underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})^{\mathrm{op}}$.*

Proof. Let $j : \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ be the $\mathcal{T}$–Yoneda embedding. Then the $\mathcal{T}$–functor

$$j_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$$

which is $\mathcal{T}$–fully faithful by Corollary 3.4.6 has essential image consisting of those parametrised functors $\varphi : \underline{\mathcal{C}} \times \underline{\mathcal{D}}^{\mathrm{op}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ such that for all $c \in \underline{\mathcal{C}}$, $\varphi(c, -) : \underline{\mathcal{D}}^{\mathrm{op}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ is representable. The essential image under j_* of $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ will then be those parametrised functors as above which moreover satisfy that for all $d \in \underline{\mathcal{D}}$, $\varphi(-, d) : \underline{\mathcal{C}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ is corepresentable - this is since $\mathcal{T}$–adjunctions can be constructed objectwise by Proposition 3.3.9. Let $\underline{\mathcal{E}} \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ be the $\mathcal{T}$–full subcategory spanned by those functors satisfying these two properties, so that $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \mathcal{E}$.

On the other hand, repeating the above for $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{C}}^{\mathrm{op}})$ gives

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{C}}^{\mathrm{op}}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}} \times \underline{\mathcal{C}}, \underline{\mathcal{S}}_{\mathcal{T}})$$

where the essential image of $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{C}}^{\mathrm{op}})$ will be precisely those that satisfy the two properties, and so also $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{C}}^{\mathrm{op}}) \xrightarrow{\simeq} \underline{\mathcal{E}}$. Thus, combining with $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{C}}^{\mathrm{op}}) \simeq \underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})^{\mathrm{op}}$ from Observation 2.1.15, we obtain the desired result. ☐

3.4 (Full) faithfulness

In this subsection we provide the parametrised analogue of the Lurie-Thomason formula for limits in categories, Theorem 3.4.2, as well as show that parametrised functor categories preserve (fully) faithfulness in Corollary 3.4.6.

Notation 3.4.1. For $p : \underline{\mathcal{C}} \to \underline{I}$ a $\mathcal{T}$–functor which is also a cocartesian fibration, we will write $\underline{\Gamma}^{\text{cocart}}_{\mathcal{T}}(p)$ for the $\mathcal{T}$–category of cocartesian sections of p. In other words, it is the $\mathcal{T}$–category $\underline{\text{Fun}}^{\text{cocart}}_{T}(\underline{I}, \underline{\mathcal{C}}) \times_{\underline{\text{Fun}}_{\mathcal{T}}(\underline{I},\underline{I})} \underline{*}$ where $\underline{\text{Fun}}^{\text{cocart}}_{T}(\underline{I}, \underline{\mathcal{C}})$ means the full $\mathcal{T}$–subcategory of those that parametrised functors that preserve cocartesian morphisms over $\underline{I}$, and the $\mathcal{T}$–functor $\underline{*} \to \underline{\text{Fun}}_{\mathcal{T}}(\underline{I}, \underline{I})$ is the section corresponding to the identity on $\underline{I}$.

The following proof is just a parametrisation of the unparametrised proof that we learnt from [HW21, Prop. I.36].

Theorem 3.4.2 (Lurie-Thomason formula)**.** *Given a $\mathcal{T}$–diagram $F : \underline{I} \to \underline{\text{Cat}}_{\mathcal{T}}$, we get*

$$\underline{\lim}_{\underline{I}} F \simeq \underline{\Gamma}^{\underline{I}-\text{cocart}}_{T}(\text{UnStr}^{\text{cocart}}(F))$$

In particular, if it factors through $F : \underline{I} \to \underline{\mathcal{S}}_{\mathcal{T}}$, then we have $\underline{\lim}_{\underline{I}} F \simeq \underline{\Gamma}_{T}(\text{UnStr}^{\text{cocart}}(F))$.

Proof. Let $d : \underline{I} \to \underline{*}$ be the unique map. Since $\underline{\text{Cat}}_{\mathcal{T}}$ has all $\mathcal{T}$–limits, we know abstractly that we have the $\mathcal{T}$–right adjoint

$$d^* : \underline{\text{Cat}}_{\mathcal{T}} \rightleftarrows \underline{\text{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\text{Cat}}_{\mathcal{T}}) : d_* =: \underline{\lim}_{\mathcal{T}}$$

so now we just need to understand the fibrewise right adjoint formula (by virtue of Corollary 2.2.7). Without loss of generality, we work with global sections and we want to describe the right adjoint in

$$d^* : \text{Fun}(\mathcal{T}^{\text{op}}, \text{Cat}) \rightleftarrows \text{Fun}_{\mathcal{T}}(\underline{I}, \underline{\text{Cat}}_{\mathcal{T}}) \simeq \text{Fun}(\text{Total}(\underline{I}), \text{Cat}) : d_*$$

We can now identify d^* concretely via the straightening-unstraightening equivalence to get $d^* : \text{coCart}(\mathcal{T}^{\text{op}}) \to \text{coCart}(\text{Total}(\underline{I}))$ given by

$$\underline{\mathcal{C}} \mapsto \big(\pi_{\mathcal{C}} : \text{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\text{op}}} \text{Total}(\underline{I}) \to \text{Total}(\underline{I})\big)$$

Let $(p : \mathcal{E}_F \to \text{Total}(\underline{I})) := \text{UnStr}^{\text{coCart}}(F)$ be the cocartesian fibration associated to $F : \text{Total}(\underline{I}) \to \text{Cat}$. We need to show that $\underline{\Gamma}^{\underline{I}-\text{cocart}}_{T}(p)$ satisfies a natural equivalence

$$\text{Map}_{\text{coCart}(\text{Total}(\underline{I}))}(\pi_{\mathcal{C}}, p) \simeq \text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})}(\underline{\mathcal{C}}, \underline{\Gamma}^{\underline{I}-\text{cocart}}_{T}(p))$$

for all $\underline{\mathcal{C}} \in \text{coCart}(\mathcal{T}^{\text{op}})$. First of all, by definition we have the pullback

$$\begin{array}{ccc}
\text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})_{/\text{Total}(\underline{I})}}(\pi_{\mathcal{C}}, p) & \longrightarrow & \text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})}(\text{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\text{op}}} \text{Total}(\underline{I}), \mathcal{E}_F) \\
\downarrow & \lrcorner & \downarrow \\
* & \xrightarrow{\pi_{\mathcal{C}}} & \text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})}(\text{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\text{op}}} \text{Total}(\underline{I}), \text{Total}(\underline{I}))
\end{array}$$

which by currying is the same as the pullback

$$\begin{array}{ccc}
\text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})_{/\text{Total}(\underline{I})}}(\pi_{\mathcal{C}}, p) & \longrightarrow & \text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})}(\underline{\mathcal{C}}, \underline{\text{Fun}}_{\mathcal{T}}(\underline{I}, \mathcal{E}_F)) \\
\downarrow & \lrcorner & \downarrow \\
* & \xrightarrow{\text{id}_{\underline{I}}} & \text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})}(\underline{\mathcal{C}}, \underline{\text{Fun}}_{\mathcal{T}}(\underline{I}, \underline{I}))
\end{array}$$

Now recall that $\text{Map}_{\text{coCart}(\text{Total}(\underline{I}))}(\pi_{\mathcal{C}}, p) \subseteq \text{Map}_{\text{coCart}(\mathcal{T}^{\text{op}})_{\text{Total}(\underline{I})}}(\pi_{\mathcal{C}}, p)$ consists precisely of those components of functors over $\text{Total}(\underline{I})$ (in the left diagram)

$$\begin{array}{ccc}
\text{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\text{op}}} \text{Total}(\underline{I}) & \longrightarrow & \mathcal{E}_F \\
& \searrow^{\pi_{\mathcal{C}}} & \downarrow p \\
& & \text{Total}(\underline{I})
\end{array}
\quad = \quad
\begin{array}{ccc}
\underline{\mathcal{C}} & \longrightarrow & \underline{\text{Fun}}_{\mathcal{T}}(\underline{I}, \mathcal{E}_F) \\
& \searrow^{\pi_{\mathcal{C}}} & \downarrow p \\
& & \underline{\text{Fun}}_{\mathcal{T}}(\underline{I}, \underline{I})
\end{array}$$

preserving cocartesian morphisms over $\mathrm{Total}(\underline{I})$. Since the cocartesian morphisms in the cocartesian fibration $\pi_{\mathcal{C}} : \mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{I}) \to \mathrm{Total}(\underline{I})$ are precisely the morphisms of $\mathrm{Total}(\underline{I})$ and an equivalence in $\mathcal{C}$, we see that this condition corresponds in the curried version on the right to those functors landing in $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(\underline{I}, \mathcal{E}_F)$. Finally for the statement about the case of factoring over $\underline{\mathcal{S}}_{\mathcal{T}}$ recall that unstraightening brings us to left fibrations $\mathcal{E}_F \to \mathrm{Total}(\underline{I})$, and since in left fibrations all morphisms are cocartesian, we need not have imposed the condition above. This shows us that we have a bijection of components

$$\pi_0 \, \mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\pi_{\mathcal{C}}, p) \simeq \pi_0 \, \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(\underline{\mathcal{C}}, \underline{\Gamma}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(p))$$

We now need to show that this would already imply that we have an equivalence of mapping *spaces*. For this, we will need to first construct a map of spaces realising the bijection above. First note that we have a map of cocartesian fibrations over $\mathrm{Total}(\underline{I})$

$$\varepsilon : \underline{\Gamma}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(p) \times \underline{I} \longrightarrow \mathcal{E}_F$$

from the evaluation. Therefore we get the following maps of spaces

$$\begin{aligned} &\mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(-, \underline{\Gamma}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(p)) \\ &\xrightarrow{\underline{I}\times -} \mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\underline{I} \times -, \underline{I} \times \underline{\Gamma}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(p)) \\ &\xrightarrow{\varepsilon_*} \mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\underline{I} \times -, \mathcal{E}_F) \end{aligned} \tag{1}$$

On the other hand, we know by the pullback definition of $\underline{\Gamma}_{\mathcal{T}}$ that

$$\mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(-, \underline{\Gamma}_{\mathcal{T}}(p)) \simeq \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})_{/\,\mathrm{Total}(\underline{I})}}(\underline{I} \times -, \mathcal{E}_F) \tag{2}$$

and so the comparison map (1) is induced by this equivalence. Our bijection on components then gives that the equivalence (2) restricts to an equivalence of spaces (1). This completes the proof of the result. ☐

As far as we are aware the following proof strategy first appeared in [GHN17, §5].

Proposition 3.4.3 (Mapping space formula in $\mathcal{T}$–functor categories)**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$ and $F, G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be $\mathcal{T}$–functors. Then we have an equivalence of $\mathcal{T}$–spaces*

$$\underline{\mathrm{Nat}}_T(F, G) \simeq \underline{\lim}_{(x\to y)\in \underline{\mathrm{TwAr}}_T(\underline{\mathcal{C}})} \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F(x), G(y)) \in \underline{\mathcal{S}}_{\mathcal{T}}$$

Proof. Recall from [BDG+16b, §10] that by definition, the parametrised mapping spaces are classified by the parametrised twisted arrow categories. By Theorem 3.4.2 we have

$$\underline{\lim}_{(x\to y)\in \underline{\mathrm{TwAr}}(\underline{\mathcal{C}})} \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F(x), G(y)) \simeq \underline{\Gamma}_{\mathcal{T}}\big(\underline{P} \to \underline{\mathrm{TwAr}}_T(\underline{\mathcal{C}})\big)$$

where $p : \underline{P} \to \underline{\mathrm{TwAr}}_T(\underline{\mathcal{C}})$ is the associated unstraightening. By considering the pullbacks

$$\begin{array}{ccccccc} \underline{P} & \longrightarrow & \underline{P}' & \longrightarrow & \underline{\mathrm{TwAr}}_T(\underline{\mathcal{D}}) & & \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \\ \underline{\mathrm{TwAr}}_T(\underline{\mathcal{C}}) & \xrightarrow[(s,t)]{} & \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}} & \xrightarrow[F^{\mathrm{op}}\times G]{} & \underline{\mathcal{D}}^{\mathrm{op}} \times \underline{\mathcal{D}} & \xrightarrow[\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(-,-)]{} & \underline{\mathcal{S}}_{\mathcal{T}} \end{array}$$

we get that

$$\underline{\Gamma}_{\mathcal{T}}\big(\underline{P} \to \underline{\mathrm{TwAr}}_T(\underline{\mathcal{C}})\big) \simeq \underline{\mathrm{Map}}_{/\underline{\mathcal{C}}^{\mathrm{op}}\times\underline{\mathcal{C}}}(\underline{\mathrm{TwAr}}_T(\underline{\mathcal{C}}), \underline{P}')$$

Now by the parametrised straightening of Theorem 2.1.18 we see furthermore that

$$\underline{\mathrm{Map}}_{/\underline{\mathcal{C}}^{\mathrm{op}}\times\underline{\mathcal{C}}}(\underline{\mathrm{TwAr}}_T(\underline{\mathcal{C}}), \underline{P}') \simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}, \underline{\mathrm{Map}}_{\underline{\mathcal{D}}} \circ (F^{\mathrm{op}} \times G)\big)$$

Currying $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}, \underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}))$ we see that

$$\underline{\mathrm{Nat}}_{\mathcal{T}}\big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}, \underline{\mathrm{Map}}_{\underline{\mathcal{D}}} \circ (F^{\underline{\mathrm{op}}} \times G)\big) \simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(y_{\underline{\mathcal{C}}}, F^* \circ y_{\underline{\mathcal{D}}} \circ G)\big)$$

But then now we have the sequence of equivalences

$$\begin{aligned}\underline{\mathrm{Nat}}_{\mathcal{T}}\big(y_{\underline{\mathcal{C}}}, F^* \circ y_{\underline{\mathcal{D}}} \circ G)\big) &\simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(F_! \circ y_{\underline{\mathcal{C}}}, y_{\underline{\mathcal{D}}} \circ G)\big)\\ &\simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(y_{\underline{\mathcal{D}}} \circ F, y_{\underline{\mathcal{D}}} \circ G)\big)\\ &\simeq \underline{\mathrm{Nat}}_{T}(F, G)\end{aligned}$$

where the last equivalence is by Lemma 3.3.3 and the second by the square

$$\begin{array}{ccc} \underline{\mathcal{C}} & \xrightarrow{\quad F \quad} & \underline{\mathcal{D}} \\ {\scriptstyle y_{\underline{\mathcal{C}}}}\big\downarrow & & \big\downarrow{\scriptstyle y_{\underline{\mathcal{D}}}} \\ \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) & \xrightarrow{F_!} & \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}) \end{array}$$

which commutes by functoriality of presheaves. □

Definition 3.4.4. A $\mathcal{T}$–functor is called $\mathcal{T}$-faithful if it is so fibrewise, where an ordinary functor is called faithful if it induces component inclusions on mapping spaces.

Observation 3.4.5. For $f : X \to Y$ a map of spaces, it being an inclusion of components is equivalent to the condition that for each $x \in X$, the fibre $\mathrm{fib}_{f(x)}\big(X \to Y\big)$ is contractible. On the other hand, it is an equivalence if and only if for each $y \in Y$, the fibre $\mathrm{fib}_y\big(f : X \to Y\big)$ is contractible. We learnt of this formulation and of the following proof in the unparametrised case from [Lei22, Appendix B].

Corollary 3.4.6. *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–(fully) faithful functor and $\underline{I}$ another $\mathcal{T}$–category. Then $F_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{D}})$ is again $\mathcal{T}$–(fully) faithful.*

Proof. Since $\mathcal{T}$–(fully) faithfulness was defined as a fibrewise condition, we just assume without loss of generality that $\mathcal{T}$ has a final object and work on global sections. In the faithful case, let $\varphi, \psi : \underline{I} \to \underline{\mathcal{C}}$ be two $\mathcal{T}$–functors. We need to show that

$$\underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi, \psi) \longrightarrow \underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}})}(F\varphi, F\psi)$$

is an inclusion of components. By the preceeding observation, we need to show that for each $\eta \in \mathrm{Nat}_{\mathrm{Fun}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi, \psi)$, the fibre

$$\underline{\mathrm{fib}}_{\eta}\big(\underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi, \psi) \to \underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}})}(F\varphi, F\psi)\big) \in \Gamma(\underline{\mathcal{S}}_{\mathcal{T}} \to \mathcal{T}^{\mathrm{op}})$$

is contractible. But then we are now in position to use Proposition 3.4.3:

$$\begin{aligned}&\underline{\mathrm{fib}}_{\eta}\Big(\underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi, \psi) \to \underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}})}(F\varphi, F\psi)\Big)\\ &\simeq \underline{\lim}_{(x\to y)\in \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{I})} \underline{\mathrm{fib}}_{\eta}\Big(\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(\varphi(x), \psi(y)) \to \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F\varphi(x), G\psi(y))\Big)\\ &\simeq \underline{\lim}_{(x\to y)\in \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{I})} *_{\mathcal{T}} \simeq *_{T}\end{aligned}$$

as was to be shown, where the second last step is by our hypothesis that F was $\mathcal{T}$–faithful. The case of $\mathcal{T}$–fully faithfulness can be done similarly. □

3.5 Recollections: filtered colimits and Ind-completions

Construction 3.5.1. Let κ be a regular cardinal. We define the $\mathcal{T}$*-Ind-completion* functor $\underline{\mathrm{Ind}}_\kappa \colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ to be the one obtained by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to the ordinary functor $\mathrm{Ind}_\kappa \colon \mathrm{Cat} \to \mathrm{Cat}$.

Notation 3.5.2. We will write $\underline{\mathrm{Fun}}^{\mathrm{filt}}_{\mathcal{T}}$ for the full $\mathcal{T}$–subcategory of parametrised functors preserving fibrewise ω–filtered colimits, and similarly $\underline{\mathrm{Fun}}^{\kappa\text{-filt}}_{\mathcal{T}}$ for those that preserve fibrewise κ-filtered colimits.

Remark 3.5.3. This agrees with the definition given in the recent paper [Sha22] by virtue of the paragraph after Theorem D therein. As indicated there, $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$ is the minimal $\mathcal{T}$–subcategory of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ generated by $\underline{\mathcal{C}}$ under fibrewise κ-filtered colimits. In more detail, [Sha22, Rmk. 9.4] showed that the fibrewise presheaf construction $\underline{\mathrm{PSh}}^{\mathrm{fb}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is a $\mathcal{T}$–full subcategory of the $\mathcal{T}$–presheaf $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ via the fibrewise left Kan extension. In particular, this means that $\underline{\mathrm{PSh}}^{\mathrm{fb}}_{\mathcal{T}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ preserves fibrewise colimits. On the other hand, by construction and [Lur09, Cor. 5.3.5.4], $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}^{\mathrm{fb}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the minimal $\mathcal{T}$–subcategory generated by $\underline{\mathcal{C}}$ under fibrewise κ-filtered colimits. Therefore, in total, we see that $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the $\mathcal{T}$–subcategory generated by $\mathcal{C}$ under fibrewise κ-filtered colimits.

Proposition 3.5.4 (Universal property of $\underline{\mathrm{Ind}}$, "[Lur09, Prop. 5.3.5.10]")**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$–categories where $\underline{\mathcal{C}}$ is small and $\underline{\mathcal{D}}$ has fibrewise small κ-filtered colimits. Then:*

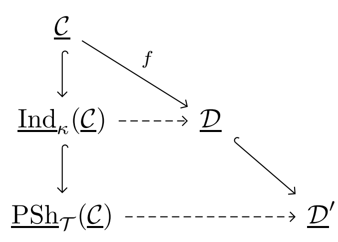
(1) $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the $\mathcal{T}$–subcategory generated by $\mathcal{C}$ under fibrewise κ-filtered colimits.

(2) The $\mathcal{T}$–inclusion $i : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$ induces an equivalence

$$i^* : \underline{\mathrm{Fun}}^{\kappa\text{-filt}}_{\mathcal{T}}(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

Proof. Part (1) is by the remark above. For part (2), we show that the $\mathcal{T}$–left Kan extension functor $i_! : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}^{\kappa\text{-filt}}_{\mathcal{T}}(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}), \underline{\mathcal{D}})$ exists and is an inverse to i^*. To do this, it will be enough to show that functors $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ can be $\mathcal{T}$–left Kan extended to $i_!F : \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}$ and that functors $F : \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}$ which preserves fibrewise κ-filtered colimits satisfy that $i_!i^*F \Rightarrow F$ is an equivalence. This will be enough since we would have shown the natural equivalence $i_!i^* \simeq \mathrm{id}$, and Proposition 3.1.14 gives that $i^*i_! \simeq \mathrm{id}$ always.

To show that the $\mathcal{T}$–left Kan extension exists, consider the diagram



where $\underline{\mathcal{D}} \subseteq \underline{\mathcal{D}}'$ is a strongly $\mathcal{T}$–colimit preserving inclusion into a $\mathcal{T}$–cocomplete $\underline{\mathcal{D}}'$ using the opposite $\mathcal{T}$–Yoneda embedding. In particular by hypothesis $\underline{\mathcal{D}}$ is closed under κ-filtered colimits in $\underline{\mathcal{D}}'$. The bottom dashed map is gotten from Theorem 3.3.7, and so strongly preserves $\mathcal{T}$–colimits. Hence restriction to $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$ lands in $\underline{\mathcal{D}}$ so we get middle dashed map, and by the following Lemma 3.5.5, this is a left Kan extension.

Now we show that if F preserves fibrewise κ-filtered colimits, then the canonical comparison $i_!i^*F \Rightarrow F$ is an equivalence. Again, by Proposition 3.1.14 we know that both sides agree on $\underline{\mathcal{C}} \subseteq \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$. Also, both sides preserve κ-filtered colimits by assumption. Hence, by statement (1) of the proposition, we see that it must be an equivalence as was to be shown. ∎

Lemma 3.5.5. *Suppose we have fully faithful functors* $\underline{\mathcal{C}} \overset{i}{\hookrightarrow} \underline{\mathcal{D}} \overset{j}{\hookrightarrow} \underline{\mathcal{E}}$ *and functors* $\underline{\mathcal{C}} \xrightarrow{f} \underline{\mathcal{A}} \overset{y}{\hookrightarrow} \underline{\mathcal{B}}$*, where* $\underline{\mathcal{B}}$ *is* $\mathcal{T}$*–(co)complete. Suppose we have a factorisation* $j^*j_!i_!(y \circ f) \colon \underline{\mathcal{C}} \xrightarrow{\overline{f}} \underline{\mathcal{A}} \overset{y}{\hookrightarrow} \underline{\mathcal{B}}$*. Then* $\overline{f} \simeq i_! f \colon \underline{\mathcal{D}} \to \underline{\mathcal{A}}$*.*

Proof. Let $\varphi \colon \underline{\mathcal{D}} \to \underline{\mathcal{A}}$. We need to show that $\underline{\mathrm{Nat}}(\overline{f}, \varphi) \simeq \underline{\mathrm{Nat}}(f, i^*\varphi)$. We compute:

$$\begin{aligned}
\underline{\mathrm{Nat}}(\overline{f}, \varphi) &\simeq \underline{\mathrm{Nat}}(y \circ \overline{f}, y \circ \varphi) \\
&= \underline{\mathrm{Nat}}(j^*j_!i_!(y \circ f), y \circ \varphi) \\
&\simeq \underline{\mathrm{Nat}}(y \circ f, i^*j^*j_*(y \circ \varphi)) \\
&\simeq \underline{\mathrm{Nat}}(y \circ f, i^*(y \circ \varphi) \simeq \underline{\mathrm{Nat}}(f, i^*\varphi)
\end{aligned}$$

where the first and last equivalences are since y was fully faithful; the fourth equivalence is since j was fully faithful and so Proposition 3.1.14 applies. The relevant Kan extensions exist since $\underline{\mathcal{B}}$ was assumed to be $\mathcal{T}$–(co)complete. □

We learnt of the following proof method from Markus Land.

Lemma 3.5.6. *For* $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$*, we have a functor* $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$ *that takes* $F : \mathcal{C} \to \mathcal{D}$ *to* $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$*.*

Proof. We know that $\mathrm{Ind}(\mathcal{E} \times \mathcal{C}) \simeq \mathrm{Ind}(\mathcal{E}) \times \mathrm{Ind}(\mathcal{C})$. In particular, we get functors

$$\Delta^n \times \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\Delta^n \times \mathcal{C}) \simeq \mathrm{Ind}(\Delta^n) \times \mathrm{Ind}(\mathcal{C})$$

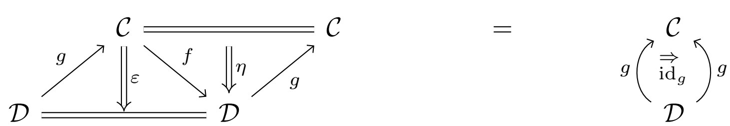
natural in both Δ^n and $\mathcal{C}$. These then induce a map of simplicial spaces

$$\mathrm{Fun}(\Delta^\bullet \times \mathcal{C}, \mathcal{D})^\simeq \longrightarrow \mathrm{Fun}(\mathrm{Ind}(\Delta^\bullet \times \mathcal{C}), \mathrm{Ind}(\mathcal{D}))^\simeq \longrightarrow \mathrm{Fun}(\Delta^\bullet \times \mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))^\simeq$$

where the first map is just by the $(\infty, 1)$–functoriality of Ind. Via the complete Segal space model of ∞-categories, we see that we have the desired functor which behaves as in the statement by looking at the case $\bullet = 0$. □

Lemma 3.5.7 (Ind adjunctions)**.** *Let* $f : \mathcal{C} \rightleftarrows \mathcal{D} : g$ *be an adjunction. Then we also have an adjunction* $F \coloneqq \mathrm{Ind}(f) : \mathrm{Ind}(\mathcal{C}) \rightleftarrows \mathrm{Ind}(\mathcal{D}) : \mathrm{Ind}(g) =: G$*.*

Proof. By [RV19, Def. 1.1.2] we know that such an adjunction is tantamount to the data of $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow gf$ and $\varepsilon : fg \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that we have the triangle identities



and the analogous other triangle. Now, we have $\mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ by Lemma 3.5.6 and so the the triangle identity on the source gets sent to a triangle identity on the target. □

Theorem 3.5.8 (Diagram decomposition, [Sha22, Thm. 8.1])**.** *Let* $\underline{\mathcal{C}}$ *be a* $\mathcal{T}$*–category,* J *a category, and* $p_\bullet : J \to (\mathrm{Cat}_{\mathcal{T}})_{/\underline{\mathcal{C}}}$ *a functor with colimit the* $\mathcal{T}$*–functor* $p : \underline{K} \to \underline{\mathcal{C}}$ *and suppose that for all* $j \in J$*, the* $\mathcal{T}$*–functor* $p_j : \underline{K}_j \to \underline{\mathcal{C}}$ *admits a* $\mathcal{T}$*–colimit* σ_j*. Then the* σ_j*'s assemble to a* $\mathcal{T}$*–functor* $\sigma_\bullet : \underline{\mathrm{const}}_{\mathcal{T}}(J) \to \underline{\mathcal{C}}$ *so that if* $\sigma_\bullet$ *admits a* $\mathcal{T}$*–colimit* σ*, then* p *admits a* $\mathcal{T}$*–colimit given by* σ*.*

Corollary 3.5.9 (Parametrised filtered colimit decomposition, "[Lur09, Cor. 4.2.3.11]")**.** *Let* $\tau \ll \kappa$ *be regular cardinals and* $\underline{\mathcal{C}}$ *be a* $\mathcal{T}$*–category admitting* τ*-small* $\mathcal{T}$*–colimits and fibrewise colimits indexed by* κ*-small* τ*-filtered posets. Then for any* κ*-small* $\mathcal{T}$*–diagram* $d : \underline{K} \to \underline{\mathcal{C}}$*, its* $\mathcal{T}$*–colimit in* $\underline{\mathcal{C}}$ *exists and can be decomposed as a fibrewise* κ*-small* τ*-filtered colimit whose vertices are* τ*-small* $\mathcal{T}$*–colimits of* $\underline{\mathcal{C}}$*.*

Proof. Let J denote the poset of τ-small $\mathcal{T}$–subcategories of $\underline{K}$. It is clearly τ-filtered and moreover it is κ-small by the hypothesis that $\tau \ll \kappa$. We can therefore apply the theorem above since the associated $\sigma_\bullet : \underline{\mathrm{const}}_{\mathcal{T}}(J) \to \underline{\mathcal{C}}$ will admit a $\mathcal{T}$–colimit by hypothesis. $\square$

Theorem 3.5.10 (Limit-filtered colimit exchange, special case of [Sha22, Thm. C])**.** *Let κ be a regular cardinal and J a κ–filtered category. Then $\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(J)} : \underline{\mathrm{Fun}}(\underline{\mathrm{const}}_{\mathcal{T}}(J), \underline{\mathcal{S}}_{\mathcal{T}}) \longrightarrow \underline{\mathcal{S}}_{\mathcal{T}}$ strongly preserves $\mathcal{T}$–κ–small $\mathcal{T}$–colimits.*

4 Preliminaries: atomic orbital base categories

Finally, we begin to impose the strictest conditions on our base category $\mathcal{T}$. From here on, $\mathcal{T}$ will be assumed to be both orbital *and* atomic.

4.1 Recollections: parametrised semiadditivity and stability

In this subsection we recall the algebraic constructions and results of [Nar16; Nar17].

Construction 4.1.1. The following list of constructions will be important in discussing $\mathcal{T}$–semiadditivity and $\mathcal{T}$–stability. See [Nar16, §4] for the original source on these constructions or [NS22, Def. 2.1.2] for a more recent treatment. Note that we have adopted the notation of Span instead of the original notation of effective Burnside categories A^{eff}.

(1) Write $\mathrm{Span}(\mathcal{T}) \coloneqq \mathrm{Span}(\mathrm{Fin}_{\mathcal{T}})$.

(2) By [Nar17, Cons. 4.8], there is a $\mathcal{T}$–category $p : \underline{\mathrm{Span}}(T) \to \mathcal{T}^{\mathrm{op}}$ whose objects are morphisms $[U \to V]$ in $\mathrm{Fin}_{\mathcal{T}}$ where $V \in \mathcal{T}$ and the cocartesian fibration p sends $[U \to V]$ to V. The morphisms in this category are spans

$$\begin{array}{ccccc} U & \longleftarrow & W & \longrightarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & V' & = & V' \end{array} \tag{3}$$

(3) From this we obtain a wide $\mathcal{T}$–subcategory $\underline{\mathrm{Fin}}_{*\mathcal{T}} \subset \underline{\mathrm{Span}}(T)$ whose morphisms are spans as in (3) such that the map $W \to U \times_V V'$ in $\mathrm{Fin}_{\mathcal{T}}$ is a summand inclusion: this makes sense since $\mathcal{T}$ was assumed to be orbital and so $\mathrm{Fin}_{\mathcal{T}}$ admits the pullback $U \times_V V'$ which will be a finite coproduct of objects of V.

(4) There is a canonical inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$ given by sending $W \to V$ to

$$\begin{array}{ccccc} V & \longleftarrow & W & = & W \\ \| & & \| & & \| \\ V & \longleftarrow & W & = & W \end{array}$$

Definition 4.1.2. Let $\underline{\mathcal{C}}$ strongly admit finite $\mathcal{T}$–coproducts and $\underline{\mathcal{D}}$ strongly admit finite $\mathcal{T}$–products. Then we say that a $\mathcal{T}$–functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is $\mathcal{T}$*-semiadditive* if it sends finite $\mathcal{T}$–coproducts to finite $\mathcal{T}$–products. We say that a $\mathcal{T}$–category $\underline{\mathcal{C}}$ strongly admitting finite $\mathcal{T}$–products and $\mathcal{T}$–coproducts is $\mathcal{T}$*–semiadditive* if the identity functor is $\mathcal{T}$–semiadditive. If moreover $\underline{\mathcal{C}}$ has fibrewise pushouts and $\underline{\mathcal{D}}$ has fibrewise pullbacks, then we say that F is $\mathcal{T}$–*linear* if it is $\mathcal{T}$–semiadditive and sends fibrewise pushouts to fibrewise pullbacks. We write $\underline{\mathrm{Fun}}^{\mathrm{sadd}}_{T}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ (resp. $\underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$) for the $\mathcal{T}$–full subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ consisting of the $\mathcal{T}$–semiadditive functors (resp. $\mathcal{T}$–linear functors).

Notation 4.1.3. For $\underline{\mathcal{C}}$ strongly admitting finite $\mathcal{T}$–limits we will denote $\mathcal{T}$*–Mackey functors* by $\underline{\mathrm{Mack}}_{\mathcal{T}}(\underline{\mathcal{C}}) := \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Span}}(T), \underline{\mathcal{C}})$ and $\mathcal{T}$*–commutative monoids* by $\underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}}) := \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}})$.

Proposition 4.1.4 ($\mathcal{T}$–semiadditivisation, [Nar16, Prop. 5.11])**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category strongly admitting finite $\mathcal{T}$–products. Then the functor $\underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ induced by the inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$ from Construction 4.1.1 (4) is an equivalence if and only if $\underline{\mathcal{C}}$ were $\mathcal{T}$–semiadditive.*

Theorem 4.1.5 ("CMon = Mackey", [Nar16, Thm. 6.5])**.** *Let $\underline{\mathcal{C}}$ strongly admit finite $\mathcal{T}$–limits. Then the defining inclusion $j : \underline{\mathrm{Fin}}_{*\mathcal{T}} \hookrightarrow \underline{\mathrm{Span}}(T)$ induces an equivalence*

$$j^* : \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Span}}(T), \underline{\mathcal{C}}) \longrightarrow \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}})$$

Notation 4.1.6. We write $\mathrm{Fun}^{\mathrm{ex}}_{\mathcal{T}}, \mathrm{Fun}^{\mathrm{lex}}_{\mathcal{T}}$, and $\mathrm{Fun}^{\mathrm{rex}}_{\mathcal{T}}$ for the category of $\mathcal{T}$–functors which strongly preserve finite $\mathcal{T}$–(co)limits, strongly preserve finite $\mathcal{T}$–limits, and strongly preserve finite $\mathcal{T}$–colimits, respectively.

Construction 4.1.7. Let $\underline{\mathrm{Sp}}^{\mathrm{pw}} \colon \mathrm{Cat}^{\mathrm{lex}}_{\mathcal{T}} \to \mathrm{Cat}^{\mathrm{lex}}_{\mathcal{T}}$ be the functor obtained by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to $\mathrm{Sp} \colon \mathrm{Cat}^{\mathrm{lex}} \to \mathrm{Cat}^{\mathrm{lex}}$. Now let $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$ strongly admitting finite $\mathcal{T}$–limits. Then we can define its $\mathcal{T}$*–stabilisation* to be $\underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{D}}) := \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\mathrm{pw}}(\underline{\mathcal{D}}))$. In particular, applying this to the case $\underline{\mathcal{D}} = \underline{\mathcal{S}}_{\mathcal{T}}$, we get $\underline{\mathrm{Sp}}_{\mathcal{T}} := \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\mathrm{pw}}(\underline{\mathcal{S}}_{\mathcal{T}}))$ which is called $\mathcal{T}$*–category of genuine $\mathcal{T}$–spectra*. Note that this is different from the notation in [Nar16, Defn. 7.3] where he used $\underline{\mathrm{Sp}}^{\mathcal{T}}$ instead, and reserved $\underline{\mathrm{Sp}}_{\mathcal{T}}$ for what we wrote as $\underline{\mathrm{Sp}}^{\mathrm{pw}}$. We prefer the notation we have adopted as it aligns well with all the parametrised subscripts $(-)_{\mathcal{T}}$ and the superscripts are reserved for modifiers such as $(-)^{\underline{\omega}}$ or $(-)^{\Delta^1}$ that we will need later.

Theorem 4.1.8 (Universal property of $\mathcal{T}$–stabilisations, [Nar16, Thm. 7.4])**.** *Let $\underline{\mathcal{C}}$ be a pointed $\mathcal{T}$–category strongly admitting finite $\mathcal{T}$–colimits and $\underline{\mathcal{D}}$ a $\mathcal{T}$–category strongly admitting finite $\mathcal{T}$–limits. Then the functor $\underline{\Omega}^{\infty} : \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{D}})) \longrightarrow \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is an equivalence of $\mathcal{T}$–categories. In particular, we see that $\underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{D}}) \simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathcal{D}})$.*

4.2 Parametrised symmetric monoidality and commutative algebras

Recollections 4.2.1. There is a notion of $\mathcal{T}$–operads mimicking the notion of ∞-operads, in the sense of [Lur17, §2.1], due to Nardin in [Nar17, §3] and further developed in [NS22, §2]. A $\mathcal{T}$–symmetric monoidal category is then a $\mathcal{T}$–category $\underline{\mathcal{C}}^{\otimes}$ equipped with a cocartesian fibration over $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ satisfying the $\mathcal{T}$–operad axioms analogous to the operad axioms of [Lur17, Definition 2.1.1.10]. Alternatively, the $\mathcal{T}$–category of $\mathcal{T}$–symmetric monoidal categories is also given as $\underline{\mathrm{CMon}}(\underline{\mathrm{Cat}})$ much like in the unparametrised setting. Furthermore, there is also the attendant notion of $\mathcal{T}$*-inert morphisms* defined as those morphisms in $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ where the the map $W \to U'$ is an equivalence (cf. the span notation in (3)). The $\mathcal{T}$–category of $\mathcal{T}$*-commutative algebras* $\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\otimes})$ of a $\mathcal{T}$–symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$ is then defined to be $\underline{\mathrm{Fun}}^{\mathrm{inert}}_{\underline{\mathrm{Fin}}_{*\mathcal{T}}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes})$ where $\underline{\mathrm{Fun}}^{\mathrm{inert}}_{\underline{\mathrm{Fin}}_{*\mathcal{T}}} \subseteq \underline{\mathrm{Fun}}_{\underline{\mathrm{Fin}}_{*\mathcal{T}}}$ is the $\mathcal{T}$–full subcategory of functors over $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ preserving $\mathcal{T}$–inert morphisms. We refer the reader to the original source [Nar17, §3.1] or to [NS22, §2] for details on this.

Terminology 4.2.2. Let $\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}$ be $\mathcal{T}$–symmetric monoidal categories. By a $\mathcal{T}$*-symmetric monoidal localisation* $L^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{D}}^{\otimes}$ we mean a $\mathcal{T}$–symmetric monoidal functor whose underlying $\mathcal{T}$–functor is a $\mathcal{T}$–Bousfield localisation. By the proof of [Nar17, Prop. 3.5], we see that the $\mathcal{T}$–right adjoint canonically refines to a $\mathcal{T}$–lax symmetric functor. Hence in this situation we obtain a relative adjunction over $\underline{\mathrm{Fin}}_{*\mathcal{T}}$

$$\begin{array}{ccc} \underline{\mathcal{C}}^{\otimes} & \overset{L^{\otimes}}{\rightleftarrows} & \underline{\mathcal{D}}^{\otimes} \\ & \searrow \qquad \swarrow & \\ & \underline{\mathrm{Fin}}_{*\mathcal{T}} & \end{array}$$

in the sense of [Lur17, §7.3.2] whose counit is moreover an equivalence.

Lemma 4.2.3 ($\mathcal{T}$–adjunction on $\mathcal{T}$–commutative algebras, "[GGN15, Lem. 3.6]")**.** *Let $\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}$ be $\mathcal{T}$–symmetric monoidal categories and $L^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{D}}^{\otimes}$ a $\mathcal{T}$–symmetric monoidal localisation. Then there is an induced $\mathcal{T}$–Bousfield localisation $L' : \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}})$ such that the diagram*

$$\begin{array}{ccc} \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) & \underset{R'}{\overset{L'}{\rightleftarrows}} & \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}}) \\ \downarrow & & \downarrow \\ \underline{\mathcal{C}} & \underset{R}{\overset{L}{\rightleftarrows}} & \underline{\mathcal{D}} \end{array}$$

commutes, where the vertical maps are given by

$$\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \coloneqq \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}}) \simeq \underline{\mathcal{C}}$$

induced by the inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$, which lands in the $\mathcal{T}$–inerts. Moreover, given $A \in \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}})$ there is a unique $\mathcal{T}$–commutative algebra structure on RLA such that the unit map $A \to RLA$ enhances to a morphism of $\mathcal{T}$–commutative algebras.

Proof. First note that we have the adjunction squares

$$\begin{array}{ccc} \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) & \underset{R'}{\overset{L'}{\rightleftarrows}} & \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) & \underset{R^{\otimes}_{*}}{\overset{L^{\otimes}_{*}}{\rightleftarrows}} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \end{array}$$

where the bottom $\mathcal{T}$–adjunction is by Theorem 3.1.10 and has the property that the counit is an equivalence. Now [Lur17, Prop. 7.3.2.5] says that relative adjunctions are stable under pullbacks and the property of being $\mathcal{T}$–functors is of course preserved by pullbacks too, and so we get the square

$$\begin{array}{ccc} \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) & & \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}}) \\ \| & & \| \\ \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} & \underset{R'}{\overset{L'}{\rightleftarrows}} & \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} & \underset{R^{\otimes}_{*}}{\overset{L^{\otimes}_{*}}{\rightleftarrows}} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \end{array}$$

Then the square in the statement of the result is just composition of this square with the one induced by the inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$ namely

$$\begin{array}{ccc} \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) & \underset{R'}{\overset{L'}{\rightleftarrows}} & \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} & \underset{R^{\otimes}_{*}}{\overset{L^{\otimes}_{*}}{\rightleftarrows}} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \\ \downarrow & & \downarrow \\ \underline{\mathcal{C}} = \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) & \underset{R_{*}}{\overset{L_{*}}{\rightleftarrows}} & \underline{\mathcal{D}} = \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) \end{array}$$

For the next part, we know already that $R'L'A$ comes with a canonical $\mathcal{T}$–commutative algebra map $\eta' : A \to R'L'A$ given by the $L' \dashv R'$ unit evaluated at A. By the square in the statement we see that this forgets to the $L \dashv R$ unit $\eta : A \to RLA$. Now if $\eta'' : A \to R'B$ is another such map of $\mathcal{T}$–commutative algebras, then by universality of η' we have an essentially unique factorisation $\phi \circ \eta' : A \to R'L'A \to R'B$. Now fgt: $\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ is conservative by [Lur17, Lem. 3.2.2.6], thus since ϕ forgets to the identity, ϕ must have been an equivalence in $\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}})$ as required. □

4.3 Indexed (co)products of categories

We now investigate various permanence properties of indexed products on categories. To begin with, recall the following for which a good summary is [QS22, Ex. 5.20].

Construction 4.3.1 (Indexed products of categories)**.** Let $f : U \to U'$ be a map of finite $\mathcal{T}$–sets. Then Construction 2.1.16 gives us the equivalences in

$$f^* : \mathrm{Fun}(\mathrm{Total}(\underline{U}'), \mathrm{Cat}) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{U}', \underline{\mathrm{Cat}}_{\mathcal{T}}) \to \mathrm{Fun}_{\mathcal{T}}(\underline{U}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{U}), \mathrm{Cat})$$

This has a right adjoint f_* (also written $\prod_f$). Thus, for $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}_{\underline{U}}$ and $\underline{\mathcal{D}} \in \underline{\mathrm{Cat}}_{\underline{\mathcal{D}}}$ we have

$$\underline{\mathrm{Fun}}_{\underline{U'}}\Big(\underline{\mathcal{D}}, f_*\underline{\mathcal{C}}\Big) \simeq \underline{\mathrm{Fun}}_{\underline{U}}(f^*\underline{\mathcal{D}}, \underline{\mathcal{C}})$$

By setting $\underline{\mathcal{D}} = \underline{V}$ we see that $f_*\underline{\mathcal{C}}$ is a $\mathcal{T}_{/U'}$–category with V–fibre given by

$$\mathrm{Fun}_{\underline{U}}(\underline{U}_{\underline{V}}, \underline{\mathcal{C}}) \simeq \prod_{O \in \mathrm{Orbit}(U \times_{U'} V)} \mathcal{C}_O$$

where $\underline{U}_{\underline{V}}$ is the model for the corepresentable $\mathcal{T}$–category associated to $U \times_{U'} V$ whose fibre over $[W \to U]$ is given by the space of commutative squares in $\mathrm{Fin}_{\mathcal{T}}$

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U' \end{array}$$

Lemma 4.3.2 (Indexed constructions preserve adjunctions)**.** *Let $f : W \to V$ be in $\mathcal{T}$. Let $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : R$ be a $\mathcal{T}_{/W}$–adjunction and $M : \underline{\mathcal{A}} \rightleftarrows \underline{\mathcal{B}} : N$ be a $\mathcal{T}_{/V}$–adjunction. Then*

$$f_*L : f_*\underline{\mathcal{C}} \rightleftarrows f_*\underline{\mathcal{D}} : f_*R \qquad\qquad f^*M : f^*\underline{\mathcal{A}} \rightleftarrows f^*\underline{\mathcal{B}} : f^*N$$

are $\mathcal{T}_{/V}$– and $\mathcal{T}_{/W}$–adjunctions respectively.

Proof. By Corollary 2.2.7, we need to show that these induce fibrewise adjunctions. This is clear for the pair (f^*M, f^*N) since fibrewise they are the same as (M, N); for (f_*L, f_*R), we use that (unparametrised) products of adjunctions are again adjunctions. □

Lemma 4.3.3 ((Co)unit of indexed products)**.** *The $\mathcal{T}$–cofree category $\underline{\mathrm{Cat}}_{\mathcal{T}}$ strongly admits $\mathcal{T}$–products, and for $f : W \to V$, $X \in \mathcal{T}_{/W}$, and $Y \in \mathcal{T}_{/V}$, we have that $(f_*\underline{\mathcal{D}})_Y \simeq \prod_{M \in \mathrm{Orbit}(Y \times_V W)} \mathcal{D}_M$ and moreover:*

- *The unit is given by $\eta = F^* : \mathcal{C}_Y \longrightarrow (f_*f^*\underline{\mathcal{C}})_Y = \prod_{M \in \mathrm{Orbit}(Y \times_V W)} \mathcal{C}_M$ where $F : Y \times_V W \to Y$ is the structure map from the pullback,*
- *The counit is given by $\varepsilon = \mathrm{proj} : (f^*f_*\underline{\mathcal{C}})_X = \prod_{N \in \mathrm{Orbit}(X \times_V W)} \mathcal{D}_N \longrightarrow \mathcal{D}_X$ the component projection (see the proof for why we have this).*

Proof. We know that $f^* : \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathrm{Cat}) \longrightarrow \mathrm{Fun}((\mathcal{T}_{/W})^{\mathrm{op}}, \mathrm{Cat})$ abstractly has a right adjoint f_* via right Kan extension, and the formula for ordinary right Kan extensions gives us the required description (which is also gotten from Construction 4.3.1).

To describe the (co)units, we have to check the triangle identities

$$\begin{array}{ccc} f^* & \xrightarrow{f^*\eta} & f^*f_*f^* \\ & \searrow^{=} & \big\downarrow{\scriptstyle \varepsilon_{f^*}} \\ & & f^* \end{array} \qquad\qquad \begin{array}{ccc} f_* & \xrightarrow{\eta_{f_*}} & f_*f^*f_* \\ & \searrow^{=} & \big\downarrow{\scriptstyle f_*\varepsilon} \\ & & f_* \end{array} \tag{4}$$

First of all we clarify why we have the counit map as stated. For this it will be helpful to write carefully the datum $\varphi : X \to W$ instead of just X. Consider

$$\begin{array}{ccccc} X & & & & \\ & \searrow^{\text{(dashed)}} & & & \\ & & X \times_V W & \longrightarrow & X \\ & {\scriptstyle \varphi} & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f\varphi} \\ & & W & \xrightarrow{f} & V \end{array}$$

This shows that X is a retract of $X \times_V W$, and so by atomicity, we get that X was an orbit in the orbit decomposition of $X \times_V W$, and so the component projection $\varepsilon : (f^*f_*\underline{\mathcal{D}})_X = \prod_{N \in \mathrm{Orbit}(X \times_V W)} \mathcal{D}_N \longrightarrow \mathcal{D}_X$ is well-defined

To check the first triangle identity, let $(\varphi : X \to W) \in \mathcal{T}_{/W}$ and consider

$$\begin{array}{ccc} \coprod_a N_a & \xrightarrow{\coprod_a \xi_a} & X \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f\varphi} \\ W & \xrightarrow{f} & V \end{array}$$

where one of the N_a's is X, by the argument above. Then we have that the composition in the first triangle in (4) is

$$\Big((f^*\underline{\mathcal{C}})_X \xrightarrow{f^*\eta} (f^*f_*f^*\underline{\mathcal{C}})_X \xrightarrow{\varepsilon_{f^*}} (f^*\underline{\mathcal{C}})_X\Big) \simeq \Big(\mathcal{C}_X \xrightarrow{\prod_a \xi_a^*} \prod_a \mathcal{C}_{N_a} \xrightarrow{\mathrm{proj}} \mathcal{C}_X\Big)$$

which is of course the identity since $\xi_a = \mathrm{id}$ in the case $N_a = X$.

The second triangle identity is slightly more intricate. Let $(\psi : Y \to V) \in \mathcal{T}_{/V}$. We consider two pullbacks (where the right square is for each b appearing in the left square)

$$\begin{array}{ccc} \coprod_b M_b & \xrightarrow{\coprod_b \zeta_b} & Y \\ {\scriptstyle \coprod_b \rho_b}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \psi} \\ W & \xrightarrow{f} & V \end{array} \qquad\qquad \begin{array}{ccc} \coprod_{c_b} \widetilde{M}_{c_b} & \xrightarrow{\coprod_{c_b} \ell_{c_b}} & M_b \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f\rho_b} \\ W & \xrightarrow{f} & V \end{array}$$

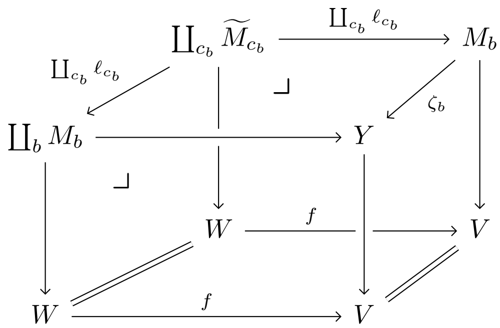
From this, the composition in the second triangle in (4) is

$$\begin{aligned} &\Big((f_*\underline{\mathcal{D}})_Y \xrightarrow{\eta_{f_*}} (f_*f^*f_*\underline{\mathcal{D}})_Y \xrightarrow{f_*\varepsilon} (f_*\underline{\mathcal{D}})_Y\Big) \\ \simeq &\Big(\prod_b \mathcal{D}_{M_b} \xrightarrow{\prod_b \prod_{c_b} \ell_{c_b}^*} \prod_b \prod_{c_b} \mathcal{D}_{\widetilde{M}_{c_b}} \xrightarrow{\prod_b \mathrm{proj}} \prod_b \mathcal{D}_{M_b}\Big) \end{aligned}$$

which is the identity map as wanted since M_b is one of the orbits in $\coprod_{c_b} \widetilde{M}_{c_b}$ by the argument above. Here we have used the diagram

$$\begin{array}{ccc} (f_*\underline{\mathcal{D}})_Y & \xrightarrow{\eta_{f_*}=\prod_b \zeta_b^*} & \prod_b (f_*\underline{\mathcal{D}})_{M_b} \\ \| & & \| \\ \prod_b \mathcal{D}_{M_b} & \xrightarrow{\prod_b \prod_{c_b} \ell_{c_b}^*} & \prod_b \prod_{c_b} \mathcal{D}_{\widetilde{M}_{c_b}} \end{array}$$

to analyse the map η_{f_*}, which in turn comes from the top square in



This finishes the proof. $\square$

4.4 Norms and adjunctions

We now recall the notion of $\mathcal{T}$–distributivity and indexed tensor products (also termed *norms*) of categories introduced in [Nar17, §3.3 and §3.4] and a nice summary of which can be found for instance in [QS22, §5.1].

Definition 4.4.1. Let $f : U \to V$ be a map in $\mathrm{Fin}_{\mathcal{T}}$, $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}_{/U}}$, $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}_{/V}}$, and $F : f_*\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}_{/V}$–functor. Then we say that F is $\mathcal{T}_{/V}$*–distributive* if for every pullback

$$\begin{array}{ccc} U' & \xrightarrow{f'} & V' \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

in $\mathrm{Fin}_{\mathcal{T}}$ and $\mathcal{T}_{/U'}$–colimit diagram $p : \underline{K}^{\triangleright} \to g'^*\underline{\mathcal{C}}$, the $\underline{V'}$–functor

$$(f'_*\underline{K})^{\triangleright} \xrightarrow{\text{can}} f'_*(\underline{K}^{\triangleright}) \xrightarrow{f'_*p} f'_*p'^*\underline{\mathcal{C}} \simeq g^*f_*\underline{\mathcal{C}} \xrightarrow{g^*F} g^*\underline{\mathcal{D}}$$

is a $\mathcal{T}_{/V'}$–colimit. We write $\mathrm{Fun}^{\delta}_{\underline{V}}(f_*\underline{\mathcal{C}}, \underline{\mathcal{D}})$ for the subcategory of $\mathcal{T}_{/V}$–distributive functors.

Construction 4.4.2 (Norms of categories)**.** Let $f : U \to V$ be a map in $\mathrm{Fin}_{\mathcal{T}}$ and $\underline{\mathcal{C}}$ a $\mathcal{T}_{/U}$–category which is $\mathcal{T}_{/U}$–cocomplete. Then we define the f*-norm* $f_{\otimes}\underline{\mathcal{C}}$, if it exists, to be a $\mathcal{T}_{/V}$–cocomplete category admitting a $\mathcal{T}_{/V}$–distributive functor $\tau : f_*\underline{\mathcal{C}} \to f_{\otimes}\underline{\mathcal{C}}$ such that for any other $\mathcal{T}_{/V}$–cocomplete category, the following functor is an equivalence

$$\tau^* : \mathrm{Fun}^{L}_{\underline{V}}(f_{\otimes}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \mathrm{Fun}^{\delta}_{\underline{V}}(f_*\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

We also write this as $f_{\otimes} = \bigotimes_f$.

Lemma 4.4.3 (Norms preserve adjunctions)**.** *Let $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ be a $\mathcal{T}_{/U}$–adjunction such that G itself admits a right adjoint and $f : U \to V$ be a map in* $\mathrm{Fin}_{\mathcal{T}}$*. Then this induces a $\mathcal{T}_{/V}$–adjunction*

$$f_{\otimes}F : f_{\otimes}\underline{\mathcal{C}} \rightleftarrows f_{\otimes}\underline{\mathcal{D}} : f_{\otimes}G$$

Proof. Recall from Lemma 4.3.2 that we have a $\mathcal{T}_{/V}$–adjunction $f_*F : f_*\underline{\mathcal{C}} \rightleftarrows f_*\underline{\mathcal{D}} : f_*G$ and since G itself has a right adjoint, both f_*F and f_*G strongly preserve $\mathcal{T}_{/V}$–colimits. Now observe that this adjunction can equivalently be encoded by the data of morphisms

$$\big(\eta\colon \mathrm{id} \Rightarrow (f_*G)\circ(f_*F)\big) \in \underline{\mathrm{Fun}}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}})$$

$$\big(\varepsilon\colon (f_*F)\circ(f_*G) \Rightarrow \mathrm{id}\big) \in \underline{\mathrm{Fun}}^L_{\underline{V}}(f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{D}})$$

whose images under the functors

$$(f_*F)_*\colon \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}}) \to \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{D}})$$

$$(f_*F)^*\colon \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{D}}) \to \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{D}})$$

respectively compose to a morphism equivalent to the identity

$$\begin{array}{ccc} f_*F & \xRightarrow{f_*F(\eta)} & (f_*F)\circ(f_*G)\circ(f_*F) \\ & \searrow^{=} & \Big\Downarrow \varepsilon_{f_*F} \\ & & f_*F \end{array}$$

and similarly for the other triangle identity. Now, we have commutative squares

$$\begin{array}{ccc} f_*\underline{\mathcal{C}} & \underset{f_*G}{\overset{f_*F}{\rightleftarrows}} & f_*\underline{\mathcal{D}} \\ \varphi\big\downarrow & & \big\downarrow\psi \\ f_\otimes\underline{\mathcal{C}} & \underset{f_\otimes G}{\overset{f_\otimes F}{\rightleftarrows}} & f_\otimes\underline{\mathcal{D}} \end{array}$$

where $\varphi : f_*\underline{\mathcal{C}} \to f_\otimes\underline{\mathcal{C}}$, $\psi : f_*\underline{\mathcal{D}} \to f_\otimes\underline{\mathcal{D}}$ are the universal distributive functors: this is since G strongly preserves $\mathcal{T}$–colimits by hypothesis. This yields

$$\begin{array}{ccccc} \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}}) & \xrightarrow{(f_*F)_*} & \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) & \xleftarrow{(f_*F)^*} & \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{D}}) \\ \varphi_*\big\downarrow & & \big\downarrow\psi_* & & \big\downarrow\psi_* \\ \mathrm{Fun}^\delta_{\underline{V}}(f_*\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{C}}) & \xrightarrow{(f_\otimes F)_*} & \mathrm{Fun}^\delta_{\underline{V}}(f_*\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{D}}) & \xleftarrow{(f_\otimes F)^*} & \mathrm{Fun}^\delta_{\underline{V}}(f_*\underline{\mathcal{D}}, f_\otimes\underline{\mathcal{D}}) \\ \varphi^*\big\uparrow\simeq & & \varphi^*\big\uparrow\simeq & & \psi^*\big\uparrow\simeq \\ \mathrm{Fun}^L_{\underline{V}}(f_\otimes\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{C}}) & \xrightarrow{(f_\otimes F)_*} & \mathrm{Fun}^L_{\underline{V}}(f_\otimes\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{D}}) & \xleftarrow{(f_\otimes F)^*} & \mathrm{Fun}^L_{\underline{V}}(f_\otimes\underline{\mathcal{D}}, f_\otimes\underline{\mathcal{D}}) \end{array}$$

Then the morphism $\big(\eta : \mathrm{id} \Rightarrow (f_*G)\circ(f_*F)\big) \in \underline{\mathrm{Fun}}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}})$ in the top left corner gets sent to a morphism $\big(\widetilde{\eta} : \mathrm{id} \Rightarrow (f_\otimes G)\circ(f_\otimes F)\big) \in \underline{\mathrm{Fun}}^L_{\underline{V}}(f_\otimes\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{C}})$ in the bottom left, and similarly for ε. Then by the characterisation of adjunctions above, since the composition of the images in the middle top term is equivalent to the identity, so is the image in the middle bottom term, that is, we have the commuting diagram

$$\begin{array}{ccc} f_\otimes F & \xRightarrow{f_\otimes F(\widetilde{\eta})} & (f_\otimes F)\circ(f_\otimes G)\circ(f_\otimes F) \\ & \searrow^{=} & \Big\Downarrow \widetilde{\varepsilon}_{f_\otimes F} \\ & & f_\otimes F \end{array}$$

and similarly for the other triangle identity. This witnesses that we have a $\mathcal{T}_{/V}$–adjunction $f_\otimes F \dashv f_\otimes G$ as required. $\square$

Remark 4.4.4. Let $V, W \in \mathcal{T}$ and $\underline{\mathcal{C}}^{\otimes}$ a $\mathcal{T}$–symmetric monoidal category. Concretely speaking, we get the structure of tensor products and norm functors as follows:

- (Tensor functor): Consider the morphism in $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ given by

$$\begin{array}{ccccc} V\coprod V & = & V\coprod V & \xrightarrow{\nabla} & V \\ \downarrow\nabla & & \downarrow\nabla & & \| \\ V & = & V & = & V \end{array}$$

  The cocartesian lifts along this morphism give us the tensor product

$$\otimes : \mathcal{C}_V \times \mathcal{C}_V \simeq \mathcal{C}_{V\coprod V} \longrightarrow \mathcal{C}_V$$

- (Norm functor): Suppose $f : V \to W$ is a morphism in $\mathcal{T}$. Consider

$$\begin{array}{ccccc} V & = & V & \xrightarrow{f} & W \\ \downarrow f & & \downarrow f & & \| \\ W & = & W & = & W \end{array}$$

  The cocartesian lifts along this morphism give us the norm functor

$$\mathrm{N}^f : \mathcal{C}_V \simeq \mathcal{C}^{\otimes}_{[f:V\to W]} \longrightarrow \mathcal{C}^{\otimes}_{[W=W]} \simeq \mathcal{C}_W$$

  Note that it might have been tempting to define the norm functor as the pushforward along the more obvious morphism

$$\begin{array}{ccccc} V & = & V & \xrightarrow{f} & W \\ \| & & \| & & \| \\ V & = & V & \xrightarrow{f} & W \end{array}$$

  instead, but the problem is that this is not a morphism in $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ because by definition the bottom right map needs to be the identity!

5 Parametrised smallness adjectives

We now introduce the notion of $\mathcal{T}$–compactness and $\mathcal{T}$–idempotent-completeness. Not only are these notions crucial in proving the characterisations of $\mathcal{T}$–presentables in Theorem 6.1.2, they are also fundamental for the applications we have in mind for parametrised algebraic K–theory in [Hil22b]. The moral of this section is that these are essentially fibrewise notions and should present no conceptual difficulties to those already familiar with the unparametrised versions. Recall that we will assume throughout that $\mathcal{T}$ is orbital.

5.1 Parametrised compactness

Recall that an object X in a category $\mathcal{C}$ is compact if $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ commutes with filtered colimits (cf. [Lur09, §5.3.4]). In this subsection we introduce the parametrised analogue of this notion and study its interaction with Ind-completions.

Definition 5.1.1. Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category and $V \in \mathcal{T}$. A V-object in $\underline{\mathcal{C}}$ (ie. an object in $\mathrm{Fun}_{\mathcal{T}}(\underline{V}, \underline{\mathcal{C}})$) is $\mathcal{T}_{/V}$–κ-*compact* if it is fibrewise κ-compact. We will also use the terminology *parametrised-κ-compact objects* when we allow V to vary. We write $\underline{\mathcal{C}}^{\underline{\kappa}}$ for the $\mathcal{T}$–subcategory of parametrised-κ-compact objects, that is, $(\underline{\mathcal{C}}^{\underline{\kappa}})_V$ is given by the full subcategory of $\mathcal{T}_{/V}$–κ-compact objects.

Notation 5.1.2. We write $\underline{\mathrm{Fun}}^{\kappa}_{\mathcal{T}}$ for the full $\mathcal{T}$–subcategory of parametrised functors preserving parametrised κ-compact objects.

Warning 5.1.3. In general, for $V \in \mathcal{T}^{\mathrm{op}}$, the inclusion $(\underline{\mathcal{C}}^{\underline{\kappa}})_V \subseteq (\mathcal{C}_V)^{\kappa}$ is not an equivalence - the point is that parametrised-κ-compactness must be preserved under the cocartesian lifts $f^* : \mathcal{C}_V \to \mathcal{C}_W$ for all $f : W \to V$, but these do not preserve κ-compactness in general.

This definition of compactness makes sense by virtue of the following:

Proposition 5.1.4 (Characterisation of parametrised-compactness)**.** *Let $\underline{\mathcal{C}}$ admit fibrewise κ-filtered $\mathcal{T}$–colimits. A $\mathcal{T}$–object $C \in \mathrm{Fun}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}})$ is κ–$\mathcal{T}$–compact in the sense above if and only if for all $V \in \mathcal{T}$ and all fibrewise κ-filtered $\mathcal{T}_{/V}$–diagram $d : \underline{\mathrm{const}}_{V}(K) \to \underline{\mathcal{C}}_{\underline{V}}$ the comparison*

$$\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{V}(K)} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}}, d) \to \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}}, \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{V}(K)} d)$$

is an equivalence.

Proof. Suppose C is κ–$\mathcal{T}$–compact. We are already provided with the comparison map above, and we just need to check that it is an equivalence, which can be done by checking fibrewise. Since $\mathrm{Total}(\underline{V}) = (\mathcal{T}_{/V})^{\mathrm{op}}$ has an initial object, we can assume that $\mathcal{T}$ has a final object. So let $W \in \mathcal{T}$. Recall that as in the proof of Lemma 2.2.9 we have

$$\Big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(C, d)\Big)_W \simeq \Big(\mathrm{Map}_{\mathcal{C}_\bullet}(C_\bullet, d_\bullet)\Big)_{\bullet \in (\mathcal{T}_{/W})^{\mathrm{op}}} \in \mathrm{Fun}((\mathcal{T}_{/W})^{\mathrm{op}}, \mathcal{S})$$

Then

$$\begin{aligned}\Big(\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{V}(K)} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}}, d)\Big)_W &\simeq \operatorname*{colim}_K \Big(\mathrm{Map}_{\mathcal{C}_\bullet}(C_\bullet, d_\bullet)\Big)_{\bullet \in (\mathcal{T}_{/W})^{\mathrm{op}}} \\ &\xrightarrow{\simeq} \Big(\mathrm{Map}_{\mathcal{C}_\bullet}(C_\bullet, \operatorname*{colim}_K d_\bullet)\Big)_{\bullet \in (\mathcal{T}_{/W})^{\mathrm{op}}}\end{aligned}$$

where the first equivalence is since fibrewise parametrised colimits are computed fibrewise, and the comparison map is an equivalence since colimits in $\mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{S})$ are computed pointwise, and C is pointwise κ-compact by hypothesis.

Now for the reverse direction, let $C \in \underline{\mathcal{C}}$ satisfy the property in the statement and $V \in \mathcal{T}$ arbitrary. We want to show that $C_V \in \mathcal{C}_V$ is κ-compact, that is: for any ordinary small κ-filtered diagram $d : K \to \mathcal{C}_V$, we have that

$$\operatorname*{colim}_K \mathrm{Map}_{\mathcal{C}_V}(C_V, d) \to \mathrm{Map}_{\mathcal{C}_V}(C_V, \operatorname*{colim}_K d)$$

is an equivalence. Now recall that $\mathcal{C}_V = \mathrm{Fun}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}}_{\underline{V}})$ by Example 2.1.4 and so by adjunction we obtain from $d : K \to \mathcal{C}_V$ a $\mathcal{T}_{/V}$–functor $\overline{d} : \underline{\mathrm{const}}_{V}(K) \longrightarrow \underline{\mathcal{C}}_{\underline{V}}$. In this case the desired comparison is an equivalence by virtue of the following diagram

$$\begin{array}{ccc} \mathrm{colim}_K \mathrm{Map}_{\mathcal{C}_V}(C_V, d) & \longrightarrow & \mathrm{Map}_{\mathcal{C}_V}(C_V, \mathrm{colim}_K d) \\ \| & & \| \\ \big(\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{V}(K)} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}}, d)\big)_V & \xrightarrow{\simeq} & \big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}}, \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{V}(K)} \overline{d})\big)_V \end{array}$$

where the bottom map is an equivalence by hypothesis. This finishes the proof. □

Observation 5.1.5. By the characterisation of $\mathcal{T}$–compactness above together with the $\mathcal{T}$–Yoneda Lemma 3.3.3, and that $\mathcal{T}$–colimits in $\mathcal{T}$–functor categories are computed in the target by Proposition 3.1.12 we see that the $\mathcal{T}$–Yoneda embedding lands in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$.

Proposition 5.1.6 ($\mathcal{T}$–compact closure, "[Lur09, Cor. 5.3.4.15]")**.** *Let κ be a regular cardinal and $\underline{\mathcal{C}}$ be $\mathcal{T}$–cocomplete. Then $\underline{\mathcal{C}}^{\underline{\kappa}}$ is closed under κ-small $\mathcal{T}$–colimits in $\underline{\mathcal{C}}$, and hence is κ–$\mathcal{T}$–cocomplete.*

Proof. Let $d : K \to \underline{\mathcal{C}}^{\underline{\kappa}}$ be a κ-small $\mathcal{T}$–diagram. Since all κ-small $\mathcal{T}$–colimits can be decomposed as κ-small fibrewise $\mathcal{T}$–colimits and $\mathcal{T}$–coproducts by the decomposition principle in Theorem 3.1.9 (3), we just have to treat these two special cases. The former case is clear by [Lur09, Cor. 5.3.4.15] since everything is fibrewise. For the latter case, let $\underline{V}$ be a corepresentable $\mathcal{T}$–category, A be a κ-filtered category, and $f : \underline{\mathrm{const}}_{\mathcal{T}}(A) \to \underline{\mathcal{C}}$ be a κ-filtered fibrewise $\mathcal{T}$–diagram. We need to show that the map in $\underline{\mathcal{S}}_{\mathcal{T}}$

$$\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, f) \longrightarrow \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}f)$$

is an equivalence. In this case, since we have for the source

$$\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, f) \simeq \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\mathrm{lim}}_{\underline{V}^{\mathrm{op}}}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(d, f)$$

and for the target

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}f) \simeq \underline{\mathrm{lim}}_{\underline{V}^{\mathrm{op}}}\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(d, f),$$

Theorem 3.5.10 gives the required equivalence, using also that $\underline{V}^{\mathrm{op}}$ is still corepresentable by Observation 2.1.8. ☐

5.2 Parametrised Ind-completions and accessibility

Proposition 5.2.1 ($\underline{\mathrm{Ind}}$ fully faithfulness, "[Lur09, Prop. 5.3.5.11]")**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and $\underline{\mathcal{D}} \in \widehat{\mathrm{Cat}}_{\mathcal{T}}$ which strongly admits fibrewise κ-filtered colimits. Suppose $F : \underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ strongly preserves fibrewise κ-filtered colimits and $f = F \circ j : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$.*

1. *If f is $\mathcal{T}$–fully faithful and the $\mathcal{T}$–essential image lands in $\underline{\mathcal{D}}^{\underline{\kappa}}$, then F is $\mathcal{T}$–fully faithful.*
2. *If f is $\mathcal{T}$–fully faithful, lands in $\underline{\mathcal{D}}^{\underline{\kappa}}$, and the $\mathcal{T}$–essential image of f generates $\underline{\mathcal{D}}$ under fibrewise κ-filtered colimits, then F is moreover a $\mathcal{T}$–equivalence.*

Proof. We prove (i) two steps. The goal is to show that

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B) \to \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA, FB)$$

is an equivalence. First suppose $A \in \underline{\mathcal{C}}$ and write $B \simeq \underline{\mathrm{colim}}_i B_i$ as a fibrewise filtered colimit where $B_i \in \underline{\mathcal{C}}$. We can equivalently compute $\underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B)$ as $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A, B)$, and so

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A, \underline{\mathrm{colim}}_i B_i) &\simeq \underline{\mathrm{colim}}_i \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A, B_i) \\ &\simeq \underline{\mathrm{colim}}_i \underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B_i)\end{aligned}$$

and

$$\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA, F\underline{\mathrm{colim}}_i B_i) \simeq \underline{\mathrm{colim}}_i \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(fA, fB_i)$$

where for the second equivalence we have used both hypotheses that F preserves fibrewise κ-filtered colimits and that the image lands in $\underline{\mathcal{D}}^{\underline{\kappa}}$. This completes this case. For a general $A \simeq \underline{\mathrm{colim}}_i A_i$ where $A_i \in \underline{\mathcal{C}}$ and the $\mathcal{T}$–colimit is fibrewise κ-filtered, we have

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A, B) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(\underline{\mathrm{colim}}_i A_i, B) &\simeq \underline{\mathrm{lim}}_i \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A_i, B) \\ &\xrightarrow{\simeq} \underline{\mathrm{lim}}_i \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA_i, FB) \\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA, FB)\end{aligned}$$

where the third equivalence is by the special case above, and so we are done. For (ii), we have shown $\mathcal{T}$–fully faithfulness, and $\mathcal{T}$–essential surjectivity is by hypothesis. ☐

Lemma 5.2.2. *Let $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then the $\mathcal{T}$–Yoneda embedding $y\colon \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ strongly preserves finite $\mathcal{T}$–colimits.*

Proof. Suppose $k : \underline{K} \to \underline{\mathcal{D}}$ is a finite $\mathcal{T}$–diagram. We need to show that the map

$$\underline{\mathrm{colim}}_{\underline{K}} \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(-, k) \to \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(-, \underline{\mathrm{colim}}_{\underline{K}} k)$$

in $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ is an equivalence. So let $\varphi \in \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ be an arbitrary object. Then mapping the morphism above into this and using Yoneda, we obtain

$$\varphi(\underline{\mathrm{colim}}_{\underline{K}} k) \longrightarrow \underline{\mathrm{lim}}_{\underline{K}^{\mathrm{op}}} \varphi(k)$$

which is an equivalence since φ is a $\mathcal{T}$–left exact functor. ☐

We thank Maxime Ramzi for teaching us the following slick proof, which is different from the standard one from [BGT13, Prop. 3.2], for instance.

Proposition 5.2.3. *Let $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then $\underline{\mathrm{Ind}}(\underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$. In particular, if $\underline{\mathcal{D}}$ were $\mathcal{T}$–stable, then $\underline{\mathrm{Ind}}(\underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{\mathcal{T}})$.*

Proof. First of all, note that $\underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}}) \subseteq \underline{\mathrm{Fun}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ is closed under fibrewise filtered colimits since fibrewise filtered colimits commutes with finite $\mathcal{T}$–limits in $\underline{\mathcal{S}}_{\mathcal{T}}$ by Theorem 3.5.10. Hence $y\colon \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ induces $\overline{y}\colon \underline{\mathrm{Ind}}(\underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ which we then know is $\mathcal{T}$–fully faithful by Proposition 5.2.1. Moreover, since y strongly preserves finite $\mathcal{T}$–colimits by Lemma 5.2.2, $\overline{y}$ strongly preserves small $\mathcal{T}$–colimits. Hence, by Theorem 6.2.1, it has a right adjoint $R : \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{D}})$ (we are free to use this result here since the present situation will not feature anywhere in the proof of adjoint functor theorem). If we can show that this right adjoint is conservative, then we would have shown that $\overline{y}$ and R are inverse equivalences. But conservativity is clear by mapping from representable functors and an immediate application of Yoneda. Finally, the statement for the $\mathcal{T}$–stable case is a straightforward consequence of Theorem 4.1.8. ☐

Proposition 5.2.4 ("[Lur09, Prop. 5.3.5.12]")**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and κ a regular cardinal. Then the canonical functor $F : \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}) \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is an equivalence.*

Proof. To see that F is an equivalence, we want to apply Proposition 5.2.1. Let $j : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}} \hookrightarrow \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}})$ be the canonical embedding. That the composite $f := F \circ j$ is $\mathcal{T}$–fully faithful and lands in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$ is clear. To see that the essential image of f generates $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ under fibrewise κ-filtered colimits, recall that any $X \in \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ can be written as a small $\mathcal{T}$–colimit of a diagram valued in $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ by Theorem 3.3.6. Then Corollary 3.5.9 gives that X can be written as a fibrewise κ-filtered colimit taking values in $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ where each object of $\underline{\mathcal{E}}$ is itself a κ-small $\mathcal{T}$–colimit of some diagram taking values in $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$. But then by Proposition 5.1.6 we know that $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$, and so this completes the proof. ☐

Proposition 5.2.5 (Characterisation of $\mathcal{T}$–compacts in $\mathcal{T}$–presheaves, "[Lur09, Prop. 5.3.4.17]")**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and κ a regular cardinal. Then a $\mathcal{T}$–object $C \in \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is κ–$\mathcal{T}$–compact if and only if it is a retract of a κ-small $\mathcal{T}$–colimit indexed in $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$.*

Proof. The if direction is clear since $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$ and by the compact closure of Proposition 5.1.6 we know that κ–$\mathcal{T}$–compacts are closed under κ-small $\mathcal{T}$–colimits and retracts.

Now suppose C is κ–$\mathcal{T}$–compact. First of all recall by Theorem 3.3.6 that $C \simeq \underline{\mathrm{colim}}_a j(B_a)$ where $j : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the $\mathcal{T}$–Yoneda embedding and $B_a \in \underline{\mathcal{C}}$. Combining this with Corollary 3.5.9 yields

$$C = \underline{\mathrm{colim}}_a j(B_a) \simeq \underline{\mathrm{colim}}_{f \in \underline{\mathrm{const}}_{\mathcal{T}}(F)} \underline{\mathrm{colim}}_{\left(p_f : K_f \to \underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})\right)} p_f$$

where F is a κ-filtered category. But then by Proposition 5.1.4 we then have that

$$\mathrm{id}_C \in \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(C,C) \simeq \underline{\mathrm{colim}}_{f\in\underline{\mathrm{const}}_{\mathcal{T}}(F)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(C, \underline{\mathrm{colim}}_{\left(p_f:K_f\to\underline{\mathcal{C}}\subseteq\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})\right)}p_f)$$

Hence we see that C is a retract of some $\underline{\mathrm{colim}}_{\left(p_f:K_f\to\underline{\mathcal{C}}\subseteq\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})\right)}p_f$ as required. □

Definition 5.2.6. Let κ be a regular cardinal and $\underline{\mathcal{C}}$ a $\mathcal{T}$–category. We say that $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–*accessible* if there is a small $\mathcal{T}$–category $\underline{\mathcal{C}}^0$ and a $\mathcal{T}$–equivalence $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}^0) \to \underline{\mathcal{C}}$. We say that $\underline{\mathcal{C}}$ is $\mathcal{T}$–*accessible* if it is κ–$\mathcal{T}$–accessible for some regular cardinal κ. A $\mathcal{T}$–functor out of a $\mathcal{T}$–accessible $\underline{\mathcal{C}}$ is said to be $\mathcal{T}$*-accessible* if it strongly preserves all fibrewise κ-filtered colimits for some regular cardinal κ.

Lemma 5.2.7 ($\mathcal{T}$–accessibility of $\mathcal{T}$–adjoints, "[Lur09, Prop. 5.4.7.7]")**.** *Let $G : \underline{\mathcal{C}} \to \underline{\mathcal{C}}'$ be a $\mathcal{T}$–functor between $\mathcal{T}$–accessibles. If G admits a right or a left $\mathcal{T}$–adjoint, then G is $\mathcal{T}$–accessible.*

Proof. The case of left $\mathcal{T}$–adjoints is clear since these strongly preserve all $\mathcal{T}$–colimits, so suppose $G \dashv F$. Choose a regular cardinal κ so that $\underline{\mathcal{C}}'$ is κ-accessible, ie. $\underline{\mathcal{C}}' = \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}}$ for some $\underline{\mathcal{D}}$ small. Consider the composite $\underline{\mathcal{D}} \xrightarrow{j} \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \xrightarrow{F} \underline{\mathcal{C}}$. Since $\underline{\mathcal{D}}$ is small there is a regular cardinal $\tau \gg \kappa$ so that both $\underline{\mathcal{C}}$ is τ-accessible and the essential image of $F\circ j$ consists of τ-$\mathcal{T}$–compact objects of $\underline{\mathcal{C}}$. We will show that G strongly preserves fibrewise τ-filtered colimits.

Since $\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is stable under small τ-filtered colimits by Proposition 3.5.4 it will suffice to prove that

$$G' : \underline{\mathcal{C}} \xrightarrow{G} \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$$

preserves fibrewise τ-filtered colimits. Since colimits in presheaf categories are computed pointwise by Proposition 3.1.12 it suffices to show this when evaluated at each $D \in \mathcal{D}_V$ for all $V \in \mathcal{T}$. Without loss of generality we just work with $D \in \mathcal{D}$, ie. a $\mathcal{T}$–object $D \in \mathrm{Fun}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}})$. In other words, by the $\mathcal{T}$–Yoneda lemma we just need to show that

$$G'_D : \underline{\mathcal{C}} \xrightarrow{G} \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}) \xrightarrow{\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})}(j(D),-)} \underline{\mathcal{S}}_{\mathcal{T}}$$

preserves fibrewise τ-filtered colimits. But G is a right adjoint and so by Lemma 2.2.9

$$\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})}(j(D), G(-)) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}}}(j(D), G(-)) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(Fj(D), -)$$

By assumption on τ, Fj lands in τ-compact objects, completing the proof. □

5.3 Parametrised idempotent-completeness

Recall that every retraction $r : X \rightleftarrows M : i$ gives rise to an idempotent self-map $i\circ r$ of X since $(i\circ r)\circ(i\circ r) \simeq i\circ(r\circ i)\circ r \simeq i\circ r$. On the other hand, in general, not every idempotent self-map of an object in a category arises in this way, and a category is defined to be idempotent-complete if every idempotent self-map of an object arises from a retraction (cf. [Lur09, §4.4.5]). We now introduce the parametrised version of this.

Definition 5.3.1. A $\mathcal{T}$–category is said to be $\mathcal{T}$*-idempotent-complete* if it is so fibrewise. A $\mathcal{T}$–functor $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is said to be a $\mathcal{T}$*-idempotent-completion* if it is fibrewise an idempotent-completion (cf. [Lur09, Def. 5.1.4.1]).

Observation 5.3.2 (Consequences of fibrewise definitions). Here are some facts we can immediately glean from our fibrewise definitions.

1. We know that for $\underline{\mathcal{C}}$ small, $\mathrm{Ind}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C})^\kappa) \simeq \mathrm{Ind}_\kappa(\mathcal{C})$, and so since $\mathcal{T}$–compactness and $\mathcal{T}$–Ind objects are fibrewise notions, we also get that for any small $\mathcal{T}$–category $\underline{\mathcal{C}}$ we have $\underline{\mathrm{Ind}}_\kappa\big(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}}\big) \simeq \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}}$. Here we have used crucially that $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}}$ is really just fibrewise compact, that is, that the cocartesian lifts of the cocartesian fibration $\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}} \to \mathcal{T}^{\mathrm{op}}$ preserve κ-compact objects. This is because $\mathrm{Ind}_\kappa(-)^\kappa$ computes the idempotent-completion by [Lur09, Lem. 5.4.2.4], which is a functor.

2. By the same token, $\underline{\mathcal{C}} \to (\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}})^{\underline{\kappa}}$ exhibits the $\mathcal{T}$–idempotent-completion of $\underline{\mathcal{C}}$ for any small $\mathcal{T}$–category $\underline{\mathcal{C}}$.

The following result will be crucial in the proof of Theorem 6.1.2.

Proposition 5.3.3 ($\mathcal{T}$–Yoneda of idempotent-complete, "[Lur09, Prop. 5.3.4.18]")**.** *Let $\underline{\mathcal{C}}$ be a small $\mathcal{T}$–idempotent-complete $\mathcal{T}$–category which is κ–$\mathcal{T}$–cocomplete. Then the $\mathcal{T}$–Yoneda embedding $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$ has a $\mathcal{T}$–left adjoint.*

Proof. By Proposition 3.3.9 we construct the adjunction objectwise. Let $\underline{\mathcal{D}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the full subcategory generated by all presheaves M where there exists $\ell M \in \underline{\mathcal{C}}$ satisfying

$$\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M, j(-)) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\ell M, -)$$

By definition, the desired left adjoint exists on this full subcategory, and hence it would suffice now to show that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}} \subseteq \underline{\mathcal{D}}$.

We first claim that $\underline{\mathcal{D}}$ is closed under retracts and inherits κ–$\mathcal{T}$–cocompleteness from $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. If $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(N, j(-))$ is a retract of $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M, j(-))$ inside $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. But then $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M, j(-))$ is in the Yoneda image from $\underline{\mathcal{C}}$, which is idempotent-complete, and hence its retract is also in the Yoneda image.

To see that $\underline{\mathcal{D}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ inherits κ–$\mathcal{T}$–cocompleteness, consider

$$\begin{aligned}
\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(\underline{\mathrm{colim}}_{\underline{K}} M_k, j(-)) &\simeq \underline{\lim}_{\underline{K}^{\mathrm{op}}} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M_k, j(-)) \\
&\simeq \underline{\lim}_{\underline{K}^{\mathrm{op}}} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\ell M_k, -) \\
&\simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{K}} \ell M_k, -)
\end{aligned}$$

where the last is since $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–cocomplete by hypothesis.

Now Proposition 5.2.5 says that everything in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is a retract of κ-small $\mathcal{T}$–colimits of the Yoneda image $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. Hence, since $\underline{\mathcal{C}} \subseteq \underline{\mathcal{D}}$ clearly, the paragraphs above yield that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}} \subseteq \underline{\mathcal{D}}$ as required. ☐

6 Parametrised presentability

We are now ready to formulate and prove two of the main results in this paper, namely the characterisations of $\mathcal{T}$–presentables in Theorem 6.1.2 and the $\mathcal{T}$–adjoint functor theorem, Theorem 6.2.1. As we shall see, given all the technology that we have, the proofs for these parametrised versions will present us with no especial difficulties either because we can mimic the proofs of [Lur09] almost word-for-word, or because we can deduce them from the unparametrised versions (as in the cases of the adjoint functor theorem or the presentable Dwyer-Kan localisation Theorem 6.3.7). In subsections §6.3 and §6.4 we will also develop the important construction of *localisation–cocompletions*. We will then prove the parametrised analogue of the correspondence between presentable categories and small idempotent-complete ones in Theorem 6.5.4 as well as record the various expected permanence properties for parametrised presentability in §6.7 and §6.6.

6.1 Characterisations of parametrised presentability

Definition 6.1.1. A $\mathcal{T}$–category $\underline{\mathcal{C}}$ is *$\mathcal{T}$–presentable* if $\underline{\mathcal{C}}$ is $\mathcal{T}$–accessible and is $\mathcal{T}$–cocomplete.

We are now ready for the Lurie-Simpson-style characterisations of parametrised presentability. Note that characterisation (7) is a purely parametrised phenomenon and has no analogue in the unparametrised world. The proofs for the equivalences between the first six characterisations is exactly the arguments in [Lur09] and so the expert reader might want to jump ahead to the parts that concern point (7).

Theorem 6.1.2 (Characterisations for parametrised presentability, "[Lur09, Thm. 5.5.1.1]")**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category. Then the following are equivalent:*

(1) $\underline{\mathcal{C}}$ is $\mathcal{T}$–presentable.

(2) $\underline{\mathcal{C}}$ is $\mathcal{T}$–accessible, and for every regular cardinal κ, $\underline{\mathcal{C}}^{\underline{\kappa}}$ is κ–$\mathcal{T}$–cocomplete.

(3) There exists a regular cardinal κ such that $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible and $\underline{\mathcal{C}}^{\underline{\kappa}}$ is κ–$\mathcal{T}$–cocomplete

(4) There exists a regular cardinal κ, a small $\mathcal{T}$–idempotent-complete and κ–$\mathcal{T}$–cocomplete category $\underline{\mathcal{D}}$, and an equivalence $\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{D}} \to \underline{\mathcal{C}}$. In fact, this $\underline{\mathcal{D}}$ can be chosen to be $\underline{\mathcal{C}}^{\underline{\kappa}}$.

(5) There exists a small $\mathcal{T}$–idempotent-complete category $\underline{\mathcal{D}}$ such that $\underline{\mathcal{C}}$ is a κ–$\mathcal{T}$–accessible Bousfield localisation of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$. By definition, this means that the image is κ–$\mathcal{T}$–accessible, and so by Lemma 5.2.7 the $\mathcal{T}$–right adjoint is also a κ–$\mathcal{T}$–accessible functor and hence the Bousfield localisation preserves κ–$\mathcal{T}$–compacts.

(6) $\underline{\mathcal{C}}$ is locally small and is $\mathcal{T}$–cocomplete, and there is a regular cardinal κ and a small set $\mathcal{G}$ of T-κ-compact objects of $\underline{\mathcal{C}}$ such that every $\mathcal{T}$–object of $\underline{\mathcal{C}}$ is a small $\mathcal{T}$–colimit of objects in $\mathcal{G}$.

(7) $\underline{\mathcal{C}}$ satisfies the left Beck-Chevalley condition (Terminology 3.1.8) and there is a regular cardinal κ such that the straightening $C : \mathcal{T}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}$ factors through $C : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Pr}_{L,\kappa}$.

Proof. That (1) implies (2) is immediate from Proposition 5.1.6. That (2) implies (3) is because by definition of $\mathcal{T}$–accessibility, there is a κ such that $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible, and since the second part of (2) says that $\underline{\mathcal{C}}^{\underline{\tau}}$ is τ-$\mathcal{T}$–cocomplete for all τ, this is true in particular for $\tau = \kappa$ so chosen. To see (3) implies (4), note that accessibility is a fibrewise condition and so we can apply the characterisation of accessibility in [Lur09, Prop. 5.4.2.2 (2)]. To see (4) implies (5), let $\underline{\mathcal{D}}$ be given by (4). We want to show that $\underline{\mathcal{C}}$ is a $\mathcal{T}$–accessible Bousfield localisation of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$. Consider the $\mathcal{T}$–Yoneda embedding (it lands in κ–$\mathcal{T}$–compacts by Observation 5.1.5)

$$j : \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})^{\underline{\kappa}}$$

This has a $\mathcal{T}$–left adjoint ℓ by Proposition 5.3.3. Define $L := \underline{\mathrm{Ind}}_{\kappa}(\ell)$ and $J := \underline{\mathrm{Ind}}_{\kappa}(j)$, so that, since $\underline{\mathrm{Ind}}_{\kappa}$ is a fibrewise construction, we have a $\mathcal{T}$–adjunction by Lemma 3.5.7

$$L : \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})^{\underline{\kappa}}) \rightleftarrows \underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{D}} : J$$

where J is $\mathcal{T}$–fully faithful by Proposition 5.2.1. But then by Proposition 5.2.4, we get $\underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})^{\underline{\kappa}}) \simeq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ and this completes this implication.

To see (5) implies (6), first of all $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is locally small and so $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is too. Moreover, Bousfield local $\mathcal{T}$–subcategories always admit $\mathcal{T}$–colimits admitted by the ambient category and so $\underline{\mathcal{C}}$ is $\mathcal{T}$–cocomplete. For the last assertion, consider the composite

$$\varphi : \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}) \xrightarrow{L} \underline{\mathcal{C}}$$

Since $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is generated by $\underline{\mathcal{D}}$ under small $\mathcal{T}$–colimits by Theorem 3.3.6 and since L preserves $\mathcal{T}$–colimits, we see that $\underline{\mathcal{C}}$ is generated under $\mathcal{T}$–colimits by $\operatorname{Im}\varphi$. To see that $\operatorname{Im}\varphi \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$, note that since by hypothesis $\underline{\mathcal{C}}$ was κ–$\mathcal{T}$–accessible, we know from Lemma 5.2.7 that the $\mathcal{T}$–right adjoint of L is automatically $\mathcal{T}$–accessible, and so L preserves κ–$\mathcal{T}$–compacts, and we are done.

To see (6) implies (1), by definition, we just need to check that $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible. Assumption (6) says that everything is a $\mathcal{T}$–colimit of $\mathcal{T}$–compacts, but we need to massage this to say that everything is a fibrewise κ-filtered $\mathcal{T}$–colimit of an essentially small subcategory - note this is where we need the assumption about $\mathcal{G}$ and not just use all of $\underline{\mathcal{C}}^{\underline{\kappa}}$, the problem being that the latter is not necessarily small. Let $\underline{\mathcal{C}}' \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$ be generated by $\mathcal{G}$ and $\underline{\mathcal{C}}' \subseteq \underline{\mathcal{C}}'' \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$ be the κ–$\mathcal{T}$–colimit closure of $\underline{\mathcal{C}}'$: here we are using that $\underline{\mathcal{C}}'' \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$ since κ–$\mathcal{T}$–compacts are closed under κ-small $\mathcal{T}$–colimits Proposition 5.1.6. Then since small $\mathcal{T}$–colimits decompose as κ-small $\mathcal{T}$–colimits and fibrewise κ-filtered colimits, we get that $\underline{\mathcal{C}}$ is generated by $\underline{\mathcal{C}}'' \subseteq \underline{\mathcal{C}}$ under κ-filtered colimits, as required.

Now to see (5) implies (7), suppose we have a $\mathcal{T}$–Bousfield localisation $F : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \rightleftarrows \underline{\mathcal{D}} : G$. For $f : W \to V$ in $\mathcal{T}$ we have

$$\begin{array}{ccc}
\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_V = \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{V}), \mathcal{S}) & \underset{G_V}{\overset{F_V}{\rightleftarrows}} & \mathcal{D}_V \\
f_! \uparrow \quad \downarrow f^* \quad \uparrow f_* & & f_! \uparrow \quad \downarrow f^* \quad \uparrow f_* \\
\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W = \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{W}), \mathcal{S}) & \underset{G_W}{\overset{F_W}{\rightleftarrows}} & \mathcal{D}_W
\end{array}$$

where all the solid squares commute. We need to show a few things, namely:

- That the dashed adjoints exist.
- That $f^* : \mathcal{D}_V \to \mathcal{D}_W$ preserves κ-compacts.
- That $f_! \dashv f^*$ on $\underline{\mathcal{D}}$ satisfies the left Beck-Chevalley conditions.

To see that the dashed arrows exist, define $f_!$ to be $F_V \circ f_! \circ G_W$. This works since

$$\begin{aligned}
\mathrm{Map}_{\mathcal{D}_V}(F_V \circ f_! \circ G_W -, -) &\simeq \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(G_W -, f^* \circ G_V -) \\
&\simeq \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(G_W -, G_W \circ f^* -) \\
&\simeq \mathrm{Map}_{\mathcal{D}_W}(-, f^* -)
\end{aligned}$$

To see that f_* exists, we need to see that f^* preserves ordinary colimits. For this, we use the description of colimits in Bousfield local subcategories. So let $\varphi : K \to \mathcal{D}_V$ be a diagram. Then

$$\begin{aligned}
f^* \operatorname*{colim}_{K \subseteq \mathcal{D}_V} \varphi &\simeq f^* F_V \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_V} G_V \circ \varphi \big) \\
&\simeq F_W f^* \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_V} G_V \circ \varphi \big) \\
&\simeq F_W \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_W} f^* \circ G_V \circ \varphi \big) \\
&\simeq F_W \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_W} G_W \circ f^* \circ \varphi \big) \\
&=: \operatorname*{colim}_{K \subseteq \mathcal{D}_W} f^* \circ \varphi
\end{aligned}$$

And hence f^* preserves colimits as required, and so by presentability, we obtain a right adjoint f_*. This completes the first point. Now to see that $f^* : \mathcal{D}_V \to \mathcal{D}_W$ preserves κ-compacts, note that $f^* : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_V \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W$ does since $f_* : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_V$ is κ-accessible by Lemma 5.2.7. Hence since $f^* F_V \simeq F_W f^*$, taking right adjoints we get $f_* G_W \simeq G_V f_*$. By hypothesis (5), G was κ-accessible and so since it is also fully faithful fibrewise, we get that $f_* : \mathcal{D}_W \to \mathcal{D}_V$ is κ-accessible, as required. For the third point, we already know from Proposition 3.1.11 that $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete, and so $f_!$ must *necessarily* give the indexed coproducts which satisfy the left Beck-Chevalley condition by Theorem 3.1.9.

Finally to see (7) implies (1), Theorem 3.1.9 says that $\underline{\mathcal{C}}$ is $\mathcal{T}$–cocomplete, and so we are left to show that it is κ–$\mathcal{T}$–accessible. But then this is just because $\underline{\mathcal{C}} \simeq \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}^{\underline{\kappa}})$ by [Lur09, Prop. 5.3.5.12] (since parametrised-compacts and ind-completion is just fibrewise ordinary compacts/ind-completion because the straightening lands in $\mathrm{Pr}_{L,\kappa}$). This completes the proof for this step and for the theorem. □

6.2 The adjoint functor theorem

We now deduce the parametrised version of the adjoint functor theorem from the unparametrised version using characterisation (7) of Theorem 6.1.2. Interestingly, and perhaps instructively, the proof shows us precisely where we need the notion of strong preservation and not just preservation (cf. Definition 3.1.3 and the discussion in Observation 3.2.1).

Theorem 6.2.1 (Parametrised adjoint functor theorem)**.** *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor between $\mathcal{T}$–presentable categories. Then:*

(1) If F strongly preserves $\mathcal{T}$–colimits, then F admits a $\mathcal{T}$–right adjoint.

(2) If F strongly preserves $\mathcal{T}$–limits and is $\mathcal{T}$–accessible, then F admits a $\mathcal{T}$–left adjoint.

Proof. We want to apply Corollary 2.2.7. To see (1), observe that the ordinary adjoint functor theorem gives us fibrewise right adjoints $F_V : \mathcal{C}_V \rightleftarrows \mathcal{D}_V : G_V$. To see that this assembles to a $\mathcal{T}$–functor G, we just need to check that the dashed square in the diagram

$$\begin{array}{ccc} \mathcal{C}_V & \overset{F_V}{\underset{G_W}{\rightleftarrows}} & \mathcal{D}_V \\ f_! \downarrow \; \uparrow f^* & & f_! \downarrow \; \uparrow f^* \\ \mathcal{C}_W & \overset{F_W}{\underset{G_W}{\rightleftarrows}} & \mathcal{D}_W \end{array}$$

commutes. But then the left adjoints of the dashed compositions are the solid ones, which we know to be commutative by hypothesis that F strongly preserves $\mathcal{T}$–colimits (and so in particular indexed coproducts, see Observation 3.2.1). Hence we are done for this case and part (2) is similar. □

We will need the following characterisation of functors that strongly preserve $\mathcal{T}$–colimits between $\mathcal{T}$–presentables in order to understand the correspondence between $\mathcal{T}$–presentable categories and small $\mathcal{T}$–idempotent-complete ones.

Proposition 6.2.2 ("[Lur09, Prop. 5.5.1.9]")**.** *Let $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor between $\mathcal{T}$–presentables and suppose $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible. Then the following are equivalent:*

(a) The functor f strongly preserves $\mathcal{T}$–colimits

(b) The functor f strongly preserves fibrewise κ-filtered colimits, and the restriction $f|_{\underline{\mathcal{C}}^{\underline{\kappa}}}$ strongly preserves κ–$\mathcal{T}$–colimits.

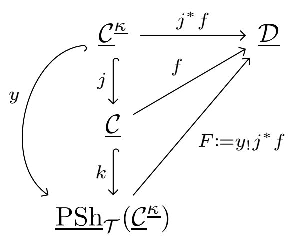
Proof. That (a) implies (b) is clear since $\underline{\mathcal{C}}^{\underline{\kappa}} \subseteq \underline{\mathcal{C}}$ creates $\mathcal{T}$–colimits by Proposition 5.1.6. Now to see (b) implies (a), let $\underline{\mathcal{C}} = \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}^{\underline{\kappa}})$ where $\underline{\mathcal{C}}^{\underline{\kappa}}$ is κ–$\mathcal{T}$–cocomplete and $\mathcal{T}$–idempotent-complete category by Proposition 5.1.6. Now by the proof of (4) implies (5) in Theorem 6.1.2 we have a $\mathcal{T}$–Bousfield adjunction

$$L : \underline{\mathrm{PSh}}(\underline{\mathcal{C}}^{\underline{\kappa}}) \rightleftarrows \underline{\mathcal{C}} : k$$

Now consider the composite

$$j^*f : \underline{\mathcal{C}}^{\underline{\kappa}} \xrightarrow{j} \underline{\mathcal{C}} \xrightarrow{f} \underline{\mathcal{D}}$$

By the universal property of $\mathcal{T}$–presheaves we get a strongly $\mathcal{T}$–colimit-preserving functor F fitting into the diagram



We know then that $f \simeq k^*y_!j^*f = k^*F$. On the other hand, we can define a functor

$$F' := f \circ L \simeq F \circ k \circ L : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}}) \longrightarrow \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{D}}$$

The $\mathcal{T}$–Bousfield adjunction unit $\mathrm{id}_{\underline{\mathrm{PSh}}_{\mathcal{T}}} \Rightarrow k \circ L$ gives us a natural transformation

$$\beta : F \Longrightarrow F' = F \circ k \circ L$$

If we can show that β is an equivalence then we would be done, since F, and so $F' = f \circ L$, strongly preserves $\mathcal{T}$–colimits. Hence since L was a $\mathcal{T}$–Bousfield localisation, f also strongly preserves $\mathcal{T}$–colimits, as required.

To see that β is an equivalence, let $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})$ be the full $\mathcal{T}$–subcategory on which β is an equivalence. Since both F and F' strongly preserve fibrewise κ-filtered colimits, we see that $\underline{\mathcal{E}}$ is stable under such. Hence it suffices to show that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}} \subseteq \underline{\mathcal{E}}$ since the inclusion will then induce the $\mathcal{T}$–functor $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}}) \simeq \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}}) \to \underline{\mathcal{E}}$ which is an equivalence by Proposition 5.2.1 (2).

Since $L \circ k \simeq \mathrm{id}$ we clearly have $\underline{\mathcal{C}}^{\underline{\kappa}} \subseteq \underline{\mathcal{E}}$, ie. that $\beta : F \Rightarrow F'$ is an equivalence on $\underline{\mathcal{C}}^{\underline{\kappa}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})$. On the other hand, by Proposition 5.1.6 we know that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}}$ is κ–$\mathcal{T}$–cocomplete, and its objects are retracts of κ-small $\mathcal{T}$–colimits valued in $\underline{\mathcal{C}}^{\underline{\kappa}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})$ by Proposition 5.2.5. Thus it suffices to show that F and F' strongly preserve κ-small $\mathcal{T}$–colimits when restricted to $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}}$. That F does is clear since it in fact strongly preserves all small $\mathcal{T}$–colimits. That F' does is because it can be written as the composition

$$F'|_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}}} : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}} \xrightarrow{L} \underline{\mathcal{C}}^{\underline{\kappa}} \xrightarrow{f} \underline{\mathcal{D}}$$

where L is a $\mathcal{T}$–left adjoint and f strongly preserves κ-small $\mathcal{T}$–colimits by assumption. Here we have crucially used that L lands in $\underline{\mathcal{C}}^{\underline{\kappa}}$ since this category is $\mathcal{T}$–idempotent-complete and κ–$\mathcal{T}$–cocomplete. □

6.3 Dwyer-Kan localisations

Terminology 6.3.1. We recall the clarifying terminology of [Hin16] in distinguishing between Bousfield localisations, as defined in Definition 2.2.3, and *Dwyer-Kan localisations*. By the latter, we will mean the following: let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category and S a class of morphisms in $\underline{\mathcal{C}}$ such that $f^*(S_W) \subseteq S_V$ for all $f\colon V \to W$ in $\mathcal{T}$. Suppose a $\mathcal{T}$–category $S^{-1}\underline{\mathcal{C}}$ exists and is equipped with a map $f : \underline{\mathcal{C}} \to S^{-1}\underline{\mathcal{C}}$ inducing the equivalence

$$f^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(S^{-1}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

for all $\mathcal{T}$–categories $\underline{\mathcal{D}}$, where $\underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}},\underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$ is the full subcategory of parametrised functors sending morphisms in S to equivalences. Such a $\mathcal{T}$–category must necessarily be unique if it exists, and this is then defined to be *the $\mathcal{T}$–Dwyer-Kan localisation of $\underline{\mathcal{C}}$ with respect to S*. The following proposition shows that being a $\mathcal{T}$–Bousfield localisation is stronger than that of being a $\mathcal{T}$–Dwyer-Kan localisation.

Proposition 6.3.2 (Bousfield implies Dwyer-Kan)**.** *Let $\underline{\mathcal{C}}, L\underline{\mathcal{C}}$ be $\mathcal{T}$–categories and $L : \underline{\mathcal{C}} \rightleftarrows L\underline{\mathcal{C}} : i$ be a $\mathcal{T}$–Bousfield localisation. Let S be the collection of morphisms in $\underline{\mathcal{C}}$ that are sent to equivalences under L. Then the functor L induces an equivalence $L^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(L\underline{\mathcal{C}},\underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$ for any $\mathcal{T}$–category $\underline{\mathcal{D}}$ so that $L\underline{\mathcal{C}}$ is a Dwyer-Kan localisation against S.*

Proof. Since $L \dashv i$ was a $\mathcal{T}$–Bousfield localisation, we know that $i^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(L\underline{\mathcal{C}},\underline{\mathcal{D}}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}) : L^*$ is also a $\mathcal{T}$–Bousfield localisation by Theorem 3.1.10, and so in particular L^* is $\mathcal{T}$–fully faithful. The image of L^* also clearly lands in $\underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$, and so we are left to show $\mathcal{T}$–essential surjectivity. By basechanging if necessary, we just show this on $\mathrm{Fun}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$. Let $\varphi : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor that inverts morphisms in S. We aim to show that $\varphi \Rightarrow \varphi \circ i \circ L$ is an equivalence. Since $L \dashv i$ was a $\mathcal{T}$–Bousfield localisation, the unit $\eta : \mathrm{id} \Rightarrow i \circ L$ gets sent to an equivalence under L, and so $\eta \in S$. Since φ inverts S by assumption, in particular it inverts η. ☐

Proposition 6.3.3. *$\mathcal{T}$–presentable categories are $\mathcal{T}$–complete.*

Proof. Let $\underline{\mathcal{C}}$ be $\mathcal{T}$–presentable so that it is a $\mathcal{T}$–Bousfield localisation of some $\mathcal{T}$–presheaf category $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ by description (5) of Theorem 6.1.2. We know that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is $\mathcal{T}$–complete and so all we need to show is that $\mathcal{T}$–Bousfield local subcategories are closed under $\mathcal{T}$–limits which exist in the ambient category. But this is clear since $\mathcal{T}$–Bousfield local subcategories can be described by a mapping-into property. ☐

Terminology 6.3.4. For $S \subseteq \underline{\mathcal{C}}$ a collection of morphisms, an object $X \in \underline{\mathcal{C}}$ is said to be S–local if $\mathrm{Map}_{\underline{\mathcal{C}}}(-, X)$ sends morphisms in S to equivalences.

In fact, as in the unparametrised case, we can give a precise description of maps that get inverted in a Bousfield localisation against an arbitrary collection of morphisms S, generalising the usual theory available for instance in [Lur09, §5.5.4].

Definition 6.3.5. Let $\overline{S}$ be a $\mathcal{T}$–collection of morphisms in a $\mathcal{T}$–category $\underline{\mathcal{C}}$. We say that it is *$\mathcal{T}$–strongly saturated* if the following conditions are satisfied:

1. (Pushout closure) Suppose we have a fibrewise pushout square in $\underline{\mathcal{C}}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ C & \longrightarrow & D. \end{array}$$

   If the left vertical is in $\overline{S}$, then the right vertical is also in $\overline{S}$.

2. ($\mathcal{T}$-colimit closure) The $\mathcal{T}$-full subcategory $\underline{\mathrm{Fun}}^{\overline{S}}(\Delta^1, \underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}(\Delta^1, \underline{\mathcal{C}})$ of morphisms in $\overline{S}$ is closed under $\mathcal{T}$-colimits.

3. (2-out-of-3) If any two of the three morphisms in

$$A \longrightarrow B \longrightarrow C \quad (\text{with the composite } A \to C)$$

   are in $\overline{S}$, then the third one is too.

For any $\mathcal{T}$–collection of morphisms S, we define its *$\mathcal{T}$–strong saturation closure* $\overline{S}$ to be the smallest $\mathcal{T}$–collection containing S which is $\mathcal{T}$–strongly saturated.

Proposition 6.3.6. *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$-presentable category and S a $\mathcal{T}$–collection of morphisms in $\underline{\mathcal{C}}$, and $\overline{S}$ its $\mathcal{T}$-strong saturation. Let $L : \underline{\mathcal{C}} \to L_S\underline{\mathcal{C}}$ be the $\mathcal{T}$-Bousfield localisation at S. Then the collection of L-equivalences consists precisely of the collection $\overline{S}$.*

Proof. We will bootstrap the parametrised statement from the unparametrised version in [Lur09, Prop. 5.5.4.15]. Let T be the collection of L-equivalences. First of all, note that we have $\overline{S} \subseteq T$ since it is straightforward to check that T is a $\mathcal{T}$–strongly saturated collection containing S and $\overline{S}$ is by definition the minimal such collection. To see the reverse inclusion, let $f : X \to Y$ be an L-equivalence and consider the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ LX & \xrightarrow[Lf]{\simeq} & LY \end{array}$$

Now since a $\mathcal{T}$-Bousfield localisation is in particular a fibrewise Bousfield localisation, we can apply [Lur09, Prop. 5.5.4.15 (1)] to see that the vertical maps in the square are in $\overline{S}$. And hence by 2-out-of-3, we see that f was also in $\overline{S}$, as desired. ☐

The following result, which will be crucial for our application in [Hil22b], is another example of the value of characterisation (7) from Theorem 6.1.2. The proof of the unparametrised result, given by Lurie in [Lur09, §5.5.4], is long and technical, and characterisation (7) allows us to obviate this difficulty by bootstrapping from Lurie's statement.

Theorem 6.3.7 (Parametrised presentable Dwyer-Kan localisations)**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$-presentable category and S a small collection of $\mathcal{T}$-morphisms of $\underline{\mathcal{C}}$ (ie. if $f : V \to W$ in $\mathcal{T}$ and $y \to z$ a morphism in S_W, then $f^*y \to f^*z$ is in S_V). Then:*

(1) Writing $S_{\underline{\amalg}} \supset S$ for the closure of S under finite indexed coproducts, the fibrewise full subcategory $S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}} \subseteq \underline{\mathcal{C}}$ of $S_{\underline{\amalg}}$-local objects assembles to a $\mathcal{T}$–full subcategory.

(2) We have a $\mathcal{T}$–accessible $\mathcal{T}$–Bousfield localisation $L : \underline{\mathcal{C}} \rightleftarrows S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}} : i$.

(3) For any $\mathcal{T}$–category $\underline{\mathcal{D}}$, the $\mathcal{T}$–functor $L^* : \underline{\mathrm{Fun}}^L_{\mathcal{T}}(S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}^{L,\overline{S}^{-1}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is an equivalence. Moreover, the inclusion $\underline{\mathrm{Fun}}^{L,\overline{S}^{-1}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}^{L,S^{-1}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is an equivalence.

Proof. For (1), we just nee to show that $S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}}$ is closed under the restriction functors in $\underline{\mathcal{C}}$. Let $f \colon V \to W$ be a map in $\mathcal{T}$ and let $x \in (S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}})_W$. We need to show that $f^*x \in (S_{\underline{\amalg}}^{-1}\underline{\mathcal{C}})_V$ is again $S_{\underline{\amalg}}$–local. So let $\varphi \colon a \to b$ be a morphism in $S_{\underline{\amalg}}$. Because $S_{\underline{\amalg}}$ is closed under finite indexed coproducts, we have the equivalence

$$\mathrm{Map}_V(b, f^*x) \simeq \mathrm{Map}_W(f_!b, x) \xrightarrow[\simeq]{(f_!\varphi)^*} \mathrm{Map}_W(f_!a, x) \simeq \mathrm{Map}_V(a, f^*x)$$

as wanted. For (2), we know from [Lur09, Prop. 5.5.4.15] that we already have fibrewise Bousfield localisations, and all we need to do is show that these assemble to a $\mathcal{T}$–Bousfield localisation via Corollary 2.2.7. Let $f : V \to W$ be in $\mathcal{T}$. We need to show that

$$\begin{array}{ccc} \mathcal{C}_V & \xrightarrow{L_V} & S_{\underline{\amalg}}^{-1}\mathcal{C}_V \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathcal{C}_W & \xrightarrow{L_W} & S_{\underline{\amalg}}^{-1}\mathcal{C}_W \end{array}$$

commutes, and for this, we first note that the diagram

$$\begin{array}{ccc} \mathcal{C}_V & \xleftarrow{i_V} & S_{\underline{\amalg}}^{-1}\mathcal{C}_V \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathcal{C}_W & \xleftarrow[i_W]{} & S_{\underline{\amalg}}^{-1}\mathcal{C}_W \end{array}$$

commutes where here f_* exists since $\underline{\mathcal{C}}$ is $\mathcal{T}$–complete by Proposition 6.3.3. Now recall by definition that $f^*(S_W) \subseteq S_V$ and so for $y \to z$ in S_W the map

$$\mathrm{Map}_W(z, f_*x) \simeq \mathrm{Map}_V(f^*z, x) \longrightarrow \mathrm{Map}_V(f^*y, x) \simeq \mathrm{Map}_V(y, f_*x)$$

is an equivalence, which implies that f_* takes $S_{\underline{\amalg}}$-local objects to $S_{\underline{\amalg}}$-local objects. Now by uniqueness of left adjoints, the first diagram commutes, as required. Finally, the first sentence of (3) is just a consequence of Proposition 6.3.2 and Proposition 6.3.6, noting also that $\overline{S} = \overline{S}_{\underline{\amalg}}$. That the inclusion is an equivalence is because if $F \in \underline{\mathrm{Fun}}_{\mathcal{T}}^{L,S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ and we have a morphism in $\overline{S}$ of the form $\underline{\mathrm{colim}}_J\varphi \colon \underline{\mathrm{colim}}_J a \to \underline{\mathrm{colim}}_J b$ for some $\underline{J}$–indexed diagram of morphisms all landing in S, then $F\underline{\mathrm{colim}}_J\varphi \simeq \underline{\mathrm{colim}}_J F\varphi$ is an equivalence by the hypothesis on F. $\square$

6.4 Localisation–cocompletions

In this subsection we formulate and prove the construction of *localisation–cocompletions* whose proof is exactly analogous to that of [Lur09]. As far as we can see, unfortunately the proof cannot be bootstrapped from the unparametrised statement as with the proof of Theorem 6.3.7 because the notion of a parametrised collection of diagrams might involve diagrams that are not fibrewise in the sense of Example 3.1.2.

Definition 6.4.1 (Parametrised collection of diagrams)**.** Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. A *parametrised collection of diagrams in* $\underline{\mathcal{C}}$ is defined to be a triple $(\underline{\mathcal{C}}, \mathcal{K}, \mathcal{R})$ where:

- $\mathcal{K}$ is a collection of small categories parametrised over $\mathcal{T}^{\mathrm{op}}$, ie. a collection $\mathcal{K}_V$ of small $\mathcal{T}_{/V}$–categories for each $V \in \mathcal{T}$.
- $\mathcal{R}$ is a parametrised collection of diagrams in $\underline{\mathcal{C}}$ whose indexing categories belong to $\mathcal{K}$, ie. for each $V \in \mathcal{T}$ a collection of coconed diagrams $\mathcal{R}_V$ indexed over categories in $\mathcal{K}_V$.

Theorem 6.4.2 ($\mathcal{T}$–localisation–cocompletions, "[Lur09, Prop. 5.3.6.2]")**.** *Let* $(\underline{\mathcal{C}}, \mathcal{K}, \mathcal{R})$ *be a parametrised collection of diagrams in* $\underline{\mathcal{C}}$*. Then there is a* $\mathcal{T}$*–category* $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ *and a* $\mathcal{T}$*–functor* $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ *such that:*

1. *The category* $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ *is* $\mathcal{K}$*–*$\mathcal{T}$*–cocomplete, ie. it strongly admits* $\mathcal{K}$*–indexed* $\mathcal{T}$*–colimits,* $\underline{\mathcal{C}_V}$ *admits* K*–indexed* $\mathcal{T}_{/V}$*–colimits.*
2. *For every* $\mathcal{K}$*-*$\mathcal{T}$*–cocomplete category* $\underline{\mathcal{D}}$*, the map* j *induces an equivalence of* $\mathcal{T}$*–categories*

   $$j^* : \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

   where the source denotes the $\mathcal{T}$*–category of functors which strongly preserve* $\mathcal{K}$*–indexed colimits and the target consists of those functors carrying each diagram in* $\mathcal{R}$ *to a parametrised colimit diagram in* $\underline{\mathcal{D}}$*.*
3. *If each member of* $\mathcal{R}$ *were already a* $\mathcal{T}$*–colimit diagram in* $\underline{\mathcal{C}}$*, then in fact* j *is* $\mathcal{T}$*–fully faithful.*

Proof. We give first all the constructions. By enlarging the universe, if necessary, we may reduce to the case where:

- Every element of $\mathcal{K}$ is small
- That $\underline{\mathcal{C}}$ is small
- The collection of diagrams $\mathcal{R}$ is small

Let $y : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the $\mathcal{T}$–yoneda embedding and let $V \in \mathcal{T}$. For a $\mathcal{T}_{/V}$–diagram $\bar{p} : K^{\trianglerighteq} \to \underline{\mathcal{C}}_{\underline{V}}$ with cone point Y, let X denote the $\mathcal{T}_{/V}$–colimit of $y \circ p : K \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{V}}$. This induces a $\mathcal{T}_{/V}$–morphism in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{V}}$

$$s : X \to y(Y)$$

Here we have used that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{V}} \simeq \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ by Construction 2.1.13. Now let S be the set of all such $\mathcal{T}_{/V}$–morphisms running over all $V \in \mathcal{T}$. This is small by our assumption and so let $L : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ denote the $\mathcal{T}$–Bousfield localisation from Theorem 6.3.7. Now we define $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) \subseteq S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ to be the smallest $\mathcal{K}$–cocomplete full $\mathcal{T}$–subcategory containing the image of $L \circ y : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. We show that this works and prove each point in turn.

Point (i) is true by construction, and so there is nothing to do. For point (ii), let $\underline{\mathcal{D}}$ be $\mathcal{K}$–$\mathcal{T}$–cocomplete. We now perform a reduction to the case when $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete. By taking the opposite Yoneda embedding we see that $\underline{\mathcal{D}}$ sits $\mathcal{T}$–fully faithfully in a $\mathcal{T}$–cocomplete category $\underline{\mathcal{D}}'$ and the inclusion strongly preserves $\mathcal{K}$–colimits. We now have a square of $\mathcal{T}$–categories (where the vertical functors are $\mathcal{T}$–fully faithful by Corollary 3.4.6)

$$\begin{array}{ccc} \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) & \xrightarrow{\phi := j^*} & \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \\ \big\downarrow & & \big\downarrow \\ \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}') & \xrightarrow{\phi' := j^*} & \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}') \end{array}$$

We claim this is cartesian in $\widehat{\mathrm{Cat}}_{\mathcal{T}}$ if ϕ' were an equivalence: given this, to prove that ϕ is an equivalence, it suffices to prove that ϕ' is an equivalence. For this, we need to show that the map into the pullback is an equivalence. That ϕ' is an equivalence ensures that the map into the pullback is fully faithful. To see essential surjectivity, let $F : \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}'$ be a strongly $\mathcal{K}$–colimit preserving functor that restricts to $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$. Then in fact F lands in $\underline{\mathcal{D}} \subseteq \underline{\mathcal{D}}'$ since $\underline{\mathcal{D}} \subseteq \underline{\mathcal{D}}'$ is stable under $\mathcal{K}$–indexed colimits, and by construction, $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ is generated under $\mathcal{K}$–indexed colimits by $\underline{\mathcal{C}}$.

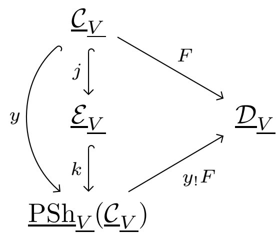
Now we turn to showing ϕ is an equivalence in the case $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete. Let $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the inverse image $L^{-1}\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ and $\overline{S}$ be the collection of all morphisms α in $\underline{\mathcal{E}}$ such that $L\alpha$ is an equivalence. Since the $\mathcal{T}$–Bousfield localisation $L : \underline{\mathrm{PSh}}(\underline{\mathcal{C}}) \rightleftarrows S^{-1}\underline{\mathrm{PSh}}(\underline{\mathcal{C}}) : i$ induces a $\mathcal{T}$–Bousfield localisation $L : \underline{\mathcal{E}} \rightleftarrows \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) : i$ we see by Proposition 6.3.2 that $L^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^{\overline{S}^{-1}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ is an equivalence. Furthermore, by using the description of colimits in $\mathcal{T}$–Bousfield local subcategories as being given by applying the localisation L to the colimit in the ambient category, we see that $f : \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}$ strongly preserves $\mathcal{K}$–colimits if and only if $f \circ L : \underline{\mathcal{E}} \to \underline{\mathcal{D}}$ does. This gives us the following factorisation of ϕ

$$\phi : \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \underset{\simeq}{\xrightarrow{L^*}} \underline{\mathrm{Fun}}^{\overline{S}^{-1}, \mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}) \xrightarrow{j^*} \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

and hence we need to show that the functor j^* is an equivalence. Since $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete, we can consider the $\mathcal{T}$–adjunction $j_! : \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \rightleftarrows \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}) : j^*$. We need to show:

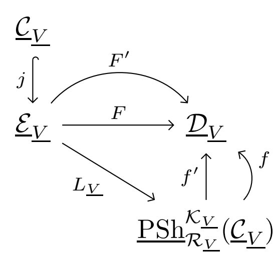
- that $j_!$ lands in $\underline{\mathrm{Fun}}^{\overline{S}^{-1}, \mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$,
- that $j_! \circ j^* \simeq \mathrm{id}$ on $\underline{\mathrm{Fun}}^{\overline{S}^{-1}, \mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ and $j^* \circ j_! \simeq \mathrm{id}$.

For the first point, fix a $V \in \mathcal{T}$. Since relative adjunctions are closed under pullbacks by Proposition 2.2.4 and since $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{\underline{V}} \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}})$ by Construction 2.1.13, we also get a $\mathcal{T}_{/V}$-adjunction $j_! : \underline{\mathrm{Fun}}_{\underline{V}}^{\mathcal{R}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}}) \rightleftarrows \underline{\mathrm{Fun}}_{\underline{V}}^{\mathcal{K}_{\underline{V}}}(\underline{\mathcal{E}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}}) : j^*$. Suppose $F : \underline{\mathcal{C}}_{\underline{V}} \to \underline{\mathcal{D}}$ is a $\underline{V}$–functor that sends $\mathcal{R}_{\underline{V}}$ to $\underline{V}$–colimit diagrams. We want to show that $j_!F : \underline{\mathcal{E}}_{\underline{V}} \to \underline{\mathcal{D}}_{\underline{V}}$ inverts maps in $\overline{S}$, ie. those maps that get inverted by $L_{\underline{V}}$. Consider



Note that $j_!F \simeq k^*y_!F$ since $j_! = \mathrm{id} \circ j_! \simeq k^*k_!j_! \simeq k^*y_!$. Now since $y_!F : \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}) \to \underline{\mathcal{D}}_{\underline{V}}$ strongly preserves $\underline{V}$–colimits and since $\underline{\mathcal{E}}_{\underline{V}}$ is stable under $\mathcal{K}$–indexed colimits in $\underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ (since $\underline{\mathrm{PSh}}_{\mathcal{R}_{\underline{V}}}^{\mathcal{K}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}})$ was closed under $\mathcal{K}$–colimits by construction) it follows that $j_!F \simeq k^*y_!F$ strongly preserves $\mathcal{K}$–colimits. Now note that the maps in $S \subseteq \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ are inverted by $y_!F$ since these were the maps comparing colimit in $\underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ and cone point in $\underline{\mathcal{C}}_{\underline{V}}$, and by hypothesis, F, and hence $y_!F$ turns these into equivalences. Therefore, by the universal property of Dwyer-Kan localisations Theorem 6.3.7, $y_!F : \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}) \to \underline{\mathcal{D}}$ factors through the Bousfield localisation L, and so in particular inverts $\overline{S}$, so that $j_!F \simeq k^*y_!F$ does too. Also $y_!F$ strongly preserves all $\underline{V}$–colimits by the universal property of presheaves, and so $j_!F \simeq k^*y_!F$ strongly preserves $\mathcal{K}$–colimits since the inclusion $k : \underline{\mathcal{E}}_{\underline{V}} \hookrightarrow \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ does.

For the second point, since j was $\mathcal{T}$–fully faithful, we have that $j^* \circ j_! \simeq \mathrm{id}$ as usual by Proposition 3.1.14. For the equivalence $j_! \circ j^* \simeq \mathrm{id}$, suppose $F \in \underline{\mathrm{Fun}}_{\mathcal{T}}^{\overline{S}^{-1},\mathcal{K}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$. Write $F' := j_!j^*F$. By universal property of Kan extensions we have $\alpha : F' = j_!j^*F \to F$ and we want to show this is an equivalence. Since F inverts $\overline{S}$ by hypothesis and $j_!j^*F$ also inverts $\overline{S}$ by the claim of the previous paragraph, we get the diagram



The transformation α induces a transformation $\beta : f' \to f$ since $\underline{\mathrm{Fun}}_{\underline{V}}^{\overline{S}^{-1}}(\underline{\mathcal{E}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}}) \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathrm{PSh}}_{\mathcal{R}_{\underline{V}}}^{\mathcal{K}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}}), \underline{\mathcal{D}}_{\underline{V}})$ and we want to show that β is an equivalence. To begin with, note that it is an equivalence on the image of the embedding $j : \underline{\mathcal{C}}_{\underline{V}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{R}_{\underline{V}}}^{\mathcal{K}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}})$. Since F and F' strongly preserve $\mathcal{K}$–colimits, hence so do f' and f. Therefore, $f' \to f$ is an equivalence on all of $\underline{\mathrm{PSh}}_{\mathcal{R}_{\underline{V}}}^{\mathcal{K}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}})$ since this $\underline{V}$-category was by construction generated under these colimits by $\underline{\mathcal{C}}$. This completes the proof of point (ii).

Finally, for point (iii), suppose every element of $\mathcal{R}$ were already a colimit diagram in $\underline{\mathcal{C}}$. The Yoneda map can be factored, by construction, as $j : \underline{\mathcal{C}} \hookrightarrow \underline{\mathcal{E}} \xrightarrow{L} \underline{\mathrm{PSh}}_{\mathcal{R}}^{\mathcal{K}}(\underline{\mathcal{C}})$ where the first map is $\mathcal{T}$–fully faithful. Since the restriction $L|_{S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})} \simeq \mathrm{id}$, it will suffice to show that j lands in $S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. That is, that $\underline{\mathcal{C}}$ is S-local, ie. for each $V \in \mathcal{T}$ and $C \in \mathcal{C}_V$, and for each

$f : W \to V$ in $\mathcal{T}$ and $s : X \to jY$ in S_W, we need to see that

$$s^* : \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(jY, jf^*C) \longrightarrow \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(X, jf^*C)$$

is an equivalence. To see this, the hypothesis of (iii) gives $Y = \underline{\mathrm{colim}}_{K\subseteq \underline{\mathcal{C}_W}}\varphi$. Then

$$\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(jY, jf^*C) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}_W}}(\underline{\mathrm{colim}}_{K\subseteq\underline{\mathcal{C}_W}}\varphi, f^*C) \simeq \underline{\lim}_{K^{\mathrm{op}}\subseteq\underline{\mathcal{C}_W}^{\mathrm{op}}}\mathrm{Map}_{\mathcal{C}}(\varphi, f^*C)$$

where the first equivalence is by Yoneda. On the other hand,

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(X, jf^*C) &\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(\underline{\mathrm{colim}}_K j\circ\varphi, jf^*C)\\ &\simeq \underline{\lim}_{K^{\mathrm{op}}}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(j\varphi, jf^*C)\\ &\simeq \underline{\lim}_{K^{\mathrm{op}}}\underline{\mathrm{Map}}_{\underline{\mathcal{C}_W}}(\varphi, f^*C)\end{aligned}$$

and so taking the section over W, one checks that these two identifications are compatible with the map s^*. This completes the proof of (iii). ☐

6.5 The presentables–idempotents equivalence

We want to formulate the equivalence between presentables and idempotent-completes in the parametrised world, and so we need to introduce some definitions. To avoid potential confusion, we will for example use the terminology *parametrised*-accessibles instead of $\mathcal{T}$–accessibles to indicate that we take $\mathcal{T}_{/V}$–accessibles in the fibre over V.

Definition 6.5.1. Let κ be a regular cardinal.

- Let $\underline{\mathrm{Acc}}_{\mathcal{T},\kappa} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the non-full $\mathcal{T}$–subcategory of κ-parametrised-accessible categories and κ-parametrised-accessible functors preserving κ-parametrised-compacts.
- Let $\underline{\mathrm{Cat}}^{\mathrm{Idem}}_{\mathcal{T}} \subseteq \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the full $\mathcal{T}$–subcategory on the small parametrised-idempotent-complete categories.
- Let $\underline{\mathrm{Cat}}^{\mathrm{rex}(\kappa)}_{\mathcal{T}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the non-full subcategory whose objects are κ-parametrised–cocomplete small categories and morphisms those parametrised–functors that strongly preserve κ-small parametrised–colimits.
- Let $\underline{\mathrm{Cat}}^{\mathrm{Idem}(\kappa)}_{\mathcal{T}} \subseteq \underline{\mathrm{Cat}}^{\mathrm{rex}(\kappa)}_{\mathcal{T}}$ be the full subcategory whose objects are κ-parametrised–cocomplete small parametrised-idempotent-complete categories.
- Let $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \subset \underline{\mathrm{Acc}}_{\mathcal{T},\kappa}$ be the non-full $\mathcal{T}$–subcategory whose objects are parametrised-presentables and whose morphisms are parametrised-left adjoints that preserve κ-parametrised-compacts.
- Let $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the non-full $\mathcal{T}$–subcategory of parametrised presentable categories and morphisms the parametrised κ-accessible functors which strongly preserve parametrised limits.

Notation 6.5.2. Let $\underline{\mathrm{Fun}}^{\kappa}_{\mathcal{T}} \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}$ be the full subcategory of κ–$\mathcal{T}$–compact-preserving functors.

Lemma 6.5.3 ("[Lur09, Prop. 5.4.2.17]")**.** *Let κ be a regular cardinal. Then $(-)^{\underline{\kappa}} : \underline{\mathrm{Acc}}_{\mathcal{T},\kappa} \longrightarrow \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ induces an equivalence to $\underline{\mathrm{Cat}}^{\mathrm{Idem}}_{\mathcal{T}}$, whose inverse $\underline{\mathrm{Cat}}^{\mathrm{Idem}}_{\mathcal{T}} \to \underline{\mathrm{Acc}}_{\mathcal{T},\kappa}$ is $\underline{\mathrm{Ind}}_{\kappa}$.*

Proof. To see $\mathcal{T}$–fully faithfulness, Proposition 3.5.4 gives

$$\begin{aligned}\underline{\mathrm{Fun}}_{\mathcal{T}}^{\kappa\text{-filt},\underline{\kappa}}(\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}},\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}}) &\xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{\underline{\kappa}}(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}},\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}})\\ &\xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}},\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{D}})^{\underline{\kappa}})\end{aligned}$$

where we have also used, by Observation 5.3.2 (1), that $\underline{\mathrm{Ind}}_\kappa(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}}) \simeq \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}}$. As for the essential image, let $\underline{\mathcal{C}}$ be a small $\mathcal{T}$–idempotent-complete category. Then by Observation 5.3.2 (2) we know that $\underline{\mathcal{C}} \simeq \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}}$, and so it is in the essential image as required. Finally to see the statement about the inverse, just note that we already have the functors and the appropriate natural transformations on compositions. Then using Observation 5.3.2 again, we see that the transformations are pointwise equivalences, and so equivalences. □

Theorem 6.5.4 ($\mathcal{T}$–presentable-idempotent correspondence, "[Lur09, Prop. 5.5.7.8 and Rmk. 5.5.7.9]")**.** *Let κ be a regular cardinal. Then $(-)^{\underline{\kappa}} : \underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \longrightarrow \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{rex}(\kappa)}$ is $\mathcal{T}$–fully faithful with essential image $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{Idem}(\kappa)}$, and inverse $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{Idem}(\kappa)} \to \underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ given by $\underline{\mathrm{Ind}}_\kappa$.*

Proof. That it is $\mathcal{T}$–fully faithful with the specified essential image is by Lemma 6.5.3 together with Proposition 5.1.6 and Proposition 6.2.2. That the inverse from Lemma 6.5.3 via $\underline{\mathrm{Ind}}_\kappa$ lands in $\mathcal{T}$–presentables is by Theorem 6.1.2 (4). □

6.6 Indexed products of presentables

The purpose of this subsection is to show that the (non-full) inclusions $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}, \underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ create indexed products.

Lemma 6.6.1 (Indexed products of $\mathcal{T}$–presentables)**.** *Let $f : W \to V$ be in $\mathcal{T}$ and $\underline{\mathcal{C}}$ be a $\mathcal{T}_{/W}$–presentable category. Then $f_*\underline{\mathcal{C}}$ is a $\mathcal{T}_{/V}$–presentable category.*

Proof. We first note that if $\underline{\mathcal{D}}$ is a $\mathcal{T}_{/W}$–category, then $f_*\underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}}) \simeq \underline{\mathrm{Fun}}_{\underline{V}}(f_!\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{V}})$. To see this, let $\underline{\mathcal{E}}$ be a $\mathcal{T}_{/V}$–category. Then

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/V}}}(\underline{\mathcal{E}}, f_*\underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}})) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(f^*\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(\underline{\mathcal{D}}, \underline{\mathrm{Fun}}_{\underline{W}}(f^*\underline{\mathcal{E}},\underline{\mathcal{S}}_{\underline{W}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(\underline{\mathcal{D}}, f^*\underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{E}},\underline{\mathcal{S}}_{\underline{V}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/V}}}(f_!\underline{\mathcal{D}}, \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{E}},\underline{\mathcal{S}}_{\underline{V}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/V}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\underline{V}}(f_!\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{V}}))\end{aligned}$$

By Theorem 6.1.2 we have a accessible $\mathcal{T}_{/W}$–Bousfield localisation $\underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}}) \rightleftarrows \underline{\mathcal{C}}$ for some small $\mathcal{T}_{/W}$–category $\underline{\mathcal{D}}$. Hence by Lemma 4.3.2, we obtain the accessible adjunction

$$\underline{\mathrm{Fun}}_{\underline{V}}(f_!\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{V}}) \simeq f_*\underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}}) \rightleftarrows f_*\underline{\mathcal{C}}$$

Therefore, $f_*\underline{\mathcal{C}}$ must be $\mathcal{T}_{/V}$–presentable, again by Theorem 6.1.2. □

Proposition 6.6.2 (Creation of indexed products for presentables)**.** *The (non-full) inclusions $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}, \underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ create indexed products.*

Proof. Let $f : W \to V$ be in $\mathcal{T}$ and $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}_{/V}$– and $\mathcal{T}_{/W}$–presentables, respectively. We know from Lemma 4.3.3 that $\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ has indexed products. We need to show that

$$\mathrm{Map}^L_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \simeq \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}},\underline{\mathcal{D}})$$

$$\mathrm{Map}^{R,\kappa\text{-filt}}_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \simeq \mathrm{Map}^{R,\kappa\text{-filt}}_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

We claim that the unit and counit in $\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ are already in both $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ and $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}}$. If we can show this then we would be done by the following pair of diagrams

$$\begin{array}{ccccc}
\mathrm{Map}^L_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) & \xrightarrow{f^*} & \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}}, f^*f_*\underline{\mathcal{D}}) & \xrightarrow{\varepsilon_*} & \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) & \xrightarrow{f^*} & \mathrm{Map}_{\underline{W}}(f^*\underline{\mathcal{C}}, f^*f_*\underline{\mathcal{D}}) & \xrightarrow{\varepsilon_*} & \mathrm{Map}_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}})
\end{array}$$

(the bottom composite being an equivalence $\simeq$)

$$\begin{array}{ccccc}
\mathrm{Map}^L_{\underline{V}}(\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) & \xleftarrow{\eta^*} & \mathrm{Map}^L_{\underline{V}}(f_*f^*\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) & \xleftarrow{f_*} & \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_{\underline{V}}(\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) & \xleftarrow{\eta^*} & \mathrm{Map}_{\underline{V}}(f_*f^*\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) & \xleftarrow{f_*} & \mathrm{Map}_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}})
\end{array}$$

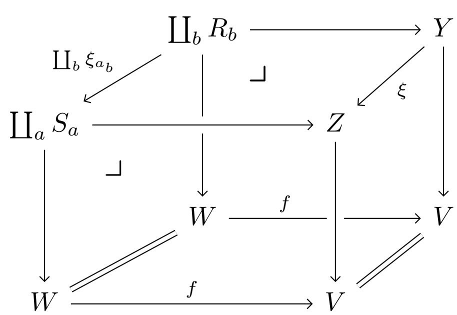
(the bottom composite being an equivalence $\simeq$)

and similarly when we replace Map^L by $\mathrm{Map}^{R,\kappa\text{-filt}}$: that the (co)units are in $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}}$ and $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ imply that the maps ε_* and η^* above takes Map^L to Map^L; that f^* and f_* also do these is by Lemma 4.3.2; and finally the bottom equivalences are inverse to each other, and so restrict to inverse equivalences to the top row of each diagram.

We now prove the claims. That they preserve κ–$\mathcal{T}$–compact objects is clear by Lemma 4.3.3 and Theorem 6.1.2. To see that the counit $\varepsilon : f^*f_*\underline{\mathcal{D}} \to \underline{\mathcal{D}}$ strongly preserves $\mathcal{T}$–(co)limits, since it is clear that they preserve fibrewise $\mathcal{T}$–(co)limits, by Proposition 3.2.2 we are left to show that they preserve the indexed (co)products. So let $\xi : Y \to Z$ be in $\mathcal{T}_{/W}$. For this we will need to know that $\underline{\mathcal{D}}$ has indexed coproducts and products (for the latter, see Proposition 6.3.3). We need to show that the squares with the dashed arrows in

$$\begin{array}{ccc}
(f^*f_*\underline{\mathcal{D}})_Z & \xrightarrow{\varepsilon} & \underline{\mathcal{D}}_Z \\
\xi_! \uparrow \; \downarrow \xi^* \; \uparrow \xi_* & & \xi_! \uparrow \; \downarrow \xi^* \; \uparrow \xi_* \\
(f^*f_*\underline{\mathcal{D}})_Y & \xrightarrow{\varepsilon} & \underline{\mathcal{D}}_Y
\end{array} \tag{5}$$

commute. We analyse this in terms of the counit formula from Lemma 4.3.3. For this, consider the diagram of orbits

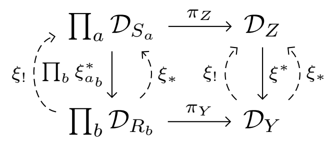


(6)

where the top square is also a pullback since we can view this diagram as

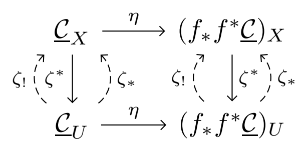
$$\begin{array}{ccccc}
\coprod_b R_b & \longrightarrow & \coprod_a S_a & \longrightarrow & W \\
\downarrow & & \downarrow & & \downarrow f \\
Y & \longrightarrow & Z & \longrightarrow & V
\end{array}$$

with the right square and the outer rectangle being pullbacks. From this we obtain that the diagram (5) is equivalent to

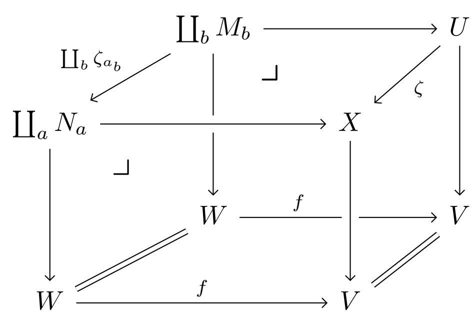


where the counits have been identified with the projections π_Z (resp. π_Y) onto the $\underline{\mathcal{D}}_Z$ (resp. $\underline{\mathcal{D}}_Y$) components by virtue of Lemma 4.3.3. Here $\prod_b \xi^*_{a_b}$ is supposed to mean forgetting about the components of $\coprod_a S_a$ that do not receive a map from $\coprod_b R_b$ and the functor $\xi^*_{a_b}$ for the other components: this makes sense because an orbit in a coproduct can only map to a unique orbit. Since $\underline{\mathcal{C}}$ was $\mathcal{T}_{/W}$–presentable, it in particular admits an $\mathcal{T}_{/W}$–initial object. And so we can easily use these, together with the adjoints $(\xi_{a_b})_!$ and fibrewise coproducts to obtain a left adjoint $\xi_!$ of $\prod_b \xi^*_{a_b}$, and similarly a right adjoint ξ_*. It is then immediate that the dashed squares also commute since the counits just project left/right adjoints from the left vertical to those on the right.

To see that the unit strongly preserves $\mathcal{T}$–(co)limits, similarly as above, we are reduced to the case of showing that it preserves indexed (co)products. Let $\zeta : U \to X$ be in $\mathcal{T}_{/V}$. And so we want the squares with the dashed arrows



to commute. For this consider the pullback comparison



where the top square is also a pullback by the argument for the previous case. Since

$$(f_*f^*\underline{\mathcal{C}})_X = \prod_a \mathcal{C}_{N_a} \quad \text{and} \quad (f_*f^*\underline{\mathcal{C}})_U = \prod_b \mathcal{C}_{M_b}$$

we see that the units η arise as restrictions along the maps $\coprod_a N_a \to X$ and $\coprod_b M_b \to U$ respectively. Then the required dashed squares commute by the Beck-Chevalley property of indexed (co)products of $\underline{\mathcal{C}}$ associated to the top pullback square. This completes the proof. □

6.7 Functor categories and tensors of presentables

In this final subsection, we record several basic results about the interaction between parametrised-presentability and functor categories, totally analogous to the unparametrised setting.

Lemma 6.7.1 (Small cotensors preserve $\mathcal{T}$–presentability)**.** *Let $\underline{\mathcal{C}}$ be a small $\mathcal{T}$-category and $\underline{\mathcal{D}}$ be $\mathcal{T}$–presentable. Then $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is also $\mathcal{T}$–presentable.*

Proof. As a special case, suppose first that $\underline{\mathcal{D}} \simeq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}')$ for a small $\mathcal{T}$–category $\underline{\mathcal{D}}'$. Then $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}'^{\underline{\mathrm{op}}}, \underline{\mathcal{S}}_{\mathcal{T}})$, and so it is also a $\mathcal{T}$–presheaf category, and so is $\mathcal{T}$–presentable. For a general $\mathcal{T}$–presentable $\underline{\mathcal{D}}$, we know that we have a κ–$\mathcal{T}$–accessible Bousfield localisation $L : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}') \rightleftarrows \underline{\mathcal{D}} : i$ for some small $\mathcal{T}$–category $\underline{\mathcal{D}}'$. Then we get a κ–$\mathcal{T}$–accessible Bousfield localisation $L_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}')) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) : i_*$ and so since $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}'))$ was $\mathcal{T}$–presentable by the first part above, by characterisation Theorem 6.1.2 (5) we get that $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is too. ◻

Lemma 6.7.2 ("[Lur09, Lem. 5.5.4.17]")**.** *Let $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ be a $\mathcal{T}$–adjunction between $\mathcal{T}$–presentables. Suppose we have a $\mathcal{T}$–accessible Bousfield localisation $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{C}}^0 : i$. Let $\underline{\mathcal{D}}^0 := G^{-1}(\underline{\mathcal{C}}^0) \subseteq \underline{\mathcal{D}}$. Then we have a $\mathcal{T}$–accessible Bousfield localisation $L' : \underline{\mathcal{D}} \rightleftarrows \underline{\mathcal{D}}^0 : i'$.*

Proof. The $\mathcal{T}$–accessibility of the Bousfield localisation $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{C}}^0 : i$ ensures that there is a small set of morphisms of $\underline{\mathcal{C}}$ such that $\underline{\mathcal{C}}^0$ are precisely the S-local objects. Then it is easy to see that $\underline{\mathcal{D}}^0 \subseteq \underline{\mathcal{D}}$ is precisely the $F(S)$-local $\mathcal{T}$–subcategory by using the adjunction. ◻

Lemma 6.7.3 ("[Lur09, Lem. 5.5.4.18]")**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–presentable category and $\{\underline{\mathcal{C}}_a\}_{a\in A}$ be a family of $\mathcal{T}$–accessible Bousfield local subcategories indexed by a small set A. Then $\bigcap_{a\in A} \underline{\mathcal{C}}_a$ is also a $\mathcal{T}$–accessible Bousfield local subcategory.*

Proof. This is because, if we write $S(a)$ for the morphisms of $\underline{\mathcal{C}}$ such that $\underline{\mathcal{C}}_a$ is the $S(a)$-local objects, then $\bigcap_{a\in A} \underline{\mathcal{C}}_a$ are the $\bigcup_{a\in A} S(a)$-local objects. ◻

For the remaining results, recall from Notation 3.3.10 that $\underline{\mathrm{Fun}}^R_{\mathcal{T}}$ and $\underline{\mathrm{Fun}}^L_{\mathcal{T}}$ denote strongly $\mathcal{T}$–limit- and $\mathcal{T}$–colimit-preserving functors, respectively, and $\underline{\mathrm{RFun}}_{\mathcal{T}}$ and $\underline{\mathrm{LFun}}_{\mathcal{T}}$ denote $\mathcal{T}$–right and $\mathcal{T}$–left adjoint functors, respectively.

Lemma 6.7.4 (Presentable functor categories, "[Lur17, Lem. 4.8.1.16]")**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$–presentables. Then $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ and $\underline{\mathrm{Fun}}^L_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ are also $\mathcal{T}$–presentable.*

Proof. By characterisation (5) of Theorem 6.1.2 and that Bousfield localisations are Dwyer-Kan Proposition 6.3.2, we know that $\underline{\mathcal{C}} \simeq S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')$ for some small $\mathcal{T}$–category $\underline{\mathcal{C}}'$ and S a small collection of morphisms in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')$. Then we have

$$\begin{aligned}\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} &\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}', \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\ &\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}'^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})\end{aligned}$$

where the first and last equivalence is by Observation 2.1.15, and the second by Proposition 3.5.4 and since $\mathcal{T}$–presentables are also $\mathcal{T}$–complete by Proposition 6.3.3. The right hand term is $\mathcal{T}$–presentable by Lemma 6.7.1, and so $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ is too by the equivalence above. Now note that we have $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{R,S^{-1}}_{T}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$: this is by virtue of the following diagram

$$\begin{aligned}\underline{\mathrm{Fun}}^R_{\mathcal{T}}((S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'))^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) &\simeq \underline{\mathrm{Fun}}^L_{\mathcal{T}}(S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\ &\xrightarrow[\simeq]{L^*} \underline{\mathrm{Fun}}^{L,S^{-1}}_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\ &\simeq \underline{\mathrm{Fun}}^{R,S^{-1}}_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})\end{aligned}$$

where we have the equivalence L^* owing to the formula for $\mathcal{T}$–colimits in $\mathcal{T}$–Bousfield local subcategories. Therefore, if for each $\alpha \in S$ we write $\underline{\mathcal{E}}(\alpha) \subseteq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ to be the $\mathcal{T}$–full subcategory of those functors which carry α to an equivalence in $\underline{\mathcal{D}}$, then $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) \simeq$

$\bigcap_{\alpha\in S}\underline{\mathcal{E}}(\alpha)\subseteq\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}},\underline{\mathcal{D}})$. Hence to show $\underline{\mathrm{Fun}}^R(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}})$ is a $\mathcal{T}$–accessible Bousfield localisation of $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}},\underline{\mathcal{D}})$, it will be enough to show it, by Lemma 6.7.3, for each $\underline{\mathcal{E}}(\alpha)$. Now these α's are morphisms in the various fibres over $\mathcal{T}^{\mathrm{op}}$ but since everything interacts well with basechanges, we can just assume without loss of generality that $\mathcal{T}^{\mathrm{op}}$ has an initial object and that α is a morphism in the fibre of this initial object. Given this, it is clear that we have the pullback

$$\begin{array}{ccc}\underline{\mathcal{E}}(\alpha) & \longrightarrow & \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}},\underline{\mathcal{D}}) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{ev}_\alpha} \\ \underline{\mathcal{E}} & \hookrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(\Delta^1),\underline{\mathcal{D}})\end{array}$$

where $\underline{\mathcal{E}}$ is the full subcategory spanned by the equivalences. Hence by Lemma 6.7.2 it will suffice to show that $\underline{\mathcal{E}}\subseteq\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(\Delta^1),\underline{\mathcal{D}})$ is a $\mathcal{T}$–accessible Bousfield localisation. But this is clear since it is just given by the $\mathcal{T}$–left Kan extension along $*\to\Delta^1$.

The statement for $\underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$ is proved analogously, but without having to take opposites in showing that $L^*:\underline{\mathrm{Fun}}^L_{\mathcal{T}}(S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'),\underline{\mathcal{D}})\to\underline{\mathrm{Fun}}^{L,S^{-1}}_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'),\mathcal{D})$ is an equivalence. □

The following result was stated as Example 3.26 in [Nar17] without proof, and so we prove it here. Here the tensor product is the one constructed in [Nar17, §3.4].

Proposition 6.7.5 (Formula for presentable $\mathcal{T}$–tensors)**.** *Let $\mathcal{T}$ be an atomic orbital category, and let $\underline{\mathcal{C}},\underline{\mathcal{D}}$ be $\mathcal{T}$–presentable categories. Then $\underline{\mathcal{C}}\otimes\underline{\mathcal{D}}\simeq\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}})$.*

Proof. This is just a consequence of the universal property of the tensor product. To wit, let $\underline{\mathcal{E}}$ be an arbitrary $\mathcal{T}$–presentable category and write $\underline{\mathrm{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}$ for $\mathcal{T}$–accessible strongly $\mathcal{T}$–limit preserving functors. Then

$$\begin{aligned}\underline{\mathrm{Fun}}^{L,L}(\underline{\mathcal{C}}\times\underline{\mathcal{D}},\underline{\mathcal{E}}) &\simeq \underline{\mathrm{Fun}}^{L}(\underline{\mathcal{C}},\underline{\mathrm{Fun}}^{L}(\underline{\mathcal{D}},\underline{\mathcal{E}}))\\ &\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathcal{D}},\underline{\mathcal{E}})^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}}\\ &\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathcal{D}},\underline{\mathcal{E}})^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}}\\ &\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{E}},\underline{\mathcal{D}}))^{\underline{\mathrm{op}}}\\ &\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}(\underline{\mathcal{E}},\underline{\mathcal{D}}))^{\underline{\mathrm{op}}}\\ &\simeq \underline{\mathrm{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}(\underline{\mathcal{E}},\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}}))^{\underline{\mathrm{op}}}\\ &\simeq \underline{\mathrm{RFun}}_{\mathcal{T}}(\mathcal{E},\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}}))^{\underline{\mathrm{op}}}\\ &\simeq \underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}}),\underline{\mathcal{E}})\\ &\simeq \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}}),\underline{\mathcal{E}})\end{aligned}$$

where the second equivalence is by Observation 2.1.15; the third, fifth, seventh, and ninth equivalence is by the adjoint functor Theorem 6.2.1; the fourth and eighth are from Proposition 3.3.11. In the seventh and ninth equivalence, we have also used that $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}})$ is $\mathcal{T}$–presentable, which is provided by Lemma 6.7.4. Therefore, $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{D}})$ satisfies the universal property of $\underline{\mathcal{C}}\otimes\underline{\mathcal{D}}$. □

References

[BDG+16a] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. “Parametrized Higher Category Theory and Higher Algebra: A General Introduction”. In: arXiv:1608.03654 (2016).

[BDG+16b] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. "Parametrized Higher Category Theory and Higher Algebra: Exposé I – Elements of Parametrized Higher Category Theory". In: arXiv:1608.03657 (2016).

[BGN14] C. Barwick, S. Glasman, and D. Nardin. "Dualizing Cartesian and Cocartesian Fibrations". In: arXiv:1409.2165 (2014).

[BGT13] A. Blumberg, D. Gepner, and G. Tabuada. "A Universal Characterization of Higher Algebraic K-Theory". In: Geom. Top. 17.2 (2013), pp. 733–838.

[BH21] Tom Bachmann and Marc Hoyois. "Norms in Motivic Homotopy Theory". In: Astérisque 425 (2021), p. 207.

[GGN15] D. Gepner, M. Groth, and T. Nikolaus. "Universality of Multiplicative Infinite Loop Space Machines". In: Alg. Geom. Top. 15 (2015), pp. 3107–3153.

[GHN17] D. Gepner, R. Haugseng, and T. Nikolaus. "Lax Colimits and Free Fibrations in ∞-Categories". In: Doc. Math. 22 (2017), pp. 1225–1266.

[Hil22a] K. Hilman. "Norms and periodicities in genuine equivariant hermitian K–theory". In: PhD Thesis, University of Copenhagen, available on the author's webpage (2022).

[Hil22b] K. Hilman. "Parametrised Noncommutative Motives and Cubical Descent in Equivariant Algebraic K-theory". In: arXiv (2022).

[Hin16] V. Hinich. "Dwyer-Kan Localization Revisited". In: Homology Homotopy Appl. 18.1 (2016), pp. 27–48.

[HW21] F. Hebestreit and F. Wagner. "Lecture Notes for Algebraic and Hermitian K-Theory". In: available on author's webpage (2021).

[Lei22] M. Leip. "On the Hochschild Homology of Hypersurfaces as a Mixed Complex". In: PhD Thesis, University of Copenhagen (2022).

[Lur09] J. Lurie. "Higher Topos Theory". In: Ann. of Math. Studies 170 (2009), pp. xviii+925.

[Lur17] J. Lurie. "Higher Algebra". In: available on the author's webpage (2017).

[MW22] L. Martini and S. Wolf. "Presentable categories internal to an ∞-topos". In: arXiv:2209.05103 (2022).

[Nar16] D. Nardin. "Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞–category". In: arXiv:1608.07704 (2016).

[Nar17] D. Nardin. "Stability and Distributivity over Orbital ∞-categories". In: PhD Thesis, MIT (2017).

[NS22] D. Nardin and J. Shah. "Parametrized and equivariant higher algebra". In: arXiv:2203.00072 (2022).

[QS22] J.D. Quigley and J. Shah. "On the parametrized Tate construction". In: arXiv:2110.07707 (2022).

[RV19] E. Riehl and D. Verity. "Infinity Category Theory from Scratch". In: arXiv:1608.05314 (2019).

[RV22] E. Riehl and D. Verity. "Elements of ∞-Category Theory". In: Cambridge Studies in Advanced Mathematics 194 (2022), pp. xix+759.

[Sha22] J. Shah. "Parametrized Higher Category Theory II: Universal Constructions". In: arXiv:2109.11954 (2022).

[Sha23] J. Shah. "Parametrized Higher Category Theory". In: Alg. Geom. Top. 23.2 (2023), pp. 509–644.

Parametrised Poincaré duality and equivariant fixed points methods

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In this article, we introduce and develop the notion of parametrised Poincaré duality in the formalism of parametrised higher category theory by Martini–Wolf, in part generalising Cnossen's theory of twisted ambidexterity to the nonpresentable setting. We prove several basechange results, allowing us to move between different coefficient categories and ambient topoi. We then specialise the general framework to yield a good theory of equivariant Poincaré duality spaces for compact Lie groups and apply our basechange results to obtain a suite of isotropy separation methods. Finally, we employ this theory to perform various categorical Smith–theoretic manoeuvres to prove, among other things, a generalisation of a theorem of Atiyah–Bott and Conner–Floyd on group actions with single fixed points.

Contents

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1. Introduction

Poincaré duality has a distinguished history that goes back right to the birth of algebraic topology at the hands of Henri Poincaré. Broadly speaking, it says that there is often a hidden symmetry between homology and cohomology, and arguably beyond Poincaré's wildest dreams, it is a phenomenon that is not just endemic to algebraic topology but also pervasive in fields as far as algebraic geometry, arithmetic geometry, and even representation theory. Essentially, it tends to show up in any context in which homological algebra is present. Perhaps a reason as to why it is such a useful principle is that it may be exploited both computationally as well as theoretically: the former because it halves the amount of homological computations to be made and the latter because, for instance, it may be used to produce "wrong–way" maps which opens the way to powerful transfer arguments. No less importantly in the way of theoretical significance, it would also be remiss of us not to mention that Poincaré duality constitutes one of the starting points of the surgery theory of manifolds. In either case, it would be fair to summarise that Poincaré duality provides strong structural constraints on homological invariants which lends a rigidity not seen for a bare homotopy type.

On another front, group actions on manifolds have attracted the attention of topologists nearly since the beginning of the subject. A deep vein in this line of work is the hope of finding algebro–topological constraints on group actions in order to rule out the existence of certain group actions on manifolds. Smith theory, for example, predicts that if a p-group G acts on an $\mathbb{F}_p$-homology sphere, then the fixed points of the action must also be an $\mathbb{F}_p$-homology sphere. In a similar spirit, the Conner–Floyd conjecture, resolved affirmatively by Atiyah–Bott, predicts in a simple version that if p is an odd prime and the group C_{p^k} acts smoothly on a smooth, closed, orientable, positive–dimensional manifold, then the fixed point set of the action cannot consist of a single isolated point.

Given the motivations above, it should come as no surprise that a theory of equivariant Poincaré duality is desirable in order to incorporate the strong homological constraints aforementioned in the equivariant context. Indeed, so natural is this a question that there is a very rich corpus of contributions – too many to mention exhaustively – in this line of investigation coming from a wide variety of schools of thought. From our point of view, the strand of work that is most pertinent to us (and on which we build, either directly or indirectly), is the parametrised category theoretic one of Costenoble–Waner [CW92; CW16] and May–Sigurdsson [MS06], which were informed by the work of tom Dieck in [Die87]. More specifically, our work builds heavily upon Cnossen's work on twisted ambidexterity [Cno23], which is, in turn, built upon the insights of the preceding work. A more detailed account of the relationships between the present work and the ones just mentioned will be given at the end of the introduction.

The main goal of this article is to develop the theory of equivariant Poincaré duality for compact Lie groups from an ∞–categorical perspective. As a proof of concept, we will then apply it to a selection of concrete problems in equivariant geometric topology, some of which have been resolved through different methods before. Our categorical formalism of choice is the parametrised ∞–category theory of [Mar22a; Mar22b; MW22; MW24] and a central role will be played by equivariant stable homotopy theory as first systematically developed in [LMS86] and later in [GM95; MNN17]. The rest of the introduction will give an overview of our methods and highlighted results.

Notations and conventions: We work in the setting of ∞–categories as developed by Joyal and Lurie without referring to any particular model such as quasicategories. To avoid notational clutter, we will refer to ∞–categories as just categories, while classical categories (for which there is a set of morphisms between objects as opposed to a space of such) will be referred to as 1-categories. We fix three Grothendieck universes $U \in V \in W$ called small, large and very large. We denote the large category of small categories by Cat and the very large category of large categories by $\widehat{\mathrm{Cat}}$. The term "category" will be reserved for small categories. Furthermore, left adjoints will always be written on top of right adjoints in our diagrams.

Equivariant and parametrised homotopy theory

Let G be compact Lie group. It has long been understood that in order to have good access to inductive methods in equivariant homotopy theory, the fixed points spaces for all closed subgroups of G should be recorded as part of the structure of a G–space. The earliest categorical articulation of this principle is the theorem of Elmendorf's which says that there is an equivalence of (∞–)categories $\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathcal{S})$ between the category of G–spaces and presheaves on the orbit category of G.

In the same way that the category of spectra is the universal homology theory on spaces, it has been identified in [Seg70] and fully developed in [LMS86] that the appropriate replacement of spectra in the equivariant setting is the stable category Sp_G of *genuine G–spectra*. In the finite group case, we may even view Sp_G as G–Mackey functors valued in spectra. To each G–space $X \in \mathcal{S}_G$ we may associate an "equivariant stable homotopy type" $\Sigma^\infty_+ X \in \mathrm{Sp}_G$. From this, we may, among other things, recover the stable homotopy type of all fixed point spaces X^H for subgroups $H \leq G$ via the *geometric fixed points functor* $\Phi^H \colon \mathrm{Sp}_G \to \mathrm{Sp}$ to obtain $\Phi^H(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+(X^H)$. In fact, more generally, there is a geometric fixed points functor associated to a *family* $\mathcal{F}$ of closed subgroups of G, and it should be thought as a functor which universally kills equivariant cells G/H where $H \in \mathcal{F}$.

It turns out to be fruitful to treat questions about G–spaces not only through a categorical lense, but rather work with equivariant versions of categories themselves. Two equivalent approaches have been developed, the first by Barwick–Dotto–Glasman–Nardin–Shah [BDG+16a; BDG+16b; Sha23; Nar17; NS22] and the second by Martini–Wolf [Mar22a; Mar22b; MW22; MW24]. Each formalism has their advantages, and for our purposes in this article, we have chosen to work mainly in the second one since it affords us the flexibility of working over an arbitrary topos: this will allow us to give uniform and streamlined proofs. In either case, the appropriate replacement for Cat in the equivariant setting is the category $\mathrm{Cat}_G := \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Cat})$ of *G–categories*. We will write $\underline{\mathcal{C}}$ for an object in Cat_G and $\mathcal{C}(G/H)$ for the evaluation of $\underline{\mathcal{C}}$ at G/H. Some G–categories of special interest to us, viewed as presheaves on the orbit category, are

$\underline{\mathcal{S}}$:	$G/H \mapsto \mathcal{S}_H$	the G–category of G–spaces;
$\underline{\mathrm{Sp}}$:	$G/H \mapsto \mathrm{Sp}_H$	the G–category of G–spectra;
$\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathrm{Sp}})$:	$G/H \mapsto \mathcal{P}\mathrm{ic}(\mathrm{Sp}_H)$	the G–space of invertible G–spectra.

Crucially, for a large part of this article, we will rely upon a good theory of parametrised presentable categories, whereupon we may speak of, for instance, the category $\mathrm{Pr}_G^{L,G-\mathrm{st}}$ of G–stable presentable G–categories (in which $\underline{\mathrm{Sp}}_G$ is the symmetric monoidal unit). Now for a closed normal subgroup $N \leq G$ and writing $Q := G/N$ for the quotient group, there is a fully faithful inclusion $\mathrm{incl}\colon \widehat{\mathrm{Cat}}_Q \hookrightarrow \widehat{\mathrm{Cat}}_G$ of large Q–categories into large G–categories and it admits a left adjoint $(-)^N$ given by forgetting all information from the subgroups of G that do not contain N. These two functors restrict to give functors $\mathrm{incl}\colon \mathrm{Pr}_Q^{L,Q-\mathrm{st}} \hookrightarrow \mathrm{Pr}_G^{L,G-\mathrm{st}}$ and $(-)^N\colon \mathrm{Pr}_G^{L,G-\mathrm{st}} \to \mathrm{Pr}_Q^{L,Q-\mathrm{st}}$ respectively. However, the adjunction on large categories does *not* descend to an adjunction on the presentable categories because the adjunction unit in $\widehat{\mathrm{Cat}}_G$ is not a morphism in $\mathrm{Pr}_G^{L,G-\mathrm{st}}$. Nevertheless, we show the following:

Theorem A (Theorem 2.2.26 and Proposition 2.2.29)**.** *Let G and Q be as above. Then the inclusion $\mathrm{Pr}_Q^{L,Q-\mathrm{st}} \hookrightarrow \mathrm{Pr}_G^{L,G-\mathrm{st}}$ admits a left adjoint Φ^N which is a smashing localisation.*

We call the functor Φ^N above the *Brauer quotient* functor, borrowing the term from classical Mackey functor theory. In fact, in the precise versions of the result, we prove it more generally for families and we also prove this for small G–stable categories when the group is finite. The result above should be viewed as a categorification of the geometric fixed points functors aforementioned. We may indeed recover the usual geometric fixed points functors by considering the adjunction unit evaluated on $\underline{\mathrm{Sp}}$. Using Theorem A, we may functorially construct geometric fixed points for *any* G–stable category, a construction that will be important to us in performing isotropy separation arguments for equivariant Poincaré duality, as we shall see below.

Equivariant and parametrised Poincaré duality

Poincaré duality is usually formulated as the statement that for a closed d-manifold M, there exists an infinite cyclic local coefficient system $\mathcal{O}$ on M and a class $[M] \in H_d(M;\mathcal{O})$ such that the the cap product with $[M]$ induces, for every local coefficient system η on M, isomorphisms

$$[M] \cap -\colon H^*(M;\eta) \longrightarrow H_*(M;\eta \otimes \mathcal{O}).$$

We will briefly recall a different formulation due to [Kle01] in terms of local systems of spectra (c.f. [Lan22, App. A] for a nice and detailed exposition of this point of view). It will let us arrive at an equivariant version (even a parametrised one, in general) with little creativity.

Following the notation of [Cno23], let $M: M \to *$ be the unique map. We get adjunctions

$$\mathrm{Sp}^M \underset{M_*}{\overset{M_!}{\rightleftarrows}} \mathrm{Sp} \qquad \mathrm{Sp}^M \xleftarrow{M^*} \mathrm{Sp}$$

where $M_!$ (resp. M_*) associates to each local system $\xi \in \mathrm{Sp}^M$ its colimit (resp. limit), and the resulting spectrum should be viewed as the homology (resp. cohomology) of M with coefficients in ξ. For a smooth manifold M, the Spivak normal fibration can be used to construct a local system $D_M \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})^M \subset \mathrm{Sp}^M$. The stable Pointryagin–Thom collapse map can be viewed as a map $c\colon \mathbb{S} \to M_! D_M$, which deserves the name "fundamental class" for M. It is possible[1] to describe the cap product with the fundamental class as a morphism in $\mathrm{Fun}(\mathrm{Sp}^M, \mathrm{Sp})$

$$c \cap -\colon M_*(-) \to M_!(- \otimes D_M) \tag{1}$$

and Poincaré duality may be interpreted as demanding that this transformation be an equivalence. It turns out that using general Morita theory, it is possible to construct a unique $D_X \in \mathrm{Sp}^X$ and $c\colon \mathbb{S} \to X_! D_X$ for *any compact space* X such that the associated map $c \cap -\colon X_*(-) \to X_!(- \otimes D_X)$ is an equivalence (since it is such a crucial property, we term the property of this map being an equivalence as *twisted ambidexterity*, inspired by [Cno23]). The local system D_X is referred to as the dualising sheaf of X. We call the compact space X a *Poincaré space* if the dualising sheaf D_X takes values in invertible spectra, i.e. in $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$. Poincaré duality, as formulated above, shows that closed manifolds are Poincaré spaces. It bears mentioning that Wall, in his seminal paper [Wal67], introduced the notion of *Poincaré complexes* and we show in Example 3.2.15 that his notion coincides with Poincaré spaces as defined above (see also [Lan22, Prop. A.12] for a proof for finite spaces).

Now the theory of parametrised higher categories as introduced in [MW22; MW24] affords us the latitude of considering the situation just presented, but working internally to an arbitrary base topos $\mathcal{B}$ (e.g. the topos $\mathcal{S}_G$ of G–spaces). In this setting, one may speak of $\mathcal{B}$–functor categories, $\mathcal{B}$–adjunctions, $\mathcal{B}$–Kan extensions, $\mathcal{B}$–(co)limits, $\mathcal{B}$–Morita theory, etc. For example, in the equivariant situation, we may take parametrised colimits with respect to a diagram indexed by a G–category (and so in particular, diagrams indexed by G–spaces). In light of this, we may just transpose the discussions of the previous paragraph into the parametrised setting with relative ease and make sense of the notion of parametrised Poincaré duality.

However, in our theory, we have chosen to strictly generalise the well–known presentation above in two main ways: (1) we do not just consider the coefficient category of spectra (or rather $\mathcal{B}$–spectra), but we allow for arbitrary symmetric monoidal coefficient categories (which was also done in [Cno23] in the twisted ambidextrous setting); (2) we do not just consider *presentable* coefficient $\mathcal{B}$–categories, as in [Cno23], but also arbitrary $\mathcal{B}$–categories. As we shall see in our geometric applications later, both of these extra flexibilities will play important and conceptually natural roles. Crucially, point (2) precludes us from having access to

[1] This is conveniently descibed in [Lur11] or [Lan22], which we generalise in §3.1.

Morita theory, by which token being a $\mathcal{B}$–Poincaré space is a property. Hence, we will have to declare more structures in order to be able to speak of Poincaré duality in arbitrary coefficient categories. We axiomatise this as follows:

Definition (Spivak data, Definition 3.1.1)**.** Let $\underline{X}$ be a $\mathcal{B}$–space and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}$–category admitting $\underline{X}$–shaped colimits. A *$\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$* is defined to be a pair (ξ, c) where $\xi \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ is called the *dualising sheaf* and $c \colon \mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ is a morphism in $\underline{\mathcal{C}}$ called the *fundamental class.*

From this datum, provided $\underline{\mathcal{C}}$ satisfies a standard condition called the $\underline{X}$–projection formula (c.f. Terminology 2.1.13), we may construct from (ξ, c) a transformation

$$c \cap_\xi - \colon X_*(-) \longrightarrow X_!(- \otimes \xi) \tag{2}$$

as in (1), called the capping transformation. This is a morphism in $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}})$. We then say that the $\underline{\mathcal{C}}$–Spivak datum (ξ, c) is *twisted ambidextrous* if (2) is an equivalence, and we say that it is *Poincaré* if additionally $\xi \colon \underline{X} \to \underline{\mathcal{C}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$. As will become clear in the article, one advantage of studying such a structural axiomatisation of the situation is that it provides us with a finer control over the specific fundamental class and capping equivalences at play. Also note that this approach is very close to traditional formulations of Poincaré duality and even covers general duality groups in the sense of Bieri-Eckmann [BE73], which we hope helps clarify the relation of modern works such as Cnossen's [Cno23] with classical literature.

Having set up the primitive notions of the paper, we focus on the equivariant setting (i.e. working over the base topos $\mathcal{B} = \mathcal{S}_G$ of G–spaces) for the rest of the introduction and point out the more general parametrised versions along the way, as appropriate.

We now state one of the main theorems of our abstract equivariant Poincaré duality theory. For ease of statement in this introductory section, we only state it for presentable G–categories, where being Poincaré is a property of a G–space (i.e. the Spivak datum is unique, if it exists).

Theorem B (Poincaré isotropy basechange, Theorem 4.2.7)**.** *Let G be a compact Lie group, N a closed normal subgroup, $\underline{X}$ a G–space, and $\underline{\mathcal{C}}$ a presentably symmetric monoidal fibrewise stable G–category. If $\underline{X}$ is Poincaré with coefficient $\underline{\mathcal{C}}$, then the G/N–fixed points space $\underline{X}^N$ is Poincaré with coefficient in the fibrewise stable Brauer quotient G/N–category $\Phi^N\underline{\mathcal{C}}$ of Theorem A.*

In the full statement, the result above works in the generality of a fixed family of closed subgroups and we also provide a version of the theorem for small categories with a weaker conclusion, but which is nevertheless strong enough for our applications in §5.1. Furthermore, Theorem B will be the key tool for our categorified Smith–theoretic proof of Theorem H.

It should also be mentioned that the theorem above is an immediate consequence of a much more general set of basechange results for arbitrary base topoi (c.f. Theorems 3.3.5, 3.3.8 and 3.3.12). These general results constitute the main theorems in our theory of parametrised Poincaré duality. The operating philosophy of these results, and thus of the paper by extension, is that many important inductive manoeuvres on Poincaré duality may be casted as instances of basechanging the coefficient categories and basechanging the ambient topoi.

The most important coefficient category for us will be that of genuine G–spectra $\underline{\mathrm{Sp}}$, and we say that a G–space is G–Poincaré if it is Poincaré with respect to $\underline{\mathrm{Sp}}$. As a straightforward consequence of Theorem B, we obtain the following result which says that being Poincaré is compatible with taking fixed points. It should be viewed as a spectral enhancement of the homological statement [CW17, Prop. 2.4] of Costenoble–Waner.

Theorem C (Theorem 4.2.9)**.** *Let G be a compact Lie group, $H \leq G$ a closed subgroup, and $N \leq G$ a closed normal subgroup. If $\underline{X}$ is a G–Poincaré space, then X^H is a Poincaré space and $\underline{X}^N$ is a G/N–Poincaré space.*

In fact, we also provide a conditional converse to the result above in Theorem 4.2.10 where we give a recognition principle for equivariant Poincaré duality in terms of nonequivariant Poincaré duality by way of the geometric fixed points functors. We warn the reader that the converse - that a compact G-space $\underline{X}$ is G-Poincaré if X^H is Poincaré for each closed subgroup $H \leq G$ - is *not* true, and we will comment on this below.

Next, to justify the theory of equivariant Poincaré spaces, we first give a large collection of examples of such as encapsulated by the following (the smooth manifolds case is certainly not new and has been proven in various forms for example in [MS06; CW16; CW17; HKZ24]):

Theorem D (Proposition 4.4.2 and Theorem 4.4.8)**.** *Let G be a compact Lie group. Then smooth closed G–manifolds and tom Dieck's generalised homotopy representations are G–Poincaré spaces.*

Here, by tom Dieck's generalised homotopy representations, we mean a compact G–space $\underline{\mathcal{V}}$ such that all fixed points $\mathcal{V}^H$ have the homotopy type of a sphere of some dimension. They are a class of G–spaces strictly distinct from smooth G–manifolds. For example, Bredon [Bre72] gave an example of a generalised C_2–homotopy representation $\underline{\mathcal{V}}$ such that $\mathcal{V}^{C_2}$ and $\mathcal{V}^e$ are spheres of the same dimension, although the map $\mathcal{V}^{C_2} \to \mathcal{V}^e$ is not an equivalence. Of course, this cannot arise as the underlying C_2–space of a smooth closed C_2–manifold.

With plenty of naturally interesting examples in hand, we then provide a suite of construction principles to construct new examples of equivariant Poincaré spaces from old ones in §4.3. Among other things, we show that equivariant Poincaré duality is preserved under various standard equivariant operations such as inflations, restrictions, inductions, coinductions, and Borelifications. We also make contact with the nilpotence theory of Mathew–Naumann–Noel [MNN17; MNN19] and show that Poincaré duality interacts nicely with nilpotence with respect to families. Furthermore, we also show the following equivariant generalisation of Klein's well–known result [Kle01, Cor. F].

Theorem E (Theorem 4.3.12)**.** *Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal G–category and $f\colon \underline{X} \to \underline{Y}$ a map of G–spaces. If $\underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré and for every closed subgroup $H \leq G$, the fibres of f over every H–point of $\underline{Y}$ is $\mathrm{Res}^G_H \underline{\mathcal{C}}$–Poincaré, then $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré too.*

Having set up a robust and nonempty abstract theory, we now ask ourselves: what does it all mean and what is it useful for?

Phenomena and applications

It turns out that equivariant Poincaré duality for a G–space $\underline{X}$ offers quite a lot "hidden" homotopical information about $\underline{X}$ that is not obvious from merely having all its fixed points satisfying Poincaré duality. To put it in a slogan, this is essentially because there is a global fundamental class which ties together the local fundamental classes of the various fixed points spaces in nontrivial ways. This is certainly not a new observation and is one that has been appreciated by many of the forerunners to this story. For the remainder of this introduction, we highlight three applications of a geometric flavour of our theory which exploit this principle in one form or another and which illustrate the rigidity of Poincaré spaces hinted at before.

Let p be a prime and G a finite group. Now for a G–space $\underline{X}$, we may view the cohomology group $H^*(X^G;\mathbb{F}_p)$ as a count of the fixed points of $\underline{X}$ mod p. As aptly interpreted by Browder in [Bro87], if a map of G–spaces $f\colon \underline{X} \to \underline{Y}$ induces an injection on $H^*(-;\mathbb{F}_p)$, then we may "pull back" the mod p fixed points of $\underline{Y}$ to those of $\underline{X}$ and, among many things, he studied situations in which one can upgrade this cohomological statement to an actual surjection on the fixed points as topological spaces. Cohomological injection results of this type were first proved by Bredon as [Bre73, Thm. 5.1] for the group $G = C_p$ purely homotopy–theoretically and later on generalised by Browder as [Bro72, Thm. 1.1] to arbitrary finite abelian p–groups under stronger manifold assumptions. This question has also been studied for instance in [ES86; HP06]. In this line, we employ our categorical technology in the generality of Poincaré duality for small, non–presentable coefficient categories to prove the following version of the aforementioned results:

Theorem F (Theorem 5.1.1)**.** *Let A be an elementary abelian p–group. Let $f\colon \underline{X} \to \underline{Y}$ be a map of compact A–spaces. Suppose X^e, Y^e are $\mathrm{H}\mathbb{F}_p$–Poincaré spaces such that $f^e\colon X^e \to Y^e$ is of degree one. Then for any $\mathrm{H}\mathbb{F}_p$–local system $\zeta \in \mathrm{Fun}(Y^A, \mathrm{Perf}_{\mathrm{H}\mathbb{F}_p})$ for the fixed point space Y^A, the map f^A induces an injection $H^*(Y^A;\zeta) \to H^*(X^A; f^*\zeta)$.*

Unlike the cited works above, our methods avoid manifold assumptions altogether and apart from one preliminary standard argument, we also avoid spectral sequences entirely and use instead formal categorical and stable homotopy theoretic manipulations. It should be noted also that our result works for arbitrary twisted coefficient systems, which as far as we know, is new and depends crucially on the categorical nature of our approach.

As an application of Theorem F (in fact, the version proved by Bredon suffices), we obtain the following rigidity result for equivariant Poincaré duality spaces.

Theorem G (Theorem 5.1.14)**.** *Let G be a solvable finite group (e.g. a group of odd order) and $\underline{X} \in \mathcal{S}^{\omega}_{G}$ a compact G–Poincaré space with $X^e \simeq * \in \mathcal{S}$. Then $\underline{X} \simeq \underline{*} \in \mathcal{S}_G$.*

This is a slightly surprising result in light of Bredon's examples mentioned after Theorem D which demonstrated that Poincaré spaces can be rather counterintuitive when the underlying space is noncontractible. Combining this with the celebrated theorem of Jones [Jon71] on the converse to Smith theory, we construct an example of a compact C_p–space whose underlying and fixed points spaces are Poincaré but which is not itself C_p–Poincaré, making good on our warning after Theorem C.

We mention one more application, whose investigation was one of the main goals of this project. In [CF64], Conner–Floyd made the following conjecture:

> *There cannot exist a periodic differentiable map of odd prime power period acting on a closed oriented manifold V^n, $n > 0$ preserving the orientation and possessing exactly one fixed point.*

The first proof of this statement (in fact, a slightly more general version) was given by Atiyah-Bott in [AB68] and soon after by Conner–Floyd [CF66] themselves. Many variations have been proven since then, and we mention [Lüc88; ABK92] as further examples. Atiyah-Bott's argument uses Atiyah–Singer's index theory, whereas Conner–Floyd's proof uses a particular bordism spectrum. In all these cases, local structures in the geometric settings were used in essential ways.

Inspired by the notion of a "gluing class" due to Lück which measures how the singular part of a G–space is glued into the whole space, we consider such a construction in our setting and use it to prove a very general version of the Conner–Floyd conjecture which in particular yields the theorem of Atiyah–Bott as an immediate corollary.

Theorem H (Theorem 5.2.2)**.** *Let p be an odd prime, $G = C_{p^k}$ for some k, and suppose $\underline{X} \in \mathcal{S}_G^\omega$ is a G–Poincare space such that the underlying space $X^e \in \mathcal{S}^\omega$ is connected, $\mathbb{Z}$–orientable, and has formal dimension (in the classical sense) $d > 0$. Then $X^G \not\simeq *$.*

Our proof uses categorified Smith–theoretic methods afforded to us by Theorem B which reduces the problem to various forms of Tate cohomology considerations. In particular, it is fully homotopical and thus is of a "global" nature. We hazard a suggestion here that, apart from being a new generalisation of a very classical result, Theorem H locates the explanation of such phenomena in the global realm of homotopy theory as opposed to the local one of geometry. Finally, let us point out that when paired together, Theorems G and H give a curious partial dichotomy for C_{p^k}–Poincaré spaces delineated by whether or not the underlying space is contractible.

Before closing the introduction, we mention that since the theory of Poincaré duality here was developed in the generality of Martini–Wolf's parametrised category theory, it might be interesting to explore the theory presently developed in the context of topoi other than the equivariant ones. It could be said that the defining feature of our work is in exploiting various kinds of geometric morphisms of topoi central to equivariant homotopy theory, and one can imagine that this might also lead to fruitful lines of pursuit in other contexts.

Relations to other work

The following works are some of the milestones that made this article possible. Wall introduced the notion of a Poincaré space motivated by surgery theory, and developed their theory in [Wal67]. Klein built up an impressive amount of theory related to Poincaré spaces, one of his most influential concepts being that of the dualising spectrum [Kle01; Kle07]. His approach was revisited by Lurie [Lur11, Lecture 26], Nikolaus-Scholze [NS18, Sec. I.4.1.] and Land [Lan22, Appendix A], also providing an account of the "universality" of Klein's construction. An advantage of Klein's approach is that the stable Spivak fibration of a Poincaré space admits a categorically more natural description (as the dualising spectrum) than in Wall's original work, where it had to be constructed. The theory of dualising spectra in general was coined "twisted ambidexterity" by Cnossen in [Cno23].

Cnossen develops twisted ambidexterity in a general topos in terms of parametrised homotopy theory, and his approach is what we most closely follow. His motivation is a characterisation of the G-category of G-spectra as the initial presentable, fibrewise stable G-category in which all compact G-spaces are twisted ambidextrous. An important predecessor in the equivariant context is the book of May-Sigurdsson [MS06], which also gives an account of equivariant Poincaré duality. A more classical approach to equivariant Poincaré duality in terms of equivariant homology and cohomology (over the Burnside ring), much in the spirit of Wall's original definition can be found in the work of Costenoble-Waner [CW92; CW17]. For finite groups G, a nonabelian version of equivariant Poincaré duality for so–called "V–framed manifolds" has also been studied in [Zou23; HKZ24] which in particular implies the homological version of Poincaré duality for such objects, c.f. [HKZ24, Prop. 4.1.4]. An approach to Poincaré duality in the context of six-functor-formalisms is to be found for instance in [Sch23, Lecture V].

Organisation of the paper

In §2, we introduce and develop the categorical underpinnings that will support the later sections. In more detail, we recall in §2.1 the Martini–Wolf theory of parametrised higher categories and take the opportunity to record some elementary observations about geometric morphisms that we need. In §2.2, we specialise the preceding discussions to the equivariant context and recall the standard gamut of equivariant operations on categories; this will lead to the proof of Theorem A, categorifying the well–known geometric fixed points functor. This will be used later to articulate our results about fixed points of equivariant Poincaré spaces.

Having set up the requisite language, we turn to the matter of defining and studying Poincaré duality in §3 in the general context of parametrising over arbitrary topoi. We define and work out the basic properties of Spivak data in §3.1; we then use this structure to define twisted ambidexterity and Poincaré duality in §3.2 with respect to arbitrary coefficient $\mathcal{B}$–categories. In §3.3, we give several constructions one can perform on Spivak data and prove the main results of the section in the form of Theorems 3.3.5 and 3.3.8 on basechanging coefficient categories and Theorem 3.3.12 on basechanging the base topoi. Finally, we set up a theory of degrees for maps between Poincaré spaces in §3.4.

We specialise the general parametrised theory in §3 to the equivariant situation in §4 for compact Lie groups. After recording the specialisations of the notions in §4.1, we state and prove several isotropy separation statements including Theorems B and C in §4.2, which will form our main suite of techniques for dealing with fixed points of equivariant Poincaré spaces. Following that, we provide a set of construction principles in §4.3 to generate new equivariant Poincaré spaces from old ones and we supply in §4.4 geometrically natural examples of Poincaré spaces. We then introduce the notion of gluing classes in §4.5 that will form the main obstruction class for our applications in §5.2, and we lay down a rudimentary theory of equivariant degrees in §4.6.

In the final §5, we use categorified Smith–theoretic methods supported by the abstract theory developed in the article to give two strands of applications: in §5.1, we use degree theory to show Theorem F, which is in turn used to show Theorem G; then, in §5.2, we use the gluing classes to prove Theorem H.

Beyond the main body of the article, we record in Appendix A several characterisations of G–stability for presentable categories when G is a compact Lie group, and we prove a standard observation about reflecting pushout squares in Appendix B.

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2. Preliminaries

The present section reviews the techniques that are essential to our approach to parametrised Poincaré duality. In §2.1, we first recall some preliminaries on category theory parametrised by a topos $\mathcal{B}$. Special attention is given to presentable $\mathcal{B}$-categories, and basechange methods that allow us to switch the base topos along a geometric morphism. These basechange results will be essential for the isotropy separation arguments in the equivariant context.

After that, §2.2 specialises to topoi related to the category $\mathcal{S}_G$ of G-spaces, where G is a compact Lie group. This also features various change-of-group functors like induction, restriction and coinduction along a homomorphism of compact Lie groups $\alpha\colon H \to G$, as well as the theory of families. We give a quick recollection on the basics of G-spaces. After that, we record some facts on equivariant stability, preservation of equivariant stability under change-of-group functors, and multiplicative properties of these constructions. We then prove the main result of this section, namely Theorem A on Brauer quotients which categorifies the geometric fixed points functors. The section ends with some remarks on free actions that will be used later on.

2.1. Parametrised category theory

For the rest of this section, let $\mathcal{B}$ be a topos. For us, most topoi of interest will actually be presheaf topoi, and for the purpose of this article the most important such will be that of presheaves on the orbit category $\mathcal{O}(G)$ for a compact Lie group G. As category theory internal to $\mathcal{B}$ is essential to our considerations, we give a short recollection of the formalism developed by Martini and Martini-Wolf in the series of articles [Mar22b; Mar22a; MW22; MW24]. Let us mention here that the theory of categories parametrised by a topos was preceded by about a decade by the theory of categories parametrised by presheaf topoi, pioneered by Barwick–Dotto–Glasman–Nardin–Shah in [BDG+16b; BDG+16a; Nar16; Sha23].

Definition 2.1.1. A $\mathcal{B}$*–category* $\underline{\mathcal{C}}$ is a limit preserving functor $\underline{\mathcal{C}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$, i.e. a sheaf of categories on $\mathcal{B}$. Denote by $\mathrm{Cat}_{\mathcal{B}} \subseteq \mathrm{PSh}_{\mathrm{Cat}}(\mathcal{B})$ the full subcategory on $\mathcal{B}$–categories. Maps in $\mathrm{Cat}_{\mathcal{B}}$ are called $\mathcal{B}$*-functors*.

In [Mar22b], Martini produces an equivalence of categories

$$\mathrm{PSh}_{\mathrm{Cat}}(\mathcal{B}) \supseteq \{\mathrm{Cat}\text{-valued sheaves on } \mathcal{B}\} \simeq \{\text{Complete Segal objects in } \mathcal{B}\} \subseteq s\mathcal{B}.$$

It is worthwhile to study $\mathcal{B}$–categories from both perspectives, the *parametrised* point of view on the left as well as the *internal* point of view on the right.

As our arguments will often require us to work with unparametrised categories and $\mathcal{B}$–categories at the same time, we follow the convention of underlining $\mathcal{B}$ categories, so a generic $\mathcal{B}$ category is denoted $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{\mathcal{E}}, \ldots$ and so on. For example, the category of G–spaces will be denoted by $\mathcal{S}_G$ while the G–category of G–spaces is written $\underline{\mathcal{S}}$ (or $\underline{\mathcal{S}}_G$ if we want to emphasise that this is happening for the group G).

Example 2.1.2 (Presheaf topoi)**.** For a small category T, consider the presheaf topos $\mathrm{PSh}(T) = \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{S})$. We write $\mathrm{Cat}_T := \mathrm{Cat}_{\mathrm{PSh}(T)}$ and call it the category of T*–categories*. Restriction along the Yoneda embedding $T \hookrightarrow \mathrm{PSh}(T)$ induces an equivalence $\mathrm{Cat}_T \xrightarrow{\simeq} \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat})$ so a T–category is simply a functor $T^{\mathrm{op}} \to \mathrm{Cat}$. In particular for $T = *$, we see $\mathrm{Cat}_{\mathcal{S}} \simeq \mathrm{Cat}$ and an $\mathcal{S}$–category is just an ordinary (∞-)category. In the special case where G is a finite group (or a compact Lie group) and $T = \mathcal{O}(G)$ is its orbit category, the category of transitive G-sets (or homogeneous G–spaces), we obtain the category $\mathrm{Cat}_G := \mathrm{Cat}_{\mathcal{O}(G)}$ of G–categories.

Example 2.1.3 ($\mathcal{B}$-groupoids)**.** The Yoneda embedding $\mathcal{B} \to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat})$ restricts to a limit preserving fully faithful functor $\mathcal{B} \to \mathrm{Cat}_{\mathcal{B}}$. An object in the essential image will be referred to as a $\mathcal{B}$*-groupoid*.

Switching to the internal picture, Martini [Mar22b, Section 3.1] also characterised $\mathcal{B}$-groupoids as the complete Segal objects that are equivalent to constant simplicial objects in $\mathcal{B}$. We will not distinguish between $\mathcal{B}$-groupoids and objects in $\mathcal{B}$, so to avoid confusion we also denote both with an underline by $\underline{X}, \underline{Y}, \underline{Z}, \ldots$, except in the spectial case $\mathcal{B} = \mathcal{S}$.

With equivariant applications in mind, it will be important for us to change the base topos $\mathcal{B}$. This can be done along *geometric morphisms* of topoi, defined as:

Definition 2.1.4. A geometric morphism is an adjunction between topoi $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ whose left adjoint f^* is left exact, i.e. commutes with finite limits.

Construction 2.1.5 (Basechange along geometric morphisms)**.** A geometric morphism $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ of topoi induces an adjoint pair $f^* \colon \mathrm{Cat}_{\mathcal{B}} \rightleftharpoons \mathrm{Cat}_{\mathcal{B}'} : f_*$ where the right adjoint f_* is given by restriction along $(f^*)^{\mathrm{op}} \colon \mathcal{B}^{\mathrm{op}} \to (\mathcal{B}')^{\mathrm{op}}$.

In the internal picture, the functor $f^* \colon \mathcal{B} \to \mathcal{B}'$ applied entrywise induces a functor on simplicial objects $s\mathcal{B} \to s\mathcal{B}'$, which commutes with finite limits. By [Mar22b, Lem. 3.3.1], it restricts to a functor on complete Segal objects, and this is how to obtain $f^* \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}'}$.

Lemma 2.1.6. *For a geometric morphism $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ of topoi, the functor $f^* \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$ preserves finite limits. In fact, it preserves all limits if $f^* \colon \mathcal{B} \to \mathcal{B}'$ does.*

Proof. The induced functor $sf^* \colon s\mathcal{B} \to s\mathcal{B}'$ commutes with finite limits, and complete Segal objects are closed under limits. Thus, f^* preserves finite limits by being a restriction of a finite limit preserving functor to a category closed under (finite) limits. ☐

Example 2.1.7 (Geometric morphisms from presheaves)**.** A good supply of examples of geometric morphisms is given by considering a functor $f \colon S \to T$ of small categories. Then restriction and right Kan extension along $f^{\mathrm{op}} \colon S^{\mathrm{op}} \to T^{\mathrm{op}}$ induce a geometric morphism $f^* \colon \mathrm{PSh}(T) \rightleftharpoons \mathrm{PSh}(S) : f_*$.

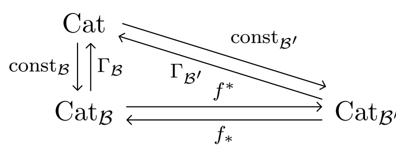
Example 2.1.8 (Étale geometric morphisms)**.** If $\mathcal{B}$ is a topos and $\underline{X} \in \mathcal{B}$, then so is the slice category $\mathcal{B}_{/\underline{X}}$. We have an ajunction

$$(\pi_{\underline{X}})^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/\underline{X}} : (\pi_{\underline{X}})_* \tag{3}$$

whose left adjoint takes $\underline{A} \in \mathcal{B}$ to $\underline{A} \times \underline{X} \to \underline{X}$. Now $\mathcal{B}_{/\underline{X}}$ itself is a topos, and the adjunction above is in fact a geometric morphism of topoi. Geometric morphisms equivalent (in the category of topoi) equivalent to such of this kind are called *étale geometric morphisms*, see [Lur09, Sec. 6.3.5.] for a detailed account. A special feature of étale geometric morphisms is that in the adjunction (3) a further left adjoint exists, and so $(\pi_{\underline{X}})^*$ commutes with all limits. The further left adjoint forgets the map to $\underline{X}$ and is denoted by $(\pi_{\underline{X}})_!$. A useful characterisation of étale geometric morphisms is given in [Lur09, Prop. 6.3.5.11.].

Note that if a geometric morphism $f^* \colon \mathrm{Cat}_{\mathcal{B}} \rightleftharpoons \mathrm{Cat}_{\mathcal{B}'} : f_*$ is étale, then Lemma 2.1.6 shows that $f^* \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}'}$ preserves all limits.

Example 2.1.9 (Constant categories and global sections)**.** Recall from [Lur09, Prop. 6.3.4.1] that there is a unique geometric morphism $\mathrm{Const} \colon \mathcal{S} \rightleftharpoons \mathcal{B} \colon \Gamma$. It induces an adjunction $\mathrm{Const} \colon \mathrm{Cat} \rightleftharpoons \mathrm{Cat}_{\mathcal{B}} \colon \Gamma$. We refer to Γ as the *global sections* functor. Explicitly, it is given by evaluation at the terminal object in $* \in \mathcal{B}$. Since geometric morphisms from $\mathcal{S}$ are unique, for any geometric morphism $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$, we get a triangle of geometric morphisms



In particular, note that $\mathrm{const}_{\mathcal{B}'} \simeq f^* \,\mathrm{const}_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}} f_* \simeq \Gamma_{\mathcal{B}'}$.

Example 2.1.10 (Internal functor categories and 2-categorical structures)**.** The category $\mathrm{Cat}_{\mathcal{B}}$ is cartesian closed. This means that for any $\mathcal{B}$–category $\underline{\mathcal{C}}$ the product functor $- \times \underline{\mathcal{C}} \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$ admits a right adjoint $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}, -) \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$. We call $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ the *$\mathcal{B}$–category of $\mathcal{B}$-functors* and denote its global sections (in the sense of Example 2.1.9) by $\mathrm{Fun}_{\mathcal{B}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$. Maps in $\mathrm{Fun}_{\mathcal{B}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are called *$\mathcal{B}$-natural transformations*. $\mathrm{Fun}_{\mathcal{B}}$ can be enhanced to a Cat-enrichment of $\mathrm{Cat}_{\mathcal{B}}$ making $\mathrm{Cat}_{\mathcal{B}}$ into a 2-category, see [Mar22b, Remark 3.4.3].

Definition 2.1.11 (Adjoint functors)**.** Using the 2-categorical structure on $\mathrm{Cat}_{\mathcal{B}}$, one can define an adjoint pair of $\mathcal{B}$-functors as an internal adjunction in $\mathrm{Cat}_{\mathcal{B}}$. Explicitly, an adjunction consists of a pair of $\mathcal{B}$-functors $L \colon \underline{\mathcal{C}} \rightleftharpoons \underline{\mathcal{D}} \colon R$ as well as a pair of natural transformations $\eta \colon \mathrm{id}_{\underline{\mathcal{C}}} \to RL$, $\epsilon \colon LR \to \mathrm{id}_{\underline{\mathcal{D}}}$. satisfying the triangle identities in the sense that $\epsilon_L \circ L\eta$ and $R\epsilon \circ \eta_R$ are equivalent to the respective identities.

We recall here the key standard categorical concept that will underpin most this article.

Construction 2.1.12. Suppose we have a commuting square of $\mathcal{B}$–categories

$$\begin{array}{ccc} \underline{\mathcal{C}} & \xrightarrow{f} & \underline{\mathcal{D}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \underline{\mathcal{C}}' & \xrightarrow{f'} & \underline{\mathcal{D}}' \end{array}$$

such that f, f' admit a right adjoints h, h' respetively. Then we obtain a transformation

$$\mathrm{BC}_*\colon gh \xrightarrow{\eta_{gh}} h'f'gh \simeq h'g'fh \xrightarrow{h'g'\epsilon} h'g'$$

called the *right Beck–Chevalley transformation*. Similarly, if f, f' admit left adjoints ℓ, ℓ' respectively, then we obtain a transformation

$$\mathrm{BC}_!\colon \ell'g' \xrightarrow{\ell'g'\eta} \ell'g'f\ell \simeq \ell'f'g\ell \xrightarrow{\epsilon_{g\ell}} g\ell$$

called the *left Beck–Chevalley transformation*. We will often omit the words "left" and "right" when the context is clear. These transformations enjoy excellent functoriality properties, and we refer the reader to [CSY22, §2.2] for a good source on these matters.

The following is an important class of Beck–Chevalley transformations.

Terminology 2.1.13 (Projection formula)**.** Let $\underline{J}$ be a $\mathcal{B}$–category and $r\colon \underline{J} \to \underline{*}$ the unique map. We say that a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ *satisfies the $\underline{J}$-projection formula* if it admits $\underline{J}$-shaped colimits and the Beck–Chevalley transformation

$$\mathrm{PF}^J_!\colon r_!(\xi \otimes r^*(-)) \to r_!\xi \otimes (-)$$

of functors $\underline{\mathcal{C}} \to \underline{\mathcal{C}}$ is an equivalence for all $\xi \in \underline{\mathcal{C}}^{\underline{J}}$.

In [MW24, Prop. 3.2.9.] it is shown, using work on relative adjunctions due to Lurie, that a functor $R\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ admits a left adjoint if and only if the following conditions are satisfied:

1. For every object $\underline{X} \in \mathcal{B}$ the map $R(\underline{X})\colon \mathcal{C}(\underline{X}) \to \mathcal{D}(\underline{X})$ admits a left adjoint $L(\underline{X})$.
2. For every map $f\colon X \to Y$ in $\mathcal{B}$ the Beck–Chevalley transformation $f^*L(\underline{X}) \to L(\underline{Y})f^*$ is an equivalence.

Example 2.1.14 (Limits and colimits)**.** A $\mathcal{B}$–category $\underline{\mathcal{C}}$ is said to admit *$\underline{I}$-shaped $\mathcal{B}$-(co)limits* if the restriction functor $I^*\colon \underline{\mathcal{C}} \to \underline{\mathrm{Fun}}(\underline{I}, \underline{\mathcal{C}})$ along $I\colon \underline{I} \to \underline{*}$ admits a right (resp. left) adjoint. The right adjoint will usually be denoted by I_*, the left adjoint by $I_!$. Note that for example, the adjunction unit for $I_! \dashv I^*$ produces for each $F \in \mathrm{Fun}_{\mathcal{B}}(\underline{I}, \underline{\mathcal{C}})$ a natural transformation

$$F \to I^*I_!F \in \mathrm{Fun}_{\mathcal{B}}(\underline{I}, \underline{\mathcal{C}})$$

which should be thought of as analogous to the diagram defining a colimit in unparametrised category theory.

Example 2.1.15 (Symmetric monoidal categories)**.** A *symmetric monoidal $\mathcal{B}$–category* is a commutative monoid in $\mathrm{Cat}_{\mathcal{B}}$. So $\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ is the category of symmetric monoidal $\mathcal{B}$–categories and symmetric monoidal functors. Notice that $\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ is equivalent to the category of $\mathrm{CMon}(\mathrm{Cat})$–valued sheaves on $\mathcal{B}$.

Geometric and étale morphisms of topoi

We record here further miscellaneous elementary observations about geometric and étale morphisms that will be relevant to us later. Since this will just be a litany of minor technical results, the reader is advised to skip this on first reading and return to it as needed.

Lemma 2.1.16. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. There is an equivalence, natural in $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}'}$ of functors $f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}})$. Moreover, if $\underline{\mathcal{C}}$ were a symmetric monoidal $\mathcal{B}$–category, then this equivalence naturally upgrades to a symmetric monoidal one.*

Proof. Note that the diagram

$$\begin{array}{ccccc} \mathrm{Cat}_{\mathcal{B}'} & \xrightarrow{(\pi_{f^*\underline{X}})^*} & \mathrm{Cat}_{\mathcal{B}'_{/f^*\underline{X}}} & \xrightarrow{(\pi_{f^*\underline{X}})_*} & \mathrm{Cat}_{\mathcal{B}'} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Cat}_{\mathcal{B}} & \xrightarrow{(\pi_{\underline{X}})^*} & \mathrm{Cat}_{\mathcal{B}_{/\underline{X}}} & \xrightarrow{(\pi_{\underline{X}})_*} & \mathrm{Cat}_{\mathcal{B}} \end{array} \tag{4}$$

commutes as the corresponding diagram

$$\begin{array}{ccccc} \mathcal{B}' & \xrightarrow{(\pi_{f^*\underline{X}})^*} & \mathcal{B}'_{/f^*\underline{X}} & \xrightarrow{(\pi_{f^*\underline{X}})_*} & \mathcal{B}' \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{B} & \xrightarrow{(\pi_{\underline{X}})^*} & \mathcal{B}_{/\underline{X}} & \xrightarrow{(\pi_{\underline{X}})_*} & \mathcal{B} \end{array}$$

of topoi commutes (this can be checked after passing to left adjoints everywhere where it is easy to see, see e.g. [Lur09, Remark 6.3.5.8.]). Now the composite $\mathrm{Cat}_{\mathcal{B}} \xrightarrow{(\pi_{\underline{X}})^*} \mathrm{Cat}_{\mathcal{B}_{/\underline{X}}} \xrightarrow{(\pi_{\underline{X}})_*} \mathrm{Cat}_{\mathcal{B}}$ sends $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}}$ to $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) = \lim_{\underline{X}} \underline{\mathcal{C}}$ which proves the first part of the statement.

For the part about symmetric monoidality, note that all functors in (4) are finite limit preserving. As the forgetful functors $\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) \to \mathrm{Cat}_{\mathcal{B}}$ are limit preserving, this shows that the equivalence $f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}})$ from the first part naturally refines to a symmetric monoidal one. ☐

Lemma 2.1.17. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be an étale morphism of topoi. There is an equivalence, natural in $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}}$ of functors $f^*\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{C}})$. Moreover, if $\underline{\mathcal{C}}$ were a symmetric monoidal $\mathcal{B}$–category, then this equivalence naturally upgrades to a symmetric monoidal one.*

Proof. The proof is similar to Lemma 2.1.16. As f is étale, it is equivalent to a functor of the form $(\pi_{\underline{Y}})^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/\underline{Y}} : (\pi_{\underline{Y}})_*$ for some $\underline{Y} \in \mathcal{B}$. Observe that there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Cat}_{\mathcal{B}} & \xrightarrow{(\pi_{\underline{X}})^*} & \mathrm{Cat}_{\mathcal{B}_{/\underline{X}}} & \xrightarrow{(\pi_{\underline{X}})_*} & \mathrm{Cat}_{\mathcal{B}} \\ \downarrow{\scriptstyle (\pi_{\underline{Y}})^*} & & \downarrow{\scriptstyle (\pi_{\underline{Y}})^*} & & \downarrow{\scriptstyle (\pi_{\underline{Y}})^*} \\ \mathrm{Cat}_{\mathcal{B}_{/\underline{Y}}} & \xrightarrow[(\pi_{\underline{X}})^*]{} & \mathrm{Cat}_{\mathcal{B}_{/\underline{X}\times\underline{Y}}} & \xrightarrow[(\pi_{\underline{X}})_*]{} & \mathrm{Cat}_{\mathcal{B}_{/\underline{Y}}} \end{array} \tag{5}$$

coming from the commutative diagram of topoi

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{(\pi_{\underline{X}})^*} & \mathcal{B}_{/\underline{X}} & \xrightarrow{(\pi_{\underline{X}})_*} & \mathcal{B} \\ \big\downarrow{\scriptstyle (\pi_{\underline{Y}})^*} & & \big\downarrow{\scriptstyle (\pi_{\underline{Y}})^*} & & \big\downarrow{\scriptstyle (\pi_{\underline{Y}})^*} \\ \mathcal{B}_{\underline{Y}} & \xrightarrow[(\pi_{\underline{X}})^*]{} & \mathcal{B}_{/\underline{X\times Y}} & \xrightarrow[(\pi_{\underline{X}})_*]{} & \mathcal{B}_{/\underline{Y}}. \end{array}$$

The left square here obviously commutes and commutativity of the right square is easy to check after passing to left adjoints. Now the top right composite sends $\underline{\mathcal{C}}$ to $f^*\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ while the bottom left composite sends it to $\underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{C}})$. For the statement about symmetric monoidality, again note that all functors in (5) are product preserving and use the same argument as in the proof of Lemma 2.1.16. $\square$

Lemma 2.1.18 (Pushforward of parametrised (co)limits)**.** *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. Consider $X \in \mathcal{B}$ and a $\mathcal{B}'$–category $\underline{\mathcal{C}}$ which admits $f^*\underline{X}$-shaped limits and colimits. Then $f_*\underline{\mathcal{C}}$ admits $\underline{X}$-shaped limits and colimits. Furthermore, the equivalence from Lemma 2.1.16 induces an identification of adjoint triples*

$$\begin{array}{ccc} f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{C}}) & \overset{\xrightarrow{f_*r_!}}{\underset{\xrightarrow[f_*r_*]{}}{\xleftarrow{f_*r^*}}} & f_*\underline{\mathcal{C}} \\ \Big\| & & \Big\| \\ \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}}) & \overset{\xrightarrow{r_!}}{\underset{\xrightarrow[r_*]{}}{\xleftarrow{r^*}}} & f_*\underline{\mathcal{C}}. \end{array} \tag{6}$$

Proof. First note that (6) commutes with the leftwards pointing arrows. Since the functor $f_*\colon \mathrm{Cat}_{\mathcal{B}'} \to \mathrm{Cat}_{\mathcal{B}}$ preserves adjunctions (see e.g. [MW24, Cor. 3.1.9.]), it follows that $f_*r_!$ and f_*r_* define left and right adjoints to r^* and (6) also commutes with the rightwards pointing arrows. $\square$

Lemma 2.1.19 (Pullback of parametrised (co)limits)**.** *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be an étale morphism of topoi. Consider $X \in \mathcal{B}$ and a $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$-shaped limits and colimits. Then $f^*\underline{\mathcal{C}}$ admits $f^*\underline{X}$-shaped limits and colimits. Furthermore, the equivalence from Lemma 2.1.17 induces an identification of adjoint triples*

$$\begin{array}{ccc} f^*\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) & \overset{\xrightarrow{f^*r_!}}{\underset{\xrightarrow[f^*r_*]{}}{\xleftarrow{f^*r^*}}} & f^*\mathcal{C} \\ \Big\| & & \Big\| \\ \underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{C}}) & \overset{\xrightarrow{r_!}}{\underset{\xrightarrow[r_*]{}}{\xleftarrow{r^*}}} & f^*\mathcal{C}. \end{array} \tag{7}$$

Proof. The proof is identical to Lemma 2.1.18, using Lemma 2.1.17 instead of Lemma 2.1.16. $\square$

Lemma 2.1.20. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. Let $\underline{J} \in \mathrm{Cat}_{\mathcal{B}}$ and $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{B}'}$, and let $\alpha\colon \underline{\mathrm{Fun}}(f^*\underline{J}, \underline{\mathcal{C}}) \times \mathrm{const}_{\mathcal{B}'} \Delta^1 \to \underline{\mathcal{D}}$ be a natural transformation and $f_*\alpha \circ (\mathrm{id} \times \eta)\colon \underline{\mathrm{Fun}}(\underline{J}, f_*\underline{\mathcal{C}}) \times \mathrm{const}_{\mathcal{B}} \Delta^1 \to \underline{\mathrm{Fun}}(\underline{J}, f_*\underline{\mathcal{C}}) \times f_*f^* \mathrm{const}_{\mathcal{B}} \Delta^1 \xrightarrow{f_*\alpha} f_*\underline{\mathcal{D}}$ the* *associated transformation. Then α is a natural equivalence if and only if $f_*\alpha \circ (\mathrm{id} \times \eta)$ is a natural equivalence.*

Proof. By using that $\Gamma_{\mathcal{B}} f_* \simeq \Gamma_{\mathcal{B}'}$ from Example 2.1.9, the two putatively equivalent statements are equivalent to the condition that the natural transformation $\mathrm{Fun}_{\mathcal{B}'}(f^*\underline{J}, \underline{\mathcal{C}}) \times \Delta^1 \to \Gamma_{\mathcal{B}'}\underline{\mathcal{D}}$ of unparametrised categories is a natural equivalence. □

Notation 2.1.21. Recall that there is the *Picard space functor* $\mathcal{P}\mathrm{ic}\colon \mathrm{CMon}(\mathrm{Cat}) \to \mathrm{CGrp}(\mathcal{S})$ which takes as input a symmetric monoidal category and outputs a space of invertible objects. This functor is right adjoint to the inclusion $\mathrm{CGrp}(\mathcal{S}) \hookrightarrow \mathrm{CMon}(\mathcal{S}) \hookrightarrow \mathrm{CMon}(\mathrm{Cat})$ and is corepresented as $\mathcal{P}\mathrm{ic}(-) \simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{Cat})}(\Omega^\infty \mathbb{S}, -)$.

Lemma 2.1.22. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism. Then we have an equivalence of functors $\underline{\mathcal{P}\mathrm{ic}}(f_*-) \simeq f_*\underline{\mathcal{P}\mathrm{ic}}(-)\colon \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'}) \to \mathrm{CGrp}(\mathcal{B})$. If $f^* \dashv f_*$ were moreover étale, then we also have an equivalence $f^*\underline{\mathcal{P}\mathrm{ic}}(-) \simeq \underline{\mathcal{P}\mathrm{ic}}(f^*-)\colon \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) \to \mathrm{CGrp}(\mathcal{B}')$.*

Proof. The first part is an immediate consequence of the fact that the diagram of left adjoints

$$\begin{array}{ccc} \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) & \hookleftarrow & \mathrm{CGrp}(\mathcal{B}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'}) & \hookleftarrow & \mathrm{CGrp}(\mathcal{B}') \end{array}$$

commutes, which is clear. For the second part, we note that $\underline{\mathcal{P}\mathrm{ic}}(-)\colon \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) \to \mathrm{CGrp}(\mathcal{B})$ is given by $\underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}}\,\Omega^\infty \mathbb{S}, -)$. Thus, since $f^* \dashv f_*$ was étale, we get that $f^*\underline{\mathcal{P}\mathrm{ic}}(-) \simeq f^*\underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}}\,\Omega^\infty \mathbb{S}, -) \simeq \underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{B}'}}(\mathrm{const}_{\mathcal{B}'}\,\Omega^\infty \mathbb{S}, f^*-) \simeq \underline{\mathcal{P}\mathrm{ic}}(f^*-)$. □

Corollary 2.1.23. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi, $\underline{X} \in \mathcal{B}$, $\underline{\mathcal{D}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'})$, and $\underline{\mathcal{E}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$.*

(1) A functor $\underline{X} \to f_*\underline{\mathcal{D}}$ has the property that it factors through $\underline{\mathcal{P}\mathrm{ic}}(f_*\underline{\mathcal{D}}) \hookrightarrow f_*\underline{\mathcal{D}}$ if and only if the associated functor $f^*\underline{X} \to \underline{\mathcal{D}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}}) \hookrightarrow \underline{\mathcal{D}}$,

(2) Suppose $f^* \dashv f_*$ is moreover étale. If a functor $\underline{X} \to \underline{\mathcal{E}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{E}})$, then $f^*\underline{X} \to f^*\underline{\mathcal{E}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(f^*\underline{\mathcal{E}})$.

Proof. Part (1) is an immediate consequence of the equivalence

$$\mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}}\,*, f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}}))) \simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}'}}(\mathrm{const}_{\mathcal{B}'}\,*, \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}})))$$

and the computation

$$\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(f_*\underline{\mathcal{D}})) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}})) \simeq f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}}))$$

where the first equivalence is by Lemma 2.1.22 and the second by Lemma 2.1.16. For part (2), if we have a factorisation $\underline{X} \to \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{E}}) \hookrightarrow \underline{\mathcal{E}}$, then applying f^* to this and using the second part of Lemma 2.1.22 gives the required factorisation. □

Proposition 2.1.24. *Let $f^*: \mathcal{B} \rightleftarrows \mathcal{B}' : f_*$ be a geometric morphism of topoi, $\underline{X} \in \mathcal{B}$, $\underline{\mathcal{D}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'})$, and $\underline{\mathcal{E}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$.*

(1) The symmetric monoidal $\mathcal{B}'$–category $\underline{\mathcal{D}}$ satisfies the $f^*\underline{X}$–projection formula if and only if the symmetric monoidal $\mathcal{B}$–category $f_*\underline{\mathcal{D}}$ satisfies the $\underline{X}$–projection formula,

(2) If f_* is fully faithful, then the colimit $X_!: \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \to f_*\underline{\mathcal{D}}$ (resp. limit X_*) exists if and only if the colimit $(f^*X)_!: \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \to \underline{\mathcal{D}}$ (resp. limit $(f^*X)_*$) does,

(3) If $f^* \dashv f_*$ is moreover étale, then if $\underline{\mathcal{E}}$ satisfies the $\underline{X}$–projection formula, then $f^*\underline{\mathcal{E}}$ satisfies the $f^*\underline{X}$–projection formula.

Proof. For (1), by the symmetric monoidal identification Lemma 2.1.16 and the identification of adjunctions Lemma 2.1.18, we see that for a fixed $A \in \underline{\mathcal{D}}$, applying f_* to the projection formula transformation on the left in

$$\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \underset{(f^*X)_!(-)\otimes A}{\overset{(f^*X)_!(-\otimes (f^*X)^*A)}{\rightrightarrows}} \underline{\mathcal{D}} \;\; \Downarrow \mathrm{PF} \qquad \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \underset{X_!(-)\otimes f_*A}{\overset{X_!(-\otimes X^*A)}{\rightrightarrows}} f_*\underline{\mathcal{D}} \;\; \Downarrow \mathrm{PF}$$

yields the projection formula transformation on the right. Thus, by Lemma 2.1.20, we see that the left projection formula transformation is an equivalence if and only if the right one is.

For (2), that the existence of $(f^*X)_!$ implies the existence of $X_!$ is by Lemma 2.1.18. For the converse, we use again the diagram Lemma 2.1.18 together with the fact f^* preserves adjunctions by [MW24, Cor. 3.1.9] and that $f^*f_* \simeq \mathrm{id}$ by fully faithfulness.

Part (3) is proved similarly as in (1), but using Lemma 2.1.17 and Lemma 2.1.19 instead. ☐

Presentability

Presentable categories are useful for many reasons, among them being that they have all (co)limits, fulfill the adjoint functor theorem, and have a symmetric monoidal structure coming from the Lurie tensor product. Presentability in the parametrised context was first studied in [Nar17] and later on in [Hil22]. Subsequently, Martini-Wolf [MW22] introduced and developed a much more general theory for $\mathcal{B}$–categories, and this is the theory that we will use. Recall that presentable categories are usually large categories. To talk about presentable $\mathcal{B}$–categories, we define a *large $\mathcal{B}$–category* to be a sheaf of large categories on $\mathcal{B}$, i.e. a limit preserving functor $\mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$. The very large category of large $\mathcal{B}$–categories will be denoted by $\widehat{\mathrm{Cat}}_{\mathcal{B}}$.

Definition 2.1.25. A $\mathcal{B}$–category $\underline{\mathcal{C}}$ is called *fibrewise presentable* if the map $\underline{\mathcal{C}}: \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ factors through $\mathrm{Pr}^L \subset \widehat{\mathrm{Cat}}$. Furthermore, $\underline{\mathcal{C}}$ is called *presentable* if it is fibrewise presentable and the following conditions hold:

1. For any map $f: X \to Y$ in $\mathcal{B}$ the map $f^*: \mathcal{C}(\underline{Y}) \to \mathcal{C}(\underline{X})$ admits a left adjoint $f_!$.

2. For any pullback square

$$\begin{array}{ccc} \underline{X'} & \xrightarrow{g'} & \underline{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \underline{Y'} & \xrightarrow{g} & \underline{Y} \end{array} \tag{8}$$

   in $\mathcal{B}$ the Beck–Chevalley transformation $f'_!(g')^* \to g^* f_!$ between functors $\mathcal{C}(\underline{X}) \to \mathcal{C}(\underline{Y'})$ is an equivalence.

The above definition was chosen because it is easy to state, but there are many useful ways to characterise presentable $\mathcal{B}$–categories, see [MW22, Thm. 6.2.4].

Definition 2.1.26. A map $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ between presentable $\mathcal{B}$–categories is said to *preserve $\mathcal{B}$-colimits* if it satisfies the following conditions:

1. For any object $\underline{X} \in \mathcal{B}$ the map $F(\underline{X})\colon \mathcal{C}(\underline{X}) \to \mathcal{D}(\underline{X})$ preserves colimits.

2. For any map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ the Beck–Chevalley transformation $f_! F(\underline{X}) \to F(\underline{Y}) f_!$ is an equivalence.

If $\underline{\mathcal{C}}$ is presentable, then $f^*\colon \mathcal{C}(\underline{Y}) \to \mathcal{C}(\underline{X})$ also admits a right adjoint f_*. By passing to right adjoints one can see that for any pull back square (8) the Beck–Chevalley transformation $g^* f_* \to (f')_*(g')^*$ is an equivalence, see e.g. [Hai22, Observation 1.6.2].

Definition 2.1.27. Denote by $\mathrm{Pr}^L_{\mathcal{B}}$ the (nonfull) subcategory of $\widehat{\mathrm{Cat}}_{\mathcal{B}}$ of presentable $\mathcal{B}$–categories and $\mathcal{B}$-colimit preserving functors. We write $\underline{\mathrm{Fun}}^L_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ for the full $\mathcal{B}$-subcategory of $\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$ of colimit preserving functors. The subcategories $\mathrm{Pr}^L_{\mathcal{B}_{/X}} \subset \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}}$ assemble into the $\mathcal{B}$–category $\underline{\mathrm{Pr}}^L_{\mathcal{B}} \subset \underline{\widehat{\mathrm{Cat}}}_{\mathcal{B}}$ of presentable $\mathcal{B}$–categories.

The $\mathcal{B}$–category $\underline{\mathrm{Pr}}^L_{\mathcal{B}}$ admits all $\mathcal{B}$-limits and $\mathcal{B}$-colimits [MW22, Cor. 6.4.11.]. Moreover, for two presentable $\mathcal{B}$–categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$, their functor category $\underline{\mathrm{Fun}}^L(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is presentable.

Construction 2.1.28 (Tensor product of presentable categories)**.** Given two presentable $\mathcal{B}$–categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$, their tensor product $\underline{\mathcal{C}} \otimes \underline{\mathcal{D}}$ is a presentable $\mathcal{B}$–category together with a functor $\underline{\mathcal{C}} \times \underline{\mathcal{D}} \to \underline{\mathcal{C}} \otimes \underline{\mathcal{D}}$ which preserves colimits in each variable such that precomposition along it induces an equivalence $\mathrm{Fun}^L_{\mathcal{B}}(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}}, \underline{\mathcal{E}}) \xrightarrow{\simeq} \mathrm{Fun}^L_{\mathcal{B}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^L(\underline{\mathcal{D}}, \underline{\mathcal{E}}))$ for any presentable $\mathcal{B}$–category $\underline{\mathcal{E}}$. This equips $\mathrm{Pr}^L_{\mathcal{B}}$ with the structure of a closed symmetric monoidal category. It can even be extended to a symmetric monoidal structure on the $\mathcal{B}$–category $\underline{\mathrm{Pr}}^L_{\mathcal{B}}$, see [MW22, Proposition 8.2.9]. Furthermore, the tensor product $- \otimes -\colon \underline{\mathrm{Pr}}^L_{\mathcal{B}} \times \underline{\mathrm{Pr}}^L_{\mathcal{B}} \to \underline{\mathrm{Pr}}^L_{\mathcal{B}}$ preserves $\mathcal{B}$-colimits in each variable.

Definition 2.1.29. A *presentably symmetric monoidal $\mathcal{B}$–category* is a commutative algebra $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$. Explicitly, this means that $\underline{\mathcal{C}}$ is a symmetric monoidal $\underline{B}$–category which is presentable such that the tensor product $- \otimes -\colon \underline{\mathcal{C}} \times \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ preserves $\mathcal{B}$-colimits in both variables. The latter condition means that:

1. For all $\underline{X} \in \mathcal{B}$ the tensor product $\mathcal{C}(\underline{X}) \times \mathcal{C}(\underline{X}) \to \mathcal{C}(\underline{X})$ preserves colimits.

2. For all maps $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and all $A \in \mathcal{C}(\underline{X})$ and $B \in \mathcal{C}(\underline{Y})$ the Beck–Chevalley transformation $f_!(A \otimes f^*B) \to f_!A \otimes B$ is an equivalence.

To construct examples of presentable $\mathcal{B}$–categories, the following proposition is useful.

Proposition 2.1.30 ([MW22, Section 8.3])**.** *There is a symmetric monoidal colimit preserving fully faithful functor* $-\otimes_{\mathcal{B}} \Omega\colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L) \hookrightarrow \mathrm{Pr}^L_{\mathcal{B}}$ *whose right adjoint* Γ^{lin} *refines the global sections functor* $\Gamma\colon \mathrm{Pr}^L_{\mathcal{B}} \to \mathrm{Pr}^L$.

Lemma 2.1.31 (Presentablility and basechange)**.** *Let* $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ *be a geometric morphism. Then the functor* $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ *restricts to a functor* $f_*\colon \mathrm{Pr}^L_{\mathcal{B}'} \to \mathrm{Pr}^L_{\mathcal{B}}$. *It admits a unique lax symmetric monoidal refinement* $f_*^{\otimes}\colon \mathrm{Pr}^{L,\otimes}_{\mathcal{B}'} \to \mathrm{Pr}^{L,\otimes}_{\mathcal{B}}$ *lifting the lax symmetric monoidal functor* $\mathrm{Pr}^{L,\otimes}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}'} \xrightarrow{f_*} \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}}$ *along the lax symmetric monoidal functor* $\mathrm{Pr}^{L,\otimes}_{\mathcal{B}} \to \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}}$.

Proof. Recall that $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ is given by precomposition with $f^*\colon \mathcal{B}^{\mathrm{op}} \to (\mathcal{B}')^{\mathrm{op}}$. Hence, the first condition in Definition 2.1.25 is immediate. The second condition follows from the fact that $f^*\colon \mathcal{B}^{\mathrm{op}} \to (\mathcal{B}')^{\mathrm{op}}$ also preserves finite limits, and in particular pullbacks.

For the statement about lax symmetric monoidality, we will freely use the terminologies from [MW22]. First observe that as $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ preserves products, it is symmetric monoidal with respect to the cartesian symmetric monoidal structure on both sides. Recall from [MW22, Section 8.2] that $\mathrm{Pr}^{L,\otimes}_{\mathcal{B}'} \looparrowright \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}'}$ is the subcategory generated by presentable $\mathcal{B}'$–categories and locally multilinear functors. We know from the first part that $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ preserves presentable categories and multilinear functors between those. As $f^*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}} \to \widehat{\mathrm{Cat}}_{\mathcal{B}'}$ preserves colimits and finite limits it preserves effective epimorphisms. From this it follows that f_* also preserves locally multilinear functors. ☐

Lemma 2.1.32 (Presentability and fully faithful basechange)**.** *Let* $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ *be a geometric morphism of topoi and assume that* f_* *is fully faithful. Then the square*

$$\begin{array}{ccc} \mathrm{Pr}^L_{\mathcal{B}'} & \xrightarrow{f_*} & \mathrm{Pr}^L_{\mathcal{B}} \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Cat}}_{\mathcal{B}'} & \xhookrightarrow{f_*} & \widehat{\mathrm{Cat}}_{\mathcal{B}} \end{array}$$

is cartesian. In particular, $f_*\colon \mathrm{Pr}^L_{\mathcal{B}'} \to \mathrm{Pr}^L_{\mathcal{B}}$ *is fully faithful.*

If, in addition, the image of $f_*\colon \mathrm{Pr}^L_{\mathcal{B}'} \to \mathrm{Pr}^L_{\mathcal{B}}$ *is closed under* $\otimes$, *then the maps* $f_*\underline{\mathcal{C}} \otimes f_*\underline{\mathcal{D}} \to f_*(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}})$ *coming from the lax symmetric monoidal structure are equivalences.*

Proof. We have to show that a $\mathcal{B}'$–category $\underline{\mathcal{C}}$ is presentable if $f_*\underline{\mathcal{C}}$ is presentable and similarly that a $\mathcal{B}'$-functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is $\mathcal{B}'$-colimit preserving if f_*F is $\mathcal{B}$-colimit preserving. Notice that the counit map $f^*f_* \to \mathrm{id}$ is an equivalence as f_* is fully faithful. This implies that for $\underline{X} \in \mathcal{B}$ we have $f_*\underline{\mathcal{C}}(f_*\underline{X}) = \mathcal{C}(f^*f_*\underline{X}) \simeq \mathcal{C}(\underline{X})$. The statements about presentablility and colimit preservation now directly follow from the definitions.

For the statement about the lax monoidal multiplication map, observe that for presentable $\mathcal{B}'$–categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{\mathcal{E}}$, the functor f_* induces an equivalence between $\mathcal{B}'$-multilinear functors

$\underline{\mathcal{C}} \times \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ and $\mathcal{B}$-multilinear functors $f_*\underline{\mathcal{C}} \times f_*\underline{\mathcal{D}} \to f_*\underline{\mathcal{E}}$: It is clear that if g is multilinear, then f_*g is multilinear while the converse follows from essential surjectivity of f^*. In particular, precomposition along the map $f_*\underline{\mathcal{C}} \otimes f_*\underline{\mathcal{D}} \to f_*(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}})$ induces an equivalence

$$\mathrm{Fun}^L(f_*(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}}), f_*\underline{\mathcal{E}}) \xrightarrow{\simeq} \mathrm{Fun}^L(f_*\underline{\mathcal{C}} \otimes f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{E}})$$

from which the claim follows. □

Lemma 2.1.33 (Presentability and étale basechange)**.** *For $X \in \mathcal{B}$, the basechange adjunction $\pi_X^* \colon \widehat{\mathrm{Cat}}_{\mathcal{B}} \rightleftharpoons \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}} : (\pi_X)_*$ along the étale geometric morphism $\pi_X^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/X} : (\pi_X)_*$ restricts to an adjunction $\pi_X^* \colon \mathrm{Pr}^L_{\mathcal{B}} \rightleftharpoons \mathrm{Pr}^L_{\mathcal{B}_{/X}} : (\pi_X)_*$. The left adjoint $\pi_X^* \colon \mathrm{Pr}^L_{\mathcal{B}} \to \mathrm{Pr}^L_{\mathcal{B}_{/X}}$ admits a unique symmetric monoidal refinement which lifts the lax symmetric monoidal functor $\mathrm{Pr}^L_{\mathcal{B}} \to \widehat{\mathrm{Cat}}_{\mathcal{B}} \xrightarrow{\pi_X^*} \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}}$.*

Proof. That $\pi_X^* \colon \widehat{\mathrm{Cat}}_{\mathcal{B}} \rightleftharpoons \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}} : (\pi_X)_*$ restricts to an adjuction $\pi_X^* \colon \mathrm{Pr}^L_{\mathcal{B}} \rightleftharpoons \mathrm{Pr}^L_{\mathcal{B}_{/X}} \colon (\pi_X)_*$ is shown in [Cno23, Corollary 2.14]. For the statement about symmetric monoidality, note that $(\pi_X)_* \colon \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ is product preserving as it admits the left adjoint $(\pi_X)_!$. In particular, we obtain the symmetric monoidal unit map $\underline{\mathrm{Pr}}^L_{\mathcal{B}} \to (\pi_X)_*\pi_X^*\underline{\mathrm{Pr}}^L_{\mathcal{B}}$ which on global sections gives the desired symmetric monoidal refinement of $(\pi_X)^* \colon \mathrm{Pr}^L_{\mathcal{B}} \to \mathrm{Pr}^L_{\mathcal{B}_{/X}}$. □

$\underline{\mathcal{C}}$-linear categories

Here we recall some facts about $\underline{\mathcal{C}}$-linear categories and the classification of $\underline{\mathcal{C}}$-linear functors from [Cno23, Section 2.2].

Definition 2.1.34. ($\underline{\mathcal{C}}$-linear categories) Consider a presentably symmetric monoidal category $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$. A *$\underline{\mathcal{C}}$-linear category* is a left $\underline{\mathcal{C}}$-module in $\mathrm{Pr}^L_{\mathcal{B}}$. The category of $\underline{\mathcal{C}}$-linear categories and $\underline{\mathcal{C}}$-linear functors is defined as the category $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$.

The categories $\mathrm{Mod}_{\pi_X^*\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}_{/X}})$ assemble into the $\mathcal{B}$–category $\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$. The relative tensor product from [MW22, Proposition 7.2.7] equips this with the structure of a symmetric monoidal $\mathcal{B}$–category which is $\mathcal{B}$-complete and $\mathcal{B}$-cocomplete such that the tensor product is bilinear. This symmetric monoidal structure on $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ is closed and we denote the internal mapping object by $\underline{\mathrm{Fun}}_{\underline{\mathcal{C}}}(-,-)$. As in [Mar22b, Remark 3.4.3] it endows $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ with a 2-categorical structure. This allows us to talk about internal adjunctions in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$. The following lemma gives convenient criteria for a $\underline{\mathcal{C}}$-linear functor to be an internal left adjoint.

Lemma 2.1.35 ([Cno23], Lem. 2.21)**.** *A $\underline{\mathcal{C}}$-linear functor $F \colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ if and only if its right adjoint G preserves fibrewise colimits and satisfies the projection formula, i.e. for each $\underline{X} \in \mathcal{B}$ and $e \in \underline{\mathcal{E}}(\underline{X})$ the map $\mathrm{PF}_* \colon c \otimes G(e) \to G(c \otimes e)$ is an equivalence.*

Definition 2.1.36 (Free and cofree categories, [Cno23, Definition 2.23])**.** For $\underline{\mathcal{D}} \in \mathrm{Mod}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$ and an object $\underline{X} \in \mathcal{B}$ we define the *cofree $\underline{\mathcal{C}}$-linear $\mathcal{B}$–category on $\underline{X}$* by $\underline{\mathcal{D}}^{\underline{X}} := \lim_{\underline{X}} \underline{\mathcal{D}}$ where the $\mathcal{B}$-limit is formed over the constant diagram $\underline{X} \to \underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$ with value $\underline{\mathcal{D}}$.

Note that after forgetting the $\underline{\mathcal{C}}$-linear structure, $\underline{\mathcal{D}}^{\underline{X}}$ is given by $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}})$ as the forgetful functor $\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}}) \to \underline{\mathrm{Cat}}_{\mathcal{B}}$ preserves $\underline{\mathcal{B}}$-limits. If $\underline{\mathcal{D}} \in \mathrm{CAlg}(\mathrm{Mod}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}}))$ is a $\underline{\mathcal{C}}$-algebra, then $\underline{\mathcal{D}}^{\underline{X}}$ has a canonical pointwise symmetric monoidal structure as the forgetful functor $\underline{\mathrm{CAlg}}(\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})) \to \underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$ preserves $\mathcal{B}$-limits.

The following theorem is crucial for the development of Poincaré duality in a presentable context, even in classical terms. It should be viewed as a generalisation of the fact that for a small category for a space X there is an equivalence $\mathrm{Fun}^L(\mathcal{S}^X, \mathcal{S}) \simeq \mathcal{S}^X$, that is sometimes referred to as Morita theory.

Theorem 2.1.37 (Classification of $\underline{\mathcal{C}}$-linear functors, [Cno23, Theorem 2.32])**.** *Let* $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ *and* $\underline{X} \in \underline{\mathcal{C}}$. *Then there is an equivalence of* $\underline{\mathcal{C}}$*-linear* $\mathcal{B}$*–categories*

$$\underline{\mathcal{C}}^{\underline{X}} \to \underline{\mathrm{Fun}}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}}), \quad \zeta \mapsto r_!(- \otimes \zeta).$$

Here, the map in the statement of the theorem is adjoint to the map $\underline{\mathcal{C}}^{\underline{X}} \otimes \underline{\mathcal{C}}^{\underline{X}} \to \underline{\mathcal{C}}^{\underline{X}} \xrightarrow{r_!} \underline{\mathcal{C}}$.

Proposition 2.1.38 (Basechange of module categories, [MW22, Prop. 7.2.7])**.** *Suppose that* $f\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ *is a map in* $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$. *Then the restriction functor* $f^*\colon \underline{\mathrm{Mod}}_{\underline{\mathcal{D}}}(\mathrm{Pr}^L_{\mathcal{B}}) \to \underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ *admits a symmetric monoidal left adjoint* $- \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}\colon \underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L(\mathcal{B})) \to \underline{\mathrm{Mod}}_{\underline{\mathcal{D}}}(\mathrm{Pr}^L(\mathcal{B}))$

Proof. Apply [MW22, Proposition 7.2.7] to the case $R = \underline{\mathcal{D}} \in \mathrm{CAlg}(\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}}))$. ☐

2.2. Equivariant categories and the theory of families

Change of group functors

We now specialise the previous general theory to the equivariant setting for a compact Lie group G. We set $\mathcal{S}_G \coloneqq \mathrm{PSh}(\mathcal{O}(G))$, the *category of* G*–spaces*, where $\mathcal{O}(G)$ is the *orbit category of* G. This is a topos, and we write $\mathrm{Cat}_G \coloneqq \mathrm{Cat}_{\mathcal{S}_G} \simeq \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Cat})$ for the category of G–categories. The value of a G-category $\underline{\mathcal{C}}$ at an orbit G/H will be denoted by $\mathcal{C}(\underline{G/H})$ or $\mathcal{C}^H$.

Recollections 2.2.1. Recall that the category of locally compact Hausdorff topological G–spaces is enriched over topological spaces by employing the compact–open topology on morphism sets. The full subcategory on the homogenous G–spaces, that is Hausdorff spaces with a transitive G-action, is equivalent to the full subcategory spanned by the orbits G/H, where $H \leq G$ is a closed subgroup. By $\mathcal{O}(G)$ we denote the associated (∞-) category which we call the *orbit category of* G.

Later we will need the following standard facts: For any morphism $\alpha\colon H \to G$ of compact Lie groups, there is an induction functor

$$\mathrm{Ind}^{\mathcal{O}}_{\alpha}\colon \mathcal{O}(H) \longrightarrow \mathcal{O}(G), \quad S \mapsto G \times_H S.$$

If $\alpha\colon H \longrightarrow G$ is an epimorphism, this admits the restriction functor

$$\mathrm{Res}^{\mathcal{O}}_{\alpha}\colon \mathcal{O}(G) \to \mathcal{O}(H)$$

as a fully faithful right adjoint. Both functors and the adjunction can be constructed on the level of topological categories. For a closed subgroup $H \leq G$, induction induces an equivalence of categories $\mathcal{O}(H) \xrightarrow{\simeq} \mathcal{O}(G)_{/(G/H)}$ whose inverse sends $T \to G/H$ to the homogeneous H-space given as the fibre over $eH \in G/H$.

More information on orbit categories of compact Lie groups can be found in [LNP22, Sec. 6] or [Bre72, Chapters I.3 and I.4].

Notation 2.2.2 (Restrictions, (co)inductions, and (co)inflations)**.** Consider a continuous homorphism $\alpha\colon K \to G$ of compact Lie groups. We obtain the two adjunctions

$$\mathrm{Cat}_H \; \begin{array}{c} \xrightarrow{\mathrm{Ind}_\alpha} \\ \xleftarrow{\mathrm{Res}_\alpha} \\ \xrightarrow[\mathrm{Coind}_\alpha]{} \end{array} \; \mathrm{Cat}_G,$$

called the *induction, restriction*, and *coinduction* functors, respectively. Here, Res_α is given by restriction along $\mathrm{Ind}^{\mathcal{O}}_\alpha\colon \mathcal{O}(K) \to \mathcal{O}(G)$ and Ind_α and Coind_α are given by left and right Kan extensions. The functors Res_α and Coind_α restrict to a geometric morphism of topoi $\mathrm{Res}_\alpha\colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_K : \mathrm{Coind}_\alpha$. The two main classes of examples are:

(a) If α were an injection $\iota\colon H \rightarrowtail G$, then the geometric morphism $\mathrm{Res}_\iota\colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_H : \mathrm{Coind}_\iota$ is étale. We will often also write Ind_ι, Res_ι, and Coind_ι as Ind^G_H, Res^G_H, and Coind^G_H respectively;

(b) If α were an epimorphism $\theta\colon G \twoheadrightarrow G/N =: Q$ (so that $N \leq G$ is a closed normal subgroup), then Coind_θ admits a further right adjoint which we write as Coinfl_θ given by right Kan extension along the fully faithful right adjoint $\mathrm{Res}^{\mathcal{O}}_\theta$ to $\mathrm{Ind}^{\mathcal{O}}_\theta$. In particular, $\mathrm{Res}_\theta = (\mathrm{Ind}^{\mathcal{O}}_\theta)^* \simeq (\mathrm{Res}^{\mathcal{O}}_\theta)_!$ and $\mathrm{Coinfl}_\theta = (\mathrm{Res}^{\mathcal{O}}_\theta)_*$ are fully faithful in this epimorphic case. Note also that in this case, $\mathrm{Coind}_\theta \simeq (\mathrm{Res}^{\mathcal{O}}_\theta)^*$, i.e. Coind_θ may be computed by restricting along $\mathrm{Res}^{\mathcal{O}}_\theta\colon \mathcal{O}(G) \to \mathcal{O}(K)$. We summarise in the following diagram the special notations we will also use in the epimorphic case as follows:

$$\mathrm{Cat}_Q \; \begin{array}{c} \xleftarrow{N\backslash(-):=\mathrm{Ind}_\theta} \\ \xhookrightarrow{\mathrm{Infl}_\theta:=\mathrm{Res}_\theta} \\ \xleftarrow{(-)^N:=\mathrm{Coind}_\theta} \\ \xhookrightarrow[\mathrm{Coinfl}_\theta]{} \end{array} \; \mathrm{Cat}_G,$$

The maps $N\backslash(-)$, Infl_θ, $(-)^N$, and Coinfl_θ are called the *genuine quotient, inflation, genuine fixed points,* and *coinflation*, respectively. We often also write Infl_θ and Coinfl_θ as Infl^Q_G and Coinfl^Q_G respectively.

Remark 2.2.3. From the left Kan extension formula defining the genuine quotient, we obtain

$$(N\backslash\underline{\mathcal{C}})(\underline{Q/H}) \simeq \operatorname*{colim}_{G/K,\, \underline{Q/H} \to N\backslash(\underline{G/K})} \mathcal{C}(\underline{G/K}). \tag{9}$$

Stability

In the parametrised theory, the theory of stability is more subtle than in the nonparametrised setting. The most naive version is the following, equivalent characterisations of which can be found in [MW22, Section 7.3].

Definition 2.2.4. A $\mathcal{B}$–category $\underline{\mathcal{C}}$ is called fibrewise pointed (resp. stable) if the functor $\underline{\mathcal{C}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ factors through the subcategory $\mathrm{Cat}_* \subset \mathrm{Cat}$ of pointed categories and pointed functors (resp. $\mathrm{Cat}^{\mathrm{st}} \subset \mathrm{Cat}$ of stable categories and exact functors). We denote by $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{ptd}}$ (resp. $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{st}}$) the category of fibrewise pointed (resp. stable) categories and pointed (resp. exact) functors.

As a parametrised analogue of the category of spectra, there is the G-category of G-spectra $\underline{\mathrm{Sp}}_G$ whose value at an orbit G/H is given by the category $\underline{\mathrm{Sp}}_G(\underline{G/H}) = \mathrm{Sp}_H$ of genuine H-spectra together with restriction maps between them, see Definition A.0.1 for a definition. If G is clear form the context, we will also just write $\underline{\mathrm{Sp}}$ for $\underline{\mathrm{Sp}}_G$. In addition to being fibrewise stable, $\underline{\mathrm{Sp}}_G$ satsifies some form of the Wirthmüller isomorphism in the sense that indexed products and coproducts over orbits G/H are canonically equivalent. This was used by Nardin in [Nar17] to define the notion of G-stability for finite groups G. We will not recall the definition here and refer the interested reader to [Nar17] or [Hil22, Section 4.1] for an exposition of this theory. In Appendix A we generalise this to define G-stability for presentable G-categories. This will be sufficient for our purposes.

Notation 2.2.5. For a compact Lie group G, we denote by $\mathrm{Pr}_G^{L,G-\mathrm{st}} \subseteq \mathrm{Pr}_G^{L,\mathrm{st}} \subseteq \mathrm{Pr}_G^{L}$ the full subcategories on G-stable and fibrewise stable presentable G-categories. For a finite group G, we also denote by $\mathrm{Cat}_G^{G-\mathrm{st}} \subset \mathrm{Cat}_G$ the subcategory of G-stable G-categories and G-exact functors.

Now we study the behaviour of G-stability with respect to standard equivariant operations.

Lemma 2.2.6 (Coinduction and stability)**.** *Let $\alpha\colon K \to G$ be a continous group homomorphism of compact Lie groups. The lax symmetric monoidal functor $\mathrm{Coind}_\alpha\colon \mathrm{Pr}_K^L \to \mathrm{Pr}_G^L$ from Lemma 2.1.31 restricts to a lax symmetric monoidal functor $\mathrm{Coind}_\alpha\colon \mathrm{Pr}_K^{L,K-\mathrm{st}} \to \mathrm{Pr}_G^{L,G-\mathrm{st}}$*

Proof. We apply Theorem A.0.5 (3) to show that Coind_α sends K-stable to G-stable categories. Suppose that $\underline{\mathcal{C}}$ is a K-stable category and V is a finite dimensional G-representation. By Lemma 2.1.18 we can identify the maps $S^V \otimes -$ and $\mathrm{Coind}_\alpha(S^{\mathrm{Res}_\alpha V} \otimes -)$ on $\mathrm{Coind}_\alpha \underline{\mathcal{C}}$. But as $\mathrm{Res}_\alpha V$ is a finite dimensional K-representation, K-stability of $\underline{\mathcal{C}}$ implies that the second map is invertible. ☐

Lemma 2.2.7 (Restriction and stability)**.** *For an injective continuous homomorphism $\alpha\colon H \rightarrowtail G$ of compact Lie groups, the adjunction $\mathrm{Res}_\alpha\colon \mathrm{Pr}_G^L \rightleftharpoons \mathrm{Pr}_H^L : \mathrm{Coind}_\alpha$ from Lemma 2.1.33 restricts to an adjunction $\mathrm{Res}_\alpha\colon \mathrm{Pr}_G^{L,G-\mathrm{st}} \rightleftharpoons \mathrm{Pr}_H^{L,H-\mathrm{st}} : \mathrm{Coind}_\alpha$ with symmetric monoidal left adjoint*

Proof. By Lemma 2.2.6 Coind_H^G restricts to a functor between equivariantly stable categories. To show that restriction Res_H^G preserves equivariantly stable categories we employ Theorem A.0.5. Recall that, by the Peter-Weyl theorem, for any finite dimensional H-representation

W there is a finite dimensional G-representaiton V such that W is a summand of $\mathrm{Res}^G_H V$. Now, if $\underline{\mathcal{C}}$ is a G-stable category, then $S^V \otimes -$ is invertible on $\underline{\mathcal{C}}$. By Lemma 2.1.19, we can identify the two maps $\mathrm{Res}^G_H(S^V \otimes -)$ and $S^{\mathrm{Res}^G_H V} \otimes -$ on $\mathrm{Res}^G_H \underline{\mathcal{C}}$. This shows that $S^{\mathrm{Res}^G_H V}$ and thus also S^W act invertibly on $\mathrm{Res}^G_H \underline{\mathcal{C}}$. □

Lemma 2.2.8 (Coinflation and stability)**.** *Let* $\theta\colon G \twoheadrightarrow Q = G/N$ *be a continuous epimorphism of compact Lie groups. Then the lax symmetric monoidal functor* $\mathrm{Coinfl}_\theta\colon \mathrm{Pr}^L_Q \to \mathrm{Pr}^L_G$ *from Lemma 2.1.31 restricts to a lax symmetric monoidal functor* $\mathrm{Coinfl}_\theta\colon \mathrm{Pr}^{L,Q-\mathrm{st}}_Q \to \mathrm{Pr}^{L,G-\mathrm{st}}_G$.

Proof. We apply Theorem A.0.5 to show that Coinfl_θ sends Q-stable to G-stable categories. Suppose that $\underline{\mathcal{C}}$ is a Q-stable category and V is a finite dimensional G-representation. By Lemma 2.1.18 we can identify the maps $S^V \otimes -$ and $\mathrm{Coinfl}_\theta(\mathrm{Coind}_\theta S^V \otimes -)$ on $\mathrm{Coinfl}_\theta \underline{\mathcal{C}}$. Note that $\mathrm{Coind}_\theta S^V \simeq S^{V^N}$ is the representation sphere of the finite dimensional Q-representation carrying the residual action. By Q-stability of $\underline{\mathcal{C}}$, this map is invertible. □

Construction 2.2.9 (Spectral restriction)**.** Let $\alpha\colon K \to G$ be a homomorphism of compact Lie groups. Lemma 2.2.6 endows $\mathrm{Coind}_\alpha\colon \mathrm{Pr}^{L,K-\mathrm{st}}_K \to \mathrm{Pr}^{L,G-\mathrm{st}}_G$ with a lax symmetric monoidal structure. This endows $\mathrm{Coind}_\alpha \underline{\mathrm{Sp}}_K$ with the structure of a commutative algebra in $\mathrm{Pr}^{L,G-\mathrm{st}}_G$. In particular, using that $\underline{\mathrm{Sp}}_G$ is the initial commutative algebra in $\mathrm{Pr}^{L,G-\mathrm{st}}_G$, we obtain the symmetric monoidal G-colimit preserving functor $\mathrm{Res}_\alpha\colon \underline{\mathrm{Sp}}_G \to \mathrm{Coind}_\alpha \underline{\mathrm{Sp}}_K$ called the *restriction map*. If $\alpha = \theta\colon G \twoheadrightarrow Q$ is an epimorphism, we also call $\mathrm{Res}_\theta = \mathrm{Infl}_\theta\colon \underline{\mathrm{Sp}}_Q \to \mathrm{Coind}_\theta \underline{\mathrm{Sp}}_G$ the *inflation map*.

Categorical isotropy separation

At various places in this article we will use isotropy separation arguments. For this, we recall here some constructions on G–categories given a family $\mathcal{F}$ of subgroups of G. Recall that a family of subgroups of a compact Lie group G is a collection of closed subgroups of G which is closed under subgroups and conjugation.

Note that conjugacy classes of subgroups of G correspond bijectively to isomorphism classes of objects in $\mathcal{O}(G)$. Given any collection S of closed subgroups of G that is closed under conjugacy, we set $\mathcal{O}_S(G) \subset \mathcal{O}(G)$ to be the full subcategory on those G/H with $H \in S$. One important example of this is the collection $S = \mathcal{F}^c$ given by the collection of all subgroups which lie in the complent of a family $\mathcal{F}$. This never forms a family, except in the extreme cases of the empty family or the family of all subgroups.

Example 2.2.10 (A family for quotients)**.** Suppose that $N \le G$ is a closed normal subgroup of G. An interesting family is provided by $\Gamma_N \coloneqq \{H \le G \mid N \not\le H\}$. Then Γ_N^c consists of those $H \le G$ with $N \le H$. Let $\alpha\colon G \to G/N$ denote the quotient homomorphism. Observe that the adjunction $\mathrm{Ind}^{\mathcal{O}}_\alpha \dashv \mathrm{Res}^{\mathcal{O}}_\alpha$ restricts to an equivalence of categories

$$\mathrm{Ind}^{\mathcal{O}}_\alpha\colon \mathcal{O}_{\Gamma^c_N}(G) \simeq \mathcal{O}(G/N)\colon\ \mathrm{Res}^{\mathcal{O}}_\alpha\,.$$

Example 2.2.11 (A family for free actions)**.** Suppose again that $N \le G$ is a closed normal subgroup of G. Another family is given as $\mathcal{F}_N \coloneqq \{H \subset G \mid H \cap N = \{1\}\}$. Note that when $N \neq \{1\}$, there is an inclusion of families $\mathcal{F}_N \subseteq \Gamma_N$. Thus, there is an inclusion $\Gamma^c_N \subseteq \mathcal{F}^c_N$.

Definition 2.2.12. Let G be a compact Lie group and S a collection of subgroups, closed under conjugacy. Then we write $\mathrm{Cat}_{G,S} \coloneqq \mathrm{Fun}(\mathcal{O}_S(G)^{\mathrm{op}}, \mathrm{Cat})$ for the *category of S–categories.*

If $\mathcal{F}$ is a family of closed subgroups of G, we have the following variant of the standard isotropy separation sequence relating the categories Cat_G, $\mathrm{Cat}_{G,\mathcal{F}}$ and $\mathrm{Cat}_{G,\mathcal{F}^c}$. This will allow us to "separate" our problems into orthogonal pieces, one part concentrated in $\mathcal{F}^c$ and the part which is $\mathcal{F}$–local.

Construction 2.2.13 (Isotropy separation for G–categories)**.** Let $\mathcal{F}$ be a family of subgroups of a compact Lie group G and denote by $b\colon \mathcal{O}_{\mathcal{F}}(G) \hookrightarrow \mathcal{O}(G)$ and $s\colon \mathcal{O}_{\mathcal{F}^c}(G) \hookrightarrow \mathcal{O}(G)$ the inclusions. We obtain the adjoint triples

$$\mathrm{Cat}_{G,\mathcal{F}^c} \xleftarrow{\;s^*\;} \mathrm{Cat}_G \quad (s_! \colon \mathrm{Cat}_{G,\mathcal{F}^c} \to \mathrm{Cat}_G,\; s_* \colon \mathrm{Cat}_{G,\mathcal{F}^c} \to \mathrm{Cat}_G) \qquad \mathrm{Cat}_G \xrightarrow{\;b^*\;} \mathrm{Cat}_{G,\mathcal{F}} \quad (b_! \colon \mathrm{Cat}_{G,\mathcal{F}} \to \mathrm{Cat}_G,\; b_* \colon \mathrm{Cat}_{G,\mathcal{F}} \to \mathrm{Cat}_G)$$

by restriction and Kan extension along s and b. Without making this precise, let us mention that these can be made into an unstable recollement using the cofibre sequence $\underline{E\mathcal{F}_+} \xrightarrow{b} \underline{S}^0 \to \underline{\widetilde{E\mathcal{F}}}$ of pointed G–spaces. For example, the map $b^*\colon \mathrm{Cat}_G \to \mathrm{Cat}_{G,\mathcal{F}}$ is equivalently given by taking the global sections on the map $b^*\colon \underline{\mathrm{Cat}} \to \underline{\mathrm{Fun}}(\underline{E\mathcal{F}}, \underline{\mathrm{Cat}})$.

Unwinding the right Kan extension formula, one obtains for example that s_* is given by

$$(s_*\mathcal{C})^H = \begin{cases} \mathcal{C}^H, & H \notin \mathcal{F} \\ *, & H \in \mathcal{F}. \end{cases}$$

Notice that the adjunction $b^* \dashv b_*$ is the basechange adjunction associated to the étale morphism $\pi_{E\mathcal{F}}^*\colon \mathcal{S}_G \rightleftharpoons (\mathcal{S}_G)_{/E\mathcal{F}} : (\pi_{E\mathcal{F}})_*$. In particular, it restricts to an adjunction $\mathrm{Pr}^L_G \rightleftharpoons \mathrm{Pr}^L_{G,\mathcal{F}}$ by Lemma 2.1.33. Similarly, the adjunction $s^* \dashv s_*$ is the basechange adjunction associated to the geometric morphism $s^*\colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_{G,\mathcal{F}^c} : s_*$.

Example 2.2.14 (Isotropy separation and coinduction)**.** Consider a continuous epimorphism $\theta\colon G \twoheadrightarrow G/N$ of compact Lie groups. Recall from Example 2.2.10 that there is the functor $\mathrm{Res}^{\mathcal{O}}_\theta\colon \mathcal{O}(Q) \hookrightarrow \mathcal{O}(G)$ which restricts to an equivalence $\mathcal{O}(G/N) \simeq \mathcal{O}_{\Gamma^c_N}(G)$. This identifies the adjunctions $\mathrm{Coind}_\theta\colon \mathrm{Cat}_G \rightleftharpoons \mathrm{Cat}_Q : \mathrm{Coinfl}_\theta$ and $s^*\colon \mathrm{Cat}_G \rightleftharpoons \mathrm{Cat}_{G,\Gamma^c_N} : s_*$.

Construction 2.2.15 (Singular part)**.** Consider the inclusion $s\colon \mathcal{O}_{\mathcal{F}^c}(G) \hookrightarrow \mathcal{O}(G)$. Then we get the Bousfield colocalisation $s_!\colon \mathcal{S}_{G,\mathcal{F}^c} \rightleftharpoons \mathcal{S}_G : s^*$ such $s_!s^*(\underline{G/H}) = \underline{G/H}$ for every and $s_!s^*(\underline{G/H}) = \varnothing$. Since $s_!s^*$ picks out the isotropy of a G–space $\underline{X}$ not in $\mathcal{F}$, we shall also use the notations (which will be part of a larger notational package in Notation 2.2.28)

$$\underline{X}_{\mathcal{F}^c} \coloneqq s^*\underline{X} \in \mathcal{S}_{\mathcal{F}^c} \qquad \underline{X}_{\widetilde{\mathcal{F}}} \coloneqq s_!s^*\underline{X} = s_!\underline{X}_{\mathcal{F}^c} \in \mathcal{S}_G.$$

The adjunction counit $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ thus admits the classical interpretation as the inclusion of the $\mathcal{F}$–singular part of the G–space $\underline{X}$. It is the identity map on G/H for $H \notin \mathcal{F}$ and the map $\varnothing \to \underline{G/H}$ for $H \in \mathcal{F}$. We refer to $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ as the *inclusion of the $\mathcal{F}$–singular part of $\underline{X}$.*

Example 2.2.16. For $\mathcal{F} = \mathcal{P}$, the family of proper subgroups, $\underline{X}_{\widetilde{\mathcal{P}}}$ is given by the fixed points space X^G, considered as a G–space with trivial action. For $\mathcal{F} = \{e\}$, the trivial family, the intuition for $\underline{X}_{\widetilde{\mathcal{F}}}$ is that is gives the G–space of all points in $\underline{X}$ with nontrivial isotropy.

Having recounted the constructions relevant to the complementary part $\mathcal{F}^c$, we now recall some language associated to the $\mathcal{F}$–local part. Recall that for a family $\mathcal{F}$, we denoted by $b\colon \underline{E\mathcal{F}} \to \underline{*}$ the unique map.

Definition 2.2.17. We say that a G–category $\underline{\mathcal{C}}$ is *$\mathcal{F}$–Borel* if that the map $\underline{\mathcal{C}} \to b_*b^*\underline{\mathcal{C}}$ is an equivalence. A G–category $\underline{\mathcal{C}}$ will be called *$\mathcal{F}$–coBorel* if the map $b_!b^*\underline{\mathcal{C}} \to \underline{\mathcal{C}}$ is an equivalence.

Example 2.2.18 (Borel categories)**.** For the trivial family $\mathcal{F} = \{1\}$, we will also write $\underline{E\mathcal{F}}$ as $\underline{EG}$ and write $\underline{\mathrm{Bor}} := b_*\colon \mathrm{Cat}_{G,\{1\}} \simeq \mathrm{Cat}^{BG} \hookrightarrow \mathrm{Cat}_G$. We call $\underline{\mathrm{Bor}}(\mathcal{C})$ the *Borel-G-category* associated to $\mathcal{C}$. Explicitly, $\underline{\mathrm{Bor}}(\mathcal{C})(\underline{G/H}) = \mathcal{C}^{hH}$. In this case, the adjunctions Construction 2.2.13 produce the Borelification Bousfield (co)localisations studied in [Hil24, §2.4]. While we will not need it in this article, we mention that there, it was shown in the case of finite groups G that $b^*\colon \mathrm{Cat}_G \to \mathrm{Cat}^{BG}$ naturally assemble to a G–symmetric monoidal Bousfield localisation and thus interacts well with the multiplicative norms.

Fact 2.2.19. An alternative description for $b_!b^*\underline{\mathcal{C}}$ and $b_*b^*\underline{\mathcal{C}}$ are $\underline{E\mathcal{F}} \times \underline{\mathcal{C}}$ and $\underline{\mathrm{Fun}}(\underline{E\mathcal{F}}, \underline{\mathcal{C}})$ respectively.

Notation 2.2.20. For a subgroup $K \leq G$, we write $\mathcal{F}_K$ for the family of subgroups of K which belong to $\mathcal{F}$. Note that, in particular, the equivalence $\mathcal{O}(G)_{/(G/K)} \simeq \mathcal{O}(K)$ induces an equivalence $\mathcal{O}_{\mathcal{F}}(G)_{/(G/K)} \simeq \mathcal{O}_{\mathcal{F}_K}(K)$.

Proposition 2.2.21 (Characterisations of (co)Borelness)**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_G$. Then:*

(a) $\underline{\mathcal{C}}$ is $\mathcal{F}$–coBorel if and only if $\mathcal{C}^H \simeq \varnothing$ for all $H \in \mathcal{F}^c$,

(b) $\underline{\mathcal{C}}$ is $\mathcal{F}$–Borel if and only if for all $K \leq G$, the canonical map $\mathcal{C}^K \to \lim_{G/H \in \mathcal{O}_{\mathcal{F}_K}(K)^{\mathrm{op}}} \mathcal{C}^H$ induced by restrictions is an equivalence.

Proof. Part (a) is immediate using the description $b_!b^*\underline{\mathcal{C}} \simeq \underline{E\mathcal{F}} \times \underline{\mathcal{C}}$. For part (b), the comma category used to compute the value of the right Kan extension $b_*\colon \mathrm{Fun}(\mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Cat})$ at G/K is

$$\big(\mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}\big)_{(G/K)/} \simeq \big(\mathcal{O}_{\mathcal{F}}(G)_{/(G/K)}\big)^{\mathrm{op}} \simeq \mathcal{O}_{\mathcal{F}_K}(K)^{\mathrm{op}}$$

whence the claim. ◻

Example 2.2.22 (Modules over $\mathcal{F}$–nilpotent rings)**.** Suppose G is a finite group and $\mathcal{F}$ is a family of subgroups. By [MNN17, Prop. 6.38 (1), Thm. 6.42] and the concrete characterisation of $\mathcal{F}$–Borelness from Proposition 2.2.21 (b), we learn that if $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ is $\mathcal{F}$–nilpotent, then $\underline{\mathrm{Mod}}_{\underline{\mathrm{Sp}}_G}(R)$ is an $\mathcal{F}$–Borel G–category.

Categorified Brauer quotients

Consider a family $\mathcal{F}$ of closed subgroups of G. The adjunction $s^*\colon \mathrm{Cat}_G \rightleftharpoons \mathrm{Cat}_{G,\mathcal{F}^c} : s_*$ from Construction 2.2.13 does *not* restrict to an adjunction between presentable or (fibrewise) stable categories as the adjunction unit does not preserve G-colimits. The main result of this section shows that the restriction $s_*\colon \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Pr}^{L,\mathrm{st}}_{G}$ (which is fully faithful by Lemma 2.1.32) admits a symmetric monoidal left adjoint $\widetilde{s}^*$. We do this by showing that it is a smashing localisation.

Construction 2.2.23 ((Co)tensoring over pointed groupoids)**.** Let $\underline{\mathcal{E}}$ be a pointed $\mathcal{B}$–category admitting all parametrised (co)limits. Then $\underline{\mathcal{E}}$ is naturally tensored and cotensored over pointed $\mathcal{B}$–groupoids $\mathcal{B}_*$ as follows: for $\underline{*} \to \underline{X}$ in $\mathcal{B}_*$ and $E \in \underline{\mathcal{E}}$, we define

$$\underline{X} \wedge E := \mathrm{cofib}\left(E \simeq \operatorname*{colim}_{\underline{*}} E \to \operatorname*{colim}_{\underline{X}} E\right) \qquad \underline{\mathrm{hom}}_*(\underline{X}, E) := \mathrm{fib}\left(\lim_{\underline{X}} E \to \lim_{\underline{*}} E \simeq E\right)$$

These exhibit $\underline{\mathcal{E}}$ as being tensored and cotensored over $\mathcal{B}_*$, respectively, since for example, for a fixed $F \in \underline{\mathcal{E}}$, we have

$$\begin{aligned}
\mathrm{Map}_{\underline{\mathcal{E}}}(\underline{X} \wedge E, F) &\simeq \mathrm{fib}\left(\mathrm{Map}_{\underline{\mathcal{E}}}(\operatorname*{colim}_{\underline{X}} E, F) \to \mathrm{Map}_{\underline{\mathcal{E}}}(E, F)\right)\\
&\simeq \mathrm{Map}_{\mathcal{B}}(\underline{X}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E,F)) \times_{\mathrm{Map}_{\mathcal{B}}(\underline{*}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E,F))} \{*\}\\
&\simeq \mathrm{Map}_{\mathcal{B}_*}(\underline{X}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E,F))
\end{aligned}$$

Observe also that these constructions give us an adjunction $\underline{X} \wedge - \colon \underline{\mathcal{E}} \rightleftharpoons \underline{\mathcal{E}} : \underline{\mathrm{hom}}_*(\underline{X}, -)$. Moreover, it is easy to see that for $\underline{X}, \underline{Y} \in \mathcal{B}_*$, we have $\underline{X} \wedge (\underline{Y} \wedge E) \simeq (\underline{X} \wedge \underline{Y}) \wedge E$ where $\underline{X} \wedge \underline{Y} \simeq \mathrm{cofib}(\underline{X} \vee \underline{Y} \to \underline{X} \times \underline{Y})$.

Observation 2.2.24. Let $\mathcal{F}$ be a family of closed subgroups of G and $s^*\colon \mathrm{Cat}_{G,*} \rightleftharpoons \mathrm{Cat}_{\mathcal{F}^c,*} : s_*$ the associate Bousfield localisation of the geometric morphism $s^*\colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_{\mathcal{F}^c}$. Let $\underline{X} \in \mathcal{S}_{G,*}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{F}^c,*}$. Then there is an equivalence $\underline{\mathrm{hom}}_*(\underline{X}, s_*\underline{\mathcal{C}}) \simeq s_*\underline{\mathrm{hom}}_*(s^*\underline{X}, \underline{\mathcal{C}})$ by virtue of the following computation

$$\underline{\mathrm{hom}}_*(\underline{X}, s_*\underline{\mathcal{C}}) \simeq \mathrm{fib}(\lim_{\underline{X}} s_*\underline{\mathcal{C}} \to \lim_{\underline{*}} s_*\underline{\mathcal{C}}) \simeq s_*\,\mathrm{fib}(\lim_{s^*\underline{X}} \underline{\mathcal{C}} \to \lim_{\underline{*}} \underline{\mathcal{C}}) \simeq s_*\underline{\mathrm{hom}}_*(s^*\underline{X}, \underline{\mathcal{C}}).$$

Here we have used that s_* commutes with limits and the equivalence $\lim_{\underline{X}} s_*\underline{\mathcal{C}} \simeq s_* \lim_{s^*\underline{X}} \underline{\mathcal{C}}$ coming from the identifications of adjunctions in Lemma 2.1.18.

We introduce now the key notion of *Brauer quotients* of categories with respect to a fixed family. As will be clear from the next terminology, they will be a special case of the standard categorical construction of Verdier quotients. However, since they will play such a key role in this article and are so specific to the equivariant situation, we have chosen to dignify them with a special name, borrowing from the classical theory of Mackey functors.

Terminology 2.2.25 ($\mathcal{F}$–Brauer quotients)**.** For a finite group G, we define the $\mathcal{F}$–Brauer quotient $\underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ of a small G–stable category $\underline{\mathcal{D}}$ as a G–stable category admitting a G–exact functor $\Phi^{\mathcal{F}}\colon \underline{\mathcal{D}} \to \underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ which, for all G–stable categories $\underline{\mathcal{E}}$, induces an equivalence

$$(\Phi^{\mathcal{F}})^*\colon \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}/\langle\mathcal{F}\rangle, \underline{\mathcal{E}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\mathrm{ex},\mathcal{F}=0}(\underline{\mathcal{D}}, \underline{\mathcal{E}})$$

where $\underline{\mathrm{Fun}}^{\mathrm{ex},\mathcal{F}=0}(\underline{\mathcal{D}},\underline{\mathcal{E}}) \subseteq \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}},\underline{\mathcal{E}})$ is the full G–subcategory of G–exact functors $F\colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ such that $\mathrm{Res}^G_H F\colon \mathrm{Res}^G_H \underline{\mathcal{D}} \to \mathrm{Res}^G_H \underline{\mathcal{E}}$ is the zero functor for all $H \in \mathcal{F}$. Observe that $\underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ must be unique if it exists. We denote by $\mathrm{Cat}^{G-\mathrm{st}}_{G,\mathcal{F}^c} \subseteq \mathrm{Cat}^{G-\mathrm{st}}_{G}$ the full subcategory given by those G-stable categories lying in the image of $s_*\colon \mathrm{Cat}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Cat}_G$, i.e. those with value 0 on $\mathcal{O}_{\mathcal{F}}(G)$.

Analogously in the presentable setting, for a compact Lie group G and a family $\mathcal{F}$ of closed subgroups, we may define the $\mathcal{F}$–Brauer quotient of an object $\underline{\mathcal{C}} \in \mathrm{Pr}^{L,\mathrm{st}}_G$ as a presentable G–category $\underline{\mathcal{C}}/\langle\mathcal{F}\rangle$ equipped with a parametrised colimit–preserving functor $\Phi^{\mathcal{F}}\colon \underline{\mathcal{C}} \to \underline{\mathcal{C}}/\langle\mathcal{F}\rangle$ inducing for every fibrewise stable presentable G–category $\underline{\mathcal{E}}$ an equivalence

$$(\Phi^{\mathcal{F}})^*\colon \underline{\mathrm{Fun}}^{L}(\underline{\mathcal{C}}/\langle\mathcal{F}\rangle,\underline{\mathcal{E}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{L,\mathcal{F}=0}(\underline{\mathcal{C}},\underline{\mathcal{E}})$$

Theorem 2.2.26 (Categorified Brauer quotients)**.** *Let G be a compact Lie group, H a finite group, $\mathcal{F}$ a family of closed subgroups of G and $\mathcal{E}$ a family of subgroups of H. Then the fully faithful inclusions*

$$s_*\colon \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Pr}^{L,\mathrm{st}}_{G} \qquad s_*\colon \mathrm{Cat}^{H-\mathrm{st}}_{H,\mathcal{E}^c} \hookrightarrow \mathrm{Cat}^{H-\mathrm{st}}_{H}$$

all admit symmetric monoidal left adjoints $\widetilde{s}^*$ which are smashing localisations at the idempotent algebra $\widetilde{EF}$. In the first case, the induced lax symmetric monoidal structure on s_* agrees with the one from Lemma 2.1.31. Moreover, $\widetilde{s}^*$ satisfies the universal property of the $\mathcal{F}$–Brauer quotient.

Proof. We only prove the first case since the second one can be done entirely analogously. $\underline{\mathrm{Pr}}^{L,\mathrm{st}}_G$ is a pointed category admitting all parametrised (co)limits and is thus tensored over $\mathcal{S}_{G,*}$ by Construction 2.2.23 (in the presentable case, this also comes from pointedness being classified by the idempotent algebra $\mathcal{S}_{G,*} \in \mathrm{Pr}^{L}_G$).

We claim that the left adjoint to s_* is given by $\underline{\widetilde{E\mathcal{F}}} \wedge -$. To see that this functor does indeed take values in $\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Pr}^{L,\mathrm{st}}_{G}$ note that for all $K \in \mathcal{F}$ and $\underline{\mathcal{C}} \in \mathrm{Pr}^{L,\mathrm{st}}_{G}$ we have

$$\mathrm{Res}^G_K(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}) \simeq \mathrm{Res}^G_K \underline{\widetilde{E\mathcal{F}}} \wedge \mathrm{Res}^G_K \underline{\mathcal{C}} \simeq * \wedge \mathrm{Res}^G_K \underline{\mathcal{C}} \simeq 0.$$

To see that it is the left adjoint to s_*, let $\underline{\mathcal{D}} \in \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}$. Observe that since $s^*\underline{\widetilde{E\mathcal{F}}} \simeq \underline{S}^0$, we have equivalences $\underline{\hom}_*(\underline{\widetilde{E\mathcal{F}}}, s_*\underline{\mathcal{D}}) \simeq s_*\underline{\hom}_*(\underline{S}^0,\underline{\mathcal{D}}) \simeq s_*\underline{\mathcal{D}}$, where the first equivalence is by Observation 2.2.24. Thus, the computation

$$\mathrm{Map}_{\mathrm{Pr}^{L,\mathrm{st}}_G}(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}, s_*\underline{\mathcal{D}}) \simeq \mathrm{Map}_{\mathrm{Pr}^{L,\mathrm{st}}_G}(\underline{\mathcal{C}}, \underline{\hom}_*(\underline{\widetilde{E\mathcal{F}}}, s_*\underline{\mathcal{D}})) \simeq \mathrm{Map}_{\mathrm{Pr}^{L,\mathrm{st}}_G}(\underline{\mathcal{C}}, s_*\underline{\mathcal{D}}).$$

shows that $\underline{\widetilde{E\mathcal{F}}} \wedge -$ is indeed the left adjoint to s_* as claimed.

Next, since $\underline{\widetilde{E\mathcal{F}}}$ is an idempotent algebra in $\mathcal{S}_{G,*}$, the left adjoint $\widetilde{s}^*(-) \simeq \underline{\widetilde{E\mathcal{F}}} \wedge -$ is a smashing localisation and in particular attains a canonical symmetric monoidal structure. To show that the induced lax symmetric monoidal structure on s_* is equivalent to the one from Lemma 2.1.31, by Lemma 2.1.32 we only have to show that the map $u\colon \widetilde{s}^*\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_G} \to \mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}}$ adjoint to the lax unit $\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_G} \to s_*\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}}$ is an equivalence. But by construction of the lax symmetric monoidal structure on s_*, we have for $\underline{\mathcal{C}} \in \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}$ the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Fun}^L_{\mathcal{F}}\left(\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}}, \underline{\mathcal{C}}\right) & & \xrightarrow{\quad\simeq\quad} & & \Gamma(\underline{\mathcal{C}}) \\
\downarrow{\scriptstyle u^*} & & & & \downarrow{\scriptstyle \simeq} \\
\mathrm{Fun}^L_{\mathcal{F}}\left(\widetilde{s}^*\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G}}, \underline{\mathcal{C}}\right) & \xrightarrow{\simeq} & \mathrm{Fun}^L_{G}\left(\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G}}, s_*\underline{\mathcal{C}}\right) & \xrightarrow{\simeq} & \Gamma(s_*\underline{\mathcal{C}})
\end{array}$$

showing that the left vertical map is an equivalence.

Finally, for the statement about $\mathcal{F}$–Brauer quotients, notice that the unit map $\underline{\mathcal{C}} \to \underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}$ has trivial restriction to each group $H \in \mathcal{F}$ as $\mathrm{Res}^G_H \underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}} \simeq \mathrm{Res}^G_H \underline{\widetilde{E\mathcal{F}}} \wedge \mathrm{Res}^G_H \underline{\mathcal{C}} \simeq * \wedge \mathrm{Res}^G_H \underline{\mathcal{C}} \simeq 0$. We have to show that for all $\underline{\mathcal{D}} \in \mathrm{Pr}^{L,\mathrm{st}}_G$, the induced map $\underline{\mathrm{Fun}}^L(s_*\widetilde{s}^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^{L,\mathcal{F}=0}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is an equivalence. From the cofibre sequence $\underline{E\mathcal{F}}_+ \to \underline{S}^0 \to \underline{\widetilde{E\mathcal{F}}}$ in $\mathcal{S}_{G,*}$ we obtain the fibre sequence

$$\underline{\mathrm{Fun}}^L(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^L(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^L(\underline{E\mathcal{F}}_+ \wedge \underline{\mathcal{C}}, \underline{\mathcal{D}})$$

in $\mathrm{Pr}^{L,\mathrm{st}}_G$. This shows that $\underline{\mathrm{Fun}}^L(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}, \underline{\mathcal{D}})$ is a full G-subcategory of $\underline{\mathrm{Fun}}^L(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ (this is true for any fibre sequence of stable categories). Now suppose that $f\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ vanishes on $\mathcal{F}$. Denote by $\langle \mathrm{Im}(f)\rangle \subseteq \underline{\mathcal{D}}$ the full presentable fibrewise stable G-subcategory generated by the image of f. The assumption on f guarantees that $\mathrm{Res}^G_H \langle \mathrm{Im}(f)\rangle = 0$ whenever $H \in \mathcal{F}$, i.e. $\langle \mathrm{Im}(f)\rangle$ lies in the image of s_*. Consider the commutative diagram

$$\begin{array}{ccccc}
\underline{\mathcal{C}} & \xrightarrow{\quad f\quad} & \langle \mathrm{Im}(f)\rangle & \hookrightarrow & \underline{\mathcal{D}} \\
\downarrow & & \downarrow{\scriptstyle\simeq} & \nearrow{\scriptstyle \text{(dashed)}} & \\
\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}} & \xrightarrow{f} & \underline{\widetilde{E\mathcal{F}}} \wedge \langle \mathrm{Im}(f)\rangle & &
\end{array}.$$

As $\langle \mathrm{Im}(f)\rangle$ lies in the image of s_*, the first part shows that the middle vertical arrow is an equivalence from which we obtain the dashed factorisation. This concludes the proof of the theorem. ☐

Remark 2.2.27. In the setting of finite groups G, for a G–stable category $\underline{\mathcal{D}}$, since $\widetilde{s}^*\underline{\mathcal{D}}$ is the $\mathcal{F}$–Brauer quotient, it satisfies the universal property of the Verdier quotient articulated in [QS22, Thm. 5.23]. Thus, by [QS22, Def. 5.21], it may alternatively be described as a fibrewise Verdier quotient in the nonequivariant sense.

Notation 2.2.28. Now that we have all the fixed points functors that will concern us, let us collect and summarise them, introducing some new notations along the way. While the notations $\{s_!, s^*, s_*, \widetilde{s}^*\}$ are compact and lithe, useful to prove results, we believe that the notations presently introduced have more intuitive appeal. The starting point will be the inclusion $s\colon \mathcal{O}_{\mathcal{F}^c}(G)^{\mathrm{op}} \hookrightarrow \mathcal{O}(G)^{\mathrm{op}}$ from Construction 2.2.13.

(a) Recall the notations from Construction 2.2.15 which gives us the top adjunctions in

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{F}^c} & \begin{array}{c} \xrightarrow{(-)_{\widetilde{\mathcal{F}}}:=s_!} \\ \xleftarrow{(-)^{\mathcal{F}^c}:=s^*} \\ \xrightarrow{(-)^{\widetilde{\mathcal{F}}}:=s_*} \end{array} & \mathcal{S}_G \\ \big\downarrow & & \big\downarrow \\ \mathrm{Cat}_{G,\mathcal{F}^c} & \begin{array}{c} \xrightarrow{(-)_{\widetilde{\mathcal{F}}}:=s_!} \\ \xleftarrow{(-)^{\mathcal{F}^c}:=s^*} \\ \xrightarrow{(-)^{\widetilde{\mathcal{F}}}:=s_*} \end{array} & \mathrm{Cat}_G \end{array}$$

and that we have the commuting squares of adjunctions is an easy check using that s^* commutes with the vertical maps and their adjoints. Since $(-)_{\widetilde{\mathcal{F}}}$ and $(-)^{\widetilde{\mathcal{F}}}$ are fully faithful, we also write $(-)_{\widetilde{\mathcal{F}}}$ and $(-)^{\widetilde{\mathcal{F}}}$ for $s_!s^*$ and s_*s^* respectively. In particular, for $\underline{X} \in \mathcal{S}_G$, the counit gives us a map $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} = s_!s^*\underline{X} \to \underline{X}$ as in Construction 2.2.15. Moreover, by Construction 2.2.13, we have for $\underline{\mathcal{C}} \in \mathrm{Cat}_{G,\mathcal{F}^c}$ the description

$$\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}} = \begin{cases} \mathcal{C}(\underline{G/H}) & \text{if } H \in \mathcal{F}^c; \\ * & \text{if } H \in \mathcal{F}. \end{cases}$$

(b) We also have the following solid commuting squares

$$\begin{array}{ccc} \mathrm{Pr}^{\mathrm{st}}_G & \begin{array}{c} \xrightarrow{\Phi^{\mathcal{F}}(-):=\widetilde{s}^*} \\ \xleftarrow[(-)^{\Phi\widetilde{\mathcal{F}}}:=s_*]{} \end{array} & \mathrm{Pr}^{\mathrm{st}}_{G,\mathcal{F}^c} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathrm{Cat}}_G & \xleftarrow{(-)^{\widetilde{\mathcal{F}}}:=s_*} & \widehat{\mathrm{Cat}}_{G,\mathcal{F}^c} \end{array} \qquad \begin{array}{ccc} \mathrm{Cat}^{\mathrm{ex}}_G & \begin{array}{c} \xrightarrow{\Phi^{\mathcal{F}}(-):=\widetilde{s}^*} \\ \xleftarrow[(-)^{\Phi\widetilde{\mathcal{F}}}:=s_*]{} \end{array} & \mathrm{Cat}^{\mathrm{ex}}_{G,\mathcal{F}^c} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Cat}_G & \xleftarrow{(-)^{\widetilde{\mathcal{F}}}:=s_*} & \mathrm{Cat}_{G,\mathcal{F}^c} \end{array}$$

where the top maps admit the dashed left adjoints. Here, the left diagram holds for general compact Lie groups G whereas the right diagram is only defined for finite groups G from Theorem 2.2.26. As above, since the functors $(-)^{\Phi\widetilde{\mathcal{F}}}$ are fully faithful, we will also write $(-)^{\Phi\widetilde{\mathcal{F}}}$ to denote $s_*\widetilde{s}^*$. The adjunction unit $\mathrm{id} \to s_*\widetilde{s}^*$ will be denoted by $\Phi^{\mathcal{F}}\colon (-) \to (-)^{\Phi\widetilde{\mathcal{F}}}$ or just $\Phi\colon (-) \to (-)^{\Phi\widetilde{\mathcal{F}}}$ when the family $\mathcal{F}$ is understood.

Stability for quotient groups

Let $N \leq G$ be a closed normal subgroup of the compact Lie group G and denote by $\theta\colon G \twoheadrightarrow G/N = Q$ the quotient map. We will use the categorified Brauer quotient from Theorem 2.2.26 for the family Γ_N from Example 2.2.10 to relate $G-$ and $Q-$stable categories.

Proposition 2.2.29. *Suppose that $\theta\colon G \twoheadrightarrow G/N = Q$ is a continuous epimorphism of compact Lie groups. Then there is an adjunction*

$$\mathrm{Pr}^{L,G-\mathrm{st}}_G \underset{\mathrm{Coinfl}_\alpha}{\overset{\mathrm{Coind}_{\widetilde{\alpha}}}{\rightleftarrows}} \mathrm{Pr}^{L,Q-\mathrm{st}}_Q$$

which is a smashing localisation. The lax symmetric monoidal structure on Coinfl_α from Lemma 2.2.8 is equivalent to the lax symmetric monoidal structure from this smashing localisation. We thus may view $G/N-$stable presentable categories precisely as $G-$stable categories which vanish for all subgroups $H \leq G$ not containing N.

Proof. By combining Example 2.2.14 and Theorem 2.2.26, $\mathrm{Coinfl}_\alpha\colon \mathrm{Pr}_Q^{L,\mathrm{st}} \hookrightarrow \mathrm{Pr}_G^{L,\mathrm{st}}$ admits a symmetric monoidal left adjoint $\mathrm{Coind}^{\sim}_\alpha = \mathrm{Coind}_\alpha(\underline{\widetilde{E\Gamma}_N} \wedge -)$ which is a smashing localisation. We only have to show that this restricts to an adjunction between G- and Q-stable categories. But this follows by combining Lemma 2.2.6, Lemma 2.2.8 and observing that $\underline{\widetilde{E\Gamma}_N} \wedge -$ preserves G-stable categories. □

Corollary 2.2.30. *Writing $\theta\colon G \twoheadrightarrow G/N$ for the quotient map by a closed normal subgroup, the symmetric monoidal unit map $\underline{\mathrm{Sp}}_{G/N} \to \mathrm{Coind}^{\sim}_\alpha \underline{\mathrm{Sp}}_G$ is an equivalence.*

Proof. This is a direct consequence of symmetric monoidality of the adjunction in Proposition 2.2.29. □

Construction 2.2.31 (Geometric fixed points)**.** Let $\theta_G\colon G \twoheadrightarrow 1$ be the quotient map. The symmetric monoidal G-colimit preserving unit map

$$\Phi^G\colon \underline{\mathrm{Sp}}_G \longrightarrow \mathrm{Coinfl}_{\theta_G} \mathrm{Coind}^{\sim}_{\theta_G} \underline{\mathrm{Sp}}_G \simeq \mathrm{Coinfl}_{\theta_G} \mathrm{Sp}$$

restricts to a symmetric monoidal colimit preserving functor $\Phi^G\colon \mathrm{Sp}_G \to \mathrm{Sp}$. There is an equivalence equivalence $\Phi^G \circ \Sigma^\infty_G(-) \simeq \Sigma^\infty(-)^G$ as, by construction, Φ_G is $\mathcal{S}_{G,*}$-linear and sends the unit to the unit. This shows that Φ^G recovers the classical geometric fixed points functor which is uniquely determined by these properties.

If $H \leq G$ is a closed subgroup, we have the symmetric monoidal G-colimit preserving functor $\Phi^H\colon \underline{\mathrm{Sp}}_G \to \mathrm{Coind}_H^G \underline{\mathrm{Sp}}_H \to \mathrm{Coind}_H^G \mathrm{Coinfl}_{\theta_H} \mathrm{Sp}$ which on global sections recovers the classical geometric fixed point functors $\Phi^H\colon \mathrm{Sp}_G \to \mathrm{Sp}$.

Definition 2.2.32. A collection of $\mathcal{B}$–functors $\{F_s\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ is *jointly conservative* if for all $X \in \mathcal{B}$, the collection $\{F_s(X)\colon \mathcal{C}(\underline{X}) \to \mathcal{D}_s(\underline{X})\}_{s\in S}$ is jointly conservative.

Observation 2.2.33. Let $\{F_s\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ be a jointly conservative collection of $\mathcal{B}$–functors and $\underline{X} \in \mathcal{B}$. Then the collection $\{F_s\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}}_s)\}_{s\in S}$ is also a jointly conservative collection. This is an immediate consequence of the definition and that the evaluation at $Y \in \mathcal{B}$ for the $\mathcal{B}$–category $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ is $\mathcal{C}(\underline{Y} \times \underline{X})$.

Remark 2.2.34. If $\{\underline{X}_i\}_{i\in I}$ is a set of objects generating $\mathcal{B}$ under colimits, then a collection of $\mathcal{B}$–functors $\{F_s\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ is jointly conservative if $\{F_s(X)\colon \mathcal{C}(\underline{X}_i) \to \mathcal{D}_s(\underline{X}_i)\}_{s\in S}$ is jointly conservative for each i. Indeed, let $\underline{X}$ be an object. Then $\mathcal{C}(\underline{X}) \simeq \lim_{(i,f\colon \underline{X}_i \to \underline{X})} \mathcal{C}(\underline{X}_i)$ and the collection $T = \{\mathcal{C}(\underline{X}) \xrightarrow{f^*} \mathcal{C}(\underline{X}_i) \mid i \in I, f\colon \underline{X}_i \to \underline{X}\}$ is jointly conservative. Suppose that h is a morphism in $\mathcal{C}(\underline{X})$ which maps to an equivalence in $\mathcal{D}_s(\underline{X})$ for each $s \in S$. For $(i, f) \in T$, we observe that in the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\underline{X}) & \xrightarrow{f^*} & \mathcal{D}_s(\underline{X}) \\ \downarrow & & \downarrow \\ \mathcal{C}(\underline{X}_i) & \longrightarrow & \mathcal{D}_s(\underline{X}_i) \end{array}$$

the morphism h maps to an equivalence in the lower right corner for each $s \in S$, so by assumption it mapped to an equivalence in the lower left corner. As that holds true for each $(i,f) \in T$ and the collection T was jointly conservative, we see that h was an equivalence to start with, as desired.

Proposition 2.2.35 (Joint conservativity of geometric fixed points)**.** *The collection of G–functors*

$$\left\{ \Phi^H : \underline{\mathrm{Sp}} \xrightarrow{\eta} \mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathrm{Sp}} \xrightarrow{\mathrm{Coind}_H^G \mathrm{Res}_H^G \Phi^{\mathcal{P}_H}} \mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathrm{Sp}}^{\Phi\widetilde{\mathcal{P}_H}} \quad \middle| \quad H \leq G \text{ closed} \right\}$$

is jointly conservative.

Proof. By Remark 2.2.34, it suffices to show that the collection is a jointly conservative collection of functors when evaluated at each $\underline{G/K} \in \mathcal{S}_G$. Let $H \leq K$ be a subgroup. Since $\underline{G/H} \simeq \mathrm{Ind}_H^G \underline{*}$, by the triangle identity, the counit $\mathrm{Ind}_H^G \mathrm{Res}_H^G \underline{G/H} \xrightarrow{\epsilon_{G/H}} \underline{G/H}$ admits a section. Using this and the map $\underline{G/H} \to \underline{G/K}$ we obtain in total a map $h\colon \underline{G/H} \to \mathrm{Ind}_H^G \mathrm{Res}_H^G \underline{G/K}$. Hence, since we have $(\mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathcal{C}})(G/K) \simeq \mathrm{Fun}_G(\mathrm{Ind}_H^G \mathrm{Res}_H^G \underline{G/K}, \underline{\mathcal{C}})$ and $\mathcal{C}(G/H) \simeq \mathrm{Fun}_G(\underline{G/H}, \underline{\mathcal{C}})$ for any G–category $\underline{\mathcal{C}}$, we get a transformation $h^*\colon (\mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathcal{C}})(G/K) \to \mathcal{C}(G/H)$ natural in $\underline{\mathcal{C}}$. Therefore, for $H \leq K$, evaluating the functor Φ^H in the statement at G/K together with the transformation above gives the following commuting diagram

$$\begin{array}{ccccc}
\mathrm{Sp}_K & \xrightarrow{\eta} & (\mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathrm{Sp}})(G/K) & \xrightarrow{\mathrm{Coind}_H^G \mathrm{Res}_H^G \Phi^{\mathcal{P}_H}} & (\mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathrm{Sp}}^{\Phi\widetilde{\mathcal{P}_H}})(G/K) \\
\| & & \downarrow h^* & & \downarrow h^* \\
\mathrm{Sp}_K & \xrightarrow{\mathrm{Res}_H^K} & \mathrm{Sp}_H & \xrightarrow{\Phi^H} & \mathrm{Sp}
\end{array}$$

But since the bottom compositions are jointly conservative when we let H vary over all closed subgroups of K (this is well–known, see for example [Sch18, Prop. 3.3.10]), we thus get similarly that the top compositions are too. This completes the proof. ☐

Free actions

Here we review a few geometric facts on G-spaces on which a normal subgroup N acts freely. It will be needed later on to argue for example, if $\underline{X}$ is a G-Poincaré space with free N-action, then also the quotient $N\backslash\underline{X}$ is G/N-Poincaré.

Definition 2.2.36 (Free actions)**.** Consider a group G together with a normal subgroup $N \leq G$. We say that the action of N on $\underline{X} \in \mathcal{S}_G$ is *free* if $\underline{X}$ is coBorel with respect to the family $\mathcal{F}_N$.

That is, N acts freely on $\underline{X}$ if whenever $N \cap H \neq \{1\}$ we have $X^H = \varnothing$ as is shown in Fact 2.2.19.

Remark 2.2.37 (Quotients of free G–spaces)**.** Consider $\underline{X} \in \mathcal{S}_G$ and isotropy separation with respect to the trivial family $\mathcal{F} = 1$ (which is the case $N = G$ in Definition 2.2.36). Recalling the operation of genuine quotients from Notation 2.2.2, note that for this family we obtain $G\backslash b_!(-) \simeq (-)_{hG}$ as the first functor is left adjoint to the composite $b^* \mathrm{Infl}_G^1 \colon \mathcal{S} \to \mathcal{S}^{BG}$ which is the restriction functor along the projection $BG \to *$. In particular, we obtain a map

$$X_{hG} \simeq G\backslash(b_! b^* \underline{X}) \to G\backslash\underline{X}.$$

This is an equivalence if G acts freely on $\underline{X}$ since $X \simeq b_! Y$ for some $Y \in \mathcal{S}^{BG}$.

We will now prove two lemmas about quotients by free actions. Together, they are useful in studying the fibres of the map $\underline{X} \to \mathrm{Infl}_G^Q N \backslash \underline{X}$, as we will see in Corollary 2.2.40.

Lemma 2.2.38. *Let G be a group and let $N \subset G$ be a closed normal subgroup. Let $f\colon \underline{X} \to \underline{Y}$ be a map of G–spaces, where $\underline{X}$ and $\underline{Y}$ have free N-actions. Then the square*

$$\begin{array}{ccc} \underline{X} & \longrightarrow & \underline{Y} \\ \downarrow & & \downarrow \\ \mathrm{Infl}_G^{G/N} N\backslash \underline{X} & \longrightarrow & \mathrm{Infl}_G^{G/N} N\backslash \underline{Y} \end{array} \tag{10}$$

is cartesian.

Proof. We want to check that the square (10) is cartesian and we will do so in three steps[2]. First, a computation shows that it is cartesian whenever $\underline{X}$ and $\underline{Y}$ are N-free G-orbits. Second, writing $\underline{Y} = \mathrm{colim}_{H \leq G,\, \underline{G/H} \to \underline{Y}} \underline{G/H}$ an application of [Lur09, 6.1.3.9.(4)] shows that the claim is true for $\underline{X}$ an N-free G-orbit and $\underline{Y}$ an N-free G–space. It is easy to see that the claim now also holds when $\underline{X}$ is a disjoint union of N-free G-orbits. For the general statement, note that by Lemma 2.2.39, we may find a collection of N-free G-orbits $\underline{G/H_i}$ together with maps $q_i\colon \underline{G/H_i} \to \underline{X}$ such that the map

$$\coprod \mathrm{Infl}_G^{G/N} N\backslash q_i\colon \coprod \mathrm{Infl}_G^{G/N} N\backslash \underline{G/H_i} \to \mathrm{Infl}_G^{G/N} N\backslash \underline{X}$$

induces a π_0-surjection on all fixed points[3]. In the diagram

$$\begin{array}{ccccc} \coprod \underline{G/H_i} & \longrightarrow & \underline{X} & \longrightarrow & \underline{Y} \\ \downarrow & & \downarrow & & \downarrow \\ \coprod \mathrm{Infl}_G^{G/N} N\backslash \underline{G/H_i} & \longrightarrow & \mathrm{Infl}_G^{G/N} N\backslash \underline{X} & \longrightarrow & \mathrm{Infl}_G^{G/N} N\backslash \underline{Y} \end{array}$$

we know that the outer square and left square are cartesian. As the bottom left map induces a π_0-surjection on all fixed points, this implies that the right square is cartesian as well. ☐

Lemma 2.2.39. *Let $\underline{X}$ be a G–space with free N-action. Then, for each map $f\colon \underline{Q/H} \to N\backslash \underline{X}$ there exists a subgroup $K \in \mathcal{F}_N$ and a commutative diagram*

$$\begin{array}{ccccc} & & \underline{G/K} & \longrightarrow & \underline{X} \\ & & \downarrow & & \downarrow \\ \mathrm{Infl}_G^Q \underline{Q/H} & \longrightarrow & \mathrm{Infl}_G^Q N\backslash \underline{G/K} & \longrightarrow & \mathrm{Infl}_G^Q N\backslash \underline{X} \end{array}$$

[2] An (arguably shorter) proof is possible if one recalls the model from [Sch18, Prop. B.7.] and observes that given an N-free topological G-CW space $\mathcal{X}$, the map $\mathcal{X} \to \mathrm{Infl}_G^{G/N} N\backslash \mathcal{X}$ is a fibration with point-set fibre N.

[3] Such morphisms are effective epimorphisms.

where the lower composition is $\mathrm{Infl}_G^Q(f)$.

Proof. Using the explicit formula from (9) we compute

$$\mathrm{Map}_{\mathcal{S}_Q}(\underline{Q/H}, N\backslash\underline{X}) \simeq \operatorname*{colim}_{G/K,\, Q/H\to N\backslash(G/K)} \mathrm{Map}_{\mathcal{S}_G}(\underline{G/K}, \underline{X}).$$

The map from the left hand side takes $g\colon \underline{G/K} \to \underline{X}$, applies $N\backslash(-)$ to it and precomposes with $Q/H \to N\backslash\underline{G/K}$. Thus, f eviently factors through some map $N\backslash\underline{G/K} \to N\backslash\underline{X}$ that is of the from $N\backslash g$ for some $g\colon \underline{G/H} \to \underline{X}$. Now as $\mathrm{Map}(\underline{G/H}, \underline{X})$ can only be nonempty if $H \in \mathcal{F}$, we have proved the assertion. ☐

Corollary 2.2.40. *Let* $\underline{X}$ *be a* G*–space on which the closed normal subgroup* $N \leq G$ *acts freely. Consider any map* $f: \underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{X}$. *Then there exists a cartesian diagram*

$$\begin{array}{ccc} \underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X} & \longrightarrow & \underline{G/K_0} \\ \downarrow{\scriptstyle proj} & & \downarrow \\ \underline{G/H} & \longrightarrow & \underline{G/K_1} \end{array}$$

where $K_0 \in \mathcal{F}$ *and* $\underline{G/K_1} \simeq \mathrm{Infl}\, N\backslash\underline{G/K_0}$.

Proof. Set $H' = H/(H\cap N) \subset Q$. The map $f: \underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{X}$ factors through the adjunction unit $\underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{G/H} \simeq \mathrm{Infl}_G^Q \underline{Q/H'}$. As the functor Infl_G^Q is fully faithful, we can apply Lemma 2.2.39 to the corresponding map $\underline{Q/H'} \to N\backslash\underline{X}$ and obtain a commutative diagram

$$\begin{array}{ccccccc} & & & & \underline{G/K} & \longrightarrow & \underline{X} \\ & & & & \downarrow & & \downarrow \\ \underline{G/H} & \longrightarrow & \mathrm{Infl}_G^Q \underline{Q/H} & \longrightarrow & \mathrm{Infl}_G^Q N\backslash\underline{G/K} & \longrightarrow & \mathrm{Infl}_G^Q N\backslash\underline{X} \end{array}$$

in which the square is cartesian by Lemma 2.2.38. Completing the cospan involving $\underline{G/K}$ and $\underline{G/H}$ to a pullback gives the desired pullback. ☐

3. Parametrised Poincaré duality

In this section we start developing the basic formalism of Poincaré duality within the context of categories parametrised over a topos as summarised in §2.1. This general theory will later be specialised to the equivariant setting for compact Lie groups in §4.

As a motivation for the definitions appearing in this section recall that, for a closed smooth manifold M^d, an embedding $M \hookrightarrow \mathbb{R}^N$ gives rise to a collapse map

$$c\colon S^N \to \mathrm{Th}(\nu_{M\subset\mathbb{R}^N})$$

where $\nu_{M\subset\mathbb{R}^N}$ is the normal bundle of M in $\mathbb{R}^N$. It turns out that neither the stable homotopy type of the Thom space $\mathrm{Th}(\nu_{M\subset\mathbb{R}^N})$ nor the stable homotopy class of the collapse map c depend on the choice of embedding. The collapse map defines a class $[c] \in H_N(\mathrm{Th}(\nu_{M\subset\mathbb{R}^N})) \simeq H_d(M;\mathcal{O}_\nu)$, where the isomorphism is the Thom isomorphism and $\mathcal{O}_\nu$ denotes the orientation local system of ν. Classical Poincaré duality now says that

$$[c]\cap -\colon H^k(M)\to H_{k-d}(M;\mathcal{O}_\nu) \tag{11}$$

is an isomorphism.

We start by axiomatising in §3.1 such stable collapse maps as *Spivak data* with respect to a fixed coefficient category, upon which we may demand the further condition of being twisted ambidextrous and Poincaré in §3.2, generalising the situation sketched above. We then investigate in §3.3 various operations one can perform on Spivak data, proving along the way the main results of the section (c.f. Theorems 3.3.5 and 3.3.8) about basechanging coefficient categories, which will be the key inputs to our equivariant theory. We then end the section with a discussion of degree theory which will serve as the foundation for our theory of equivariant degrees in §4.6 and our geometric applications in §5.

3.1. Spivak data

For an object $\underline{X}\in\mathcal{B}$ we denote by $X\colon \underline{X}\to\underline{*}$ the map to the final object. Recall that a $\mathcal{B}$–category $\underline{\mathcal{C}}$ admits $\underline{X}$–shaped limits (resp. colimits) if $X^*\colon \underline{\mathcal{C}}\simeq \underline{\mathrm{Fun}}(\underline{*},\underline{\mathcal{C}})\to\underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}})$ admits a right adjoint X_* (resp. left adjoint $X_!$).

Definition 3.1.1. Let $\underline{X}\in\mathcal{B}$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}$–category which admits $\underline{X}$-shaped colimits. A *$\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$* consists of

(1) an object $\xi\in\underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}})$ called the dualising sheaf;

(2) a map $c\colon \mathbb{1}_{\underline{\mathcal{C}}}\to X_!\xi$ in $\underline{\mathcal{C}}$, called the fundamental class (or collapse map).

The importance of Spivak data comes from the following construction, which allows us to compare the $\underline{X}$–shaped limit functor with a twisted $\underline{X}$–colimit functor. It is a generalisation of the map (11) given by capping with the fundamental class appearing in classical Poincaré duality.

Construction 3.1.2 (Capping map)**.** Let $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}$–category which admits $\underline{X}$–shaped limits and colimits and satisfies the $\underline{X}$–projection formula (c.f. Terminology 2.1.13). For each $\underline{\mathcal{C}}$–Spivak datum (ξ,c) on $\underline{X}$ we can construct a natural transformation

$$c\cap_\xi -\colon X_*(-)\xrightarrow{c\otimes -} X_!\xi\otimes X_*(-)\xleftarrow[\simeq]{\mathrm{PF}^X} X_!(\xi\otimes X^*X_*(-))\xrightarrow{X_!(\mathrm{id}\otimes\epsilon)} X_!(\xi\otimes -)$$

which is a morphism in $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\underline{X}},\underline{\mathcal{C}})$ where $\epsilon\colon X^*X_*\to\mathrm{id}$ denotes the adjunction counit. To avoid notational clutter, we will often omit the ξ from $c\cap_\xi -$ when the context is clear.

There is also a construction in the other direction, which produces a fundamental class for ξ from a natural transformation $X_*(-)\to X_!(\xi\otimes -)$.

Construction 3.1.3. Given a natural transformation $t\colon X_*(-) \to X_!(\xi \otimes -)$ and writing $\eta\colon \mathrm{id} \to X_*X^*$ for the adjunction unit, we obtain a collapse map as the composite

$$\mathrm{clps}_\xi(t)\colon \mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{\eta} X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{t} X_!(\xi \otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) \simeq X_!\xi.$$

Lemma 3.1.4. *There is an equivalence* $\mathrm{clps}_\xi(c \cap_\xi -) \simeq c \in \mathrm{Map}_{\underline{\mathcal{C}}}(\mathbb{1}_{\underline{\mathcal{C}}}, X_!\xi)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathbb{1}_{\underline{\mathcal{C}}} & \xrightarrow{\quad\eta\quad} & X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} & & \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c\otimes -} & & \\
X_!\xi & \xrightarrow{-\otimes\eta} & X_!\xi \otimes X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} & & \\
\simeq\uparrow{\scriptstyle \mathrm{PF}^X} & & \simeq\uparrow{\scriptstyle \mathrm{PF}^X} & & \\
X_!(\xi \otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) & \xrightarrow{X^*\eta} & X_!(\xi \otimes X^*X_*X^*\mathbb{1}_{\underline{\mathcal{C}}}) & \xrightarrow{\epsilon_{X^*}} & X_!(\xi \otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) \simeq X_!\xi.
\end{array}$$

The composite $\mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ going through the upper right corner of the rectangle is by definition equal to $\mathrm{clps}_\xi(c \cap_\xi -)$. The composite $\mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ going through the bottom left corner of the rectangle is equivalent to c using the triangle identity $\epsilon_{X^*} \circ X^*\eta \simeq \mathrm{id}$. ☐

Intertwining capping with module maps

As we shall see throughout the article, the capping maps produced from Spivak data often intertwine the left and right Beck–Chevalley transformations. Our aim now is to give the first expression of this principle in the form of Proposition 3.1.9, the other one being Lemma 3.4.6.

Setting 3.1.5 (Module pushforwards from multiplicative basechanges)**.** Suppose we have:

- symmetric monoidal $\mathcal{B}$–categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$,
- a symmetric monoidal parametrised colimit–preserving functor $U\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ as well as a $\underline{\mathcal{C}}$–linear functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ using the $\underline{\mathcal{C}}$–linear structure on $\underline{\mathcal{D}}$ coming from U,
- a map $r\colon \underline{J} \to \underline{K}$ in $\mathrm{Cat}_{\mathcal{B}}$ (to disambiguate notations, we will write $\rho := r$ when we use it in the context of the category $\underline{\mathcal{D}}$),
- $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ admit left Kan extensions along $\underline{J} \to \underline{K}$.

For (ξ, c) a $\underline{\mathcal{C}}$–Spivak datum for r, we define

$$(\zeta,\ d) := \Big(U(\xi),\ U(c)\colon \mathbb{1}_{\underline{\mathcal{D}}} \to U(r_!\xi) \simeq \rho_!\zeta\Big)$$

as the associated $\underline{\mathcal{D}}$–Spivak datum for ρ. From the data above, we also obtain symmetric monoidal functors $U\colon \underline{\mathcal{C}}^{\underline{K}} \to \underline{\mathcal{C}}^{\underline{J}}$ and $U\colon \underline{\mathcal{D}}^{\underline{K}} \to \underline{\mathcal{D}}^{\underline{J}}$, using which we may upgrade the functors $F\colon \underline{\mathcal{C}}^{\underline{K}} \to \underline{\mathcal{D}}^{\underline{K}}, F\colon \underline{\mathcal{C}}^{\underline{J}} \to \underline{\mathcal{D}}^{\underline{J}}$ to a $\underline{\mathcal{C}}^{\underline{K}}$– and a $\underline{\mathcal{C}}^{\underline{J}}$–linear one, respectively. Note that by virtue of $\underline{\mathcal{C}}$–linearity in all its guises as explained in the previous sentence, we have for any $A \in \{\underline{\mathcal{C}}, \underline{\mathcal{C}}^{\underline{J}}, \underline{\mathcal{C}}^{\underline{K}}\}$ a natural map $UA \otimes F(-) \to F(A \otimes -)$ which is an equivalence. Furthermore, note also that we clearly have equivalences $\rho^*F \simeq Fr^*$. Since U was parametrised colimit preserving, we have an equivalence $Ur_! \simeq \rho_!U$.

Example 3.1.6. The following will be the examples of the abstract Setting 3.1.5 that will be important for us:

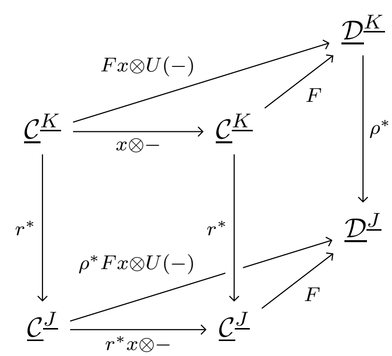
(a) In the case $F = U$, the $\underline{\mathcal{C}}$–linear structure on $F = U$ will be given by the symmetric monoidality structure $UA \otimes U(-) \xrightarrow{\simeq} U(A \otimes -)$;

(b) In the case when $\underline{\mathcal{D}} = \underline{\mathcal{C}}$, $U = \mathrm{id}_{\underline{\mathcal{C}}}$, and $F = a \otimes -$ for some fixed object $a \in \underline{\mathcal{C}}$, the $\underline{\mathcal{C}}$–linear structure on F is the tautological one given by $\mathrm{id}(A) \otimes a \otimes - \simeq a \otimes A \otimes -$ coming from the symmetric monoidal structure on $\underline{\mathcal{C}}$.

Lemma 3.1.7. *Suppose we are in the Setting 3.1.5. For all $A \in \underline{\mathcal{C}}^{\underline{J}}$, writing $B \coloneqq U(A) \in \underline{\mathcal{D}}^{\underline{J}}$, we have a commuting diagram*

$$\begin{array}{ccc}
F(-) \otimes \rho_! B & \xrightarrow[\simeq]{\text{linearity}} & F(- \otimes r_! A) \\
 & & \uparrow{\scriptstyle F(\mathrm{BC}_!)} \\
{\scriptstyle \mathrm{BC}_!}\uparrow & & Fr_!(r^*(-) \otimes A) \\
 & & \uparrow{\scriptstyle \mathrm{BC}_!} \\
\rho_!(\rho^* F(-) \otimes B) & \xrightarrow[\simeq]{\rho_!(\text{linearity})} & \rho_! F(r^*(-) \otimes A)
\end{array}$$

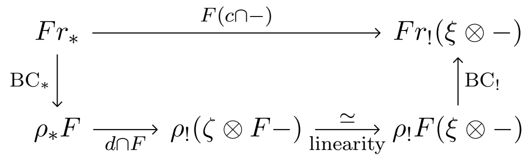
Proof. Let $x \in \underline{\mathcal{C}}^{\underline{K}}$ be an arbitrary object. Consider the diagram



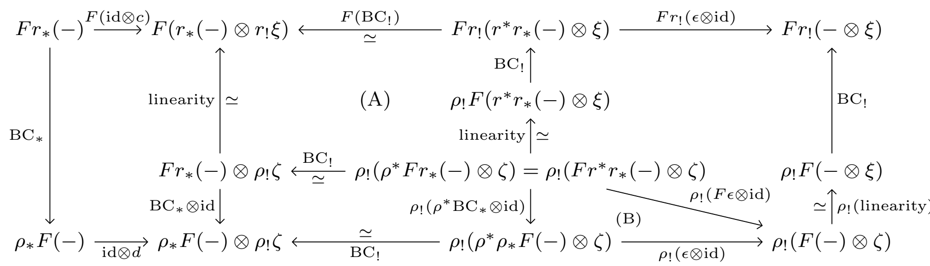
where the commuting triangles come from the $\underline{\mathcal{C}}$–linearity of the functor F with the $\underline{\mathcal{C}}$–linear structure on $\underline{\mathcal{D}}$ coming from the symmetric monoidal colimit–preserving functor $U : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$. By passing to the left adjoints $r_! \dashv r^*$ and $\rho_! \dashv \rho^*$ of the vertical functors and Beck–Chevalley pasting [CSY22, Lem. 2.2.4], we obtain the required commuting diagram. ☐

Observation 3.1.8. A funny consequence of the preceding lemma is that if we supposed that $\underline{\mathcal{C}}$ satisfied the r–projection formula and $\underline{\mathcal{D}}$ the ρ–projection formula so that the left vertical $\mathrm{BC}_!$ map and $F(\mathrm{BC}_!)$ are equivalences, then $\mathrm{BC}_! : \rho_! F(r^*(-) \otimes A) \to Fr_!(r^*(-) \otimes A)$ is automatically an equivalence.

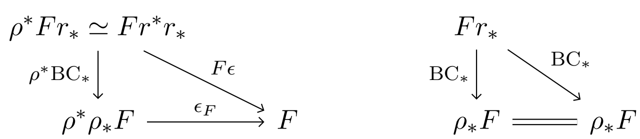
Proposition 3.1.9 (Linear intertwining principle)**.** *Suppose we are as in Setting 3.1.5 and that $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ admit right Kan extensions along $\underline{J} \to \underline{K}$. Then we have a commuting square*



Proof. Consider the following large commuting diagram



where three of the squares clearly commute, square (A) commute by Lemma 3.1.7, and triangle (B) commutes since the left triangle in the diagram



is adjoint to the right one, which clearly commutes. Now we may take the outer square of the large diagram to conclude. ☐

3.2. Twisted ambidexterity and Poincaré duality

Our aim in this subsection is to introduce the notion of Poincaré duality for Spivak data. To this end, it would be beneficial first to isolate a property that we will demand Poincaré Spivak data to satisfy, namely that of *twisted ambidexterity*, i.e. that the associated capping map is an equivalence. This notion gives the equivalence of homology with cohomology necessary for Poincaré duality. While our definition makes sense in more generality – a level of flexibility we will need for some of our applications – we show in Remark 3.2.6 that our notion of twisted ambidexterity nevertheless coincides with the one given in [Cno23] for presentably symmetric monoidal coefficient categories.

For this subsection, we consider $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}$–category which admits $\underline{X}$-shaped limits and colimits and satisfies the $\underline{X}$-projection formula. Notice that these conditions are satisfied whenever $\underline{\mathcal{C}}$ is a presentably symmetric monoidal $\mathcal{B}$–category.

Twisted ambidexterity

Definition 3.2.1. A $\underline{\mathcal{C}}$–Spivak datum (ξ, c) for $\underline{X}$ is *twisted ambidextrous* if the capping transformation $c \cap_\xi (-) \colon X_*(-) \to X_!(\xi \otimes -)$ from Construction 3.1.2 is an equivalence.

There is also the following relative version of this definition. Recall that associated to an object $\underline{Y} \in \mathcal{B}$ there is the basechange adjunction $\pi_Y^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/Y} : (\pi_Y)_*$.

Definition 3.2.2. (Twisted ambidextrous maps) Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ such that the $\mathcal{B}_{/Y}$–category $(\pi_Y)^*\underline{\mathcal{C}}$ admits f shaped limits and colimits and satisfies the f-projection formula. A *$\underline{\mathcal{C}}$-Spivak datum* for f is a $(\pi_Y)^*\underline{\mathcal{C}}$-Spivak datum for $f \in \mathcal{B}_{/Y}$. We say that such a Spivak datum exhibits f as a *$\underline{\mathcal{C}}$–twisted ambidextrous map* if it exhibits $f \in \mathcal{B}_{/Y}$ as $(\pi_Y)^*\underline{\mathcal{C}}$–twisted ambidextrous object.

We will see in Proposition 3.3.14 that f being $\underline{\mathcal{C}}$–twisted ambidextrous is closely related to the fibres of f being $\underline{\mathcal{C}}$–twisted ambidextrous, see also [Cno23, Prop. 3.13].

Next we set out to show that in the presentable case, twisted ambidextrous Spivak data are unique and demonstrate that our notion of twisted ambidexterity is equivalent to the one defined in [Cno23, Def. 3.4].

Lemma 3.2.3. *Let (ξ, c) be a twisted ambidextrous $\underline{\mathcal{C}}$–Spivak datum for $\underline{X} \in \mathcal{B}$. The adjunction $X^* \dashv X_*$ induces an adjunction $X^* \dashv X_!(\xi \otimes -)$ whose unit is given by*

$$\mathrm{id}(-) \xrightarrow{\mathrm{id}\otimes c} \mathrm{id}(-) \otimes X_!\xi \xleftarrow[\simeq]{\mathrm{BC}_!} X_!(X^*(-) \otimes \xi) = X_!(- \otimes \xi) \circ X^*(-),$$

Proof. That the equivalence $X_*(-) \simeq X_!(\xi \otimes -)$ induces an adjunction $X^* \dashv X_!(\xi \otimes -)$ is clear. For the description of the adjunction unit, observe that we have the commuting diagram

$$\begin{array}{ccccccc}
(-) & \xrightarrow{\mathrm{id}\otimes c} & (-) \otimes X_!\xi & \xleftarrow[\simeq]{\mathrm{BC}_!} & X_!(X^*(-) \otimes \xi) & & \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}} & & \downarrow{\scriptstyle X_!(X^*\eta\otimes\mathrm{id})} & \searrow{=} & \\
X_*X^*(-) & \xrightarrow[\mathrm{id}\otimes c]{} & X_*X^*(-) \otimes X_!\xi & \xleftarrow[\mathrm{BC}_!]{\simeq} & X_!(X^*X_*X^*(-) \otimes \xi) & \xrightarrow[r_!(\epsilon_{X^*}\otimes\mathrm{id})]{} & X_!(X^*(-) \otimes \xi)
\end{array}$$

where the bottom composite is the capping equivalence and the right triangle is by the triangle identity. This shows that the claimed map is compatible with the unit $\eta\colon \mathrm{id} \to X_*X^*$ under the capping equivalence $c \cap -\colon X_*(-) \xrightarrow{\simeq} X_!(\xi \otimes -)$ as required. ☐

Observation 3.2.4. Let $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$. If $X^*\colon \underline{\mathcal{C}} \to \underline{\mathcal{C}}^{\underline{X}}$ is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$, then its right adjoint must be of the form $X_!(D_{\underline{X}} \otimes -)$ for a unique D_X by Theorem 2.1.37

Proposition 3.2.5 (The presentable case)**.** *Let $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ be a presentably symmetric monoidal $\mathcal{B}$–category and $\underline{X} \in \mathcal{B}$.*

(1) If X^* is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ with right adjoint $X_!(D_{\underline{X}} \otimes -)$, then the unit map $c : \mathbb{1}_{\underline{\mathcal{C}}} \to X_!(X^*\mathbb{1}_{\underline{\mathcal{C}}} \otimes D_{\underline{X}}) = X_!D_{\underline{X}}$ forms a $\underline{\mathcal{C}}$–twisted ambidextrous Spivak datum $(D_{\underline{X}}, c)$ for $\underline{X}$.

(2) If (ξ, c) is a $\underline{\mathcal{C}}$-twisted ambidextrous Spivak datum for $\underline{X}$, then the map

$$(-) \xrightarrow{\mathrm{id}\otimes c} (-) \otimes X_!\xi \simeq X_!(X^*(-) \otimes \xi)$$

is the unit map of a $\underline{\mathcal{C}}$-linear adjunction $X^* \dashv X_!(- \otimes \xi)$.

In particular, if (ξ, c) and (ξ', c') are twisted ambidextrous Spivak data, then there is an equivalence $\xi \simeq \xi'$ so that the composition $\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c} X_!\xi \simeq X_!\xi'$ is equivalent to c'.

Proof. For point (1), suppose that X^* is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$. This means that X_* has a $\underline{\mathcal{C}}$-linear structure making the adjunction $X^* \dashv X_*$ into a $\underline{\mathcal{C}}$-linear one. By Theorem 2.1.37, there is an object $D_{\underline{X}} \in \underline{\mathcal{C}}^{\underline{X}}$ together with a $\underline{\mathcal{C}}$-linear equivalence $\phi\colon X_* \simeq X_!(-\otimes D_{\underline{X}})$. In light of the $\underline{\mathcal{C}}$–linearity of the adjunction $X^* \dashv X_*$ and $X_! \dashv X^*$, we see that the capping transformation

$$c \cap (-)\colon X_*(-) \xrightarrow{-\otimes c} X_*(-) \otimes X_!D_{\underline{X}} \xleftarrow{\simeq} X_!(X^*X_*(-) \otimes D_{\underline{X}}) \to X_!(-\otimes D_{\underline{X}}) \tag{12}$$

refines to a $\underline{\mathcal{C}}$–linear transformation because each constituent map refines canonically to a $\underline{\mathcal{C}}$–linear transformation: the first map is clear; the second map is so since the Beck–Chevalley equivalence $X_!(X^*(-) \otimes -) \to - \otimes X_!(-)$ is canonically a $\underline{\mathcal{C}}$-linear equivalence; the third map is so since it is the counit to a $\underline{\mathcal{C}}$-linear adjunction $X^*X_* \to \mathrm{id}$. We claim that $c \cap (-)$ is equivalent to the equivalence ϕ, which would prove the statement. By standard adjunction arguments, it suffices to show that the transformations $(-) \xrightarrow{\eta} X_*X^*(-) \xrightarrow{c\cap X^*(-)} X_!(X^*(-)\otimes D_X)$ and $(-) \xrightarrow{\eta} X_*X^*(-) \xrightarrow{\phi(X^*(-))} X_!(X^*(-) \otimes D_X)$ are equivalent. Employing Theorem 2.1.37, we can test this after evaluating at $\mathbb{1}_{\underline{\mathcal{C}}}$. Now the composite $\mathbb{1}_{\underline{\mathcal{C}}} \to X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{\phi} X_!D_X$ is by definition the collapse map c. The composite $\mathbb{1}_{\underline{\mathcal{C}}} \to X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c\cap X^*\mathbb{1}_{\underline{\mathcal{C}}}} X_!D_X$ is also equivalent to c by Lemma 3.1.4.

Next, for point (2), suppose that (ξ, c) is a $\underline{\mathcal{C}}$-twisted ambidextrous Spivak datum for $\underline{X}$. Then X_* is $\underline{\mathcal{B}}$-colimit preserving. First, we check the condition in [Cno23, Prop. A.5] which guarantees that the adjunction $X^* \dashv X_*$ is $\underline{\mathcal{C}}$–linear. For this, we need to show that for $a \in \underline{\mathcal{C}}$ and $E \in \underline{\mathcal{C}}^{\underline{X}}$, the Beck–Chevalley map $\mathrm{BC}_*\colon a \otimes X_*E \to X_*(X^*a \otimes E)$ is an equivalence. By the intertwining square in Proposition 3.1.9 applied to Example 3.1.6 (b), we see that BC_* is an equivalence because $\mathrm{BC}_!\colon X_!(X^*a \otimes E \otimes \xi) \to a \otimes X_!(E \otimes \xi)$ is an equivalence by presentably symmetric monoidality of $\underline{\mathcal{C}}$. As in part (1) we see that the capping equivalence (12) refines to a $\underline{\mathcal{C}}$-linear equivalence from which we obtain a $\underline{\mathcal{C}}$-linear adjunction $X^* \dashv X_!(-\otimes\xi)$ The claimed description of the adjunction unit comes from Lemma 3.2.3.

For the final statement, since both $X_!(\xi \otimes -)$ and $X_!(\xi' \otimes -)$ are $\underline{\mathcal{C}}$-linear right adjoints to X^* by (2),we see by (1) that there is an equivalence $\xi \simeq D_X \simeq \xi'$. To see the coincidence of c and c', we use Lemma 3.1.4 to obtain the two commuting triangles in

$$\begin{array}{ccccc} X_!\xi' & \xleftarrow[\simeq]{c'\cap_{\xi'} X^*\mathbb{1}} & X_*X^*\mathbb{1} & \xrightarrow[\simeq]{c\cap_{\xi} X^*\mathbb{1}} & X_!\xi \\ & {\scriptstyle c'}\nwarrow & \uparrow{\scriptstyle \eta} & \nearrow{\scriptstyle c} & \\ & & \mathbb{1} & & \end{array}$$

witnessing that $c \simeq c'$ as required. ☐

Remark 3.2.6. By combining Proposition 3.2.5 and [Cno23, Prop. 3.8], we see that $\underline{X}$ is $\underline{\mathcal{C}}$-twisted ambidextrous in the sense of Definition 3.2.1 if and only if it is so in the sense of [Cno23, Def. 3.4]. If that is the case, the twisted norm map $\widetilde{\mathrm{Nm}}_{\underline{X}}\colon X_!(-\otimes D_{\underline{X}}) \to X_*(-)$ constructed in [Cno23, Def. 3.3] is an equivalence with inverse the map $\widetilde{\mathrm{Nm}}_{\underline{X}}^{-1}(\mathbb{1}) \cap_{D_{\underline{X}}} (-)$.

Definition 3.2.7. Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. An object of $\mathcal{B}$ is called *$\underline{\mathcal{C}}$–twisted ambidextrous* if it admits a (necessarily unique) twisted ambidextrous Spivak datum with coefficients in $\underline{\mathcal{C}}$.

Notation 3.2.8. The twisted ambidextrous Spivak datum of a twisted $\underline{\mathcal{C}}$-ambidextrous object $\underline{X} \in \mathcal{B}$ will be denoted by $(D^{\underline{\mathcal{C}}}_{\underline{X}}, c)$. If $\underline{\mathcal{C}}$ is clear from the context, we will sometimes abbreviate this to $(D_{\underline{X}}, c)$.

Poincaré duality

We now come to the definition of Poincaré duality in the parametrised setting.

Definition 3.2.9. A Spivak datum (ξ, c) for $\underline{X}$ with coefficients in $\underline{\mathcal{C}}$ is *Poincaré* if it is twisted ambidextrous and ξ takes values in $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$.

Definition 3.2.10. Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. An object $\underline{X} \in \mathcal{B}$ is called *$\underline{\mathcal{C}}$–Poincaré* if it is twisted $\underline{\mathcal{C}}$–ambidextrous and the unique twisted ambidextrous Spivak datum $(D_{\underline{X}}, c)$ from Proposition 3.2.5 is Poincaré.

Remark 3.2.11. In [Qui72], Quinn defines the notion of a *normal space* to be a space together with (the unstable analog of) a Spivak datum (ξ, c), where ξ takes values in $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$. He does not require the Spivak datum to be twisted ambidextrous though.

We again have the following relative version.

Definition 3.2.12. (Poincaré duality maps) Consider a map $f \colon X \to Y$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ such that the $\mathcal{B}_{/Y}$–category $(\pi_Y)^*\underline{\mathcal{C}}$ admits f-shaped limits and colimits and satisfies the f-projection formula. We say that a $\underline{\mathcal{C}}$-Spivak datum for f exhibits f as a *$\underline{\mathcal{C}}$–Poincaré duality map* if it exhibits $f \in \mathcal{B}_{/Y}$ as a $(\pi_Y)^*\underline{\mathcal{C}}$–Poincaré duality object.

Using Costenoble-Waner duality, one can show the following standard result saying that dualisability of the dualising object implies its invertibility. We will not use it anywhere in the rest of this article but include it for completeness. In the setting of $\mathcal{B}$-categories, Costenoble-Waner duality was introduced in [Cno23, Section 3.3] and we follow the notation used there. In the nonparametrised context, the following result appears in [Lan22, Remark A.9].

Proposition 3.2.13 (Invertibility of dualising objects)**.** *Let $\underline{\mathcal{C}}$ be a presentable symmetric monoidal $\mathcal{B}$–category. Suppose that $\underline{X} \in \mathcal{B}$ is $\underline{\mathcal{C}}$-twisted ambidextrous and that $D_{\underline{X}} \in \underline{\mathcal{C}}^{\underline{X}}$ is dualisable. Then $D_{\underline{X}}$ is invertible, i.e. $\underline{X}$ is Poincaré.*

Proof. By [Cno23, Proposition 3.29], the unit $\mathbb{1}_X \in \mathcal{C}(\underline{X} \times *)$ is left Costenoble-Waner dualisable with left dual $D_{\underline{X}} \in \mathcal{C}(* \times \underline{X})$. By [Cno23, Proposition 3.30], this implies that for $F \in \mathcal{C}(\underline{X} \times Y)$ and $E \in \mathcal{C}(Y)$ we have equivalences

$$\begin{aligned}\mathrm{Hom}(E, X_! F) &\simeq \mathrm{Hom}(E, F \odot \mathbb{1}_X) \simeq \mathrm{Hom}(E \odot D_{\underline{X}}, F)\\ &= \mathrm{Hom}(X^*E \otimes D_{\underline{X}}, F) \simeq \mathrm{Hom}(X^*E, D_{\underline{X}}^\vee \otimes F)\\ &\simeq \mathrm{Hom}(E, X_*(D_{\underline{X}}^\vee \otimes F)) \simeq \mathrm{Hom}(E, X_!(D_{\underline{X}} \otimes D_{\underline{X}}^\vee \otimes F)).\end{aligned}$$

giving a $\underline{\mathcal{C}}$–linear equivalence $X_!(-) \simeq X_!(- \otimes D_{\underline{X}} \otimes D_{\underline{X}}^\vee)$. It now follows from Theorem 2.1.37 that $D_{\underline{X}} \otimes D_{\underline{X}}^\vee \simeq \mathbb{1}_X$ so that $D_{\underline{X}}$ is invertible. ☐

Example 3.2.14. The phenomenon of higher semiadditivity introduced by [HL13] provides many instances of Poincaré duality with trivial dualising sheaf.

(1) For any topos $\mathcal{B}$ and any symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$, the terminal object $\underline{*}$ has the tautological Poincaré $\underline{\mathcal{C}}$–Spivak datum $(\mathbb{1}, \mathrm{id}_{\mathbb{1}})$.

(2) If $\underline{\mathcal{C}}$ is pointed, then by [HL13, Rmk. 4.4.6], the map $\underline{\varnothing} \to \underline{X}$ in $\mathcal{B}$ is $\underline{\mathcal{C}}$–Poincaré.

(3) If $\underline{\mathcal{C}}$ is semiadditive, then by [HL13, Prop. 4.4.9], any finite fold map $\nabla\colon \coprod_{i=1}^{n} \underline{X} \to \underline{X}$ is $\underline{\mathcal{C}}$–Poincaré.

(4) More generally, a good supply of Poincaré spaces with trivial dualising sheaf comes from the theory of higher semiadditivity of [HL13; CSY22], as worked out in [Cno23].

Example 3.2.15 (Wall's Poincaré complexes)**.** Next, we recount some parts of the classical story that began from Wall's seminal paper [Wal67]. In this setting, our base topos $\mathcal{B}$ will be the category $\mathcal{S}$ of spaces. Wall defined a Poincaré complex (he used the word complex, because he worked with CW-complexes) to be a compact space X together a Spivak datum $(\xi \in \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathrm{H}\mathbb{Z}})^X, c\colon \mathrm{H}\mathbb{Z} \to X_!\xi)$ such that for each $\psi \in (\mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^{\heartsuit})^X$ the map

$$c \cap_\xi \psi\colon X_*\psi \longrightarrow X_!(\xi \otimes \psi) \tag{13}$$

is an equivalence. As X was assumed to be compact, both sides of (13) commute with all (co)limits and so this also implies that the same transformation is an equivalence for arbitrary $\psi \in \mathrm{Mod}_{\mathbb{Z}}$. On the other hand, to compute the value of the Sp–dualising sheaf D_X of a space X at a point $x\colon * \to X$, one calculates

$$D_X(x) = x^*D_X \simeq X_!x_!x^*D_X \simeq X_!(D_X \otimes x_!\mathbb{S}) \simeq X_*x_!\mathbb{S}.$$

Note that $x_!$ preserves connective objects while X_* preserves bounded below objects if it is a retract of a space admitting a finite-dimensional cell structure . So we see that if X is compact (i.e. a retract of a space having a finite cell structure), then D_X is pointwise bounded below. This implies that if $D_X \otimes \mathbb{Z} \in \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathbb{Z}})^X$, then D_X is pointwise given by shifts of spheres and in particular, $D_X \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})^X$. In conclusion, by combining the points above, a space X is a Poincaré complex in the sense of Wall if and only if it is compact and Sp–Poincaré in the sense of Definition 3.2.10. See also [Lan22, Prop. A.12] for a proof in the case of finite spaces.

Example 3.2.16 (Weak Poincaré spaces)**.** After Wall, some authors subsequently relaxed the compactness condition in the definition of Poincaré complexes. For example, in group theory it is not unusual to completely drop it. We say that a space X is weakly Poincaré if it admits a Spivak datum $(\xi \in \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathrm{H}\mathbb{Z}})^X, c\colon \mathrm{H}\mathbb{Z} \to X_!\xi)$ such that for each $\psi \in \mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^{\heartsuit}$ the map in (13) is an equivalence. As X_* preserves coconnectivity, $X_!$ preserves connectivity, and both preserve fibre sequences, we see that they restrict to functors

$$X_*, X_!(- \otimes \xi)\colon (\mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^b)^X \to \mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^b$$

where $\mathrm{Mod}_{\mathbb{Z}}^b$ denotes the category of bounded $\mathbb{Z}$–chain complexes. Being weakly Poincaré is seen to be equivalent to admitting a Poincaré Spivak datum in the sense of Definition 3.2.9 with respect to the symmetric monoidal stable category $\mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^b$.

Remark 3.2.17. Having now established the three notions and implications

$$\mathrm{Sp}\text{–Poincaré and compact} \implies \mathrm{Sp}\text{–Poincaré} \implies \text{weakly Poincaré},$$

we cannot give a conclusive answer about their precise relation. In [Bro72], Browder notes that if X is weakly Poincaré with finitely presented fundamental group, then it is even compact, and so by Example 3.2.15, also Sp–Poincaré. On the other hand, Davis shows in [Dav98] that there are weakly Poincaré spaces whose fundamental groups do not admit a finite presentation.

From here on, we will reserve the term *Poincaré space* for what we referred to as Sp-Poincaré spaces above. In particular, we slightly deviate from Wall's definition. It is useful to try and port concepts from manifold theory to the theory of Poincaré spaces. One concept that has a straighforward analog for Poincaré spaces is the dimension of a manifold.

Terminology 3.2.18 (Formal dimensions)**.** Let $X \in \mathcal{S}$ be a Poincaré space. We say that it has *formal dimension* d if for every point $x\colon * \to X$, we have $x^*D_X \simeq \Sigma^{-d}\mathbb{S} \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})$. If for every point $x\colon * \to X$, we have $x^*D_X \simeq \Sigma^{-k}\mathbb{S}$ for some $0 \leq k \leq d$, then we will say that it has *formal dimension at most* d.

Fact 3.2.19. Here are some classical facts about nonequivariant Poincaré spaces that will be relevant to our investigations later.

(1) Let $X \in \mathcal{S}^\omega$ be a connected Poincaré space of formal dimension $d = 0$. Then by [Wal67, Thm. 4.2], we have $X \simeq *$. In fact, in the aforementioned theorem, Wall even provided classifications of Poincaré spaces up to formal dimension 3.

(2) Every connected Poincaré space has formal dimension a nonnegative number. This is since if X has formal dimension d, then taking $\mathbb{F}_2$–homology, we get $H_0(X;\mathbb{F}_2) \cong H^d(X;\mathbb{F}_2)$. Thus if $d < 0$, then $H_0(X;\mathbb{F}_2) = 0$, i.e. X was the empty space.

3.3. Constructions with Spivak data

This subsection constitutes the heart of our parametrised Poincaré duality theory. We begin by studying compositions of Spivak data. Next, we shall study two types of basechange results, namely basechanging coefficient categories (Theorems 3.3.5 and 3.3.8) and basechanging the underlying topos Theorem 3.3.12. These are the main abstract results of this article and they will play a fundamental role in much of our equivariant work in §§4 and 5. We then end this subsection by proving a descent result for Poincaré duality.

Compositions

Construction 3.3.1 (Compositions of Spivak data)**.** Let $f\colon \underline{X} \to \underline{Y}$ and $g\colon \underline{Y} \to \underline{Z}$ be maps in $\mathcal{B}$ equipped with Spivak data

$$\Big(\xi_f \in \underline{\mathcal{C}}^{\underline{X}}, \quad \mathbb{1}_{\underline{\mathcal{C}}^{\underline{Y}}} \xrightarrow{c_f} f_!\xi_f\Big) \qquad \Big(\xi_g \in \underline{\mathcal{C}}^{\underline{Y}}, \quad \mathbb{1}_{\underline{\mathcal{C}}^{\underline{Z}}} \xrightarrow{c_g} g_!\xi_g\Big)$$

We may then define the *composition Spivak datum for the map* $gf\colon \underline{X} \to \underline{Z}$ as

$$\Big(\xi_{gf} \coloneqq \xi_f \otimes f^*\xi_g \in \underline{\mathcal{C}}^{\underline{X}}, \quad c_{gf}\colon \mathbb{1}_{\underline{\mathcal{C}}^{\underline{X}}} \xrightarrow{c_g} g_!\xi_g \xrightarrow{g_!(c_f\otimes \mathrm{id})} g_!(f_!\xi_f \otimes \xi_g) \xleftarrow[\simeq]{\mathrm{BC}} (gf)_!(\xi_f \otimes f^*\xi_g)\Big)$$

Lemma 3.3.2 (Capping map for compositions)**.** *The capping map for the composition Spivak datum is equivalent to the composition of the constituent capping maps. That is, in the situation of Construction 3.3.1, we have a commuting diagram*

$$\begin{array}{ccc} (gf)_*(-) & \xrightarrow{c_{gf}\cap(-)} & (gf)_!(\xi_f \otimes f^*\xi_g) \\ {\scriptstyle c_g\cap f_*(-)}\downarrow & & \downarrow{\scriptstyle \simeq\,\mathrm{BC}} \\ g_!(f_*(-)\otimes \xi_g) & \xrightarrow[g_!((c_f\cap(-))\otimes \mathrm{id})]{} & g_!(f_!(-\otimes\xi_f)\otimes\xi_g) \end{array}$$

Proof. First note that we have the commuting diagram

$$\begin{array}{ccccc} g_*f_*(-) & & & & \\ \downarrow{\scriptstyle c_g\otimes\mathrm{id}} & & & & \\ g_!\xi_g\otimes g_*f_*(-) & \xleftarrow[\simeq]{\mathrm{BC}} & g_!(\xi_g\otimes g^*g_*f_*(-)) & \xrightarrow{g_!(\mathrm{id}\otimes\epsilon)} & g_!(\xi_g\otimes f_*(-)) \\ {\scriptstyle g_!(c_f\otimes\mathrm{id})}\downarrow & & {\scriptstyle g_!(c_f\otimes\mathrm{id})}\downarrow & & \downarrow{\scriptstyle g_!(c_f\otimes\mathrm{id})} \\ g_!(\xi_g\otimes f_!\xi_f)\otimes g_*f_*(-) & \xleftarrow[\simeq]{\mathrm{BC}} & g_!(\xi_g\otimes f_!\xi_f\otimes g^*g_*f_*(-)) & \xrightarrow{g_!(\mathrm{id}\otimes\epsilon)} & g_!(\xi_g\otimes f_!\xi_f\otimes f_*(-)) \\ & & {\scriptstyle g_!(\mathrm{BC})}\uparrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\uparrow{\scriptstyle g_!(\mathrm{BC})} \\ & & g_!f_!(f^*\xi_g\otimes\xi_f\otimes f^*g^*g_*f_*(-)) & \xrightarrow{g_!f_!(\mathrm{id}\otimes\epsilon)} & g_!f_!(f^*\xi_g\otimes\xi_f\otimes f^*f_*(-)) \\ & & & \searrow{\scriptstyle g_!f_!(\mathrm{id}\otimes\epsilon)} & \downarrow{\scriptstyle g_!f_!(\mathrm{id}\otimes\epsilon)} \\ & & & & g_!f_!(f^*\xi_g\otimes\xi_f\otimes -). \end{array}$$

The required commuting square is then obtained by taking the outer diagram. ☐

Proposition 3.3.3 (Duality composition formula)**.** *Let $f\colon \underline{X}\to\underline{Y}$ and $g\colon\underline{Y}\to\underline{Z}$ be maps in $\mathcal{B}$ and $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}$–category satisfying the f– and g–projection formulas. Suppose f and g are equipped with Spivak data (ξ_f, c_f) and (ξ_g, c_g) respectively. Then under the composition Spivak datum on $gf\colon\underline{X}\to\underline{Z}$ from Construction 3.3.1 with dualising sheaf*

$$\xi_{gf} \coloneqq \xi_f\otimes f^*\xi_g,$$

we have that:

(1) if f and g are twisted ambidextrous, then so is gf,

(2) if f, g, gf are all twisted ambidextrous and g is furthermore Poincaré duality, then f is Poincaré duality if and onlf if gf is.

Proof. By Lemma 3.3.2, we have the commuting square

$$\begin{array}{ccc} (gf)_*(-) & \xrightarrow{c_{gf}\cap(-)} & (gf)_!(\xi_f \otimes f^*\xi_g) \\ {\scriptstyle c_g\cap f_*(-)}\downarrow & & \downarrow{\scriptstyle \simeq\,g_!\mathrm{BC}_!} \\ g_!(f_*(-)\otimes \xi_g) & \xrightarrow[g_!((c_f\cap(-))\otimes \mathrm{id})]{} & g_!(f_!(-\otimes\xi_f)\otimes\xi_g). \end{array}$$

Hence, if the left vertical and bottom horizontal maps are equivalences, then so is the top horizontal map. It is also clear that from the formula $\xi_{gf}=\xi_f\otimes f^*\xi_g$ that if two ξ_g is invertible (and so also $f^*\xi_g$), then ξ_{gf} is invertible if and only if ξ_f is. ☐

Change of coefficients

Construction 3.3.4 (Basechanging Spivak data)**.** Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{B}$–categories admitting $\underline{X}$-shaped (co)limits and satisfying the $\underline{X}$-projection formula. Suppose $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a symmetric monoidal functor of $\mathcal{B}$–categories which preserves $\underline{X}$–shaped colimits. We define a new $\underline{\mathcal{D}}$–Spivak datum for $\underline{X}$ as follows

$$F(\xi, c) \coloneqq \left(F\xi\colon \underline{X} \xrightarrow{\xi} \underline{\mathcal{C}} \xrightarrow{F} \underline{\mathcal{D}},\;\; Fc\colon \mathbb{1}_{\underline{\mathcal{D}}} \simeq F(\mathbb{1}_{\underline{\mathcal{C}}}) \xrightarrow{Fc} F(r_!\xi) \simeq r_!F\xi\right).$$

Theorem 3.3.5 (Poincaré basechange - presentable version)**.** *Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a functor of presentably symmetric monoidal $\mathcal{B}$–categories. Suppose that (ξ, c) is a twisted ambidextrous Spivak datum with coefficients in $\underline{\mathcal{C}}$ for the object $\underline{X} \in \mathcal{B}$. Then $F(\xi, c)$ is a twisted ambidextrous Spivak datum with coefficients in $\underline{\mathcal{D}}$ for $\underline{X}$. In particular, if $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré, then $\underline{X}$ is also $\underline{\mathcal{D}}$–Poincaré.*

Proof. Recall from Proposition 2.1.38 that $- \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}\colon \underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}}) \to \underline{\mathrm{Mod}}_{\underline{\mathcal{D}}}(\mathrm{Pr}^L_{\mathcal{B}})$ is symmetric monoidal $\mathcal{B}$-colimit preserving. Using that $\underline{\mathcal{C}}^{\underline{X}} \in \mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L(\mathcal{B}))$ is self dual (see [Cno23, Corollary 2.27]), one sees that the coassembly map $(\lim_{\underline{X}} \underline{\mathcal{C}}) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \to (\lim_{\underline{X}} \underline{\mathcal{C}}) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}$ is an equivalence (even a symmetric monoidal one). The commutative diagram

$$\begin{array}{ccccc}
\underline{\mathcal{C}}^{\underline{X}} & \xrightarrow{-\otimes_{\underline{\mathcal{C}}}\xi} & \underline{\mathcal{C}}^{\underline{X}} \otimes \underline{\mathcal{C}}^{\underline{X}} & \xrightarrow{-\otimes_{\underline{\mathcal{C}}}-} & \underline{\mathcal{C}}^{\underline{X}} \\
\downarrow F & & \downarrow F & & \downarrow F \\
\underline{\mathcal{D}}^{\underline{X}} & \xrightarrow[-\otimes_{\underline{\mathcal{D}}}F\xi]{} & \underline{\mathcal{D}}^{\underline{X}} \otimes \underline{\mathcal{D}}^{\underline{X}} & \xrightarrow[-\otimes_{\underline{\mathcal{C}}}-]{} & \underline{\mathcal{D}}^{\underline{X}}
\end{array}$$

together with the equivalence $\mathrm{Fun}_{\underline{\mathcal{D}}}(\underline{\mathcal{D}}^{\underline{X}}, \underline{\mathcal{D}}^{\underline{X}}) \simeq \mathrm{Fun}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{D}}^{\underline{X}})$ then gives us an equivalence $(- \otimes \xi) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq (- \otimes F\xi)$.

By standard arguments, the functor $- \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}\colon \mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}}) \to \mathrm{Mod}_{\underline{\mathcal{D}}}(\mathrm{Pr}^L_{\mathcal{B}})$ preserves internal adjunctions. Hence, we see that $(X_{\underline{\mathcal{C}}})_! \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}\colon \underline{\mathcal{C}}^{\underline{X}} \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq \underline{\mathcal{D}}^{\underline{X}} \to \underline{\mathcal{C}} \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq \underline{\mathcal{D}}$ is an internal left adjoint of $(X_{\underline{\mathcal{D}}})^* = (X_{\underline{\mathcal{C}}})^* \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}$ from which we obtain an equivalence $(X_{\underline{\mathcal{C}}})_! \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq (X_{\underline{\mathcal{D}}})_!$. Together with the first part, the internal right adjoint $(X_{\underline{\mathcal{C}}})_!(\xi \otimes -)$ to $(X_{\underline{\mathcal{C}}})^*$ basechanges to an internal right adjoint

$$(X_{\underline{\mathcal{C}}})_!(\xi \otimes -) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq (X_{\underline{\mathcal{D}}})_!(F\xi \otimes -)\colon \underline{\mathcal{D}}^{\underline{X}} \longrightarrow \underline{\mathcal{D}}$$

of $(X_{\underline{\mathcal{D}}})^*$. Because the internal adjunction $(X_{\underline{\mathcal{D}}})^* \dashv (X_{\underline{\mathcal{D}}})_!(F\xi \otimes -)$ on $\underline{\mathcal{D}}$ is basechanged from the internal adjunction $X^*_{\underline{\mathcal{C}}} \dashv (X_{\underline{\mathcal{C}}})_!(\xi \otimes -)$ on $\underline{\mathcal{C}}$, we see that the $\underline{\mathcal{D}}$–fundamental class, which is the unit of the former internal adjunction, is given by the composite

$$\mathbb{1}_{\underline{D}} \simeq F(\mathbb{1}_{\underline{\mathcal{C}}}) \xrightarrow{Fc} F(X_{\underline{\mathcal{C}}})_!\xi \simeq (X_{\underline{\mathcal{D}}})_!F\xi.$$

The final statement about Poincaré duality is clear since F is symmetric monoidal and so preserves invertibility. ☐

Corollary 3.3.6. *Let $\Phi\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a symmetric monoidal functor of presentably symmetric monoidal $\mathcal{B}$–categories and let $\underline{X} \in \mathcal{B}$ be a $\underline{\mathcal{C}}$–twisted ambidextrous space. Then the Beck–Chevalley transformation $\mathrm{BC}_*\colon \Phi X_*(-) \to X_*\Phi(-)$ is an equivalence.*

Proof. To disambiguate notations, we will denote by $X_!$ and X_* for the $\underline{X}$–colimit and limit for the category $\underline{\mathcal{D}}$. Now, by Proposition 3.1.9 applied to the case of Example 3.1.6 (1), we obtain a commuting square

$$\begin{array}{ccc} \Phi X_*(-) & \xrightarrow[\simeq]{\Phi(c\cap_{D_X}-)} & \Phi X_!(D_X\otimes -) \\ \big\downarrow{\scriptstyle \mathrm{BC}_*} & & {\scriptstyle \mathrm{BC}_!}\big\uparrow{\scriptstyle\simeq} \\ X_*\Phi(-) & \xrightarrow[\Phi c\,\cap_{\Phi D_X}\Phi-]{\simeq} & X_!\Phi(D_X\otimes -)\simeq X_!(\Phi D_X\otimes\Phi(-)) \end{array}$$

where the top map is an equivalence by $\underline{\mathcal{C}}$–twisted ambidexterity, the bottom an equivalence by Theorem 3.3.5, and the right vertical is an equivalence since Φ preserves parametrised colimits by hypothesis. Thus the left vertical map is an equivalence too, as desired. ☐

In a limited sense, it is possible to exploit that an object $\underline{X}\in\mathcal{B}$ admits a Poincaré Spivak datum with coefficients in $\underline{\mathcal{C}}$ to get a Spivak datum with coefficients in $\underline{\mathcal{D}}$ with interesting properties. To this end, it would be convenient to establish the following terminology:

Terminology 3.3.7. Let $\underline{X}\in\mathcal{B}, \underline{\mathcal{D}}\in\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ satisfying the $\underline{X}$–projection formula, and $\Phi\colon\underline{\mathcal{C}}\to\underline{\mathcal{D}}$ a functor of $\mathcal{B}$–categories. Suppose we have a $\underline{\mathcal{D}}$–Spivak datum (ζ,d) for $\underline{X}$. We say that the Spivak datum (ζ,d) is:

(a) Φ–*twisted ambidextrous* if the capping transformation

$$X_*\Phi(-)\xrightarrow{d\cap_\zeta\Phi-}X_!\big(\zeta\otimes\Phi(-)\big)$$

of functors $\underline{\mathcal{C}}^{\underline{X}}\to\underline{\mathcal{D}}$ is an equivalence,

(b) Φ–*Poincaré duality* if it is Φ–twisted ambidextrous and ζ takes values in invertible objects.

Theorem 3.3.8 (Poincaré basechange - general version)**.** *Suppose $\Phi\colon\underline{\mathcal{C}}\to\underline{\mathcal{D}}$ is a symmetric monoidal functor of $\mathcal{B}$–categories such that*

- *the $\mathcal{B}$–categories $\underline{\mathcal{C}},\underline{\mathcal{D}}$ admit $\underline{X}$-shaped (co)limits and satisfy the $\underline{X}$-projection formula;*
- *the $\mathcal{B}$-functor Φ preserves $\underline{X}$-shaped limits and colimits.*

If (ξ,c) is a twisted ambidextrous $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$, then $\Phi(\xi,c)$ is a twisted ambidextrous Φ–Spivak datum for $\underline{X}$. In particular, if (ξ,c) is a Poincaré duality $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$, then $\Phi(\xi,c)$ is a Poincaré duality Φ–Spivak datum for $\underline{X}$.

Proof. To distinguish from the Kan extensions $X_*, X_!$ associated to the category $\underline{\mathcal{C}}$, we will write $X_*, X_!$ for the functors $\underline{\mathcal{D}}^{\underline{X}}\to\underline{\mathcal{D}}$. By Proposition 3.1.9, we have a commuting square

$$\begin{array}{ccc} \Phi X_*(-) & \xrightarrow[\simeq]{\Phi(c\cap_\xi-)} & \Phi X_!(\xi\otimes -) \\ {\scriptstyle\simeq}\big\downarrow{\scriptstyle \mathrm{BC}_*} & & {\scriptstyle \mathrm{BC}_!}\big\uparrow{\scriptstyle\simeq} \\ X_*\Phi(-) & \xrightarrow[\Phi c\,\cap_{\Phi\xi}\Phi-]{} & X_!\Phi(\xi\otimes -)\simeq X_!(\Phi\xi\otimes\Phi(-)) \end{array}$$

from which the desired result is immediate. The final statement about Poincaré duality is an immediate consequence of the fact that Φ is symmetric monoidal and so preserves invertible objects. ☐

We will exploit this later to reprove an injectivity result of Bredon and Browder as Theorem 5.1.1.

Change of base topoi

Notation 3.3.9. To state the next construction and result, it will be convenient to adopt the following notation: let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. If $\underline{\mathcal{C}} \xrightarrow{F} \underline{\mathcal{D}} \xrightarrow{G} \underline{\mathcal{E}}$ are functors of $\mathcal{B}'$–categories, we write $f_*[G \circ F]$ to mean the composite $f_*\underline{\mathcal{C}} \xrightarrow{f_*F} f_*\underline{\mathcal{D}} \xrightarrow{f_*G} f_*\underline{\mathcal{E}}$ and similarly for f^*. Furthermore, since both f^* and f_* are product–preserving functors and so enhance to symmetric monoidal functors, we see that for an object $A \in \underline{\mathcal{D}}$, writing $f_*A \in f_*\underline{\mathcal{D}}$ under the equivalence $\mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}}\,*, f_*\underline{\mathcal{D}}) \simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}'}}(\mathrm{const}_{\mathcal{B}'}\,*, \underline{\mathcal{D}})$, the map $-\otimes A\colon \underline{\mathcal{D}} \to \underline{\mathcal{D}}$ is sent to $-\otimes f_*A\colon f_*\underline{\mathcal{D}} \to f_*\underline{\mathcal{D}}$, and similarly in the case when we apply f^*.

Construction 3.3.10 (Pushing Spivak data along geometric morphisms)**.** Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi and consider $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}'$–category which admits $f^*\underline{X}$–indexed colimits. By Lemma 2.1.18, we know that $f_*\underline{\mathcal{C}}$ admits $\underline{X}$–colimits.

Suppose we are given a $\underline{\mathcal{C}}$-Spivak datum (ξ, c) for $f^*\underline{X}$ and a $f_*\underline{\mathcal{C}}$–Spivak datum (ζ, d) for $\underline{X}$. Using the symmetric monoidal identification from Lemma 2.1.16, we obtain a $f_*\underline{\mathcal{C}}$–Spivak datum $f_*(\xi, c)$ for $\underline{X}$ and a $\underline{\mathcal{C}}$–Spivak datum $f^*(\zeta, d)$ for $f^*\underline{X}$. Observe in particular that, by construction, we have $f^*f_*(\xi, c) \simeq (\xi, c)$ and $f_*f^*(\zeta, d) \simeq (\zeta, d)$.

Here, for instance, $f_*\xi$ corresponds to ξ under the equivalence $f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}})$ and $f_*c\colon \mathbb{1}_{f_*\underline{\mathcal{C}}} \to X_! f_*\xi$ corresponds to c under the identification of adjunctions in (6). Explicitly, these new Spivak data are given by

$$f_*(\xi, c) := \Big(f_*\xi\colon \underline{X} \xrightarrow{\eta} f_*f^*\underline{X} \xrightarrow{f_*\xi} f_*\underline{\mathcal{C}},\ f_*c\colon \mathbb{1}_{f_*\underline{\mathcal{C}}} = f_*[\mathbb{1}_{\underline{\mathcal{C}}}] \to X_! f_*\xi = f_*[(f^*X)_!\xi]\Big),$$

$$f^*(\eta, d) := \Big(f^*\underline{X} \xrightarrow{f^*\zeta} f^*f_*\underline{\mathcal{C}} \xrightarrow{\epsilon} \underline{\mathcal{C}},\ f^*d\colon \mathbb{1}_{\underline{\mathcal{C}}} = f^*[\mathbb{1}_{f_*\underline{\mathcal{C}}}] \to (f^*X)_! f^*\zeta = f^*[X_!\zeta]\Big).$$

Construction 3.3.11 (Pushing Spivak data along étale morphisms)**.** Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be an étale morphism of topoi, $\underline{\mathcal{C}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$, and $\underline{X} \in \mathcal{B}$. Suppose (ξ, c) is a $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$. Then we can construct a $f^*\underline{\mathcal{C}}$–Spivak datum $f^*(\xi, c)$ for $f^*\underline{X}$ given by

$$\Big(f^*\xi\colon f^*\underline{X} \xrightarrow{f^*\xi} f^*\underline{\mathcal{C}},\ f^*c\colon \mathbb{1}_{f^*\underline{\mathcal{C}}} \simeq f^*[\mathbb{1}_{\underline{\mathcal{C}}}] \xrightarrow{f^*[c]} (f^*X)_! f^*\xi \simeq f^*[X_!\xi]\Big)$$

where in the last equivalence, we have used $f^*[X_!] \simeq (f^*X)_!$ from Lemma 2.1.19.

For part (e) of the next result, see Terminology 3.3.7.

Theorem 3.3.12 (Omnibus geometric basechange of Spivak data)**.** *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi, $\underline{X} \in \mathcal{B}$, $\underline{\mathcal{D}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'})$ satisfying the $f^*\underline{X}$–projection formula, and $\underline{\mathcal{E}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ satisfying the $\underline{X}$–projection formula. Let (ξ, c) be a $\underline{\mathcal{D}}$–Spivak datum for $f^*\underline{X}$ and (ζ, d) a $f_*\underline{\mathcal{D}}$–Spivak datum for $\underline{X}$. Then:*

(a) There is a commuting square of capping maps

$$\begin{array}{ccc} X_*(-) & \xrightarrow{f_*c\cap_{f_*\xi}-} & X_!(f_*\xi\otimes -) \\ \simeq\downarrow & & \downarrow\simeq \\ f_*\big[(f^*X)_*(-)\big] & \xrightarrow{f_*[c\cap_\xi -]} & f_*\big[(f^*X)_!(\xi\otimes -)\big] \end{array}$$

of functors $\underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \simeq f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \to f_*\underline{\mathcal{D}}$

(b) Suppose that $f_*\colon \mathrm{Cat}_{\mathcal{B}'} \to \mathrm{Cat}_{\mathcal{B}}$ *is fully faithful (resp. that* $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ *is étale). Then there is a commuting square*

$$\begin{array}{ccc} f^*\big[X_*(-)\big] & \xrightarrow{f^*[d\cap_\zeta -]} & f^*\big[X_!(\zeta\otimes -)\big] \\ \simeq\downarrow & & \downarrow\simeq \\ (f^*X)_*(-) & \xrightarrow{f^*d\cap_{f^*\zeta}-} & (f^*X)_!(f^*\zeta\otimes -) \end{array}$$

of functors $\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \simeq f^*\underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \to \underline{\mathcal{D}}$ *(resp.* $\underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{E}}) \to f^*\underline{\mathcal{E}}$*),*

(c) If (ξ, c) *is a twisted ambidextrous (resp. Poincaré)* $\underline{\mathcal{D}}$*–Spivak datum for* $f^*\underline{X}$*, then* $f_*(\xi, c)$ *is a twisted ambidextrous (resp. Poincaré)* $f_*\underline{\mathcal{D}}$*–Spivak datum for* $\underline{X}$*.*

(d) Suppose either that f_* *is fully faithful or that* $\underline{\mathcal{D}}$ *is presentably symmetric monoidal. If* (ζ, d) *is a twisted ambidextrous (resp. Poincaré)* $f_*\underline{\mathcal{D}}$*–Spivak datum for* $\underline{X}$*, then* $f^*(\zeta, d)$ *is a twisted ambidextrous (resp. Poincaré)* $\underline{\mathcal{D}}$*–Spivak datum for* $f^*\underline{X}$*.*

(e) More generally: suppose f_* *is fully faithful. Let* $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}}$ *and* $\Phi\colon \underline{\mathcal{C}} \to f_*\underline{\mathcal{D}}$ *be a functor between* $\mathcal{B}$*–categories. Then* (ζ, d) *is* Φ*–twisted ambidextrous (resp. –Poincaré) for* $\underline{X}$ *if and only if* $f^*(\zeta, d)$ *is* $(f^*\Phi\colon f^*\mathcal{C} \to \underline{\mathcal{D}})$*–twisted ambidextrous (resp. –Poincaré) for* $f^*\underline{X}$*,*

(f) Suppose the geometric morphism $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ *is étale. If* (ζ, d) *is a twisted ambidextrous (resp. Poincaré)* $\underline{\mathcal{E}}$*–Spivak datum for* $\underline{X}$*, then* $f^*(\zeta, d)$ *is a twisted ambidextrous (resp. Poincaré)* $f^*\underline{\mathcal{E}}$*–Spivak datum for* $f^*\underline{X}$*.*

Proof. First note by Proposition 2.1.24 (1, 3) that $f_*\underline{\mathcal{D}}$ satisfies the $\underline{X}$–projection formula and $f^*\underline{\mathcal{E}}$ satisfies the $f^*\underline{X}$–projection formula, and so the squares in (a) and (b) make sense. Now to prove part (a), we have the commutative diagram

$$\begin{array}{ccc} X_*(-) & \xrightarrow{\simeq} & f_*\big[(f^*X)_*(-)\big] \\ {\scriptstyle \mathrm{id}\otimes f_*c}\downarrow & & \downarrow{\scriptstyle f_*[\mathrm{id}\otimes c]} \\ X_*(-)\otimes X_!f_*\xi & \xrightarrow{\simeq} & f_*\big[(f^*X)_*(-)\otimes (f^*X)_!\xi\big] \\ {\scriptstyle \mathrm{BC}_!}\uparrow\simeq & & \simeq\uparrow{\scriptstyle f_*\mathrm{BC}_!} \\ X_!(X^*X_*(-)\otimes f_*\xi) & \xrightarrow{\simeq} & f_*\big[(f^*X)_!((f^*X)^*(f^*X)_*(-)\otimes \xi)\big] \\ {\scriptstyle \epsilon_*}\downarrow & & \downarrow{\scriptstyle f_*\epsilon_*} \\ X_!(-\otimes f_*\xi) & \xrightarrow{\simeq} & f_*\big[(f^*X)_!(-\otimes \xi)\big] \end{array}$$

The horizontal arrows come from the identification in Lemma 2.1.18; the top square commutes by symmetric monoidality of the identification; the middle and bottom squares commute as the respective adjunction (co-)units are identified by (6). The required square is now obtained by extracting the outer square of the diagram above.

The proof for (b) in the case that f_* is fully faithful is done similarly as for (a), but using now the commuting squares of adjunctions obtained by applying f^* to Lemma 2.1.18 (f^* preserves adjunctions by [MW24, Cor. 3.1.9]) and that $f^* f_* \simeq \mathrm{id}$. The case of étale morphisms is also done similarly, using instead the squares of adjunctions from Lemma 2.1.19.

Next, we prove part (c). If (ξ, c) is twisted ambidextrous, then by Lemma 2.1.20, the bottom map in the square from (a) is an equivalence, and so the top map is an equivalence too, i.e. $f_*(\xi, c)$ is twisted ambidextrous. The statements about being Poincaré is a straightforward consequence of the twisted ambidexterity statements we just proved and the characterisation of factoring through invertible objects in Corollary 2.1.23 (1).

For the proof of (d), suppose now that f_* is fully faithful and that (ζ, d) is twisted ambidextrous. Then since $f^* f_* \simeq \mathrm{id}$, the top map in the square from (b) is an equivalence, and so the bottom map is an equivalence too, i.e. $f^*(\zeta, d)$ is twisted ambidextrous. Poincaré duality is then handled similarly as in (c).

Next, assume that $\underline{\mathcal{D}}$ is presentably symmetric monoidal. We show that the capping transformation $f^*d \cap_{f^*\zeta} (-) \colon \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \to \underline{\mathcal{D}}^{\Delta^1}$ is a natural equivalence in unparametrised categories when evaluated at every $W \in \mathcal{B}'$. Firstly, note that the $\underline{\mathcal{D}}$–Spivak datum for $f^*\underline{X}$ at level W is obtained via the symmetric monoidal biadjoint $\mathcal{B}'$–functor $W^* \colon \underline{\mathcal{D}} \to \underline{\mathcal{D}}^{\underline{W}}$, and so the transformation evaluated at $W \in \mathcal{B}'$ is given by applying global sections $\Gamma_{\mathcal{B}'}$ to

$$W^*(f^*d) \cap_{W^* \circ f^*\zeta} (-) \colon \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}^{\underline{W}}) \longrightarrow (\underline{\mathcal{D}}^{\underline{W}})^{\Delta^1}. \tag{14}$$

Next, applying Theorem 3.3.5 along $f_*[W^*] \colon f_*\underline{\mathcal{D}} \to f_*(\underline{\mathcal{D}}^{\underline{W}})$ shows that the $f_*(\underline{\mathcal{D}}^{\underline{W}})$–Spivak datum $f_*[W^*](\zeta, d)$ is twisted ambidextrous, i.e.

$$f_*[W^*](d) \cap_{f_*[W^*] \circ \zeta} (-) \colon \underline{\mathrm{Fun}}(\underline{X}, f_*(\underline{\mathcal{D}}^{\underline{W}})) \to (f_*(\underline{\mathcal{D}}^{\underline{W}}))^{\Delta^1}$$

is a natural equivalence. But then by the square in part (a), this capping transformation is equivalent to $f_*[W^*(f^*d) \cap_{W^* \circ f^*\zeta} (-)] \colon f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}^{\underline{W}}) \to f_*((\underline{\mathcal{D}}^{\underline{W}})^{\Delta^1})$, where we have also used that $f_* f^*(\zeta, d) \simeq (\zeta, d)$ from Construction 3.3.10. Thus, by using that $\Gamma_{\mathcal{B}} f_* \simeq \Gamma_{\mathcal{B}'}$ from Example 2.1.9, we may apply $\Gamma_{\mathcal{B}}$ to $f_*[W^*(f^*d) \cap_{W^* \circ f^*\zeta} (-)]$ to get that applying $\Gamma_{\mathcal{B}'}$ to (14) yields a natural equivalence, as desired. And as usual, Poincaré duality is handled by Corollary 2.1.23 (1).

Now, the proof of parts (c, d) clearly goes through straightforwardly to yield a proof of (e). Finally, the proof of (f) in the twisted ambidexterity case is done similarly as in the proof of (c) using the square from (b), and the Poincaré duality case is handled by Corollary 2.1.23 (2). ☐

Descent

For the next result, we briefly recall the notion of effective epimorphisms in a topos $\mathcal{B}$. Given a morphism $f \colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$, its *Čech nerve* is the simplicial object

$$\check{\mathrm{C}}(f) \colon \Delta^{\mathrm{op}} \to \mathcal{B}, \quad [n] \mapsto \underline{X} \times_{\underline{Y}} \underline{X} \times_{\underline{Y}} \cdots \times_{\underline{Y}} \underline{X} \quad (n+1 \text{ factors}).$$

Now f is called an *effective epimorphism* if the canonical map $\operatorname{colim}_{\Delta^{\mathrm{op}}} \check{\mathrm{C}}(f) \to Y$ is an equivalence.

Example 3.3.13. A map in the topos $\mathcal{S}$ of spaces is an effective epimorphism if and only if it is a π_0-surjection (see e.g. [Lur09, Corollary 7.2.1.15]). Applying this criterion pointwise, one sees that a map $f\colon X \to Y$ in a presheaf topos $\mathrm{PSh}(T)$ over some category T is an effective epimorphism if and only if for all $t \in T$ the map $f(t)\colon X(t) \to Y(t)$ is a π_0-surjection. For example, a map of G–spaces $f\colon \underline{X} \to \underline{Y}$ is an effective epimorphism if and only if for each closed subgroup $H \leq G$ the map $f^H\colon X^H \to Y^H$ is a surjection on path components.

Proposition 3.3.14 (Poincaré duality and descent)**.** *Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. Consider a pullback square*

$$\begin{array}{ccc} P & \xrightarrow{f'} & Z \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$

in $\mathcal{B}$. If f is $\underline{\mathcal{C}}$-twisted ambidextrous, then f' is $\underline{\mathcal{C}}$-twisted ambidextrous. Furthermore, there is an equivalence $(g')^*D_f \simeq D_{f'}$ where $D_f \in \mathcal{C}(X)$ denotes the dualising object. In particular, if f is a $\underline{\mathcal{C}}$–Poincaré duality map, then f' is a $\underline{\mathcal{C}}$–Poincaré duality map.

The converse to both statements is true if g is an effective epimorphism.

Proof. Basechange along g defines an étale morphism of topoi $g^*\colon \mathcal{B}_{/Y} \to \mathcal{B}_{/Z}\colon g_*$ where g^* is given by pullback along g. Now suppose that f is $\underline{\mathcal{C}}$-twisted ambidextrous. This means that $f \in \mathcal{B}_{/Y}$ is $(\pi_Y)^*\underline{\mathcal{C}}$-twisted ambidextrous. Applying Theorem 3.3.12 shows that $g^*f = f' \in \mathcal{B}_{/Z}$ is $g^*\pi_Y^*\underline{\mathcal{C}} = \pi_Z^*\underline{\mathcal{C}}$-twisted ambidextrous with Spivak datum $(g^*\xi, g^*c)$.

If f is a $\underline{\mathcal{C}}$–Poincaré duality map, then D_f is invertible. By symmetric monoidality of the restriction map $(g')^*\colon \mathcal{C}(X) \to \mathcal{C}(P)$ we obtain that $D_{f'} = (g')^*D_f$ is invertible.

Now suppose that g is an effective epimorphism and that f' is $\underline{\mathcal{C}}$-twisted ambidextrous. It is shown in [Cno23, Proposition 3.13 (5)] that f is $\underline{\mathcal{C}}$-twisted ambidextrous. As $g'\colon P \to X$ is an effective epimorphism and the map $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathbb{\Delta}^{\mathrm{op}}$ is colimit cofinal, the map $\operatorname{colim}_{\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{inj}}} \check{\mathrm{C}}(g') \to X$ is an equivalence from which we obtain the symmetric monoidal equivalence $\mathcal{C}(X) \xrightarrow{\simeq} \lim_{\mathbb{\Delta}_{\mathrm{inj}}} \mathcal{C}(\check{\mathrm{C}}(g'))$. Next, suppose furthermore that f' is a Poincaré duality map. As invertibility in limits can be checked pointwise, we have to show that each restriction of D_f to $\mathcal{C}(\check{\mathrm{C}}_n(g'))$ is invertible. Note that a restriction map $\mathcal{C}(X) \to \mathcal{C}(\check{\mathrm{C}}_n(g'))$ factors into the symmetric monoidal restriction maps $\mathcal{C}(X) \to \mathcal{C}(P) \to \mathcal{C}(\check{\mathrm{C}}_n(g'))$ and the first part shows the restriction $D_{f'} = (g')^*D_f$ to P is invertible. $\square$

Corollary 3.3.15 (Finite products)**.** *Let $\underline{X}, \underline{Y} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. If $\underline{X}$ and $\underline{Y}$ are $\underline{\mathcal{C}}$–Poincaré, then $\underline{X} \times \underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré and there is an equivalence*

$$D_{X \times Y} \simeq \mathrm{pr}_X^* D_X \otimes \mathrm{pr}_Y^* D_Y$$

where $\mathrm{pr}_X\colon \underline{X} \times \underline{Y} \to \underline{X}$ and $\mathrm{pr}_Y\colon \underline{X} \times \underline{Y} \to \underline{Y}$ denote the projections.

Proof. As $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré, it follows from Proposition 3.3.14 that the map $\mathrm{pr}_Y\colon \underline{X}\times\underline{Y}\to\underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré map. Since $\underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré, Proposition 3.3.3 implies that the composite $\underline{X}\times\underline{Y}\to\underline{Y}\to *$ is a $\underline{\mathcal{C}}$–Poincaré map showing that $\underline{X}\times\underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré. Propositions 3.3.3 and 3.3.14 then give the identifications $D_{X\times Y}\simeq \mathrm{pr}_Y^* D_Y\otimes D_{\mathrm{pr}_Y}\simeq \mathrm{pr}_Y^* D_Y\otimes \mathrm{pr}_X^* D_X$. ☐

Lemma 3.3.16. *Let $\underline{\mathcal{C}}\in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ be a symmetric monoidal $\mathcal{B}$–category and $(f_i\colon \underline{X}_i\to\underline{Y}_i)_{i\in I}$ be a collection of maps in $\mathcal{B}$. Then the map $f=\coprod_i f_i\colon \coprod_i \underline{X}_i\to\coprod_i\underline{Y}_i$ is $\underline{\mathcal{C}}$-twisted ambidextrous (or $\underline{\mathcal{C}}$–Poincaré duality) if and only if for all $i\in I$ the map f_i is $\underline{\mathcal{C}}$-twisted ambidextrous (or $\underline{\mathcal{C}}$–Poincaré duality). If this is so, then under the identification $\mathcal{C}(\coprod_i X_i)\simeq\prod_i\mathcal{C}(X_i)$, we have an equivalence $D_f\simeq (D_{f_i})_i$.*

Proof. The "only if"-direction follows from Proposition 3.3.14. The "if"-direction in the twisted ambidexterity case is [Cno23, Proposition 3.13(3)]. If in addition all f_i are Poincaré duality maps, then $\coprod_i f_i$ is a Poincaré duality map as $D_f=(D_{f_i})_i$ under the equivalence $\mathcal{C}(\coprod_i \underline{X}_i)=\prod_i\mathcal{C}(\underline{X}_i)$. ☐

Corollary 3.3.17. *Let $\underline{\mathcal{C}}\in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ be a symmetric monoidal $\mathcal{B}$–category which is semiadditive and $\{\underline{X}_i\}_i$ a finite collection of objects in $\mathcal{B}$. Then $\coprod_i\underline{X}_i$ is $\underline{\mathcal{C}}$–Poincaré duality if and only if each $\underline{X}_i$ is $\underline{\mathcal{C}}$–Poincaré duality. In this case, under the identification $\mathcal{C}(\coprod_i X_i)\simeq\prod_i\mathcal{C}(X_i)$, we have an equivalence $D_{\coprod_i \underline{X}_i}\simeq (D_{\underline{X}_i})_i$.*

Proof. Suppose $\coprod_i\underline{X}_i$ is $\underline{\mathcal{C}}$–Poincaré duality. By Example 3.2.14 (1) and Lemma 3.3.16, we see that the inclusion $\underline{X}_j\hookrightarrow\coprod_i\underline{X}_i$ is Poincaré duality for each j. An immediate application of Proposition 3.3.3 using the triple of maps $\underline{X}_j\hookrightarrow\coprod_i\underline{X}_i, \coprod_i\underline{X}_i\to\underline{*}$, and $\underline{X}_j\to\underline{*}$ then shows that $\underline{X}_j$ is also Poincaré duality. Next, suppose each $\underline{X}_j$ is Poincaré duality. By semiadditivity and Example 3.2.14 (2), the map $\nabla\colon\coprod_i\underline{*}\to\underline{*}$ is $\underline{\mathcal{C}}$–Poincaré duality with dualising sheaf $(\mathbb{1}_{\underline{\mathcal{C}}})_i\in\prod_i\mathcal{C}(*)$. Thus, a simple combination of Proposition 3.3.3 and Lemma 3.3.16 using the triple of maps $\sqcup r_i\colon\coprod_i\underline{X}_i\to\coprod_i\underline{*}$, ∇, and $\nabla\circ(\sqcup_i r_i)$ yields the desired conclusion. ☐

3.4. Degree theory

In this subsection we introduce the notion of the degree of a map between Poincaré spaces (or more generally objects with Spivak data). We use this to construct Umkehr squares which will important for our geometric applications. In §4.6 we will specialise this to the case of G–spaces for a finite group G which generalises classical constructions of the equivariant degree.

As a motivation for the definition, recall that given a map $f\colon X\to Y$ between closed connected manifolds of the same dimension, one can assign to it a degree if f is compatible with the orientation behaviour of X and Y: given an identification $\mathcal{O}_X\simeq f^*\mathcal{O}_Y$ of orientation local systems, the degree is given by the image of $[X]$ under $f_*\colon H_n(X;\mathcal{O}_X)\to H_n(Y;\mathcal{O}_Y)\simeq\mathbb{Z}$. In our setting, we will replace orientation local systems and the fundamental classes $[X]$ above with the dualising sheaves and fundamental classes from Definition 3.1.1.

The definition

Construction 3.4.1 ((Co)homological functoriality)**.** Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped limits and colimits. We obtain transformations

$$\mathrm{BC}^f_!\colon X_! f^* \longrightarrow Y_! \qquad\qquad \mathrm{BC}^f_*\colon Y_* \longrightarrow X_* f^*$$

of functors $\underline{\mathcal{C}}^{\underline{Y}} \to \underline{\mathcal{C}}$ coming from the left and right Beck–Chevalley transformations, respectively, associated to the commuting triangle

$$\begin{array}{ccc} \underline{\mathcal{C}}^{\underline{X}} & \xleftarrow{f^*} & \underline{\mathcal{C}}^{\underline{Y}} \\ {\scriptstyle X^*}\uparrow & \nearrow{\scriptstyle Y^*} & \\ \underline{\mathcal{C}} & & \end{array}$$

We call $\mathrm{BC}^f_!$ the *homological functoriality map* and BC^f_* the *cohomological functoriality map.*

Definition 3.4.2 (Degree of a map)**.** Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped limits and colimits and satisfies the $\underline{X}$- and $\underline{Y}$-projection formula. Suppose we are given Spivak data $(\xi_{\underline{X}}, c_{\underline{X}})$ for $\underline{X}$ and $(\xi_{\underline{Y}}, c_{\underline{Y}})$ for $\underline{Y}$. A *$\underline{\mathcal{C}}$-degree datum* for f is an equivalence $\alpha\colon \xi_{\underline{X}} \xrightarrow{\simeq} f^*\xi_{\underline{Y}}$ in $\mathrm{Fun}_{\mathcal{B}}(\underline{X}, \underline{\mathcal{C}})$. We define the *$\underline{\mathcal{C}}$-degree* of (f,α) as the point $\deg_{\underline{\mathcal{C}}}(f,\alpha) \in \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}})$ given by the composite

$$\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c_{\underline{X}}} X_!\xi_{\underline{X}} \xrightarrow{\alpha} X_! f^*\xi_{\underline{Y}} \xrightarrow{\mathrm{BC}^f_!} Y_!\xi_{\underline{Y}}.$$

We say that an equivalence $c_{\underline{Y}} \simeq \deg_{\underline{\mathcal{C}}}(f,\alpha)$ exhibits f as a map of *$\underline{\mathcal{C}}$-degree one.*

Remark 3.4.3. Note that an equivalence $c_{\underline{Y}} \simeq \deg_{\underline{\mathcal{C}}}(f,\alpha)$ is the same datum as a homotopy rendering the following diagram commutative

$$\begin{array}{ccccc} \mathbb{1}_{\underline{\mathcal{C}}} & \xrightarrow{c_{\underline{X}}} & X_!\xi_{\underline{X}} & \underset{\simeq}{\xrightarrow{X_!\alpha}} & X_! f^*\xi_{\underline{Y}} \\ & \searrow^{c_{\underline{Y}}} & & & \downarrow{\scriptstyle \mathrm{BC}^f_!} \\ & & & & Y_!\xi_{\underline{Y}}. \end{array}$$

Construction 3.4.4. If the Spivak datum $(\xi_{\underline{Y}}, c_{\underline{Y}})$ is $\underline{\mathcal{C}}$-twisted ambidextrous, then the equivalence $c_{\underline{Y}} \cap_{\xi_{\underline{Y}}} \mathbb{1}_{\underline{\mathcal{C}}}\colon Y_*Y^*\mathbb{1}_{\underline{\mathcal{C}}} \simeq Y_!\xi_{\underline{Y}}$ endows $Y_!\xi_{\underline{Y}}$ with the structure of a commutative algebra in $\underline{\mathcal{C}}$. This gives $\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}})$ the structure of a commutative monoid in $\mathcal{S}$ with unit $c_{\underline{Y}}$. This explains the name "degree one" in the previous definition.

Example 3.4.5. Here are some well-known sources of degree data in the case $\mathcal{B} = \mathcal{S}$ with respect to a presentably symmetric monoidal coefficients $\mathcal{C}$. Let $f\colon X \to Y$ be a map of connected Poincaré spaces of the same formal dimension d (c.f. Terminology 3.2.18). We consider situations when a degree datum exists for the map f with the Poincaré Spivak data (D_X, c_X) and (D_Y, c_Y) for X resp. Y.

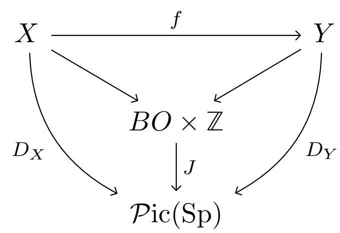
(1) For $\mathcal{C} = \mathrm{Mod}_{\mathbb{F}_2}$, a degree datum exists uniquely since $\mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathbb{F}_2}) \simeq \mathbb{Z} \times B\mathrm{Aut}(\mathbb{F}_2) \simeq \mathbb{Z} \times *$ has contractible components.

(2) For $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$, writing $w_1(-)$ for the first Stiefel–Whitney class of a space, a degree datum exists if and only if $f^* w_1(Y) = w_1(X) \in H^1(X; \mathbb{F}_2)$. On homotopy groups, the composite $X_! D_X \simeq X_! f^* D_Y \to Y_! D_Y$ then identifies with

$$f_* \colon H_{d+*}(X; \mathcal{O}_X) \to H_{d+*}(Y; \mathcal{O}_Y)$$

where $\mathcal{O}_X$ and $\mathcal{O}_Y$ denote the orientation local systems. The degree defined above is then given by $f_*[X] \in H_d(Y; \mathcal{O}_Y) \simeq \pi_0(\mathrm{Map}(\mathbb{1}_{\mathrm{Mod}_{\mathbb{Z}}}, Y_! D_Y))$ and agrees with the classical definition of the degree.

(3) In surgery theory, one is often provided with a *normal map*, i.e. a commuting diagram



and of course restricting to the outer commutative triangle gives rise to a degree datum.

Homological Umkehr squares

Classically, given a map $f \colon M \to N$ of degree one between closed oriented manifolds, one can construct a "homological Umkehr map" $f^! \colon H_*(N) \to H_*(M)$ going the "wrong way" using Poincaré duality. The following result is a generalisation of this.

Lemma 3.4.6 (Umkehr square). *Consider a map $f \colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped limits and colimits and satisfies the $\underline{X}$- and $\underline{Y}$-projection formula. Suppose that there is a degree datum α for f which is of degree one. Then the diagram*

$$\begin{array}{ccccc}
Y_*(-) & & \xrightarrow{\mathrm{BC}^f_*} & & X_* f^*(-) \\
\Big\downarrow{\scriptstyle c_{\underline{Y}} \cap_{\xi_{\underline{Y}}} (-)} & & & & \Big\downarrow{\scriptstyle c_{\underline{X}} \cap_{\xi_{\underline{X}}} f^*(-)} \\
Y_!(\xi_{\underline{Y}} \otimes (-)) & \xleftarrow[\mathrm{BC}^f_!]{} & X_!(f^*\xi_{\underline{Y}} \otimes f^*(-)) & \xleftarrow[\alpha]{} & X_!(\xi_{\underline{X}} \otimes f^*(-))
\end{array}$$

commutes.

Proof. Consider the diagram

$$
\begin{array}{ccccc}
 & {\scriptstyle c_{\underline{Y}}\otimes -}\swarrow & Y_*(-) & \xrightarrow{\mathrm{BC}_*^f} & X_*f^*(-) \\
 & & \downarrow{\scriptstyle \alpha c_{\underline{Y}}\otimes -} & & \downarrow{\scriptstyle \alpha c_{\underline{Y}}\otimes -} \\
Y_!\xi_{\underline{Y}}\otimes Y_*(-) & \xleftarrow[\mathrm{BC}_!^f\otimes\mathrm{id}]{} & X_!f^*\xi_{\underline{Y}}\otimes Y_*(-) & \xrightarrow{\mathrm{id}\otimes\mathrm{BC}_*^f} & X_!f^*\xi_{\underline{Y}}\otimes X_*f^*(-) \\
{\scriptstyle \mathrm{PF}_!^{\underline{Y}}}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \mathrm{PF}_!^{\underline{X}}}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \mathrm{PF}_!^{\underline{X}}}\uparrow{\scriptstyle \simeq} \\
Y_!(\xi_{\underline{Y}}\otimes Y^*Y_*(-)) & \xleftarrow[\mathrm{BC}_!^f]{} & X_!(f^*\xi_{\underline{Y}}\otimes X^*Y_*(-)) & \xrightarrow{\mathrm{BC}_*^f} & X_!(f^*\xi_{\underline{Y}}\otimes X^*X_*f^*(-)) \\
\downarrow{\scriptstyle \epsilon_{\underline{Y}}} & & \downarrow{\scriptstyle \epsilon_{\underline{Y}}} & \swarrow{\scriptstyle \epsilon_{\underline{X}}} & \\
Y_!(\xi_{\underline{Y}}\otimes(-)) & \xleftarrow[\mathrm{BC}_!^f]{} & X_!(f^*\xi_{\underline{Y}}\otimes f^*(-)) & &
\end{array}
$$

The degree one datum makes the top left triangle commute. The bottom right triangle commutes using the definition of the restriction map and the triangle identities. The top right and bottom left squares commute by naturality of BC. The middle two squares commute by naturality of By definition, the composite of the blue arrows is given by $c_{\underline{Y}} \cap_{\xi_{\underline{Y}}} (-)$ and the composite of the red arrows is given by $\alpha c_{\underline{X}} \cap_{f^*\xi_{\underline{Y}}} (-)$. To finish, observe that the diagram

$$
\begin{array}{ccc}
X_*(-) & \searrow{\scriptstyle c_{\underline{X}}\cap_{\xi_{\underline{X}}}(-)} & \\
{\scriptstyle \alpha c_{\underline{X}}\cap_{f^*\xi_{\underline{Y}}}(-)}\downarrow & & \\
X_!(f^*\xi_{\underline{Y}}\otimes(-)) & \xleftarrow[\alpha]{} & X_!(\xi_{\underline{X}}\otimes(-))
\end{array}
$$

commutes. ☐

Basechange

Construction 3.4.7. Suppose that $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a symmetric monoidal colimit preserving functor of presentably symmetric monoidal $\mathcal{B}$–categories. Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and consider a $\underline{\mathcal{C}}$-degree datum α for f. We obtain a $\underline{\mathcal{D}}$-degree datum $F(\alpha)\colon F\xi_{\underline{X}} \xrightarrow{\simeq} Ff^*\xi_{\underline{Y}} \simeq f^*F\xi_Y$ for f and the Spivak data $F(\xi_{\underline{X}}, c_{\underline{X}})$ and $F(\xi_{\underline{Y}}, c_{\underline{Y}})$ from Construction 3.3.4.

Lemma 3.4.8. *In the situation of Construction 3.4.7, the image of* $\deg_{\underline{\mathcal{C}}}(f,\alpha)$ *under the map*

$$\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}}) \xrightarrow{F} \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{D}}}, Y_!F\xi_{\underline{Y}}) \tag{15}$$

is equivalent to $\deg_{\underline{\mathcal{D}}}(f, F(\alpha))$. *Furthermore, if* $(\xi_{\underline{Y}}, c_{\underline{Y}})$ *is* $\underline{\mathcal{C}}$*-twisted ambidextrous, then the map (15) refines to a map of commutative monoids for the commutative monoid structures from Construction 3.4.4 and we have a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}}) & \xrightarrow{F} & \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{D}}}, Y_!F\xi_{\underline{Y}}) \\
{\scriptstyle\simeq}\uparrow{\scriptstyle c_Y} & & {\scriptstyle\simeq}\uparrow{\scriptstyle F(c_Y)} \\
\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_*Y^*\mathbb{1}_{\underline{\mathcal{C}}}) & \xrightarrow{F} & \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{D}}}, Y_*Y^*\mathbb{1}_{\underline{\mathcal{D}}}).
\end{array}
$$

Proof. Note that as F is colimit preserving, it commutes with the homological functoriality constructed in Construction 3.4.1. Thus it sends

$$\deg_{\underline{\mathcal{C}}}(f,\alpha)\colon \mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c_{\underline{X}}} X_!\xi_{\underline{X}} \underset{\simeq}{\xrightarrow{\alpha}} X_!f^*\xi_{\underline{Y}} \xrightarrow{\mathrm{BC}_!} Y_!\xi_{\underline{Y}}$$

$$\text{to}\qquad \deg_{\underline{\mathcal{D}}}(f,F(\alpha))\colon \mathbb{1}_{\underline{\mathcal{D}}} \xrightarrow{F(c_{\underline{X}})} X_!F\xi_{\underline{X}} \underset{\simeq}{\xrightarrow{\alpha}} X_!f^*F\xi_{\underline{Y}} \xrightarrow{\mathrm{BC}_!} Y_!F\xi_{\underline{Y}}$$

as desired. Next, suppose that $(\xi_{\underline{Y}}, c_{\underline{Y}})$ is $\underline{\mathcal{C}}$-twisted ambidextrous. By Proposition 3.1.9 applied to Example 3.1.6 (1), F sends the equivalence $Y_*Y^*\mathbb{1}_{\underline{\mathcal{C}}} \simeq Y_!\xi_{\underline{Y}}$ induced by $(\xi_{\underline{Y}}, c_{\underline{Y}})$ to the equivalence $Y_*Y^*\mathbb{1}_{\underline{\mathcal{D}}} \simeq Y_!F(\xi_{\underline{Y}})$ induced by $F(\xi_{\underline{Y}}, c_{\underline{Y}})$. Thus $Y_!F\xi_{\underline{Y}} \simeq FY_!\xi_{\underline{Y}}$ as commutative algebras in $\underline{\mathcal{D}}$. ☐

Construction 3.4.9. Let $f\colon \underline{X} \to \underline{Y}$ be a map in $\mathcal{B}$ and consider a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$-shaped limits and colimits and satisfies the $\underline{X}$-projection formula. Furthermore, assume that the map $f^*\colon \underline{\mathrm{Fun}}(\underline{Y},\underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}})$ is an equivalence. It canonically refines to a symmetric monoidal equivalence. Hence, by Construction 3.4.1, we obtain canonical equivalences $\mathrm{BC}^f_!\colon X_!f^* \xrightarrow{\simeq} Y_!$ and $\mathrm{BC}^f_*\colon Y_* \xrightarrow{\simeq} X_*f^*$. Thus, for any Spivak datum $(\xi_{\underline{X}}, c_{\underline{X}})$ for $\underline{X}$ we obtain the Spivak datum

$$\left(\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c_{\underline{X}}} X_!\xi_{\underline{X}} \simeq Y_!(f^*)^{-1}\xi_{\underline{X}}\right)$$

for $\underline{Y}$. It is twisted ambidextrous (or Poincaré) if and only if $(\xi_{\underline{X}}, c_{\underline{X}})$ is. Furthermore, note that with respect to these Spivak data, the map f is clearly of degree one.

Lemma 3.4.10. *Consider a geometric morphism $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}'\colon f_*$ of topoi and $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}'$-category. Suppose that we are given a map $g\colon \underline{X} \to \underline{Y}$ in $\underline{\mathcal{B}}$ together with $\underline{\mathcal{C}}$-Spivak data for $f^*\underline{X}$ and $f^*\underline{Y}$. Then a $\mathcal{C}$-degree (one) datum for $f^*g\colon f^*\underline{X} \to f^*\underline{Y}$ is equivalent to a $f_*\mathcal{C}$-degree (one) datum for g, where we endow $\underline{X}$ and $\underline{Y}$ with the $f_*\underline{\mathcal{C}}$-Spivak data from Construction 3.3.10.*

Proof. The equivalence of degree data follows from Lemma 2.1.16. The statement about degree one data being equivalent follows from Lemma 2.1.18. ☐

4. Equivariant Poincaré duality: elements

In this section we will apply the abstract theory of parametrised Poincaré duality developed in §3 to the topos $\mathcal{S}_G$ of G–spaces for a compact Lie group G and use this as our definition of equivariant Poincaré duality spaces. We begin in §4.1 by explaining the definition in this special case in more detail, and then come to one of the key components of the theory in §4.2, namely fixed points methods. After that, in §4.3 we study how Poincaré duality interacts with various kinds of equivariant and homotopical operations such as restrictions, inflations, (co)inductions, fibre sequences, and quotients, and we then give natural examples of G–Poincaré spaces in §4.4. Lastly, we will round off this section by explaining some geometrically meaningful ramifications of the theory of fundamental classes in §§4.5 and 4.6.

The reader who is not too familiar with the abstract categorical language can in most situations safely replace G by a finite group and a presentably symmetric monoidal G–category $\underline{\mathcal{C}}$ by the G–category $\underline{\mathrm{Sp}}$ of genuine G–spectra or even $\underline{\mathrm{Mod}}_{A(G)}(\underline{\mathrm{Sp}}) \simeq D(\underline{\mathrm{Mack}}_G(\mathrm{Ab}))$ (c.f. [GS14, §5.2] or [PSW22, §5] for this equivalence), the derived category of G-Mackey functors with values in abelian groups. As a sanity check for constructions not involving a change of groups, it might also be helpful to first read the statements for $G = 1$.

4.1. Setting the stage

We specialise the definitions in §§3.1 and 3.2 from the abstract parametrised setting to the equivariant situation for a compact Lie group G. After giving the formal definitions, we will provide more intuition for them by unraveling what these notions mean in Remark 4.1.5.

Definition 4.1.1. Let $\underline{X} \in \mathcal{S}_G$ and $\underline{\mathcal{C}}$ a symmetric monoidal G–category admitting $\underline{X}$–shaped colimits. A $\underline{\mathcal{C}}$*–Spivak datum* for $\underline{X}$ is a pair (ξ, c) where $\xi \in \mathrm{Fun}_G(\underline{X}, \underline{\mathcal{C}})$ is called the dualising sheaf and $c\colon \mathbb{1}_{\mathcal{C}} \to X_!\xi$ is a morphism in $\mathcal{C}$ called the fundamental class.

Now let $\underline{\mathcal{C}}$ be a symmetric monoidal G–category and $\underline{X} \in \mathcal{S}_G$ a G–space. Suppose that $\underline{\mathcal{C}}$ admits $\underline{X}$–shaped limits and colimits and satisfies the $\underline{X}$–projection formula (c.f. Terminology 2.1.13). For example, if either: (a) the G–category $\underline{\mathcal{C}}$ were presentably symmetric monoidal, i.e. an object in $\mathrm{CAlg}(\mathrm{Pr}^L_G)$, or (b) if G were a finite group, $\underline{X} \in \mathcal{S}_G$ were compact, and $\underline{\mathcal{C}}$ were a small G–stably symmetric monoidal category, i.e. an object in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{ex}}_G)$, then these conditions are satisfied. Under these conditions, given a $\underline{\mathcal{C}}$–Spivak datum, Construction 3.1.2 provides a a morphism in $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}})$

$$c \cap_\xi -\colon X_*(-) \longrightarrow X_!(- \otimes \xi) \tag{16}$$

called the capping transformation. We refer the reader to the preamble of §3 for the motivation for these notations.

Definition 4.1.2. A $\underline{\mathcal{C}}$–Spivak datum (ξ, c) for $\underline{X}$ is *twisted ambidextrous* if the capping map (16) is an equivalence. It is *Poincaré* if additionally, ξ takes values in the subcategory $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$.

If we take a presentably symmetric monoidal G–category $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_G)$ as coefficient system, then the situation again simplifies a little, for example because a twisted ambidextrous Spivak datum is unique, if it exists, by Proposition 3.2.5.

Definition 4.1.3. If $\underline{X}$ is a G–space and $\underline{\mathcal{C}}$ is a presentably symmetric monoidal G–category, we say that $\underline{X}$ is $\underline{\mathcal{C}}$*–twisted ambidextrous* if it admits a twisted ambidextrous $\underline{\mathcal{C}}$–Spivak datum $(D_{\underline{X}}, c)$. Furthermore, $\underline{X}$ is $\underline{\mathcal{C}}$*–Poincaré* if additionally $D_{\underline{X}}$ takes values in $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$.

Terminology 4.1.4. In the special case where $\underline{\mathcal{C}} = \underline{\mathrm{Sp}}$, we just say that $\underline{X}$ is G–twisted ambidextrous or G–Poincaré.

Understanding the case of $\underline{\mathrm{Sp}}$–Poincaré duality is our main motivation for this article. Because of its importance, we give a few explanations about this particular space.

Remark 4.1.5 (Unraveling the definition of $\underline{\mathrm{Sp}}$-Poincaré duality)**.** To set up a good formalism, we needed to work in a generality that runs the risk of seeming overly abstract. We stress that the task of checking if a space $\underline{X}$ is G–Poincaré closely resembles classical Poincaré duality.

First, one has to find the correct analog of a local system with respect to which $\underline{X}$ is supposed to satisfy Poincaré duality. We required a $\xi \in \underline{\mathrm{Fun}}_G(\underline{X}, \underline{\mathrm{Sp}})$ that lands in $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathrm{Sp}})$, which unravels to providing for each closed subgroup $H \leq G$ a local system of invertible H–spectra $\xi^H : X^H \to \mathcal{P}\mathrm{ic}(\mathrm{Sp}_H)$ together with compatibilities that amount to providing for each map $G/K \to G/H$ a homotopy in the diagram

$$\begin{array}{ccc} X^K & \xrightarrow{\xi^K} & \mathcal{P}\mathrm{ic}(\mathrm{Sp}_K) \\ {\scriptstyle \mathrm{Res}^K_H}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^K_H} \\ X^H & \xrightarrow{\xi^H} & \mathcal{P}\mathrm{ic}(\mathrm{Sp}_H) \end{array}$$

plus higher coherences between these homotopies. Additionally, we required a fundamental class $c \colon \mathbb{S}_G \to X_!(\xi)$ which the reader should think of as an equivariant homology class of $\underline{X}$ with coefficients in the local system ξ. The capping map $c \cap_\xi - \colon X_*(-) \to X_!(- \otimes \xi)$ should then be thought of as the cap product with the homology class c. Recall from the preamble to §3 that classical Poincaré duality is the statement that capping with a certain "fundamental class" induces an isomorphism between cohomology and homology, and what we ask here is exactly the same condition.

In the presentable setting, we are in the pleasant situation where we can identify a large class of twisted ambidextrous objects.

Proposition 4.1.6. *Every compact G–space $\underline{X}$ is $\underline{\mathrm{Sp}}$–twisted ambidextrous. Consequently, every compact G–space $\underline{X}$ is $\underline{\mathcal{C}}$–twisted ambidextrous for any G-stable presentably symmetric monoidal G–category.*

Proof. The first part is an immediate consequence of [Cno23, Thm. 4.8 (5)] and Remark 3.2.6, and the second part is by Theorem 3.3.5. ☐

Next, as may be expected of a well–behaved equivariant notion, equivariant Poincaré duality is preserved under restrictions. To show this, first recall from Recollection 2.2.1 that for a closed subgroup $H \leq G$ there is an identification $\mathcal{O}(H) \simeq \mathcal{O}(G)_{/(G/H)}$ so that the induction $\mathcal{S}_H \to \mathcal{S}_G$ can be identified with the étale geometric morphism $\mathcal{S}_G \rightleftharpoons (\mathcal{S}_G)_{/(\underline{G/H})}$.

Construction 4.1.7 (Pushing Spivak data along restrictions)**.** Let $\underline{X} \in \mathcal{S}_G, \underline{\mathcal{C}} \in \mathrm{CMon}(\mathrm{Cat}_G)$, and (ξ, c) a $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$. Then by Construction 3.3.11, we obtain a $\mathrm{Res}^G_H \underline{\mathcal{C}}$–Spivak datum $\mathrm{Res}^G_H(\xi, c)$ for $\mathrm{Res}^G_H \underline{X}$ given by

$$\Big(\mathrm{Res}^G_H \xi \colon \mathrm{Res}^G_H \underline{X} \xrightarrow{\mathrm{Res}^G_H \xi} \mathrm{Res}^G_H \underline{\mathcal{C}},\ \mathrm{Res}^G_H c \colon \mathrm{Res}^G_H \mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{\mathrm{Res}^G_H [c]} (\mathrm{Res}^G_H X)_! \mathrm{Res}^G_H \xi\Big)$$

Proposition 4.1.8 (Restriction stability of Poincaré duality)**.** *Let $\underline{X} \in \mathcal{S}_G$, $\underline{\mathcal{C}}$ be a symmetric monoidal G–category, and (ξ, c) be a Poincaré $\underline{\mathcal{C}}$–Spivak datum. Then for all closed subgroups $H \leq G$, $\mathrm{Res}^G_H(\xi, c)$ is a Poincaré $\mathrm{Res}^G_H \underline{\mathcal{C}}$–Spivak datum for the H-space $\mathrm{Res}^G_H \underline{X}$.*

Proof. This is a direct consequence of part (e) of Theorem 3.3.12 applied to étale the geometric morphism $\mathcal{S}_G \rightleftharpoons (\mathcal{S}_G)_{/(\underline{G/H})} \simeq \mathcal{S}_H$. ☐

4.2. Fixed points methods

Let G be a compact Lie group. In this subsection, we study how G–Poincaré duality for a G–space $\underline{X}$ relates to Poincaré duality for its fixed points. In fact, we shall build upon the theory set up in §2.2 and discuss these questions in the generality of isotropy separations with respect to a family of subgroups, of which the case of fixed points against a subgroup is a special case. Therefore, let us fix a family $\mathcal{F}$ of closed subgroups of G throughout this subsection. Recall the notational package from Notation 2.2.28.

An important family to keep in mind as an intuitional guide is the following:

Example 4.2.1 (Proper family)**.** Denote by $\mathcal{P}$ the family of proper closed subgroups of G, so that $\mathcal{P}^c = \{G\}$ and $s\colon * \simeq \mathcal{O}_{\mathcal{P}^c}(G)^{\mathrm{op}} \hookrightarrow \mathcal{O}(G)^{\mathrm{op}}$ is the inclusion of the orbit G/G. Note that for any $\underline{J} \in \mathrm{Cat}_G$, we thus have $\underline{J}^{\mathcal{P}^c} = s^*\underline{J} \simeq J^G \in \mathrm{Cat}$. In this special case, we know that the adjunction unit $\Phi\colon \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}^{\Phi\widetilde{\mathcal{P}}} \simeq s_*\widetilde{s}^*\underline{\mathrm{Sp}}$ adjoints to the geometric fixed points functor $\Phi^G\colon s^*\underline{\mathrm{Sp}} = \underline{\mathrm{Sp}}^{\mathcal{P}^c} \simeq \mathrm{Sp}_G \to \Phi^{\mathcal{P}}\underline{\mathrm{Sp}} \simeq \mathrm{Sp}$.

Observation 4.2.2. Consider the case of the family of proper closed subgroups $\mathcal{P}$ of G. In particular, we have that $\underline{\mathrm{Fun}}(-,-)^{\mathcal{P}^c} \simeq \underline{\mathrm{Fun}}(-,-)^G \simeq \mathrm{Fun}_G(-,-)$. For a fixed $\mathcal{C} \in \mathrm{Cat}$ having the approparite (co)limits and a G-space $\underline{X}$, applying $(-)^{\mathcal{P}^c}$ to the commuting diagram in Lemma 2.1.18 and using that $(-)^{\widetilde{\mathcal{P}}}$ is fully faithful yields the left commuting diagram

$$\begin{array}{ccc} \mathrm{Fun}(X^G, \mathcal{C}) & \overset{X_!^G}{\underset{X_*^G}{\rightrightarrows}} & \mathcal{C} \\ \| & & \| \\ \mathrm{Fun}_G(\underline{X}, \mathcal{C}^{\widetilde{\mathcal{P}}}) & \overset{X_!}{\underset{X_*}{\rightrightarrows}} & \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}}) & \xrightarrow{X_!} & \mathrm{Sp}_G \\ \downarrow{\scriptstyle \Phi^G} & & \downarrow{\scriptstyle \Phi^G} \\ \mathrm{Fun}(X^G, \mathrm{Sp}) & \xrightarrow{X_!^G} & \mathrm{Sp}. \end{array}$$

That is, parametrised (co)limits in G–categories of the form $\mathcal{C}^{\widetilde{\mathcal{P}}}$ is given by the ordinary (co)limits of the fixed points of the indexing diagram. In particular, since $\Phi^G\colon \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}^{\Phi\widetilde{\mathcal{P}}}$ preserves parametrised colimits, the identifications above yield the right commuting square in the diagram above.

Lemma 4.2.3. *Let $\underline{X} \in \mathcal{S}_G$ and consider a family $\mathcal{F}$ of subgroups of G. Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{G,\mathcal{F}^c}$. Then the map $\epsilon^*\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}) \to \underline{\mathrm{Fun}}(\underline{X}_{\widetilde{\mathcal{F}}}, \underline{\mathcal{C}}^{\widetilde{\mathcal{F}}})$ is an equivalence.*

Proof. By Lemma 2.1.16, the equivalence $\underline{\mathrm{Fun}}_G(\underline{X}, s_*\underline{\mathcal{C}}) \simeq s_*\underline{\mathrm{Fun}}_{\mathcal{F}^c}(s^*\underline{X}, \underline{\mathcal{C}})$ identifies restriction along $\epsilon\colon s_!s^*\underline{X} \to \underline{X}$ on the left side with restriction along $s^*\epsilon\colon s^*s_!s^*\underline{X} \to s^*\underline{X}$ on the right side. But $s^*\epsilon$ is an equivalence. ☐

Construction 4.2.4 (Isotropy separation for Spivak data)**.** Let $\underline{X} \in \mathcal{S}_G$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{F}^c$–category which admits $\underline{X}^{\mathcal{F}^c}$–indexed colimits. By Lemma 2.1.18, we know that $\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}$ admits $\underline{X}$–colimits. Suppose we are given a $\underline{\mathcal{C}}$-Spivak datum (ξ, c) for $\underline{X}^{\mathcal{F}^c}$ and a $\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}$–Spivak datum (ζ, d) for $\underline{X}$. By Construction 3.3.10, we obtain a $\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}$–Spivak datum $(\xi, c)^{\widetilde{\mathcal{F}}}$ for $\underline{X}$ and a $\underline{\mathcal{C}}$–Spivak datum $(\zeta, d)^{\mathcal{F}^c}$ for $\underline{X}^{\mathcal{F}^c}$. Observe in particular that, by construction, we have $((\xi, c)^{\widetilde{\mathcal{F}}})^{\mathcal{F}^c} \simeq (\xi, c)$ and $((\zeta, d)^{\mathcal{F}^c})^{\widetilde{\mathcal{F}}} \simeq (\zeta, d)$.

Corollary 4.2.5 (Inclusion of singular part is degree one)**.** *Let $\underline{\mathcal{D}}$ be a symmetric monoidal $\mathcal{F}^c$–category and $\underline{X} \in \mathcal{S}_G$. Suppose $\underline{X}$ is equipped with a $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–Spivak datum. Then $\underline{X}_{\widetilde{\mathcal{F}}} \in \mathcal{S}_G$ inherits a $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–Spivak datum under which the inclusion $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ is of $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–degree one.*

Proof. By Lemma 4.2.3, the map $\epsilon^*\colon \underline{\mathrm{Fun}}(\underline{X}, s_*\underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}(s_!s^*\underline{X}, s_*\underline{\mathcal{D}})$ is an equivalence. The result now follows immediately from Construction 3.4.9. □

Lemma 4.2.6. *Let $\underline{X} \in \mathcal{S}_G$, $\mathcal{F}$ be a family of closed subgroups of G, and $\underline{\mathcal{D}}$ a symmetric monoidal $\mathcal{F}^c$–category. Then a $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–Spivak datum (ξ, c) for $\underline{X}$ is Poincaré if and only if the $\underline{\mathcal{D}}$–Spivak datum $(\xi, c)^{\mathcal{F}^c}$ for $\underline{X}_{\mathcal{F}^c}$ is Poincaré.*

Proof. This is a special case of Theorem 3.3.12 (c). □

We now come to the main result of this subsection which says that we may perform isotropy separation on equivariant Poincaré spaces by appropriately isotropy–separating the coefficient category. For the second part of the result, we will need to recall Terminology 3.3.7.

Theorem 4.2.7 (Poincaré isotropy basechange)**.** *Let $\underline{X} \in \mathcal{S}_G$, $\underline{Y} \in \mathcal{S}_G^{\omega}$, $\underline{\mathcal{C}}$ be a presentably symmetric monoidal fibrewise stable G–category, and $\underline{\mathcal{D}}$ be a G–stably symmetric monoidal category.*

(1) If $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré, then $\underline{X}^{\mathcal{F}^c}$ is $\Phi^{\mathcal{F}}\underline{\mathcal{C}}$–Poincaré;

(2) If (ξ, c) is a Poincaré $\underline{\mathcal{D}}$–Spivak datum for $\underline{Y}$, then the Spivak datum $(\Phi\xi, \Phi c)^{\mathcal{F}^c}$ is a Poincaré $(\Phi\colon \underline{\mathcal{D}}^{\mathcal{F}^c} \to \Phi^{\mathcal{F}}\underline{\mathcal{D}})$–Spivak datum for $\underline{Y}^{\mathcal{F}^c}$.

Proof. For (1), applying the basechange result Theorem 3.3.5 along the symmetric monoidal G-colimit preserving unit map $\underline{\mathcal{C}} \to \underline{\mathcal{C}}^{\Phi\widetilde{\mathcal{F}}}$ shows that $\underline{X}$ is $\underline{\mathcal{C}}^{\Phi\widetilde{\mathcal{F}}}$–Poincaré. Thus Lemma 4.2.6 shows that $\underline{X}^{\mathcal{F}^c}$ is $\Phi^{\mathcal{F}}\underline{\mathcal{C}}$–Poincaré. Point (2) is an immediate consequence of Theorem 3.3.8 and Lemma 4.2.6. □

Having set up a general theory of equivariant fixed points for Poincaré spaces, we now specialise to the most important coefficient category, namely the presentably symmetric monoidal G–stable category $\underline{\mathrm{Sp}}$ of genuine G–spectra.

Construction 4.2.8 (Pushing Spivak data along geometric fixed points)**.** Let $\underline{X} \in \mathcal{S}_G$, (ξ, c) a $\underline{\mathrm{Sp}}_G$–Spivak datum for $\underline{X}$, and $H \leq G$ a closed subgroup. By Construction 4.1.7, we obtain a $\underline{\mathrm{Sp}}_H$–Spivak datum $\mathrm{Res}^G_H(\xi, c)$ for $\mathrm{Res}^G_H \underline{X}$. On the other hand, we may apply Construction 4.2.4 to $\mathrm{Res}^G_H(\xi, c)$ along the symmetric monoidal map $\Phi^H\colon \underline{\mathrm{Sp}}_H \to s_*\mathrm{Sp}$ to get a nonequivariant Sp–Spivak datum $\Phi^H(\xi, c)$ for X^H. Explicitly, this is given by

$$\left(\Phi^H\xi\colon X^H \xrightarrow{\xi^H} \mathrm{Sp}_H \xrightarrow{\Phi^H} \mathrm{Sp},\ \Phi^H c\colon \mathbb{1}_{\mathrm{Sp}} = \Phi^H \mathbb{1}_{\mathrm{Sp}_H} \xrightarrow{\Phi^H c} \Phi^H(\mathrm{Res}^G_H X)_! \mathrm{Res}^G_H \xi \simeq X^H_! \Phi^H \xi\right)$$

where the last equivalence is by Observation 4.2.2.

Next, we unwind the general Theorem 4.2.7 (1) for the geometric fixed points functor on spectra to show that the fixed points of an equivariant Poincaré space are Poincaré with the residual Weyl group action (c.f. [CW17, Prop. 2.4] for the homological shadow of this).

Theorem 4.2.9 (Fixed points of Poincaré spaces)**.** *Suppose* $\underline{X} \in \mathcal{S}_G$ *is* G*–Poincaré. Then for any closed* $H \leq G$*,* $\underline{X}^H \in \mathcal{S}_{W_G H}$ *is a* $W_G H$*–Poincaré space. In particular,* $X^H \in \mathcal{S}$ *is a nonequivariant Poincaré space with dualising sheaf* $X^H \xrightarrow{D_{\underline{X}}} \mathrm{Sp}_H \xrightarrow{\Phi^H} \mathrm{Sp}$.

Proof. First consider the case where H is normal in G. We apply Theorem 4.2.7 (1) in the case $\underline{\mathcal{C}} = \underline{\mathrm{Sp}}_G$ and for the family $\Gamma_H \coloneqq \{K \leq G \mid H \not\leq K\}$ of subgroups of G not containing H. Thus, if $\underline{X}$ is a G–Poincaré space, then $s^*\underline{X}$ is a $\widetilde{s}^*\underline{\mathrm{Sp}}_G$ Poincaré space. In Proposition 2.2.29 we saw that $\mathrm{Coind}\colon \mathcal{S}_{G/H} \to \mathcal{S}_G$ induces an equivalence $\mathcal{S}_{G/H} \simeq \mathcal{S}_{\Gamma_H^c}$ endowing $s^*\underline{X}$ with a G/H action. It also follows from Corollary 2.2.30 that $\widetilde{s}^*\underline{\mathrm{Sp}}_G \simeq \underline{\mathrm{Sp}}_{G/N}$ which completes the proof of this case.

Now suppose that $H \leq G$ is a general subgroup. We can apply Proposition 4.1.8 to obtain that $\mathrm{Res}^G_{N_G H}\underline{X}$ is a $N_G H$–Poincaré duality space. Then the normal subgroup case from above shows that $\mathrm{Res}^G_H \underline{X} = \mathrm{Res}^{N_G H}_H \mathrm{Res}^G_{N_G H} \underline{X}$ is a $W_G H = N_G H/H$–Poincaré duality space.

The first part in particular shows that X^G is a Sp–Poincaré duality space. Applying this to $X^H \in \mathcal{S}_{W_G H}$, we see that X^H is a nonequivariant Sp–Poincaré duality space. ☐

To end our discussion on general fixed points methods, we provide a sort of converse to the previous statement. By Proposition 3.2.5, we know that in the presentable setting, a twisted ambidextrous Spivak datum is unique if it exists. Via the geometric fixed points functors, the following result gives a full characterisation for a candidate invertible Spivak datum to be the unique one for a twisted ambidextrous space in terms of nonequivariant Poincaré duality. It will be essential for constructing examples of equivariant Poincaré duality spaces in §4.4.

Theorem 4.2.10 (Fixed point recognition principle of Poincaré spaces)**.** *Suppose that* $\underline{X} \in \mathcal{S}_G$ *is a twisted ambidextrous* G*–space (e.g. a compact* G*–space) and let* (ξ, c) *be a* $\underline{\mathrm{Sp}}_G$*-Spivak datum for* $\underline{X}$ *such that* $\xi\colon \underline{X} \to \underline{\mathrm{Sp}}_G$ *takes values in* $\underline{\mathrm{Pic}}(\underline{\mathrm{Sp}}_G)$*. Then* (ξ, c) *exhibits* $\underline{X}$ *as a* G*–Poincaré duality space if and only if for all closed subgroups* $H \leq G$*, the Spivak datum* $\Phi^H(\xi, c)$ *from Construction 4.2.8 exhibits* X^H *as a nonequivariant* Sp*–Poincaré space.*

Proof. The "only if" direction is a consequence of Theorem 4.2.9. For the other direction, we have to show that the Spivak datum (ξ, c) is twisted ambidextrous as ξ is invertible by assumption. By Observation 2.2.33 and Proposition 2.2.35, the collection

$$\left\{ \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}}) \xrightarrow{\Phi^H} \underline{\mathrm{Fun}}(\underline{X}, \mathrm{Coind}^G_H s_* \widetilde{s}^* \underline{\mathrm{Sp}}) \simeq \prod_H^G s_* \mathrm{Fun}(X^H, \mathrm{Sp}) \mid H \leq G \text{ closed subgroups} \right\}$$

is jointly conservative. Thus, it suffices to show that the transformations

$$\Phi^H(c \cap_\xi -)\colon \Phi^H X_*(-) \to \Phi^H X_!(- \otimes \xi) \tag{17}$$

are equivalences. By passing to the adjoint $\underline{\mathrm{Fun}}(\mathrm{Res}^G_H \underline{X}, \mathrm{Res}^G_H \underline{\mathrm{Sp}}) \xrightarrow{\Phi^H} s_* \mathrm{Fun}(X^H, \mathrm{Sp})$ to consider everything as H–categories, we may without loss of generality just consider the case Φ^G. By Proposition 3.1.9 applied to the case of Example 3.1.6 (1), the symmetric monoidal functor of presentably symmetric monoidal G–categories $\Phi^G\colon \underline{\mathrm{Sp}} \to s_*\mathrm{Sp}$ yields a square

$$\begin{array}{ccc} \Phi^G X_*(-) & \xrightarrow{\Phi^G(c\cap_\xi -)} & \Phi^G X_!(\xi\otimes -) \\ \simeq\downarrow \mathrm{BC}_* & & \mathrm{BC}_!\uparrow\simeq \\ X_*\Phi^G(-) & \xrightarrow[\Phi^G c\cap_{\Phi^G\xi}\Phi^G -]{} & X_!\Phi^G(\xi\otimes -)\simeq X_!(\Phi^G\xi\otimes\Phi^G(-)) \end{array}$$

where the vertical Beck–Chevalley maps are equivalences, the right one by Observation 4.2.2 and the left one by Corollary 3.3.6 since $\underline{X}$ was assumed to be twisted ambidextrous. By Observation 4.2.2, the bottom map identifies with $\Phi^G c\cap_{\Phi^G\xi}\Phi^G-\colon X^G_*\Phi^G(-)\to X^G_!(\Phi^G\xi\otimes\Phi^G-)$, which is an equivalence by hypothesis. Thus, in total, we see that the top horizontal map in the square above is an equivalence, as was to be shown. ☐

4.3. Construction principles

In this section we will study various results on how to build new Poincaré duality spaces out of old ones.

Change of groups

We begin by studying the effect of standard equivariant operations on $\underline{X}$. Recall the constructions and notations from Notation 2.2.2 and Construction 2.2.13.

Proposition 4.3.1 (Poincaré duality and restriction)**.** *Suppose that $\alpha\colon H\to G$ is a continuous homomorphism of compact Lie groups and $\underline{X}\in\mathcal{S}_G$. If $\underline{X}$ is a G–Poincaré space, then $\mathrm{Res}_\alpha\underline{X}$ is a H–Poincaré space with Spivak datum $(\mathrm{Res}_\alpha c,\mathrm{Res}_\alpha D_{\underline{X}})$ where*

1. *the local system $\mathrm{Res}_\alpha D_{\underline{X}}$ is $\mathrm{Res}_\alpha\underline{X}\to\mathrm{Res}_\alpha\underline{\mathrm{Sp}}_G\to\underline{\mathrm{Sp}}_H$ and*
2. *the collapse $\mathrm{Res}_\alpha c$ is $\mathbb{1}_{\mathrm{Sp}_H}=\mathrm{Res}_\alpha\mathbb{1}_{\mathrm{Sp}_G}\to\mathrm{Res}_\alpha X_!D_{\underline{X}}\simeq r_!^{\mathrm{Res}_\alpha\underline{X}}\mathrm{Res}_\alpha D_{\underline{X}}$.*

Proof. If $\underline{X}$ is $\underline{\mathrm{Sp}}_G$–Poincaré, then applying Theorem 3.3.5 for the symmetric monoidal G-colimit preserving functor $\mathrm{Res}_\alpha\colon\underline{\mathrm{Sp}}_G\to\mathrm{Coind}_\alpha\underline{\mathrm{Sp}}_H$ from Construction 2.2.9 shows that $\underline{X}$ is $\mathrm{Coind}_\alpha\underline{\mathrm{Sp}}_H$–Poincaré. By Theorem 3.3.12 (d) we see that $\mathrm{Res}_\alpha\underline{X}$ is $\underline{\mathrm{Sp}}_H$–Poincaré with claimed Spivak datum. ☐

Proposition 4.3.2 (Poincaré duality and inflation)**.** *Consider a closed normal subgroup $N\leq G$ and a G/N-space $\underline{X}$. Then $\underline{X}$ is a G/N–Poincaré duality space if and only if $\mathrm{Infl}_G^{G/N}\underline{X}$ is a G–Poincaré duality space.*

Proof. One direction is a consequence of Proposition 4.3.1 while the other one follows from Theorem 4.2.9. ☐

Proposition 4.3.3 (Poincaré duality and induction)**.** *Let $\iota\colon H\to G$ be an injective homomorphism of compact Lie groups. If $\underline{X}$ is a H–Poincaré space, then $\mathrm{Ind}_H^G\underline{X}$ is G–Poincaré space.*

Proof. We first claim that the map $\mathrm{Ind}_H^G \underline{X} \to \underline{G/H}$ is a G–Poincaré map. Using the equivalence $(\mathcal{S}_G)_{/\underline{G/H}} \simeq \mathcal{S}_H$, this is equivalent to $\mathrm{Res}_H^G \mathrm{Ind}_H^G \underline{X}$ being a H–Poincaré space. Observe that $\mathrm{Res}_H^G \mathrm{Ind}_H^G \underline{X} \simeq \underline{X} \times \mathrm{Res}_H^G \underline{G/H}$. As $\underline{G/H}$ is a G–Poincaré space, the claim follows from Corollary 3.3.15 and Proposition 4.3.1. Now Proposition 3.3.3 implies that the composite $\mathrm{Ind}_H^G \underline{X} \to \underline{G/H} \to \underline{*}$ is a G–Poincaré map meaning that $\mathrm{Ind}_H^G \underline{X}$ is a G–Poincaré space. ☐

For the next result, we will need to restrict to the case of finite groups since we will need to invoke the theory of G–symmetric monoidal structures as introduced in [Nar17] and further developed in [NS22].

Recollections 4.3.4 (Multiplicative norms)**.** Nardin constructed in [Nar17] a G–symmetric monoidal structure for the G–category of genuine G–spectra $\underline{\mathrm{Sp}}$, packaging the multiplicative norms of [GM97; HHR16] coherently. For a finite G–set, $U = \coprod_i G/H_i$, we write $\mathrm{Cat}_U := \prod_i \mathrm{Cat}_{H_i}$ and write $\underline{\mathrm{Sp}}_U := (\underline{\mathrm{Sp}}_{H_i})_i \in \mathrm{Cat}_U$. For a map of finite G–sets $f\colon U \to V$, we get an adjunction $f^*\colon \mathrm{Cat}_V \rightleftharpoons \mathrm{Cat}_U : f_*$ where f^* is given by restrictions and f_* is given by coinductions. As part of the G–symmetric monoidal structure on $\underline{\mathrm{Sp}}$, we have a map $f_\otimes\colon f_*\underline{\mathrm{Sp}}_U \to \underline{\mathrm{Sp}}_V$ encoding the multiplicative norm along f. For example, when f is the map $f\colon G/H \to G/G$, this encodes a map $f_\otimes\colon \mathrm{Coind}_H^G \underline{\mathrm{Sp}}_H \to \underline{\mathrm{Sp}}_G$, upon applying the functor $(-)^G$ to which yields the multiplicative norm $\mathrm{N}_H^G\colon \mathrm{Sp}_H \to \mathrm{Sp}_G$. By [NS22, §3.3], for a fixed $\underline{X} \in \mathcal{S}_G$, we may obtain a pointwise G–symmetric monoidal structure on the functor category $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}})$. From this, we ma for example extract the pointwise multiplicative norm functor

$$f_\otimes\colon f_*\underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \longrightarrow f_\otimes\underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \simeq \underline{\mathrm{Fun}}(f_*\underline{X}, \underline{\mathrm{Sp}})$$

where the equivalence is by [Hil24, Cor. 2.2.20].

Proposition 4.3.5 (Poincaré duality and coinductions)**.** *Let G be a finite group and $\{H_i\}_i$ a finite collection of subgroups of G. Suppose for each i, we have a H_i–Poincaré space $\underline{X}_i \in \mathcal{S}_{H_i}$ with dualising sheaf $D_{\underline{X}_i}$. Then $\prod_i \mathrm{Coind}_{H_i}^G \underline{X}_i \in \mathcal{S}_G$ is a G–Poincaré space with dualising sheaf $\bigotimes_i \mathrm{N}_{H_i}^G D_{\underline{X}_i} \in \underline{\mathrm{Fun}}(\prod_i \mathrm{Coind}_{H_i}^G \underline{X}, \underline{\mathrm{Sp}})$.*

Proof. We consider a map of finite G–sets $f\colon U = \coprod_i G/H_i \to V = G/G$ as in Recollection 4.3.4. Writing $\underline{X} := (\underline{X}_i)_i \in \mathrm{Cat}_U$, we have an equivalence of the two functors

$$\underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xrightarrow[D_{\underline{X}}\otimes -]{} \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xrightarrow[X_!]{} f^*\underline{\mathrm{Sp}}, \qquad \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xrightarrow{X_*} f^*\underline{\mathrm{Sp}} \tag{18}$$

by our hypothesis. Now writing $f_*X\colon f_*\underline{X} \to \underline{*}$ for the unique map, note that since X_* itself has a right adjoint, we may use [Hil22, Lem. 4.4.3] to see that applying $f_\otimes$ preserves the adjunctions $X_! \dashv X^* \dashv X_*$ in the sense that we have the adjunctions

$$f_\otimes\underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \;\overset{f_\otimes(X_!)}{\underset{f_\otimes(X_*)}{\rightrightarrows}}\; f_\otimes f^*\underline{\mathrm{Sp}}, \qquad f_\otimes\underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xleftarrow{f_\otimes(X^*)} f_\otimes f^*\underline{\mathrm{Sp}}$$

But then, since $\underline{\mathrm{Fun}}(-,\underline{\mathrm{Sp}})\colon \underline{\mathrm{Cat}}_G^{\times}\to \underline{\mathrm{Pr}}_{L,\mathrm{st}}^{\otimes}$ functorial in left Kan extensions is $\underline{\otimes}$–symmetric monoidal by [NS22, After Cor. 6.0.11] together with [Hil24, Cor. 2.2.20], we get

$$f_\otimes(X_!)\simeq (f_*X)_!\colon f_\otimes\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})\simeq \underline{\mathrm{Fun}}(f_*\underline{X},\underline{\mathrm{Sp}})\longrightarrow f_\otimes f^*\underline{\mathrm{Sp}}\simeq\underline{\mathrm{Sp}}$$

and thus consequently, also that $f_\otimes(X^*)\simeq(f_*X)^*$ and $f_\otimes(X_*)\simeq(f_*X)_*$. Next, note that the functor $D_{\underline{X}}\otimes -$ may be written as

$$f^*\underline{\mathrm{Sp}}\otimes\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})\xrightarrow{D_{\underline{X}}\otimes\mathrm{id}}\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})\otimes\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})\xrightarrow{\otimes}\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})$$

Thus applying $f_\otimes$ to this composite and using that $f_\otimes$ is itself a symmetric monoidal functor, we get the identification of $f_\otimes(D_{\underline{X}}\otimes -)$ as

$$\underline{\mathrm{Sp}}\otimes f_\otimes\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})\xrightarrow{f_\otimes D_{\underline{X}}\otimes\mathrm{id}} f_\otimes\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})\otimes f_\otimes\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})\xrightarrow{\otimes} f_\otimes\underline{\mathrm{Fun}}(\underline{X},f^*\underline{\mathrm{Sp}})$$

That is, that $f_\otimes(D_{\underline{X}}\otimes -)\simeq f_\otimes D_{\underline{X}}\otimes -$. Therefore, all in all, applying $f_\otimes$ to the identification in (18), we obtain an equivalence

$$(f_*X)_*\simeq f_\otimes(X_*)\simeq f_\otimes\big(X_!(D_{\underline{X}}\otimes -)\big)\simeq (f_*X)_!(f_\otimes D_{\underline{X}}\otimes -)$$

as was to be shown. ☐

Proposition 4.3.6 (Poincaré duality and Borelification)**.** *Let $\mathcal{C}\in\mathrm{CAlg}(\mathrm{Pr}^L)$ and $\underline{X}\in\mathcal{S}_G$ such that X^e is nonequivariantly a $\mathcal{C}$–twisted ambidextrous (resp. Poincaré) space. Then $\underline{X}$ is a $\underline{\mathrm{Bor}}(\mathcal{C})$–twisted ambidextrous (resp. Poincaré) space.*

Proof. Since $*\to BG$ is an effective epimorphism, we may apply Proposition 3.3.14 to the fibre sequence $X^e\to X_{hG}\xrightarrow{\pi} BG$ to get that π is a $\mathcal{C}$–twisted ambidextrous (resp. Poincaré) map. Writing $s\colon BG\to\mathcal{O}(G)$ for the inclusion and using the identification $\mathcal{S}_{/BG}=\mathrm{Fun}(BG,\mathcal{S})$ under which π corresponds to $s^*\underline{X}$, this means by Definition 3.2.12 that $s^*\underline{X}$ is $\pi_{BG}^*\mathcal{C}$–twisted ambidextrous (resp. Poincaré), where $\pi_{BG}^*\colon\mathrm{Cat}\to\mathrm{Cat}_{BG}$ denotes the restriction functor. Now the basechange result Theorem 3.3.12 shows that $\underline{X}$ is $s_*\pi_{BG}^*\mathcal{C}=\underline{\mathrm{Bor}}(\mathcal{C})$–twisted ambidextrous (resp. Poincaré). ☐

Lemma 4.3.7 (Degree one data and Borelification)**.** *Let $\mathcal{C}\in\mathrm{CAlg}(\mathrm{Pr}^L)$ be a presentably symmetric monoidal category and $f\colon\underline{X}\to\underline{Y}$ a map of G–spaces such that X^e and Y^e are nonequivariantly $\underline{\mathcal{C}}$–twisted ambidextrous. Suppose that $\alpha\colon D_{X^e}\simeq f^*D_{Y^e}$ is a G–equivariant $\underline{\mathcal{C}}$–degree datum for $f^e\colon X^e\to Y^e$, i.e. α is an equivalence in $\mathrm{Fun}(X^e,\mathcal{C})^{hG}$. Then there is a $\underline{\mathrm{Bor}}(\mathcal{C})$–degree datum for $f\colon\underline{X}\to\underline{Y}$. If in addition the G–equivariant degree datum for f^e is G-equivariantly of degree one, i.e. there is an equivalence $c_{Y^e}\simeq f_!c_{X^e}$ in $\mathrm{Map}(\mathbb{1}_{\mathcal{C}},(Y^e)_!D_{Y^e})^{hG}$, then the $\underline{\mathrm{Bor}}(\mathcal{C})$–degree datum for f is of degree one.*

Proof. With the notation from Proposition 4.3.6, the assumption on G-equivariance of the degree datum implies that means that α is a $\pi_{BG}^*\mathcal{C}$-degree datum for the map $f^e\colon X^e\to Y^e$ in $\mathcal{S}^{BG}$. If α is G-equivariantly of degree one, then the $\pi_{BG}^*\mathcal{C}$-degree datum for f^e is of degree one. Now Lemma 3.4.10 provides us with a $\underline{\mathrm{Bor}}(\mathcal{C})$ degree datum for $f\colon\underline{X}\to\underline{Y}$ (which is of degree one if α was G-equivariantly of degree one). ☐

Family nilpotence

We now study how Poincaré duality interacts with the $\mathcal{F}$–nilpotence theory of [MNN17].

Proposition 4.3.8. *Let $\mathcal{F}$ be a family of subgroups of G, $\underline{\mathcal{C}}$ a presentably symmetric monoidal G–category which is $\mathcal{F}$–Borel complete, and $\underline{X} \in \mathcal{S}_G$. Then $\underline{X}$ satisfies $\underline{\mathcal{C}}$–Poincare duality if and only if $\mathrm{Res}^G_H \underline{X}$ satisfies $\mathrm{Res}^G_H \underline{\mathcal{C}}$–Poincare duality for all $H \in \mathcal{F}$.*

Proof. Using that $\mathcal{C}$ is $\mathcal{F}$-Borel complete, Theorem 3.3.12 shows that $\underline{X}$ is a $\mathcal{C} \simeq b_* b^* \mathcal{C}$–Poincaré space if and only if $b^* \underline{X}$ is a $b^* \underline{\mathcal{C}}$–Poincaré space. But by definition, this is equivalent to the map $\underline{X} \to \underline{E\mathcal{F}}$ being a $\underline{\mathcal{C}}$–Poincaré duality map.

There is an effective epimorphism $\coprod_{H \in \mathcal{F}} \underline{G/H} \to \underline{E\mathcal{F}}$. The descent result Proposition 3.3.14 together with Lemma 3.3.16 now shows that $\underline{X} \to \underline{E\mathcal{F}}$ is a $\underline{\mathcal{C}}$–Poincaré duality map if and only if $\underline{G/H} \times_{\underline{E\mathcal{F}}} \underline{X} \to \underline{G/H}$ is a $\underline{\mathcal{C}}$–Poincaré duality map for all $H \in \mathcal{F}$. Note that under the equivalence $(\mathcal{S}_G)_{/\underline{G/H}} \simeq \mathcal{S}_H$ the map $\underline{G/H} \times_{\underline{E\mathcal{F}}} \underline{X} \to \underline{G/H}$ corresponds to $\mathrm{Res}^G_H \underline{X}$. Similarly, this equivalence identifies $\pi^*_{\underline{G/H}} \underline{\mathcal{C}}$ and $\mathrm{Res}^G_H \underline{\mathcal{C}}$. ☐

Recall the notion of $\mathcal{F}$–nilpotent ring G–spectra from [MNN17, Def. 6.36].

Corollary 4.3.9. *Let G be a finite group, $\mathcal{F}$ a family of subgroups, $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ an $\mathcal{F}$–nilpotent ring G–spectrum, and $\underline{X} \in \mathcal{S}_G$. Then $\underline{X}$ is an R–Poincaré space if and only if for all $H \in \mathcal{F}$, $\mathrm{Res}^G_H \underline{X}$ is $\mathrm{Res}^G_H R$–Poincaré.*

Proof. By Example 2.2.22, we know that the presentably symmetric monoidal G–stable category $\underline{\mathrm{Mod}}_R(\underline{\mathrm{Sp}})$ is $\mathcal{F}$–Borel and so we may apply Proposition 4.3.8 to conclude. ☐

Example 4.3.10. We collect here a small list of potentially interesting consequences of Corollary 4.3.9 using known nilpotence results from [MNN19, Table 2] for finite groups G. We invite the reader to consult the cited table for a quite exhaustive list of possibly interesting examples of coefficient ring G–spectra to consider.

(1) Writing KO_G and KU_G for Segal's equivariant topological K–theories, we see that a G–space $\underline{X}$ is KO_G– or KU_G–Poincaré if and only if $\mathrm{Res}^G_C \underline{X}$ is KO_C– or KU_C–Poincaré for all cyclic subgroups $C \leq G$.

(2) A G–space $\underline{X}$ is $\mathrm{Bor}_G(\mathrm{H}\mathbb{Z})$–Poincaré if and only if $\mathrm{Res}^G_E \underline{X}$ is $\mathrm{Bor}_E(\mathrm{H}\mathbb{Z})$–Poincaré for all elementary abelian p–subgroups $E \leq G$ for all primes p.

(3) A G–space $\underline{X}$ is $\mathrm{Bor}_G(\mathrm{MU})$–Poincaré if and only if $\mathrm{Res}^G_A \underline{X}$ is $\mathrm{Bor}_A(\mathrm{MU})$–Poincaré for all abelian p–subgroups $A \leq G$ for all primes p.

(4) A G–space $\underline{X}$ is $\mathrm{Bor}_G(\mathrm{MO})$–Poincaré if and only if $\mathrm{Res}^G_A \underline{X}$ is $\mathrm{Bor}_A(\mathrm{MO})$–Poincaré for all elementary abelian 2–subgroups $A \leq G$.

Poincaré integration

In this section we study the equivariant generalisation of a well known result of Klein [Kle01, Corollary F][4] which says that in a fibration $F \to E \to B$ of finitely dominated spaces, E is a Poincaré space if F and B are Poincaré spaces. Since we have $E \simeq \mathrm{colim}_B F$ by the straightening–unstraightening equivalence, the aforementioned result may be viewed as saying that integrating a Poincaré space along a diagram which is itself Poincaré yields a Poincaré space.

Terminology 4.3.11 (Fibrewise twisted ambidextrous and Poincaré maps)**.** Let $f\colon \underline{X} \to \underline{Y}$ be a map of G–spaces for G a compact Lie group and $\underline{\mathcal{C}}$ a presentably symmetric monoidal G-category. We say that it is a *fibrewise $\underline{\mathcal{C}}$–twisted ambidextrous (resp. Poincaré) map* if for all closed subgroups $H \leq G$ and all maps $y\colon \underline{G/H} \to \underline{Y}$, writing F_y for the pullback $\underline{G/H} \times_{\underline{Y}} \underline{X}$, the map $F_y \to \underline{G/H}$ is $\underline{\mathcal{C}}$–twisted ambidextrous (resp. Poincaré). Expanding Definitions 3.2.2 and 3.2.12, this means that viewed as an object in $\mathcal{S}_H \simeq (\mathcal{S}_G)_{/\underline{G/H}}$, F_y is $\mathrm{Res}^G_H \underline{\mathcal{C}}$–twisted ambidextrous (resp. Poincaré).

In the case where $\underline{\mathcal{C}} = \underline{\mathrm{Sp}}_G$ this means that F_y is $\mathrm{Res}^G_H \underline{\mathrm{Sp}}_G = \underline{\mathrm{Sp}}_H$–twisted ambidextrous (resp. Poincaré). Since $\underline{G/H}$ is G–Poincaré, we see by Proposition 3.3.3, Corollary 3.3.17, and Proposition 4.1.8 that in this case the preceding condition is also equivalent to F_y being G–twisted ambidextrous (resp. Poincaré).

Theorem 4.3.12 (Equivariant Poincaré integration)**.** *Let $f\colon \underline{X} \to \underline{Y}$ be a map of G–spaces and $\underline{\mathcal{C}}$ a presentably symmetric monoidal G-category. If $\underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré space and f is a fibrewise $\underline{\mathcal{C}}$–Poincaré map, then $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré space. Furthermore, there is an equivalence $D_{\underline{X}} \simeq f^* D_{\underline{Y}} \otimes D_f$, where $D_f \in \mathrm{Fun}_G(\underline{X}, \underline{\mathcal{C}})$ such that $y^* D_f \simeq D_{F_y}$ is the dualising sheaf of the fibres.*

Conversely, suppose that $\underline{Y}$ is $\underline{\mathcal{C}}$–twisted ambidextrous and that f is fibrewise $\underline{\mathcal{C}}$-twisted ambidextrous. Furthermore assume that for all closed subgroups $H \leq G$ the map $f^H\colon X^H \to Y^H$ is a π_0-surjection. If $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré space, then $\underline{Y}$ is also a $\underline{\mathcal{C}}$–Poincaré space and f is a fibrewise Poincaré map.

Proof. The map $\coprod_{H \leq G} \coprod_{\pi_0(Y^H)} \underline{G/H} \to \underline{Y}$ is a π_0 surjection on each fixed point space and thus an effective epimorphism in $\mathcal{S}_G$ (see Example 3.3.13). It then follows from Proposition 3.3.14 and Lemma 3.3.16 that f is a $\underline{\mathcal{C}}$–Poincaré map if and only if the map $F_y \to \underline{G/H}$ is $\underline{\mathcal{C}}$–Poincaré for all closed subgroups $H \leq G$ and all $y\colon \underline{G/H} \to \underline{Y}$. This is precisely what it means that f was fibrewise $\underline{\mathcal{C}}$–Poincaré. Moreover, Proposition 3.3.14 also provides an equivalence $y^* D_f \simeq D_{F_y}$. Since in addition $\underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré space, it follows from Proposition 3.3.3 that $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré space and there is an equivalence $D_{\underline{X}} \simeq y^* D_{\underline{Y}} \otimes D_{F_y}$ as desired.

For the converse, as in the first we conclude from Proposition 3.3.14 and Proposition 3.3.3 that there is an equivalence $D_{\underline{X}} \simeq f^* D_{\underline{Y}} \otimes D_f$. If $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré, then $D_{\underline{X}}$ is invertible which implies that $f^* D_{\underline{Y}}$ and D_f are invertible. The π_0 surjectivity hypothesis on f implies that $D_{\underline{Y}}$ is invertible so $\underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré space. From the equivalence $y^* D_f \simeq D_{F_y}$ we see that $(F_y \to \underline{G/H})$ is a $\mathrm{Res}^G_H \underline{\mathcal{C}}$–Poincaré space. □

[4] In [Kle01], Klein mentioned that the result answered a question of Wall and also attributed the result to Quinn from an unpublished announcement and Gottlieb [Got79] who proved it in the manifolds setting.

We now use the theorem above to obtain a characterisation of G–Poincaré duality for spaces with free actions in terms of Poincaré duality for a quotient group.

Corollary 4.3.13 (Poincaré duality and quotients by free actions)**.** *Let G be a compact Lie group, $N \leq G$ a closed normal subgroup and $Q := G/N$. If $\underline{X}$ is a G–space such that the action of N on $\underline{X}$ is free in the sense of Definition 2.2.36, then $\underline{X}$ is G–Poincaré duality space if and only if $N \backslash \underline{X}$ is a Q–Poincaré duality space.*

Proof. We will show that this follows from Theorem 4.3.12. To do so, it suffices to check that for each map $\underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{X}$ the space $\underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X}$ is a G–Poincaré space. But in Corollary 2.2.40 we have seen that there exists a cartesian square of the following form.

$$\begin{array}{ccc} \underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X} & \longrightarrow & \underline{G/K_0} \\ \big\downarrow{\scriptstyle \mathrm{proj}} & & \big\downarrow{\scriptstyle f} \\ \underline{G/H} & \longrightarrow & \underline{G/K_1} \end{array}$$

Let S denote the point-set fibre of the map of topological G–spaces $f\colon G/K_0 \to G/K_1$. Then S is a homogenous K_1-space, and $f = \mathrm{Ind}_{K_1}^G(S \to *)$. Now note that since the right map is a Poincaré duality map, so is the left one, and as $\underline{G/H}$ is G–Poincaré, Proposition 3.3.3 implies that $\underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X}$ is G–Poincaré, as desired. ☐

4.4. Examples

The next paragraphs will introduce two different sources of equivariant Poincaré spaces. First, we show that smooth G-manifolds are equivariantly Poincaré. Their study is one of the main motivations for a theory of equivariant Poincaré duality, and equivariant Poincaré spaces should be viewed as their homotopical analogue. Let us mention that while the proofs given here *depend* on the Wirthmüller isomorphism, the Wirthmüller isomorphism can also be proven using a different version of equivariant Poincaré duality, as is done for example in [MS06].

Our second source of examples are tom Dieck–Petrie's generalised homotopy representations. Here we will find what we consider to be the strangest equivariant Poincaré space we know: a C_p–Poincaré space $\underline{X}$ such that X^{C_p} and X^e are Poincaré of the same dimension, yet the map $X^{C_p} \to X^e$ is not an equivalence, see Example 4.4.10.

A general principle here is that Theorem 4.2.10 provides us with a clear strategy to deduce equivariant Poincaré duality from nonequivariant Poincaré duality of fixed points, provided an appropriate Spivak datum has been constructed.

Smooth G-manifolds

Let G be a compact Lie group. A *smooth G-manifold* is a smooth manifold on which G acts such that the action map $G \times M \to M$ is a smooth map. An *equivariant embedding* of smooth G-manifolds is a smooth embedding between smooth G-manifolds that is also equivariant. An *equivariant vector bundle* on M is a tuple $\xi = (E, p)$, where E is a smooth G–manifold and $p\colon E \to M$ is an equivariant map which is a vector bundle where G acts by bundle maps.

For $x \in M^H$, the vector space $E_x := p^{-1}(x)$ carries an H-action by restriction. Smooth G-manifolds port nicely into our homotopical context by virtue of [Ill83, Cor. 7.2.] which guarantees that smooth G–manifolds admit the structure of G–CW complexes which is necessarily finite for compact manifolds. We recommend [Bre72, Chapter IV] for an introduction to the theory of smooth G-manifolds.

Fact 4.4.1. We collect here some basic facts from equivariant smooth manifold theory that we will need for our purposes.

(i) The tangent bundle of a smooth G-manifold can naturally be considered as an equivariant vector bundle [Bre72, p. 303]. If $f\colon M \to N$ is an equivariant embedding of smooth G-manifolds, then the equivariant tubular neighborhood theorem provides a smooth equivariant embedding of $\nu(f) = f^*TN/TM$ into N [Bre72, Thm. VI.2.2.].

(ii) Let us denote the underlying G-homotopy type of M by $\underline{M}$. Any G-vector bundle $p\colon E \to M$ over M defines a stable equivariant spherical fibration of the G-vector bundle $p\colon E \to M$. Furthermore, we can choose a G–invariant Riemannian metric for p from which we obtain an associated unit disc bundle $D(p) \subset E$ and unit sphere bundle $S(p) \subset E$. The fibrewise collapse maps $\underline{S^{E_x}} \to \mathrm{cofib}(\underline{S(p)}_x \to \underline{D(p)}_x)$ for each $x \in M$ then assemble into a G-equivalence

$$M_!(J(p)) \xrightarrow{\simeq} \Sigma^\infty \,\mathrm{cofib}\,\big[S(p) \to D(p)\big].$$

(iii) For each G-manifold M, there exists an equivariant embedding into some G-representation V. This is the content of the Mostow-Palais theorem, see [Pal57].

Proposition 4.4.2. *Let M be a closed smooth G–manifold. Then the underlying G–space $\underline{M}$ is a G–Poincaré space with dualizing object $J(TM)^{-1}$.*

Proof. Choose an embedding $f\colon M \to V$ into some G-representation. Denote the normal bundle of f by $\nu = (p\colon E \to M)$ and pick a tubular neighborhood of M in V.

Consider the Pontryagin-Thom collapse map

$$\begin{aligned} c\colon \mathbb{S} \xrightarrow{\simeq} \mathbb{S}^V \otimes \mathbb{S}^{-V} &\to \Sigma^\infty \,\mathrm{cofib}\,\big[S^V \setminus (D(\nu)\setminus S(\nu)) \to S^V\big] \otimes \mathbb{S}^{-V} \\ &\simeq \Sigma^\infty \,\mathrm{cofib}\,\big[S(\nu) \to D(\nu)\big] \otimes \mathbb{S}^{-V} \simeq M_!(J(\nu) \otimes \mathbb{S}^{-V}). \end{aligned}$$

We claim that the Spivak datum $(J(\nu) \otimes \mathbb{S}^{-V}, c)$ is Poincaré. Since M is a G–compact space and $J(\nu)$ is invertible, by Theorem 4.2.10, it suffices to check that for every $H \subset G$, the Spivak datum $(\Phi^H J(\nu), \Phi^H c)$ is a Poincaré Spivak datum for M^H. Recall that $\Phi^H J(\nu)$ is

$$\Phi^H J(\nu)\colon M^H \to \mathcal{P}\mathrm{ic}(\mathrm{Sp}), \quad x \mapsto \Phi^H(J(\nu)(x)) = \Phi^H \Sigma^\infty S^{E_x} \simeq \Sigma^\infty S^{E_x^H}.$$

But this is just the underlying stable spherical fibration of the normal bundle of M^H in V^H. The collapse map $\Phi^H c$ identifies with the geometric Pontryagin-Thom collapse map of the smooth manifold M^H embedded in V^H. Thus, by [Lan22, Cor. A.11] the Spivak datum $(\Phi^H J(\nu), \Phi^H c)$ is Poincaré.

Now note that the equivalence $\mathrm{const}_V = TV|_M \simeq \nu \oplus TM$ shows that

$$J(\nu) \otimes \mathbb{S}^{-V} \simeq J(\nu) \otimes J(\mathrm{const}_V)^{-1} \simeq J(\mathrm{const}_V) \otimes J(TM)^{-1} \otimes J(\mathrm{const}_V)^{-1} \simeq J(TM)^{-1}$$

as claimed. □

Remark 4.4.3. We want to mention that versions of Proposition 4.4.2 are already contained in the literature so we do not claim any originality. In particular, May–Sigurdsson give an account of equivariant Poincaré duality and show that closed smooth G-manifolds satisfy Poincaré duality in their sense [MS06, Chapter 18.6.]. Depending on which proof of the Wirthmüller isomorphism the reader has in mind, the reader might complain that the proof of Proposition 4.4.2 is circular, as the Wirthmüller isomorphism for compact Lie groups itself was proved by showing that smooth G-manifolds are G-Poincaré. Another variant of Proposition 4.4.2 was given by Costenoble–Waner, see [CW17].

Generalised homotopy representations

We now turn our attention to another interesting source of equivariant Poincaré duality spaces, namely the class of generalised homotopy representations of tom Dieck–Petrie [DP82].

Definition 4.4.4. A *generalised homotopy representation* of a compact Lie group G is a compact G–space $\underline{\mathcal{V}}$ such that for each closed subgroup $H \leq G$ the space $\mathcal{V}^H$ is equivalent to $S^{n(H)}$ for some $n(H) \in \mathbb{N}$. The function $H \mapsto n(H)$ associated to a generalised homotopy representation is called its *dimension function.*[5]

Examples of generalised homotopy representations are unit spheres of finite dimensional orthogonal G-representations or one–point compactifications of finite dimensional linear G-representations.

Remark 4.4.5. While it will not play a role in this article, let us mention that [DP82; Die86] have also studied what are called *homotopy representations*, namely generalised homotopy representations for which the fixed points have CW–dimensions those of the respective spheres. A special feature of homotopy representations is that they satisfy an equivariant Hopf degree theorem, i.e. G-homotopy classes of self maps are classified by their degree, an element in a Burnside ring.

To show that generalised G-homotopy representations are indeed G–Poincaré, we first recall a construction of a Poincaré Spivak datum for the nonequivariant spheres.

Observation 4.4.6 (Spivak data for spheres)**.** We construct a Spivak datum for $S^d \in \mathcal{S}$. Let $E := \mathrm{fib}(\Sigma^\infty_+ S^d \to \Sigma^\infty_+ * \simeq \mathbb{S})$. Then $E \simeq \mathbb{S}^d \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})$. Consider the composition

$$c\colon \mathbb{S} \xrightarrow{\simeq} E \otimes E^\vee \to \Sigma^\infty_+ S^d \otimes E^\vee \simeq S^d_!(S^d)^* E^\vee.$$

We argue now that $((S^d)^*E^\vee, c)$ is a Poincaré Spivak datum for S^d. As S^d is stably parallelisable, we know that its dualising sheaf is constant with value $\mathbb{S}^{-d} \simeq E^\vee$. Assume $d \geq 1$,

[5] Beware that it is also common in the literature to shift the dimension function by one.

the case $d = 0$ being easier. Now $\pi_0 S^d_!(S^d)^* E^\vee \simeq \mathbb{Z}$, and $c \in \pi_0 S^d_!(S^d)^* E^\vee \simeq \mathbb{Z}$ gives the collapse map of a Poincaré Spivak datum if and only if it corresponds to a generator. This is indeed the case for $((S^d)^* E, c)$, so it is Poincaré as claimed.

Having this in mind, we can make an educated guess for the a Spivak datum of a generalised homotopy representation. To this end, the following terminology will be useful.

Definition 4.4.7. A *homotopical framing* for $\xi \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}})$ is a G–spectrum E together with an equivalence $\xi \xrightarrow{\simeq} X^* E$. A compact G–space $\underline{X}$ is *homotopically parallelisable* if its dualising sheaf $D_{\underline{X}} \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}})$ admits a homotopical framing.

Theorem 4.4.8. *The dualising sheaf of a generalised homotopy representation $\underline{\mathcal{V}}$ admits a canonical homotopical framing $D_{\underline{\mathcal{V}}} \xrightarrow{\simeq} \mathcal{V}^* \operatorname{fib}(\Sigma^\infty_+ \mathcal{V}^\vee \to \Sigma^\infty_+ * \simeq \mathbb{S})^\vee$. In particular, generalised homotopy spheres are homotopically parallelisable G–Poincaré spaces.*

Proof. To prove the theorem, we will construct a Poincaré Spivak datum whose underlying parametrised spectrum is constant with value $E^\vee := \operatorname{fib}(\Sigma^\infty_+ \mathcal{V}^\vee \to \Sigma^\infty_+ * \simeq \mathbb{S})^\vee$. As in Observation 4.4.6, we have a map $c\colon \mathbb{S} \to E \otimes E^\vee \to \Sigma^\infty_+ \mathcal{V} \otimes E^\vee \simeq \mathcal{V}_! \mathcal{V}^* E^\vee$. Upon taking geometric fixed points, Observation 4.4.6 identifies the composition

$$\Phi^H c\colon \Phi^H \mathbb{S} \to \Phi^H E \otimes \Phi^H E^\vee \to \Phi^H \Sigma^\infty_+ \underline{\mathcal{V}} \otimes \Phi^H E^\vee \simeq \mathcal{V}^H_! (\mathcal{V}^H)^* \Phi^H E^\vee.$$

as a Poincaré Spivak datum for $\mathcal{V}^H$. Thus, by Theorem 4.2.10, we get that $\mathcal{V}^* E^\vee$ is a Poincaré G–Spivak datum for $\underline{\mathcal{V}}$ and by Proposition 3.2.5 we get $D_{\underline{\mathcal{V}}} \simeq \mathcal{V}^* E^\vee$ as claimed. ☐

Lemma 4.4.9. *Suppose that $X \in \mathcal{S}^\omega_G$ is homotopically parallelisable and that X^H is a Poincaré space for all $H \leq G$. Then X is a G–Poincaré space.*

Proof. Since $\underline{X}$ was compact, note that $X_! D_{\underline{X}} \simeq X_* X^* \mathbb{S}_G$ is a compact G–spectrum. Now suppose that there is $E \in \mathrm{Sp}_G$ such that $D_{\underline{X}} \simeq X^* E$. As E is a retract of $X_! D_{\underline{X}} \simeq \Sigma^\infty_+ X \otimes E$ this implies that E is compact itself. If all fixed points of X are Poincaré spaces, then all geometric fixed points of E are invertible. Together this shows that E is invertible so that X is a G–Poincaré space. ☐

Example 4.4.10. In [Bre72, p. 391], Bredon constructs a curious example of a generalised homotopy representation. Namely, he constructs examples of compact C_p-spaces $\underline{X}$ which satisfy that $X^{C_p} \simeq X^e \simeq S^2$ such that the map $X^{C_p} \to X^e$ has degree $q = kp + 1$ for $k \in \mathbb{Z}$ arbitrary. Taking the unreduced suspension, examples of this type exist in arbitrary dimensions. From Smith theory we know that each generalised C_p-homotopy representation has the property that the dimension of the fixed point sphere does not exceed the dimension of the underlying space.

4.5. Gluing classes

Our next goal is to hint at nontrivial ways in which the fixed points interact. For this, we construct a certain homology class, the *gluing class*, that should be thought of as passing information between the fundamental class of a Poincaré space and fundamental classes of various fixed point spaces. The gluing class will be one of the main tools for our geometric applications. It is inspired by Lück's work on the Nielsen realisation problem, specifically by [Lüc22, Notation 1.8 (H) and Lemma 1.8 (5)]. Much of what we will present here will work for compact Lie groups too, but we nevertheless restrict our attention to finite groups G for this subsection which is sufficient for our geometric purposes later.

Construction 4.5.1 (Nonsingular part)**.** Fix $\underline{X} \in \mathcal{S}_G$, $\mathcal{F}$ a family of subgroups of G, and $\underline{\mathcal{C}}$ a G–stable category. Recall the adjunction counit $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ from Construction 2.2.15. This map then itself induces the adjunction $\epsilon_!\colon \underline{\mathrm{Fun}}(\underline{X}_{\widetilde{\mathcal{F}}}, \underline{\mathcal{C}}) \rightleftharpoons \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) :\epsilon^*$. The adjunction (co)unit of *this* adjunction then gives us functors

$$\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{c} \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})^{\Delta^1} :: \xi \mapsto (\epsilon_!\epsilon^*\xi \to \xi), \quad \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{u} \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})^{\Delta^1} :: \xi \mapsto (\xi \to \epsilon_*\epsilon^*\xi).$$

All in all, we can consider the compositions

$$\alpha\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{c} \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})^{\Delta^1} \xrightarrow{X_!} \underline{\mathcal{C}}^{\Delta^1} \xrightarrow{\mathrm{cofib}} \underline{\mathcal{C}}, \quad \beta\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{u} \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})^{\Delta^1} \xrightarrow{X_*} \underline{\mathcal{C}}^{\Delta^1} \xrightarrow{\mathrm{fib}} \underline{\mathcal{C}}.$$

Concretely, these take ξ to the objects

$$\alpha(\xi) \simeq \mathrm{cofib}\Big((X_{\widetilde{\mathcal{F}}})_!\epsilon^*\xi \longrightarrow X_!\xi\Big), \qquad \beta(\xi) \simeq \mathrm{fib}\Big(X_*\xi \longrightarrow (X_{\widetilde{\mathcal{F}}})_*\epsilon^*\xi\Big).$$

Corollary 4.5.2 (Nonsingular vanishing)**.** *Let $\underline{X} \in \mathcal{S}_G^{\omega}$ and $\mathcal{F}$ a family of subgroups of G. Let $\nu\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a G–exact functor of G–stable categories such that for all $H \in \mathcal{F}$, functor $\mathrm{Res}^G_H \nu\colon \mathrm{Res}^G_H \underline{\mathcal{C}} \to \mathrm{Res}^G_H \underline{\mathcal{D}}$ is the zero map. Then the compositions*

$$\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{\alpha} \underline{\mathcal{C}} \xrightarrow{\nu} \underline{\mathcal{D}} \qquad \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{\beta} \underline{\mathcal{C}} \xrightarrow{\nu} \underline{\mathcal{D}}$$

have the property of being the zero functors.

Proof. First of all, since ν was G–exact, we have a commuting square

$$\begin{array}{ccc} \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) & \xrightarrow{\nu} & \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \underline{\mathcal{C}} & \xrightarrow{\nu} & \underline{\mathcal{D}}. \end{array}$$

Thus it suffices to show that $\alpha\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}}) \to \underline{\mathcal{D}}$ is the zero functor. By replacing $\underline{\mathcal{D}}$ by the G-stable subcategory generated by the image of ν we can assume that that $\mathcal{D}^H = 0$ for all $H \in \mathcal{F}$. Therefore, we have that $\underline{\mathcal{D}} \simeq \underline{\mathcal{D}}^{\Phi\widetilde{\mathcal{F}}}$ and Lemma 4.2.3 shows that the functor ϵ^* is an equivalence, and so the counit $\epsilon_!\epsilon^* \to \mathrm{id}$ and unit $\mathrm{id} \to \epsilon_*\epsilon^*$ are equivalences in $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}}) = \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}}^{\Phi\widetilde{\mathcal{F}}})$. From this the claim directly follows. ☐

Notation 4.5.3. The family of relevance to us in this subsection will be the singleton family $\mathcal{T}$ consisting of the trivial subgroup. To reduce our notational cluttering, we will also write $\underline{X}^{>1}$ for $\underline{X}_{\widetilde{\mathcal{T}}}$, so that for example, for $\underline{X} \in \mathcal{S}_G$, we have the inclusion of the singular part $\epsilon\colon \underline{X}^{>1} \simeq \underline{X}_{\widetilde{\mathcal{T}}} \to \underline{X}$. The gluing class of $\underline{X}$ will live in $\pi_{-1}(X^{>1}_!\epsilon^* D_{\underline{X}})_{hG}$.

Construction 4.5.4. Let $\xi \in \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}})$ and write $Q := \mathrm{cofib}\left(X_!^{>1}\epsilon^*\xi \to X_!\xi\right)$. Consider

$$\begin{array}{ccccccc}
(X_!^{>1}\epsilon^*\xi)_{hG} & \longrightarrow & (X_!^{>1}\epsilon^*\xi)^{hG} & \longrightarrow & (X_!^{>1}\epsilon^*\xi)^{tG} & \color{blue}{\longrightarrow} & \Sigma(X_!^{>1}\epsilon^*\xi)_{hG} \\
\downarrow & \ulcorner & \downarrow & & \color{blue}{\downarrow \simeq} & & \\
(X_!\xi)_{hG} & \longrightarrow & (X_!\xi)^{hG} & \color{blue}{\longrightarrow} & (X_!\xi)^{tG} & & \\
\downarrow & & \color{red}{\downarrow} & & & & \\
Q_{hG} & \color{red}{\xrightarrow{\simeq}} & Q^{hG} & & & & \\
\color{red}{\downarrow} & & & & & & \\
\Sigma(X_!^{>1}\epsilon^*\xi)_{hG} & & & & & &
\end{array} \tag{19}$$

where the equivalence $Q_{hG} \to Q^{hG}$ is since $Q^{tG} \simeq 0$ by virtue of Corollary 4.5.2 applied to the functor $\nu\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ given by $\widetilde{EG} \otimes F(EG_+, -)\colon \underline{\mathrm{Sp}} \to \underline{\mathrm{Mod}}_{\widetilde{EG}\otimes F(EG_+,\mathbb{S})}(\underline{\mathrm{Sp}})$ and the identification $(\widetilde{EG} \otimes F(EG_+, A))^G \simeq A^{tG}$. Observe that by Lemma B.0.1, up to a sign change, the red composite is equivalent to the blue composite in (19).

Construction 4.5.5 (Gluing classes)**.** Let $\underline{X} \in \mathcal{S}_G^{\omega}$ and $D_{\underline{X}} \in \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}})$ its dualising sheaf (which in this generality, need not be invertible). From the fundamental class $\mathbb{S}_G \xrightarrow{c} X_!D_{\underline{X}}$ in Sp_G, we may extract a nonequivariant fundamental class $\mathbb{S} \xrightarrow{\mathrm{can}} \mathbb{S}_G^{hG} \xrightarrow{c^{hG}} (X_!D_{\underline{X}})^{hG}$ in Sp which we also denote by c. The *gluing class* is defined to be the composition

$$\mathbb{S} \xrightarrow{c} (r_!D_{\underline{X}})^{hG} \longrightarrow \Sigma(X_!^{>1}\epsilon^*D_{\underline{X}})_{hG}$$

obtained by postcomposing c with the blue route from (19).

Our goal now is to show Corollary 4.5.7 which says that under certain orientability assumptions, the gluing class "adds up to zero" in group homology. This supplies us with a useful obstruction class which will have meaningful geometric consequences as we shall in our applications in §5.2.

Lemma 4.5.6. *Let $\underline{X} \in \mathcal{S}_G$ and $\xi \simeq X^*W \in \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}})$ for some $W \in \mathrm{Sp}_G$. Then the composition in $Q \longrightarrow \Sigma r_!^{>1}\epsilon^*\xi \simeq \Sigma X_!^{>1}(X^{>1})^*\xi \xrightarrow{\mathrm{BC}_!^{X^{>1}}} \Sigma W$ in Sp_G is nullhomotopic.*

Proof. By functoriality of colimits, we have the following map of cofibre sequences

$$\begin{array}{ccccc}
X_!^{>1}\epsilon^*X^*W & \xrightarrow{\mathrm{BC}_!^{\epsilon}} & X_!X^*W & \longrightarrow & Q \\
{\scriptstyle \mathrm{BC}_!^{X^{>1}}}\downarrow & & \downarrow{\scriptstyle \mathrm{BC}_!^{X}} & & \downarrow \\
W & = & W & \longrightarrow & 0
\end{array}$$

Thus taking the cofibre of the right horizontal maps gives a factorisation of the composition of interest through 0. ☐

Corollary 4.5.7. *Let $\underline{X} \in \mathcal{S}_G$ and $W \in \mathrm{Sp}_G$. Then the composition*

$$(X_! X^* W)^{hG} \xrightarrow{\text{red composite in (19)}} \Sigma(X_!^{>1}(X^{>1})^* W)_{hG} \xrightarrow{\mathrm{BC}_!^{X^{>1}}} \Sigma W_{hG}$$

is nullhomotopic.

Proof. This is an immediate combination of the fact from Construction 4.5.4 that the red and blue routes in (19) agrees up to a sign with Corollary 4.5.7. □

Remark 4.5.8. The gluing class really is an essential feature of equivariant Poincaré duality. It provides some information on how the "free part" of an equivariant Poincaré space is glued to the singular part. We will exploit it in the proof of Theorem 5.2.2 and plan on clarifying it and its role in relation to Lück's work [Lüc22] in the future.

4.6. Equivariant degree theory

A nice application of equivariant Poincaré duality is a theory of equivariant mapping degrees, as developed in [Lüc88]. For simplicity, we will assume that G is a finite group throughout this section.

Recollections on the Burnside ring

Our aim is to remind the reader of the classical connection between Burnside rings and the equivariant sphere spectrum.

Recollections 4.6.1 (Character maps on the Burnside ring)**.** The *Burnside ring of finite G-sets* $A(G)$ is the group–completion of the semiring of isomorphism classes of finite G-sets, with disjoint union as addition and the cartesian product as product. For each $H \leq G$, there is a unique ring homomorphism $\chi_H \colon A(G) \to \mathbb{Z}$ sending a finite G-set S to the order of the finite set S^H, and these assemble into a ring map, called the *character map*,

$$\chi\colon A(G) \to \prod_{(H)\leq G} \mathbb{Z} \tag{20}$$

where (H) runs through all conjugacy classes of finite subgroups.

The following classical theorem may be found for instance in [Die79, Prop. 1.3.5.].

Theorem 4.6.2. *The character map (20) is an injective ring homomorphism with finite cokernel. The image can be described through explicit congruences, the Burnside congruences.*

We abstain from recalling the Burnside congruences in full generality, the reader may find them in the reference mentioned above. To give some intuition, and as we use it later in the proof of Lemma 5.1.12, we describe them in the case of the group C_p.

Example 4.6.3. If $G = C_p$ then the image of the character homomorphism consists of those pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ that satisfy the congruence

$$a \equiv b \mod p.$$

Indeed, for a finite G-set S, the orders of S and S^H agree modulo p. On the other hand, if $a + kp = b$ for integers a, b, c, then a copies of the point and k copies of C_p define an element in $A(G)$ mapping to (a, b).

Construction 4.6.4. We may obtain a similar character map for the ring $\pi_0^G \mathbb{S}_G$: for each subgroup $H \leq G$, using that $\Phi^H \mathbb{S}_G \simeq \mathbb{S} \in \mathrm{Sp}$, we may assemble the geometric fixed points functors Φ^H together with the identification $\deg\colon \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\mathbb{S}, \mathbb{S}) \xrightarrow{\cong} \mathbb{Z}$ to obtain a ring map

$$\pi_0^G \mathbb{S}_G \cong \pi_0 \operatorname{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, \mathbb{S}_G) \longrightarrow \prod_{(H)} \mathbb{Z}, \quad f \mapsto \deg(\Phi^H f) \tag{21}$$

Theorem 4.6.5 (Segal). *Let G be a finite group. The map (21) is an injective ring homomorphism whose image agrees with the image of the character map $\chi\colon A(G) \to \prod_{(H)} \mathbb{Z}$, yielding an identification $\pi_0^G \mathbb{S}_G \cong \pi_0 \operatorname{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, \mathbb{S}_G) \cong A(G)$ as commutative rings.*

Of course, this implies that the set of path components of the selfmaps of any $E \in \mathcal{P}\mathrm{ic}(\mathrm{Sp}_G)$ is equivalent to $A(G)$ by the equivalence $\operatorname{Map}_{\mathrm{Sp}_G}(E, E) \simeq \operatorname{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, \mathbb{S}_G)$.

The equivariant degree

We now want to specialise the abstract definition of the degree from §3.4 to the case of maps of G–Poincaré spaces which should roughly encode the mapping degrees on the various fixed points spaces. Recall that the definition of the degree of a map between G–spaces $f\colon \underline{X} \to \underline{Y}$ with Spivak data $(\xi_{\underline{X}}, c_{\underline{X}})$ and $(\xi_{\underline{Y}}, c_{\underline{Y}})$ depends on an equivalence $\xi_{\underline{X}} \xrightarrow{\simeq} f^*\xi_{\underline{Y}}$. The existence of such an equivalence is unreasonable to expect with coefficients in Sp_G but becomes more likely after linearising. Here we choose to work with coefficients in the Burnside Mackey functor $\underline{A}(G)$. Recall that for each subgroup $H \subset G$, restriction defines a ring homomorphism $A(G) \to A(H)$ and induction a transfer map $A(H) \to A(G)$. These assemble into a Mackey functor, and hence a G–spectrum $\underline{A}(G)$ which has values $\underline{A}(G)^H = A(H)$.

Definition 4.6.6. Let $\underline{X}$ and $\underline{Y}$ be Poincaré G–spaces. An *$\underline{A}(G)$–degree datum* is a pair (f, ψ) where $f\colon \underline{X} \to \underline{Y}$ is a map of G–spaces and $\psi\colon D_{\underline{X}} \otimes \underline{A}(G) \xrightarrow{\simeq} f^* D_{\underline{Y}} \otimes \underline{A}(G)$ is an equivalence of the $\underline{A}(G)$–linearised dualising sheaves.

In other words, a $\underline{A}(G)$–degree datum is a $\underline{\mathrm{Mod}}_{\underline{A}(G)}(\underline{\mathrm{Sp}})$–degree datum in the sense of Definition 3.4.2.

Definition 4.6.7. Let $\underline{X}, \underline{Y} \in \mathcal{S}_G$ be G–Poincaré and (f, ψ) a $\underline{A}(G)$–degree datum. We define the *equivariant degree* $\deg_G(f, \psi) \in \pi_0 \operatorname{Map}(\mathbb{S}_G, Y_* Y^* \underline{A}(G)) =: H^0(\underline{Y}; \underline{A})$ as the composite

$$\mathbb{S}_G \xrightarrow{c_X} X_!(D_{\underline{X}} \otimes \underline{A}(G)) \xrightarrow{\psi} X_!(f^* D_{\underline{Y}} \otimes \underline{A}(G)) \xrightarrow{\mathrm{BC}_!^f} Y_!(D_{\underline{Y}} \otimes \underline{A}(G)) \xleftarrow[\simeq]{c_Y \cap -} Y_* Y^* \underline{A}(G).$$

As explained in Construction 3.4.4, the commutative algebra structure on $Y_*Y^*\underline{A}(G)$ endows $H^0(\underline{Y};\underline{A})$ with the structure of a commutative ring with unit c_Y.

Our goal for the rest of this discussion is to relate the equivariant degree of a map, which lives in $H^0(\underline{X};\underline{A})$, to the various degrees induced on fixed points via a character map similar to (20) constructed from the geometric fixed points functors. To this end, first recall the Bousfield localisation $\pi_0\colon \mathcal{S}_G \rightleftharpoons \mathrm{Set}_G : \mathrm{incl}$ from Construction 5.2.10. Notice that $\Omega^\infty\underline{A}(G)$ is levelwise 0–truncated with fixed points $(\Omega^\infty\underline{A}(G))^H = A(H)$.

Lemma 4.6.8. *For $\underline{X} \in \mathcal{S}_G$, we have an equivalence $H^0(\underline{X};\underline{A}) \simeq \pi_0 \operatorname{Map}_{\mathcal{S}_G}(\tau_{\leq 0}\underline{X}, \Omega^\infty\underline{A}(G))$.*

Proof. Consider the computation of $H^0(\underline{X};\underline{A})$ as

$$\pi_0 \operatorname{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, X_*X^*\underline{A}(G)) \simeq \operatorname{Map}_{\mathrm{Sp}_G}(\Sigma^\infty_+\underline{X}, \underline{A}(G)) \simeq \operatorname{Map}_{\mathcal{S}_G}(\tau_{\leq 0}\underline{X}, \Omega^\infty\underline{A}(G))$$

where the last equivalence uses that $\Omega^\infty\underline{A}(G)$ is levelwise 0-truncated. ☐

Remark 4.6.9. This turns out to be quite simple to compute. Note that for two 0-truncated G–spaces $\underline{S}$ and $\underline{T}$, the map

$$\operatorname{Map}_{\mathcal{S}_G}(\underline{S},\underline{T}) \longrightarrow \prod_{(H)\leq G} \operatorname{Map}_{\mathcal{S}}(S^H, T^H) \simeq \operatorname{Map}_{\mathrm{Set}}(S^H, T^H),$$

is injective with image given by all collections of maps $(f^H)_{(H)}$ compatible with the restrictions coming from inclusions $K \leq H$ or inner automorphsim $K \simeq H$. Specialising this to the case of interest, we obtain an injection

$$H^0(\underline{X};\underline{A}) \hookrightarrow \prod_{(H)} \left(A(H)^{\pi_0(X^H)}\right)^{W_G H}.$$

For example, we have $\operatorname{Map}(\tau_{\leq 0}\underline{X}, \Omega^\infty\underline{A}(G)) \simeq A(G)$ if all fixed point sets of $\underline{X}$ are nonempty and connected. For a more complicated example, consider the C_2-action on S^1 given by complex conjugation. Then $(\tau_{\leq 0}\underline{X})^{C_2} \simeq * \coprod *$ while $(\tau_{\leq 0}\underline{X})^e \simeq *$. The set of equivalence classes of maps above identifies with the pullback $A(C_2) \times_{A(1)} A(C_2)$ where the two maps $A(C_2) \to A(1)$ are given by restriction along the group homomorphism $1 \to C_2$.

Recovering degrees on fixed points

Now we want to recover different degrees on fixed point spaces from the equivariant degree by base changing along the geometric fixed points functor. As it is not true that $\Phi^G\underline{A}(G) = \mathbb{Z}$, we need a small preparatory lemma. For this, denote by $\mathrm{Sp}_G^{\geq 0}$ (resp. $\mathrm{Sp}_G^{\leq 0}$) the full subcategory of all G–spectra X for which $X^H \in \mathrm{Sp}$ is connective (resp. coconnective) for each $H \leq G$. The pair $(\mathrm{Sp}_G^{\geq 0}, \mathrm{Sp}_G^{\leq 0})$ forms a t-structure on on Sp_G.

Lemma 4.6.10 (Geometric fixed points of Mackey functors)**.** *Let $X \in \mathrm{Sp}_G^{\geq 0}$. Then the canonical map $X \to \tau_{\leq 0}X$ induces an isomorphism $\pi_0\Phi^G X \xrightarrow{\cong} \pi_0\Phi^G\tau_{\leq 0}X$ of abelian groups.*

Proof. Recall that the geometric fixed points participates in an adjunction $\Phi^G \colon \mathrm{Sp}_G \rightleftharpoons \mathrm{Sp} : \Xi^G$ where the right adjoint Ξ^G is fully faithful and is given by the formula

$$(\Xi^G Y)^H = \begin{cases} Y & \text{if } H = G; \\ 0 & \text{if } H \lneq G. \end{cases}$$

Observe that Φ^G preserves connective objects. This is because Φ^G sends $\Sigma^\infty_+ G/G$ to $\mathbb{S}$ and $\Sigma^\infty_+ G/H$ to 0 for $H \lneq G$. Since connective G–spectra are built as colimits of the orbits $\{\Sigma^\infty_+ G/H\}_{H \leq G}$ and Φ^G preserves colimits, we see that connective G–spectra are sent to connective spectra. The formula for Ξ^G shows that it preserves connective and coconnective objects. In particular, both restrict to functors $\Xi^G \colon \mathrm{Sp}^\heartsuit \hookrightarrow \mathrm{Sp}^\heartsuit_G$ and $\tau_{\leq 0}\Phi^G \colon \mathrm{Sp}^\heartsuit_G \to \mathrm{Sp}^\heartsuit$ and we claim that those are adjoint. To see this, let $\underline{M} \in \mathrm{Sp}^\heartsuit_G$ and $N \in \mathrm{Sp}^\heartsuit$, and consider

$$\mathrm{Map}_{\mathrm{Sp}^\heartsuit}(\tau_{\leq 0}\Phi^G \underline{M}, N) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Phi^G \underline{M}, N) \simeq \mathrm{Map}_{\mathrm{Sp}_G}(\underline{M}, \Xi^G N) \simeq \mathrm{Map}_{\mathrm{Sp}^\heartsuit_G}(\underline{M}, \Xi^G N).$$

To conclude, since the solid square in

$$\begin{array}{ccc} \mathrm{Sp}^{\geq 0}_G & \underset{\Xi^G}{\overset{\Phi^G}{\rightleftarrows}} & \mathrm{Sp}^{\geq 0} \\ {\scriptstyle \tau_{\leq 0}} \downarrow \uparrow & & {\scriptstyle \tau_{\leq 0}} \downarrow \uparrow \\ \mathrm{Sp}^\heartsuit_G & \underset{\Xi^G}{\overset{\tau_{\leq 0}\Phi^G}{\rightleftarrows}} & \mathrm{Sp}^\heartsuit \end{array}$$

commutes, so does the dashed square of left adjoints, as was to be shown. ☐

Remark 4.6.11. Theorem 4.6.5 gives an equivalence $\tau_{\leq 0}\mathbb{S}_G = \underline{A}(G)$. By Lemma 4.6.10, we have $\pi_0 \Phi^G \underline{A}(G) = \pi_0 \Phi^G \tau_{\leq 0}\mathbb{S}_G \cong \pi_0 \Phi^G \mathbb{S}_G \cong \pi_0 \mathbb{S} \cong \mathbb{Z}$. Now note that the diagram

$$\begin{array}{ccccc} \pi_0(\mathbb{S}_G)^G & \xrightarrow{\chi^H} & \pi_0 \Phi^G \mathbb{S}_G & \xrightarrow{\simeq} & \pi_0 \tau_{\leq 0} \Phi^G \mathbb{S}_G \simeq \mathbb{Z} \\ \downarrow \cong & & \downarrow \simeq & & \downarrow \simeq \\ \pi_0 \underline{A}(G)^G & \longrightarrow & \pi_0 \Phi^G \underline{A}(G) & \longrightarrow & \pi_0 \tau_{\leq 0} \Phi^G \underline{A}(G) \simeq \mathbb{Z} \end{array} \tag{22}$$

commutes. The lower horizontal composition thus agrees with the character map.

Now we come back to the problem of relating the equivariant degree to the degree on each fixed point space. We have the symmetric monoidal colimit preserving functor

$$\begin{aligned} \phi^H \colon \underline{\mathrm{Mod}}_{\underline{A}(G)}(\underline{\mathrm{Sp}}_G) &\xrightarrow{\Phi^H} \mathrm{Coind}^G_H \mathrm{Coinfl}^1_H \mathrm{Mod}_{\Phi^H \underline{A}(H)}(\mathrm{Sp}) \\ &\to \mathrm{Coind}^G_H \mathrm{Coinfl}^1_H \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}) \end{aligned}$$

where Φ^H is the parametrised geometric fixed point functor constructed in Construction 2.2.31 and the second map is induced by the ring map $\Phi^H \underline{A}(H) \to \tau_{\leq 0}\Phi^H \underline{A}(H) \simeq \mathbb{Z}$.

Proposition 4.6.12. *For a G–Poincaré $\underline{Y} \in \mathcal{S}_G$, basechange along ϕ^H induces a ring map*

$$\chi^H \colon H^0(\underline{Y}; \underline{A}) \longrightarrow H^0(Y^H, \mathbb{Z}).$$

Given a degree datum $(f \colon \underline{X} \to \underline{Y}, \psi)$, we have

$$\chi^H(\deg_{\underline{A}(G)}(f, \psi)) = \deg_{\mathbb{Z}}(f^H \colon X^H \to Y^H, \psi^H).$$

For $\underline{Y} = \underline{*}$, χ^H agrees with the character map from (20).

Proof. Note that we have identifications

$$\begin{aligned}\operatorname{Fun}_G(\underline{Y}, \operatorname{Coind}_H^G \operatorname{Coinfl}_H^1 \operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})) &\simeq \operatorname{Fun}_H(\operatorname{Res}_H^G \underline{Y}, \operatorname{Coinfl}_H^1 \operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})) \\ &\simeq \operatorname{Fun}(Y^H, \operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})).\end{aligned}$$

Recall from Observation 4.2.2 that this identifies $Y_!$ with $Y_!^H$ (and also Y_* with Y_*^H as $\underline{Y}$ is Poincaré). Now applying Lemma 3.4.8 to basechange along ϕ^H, we obtain a ring map

$$H^0(\underline{Y}; \underline{A}) = \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\underline{A}(G)}(\mathrm{Sp}_G)}(\mathbb{1}, Y_* Y^* \mathbb{1}) \to \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})}(\mathbb{1}, Y_*^H Y^{H*} \mathbb{1}) \simeq H^0(Y^H, \mathbb{Z}).$$

The statment about the degrees follows from Lemma 3.4.8. In the case $\underline{Y} = \underline{*}$, this map identifies with the character map by (22). □

In the next corollary, we unravel Proposition 4.6.12 in a special case to illustrate how it can be used to deduce condruences between (nonequivariant) degrees between fixed point sets.

Corollary 4.6.13 (Congruences between degrees on fixed point sets)**.** *Suppose that $\underline{Y}$ is a G–Poincaré space and assume that Y^H is nonempty connected for all $H \leq G$. Given a degree datum $(f \colon \underline{X} \to \underline{Y}, \psi)$, the collection $(\deg_{\mathbb{Z}}(f^H, \psi^H))_{(H)}$ lies in the image of the character map*

$$\chi \colon A(G) \to \prod_{(H)} \mathbb{Z}.$$

Proof. Any map $f \colon \underline{X} \to \underline{Y}$ of G–Poincaré spaces induces a commutative diagram

$$\begin{array}{ccc} \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\underline{A}(G)}(\mathrm{Sp}_G)}(\mathbb{1}, Y_* Y^* \mathbb{1}) & \xrightarrow{\chi^H} & \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})}(\mathbb{1}, Y_*^H Y^{H*} \mathbb{1}) \\ \big\downarrow {\scriptstyle \mathrm{BC}_*} & & \big\downarrow {\scriptstyle \mathrm{BC}_*} \\ \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\underline{A}(G)}(\mathrm{Sp}_G)}(\mathbb{1}, X_* X^* \mathbb{1}) & \xrightarrow{\chi^H} & \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})}(\mathbb{1}, X_*^H X^{H*} \mathbb{1}). \end{array}$$

Applying this to the unique map $\underline{Y} \to \underline{*}$, the vertical maps become equivalences by the assumption on the fixed points of $\underline{Y}$. By Proposition 4.6.12, this $\chi^H \colon H^0(\underline{Y}; \underline{A}) \to \mathbb{Z}$ identifies with the character map $\chi^H \colon A(G) = H^0(\underline{G/G}; \underline{A}) \to \mathbb{Z}$. By Proposition 4.6.12, this $\chi^H \colon H^0(\underline{Y}; \underline{A}) \to \mathbb{Z}$ identifies with the character map $\chi^H \colon A(G) = H^0(\underline{G/G}; \underline{A}) \to \mathbb{Z}$. The statement about the degrees is now a consequence of Proposition 4.6.12. □

5. Equivariant Poincaré duality: applications

In this section, we employ the general theory developed in the article to investigate some problems of an equivariant geometric topological nature. In §5.1, we study cohomological injectivity statements for degree one maps and prove Theorem 5.1.1 along these lines; we then use it to obtain a rigidity result of equivariant Poincaré spaces in Theorem 5.1.14. Next, in §5.2, we prove the equivariant Poincaré generalisation of Atiyah–Bott and Conner–Floyd's theorem on single fixed points for group actions on smooth manifolds.

5.1. Pulling back twisted fixed points

Let $f\colon M \to N$ be a map closed, connected, oriented manifolds of the same dimension d. If the degree of f is nonzero, then f is surjective. The theory of equivariant degrees immediately gives an equivariant application: if $f\colon M \to N$ is a map of closed, conneced, smooth, oriented C_p-manifolds with connected fixed point sets of the same dimension, then if f is of degree coprime to p (when considered as a nonequivariant map), then also the degree of f^{C_p} is coprime to p. Thus, f^{C_p} is surjective as well.

To detect if the degree of f is coprime to p it of course suffices to check that $H^*(f^{C_p};\mathbb{F}_p)$ is nonzero in the top degree. This line of thought led Browder [Bro87] to interpret results about the injectivity of $H^*(f^{C_p};\mathbb{F}_p)$ as the "ability to pull back fixed points from N to fixed points of M". Browder's strategy is very successful to show actual surjectivity results on fixed points, even if one relaxes the conditions like smoothness, or takes G to be a more general group like an abelian p-group. Let us mention [HP06] for more information, and many interesting variations on this approach. In particular, see [HP06, Thm. 4] to see how to pass from cohomological injectivity results to surjectivity on fixed points. We will content ourselves with showing how our methods can be used to derive cohomological injectivity results of the following type, which was first studied by Bredon [Bre73] and generalised by Browder [Bro87] under stronger manifold assumptions:

Theorem 5.1.1 (Twisted Bredon–Browder injection)**.** *Let A be an elementary abelian p–group $C_p^{\times r}$. Let $f\colon \underline{X} \to \underline{Y}$ be a map of compact A–spaces. Suppose X^e, Y^e are $\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p}$–Poincaré spaces such that $f^e\colon X^e \to Y^e$ is of $\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p}$–degree one (c.f. Definition 3.4.2). Then for any $\zeta \in \mathrm{Fun}(Y^A, \mathrm{Perf}_{\mathrm{H}\mathbb{F}_p})$, the map induces an injection $H^*(Y^A;\zeta) \to H^*(X^A; f^*\zeta)$.*

Our approach is by default homotopical, and point-set techniques are avoided. As our systematic treatment of Poincaré duality allows us to also derive consequences for homology with twisted coefficients, orientability assumptions may even be relaxed. We will also illustrate the usefulness of Browder's cohomological injectivity results to prove a structural result for G-Poincaré spaces where G is a solvable finite group as Theorem 5.1.14: if X^e is contractible, then so is X^H for any $H \leq G$. This in turn can be applied to give an example of a compact C_p-space all of whose fixed points are Poincaré spaces, while itself not being C_p-Poincaré.

The basic philosophy of our proof is similar to that of [HP06], namely, we proceed via equivariant localisations using the *proper Tate construction*. To this end, we first show that the fixed points of an equivariant space which is underlying Poincaré may most naturally be viewed as a Poincaré space with coefficients in the *stable module category*, which we now recall.

Construction 5.1.2 (Proper stable module categories)**.** Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ and G be a finite group. Consider the G–stable category $\underline{\mathrm{Bor}}(\mathrm{Perf}_R) \in \mathrm{Cat}_G^{G-\mathrm{st}}$ with value $\mathrm{Fun}(BH, \mathrm{Perf}_R)$ at G/H. Recall the notion and notations of Brauer quotients from §2.2. Using the family of proper subgroups $\mathcal{P}$ of G, we may construct a new G–stable category $\underline{\mathrm{stmod}}^{\mathcal{P}}(R) \in \mathrm{Cat}_G^{G-\mathrm{st}}$ defined as $s_*\widetilde{s}^*\underline{\mathrm{Bor}}(\mathrm{Perf})$. This G–category has value

$$\mathrm{stmod}_G^{\mathcal{P}}(R) \coloneqq \mathrm{Fun}(BG, \mathrm{Perf}_R)/\langle R[G/H] \mid H \lneq G\rangle$$

at G/G and is trivial elsewhere. Furthermore, there exists a G–exact symmetric monoidal functor $\Phi\colon \underline{\mathrm{Bor}}(\mathrm{Perf}_R) \longrightarrow \underline{\mathrm{stmod}}^{\mathcal{P}}(R)$.

Construction 5.1.3 (Descending Poincaré duality from large to small coefficients)**.** Let $\underline{X} \in \mathcal{S}_G^{\omega}$ such that X^e is an R–Poincaré space. Since $\underline{X}$ was a compact G–space, the adjunctions $X_! \dashv X^* \dashv X*\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Mod}_R)) \rightleftharpoons \underline{\mathrm{Bor}}(\mathrm{Mod}_R)$ restrict to adjunctions $X_! \dashv X^* \dashv X_*\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Perf}_R)) \rightleftharpoons \underline{\mathrm{Bor}}(\mathrm{Perf}_R)$ on the full subcategories. Now by Proposition 4.3.6, we know that $\underline{X}$ is $\underline{\mathrm{Bor}}(\mathrm{Mod}_R)$–Poincaré and we write $D_{\underline{X}} \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathcal{P}\mathrm{ic}_R))$ for the dualising sheaf. Since $D_{\underline{X}} \in \underline{\mathrm{Bor}}(\mathrm{Perf}_R)$, we even obtain an equivalence

$$X_*(-) \simeq X_!(D_{\underline{X}} \otimes -)\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Perf}_R)) \longrightarrow \underline{\mathrm{Bor}}(\mathrm{Perf}_R)$$

and so $\underline{X}$ is also $\underline{\mathrm{Bor}}(\mathrm{Perf}_R)$–Poincaré. We write $D_X^G\colon X^G \to \mathrm{Perf}_R^{BG}$ for the dualising sheaf evaluated at the fixed points.

Via this construction, we may now prove the following as a simple consequence.

Proposition 5.1.4 (Proper stable module Poincaré duality)**.** *Let G be a finite group and $R \in \mathrm{CAlg}(\mathrm{Sp})$. If $\underline{X} \in \mathcal{S}_G^{\omega}$ such that the underlying space X^e is an R–Poincare duality space, then X^G is a partial $\mathrm{stmod}_G^{\mathcal{P}}(R)$–Poincare duality space, i.e. for any $\zeta \in \mathrm{Fun}(X^G, \mathrm{Perf}_R^{BG})$, we have an equivalence in $\mathrm{stmod}_G^{\mathcal{P}}(R)$*

$$\Phi c \cap_{\Phi D_X^G} \Phi\zeta\colon X_*^G(\Phi\zeta) \xrightarrow{\sim} X_!^G(\Phi D_X^G \otimes \Phi\zeta)$$

Proof. By Construction 5.1.3, we know that $\underline{X}$ is $\underline{\mathrm{Bor}}(\mathrm{Perf}_R)$–Poincaré duality. The statement of the proposition is now an immediate consequence of Theorem 4.2.7 (2). ☐

Remark 5.1.5. While the proposition above looks restrictive and artificial, it already contains some interesting content since the map $\Phi\colon\ \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Perf}_R)) \to \mathrm{Fun}(X^G, \mathrm{stmod}_G^{\mathcal{P}}(R))$ is symmetric monoidal. In particular, it holds when ζ is the tensor unit $\mathbb{1}$. This will then recover the usual untwisted cohomology of X^G.

Example 5.1.6 (Underlying Poincaré duality does not imply Poincaré duality of the fixed points)**.** There exist piecewise linear C_2-actions on the sphere S^d whose fixed point sets are submanifolds M which are not homology spheres, see e.g. [FL04, p.5] for an exposition. Let $\underline{X}$ be the (unreduced) suspension of such an action. Then $X^e \simeq S^{d+1}$ which is a Poincaré duality space. However, X^{C_2} is the unreduced suspension of a manifold which is not a homology sphere, and hence clearly not Poincaré as Poincaré duality with integer coefficients must fail.

Next, we recall the proper Tate construction. The significance of this to our proof is that combining Proposition 5.1.4 with the degree theory from §3.4, we may obtain a version of the cohomological injection in proper Tate cohomology. We then extract the desired injection from this version by "finding" $\mathbb{F}_p$–cohomology inside proper Tate cohomology.

Recollections 5.1.7 (Proper Tate)**.** Let G be a finite group and $R \in \mathrm{CAlg}(\mathrm{Sp})$. One way to define the R–*based proper Tate functor* is as the lax symmetric monoidal composite

$$(-)^{t_{\mathcal{P}}G} \colon \mathrm{Fun}(BG, \mathrm{Mod}_R) \xhookrightarrow{b_*} \mathrm{Mack}_G(\mathrm{Mod}_R) \xrightarrow{\Phi^G} \mathrm{Mod}_R$$

This functor kills the proper induced terms, i.e. those $M \in \mathrm{Mod}_R^{BG}$ such that $M \simeq \mathrm{Ind}_H^G N$ for some $H \lneq G$ and $N \in \mathrm{Mod}_R^{BH}$ since Φ^G does. Furthermore, since $(-)^{t_{\mathcal{P}}G}$ is a lax symmetric monoidal functor, $R^{t_{\mathcal{P}}G}$ canonically attains an R–algebra structure.

Now let A be an elementary abelian p–group $A = C_p^{\times r}$. With the trivial action of A on $\mathrm{H}\mathbb{F}_p$, the A–proper Tate $\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}$ is a nontrivial $\mathrm{H}\mathbb{F}_p$–algebra by [MNN19, Prop. 5.16]. Now let $T \colon \mathrm{stmod}_A^{\mathcal{P}}(\mathrm{H}\mathbb{F}_p) \longrightarrow \mathrm{Mod}_{\mathrm{H}\mathbb{F}_p}$ be the universal functor making the triangle

$$\begin{array}{ccc} \mathrm{Fun}(BA, \mathrm{Perf}_{\mathrm{H}\mathbb{F}_p}) & \xrightarrow{\mathrm{can}} & \mathrm{stmod}_A^{\mathcal{P}}(\mathrm{H}\mathbb{F}_p) \\ & {\scriptstyle (-)^{t_{\mathcal{P}}A}} \searrow & \downarrow {\scriptstyle T} \\ & & \mathrm{Mod}_{\mathrm{H}\mathbb{F}_p} \end{array}$$

commute, coming from the universal property of $\mathrm{stmod}_A^{\mathcal{P}}(\mathrm{H}\mathbb{F}_p)$.

Lemma 5.1.8 (Projection formula at dualisables)**.** *Let $\mathcal{A}, \mathcal{C}, \mathcal{D}$ be stably symmetric monoidal categories and $u \colon \mathcal{A} \to \mathcal{C}$ and $L \colon \mathcal{C} \to \mathcal{D}$ be symmetric monoidal exact functors. Suppose L admits a right adjoint R. Then for every $a \in \mathcal{A}$ dualisable and $d \in \mathcal{D}$, the canonical map $ua \otimes Rd \to R(Lua \otimes d)$ is an equivalence.*

Proof. Let $c \in \mathcal{C}$. By considering the equivalences

$$\mathrm{Map}_{\mathcal{C}}(c, ua \otimes Rd) \simeq \mathrm{Map}_{\mathcal{C}}(c \otimes ua^\vee, Rd) \simeq \mathrm{Map}_{\mathcal{D}}(Lc \otimes Lua^\vee, d) \simeq \mathrm{Map}_{\mathcal{D}}(c, R(Lua \otimes d)),$$

we obtain the desired conclusion by an application of Yoneda's lemma. ☐

Lemma 5.1.9. *Let $X \in \mathcal{S}^\omega$, $R \in \mathrm{CAlg}(\mathrm{Sp})$, and $\zeta \in \mathrm{Fun}(X, \mathrm{Perf}_R)$. Then viewing ζ as having the trivial G–action, we have an equivalence $(X_*\zeta)^{t_{\mathcal{P}}G} \simeq R^{t_{\mathcal{P}}G} \otimes_R X_*\zeta$.*

Proof. Since X was compact, we know that $X_*\zeta \in \mathrm{Perf}_R$, i.e. $X_*\zeta$ is a dualisable R–module. Setting $\mathrm{Infl} \colon \mathrm{Mod}_R \to \mathrm{Mack}_G(\mathrm{Mod}_R)$ and $b^* \colon \mathrm{Mack}_G(\mathrm{Mod}_R) \to \mathrm{Fun}(BG, \mathrm{Mod}_R)$ for the functors u and L in Lemma 5.1.8 (and writing $\mathrm{triv}_G \colon \mathrm{Mod}_R \to \mathrm{Fun}(BG, \mathrm{Mod}_R)$ for the composite), we see that by the lemma that

$$(\mathrm{triv}_G X_* R)^{t_{\mathcal{P}}G} = \Phi^G b_*\big((\mathrm{triv}_G X_* R) \otimes_R R\big) \simeq \Phi^G\big((\mathrm{Infl} X)_* R \otimes_R b_* R\big) \simeq (X_* R) \otimes_R R^{t_{\mathcal{P}}G}$$

as was to be shown. ☐

We now come to the main general proposition.

Proposition 5.1.10 (Injection after basechanging to proper Tate)**.** *Consider a finite group G, $R \in \mathrm{CAlg}(\mathrm{Sp})$, and $f\colon \underline{X} \to \underline{Y}$ a map of compact G–spaces. Suppose X^e, Y^e are Mod_R–Poincaré spaces and $f\colon \underline{X} \to \underline{Y}$ is equipped with a $\underline{\mathrm{Bor}}(\mathrm{Mod}_R)$–degree one datum (c.f. Definition 3.4.2). Then for any $\zeta \in \mathrm{Fun}(Y^G, \mathrm{Perf}_R)$, the cohomological functoriality map in Mod_R*

$$R^{t_{\mathcal{P}}G} \otimes_R Y^G_*\zeta \longrightarrow R^{t_{\mathcal{P}}G} \otimes_R X^G_* f^*\zeta$$

is a π_*–split injection.

Proof. By Proposition 5.1.4 we know that X^G and Y^G are $\mathrm{stmod}^{\mathcal{P}}_G(R)$–partial Poincaré duality. In particular, viewing ζ as an object in $\mathrm{Fun}(Y^A, \mathrm{Perf}^{BG}_R)$ under the symmetric monoidal functor $\mathrm{triv}_A\colon \mathrm{Perf}_R \to \mathrm{Perf}^{BG}_R$, we obtain using Lemma 3.4.6 the left commuting square

$$\begin{array}{ccc} Y^A_*(\zeta) & \xrightarrow{\quad \mathrm{BC}^f_* \quad} & X^A_*(f^*\zeta) \\ \simeq\big\downarrow \mathrm{PD} & & \simeq\big\downarrow \mathrm{PD} \\ Y^A_!(\Phi D_{Y^e} \otimes \Phi\zeta) & \xleftarrow{\ \mathrm{BC}^f_!\ } & X^A_!(\Phi f^* D_{Y^e} \otimes \Phi f^*\zeta) \end{array}$$

in $\mathrm{stmod}^{\mathcal{P}}_G(R)$. Hence, the map $\mathrm{BC}^f_*\colon Y^G_*\zeta \to X^G_* f^*\zeta$ is a split inclusion. Finally, applying T to this map and using that $T \circ \Phi \simeq (-)^{t_{\mathcal{P}}G}$, we conclude from Lemma 5.1.9 that the map stated in the proposition is a split inclusion in Mod_R and in particular is a π_*–split injection. ☐

We would like to apply Proposition 5.1.10 to prove Theorem 5.1.1, and for this, a small preliminary calculation will be needed.

Lemma 5.1.11. *Let G be a p–group, $f\colon \underline{X} \to \underline{Y}$ a morphism in $\mathcal{S}_G$. Suppose that X^e and Y^e are $\mathrm{H}\mathbb{F}_p$–Poincaré and that $f\colon X^e \to Y^e$ is equipped with an $\mathrm{H}\mathbb{F}_p$–degree one datum. Then this degree one datum lifts to yield a $\underline{\mathrm{Bor}}(\mathrm{Mod}_{\mathrm{H}\mathbb{Z}})$–degree one datum for the map $f\colon \underline{X} \to \underline{Y}$.*

Proof. First recall from Proposition 4.3.6 that $\underline{X}$ and $\underline{Y}$ are indeed $\underline{\mathrm{Bor}}(\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p})$–Poincaré. So by Lemma 4.3.7, we just need to find G–equivariant lifts of the equivalences $D_{X^e} \xrightarrow[\simeq]{\alpha} f^*D_{Y^e} \in \mathrm{Fun}(X^e, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p)) \simeq \mathrm{Map}(X^e, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))$ and $c_Y \simeq \mathrm{BC}^f_! \circ \alpha \circ c_X \in Y^e_! D_{Y^e}$. That is, we would like to lift these equivalences to ones in $\mathrm{Map}(X^e, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG}$ and $(Y^e_! D_{Y^e})^{hG}$ respectively. For the first problem, note that $\mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p) \simeq \mathbb{Z} \times B\mathrm{Aut}(\mathbb{F}_p) \simeq \mathbb{Z} \times B\mathbb{Z}/(p-1)$. Thus, by a standard analysis of the $(-)^{hG}$–spectral sequence

$$H^s(G; \pi_t \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))) \Rightarrow \pi_{t-s} \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG},$$

applying π_0 yields

$$\pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG} \cong (\pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p)))^G \longrightarrow \pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))$$

which in particular is an injection. Thus, since the G–equivariant lifts $D_{\underline{X}}$, $f^*D_{\underline{Y}}$ in the source get mapped to $D_{X^e} = f^*D_{Y^e} \in \pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))$, we get that $D_{\underline{X}} = f^*D_{\underline{Y}}$ in the set $\pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG}$. That is, the equivalence α lifts to a G–equivariant one, as required.

Next, note by Poincaré duality that $Y^e_! D_{Y^e} \simeq Y^e_* \mathbb{1}_{\mathrm{H}\mathbb{F}_p}$, and so since Y^e_* preserves coconnectivity, we learn that $Y^e_! D_{Y^e}$ is coconnective. Again, by looking at the spectral sequence $H^s(G; \pi_t Y^e_! D_{Y^e}) \Rightarrow \pi_{t-s}(Y^e_! D_{Y^e})^{hG}$, since no higher cohomologies may contribute to $\pi_0(Y^e_! D_{Y^e})^{hG}$ by coconnectivity, on π_0 the map $(Y^e_! D_{Y^e})^{hG} \to Y^e_! D_{Y^e}$ induces the map $(\pi_0 Y^e_! D_{Y^e})^G \to \pi_0 Y^e_! D_{Y^e}$, which is an injection. Thus by a similar argument as above, we obtain a G–equivariant lift of the equivalence $c_Y \simeq \mathrm{BC}^f_! \circ \alpha \circ c_X$, as wanted. ☐

We are now ready to assemble the pieces to prove the theorem.

Proof of Theorem 5.1.1. Since $\mathrm{H}\mathbb{F}_p$ was a field, we have the Künneth isomorphisms

$$\pi_{-*}\big(Y^A_* \zeta \otimes_{\mathrm{H}\mathbb{F}_p} \mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}\big) \cong H^*(Y^A; \zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A})$$

$$\pi_{-*}\big(X^A_* f^*\zeta \otimes_{\mathrm{H}\mathbb{F}_p} \mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}\big) \cong H^*(X^A; f^*\zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}).$$

Now consider the commuting square

$$\begin{array}{ccc} H^*(Y^A;\zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}) & \xrightarrow{f^*} & H^*(X^A; f^*\zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}) \\ \uparrow & & \uparrow \\ H^*(Y^A;\zeta) & \xrightarrow{f^*} & H^*(X^A; f^*\zeta) \end{array}$$

Here, the vertical arrows are induced by the injection $\mathbb{F}_p = \pi_{-*}(\mathrm{H}\mathbb{F}_p) \to \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A})$ and so are themselves injections: this is since we are tensoring over a field and so all modules are flat. The top horizontal map is an injection by Proposition 5.1.10 and the fact that, by Lemma 5.1.11, we have a lift of the given nonequivariant degree one datum to a $\underline{\mathrm{Bor}}(\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p})$–degree one datum for the map $f \colon \underline{X} \to \underline{Y}$. Therefore all in all, we see that the bottom map f^* is injective as desired. ☐

We end this subsection with an application of Theorem 5.1.1 where we show Theorem 5.1.14 that, when G is a p–group for an odd prime p, equivariant Poincaré spaces with contractible underlying spaces must already by G–contractible. Apart from perhaps being interesting in its own right, this result will also be a crucial ingredient in the inductive proof of the main theorem in the next subsection. We will need several preliminaries on orientations.

Lemma 5.1.12. *Let $\underline{X}$ be a C_p–Poincaré duality space, where p is an odd prime. Assume X^e is $\mathbb{Z}$–orientable. Then also X^{C_p} is $\mathbb{Z}$–orientable.*

Proof. We check that a class $w_1(X^{C_p}) \in H^1(X^{C_p}; \mathbb{Z}/2) \cong \hom(\pi_1(X^{C_p}), \mathbb{Z}/2)$, the first Stiefel-Whitney class of X^{C_p}, vanishes. Let $\gamma \colon S^1 \to X^{C_p}$ be a loop. The value of w_1 at the loop γ can be computed as the degree of $\mathrm{Mdrmy}^X_\gamma \colon D_{X^{C_p}}(\gamma(1)) \to D_{X^{C_p}}(\gamma(1)) \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})$, the induced monodromy automorphism map.

We also have the automorphism $\mathrm{Mdrmy}^{\underline{X}}_\gamma \colon D_{\underline{X}}(\gamma(1)) \to D_{\underline{X}}(\gamma(1)) \in \mathcal{P}\mathrm{ic}(\mathrm{Sp}_G)$. Using Theorem 4.2.9, we see that $\Phi^{C_p}\mathrm{Mdrmy}^{\underline{X}}_\gamma \simeq \mathrm{Mdrmy}^{X^{C_p}}_\gamma$. Now we know that

$$\deg \Phi^{C_p}\mathrm{Mdrmy}^{\underline{X}}_\gamma \equiv \deg \Phi^e \mathrm{Mdrmy}^{\underline{X}}_\gamma \mod p$$

by Theorem 4.6.5 and Example 4.6.3. But we also know that both degrees are ± 1, as $\mathrm{Mdrmy}\frac{X}{\gamma}$ is invertible. Thus, in fact we even have $\deg \Phi^{C_p}\mathrm{Mdrmy}\frac{X}{\gamma} = \deg \Phi^{e}\mathrm{Mdrmy}\frac{X}{\gamma}$ since p was odd. But $\deg \Phi^{e}\mathrm{Mdrmy}\frac{X}{\gamma}$ is the value of the first Stiefel Whithney class of the Poincaré space X^e at the loop $S^1 \xrightarrow{\gamma} X^{C_p} \to X^e$, which is 1 as X^e was assumed to be $\mathbb{Z}$–orientable. □

Proposition 5.1.13 (Rigidity of orientability)**.** *Let p be an odd prime and G be a p–group. Let $\underline{X}$ be a G–Poincaré space. Suppose that $X^e \in \mathcal{S}$ is nonequivariantly $\mathbb{Z}$–orientable. Then for each subgroup $H \leq G$, the Poincaré space X^H is also nonequivariantly $\mathbb{Z}$–orientable.*

Proof. We shall prove this by induction on the order of the subgroup, where the base case $|H| = 1$ is given by the hypothesis that X^e is nonequivariantly orientable. Suppose we know the statement for all subgroups of order p^{k-1} and consider $H \leq G$ with $|H| = p^k$. Since H is a p–group, we may find a normal subgroup $N \lneq H$ such that $H/N \cong C_p$. By Proposition 4.1.8 and Theorem 4.2.9, we know that $(\mathrm{Res}^G_H \underline{X})^N$ is a $H/N \cong C_p$–Poincaré duality space and by induction, we know that $(X^N)^e$ is orientable. Thus, by Lemma 5.1.12, we see that $X^H \simeq (X^N)^{H/N} \simeq (X^N)^{C_p}$ is also orientable as required. □

Theorem 5.1.14 (Poincaré rigidity of contractible underlying spaces)**.** *Let G be a solvable group and $\underline{X} \in \mathcal{S}^{\omega}_G$ a compact G–Poincaré space with $X^e \simeq *$. Then $\underline{X} \simeq \underline{*}$.*

Proof. We prove this reducing to the case of $G = C_p$ using the solvability assumption. To wit, let us suppose we know the statement to be true for all solvable groups with size smaller than $|G|$. Choose a normal subgroup N of G such that $G/N = C_p$. By Proposition 4.1.8, we know that $\mathrm{Res}^G_N \underline{X}$ is N–Poincaré with $(\mathrm{Res}^G_N \underline{X})^e \simeq X^e \simeq *$, and so by induction, $\mathrm{Res}^G_N \underline{X} \simeq \underline{*}$. In particular, $X^N \simeq *$. Therefore, by Theorem 4.2.9, we have that $\underline{X}^N$ is a $G/N = C_p$–Poincaré space with $(\underline{X}^N)^e \simeq X^N \simeq *$. Thus, we are left to prove that for a C_p–Poincaré space $\underline{X}$, $X^e \simeq *$ implies $X^{C^p} \simeq *$.

Observe that $X^{C_p} \neq \varnothing$ as $\underline{EC_p}$ is not compact. Now pick a map $f\colon \underline{*} \to \underline{X}$. It is an equivalence on undelying spaces with C_p-action. By Theorem 5.1.1, f induces an injection

$$f^*\colon H^*(X^{C_p}; \mathbb{F}_p) \rightarrowtail H^*(*; \mathbb{F}_p). \tag{23}$$

In degree 0, this shows that X^{C_p} is connected. Furthermore, again by Theorem 4.2.9, X^{C_p} is a Poincaré space. To conclude, by the classification of zero-dimensional Poincaré spaces (Fact 3.2.19) it suffices to show that the formal dimension of X^{C_p} is zero. Note that X^{C_p} is $\mathbb{F}_p$-orientable. In the case $p = 2$ this is clear while in the case $p \neq 2$ this follows from Proposition 5.1.13. Now, injectivity of (23) implies that the formal dimension of X^{C_p} is zero, as zero is the highest degree in which $H^*(X^{C_p}; \mathbb{F}_p)$ does not vanish. □

Remark 5.1.15. By Feit–Thompson's celebrated result, all finite groups of odd order are solvable. Hence, the Poincaré rigidity result above holds unconditionally for all odd finite groups.

Corollary 5.1.16. *Let p be an odd prime. There exists a compact C_p-space $\underline{X}$ with*

1. *the underlying space X^e is contractible and*
2. *the fixed point space X^{C_p} is Poincaré and*

3. *the C_p-space $\underline{X}$ is not C_p–Poincaré.*

Proof. Pick a noncontractible $\mathbb{F}_p$-acyclic Poincaré space K that is homotopy equivalent to a finite CW complex, for example $\mathbb{R}\mathrm{P}^d$ for $d > 0$ an even number. By [Jon71, Thm. 1.1], we may pick a finite C_p–CW complex $\underline{X}$ with $X^e \simeq *$ and $X^{C_p} \simeq K$. By Theorem 5.1.14, we see that $\underline{X}$ can not be C_p–Poincaré, as then we would have $K \simeq *$. ☐

5.2. The theorem of single fixed points

Throughout this subsection, we will fix an odd prime p.

In [CF64], Conner-Floyd conjectured that a smooth action by a cyclic group of odd prime power on a smooth, closed, orientable, positive–dimensional manifold cannot have exactly one fixed point. The first proof of this statement (in fact, a slightly more general version) was given by Atiyah-Bott in [AB68] and soon after by [CF66] themselves. Many variations have been proven since then, and we mention [Lüc88; ABK92] as further examples. Atiyah-Bott's argument uses Atiyah–Singer's index theory, whereas Conner–Floyd's proof used a particular bordism spectrum. In either case, and also in [Lüc88], local structures of smooth manifolds were used in essential ways. We exemplify such local arguments with the following corollary of Theorem 5.1.1 which answers the Conner–Floyd question for elementary abelian p–groups. As will be clear from the proof, the result holds more generally for locally smooth manifolds.

Corollary 5.2.1 (Conner–Floyd for elementary abelian groups)**.** *Let A be an elementary abelian p–group, and M a closed, orientable, positive–dimensional, smooth A–manifold. Then $M^A \neq *$.*

Proof. Suppose $M^A = *$. Writing $x \in M$ for this single fixed point, we may thus find an A–representation V equipped with a codimension zero equivariant embedding $V \subseteq M$ which sends $0 \in V$ to $x \in M$. Consider the collapse map $c\colon M \longrightarrow M/(M \setminus Y) \simeq S^V$. It is a map of A–Poincaré spaces with $D_{M^e} \otimes \mathrm{H}\mathbb{Z} \simeq c^* D_{(S^V)^e} \otimes \mathrm{H}\mathbb{Z}$ as both are orientable. Thus by Theorem 5.1.1, we have an injection $H^*((S^V)^A; \mathbb{F}_p) \to H^*(M^A; \mathbb{F}_p)$. But note that $H^*((S^V)^A; \mathbb{F}_p) \simeq H^*(* \amalg *; \mathbb{F}_p)$ while $H^*(M^A; \mathbb{F}_p) \simeq H^*(*; \mathbb{F}_p)$. This is a contradiction. ☐

In this subsection, we will employ the theory of fundamental classes developed in this article to give a fully homotopical and global proof of the following generalisation of Atiyah–Bott and Conner–Floyd's theorem for C_{p^k}–Poincaré spaces. Philosophically, this says that the equivariant fundamental class packages enough structures so as to be able to provide a global obstruction to some naturally interesting geometric questions.

Theorem 5.2.2 (Generalised Atiyah–Bott–Conner–Floyd)**.** *Let p be an odd prime, $G = C_{p^k}$ for some k, and suppose $\underline{X} \in \mathcal{S}^{\omega}_G$ is G–Poincare such that the underlying space $X^e \in \mathcal{S}^{\omega}$ is connected, $\mathbb{Z}$–orientable, and has formal dimension $d > 0$. Then $X^G \not\simeq *$.*

We obtain the theorem of Atiyah–Bott and Conner–Floyd as an immediate consequence.

Corollary 5.2.3 ([AB68, Thm. 7.1], [CF66, p. 8.3])**.** *Let p be an odd prime and $G = C_{p^k}$. Let M be a closed connected orientable smooth manifold of positive dimension equipped with a smooth G–action. Then $M^G \neq *$.*

The orientability assumption is crucial, as illustrated by the following:

Example 5.2.4. For p odd, consider the suspension of the action of $C_p \subset S^1$ on S^2 which descends to an action of C_p on $\mathbb{R}\mathrm{P}^2$ with a single fixed point.

Consequently, we see that these no–go results for single fixed points cannot purely be a product of classical Smith theory since they must incorporate orientations in some fundamental way. From this perspective, our approach may be seen as a way to encode orientations by enconcing the discussion within the formalism of equivariant Poincaré duality, where Smith–theoretic fixed points methods are also available as afforded by §4.2.

Restricting to odd prime powers is essential as well, as the following example illustrates.

Example 5.2.5 ([CF64], Chapter 45)**.** The group C_4 acts on $\mathbb{C}\mathrm{P}^2$ with a single fixed point, by letting a generator act via $[z_0 : z_1 : z_2] \mapsto [\overline{z_0} : -\overline{z_2} : \overline{z_1}]$.

To start work on Theorem 5.2.2, we record some preliminaries on Tate cohomology which will be the computational input to our proof.

Recollections 5.2.6 (Group (co)homologies)**.** Let $n \geq 2$ be an integer and A an abelian group equipped with the trivial C_n–action. Then by definition, we have

$$\pi_d HA^{hC_n} \cong H^{-d}(C_n; A), \qquad \pi_d HA^{tC_n} \cong \widehat{H}^{-d}(C_n; A), \qquad \pi_d HA_{hC_n} \cong H_d(C_n; A).$$

Moreover, using the fibre sequence of spectra $HA_{hC_n} \longrightarrow HA^{hC_n} \longrightarrow HA^{tC_n}$, we get a long exact sequence

$$\cdots \to H^{-d}(C_n; A) \longrightarrow \widehat{H}^{-d}(C_n; A) \longrightarrow H_{d-1}(C_n; A) \longrightarrow H^{-(d-1)}(C_n; A) \to \cdots$$

giving us

$$\widehat{H}^{-d}(C_n; A) = \begin{cases} H^{-d}(C_n; A) \cong A/n & \text{if } d \leq -1 \text{ and } d \text{ even;} \\ H^{-d}(C_n; A) \cong 0 & \text{if } d \leq -1 \text{ and } d \text{ odd;} \\ A/n & \text{if } d = 0; \\ H_{d-1}(C_n; A) \cong A/n & \text{if } d \geq 1 \text{ and } d \text{ even;} \\ H_{d-1}(C_n; A) \cong 0 & \text{if } d \geq 1 \text{ and } d \text{ odd.} \end{cases}$$

It will be convenient to recall the notations of [GM95] to manipulate the various forms of the Tate constructions.

Lemma 5.2.7. *Let $H \leq G$ be a subgroup of a finite group G and $A \in \mathrm{Sp}_H$. Then we have $(\mathrm{Ind}_H^G A)^{tG} \simeq A^{tH}$.*

Proof. First observe that $\mathrm{Res}^G_H \widetilde{EG} \simeq \widetilde{EH}$ and $\mathrm{Res}^G_H EG_+ \simeq EH_+$. The required result is now obtained from the computation of $(\mathrm{Ind}^G_H A)^{tG}$ as

$$\big(\widetilde{EG} \otimes F(EG_+, \mathrm{Ind}^G_H A)\big)^G \simeq \big(\mathrm{Ind}^G_H \mathrm{Res}^G_H \widetilde{EG} \otimes F(EG_+, A)\big)^G \simeq \big(\widetilde{EH} \otimes F(EH_+, A)\big)^H = A^{tH}$$

where the equivalence $(\mathrm{Ind}^G_H -)^G \simeq (-)^H$ is since $\mathrm{Ind}^G_H \simeq \mathrm{Coind}^G_H$ and we have an equivalence of their left adjoints $\mathrm{Infl}^1_H \simeq \mathrm{Res}^G_H \mathrm{Infl}^1_G$. □

Lemma 5.2.8. *Let $\underline{Y} \in \mathcal{S}^{\omega}_{C_{p^k}}$ such that $Y^{C_{p^k}} \simeq \varnothing$. Then the change of coefficients map $(Y_+ \otimes \mathrm{H}\mathbb{Z})^{tC_{p^k}} \to (Y_+ \otimes \mathrm{H}\mathbb{Z}/p^{k-1})^{tC_{p^k}}$ is an equivalence. In particular, the groups $\pi_n(Y_+ \otimes \mathrm{H}\mathbb{Z})^{tC_{p^k}}$ are p^{k-1}–torsion for all n.*

Proof. Note that the map being an equivalence is stable under retracts and finite colimits in the Y–variable. As any compact C_{p^k} space $\underline{Y}$ with $Y^{C_{p^k}} = \varnothing$ is a retract of a finite sequence of pushouts of orbits $\underline{C_{p^k}/C_{p^l}}$ with $l < k$, it thus suffices to show the desired equivalence for each of these orbits. By Lemma 5.2.7, we obtain for any $R \in \mathrm{CAlg}(\mathrm{Sp})$ the natural equivalence

$$\big((C_{p^k}/C_{p^l})_+ \otimes R\big)^{tC_{p^k}} = \Big(\mathrm{Ind}_{C_{p^l}}^{C_{p^k}} R\Big)^{tC_{p^k}} \simeq R^{tC_{p^l}}.$$

Thus, the claim follows from the fact that the quotient map $\mathrm{H}\mathbb{Z}^{tC_{p^l}} \to \big(\mathrm{H}\mathbb{Z}/p^{k-1}\big)^{tC_{p^l}}$ is an equivalence for $l < k$, see e.g. Recollection 5.2.6. ☐

For the next lemma, recall the cofibre sequence $EG_+ \to \mathbb{S}_G \to \widetilde{EG}$ in Sp_G.

Lemma 5.2.9. *Let G be an odd finite group and $P \in \mathcal{P}\mathrm{ic}(\mathrm{Sp}_G)$ with $P^e \simeq \Sigma^k\mathbb{S}$. Then there is a canonical equivalence $F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P \simeq \Sigma^k F(EG_+, \mathrm{H}\mathbb{Z}) \in \mathrm{Sp}_G$. Consequently, the map*

$$\big(\widetilde{EG} \otimes F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P\big)^G \longrightarrow \big(\Sigma EG_+ \otimes F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P\big)^G$$

from the cofibre sequence $EG_+ \to \mathbb{S}_G \to \widetilde{EG}$ may be identified with the usual connecting map $\Sigma^k \mathrm{H}\mathbb{Z}^{tG} \longrightarrow \Sigma^{1+k}\mathrm{H}\mathbb{Z}_{hG}$.

Proof. We first show that the Borelification map $F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P \to F(EG_+, \mathrm{H}\mathbb{Z} \otimes P^e)$ is an equivalence. To wit, let $Y \in \mathrm{Sp}_G$. Then

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}_G}(Y, F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P) &\simeq \mathrm{Map}_{\mathrm{Sp}_G}\big(Y \otimes P^{-1}, F(EG_+, \mathrm{H}\mathbb{Z})\big)\\
&\simeq \mathrm{Map}_{\mathrm{Sp}^{BG}}(Y^e \otimes (P^e)^{-1}, \mathrm{H}\mathbb{Z})\\
&\simeq \mathrm{Map}_{\mathrm{Sp}^{BG}}(Y^e, \mathrm{H}\mathbb{Z} \otimes P^e)\\
&\simeq \mathrm{Map}_{\mathrm{Sp}_G}\big(Y, F(EG_+, \mathrm{H}\mathbb{Z} \otimes P^e)\big)
\end{aligned}$$

as claimed. But then, since $\mathrm{H}\mathbb{Z} \otimes P^e \in \mathrm{Fun}(BG, \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{H}\mathbb{Z})))$ and G was an odd group and $B\mathrm{Aut}(\mathrm{H}\mathbb{Z}) \simeq BC_2$, we know that $\mathrm{H}\mathbb{Z} \otimes P^e \simeq \mathrm{triv}_G \Sigma^k \mathrm{H}\mathbb{Z}$, whence the first statement. The second statement is then immediate from the equivalences $\big(\widetilde{EG} \otimes F(EG_+, E)\big)^G \simeq E^{tG}$ and $\big(EG_+ \otimes F(EG_+, E)\big)^G \simeq E_{hG}$ for all $E \in \mathrm{Sp}_G$. ☐

For the proof of the theorem, it will also be helpful to record the following:

Construction 5.2.10 (Orbit–component decompositions)**.** Let $\underline{X} \in \mathcal{S}_G$. By an easy adjunction computation, we have that $(\pi_0 X^e)/G \cong \pi_0(X_{hG})$. Let $S \sqcup T$ be a decomposition of $(\pi_0 X^e)/G \cong \pi_0(X_{hG})$. By considering the triple of adjunctions

$$\mathcal{S}_G \underset{\mathrm{incl}}{\overset{\pi_0}{\rightleftarrows}} \mathrm{Set}_G = \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Set}) \underset{b_*}{\overset{b^*}{\rightleftarrows}} \mathrm{Fun}(BG, \mathrm{Set}) \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathrm{Set}, \tag{24}$$

we may obtain a decomposition $\underline{X} \simeq \underline{Y} \sqcup \underline{Z} \in \mathcal{S}_G$ such that $\pi_0 Y_{hG} \cong S$ and $\pi_0 Z_{hG} \cong T$.

We now come to the proof of our generalisation of Atiyah–Bott and Conner–Floyd's theorem. For this, recall the notion of formal dimensions from Terminology 3.2.18.

Proof of Theorem 5.2.2. We prove this by induction on k, where the base case of $k = 0$ is trivial. Now suppose we know that it is true for $k-1$. To prove the inductive step for the case of k, the strategy is to obtain a contradiction using the gluing class. For this, note first that $\underline{X}^{C_p}$ is a $\underline{\mathrm{Sp}}_{G/C_p}$–Poincaré space by Theorem 4.2.9. We claim that there is a decomposition $\underline{X}^{C_p} = \underline{*} \sqcup \underline{Y}$ of G/C_p-spaces, where $Y^{G/C_p} \simeq \varnothing$ necessarily since $* \simeq X^G = (X^{C_p})^{G/C_p} \simeq * \sqcup Y^{G/C_p}$. If the component of X^{C_p} containing $\underline{*}$ is of formal dimension larger than 0, then the induction hypothesis and Construction 5.2.10 gives such a decomposition as the G/C_p-space $\underline{X}^{C_p}$ also satisfies the conditions of the theorem. If the component of X^{C_p} containing $\underline{*}$ is of formal dimension 0, then it must be G/C_p–contractible by Fact 3.2.19 (1) and Theorem 5.1.14. Thus, again by Construction 5.2.10, we obtain the desired decomposition.

We will derive the contradiction by basechanging along $\underline{\mathrm{Sp}} \to \underline{\mathrm{Mod}}_{F(EG_+, \mathrm{H}\mathbb{Z})}$. Observe first that we may assume that $d > 0$ is even since we may replace $\underline{X}$ with $\underline{X} \times \underline{X}$ if necessary: this will still be a $\underline{\mathrm{Sp}}$–Poincaré space satisfying the hypotheses of the theorem with $(\underline{X} \times \underline{X})^G \simeq *$ and $X^e \times X^e$ having formal dimension $2d > 0$.

To set up notation, recall the map $\underline{X}^{>1} \xrightarrow{\epsilon} \underline{X}$ from Construction 2.2.15 and write $W := \Sigma^{-d}\mathrm{H}\mathbb{Z} \in \mathrm{Mod}_{\mathrm{H}\mathbb{Z}}$. We write $D^{\mathbb{Z}}_{\underline{X}} := F(EG_+, \mathrm{H}\mathbb{Z}) \otimes D_{\underline{X}} \in \underline{\mathrm{Mod}}^{\underline{X}}_{F(EG_+, \mathrm{H}\mathbb{Z})}$. By the hypothesis of $\mathbb{Z}$–orientability, we get $D^{\mathbb{Z}}_{X^e} \simeq \mathrm{H}\mathbb{Z} \otimes D_{X^e} \simeq X^* W \in \mathrm{Mod}^{X^e}_{\mathrm{H}\mathbb{Z}}$. By Proposition 5.1.13, the $\underline{\mathrm{Sp}}_{G/C_p}$–Poincaré space $\underline{X}^{C_p}$ has $\mathbb{Z}$–orientable underlying dualising sheaf. By Corollary 4.5.7, the composition giving the gluing class

$$
\begin{array}{ccccc}
\mathrm{H}\mathbb{Z} & & (X^{>1}_! \epsilon^* D^{\mathbb{Z}}_{\underline{X}})^{tG} \simeq (X^{>1}_! (X^{>1})^* W)^{tG} & \xrightarrow{\mathrm{can}} & \Sigma (X^{>1}_! \epsilon^* D^{\mathbb{Z}}_{\underline{X}})_{hG} \simeq \Sigma (X^{>1}_! (X^{>1})^* W)_{hG} \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{can}} \\
(X_! D^{\mathbb{Z}}_{\underline{X}})^{hG} & \xrightarrow{\mathrm{can}} & (X_! D^{\mathbb{Z}}_{\underline{X}})^{tG} & & \Sigma W_{hG}
\end{array}
\tag{25}
$$

is nullhomotopic. To achieve a contradiction, we show that this composition is also π_0–surjective onto a nontrivial group, assuming that $X^G \simeq *$. We do this in three steps.

(1) To this end, first note that since the composite $\underline{*} \hookrightarrow \underline{X}^{>1} \xrightarrow{\epsilon} \underline{X} \to \underline{*}$ is equivalent to the identity, by functoriality of colimits, the map in $\mathrm{Fun}(BG, \mathrm{Mod}_{\mathrm{H}\mathbb{Z}})$

$$W \longrightarrow X^{>1}_! \epsilon^* D^{\mathbb{Z}}_{X^e} \simeq X^{>1}_! (X^{>1})^* W \longrightarrow W$$

is also equivalent to the identity. Thus the rightmost vertical map can in (25) is (split) surjective on homotopy groups coming from the summand ΣW_{hG} inside $\Sigma (X^{>1}_! \epsilon^* D^{\mathbb{Z}}_{\underline{X}})_{hG} \simeq \Sigma W_{hG} \oplus \Sigma (Y_! Y^* W)_{hG}$, where we have used that $\underline{X}^{>1} \simeq \mathrm{Infl}^{G/C_p}_G \underline{X}^{C_p} \simeq \underline{*} \sqcup \mathrm{Infl}^{G/C_p}_G \underline{Y}$ from the first paragraph of the proof.

(2) Next, the Tate–to–orbit canonical map breaks up to become

$$(X^{>1}_! \epsilon^* D^{\mathbb{Z}}_{\underline{X}})^{tG} \simeq W^{tG} \oplus (Y_! Y^* W)^{tG} \xrightarrow{\mathrm{can} \oplus \mathrm{can}} \Sigma W_{hG} \oplus \Sigma (Y_! Y^* W)_{hG}$$

By Recollection 5.2.6 and since $d-1$ is odd by our assumption, $\mathrm{can}\colon W^{tG} \to \Sigma W_{hG} \simeq \Sigma^{1-d}\mathrm{H}\mathbb{Z}_{hG}$ is a π_0–isomorphism onto $\mathbb{Z}/p^k$. Moreover, by Lemma 5.2.8, the image of $\mathrm{can}\colon \pi_0(Y_!Y^*W)^{tG} \to \pi_0\Sigma(Y_!Y^*W)_{hG}$ is p^{k-1}–torsion since $Y^{G/C_p} \simeq \varnothing$.

(3) Finally, consider the commuting diagram

$$\begin{array}{ccc}
\mathrm{H}\mathbb{Z} & \xrightarrow{c} & \\
\downarrow c & & \\
\left(\widetilde{EG}\otimes X_!D^{\mathbb{Z}}_{\underline{X}}\right)^G & \xleftarrow[\simeq]{\epsilon_!} & \left(\widetilde{EG}\otimes X^{>1}_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}}\right)^G \\
\downarrow & & \downarrow \\
(X_!D^{\mathbb{Z}}_{\underline{X}})^{tG} & \xleftarrow[\simeq]{\epsilon_!} & (X^{>1}_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})^{tG} \simeq \left(\widetilde{EG}\otimes F(EG_+, X^{>1}_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})\right)^G
\end{array}$$

where the top right triangle involving the fundamental classes commutes since by Corollary 4.2.5 and Lemma 4.2.6, the map $\epsilon\colon \underline{X}^{>1} \to \underline{X}$ is $\underline{\mathrm{Mod}}_{\widetilde{EG}\otimes F(EG_+,\mathrm{H}\mathbb{Z})}$–degree one. Note importantly that the map

$$\mathrm{H}\mathbb{Z} \xrightarrow{c} (\widetilde{EG}\otimes X^{>1}_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})^G \simeq (\widetilde{EG}\otimes *_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})^G \oplus (\widetilde{EG}\otimes Y_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})^G$$

is the fundamental class of the $\widetilde{EG}\otimes F(EG_+,\mathrm{H}\mathbb{Z})$–Poincaré space $\underline{*}\sqcup \mathrm{Infl}^G_{G/C_p}\underline{Y}$, and so by Lemma 3.3.16, it hits the algebra unit 1 of $\pi_0(\widetilde{EG}\otimes *_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})^G$. Next, consider

$$\begin{array}{ccc}
\left(\widetilde{EG}\otimes *_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}}\right)^G & \longrightarrow & \left(\Sigma EG_+\otimes *_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}}\right)^G \\
\simeq\downarrow & & \downarrow\simeq \\
W^{tG}\simeq\left(\widetilde{EG}\otimes F(EG_+, *_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})\right)^G & \longrightarrow & \left(\Sigma EG_+\otimes F(EG_+, *_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})\right)^G \simeq \Sigma W_{hG}.
\end{array}$$

where the vertical equivalences are by Lemma 5.2.9. Since the bottom horizontal map is a π_0–isomorphism onto $\mathbb{Z}/p^k$ as in step (2), the composition $\mathrm{H}\mathbb{Z} \xrightarrow{c} W^{tG} \to \Sigma W_{hG}$ is a π_0–surjection onto the abelian group $\mathbb{Z}/p^k$.

All in all, putting the three steps together, we see that the image of $1 \in \pi_0\mathrm{H}\mathbb{Z}$ under the composition map $\pi_0\mathrm{H}\mathbb{Z} \to \pi_0\Sigma W_{hG} \cong \mathbb{Z}/p^k$ in (25) is of the form $1 + p\cdot a$ for some element $a \in \mathbb{Z}/p^k$, and hence is nonzero. This finishes the proof of the claim, and thus also of the theorem. ☐

Remark 5.2.11. Step (3) in the proof above might seem labyrinthine at first glance, but the basic idea leading to it is quite simple. Namely, we know always from Corollary 4.5.2 that the map $\epsilon_!\colon (X^{>1}_!\epsilon^*D^{\mathbb{Z}}_{\underline{X}})^{tG} \to (X_!D^{\mathbb{Z}}_{\underline{X}})^{tG}$ is an equivalence. However, this equivalence has no control over the fundamental class $c\colon \mathrm{H}\mathbb{Z} \to (X_!D^{\mathbb{Z}}_{\underline{X}})^{tG}$, essentially because $(-)^{tG}$ is only a lax symmetric monoidal functor. In contrast, the functor $\widetilde{EG}\otimes -$ is a symmetric monoidal one, and so it is more suited to lift the fundamental class by virtue of the theory of degree one maps as encapsulated in Corollary 4.2.5.

A. G–stability for presentable G-categories

In this section we study presentable G-stable categories for compact Lie groups. In [Nar17], Nardin defines for a finite group G-stability as a property of fibrewise stable G–categories, that roughly translates to requiring certain Wirthmüller isomorphisms to hold. Instead of developing his theory for compact Lie groups in full generality, we take a different approach following the general phenomenon that certain properties of categories can be classified through idempotent algebras. For example, a presentable category is stable if and only if it is a module over the category of spectra, see [GGN15; CSY21] for more examples of this type. We define presentable G-stable categories as those presentable G–categories which are modules over the G–category $\underline{\mathrm{Sp}}_G$ of G–spectra.

Recall that we say that a map $u\colon \mathbb{1} \to A$ exhibits an object A in a symmetric monoidal category $\mathcal{C}$ as an idempotent object if the map $A \simeq \mathbb{1} \otimes A \xrightarrow{u\otimes A} A \otimes A$ is an equivalence. An idempotent object admits a unique structure of a commutative algebra in $\mathcal{C}$ with u as its unit map, see [Lur17, Proposition 4.8.2.9]. Now given an idempotent algebra A in a symmetric monoidal category $\mathcal{C}$, it is a property of an object $X \in \mathcal{C}$ to be a module over A, in the sense that the forgetful functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is fully faithful. Its image is characterised by those $X \in \mathcal{C}$ for which the unit map $X \to A \otimes X$ is an equivalence, or equivalently admits a right inverse, see [Lur17, Proposition 4.8.2.10]. Taking this point of view, we observe that the G–category of G–spectra is an idempotent algebra in Pr^L_G and use this for the definition of presentable G-stable categories.

Definition A.0.1 ([GM23, Definition C.1, Corollary C.7], [Cno23, Definition 4.1])**.** The category of G–spectra is defined as the formal inversion

$$\mathrm{Sp}_G = \mathcal{S}_{G,*}[\{S^V\}^{-1}].$$

Here $\{S^V\}^{-1}$ denotes the collection in $\mathcal{S}_{G,*}$ consisting of representation spheres of all finite dimensional G-representations V.

This means that it comes together with a symmetric monoidal colimit preserving functor $\Sigma^\infty_G\colon \mathcal{S}_{G,*} \to \mathrm{Sp}_G$ sending all representation spheres S^V to invertible objects and is inital among those. More details on formal inversions of presentably symmetric monoidal categories can be found in [Rob15, Section 2] and [Hoy17, Section 6.1]. By [GM23, Corollary C.7], the canoncial map

$$\mathrm{Stab}_{\{S^V\}}(\mathcal{S}_{G,*}) \to \mathcal{S}_{G,*}[\{S^V\}^{-1}] \tag{26}$$

is an equivalence. Here, we denote for a presentably symmetric monoidal category $\mathcal{C}$ together with a small collection of objects $S \subseteq \mathcal{C}$ the stablisiation of $\mathcal{C}$ at S by $\mathrm{Stab}_S(\mathcal{C}) = \mathrm{colim}_{F\subseteq S \text{ finite}} \mathrm{Stab}_{\bigotimes F}(\mathcal{C})$, where for an element $x \in \mathcal{C}$ we denote

$$\mathrm{Stab}_x(\mathcal{C}) = \mathrm{colim}\left(\mathcal{C} \xrightarrow{-\otimes x} \mathcal{C} \xrightarrow{-\otimes x} \dots\right).$$

Definition A.0.2 ([Cno23, Definition 4.2])**.** We define the G–categories of pointed G–spaces and of (genuine) G–spectra as

$$\underline{\mathcal{S}}_{G,*} = \mathcal{S}_{G,*} \otimes_{\mathcal{S}_G} \Omega \qquad \text{and} \qquad \underline{\mathrm{Sp}}_G = \mathrm{Sp}_G \otimes_{\mathcal{S}_G} \Omega,$$

where $-\otimes_{\mathcal{S}_G}\Omega\colon \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L)\to \mathrm{Pr}^L_G$ is the symmetric monoidal colimit preserving embedding from Proposition 2.1.30.

Lemma A.0.3. *The G–categories $\underline{\mathcal{S}}_{G,*}$ and $\underline{\mathrm{Sp}}_G$ are idempotent algebras in Pr^L_G.*

Proof. First note that $\mathcal{S}_{G,*}\in \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L)$ is an idempotent algebra as the image of the idempotent algebra $\mathcal{S}_*\in \mathrm{Pr}^L$ under the symmetric monoidal functor $-\otimes\mathcal{S}_G\colon \mathrm{Pr}^L\to \mathrm{Mod}_{\mathcal{S}_G}$. It now follows from the definition of formal inversion that $\mathrm{Sp}_G\in \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L)$ is an idempotent algebra. This proves the claim as the symmetric monoidal functor $-\otimes_{\mathcal{S}_G}\Omega\colon \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L)\to \mathrm{Pr}^L_G$ preserves idempotent algebras. ☐

Having this at hand, we can now give our definition of G-stability.

Definition A.0.4. We say that a presentable G–category $\underline{\mathcal{C}}$ is *G-stable* if it is a module over the idempotent algebra $\underline{\mathrm{Sp}}_G\in \mathrm{CAlg}(\mathrm{Pr}^L_G)$. We denote by $\mathrm{Pr}^{L,G-\mathrm{st}}_G\subseteq \mathrm{Pr}^L_G$ the full subcategory on G-stable presentable G–categories. It is closed under all limits and colimits.

Our goal is to prove the following characterisation of presentable G-stable categories.

Theorem A.0.5 (Characterisation of G-stability)**.** *For a presentable G–category $\underline{\mathcal{C}}$ the following are equivalent:*

1. *$\underline{\mathcal{C}}$ is G-stable.*
2. *$\underline{\mathcal{C}}$ is fibrewise pointed and for all closed subgroups $H\leq G$ and all finite dimensional H-representations V tensoring with $S^V\in\mathcal{S}_{H,*}$ induces an equivalence $-\otimes S^V:\mathcal{C}^H\xrightarrow{\simeq}\mathcal{C}^H$.*
3. *$\underline{\mathcal{C}}$ is fibrewise pointed and for all finite dimensional G-representation V tensoring with $S^V\in\mathcal{S}_{G,*}$ induces an equivalence $-\otimes S^V:\underline{\mathcal{C}}\xrightarrow{\simeq}\underline{\mathcal{C}}$.*

To clarify the statement, recall that the $\underline{\mathcal{S}}_G$-module structure on $\underline{\mathcal{C}}$ restricts to a $\mathcal{S}_H$-module structure on $\mathcal{C}^H$ which refines to a $\mathcal{S}_{H,*}$-module structure as $\mathcal{C}^H$ is pointed. The map $-\otimes S^V:\mathcal{C}^H\xrightarrow{\simeq}\mathcal{C}^H$ is now just the multiplication map induced by this module structure.

For the proof of Theorem A.0.5, we need the following preliminary result.

Lemma A.0.6. *Suppose that $\underline{\mathcal{D}}$ is a presentable G–category such that $\mathcal{D}^G$ is pointed and -$-\otimes S^V:\mathcal{D}^G\to\mathcal{D}^G$ is an equivalence for any finite dimensional G-representation V. Then the restriction map $\mathrm{Fun}^L_G(\underline{\mathrm{Sp}}_G,\mathcal{D})\to\mathrm{Fun}^L_G(\underline{\mathcal{S}}_G,\mathcal{D})$ is an equivalence.*

Proof. Using that $-\otimes\underline{\mathcal{S}}_G\colon\mathrm{Pr}^L\to\mathrm{Pr}^L_G$ is left adjoint to Γ, we obtain an equivalence

$$\mathrm{Fun}^L_G(\underline{\mathcal{S}}_{G,*},\underline{\mathcal{D}})\simeq\mathrm{Fun}^L(\mathcal{S}_*,\mathcal{D}^G)\xrightarrow{\simeq}\mathrm{Fun}^L(\mathcal{S},\mathcal{D}^G)\simeq\mathrm{Fun}^L_G(\underline{\mathcal{S}}_G,\underline{\mathcal{D}})$$

where the middle equivalence uses that $\mathcal{D}^G$ is pointed. Similarly, the restriction map

$$\mathrm{Fun}^L_G(\underline{\mathrm{Sp}}_G,\underline{\mathcal{D}})\simeq\mathrm{Fun}^L_{\mathcal{S}_{G,*}}(\mathrm{Sp}_G,\mathcal{D}^G)\xrightarrow{\simeq}\mathrm{Fun}^L_{\mathcal{S}_{G,*}}(\mathcal{S}_{G,*},\mathcal{D}^G)\simeq\mathrm{Fun}^L_G(\underline{\mathcal{S}}_{G,*},\underline{\mathcal{D}})$$

is an equivalence by employing the colimit description of $\mathrm{Sp}_G=\mathcal{S}_{G,*}[\{S^V\}^{-1}]$ from (26). ☐

Proof of Theorem A.0.5. $\underline{1 \implies 2}$: Observe that $\underline{\mathrm{Sp}}_G$ is fibrewise pointed and satisfies the assumption on invertible actions of representations spheres as $\underline{\mathrm{Sp}}_G(G/H) = \mathrm{Sp}_H$ is the formal inversion of $\mathcal{S}_{H,*}$ at representation spheres of finite dimensional H-represenations. But this also holds for any G-stable category $\underline{\mathcal{C}}$ as $\mathcal{C}^H$ then is a module over Sp_H.

$\underline{2 \implies 3}$: Recall that, by the Peter-Weyl theorem, for any finite dimensional H-represenation W there is a finite dimensional G-representation V such that W is a summand of $\mathrm{Res}^G_H V$. In particular, if $- \otimes S^V : \mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^H$ is an equivalence, this implies that $- \otimes S^{\mathrm{Res}^G_H V} : \mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^H$ is an equivalence. But then also $- \otimes S^W$ is an equivalence

$\underline{3 \implies 1}$: We want to construct a right inverse $\underline{\mathcal{C}} \otimes \underline{\mathrm{Sp}}_G \to \underline{\mathcal{C}}$ to the unit map. By adjunction, this is equivalent to finding a factorisation of the unit map $\underline{\mathcal{S}}_G \to \underline{\mathrm{Fun}}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$ through the unit map $\underline{\mathcal{S}}_G \to \underline{\mathrm{Sp}}_G$. For this, we apply Lemma A.0.6 for $\underline{\mathcal{D}} = \underline{\mathrm{Fun}}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$. It thus remains to show that $\mathrm{Fun}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}}) = \mathcal{D}^G$ is pointed and tensoring with representation spheres is invertible. The assumption on $\underline{\mathcal{C}}$ being fibrewise pointed implies that $\mathrm{Fun}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$ is pointed. Furthermore, S^V acts invertibly on $\mathrm{Fun}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$ as it does so on $\underline{\mathcal{C}}$. □

B. Reflecting pushout squares

Let $A \to B \to B/A$ be a cofibre sequence in a stable category. Recall that there is a natural identification of the cofibre of $B \to B/A$ as follows, constructed as follows. Consider the diagram

$$A \longrightarrow B \longrightarrow B/A \longrightarrow \mathrm{cofib}(B \to B/A) \qquad (27)$$

and note that the two nullhomotopies of bent arrows - coming from them being the structure of the cofibre sequences - define a map $\Sigma A \to \mathrm{cofib}(B/A)$, which turns out to be an equivalence. This equivalence is natural in maps of cofibre sequences, and we will always use it to identify $\mathrm{cofib}(B \to B/A)$ with ΣA.

Lemma B.0.1. *Consider a pushout square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in a stable category. Then the two composites

$$\phi_C \colon D \to D/C \simeq B/A \to \Sigma A \quad \textit{and} \quad \phi_B \colon D \to D/B \simeq C/A \to \Sigma A$$

coming from the following diagram satisfy $\phi_B \simeq \pm\phi_C$.

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & \Sigma A \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\simeq} & & \\
C & \longrightarrow & D & \longrightarrow & D/C & & \\
\big\downarrow & & \big\downarrow & & & & \\
C/A & \xrightarrow{\simeq} & D/B & & & & \\
\big\downarrow & & & & & & \\
\Sigma A & & & & & &
\end{array}$$

Proof. To prove this, we consider the universal example of a span in a stable category. Denote by $\mathrm{Span}(\mathcal{C}) = \mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{C})$ the category of spans in the stable category $\mathcal{C}$. Note that there is an equivalence

$$\begin{aligned}
\mathrm{Span}(\mathcal{C}) &= \mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{C}) \\
&\simeq \mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{\omega}, \mathcal{C})) \\
&\simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{\omega} \otimes (\bullet \leftarrow \bullet \rightarrow \bullet), \mathcal{C}),
\end{aligned}$$

where $\mathrm{Sp}^{\omega} \otimes (\bullet \leftarrow \bullet \rightarrow \bullet)$ denotes the tensoring of $\mathrm{Cat}^{\mathrm{ex}}$ over Cat. The construction of the tensoring in [CDH+23, Section 6.4] shows that $\mathrm{Sp}^{\omega} \otimes (\bullet \leftarrow \bullet \rightarrow \bullet)$ is given by the stable subcategory $\mathrm{Cospan}(\mathrm{Sp})^{f}$ of $\mathrm{Cospan}(\mathrm{Sp})$ generated by the three objects in the span (28) (which are given as the values of the left Kan extensions of the inclusions of the individual objects in the category $(\bullet \rightarrow \bullet \leftarrow \bullet)$ at the sphere). Using this description, the equivalence $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Cospan}(\mathrm{Sp})^{f}, \mathcal{C}) \simeq \mathrm{Span}(\mathcal{C})$ is given by evaluation at the universal span

$$\begin{array}{ccc}
\left(\begin{array}{ccccc} & & \mathbb{S} & & \\ & \nearrow & & \nwarrow & \\ 0 & & & & 0 \end{array}\right) & \longrightarrow & \left(\begin{array}{ccccc} & & \mathbb{S} & & \\ & \nearrow & & \mathbin{\rotatebox[origin=c]{-45}{$=$}} & \\ 0 & & & & \mathbb{S} \end{array}\right) \\
\big\downarrow & & \\
\left(\begin{array}{ccccc} & & \mathbb{S} & & \\ & \mathbin{\rotatebox[origin=c]{45}{$=$}} & & \nwarrow & \\ \mathbb{S} & & & & 0 \end{array}\right). & &
\end{array} \tag{28}$$

It suffices to prove the claim in this specific case. Any span in $\mathcal{C}$ is the image of this universal span under an exact functor and thus also satisfies the statment of the lemma.

The possibilites for ϕ_B and ϕ_C are limited, since

$$
\begin{aligned}
&\pi_0 \operatorname{Map}\left(\begin{array}{ccc} & \mathbb{S} & \\ \mathbb{S} \;/\!\!/ & & \backslash\!\!\backslash\; \mathbb{S} \end{array} , \begin{array}{ccc} & \Sigma\mathbb{S} & \\ 0 \nearrow & & \nwarrow 0 \end{array} \right) \\
\simeq &\pi_0 \operatorname{Map}\left(\mathbb{S}, \lim \left(\begin{array}{ccc} & \Sigma\mathbb{S} & \\ 0 \nearrow & & \nwarrow 0 \end{array} \right)\right) \simeq \pi_0 \operatorname{Map}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{Z}.
\end{aligned}
$$

so ϕ_B and ϕ_C identify with integers n_B and n_C. Note that if n_B is divisible by $k \in \mathbb{Z}$, then ϕ_B is divisible by k for any pushout in any stable category. But in the case of the following pushout in Sp

$$
\begin{array}{ccc}
\mathbb{S} & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma\mathbb{S}
\end{array} \tag{29}
$$

the map ϕ_B is clearly an equivalence, so $n_B = \pm_1$. The same holds for ϕ_C, so by lack of alternatives we see $\phi_B = \pm\phi_C$. ☐

Remark B.0.2. A more careful analysis of (29) in fact yields that $\phi_B = -\phi_C$.

References

[AB68] M. F. Atiyah and R. Bott. "A Lefschetz Fixed Point Formula for Elliptic Complexes: II. Applications". In: Ann. Math. (2) 88.3 (1968), pp. 451–491.

[ABK92] Amir H. Assadi, Rebecca Barlow, and Friedrich Knop. "Abelian group actions on algebraic varieties with one fixed point". In: Math. Z. 210.1 (1992), pp. 129–136.

[BDG+16a] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. "Parametrized Higher Category Theory and Higher Algebra: A General Introduction". In: arXiv:1608.03654 (2016).

[BDG+16b] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. "Parametrized Higher Category Theory and Higher Algebra: Exposé I – Elements of Parametrized Higher Category Theory". In: arXiv:1608.03657 (2016).

[BE73] R. Bieri and B. Eckmann. "Groups with homological duality generalizing Poincaré duality". In: Invent. Math. 20 (1973), pp. 103–124.

[Bre72] Glen E. Bredon. Introduction to compact transformation groups. Vol. Vol. 46. Pure and Applied Mathematics. Academic Press, New York-London, 1972, pp. xiii+459.

[Bre73] Glen E. Bredon. "Fixed point sets of actions on Poincaré duality spaces". In: Topology 12 (1973), pp. 159–175.

[Bro72] W. Browder. “Poincaré spaces, their normal fibrations and surgery”. In: Invent. Math. 17 (1972), pp. 191–202.

[Bro87] William Browder. “Pulling back fixed points”. In: Invent. Math. 87.2 (1987), pp. 331–342.

[CDH+23] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. “Hermitian K-theory for stable ∞-categories I: Foundations”. In: Selecta Math. (N.S.) 29.1 (2023), Paper No. 10, 269.

[CF64] P. E. Conner and E. E. Floyd. Differentiable periodic maps. Vol. Band 33. Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.) Springer-Verlag, Berlin-Göttingen-Heidelberg; Academic Press, Inc., Publishers, New York, 1964, pp. vii+148.

[CF66] P. E. Conner and E. E. Floyd. “Maps of Odd Period”. In: Ann. Math. 84.2 (1966), pp. 132–156.

[Cno23] Bastiaan Cnossen. “Twisted ambidexterity in equivariant homotopy theory”. In: arXiv:2303.00736 (2023).

[CSY21] S. Carmeli, T.M. Schlank, and L. Yanovski. “Ambidexterity and height”. In: Adv. Math. 385 (2021), Paper No. 107763, 90.

[CSY22] S. Carmeli, T.M. Schlank, and L. Yanovski. “Ambidexterity in chromatic homotopy theory”. In: Invent. Math. 228.3 (2022), pp. 1145–1254.

[CW16] S.R. Costenoble and S. Waner. Equivariant ordinary homology and cohomology. Vol. 2178. Lecture Notes in Mathematics. Springer–Verlag, 2016.

[CW17] S.R. Costenoble and S. Waner. “The equivariant Spivak normal bundle and equivariant surgery for compact Lie groups”. In: arXiv:1705.10909 (2017).

[CW92] S.R. Costenoble and S. Waner. “Equivariant Poincaré duality”. In: Michigan Math. J. 39.2 (1992), pp. 325–351.

[Dav98] Michael W. Davis. “The cohomology of a Coxeter group with group ring coefficients”. In: Duke Math. J. 91.2 (1998), pp. 297–314.

[Die79] T. tom Dieck. Transformation groups and representation theory. Vol. 766. Lecture Notes in Mathematics. Springer, Berlin, 1979, pp. viii+309.

[Die86] T. tom Dieck. “Dimension functions of homotopy representations”. In: Bull. Soc. Math. Belg. Sér. A 38 (1986), pp. 103–129.

[Die87] T. tom Dieck. Transformation groups. Vol. 8. de Gruyter Studies in Mathematics. 1987.

[DP82] T. tom Dieck and Ted Petrie. “Homotopy representations of finite groups”. In: Publ. Math. I.H.E.S. 56 (1982), pp. 129–169.

[ES86] J. Ewing and R. Stong. “Group actions having one fixed point”. In: Math. Z. 191 (1986), pp. 159–164.

[FL04] F.T. Farrell and J.-F. Lafont. "Finite automorphisms of negatively curved Poincaré duality groups". In: Geom. Funct. Anal. (GAFA) 14 (2004), pp. 283–294.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. "Universality of multiplicative infinite loop space machines". In: Algebr. Geom. Topol. 15.6 (2015), pp. 3107–3153.

[GM23] David Gepner and Lennart Meier. "On equivariant topological modular forms". In: Compos. Math. 159.12 (2023), pp. 2638–2693.

[GM95] J.P.C. Greenlees and J.P. May. Generalized Tate cohomology. Vol. 113 (543). Mem. Amer. Math. Soc., 1995, pp. viii+178.

[GM97] J.P.C. Greenlees and J.P. May. "Localization and Completion Theorems for MU–module Spectra". In: Ann. of Math. (2) 146 (1997), pp. 509–544.

[Got79] D.H. Gottlieb. "Poincaré duality and fibrations". In: Proc. A.M.S. 76 (1979), pp. 148–150.

[GS14] J.P.C. Greenlees and B. Shipley. "Fixed point adjunctions for equivariant module spectra". In: Alg. Geom. Topol. 14.3 (2014), pp. 1779–1799.

[Hai22] Peter J. Haine. "From nonabelian basechange to basechange with coefficients". In: arXiv:2108.03545 (2022).

[HHR16] M. Hill, M. Hopkins, and D. Ravenel. "On the Non-Existence of Elements of Kervaire Invariant One". In: Ann. of Math. (2) 184.1 (2016), pp. 1–262.

[Hil22] K. Hilman. "Parametrised Presentability over Orbital Categories". In: arXiv:2202.02594 (2022).

[Hil24] K. Hilman. "Parametrised noncommutative motives and equivariant cubical descent in algebraic K-theory". In: arXiv:2202.02591 (2024).

[HKZ24] A. Horev, I. Klang, and F. Zou. "Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra". In: arXiv:2006.13348 (2024).

[HL13] Michael Hopkins and Jacob Lurie. "Ambidexterity in K(n)-Local Stable Homotopy Theory". In: available on author's webpage (2013).

[Hoy17] Marc Hoyois. "The six operations in equivariant motivic homotopy theory". In: Adv. Math. 305 (2017), pp. 197–279.

[HP06] B. Hanke and V. Puppe. "Equivariant Gysin Maps and Pulling Back Fixed Points". In: Trans. Amer. Math. Soc. 358.2 (2006), pp. 687–702.

[Ill83] Sören Illman. "The equivariant triangulation theorem for actions of compact Lie groups". In: Math. Ann. 262.4 (1983), pp. 487–501.

[Jon71] Lowell Jones. "The Converse to the Fixed Point Theorem of P. A. Smith: I". In: Annals of Mathematics 94.1 (1971), pp. 52–68.

[Kle01] John R. Klein. "The dualizing spectrum of a topological group". In: Math. Ann. 319.3 (2001), pp. 421–456.

[Kle07] John R. Klein. “The dualizing spectrum. II”. In: Algebr. Geom. Topol. 7 (2007), pp. 109–133.

[Lan22] Markus Land. “Reducibility of low-dimensional Poincaré duality spaces”. In: Münster J. Math. 15.1 (2022), pp. 47–81.

[LMS86] L.G. Lewis, J.P. May, and M. Steinberger. Equivariant stable homotopy theory. Vol. 1213. Lecture Notes in Mathematics. Springer–Verlag, 1986.

[LNP22] Sil Linskens, Denis Nardin, and Luca Pol. “Global homotopy theory via partially lax limits”. In: arXiv:2206.01556 (2022).

[Lüc22] Wolfgang Lück. “On Brown’s Problem, Poincaré models for the classifying spaces for proper actions and Nielsen Realization”. In: arXiv:2201.10807 (2022).

[Lüc88] Wolfgang Lück. “The equivariant degree”. In: Lecture Notes in Mathematics 1361 Konferenzbericht der Göttinger Topologie Tagung 1987 (1988).

[Lur09] Jacob Lurie. Higher topos theory. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925.

[Lur11] Jacob Lurie. “Lecture notes on algebraic L-theory and surgery”. In: Lecture notes (2011).

[Lur17] J. Lurie. “Higher Algebra”. In: available on author’s webpage (2017).

[Mar22a] Louis Martini. “Cocartesian fibrations and straightening internal to an ∞-topos”. In: arXiv:2204.00295 (2022).

[Mar22b] Louis Martini. “Yoneda’s lemma for internal higher categories”. In: arXiv:2103.17141 (2022).

[MNN17] Akhil Mathew, Niko Naumann, and Justin Noel. “Nilpotence and descent in equivariant stable homotopy theory”. In: Adv. Math. 305 (2017), pp. 994–1084.

[MNN19] A. Mathew, N. Naumann, and J. Noel. “Derived induction and restriction theory”. In: Geom. Topol. 23.2 (2019), pp. 541–636.

[MS06] J. P. May and J. Sigurdsson. Parametrized homotopy theory. Vol. 132. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006, pp. x+441.

[MW22] Louis Martini and Sebastian Wolf. “Presentable categories internal to an ∞-topos”. In: arXiv:2209.05103 (2022).

[MW24] Louis Martini and Sebastian Wolf. “Colimits and cocompletions in internal higher category theory”. In: arXiv:2111.14495 (2024).

[Nar16] D. Nardin. “Parametrized higher category theory and higher algebra: Exposé IV – Stability with respect to an orbital ∞–category”. In: arXiv:1608.07704 (2016).

[Nar17] Denis Nardin. “Stability and distributivity over orbital ∞-categories”. In: (2017). PhD Thesis.

[NS18] T. Nikolaus and P. Scholze. “On Topological Cyclic Homology”. In: Acta Math. 221.2 (2018), pp. 203–409.

[NS22] D. Nardin and J. Shah. "Parametrized and equivariant higher algebra". In: arXiv:2203.00072 (2022).

[Pal57] Richard S. Palais. "Imbedding of compact, differentiable transformation groups in orthogonal representations". In: J. Math. Mech. 6 (1957), pp. 673–678.

[PSW22] I. Patchkoria, B. Sanders, and C. Wimmer. "The spectrum of derived Mackey functors". In: Trans. Amer. Math. Soc. 375 (2022), pp. 4057–4105.

[QS22] J.D. Quigley and J. Shah. "On the parametrised Tate construction". In: arXiv:2110.07707 (2022).

[Qui72] Frank Quinn. "Surgery on Poincaré and normal spaces". In: Bull. Amer. Math. Soc. 78 (1972), pp. 262–267.

[Rob15] Marco Robalo. "K-theory and the bridge from motives to noncommutative motives". In: Adv. Math. 269 (2015), pp. 399–550.

[Sch18] Stefan Schwede. Global homotopy theory. Vol. 34. New Mathematical Monographs. Cambridge University Press, Cambridge, 2018, pp. xviii+828.

[Sch23] Peter Scholze. "Lecture notes on Six-Functor-Formalisms". In: Lecture notes (2023).

[Seg70] G.B. Segal. "Equivariant stable homotopy theory". In: Actes, Congres Intern. Math. Tome 2 (1970), pp. 59–63.

[Sha23] J. Shah. "Parametrized Higher Category Theory". In: Alg. Geom. Top. 23.2 (2023), pp. 509–644.

[Wal67] C. T. C. Wall. "Poincaré complexes. I". In: Ann. of Math. (2) 86 (1967), pp. 213–245.

[Zou23] F. Zou. "A geometric approach to equivariant factorization homology and nonabelian Poincaré duality". In: Math. Z. 303.98 (2023).