Lecture notes: an introduction to L_∞ algebras

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Abstract

In this note, we give an overview of (ordinary) Lie algebras, the universal enveloping algebra of a Lie algebra, the Chevalley-Eilenberg (co)algebras of a Lie algebra, and Lie algebra (co)homology. We will also discuss dg Lie algebras, L_∞-algebras, and the Chevalley-Eilenberg (co)algebras of an L_∞-algebra.

Contents

1 Glossary and summary chart

Recall from last week that we plan to study field theories via various "realms" of study; the perspective taken by CG in [2, 3] includes one realm given by L_∞ algebras, which we view as *homotopy Lie algebras*. The purpose of this note is to summarize the algebraic formalism of this realm, by defining L_∞ algebras, as well as Lie algebra cohomology.

Throughout this note, we primarily follow Appendix A.3 of [2] and Appendix A.1 of [3]. Fix $\mathbb{K}$ a field of characteristic 0, and R a $\mathbb{K}$-algebra. We'll begin with a sequence of definitions relating to (co)associative dg (co)algebras; this language underlies the language of L_∞ algebras, so it should be somewhat familiar before attempting the rest of the note.

Much of the information surrounding definitions of L_∞ algebras is then summarized in a chart in Subsection 1.2.

1.1 Glossary of dg algebras

Here we summarize some terms surrounding algebra in the differential-graded setting, which we'll often abbreviate into confusing initialisms.

Definition 1.1. A *ga*, or *graded R-algebra*, is a graded associative R-algebra, i.e. a $\mathbb{Z}$-graded R-module together A with a multiplication and unit map of graded R-modules

$$A \otimes A \to A \qquad\qquad R \to A$$

which are together associative and unital.

A *derivation* of an algebra A is an R-linear morphism $d : A \to A$ satisfying the *Leibniz rule*:

$$d(ab) = d(a)b + ad(b).$$

A **dga**, or *differential graded R-algebra*, or dga, is a ga equipped with a (graded) derivation $d : A \to A$ of degree 1 satisfying $d \circ d = 0$;

A ga is also known as a monoid in the category of graded R-modules, and a dga is also known as a monoid in the category of dg (differential graded) R-modules. We can impose commutativity internal to those categories as well:

Definition 1.2. A **cga**, or *graded commutative R-algebra* is a ga A satisfying graded commutativity; for $x, y \in A$ homogeneous, a gca satisfies
$$xy = (-1)^{|x||y|} yx.$$

A **cdga**, or *differential graded commutative R-algebra* is a dga whose underlying graded algebra is graded commutative.

All of these definitions naturally dualize:

Definition 1.3. A **gca**, or *graded R-coalgebra*, is a graded coassociative R-coalgebra.

A *coderivation* of a graded coalgebra A is an R-linear graded morphism $d : A \to A$ satisfying the *co-Leibniz rule*:
$$\Delta d = (d \otimes \mathrm{id} + \tau \circ (d \otimes \mathrm{id}) \circ \tau)\Delta$$
where $\tau : A^2 \to A^2$ is the map $(x, y) \mapsto (-1)^{|x||y|}(y, x)$. A **dgca**, or *differential graded R-coalgebra*, is a gca equipped with a (graded) derivation $d : A \to A$ of degree -1 satisfying $d \circ d = 0$.

A **cgca**, or *cocommutative graded R-coalgebra* is a cocommutative gca.

A **cdgca**, or *cocommutative differential graded R-coalgebra* is a dgca whose underlying gca is cocommutative.

We define an adjoint to the forgetful functor from gcas to graded modules:

Definition 1.4. For V_* a graded R-module, define the *graded symmetric algebra* $\mathrm{Sym}(V)$ by
$$\mathrm{Sym}(V_*) = \bigoplus_n \left((V_*)^{\otimes n}\right)_{S_n}$$
$$= \bigvee \left(\bigoplus V_{2n+1}\right) \otimes S\left(\bigoplus V_{2n}\right),$$
where S_n acts by the normal permutation representation on tensor powers times the Koszul sign rule, and $E(-)$ is the exterior algebra and $S(-)$ is the symmetric algebra. Define the *free cga on V_** to be the cga with underlying graded vector space given by $\mathrm{Sym}(V)$, and where multiplication is defined by the concatenation product
$$\mathrm{Sym}^n(V_*) \otimes \quad \mathrm{Sym}^m(V_*) \quad \longrightarrow \mathrm{Sym}^{n+m}(V_*)$$
$$(x_1 \cdots x_n) \otimes (x_{n+1} \cdots x_{n+m}) \longmapsto (x_1 \cdots x_{n+m}).$$

Define the *free cgca on V_** to be the cgca with underlying graded vector space given by $\mathrm{Sym}(V)$, and where comultiplication is defined by the coconcatenation product
$$\mathrm{Sym}^n(V_*) \longrightarrow \bigoplus_{i+j=n} \mathrm{Sym}^i(V_*) \otimes \mathrm{Sym}^j(V_*)$$
$$(x_1 \cdots x_n) \longmapsto \sum_{\sigma \in S_n} \sum_{1 \leq k \leq n-1} \left(x_{\sigma(1)} \cdots x_{\sigma(k)}\right) \otimes \left(x_{\sigma(k+1)} \cdots x_{\sigma(n)}\right).$$

We say that a cdga is *semifree* if its underlying cga is free (ie. it is $\mathrm{Sym}(V)$ for some graded R-module V_*), and likewise we say that a cdgca is semifree if its underlying cgca is free.

1.2 A summary chart

	Lie algebras	dg Lie algebras	L_∞ algebras
idea	$\mathbb{K}$-algebra satisfying: • skew-symmetry • jacobi identity	dg R-algebra satisfying: • graded skew-symmetry • graded jacobi identity	dg R-algebra satisfying: • graded skew-symmetry • strong homotopy jacobi identity
as operad algebras	algebra over Lie operad in $\mathrm{Vec}_\mathbb{K}$.	algebra over Lie operad in $\mathrm{Ch}_\bullet(R)$.	algebra over homotopy Lie operad (cofibrant replacement of Lie operad in $\mathrm{Ch}_\bullet(R)$).
as dg coalgebras	semifree dg cocommutative coalg. generated in deg. 1.	$*$	semifree dg cocommutative coalgebra.

2   Review of Lie algebras and their homological algebra

2.1   The definition of a Lie algebra

Definition 2.1. Let $\mathbb{K}$ be a field of characteristic 0. A *Lie algebra* over $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ equipped with a bilinear map $[,] : \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$, called its *bracket*, satisfying the following properties:

(i) *(anticommutativity)*. For all $x, y \in \mathfrak{g}$, we have

$$[x, y] = -[y, x]$$

(ii) *(jacobi identity)*. For all $x, y, z \in \mathfrak{g}$, we have

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

A homomorphism of Lie algebras is a linear map $f : \mathfrak{g} \to \mathfrak{h}$ respecting the bracket:

$$f([x, y]) = [f(x), f(y)]$$

We now give a family of examples of Lie algebras.

> **Example 2.2:**
> Let A be an associative algebra. Then, there is a Lie algebra $F(A)$ whose underlying vector space is the same as A, and whose bracket is given by
>
> $$[x, y] = xy - yx.$$
>
> Since there is an evident commutative triangle
>
> $$\mathrm{AssAlg}_{\mathbb{K}} \xrightarrow{\ F\ } \mathrm{LieAlg}_{\mathbb{K}}$$
> $$\mathit{forget} \searrow \qquad \downarrow \mathit{forget}$$
> $$\mathbf{Set}$$
>
> F is a faithful functor, and we are also justified in referring to it as a *forgetful functor*, and referring to $F(A)$ simply as A when the meaning is clear. We will see in Subsection 2.2 that F has a left adjoint, called the *universal enveloping algebra*.
>
> The Lie algebra associated with the algebra of endomorphisms of a vector space V is denoted $\mathfrak{gl}(V)$; if $V = k^n$ with a specified basis, this is instead denoted $\mathfrak{gl}_n$.

We want to do homological algebra over Lie algebras, so we need to define *modules over a Lie algebra*.

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra. A *module over $\mathfrak{g}$* or *representation of $\mathfrak{g}$* is a vector space M with a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(M)$ of Lie algebras. Equivalently, this is a bilinear map $\rho : \mathfrak{g} \otimes M \to M$ such that

$$\rho(x \otimes \rho(y \otimes m)) - \rho(y \otimes \rho(x \otimes m)) = \rho([x, y] \otimes m).$$

Usually we'll suppress the notation ρ and simply write $x \cdot m$ or $[x, m]$. A *homomorphism of $\mathfrak{g}$-modules* is a linear map $f : M \to N$ compatible with the $\mathfrak{g}$-action:

$$[x, f(m)] = f([x, m]).$$

We denote by $\mathfrak{g} - \mathbf{Mod}$ the category of $\mathfrak{g}$ modules.

> **Example 2.4:**
> The multiplication map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ gives $\mathfrak{g}$ the structure of a $\mathfrak{g}$-module; this is called the *adjoint representation* of $\mathfrak{g}$.

> **Example 2.5:**
> $\mathfrak{gl}_n$ acts on $\mathbb{K}^n$ by left-multiplication, so $\mathbb{K}^n$ is a $\mathfrak{gl}_n$-module.

In order to define Lie algebra homology and cohomology, we'll construct two natural functors, and define homology and cohomology as derived functors of these, analogously to group homology and cohomology. The first functor we construct is called the *invariants*:

$$(-)^g : \mathfrak{g} - \mathbf{Mod} \to \quad \mathrm{Vec}_{\mathbb{K}}$$
$$M \quad \mapsto \quad M^{\mathfrak{g}} := \{m \in M \mid [\mathfrak{g}, m] = 0\}$$

These are a linear version of the invariants of a module over a group. Even closer to the group case, we define the functor of *coinvariants* by

$$(-)_g : \mathfrak{g} - \mathbf{Mod} \to \quad \mathrm{Vec}_{\mathbb{K}}$$
$$M \quad \mapsto \quad M_{\mathfrak{g}} := M/\mathfrak{g}M = M/\{[x, m] \mid x \in \mathfrak{g}, m \in m\}.$$

We will define cohomology as the right derived functor of the functor of invariants, and homology as the left derived functor of the functor of coinvariants. In order to realize this concretely, we will take a detour into *universal enveloping algebras*, which embed the representation theory of Lie algebras into the representation theory of (infinite dimensional, noncommutative) associative algebras, where these derived functors will be realized as Ext and Tor functors.

2.2 Universal enveloping algebras and Lie algebra cohomology

Definition 2.6. The *universal enveloping algebra functor* is the left adjoint U to the forgetful functor from associative algebras to Lie algebras[1]

$$U : \mathrm{LieAlg}_k \quad \perp \quad \mathrm{AssAlg}_k : F$$

Remark. The universal enveloping algebra has a natural Hopf algebra structure, and hence modules over a Lie algebra have tensor products and duals. Further, modules M over $\mathfrak{g}$ have exterior powers $\Lambda^n M := (M^{\otimes n})_{S_n} = M^{\otimes n}/I$, where I is the submodule generated by $m_1 \otimes \cdots \otimes m_n$, where $m_i = m_j$ for some $i \neq j$.

This adjoint embeds the representation theory of Lie algebras within the representation theory of Hopf algebras via the natural isomorphism

$$\mathrm{Hom}(\mathfrak{g}, \mathfrak{gl}(V)) \simeq \mathrm{Hom}(U\mathfrak{g}, \mathrm{End}\, V).$$

Hence we may study the cohomology of $\mathfrak{g}$-modules via the cohomology of $U\mathfrak{g}$-modules.

> **Example 2.7:**
> We give a $U\mathfrak{g}$ bimodule structure on $\mathbb{K}$: first, we give it a trivial left action: $x \cdot \mathbb{K} = 0$. Next, realizing $U\mathfrak{g}/(\mathfrak{g}) = \mathbb{K}$ yields a right action of $U\mathfrak{g}$ on $\mathbb{K}$.

Note that

$$M_g = \mathbb{K} \otimes_{U\mathfrak{g}} M \qquad \text{and} \qquad M^g = \mathrm{Hom}_{U\mathfrak{g}}(\mathbb{K}, M).$$

Hence we may straightforwardly verify that the invariants are left exact, and the coinvariants are right exact; furthermore, their derived functors have familiar names:

Definition 2.8. For M a $\mathfrak{g}$-module, the *Lie algebra homology of M* is

$$H_*(\mathfrak{g}, M) := \mathrm{Tor}_*^{U\mathfrak{g}}(\mathbb{K}, M)$$

and the *Lie algebra cohomology of M* is

$$H^*(\mathfrak{g}, M) := \mathrm{Ext}_{U\mathfrak{g}}^*(\mathbb{K}, M).$$

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[1]This adjoint may be realized pointwise as a quotient of the tensor algebra: for $T\mathfrak{g} := \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n}$, define the two-sided ideal I generated by elements of the form

$$a \otimes b - b \otimes a - [a, b] \in \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}$$

Then, define

$$U\mathfrak{g} := T\mathfrak{g}/I.$$

Verifying the adjunction is straightforward from this definition. In fact, $T\mathfrak{g}$ is a Hopf algebra with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in V$ and $\Delta(1) = 1 \otimes 1$, counit given by $\varepsilon(x) = 0$ for $x \in V$, and antipde given in degree 1 by $S(x) = -x$; this preserves I, so it descends to a Hopf algebra structure on $U\mathfrak{g}$.

2.3 The Chevalley-Eilenberg complex for Lie algebras

Naturally, we now describe a popular choice of projective and injective resolution for $\mathbb{K}$, which we may use to explicitly compute (co)homology.

Recall that $\mathbb{K}$ is a quotient of $U\mathfrak{g}$ by $\mathfrak{g}$. We can define the resolution

$$(\cdots \to \Lambda^n \mathfrak{g} \otimes_k U\mathfrak{g} \to \cdots \to \mathfrak{g} \otimes_k U\mathfrak{g} \to U\mathfrak{g}) \xrightarrow{\cong} k,$$

where the rightmost map is the quotient by $(\mathfrak{g})$, and the remaining maps are defined by

$$(y_1 \wedge \cdots \wedge y_n) \otimes (x_1 \cdots x_n) \mapsto \sum_{k=1}^{n} (-1)^{n-k} (y_1 \wedge \cdots \widehat{y_k} \cdots \wedge y_n) \otimes (y_k \cdot x_1 \cdots x_m)$$
$$- \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} ([y_j, y_k] \wedge y_1 \wedge \cdots \widehat{y_j} \cdots \widehat{y_k} \cdots \widehat{y_n}) \otimes (x_1 \cdots x_m)$$

where $\widehat{\cdot}$ denotes omission. Taking the tensor product of or hom to this resolution yields complexes which we use to compute homology and cohomology.

Note that our resolution has underlying graded vector space $\mathrm{Sym}_k(\mathfrak{g}[1])$, where $\mathfrak{g}$ is considered as a graded vector space concentrated in degree 0 and $[n]$ is the grading shift operatorn. Further note that the dual to our resolution has underlying graded vector space $\mathrm{Sym}_k(\mathfrak{g}^\vee[-1])$. We use this notation to define complexes we use to compute (co)homology, named after Chevalley and Eilenberg, who first introduced them.

Definition 2.9. The *Chevalley-Eilenberg complex for the Lie algebra cohomology of the $\mathfrak{g}$-module M* is

$$C_*^{\mathrm{Lie}}(\mathfrak{g}, M) := (\mathrm{Sym}_{\mathbb{K}}(\mathfrak{g}[1]) \otimes_{\mathbb{K}} M, d)$$

where the differential d encodes the bracket of $\mathfrak{g}$ on itself and on M:

$$d(y_1 \wedge \cdots \wedge y_n \otimes m) = \sum_{k=1}^{n} (-1)^{n-k} y_1 \wedge \cdots \widehat{y_k} \cdots \wedge y_n \otimes [y_k, m]$$
$$- \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} [y_j, y_k] \wedge y_1 \wedge \cdots \widehat{y_j} \cdots \widehat{y_k} \cdots \widehat{y_n} \otimes m$$

We simply write $C_*^{\mathrm{Lie}}(\mathfrak{g}) := C_*^{\mathrm{Lie}}(\mathfrak{g}, k)$.

Recall that the underlying graded vector space of $C_*^{\mathrm{Lie}}(\mathfrak{g})$ has the structure of a cgca under coconcatenation. Note that the differential is the unique coderivation of degree -1 with action restricting to $\mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ given by the Lie bracket; in general, the co-Leibniz rule expresses coderivations of a (graded) coalgebra in terms of cogenerators. Hence $C_*^{\mathrm{Lie}}(\mathfrak{g})$ is a (semifree) cgdca.

Further, the property $d^2 = 0$ boils down to the Jacobi identity; for instance, in the case $n = 1$, we have

$$d(y_1 \wedge y_2 \wedge y_3) = [y_1, y_2] \wedge y_3 + [y_2, y_3] \wedge y_1 - [y_1, y_3] \wedge y_2$$

so that

$$d^2(y_1 \wedge y_2 \wedge y_3) = [[y_1, y_2], y_3] + [[y_2, y_3], y_1] - [[y_1, y_3], y_2]$$

and the requirement $d^2 = 0$ is equivalent to the Jacobi identity in degree 3 (which in turn could prove that $d^2 = 0$ in higher degree). This proves something strong: $C_*^{\mathrm{Lie}}(-)$ is (the objection function of) an equivalence of categories between Lie algebras and semifree cdgcas which are generated in degree 1.

Now that we're familiar with this language of defining coderivations on cogenerators, we can draw the dual picture concisely:

Definition 2.10. The Chevalley-Eilenberg complex for the Lie algebra cohomology of the $\mathfrak{g}$-module M is

$$C^*_{\mathrm{Lie}}(\mathfrak{g}, M) := (\mathrm{Sym}_{\mathbb{K}}(\mathfrak{g}^\vee[-1]) \otimes_{\mathbb{K}} M, d)$$

where the differential d encodes the linear dual to the bracket of $\mathfrak{g}$ on itself and on M; fixing a linear basis $\{e_k\}$ for $\mathfrak{g}$ and writing $\{e^k\}$ for the dual basis, we have

$$d(e^k \otimes m) = \sum_l e^k \wedge e^l \otimes [e_l, m] - \sum_{i<j} e^k([e_i, e_j]) e^i \wedge e^j \otimes m,$$

and d is extended to the rest of the complex as a derivation of cohomological degree 1. We simply write $C^*_{\mathrm{Lie}}(\mathfrak{g}) := C^*_{\mathrm{Lie}}(\mathfrak{g}, k)$.

Note that $C^*_{\mathrm{Lie}}(\mathfrak{g})$ is a commutative dg algebra under the concatenation product; as in the dual case, $C^*_{\mathrm{Lie}}(-)$ is the object function of an equivalence of categories between Lie algebras and semifree cdgas which are generated in degree -1.

3 Homotopical generalizations: dg Lie algebras and L_∞ algebras

We'll now introduce the appropriate notion of a homotopy Lie algebra. We extend Lie algebras to the dg setting first, then use the grading and differential to weaken the Jacobi identity into holding only up to coherent homotopy. For the remainder of this section, R is a commutative $\mathbb{K}$-algebra.

3.1 Dg Lie algebras

The following definition says that a dg Lie algebra is an algebra over the Lie operad in the category of R-chain complexes.

Definition 3.1. A *dg Lie algebra over R* is a dg R-module $(\mathfrak{g}, d)$ together with a graded bilinear map $[,] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$ such that:

(i) *(graded Leibniz rule).*
$$d[x,y] = [dx,y] + (-1)^{|x|}[x,dy].$$

(ii) *(graded antisymmetry).*
$$[x,y] = (-1)^{|x||y|+1}[y,x].$$

(iii) *(graded Jacobi identity).*
$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]].$$

The graded Leibniz rule says that this map is compatible with the differential on $\mathfrak{g} \otimes_R \mathfrak{g}$. Graded antisymmetry is a translation of antisymmetry into the graded context (where twists are applied in odd degrees). The graded Jacobi rule is a translation of the ordinary Jacobi rule to the graded context (again, twisted in odd degree).

Homological algebra of dg Lie algebras is well studied, as it appears prominently in the study of rational homotopy theory via the following example:

> **Example 3.2:**
> For X a space, the graded Abelian group $\pi_*(X)$ can be embedded with a (functorial) product $[,] : \pi_k(X) \oplus \pi_l(X) \to \pi_{k+l-1}(X)$ as follows: construct S^{k+l} as a relative CW complex over $S^k \vee S^l$ by attacking a single $k+l$-cell; the attaching map $\varphi : S^{k+l-1} \to S^k \vee S^l$ yields a pullback natural transformation
> $$\left[S^k, -\right] \oplus \left[S^l, -\right] = \left[S^k \vee S^l, -\right] \xrightarrow{\varphi^*} \left[S^{k+l-1}, -\right]$$
> This map is bilinear, graded-symmetric, and satisfies the graded Jacobi rule.
>
> We want to turn this into a dg Lie algebra structure: first, note that the grading is wrong; by bilinearity, we have maps $\pi_k(X) \otimes \pi_l(X) \to \pi_{k+l-1}(X)$, but we want the degree of the codomain to be the sum of the degrees of the domain. This is rectified by redefining the grading, so that the degree of $\pi_k(X)$ is $k-1$. With this grading, the map is instead graded skew-symmetric.
>
> Tensoring all of this with $\mathbb{Q}$ yields a graded $\mathbb{Q}$-Lie algebra. This is the object function of a functor, which was proved by Quillen to yield an equivalence from the rational homotopy category (localization of **Top** by $\pi_* \otimes \mathbb{Q}$-isomorphisms) to the homotopy category of dg Lie algebras (under dg Lie algebra morphisms whose underlying chain maps are quasi-isomorphisms, i.e., the graded Lie algebras).
>
> See [4] or other texts on rational homotopy theory for more on this example, and on dg Lie algebras in general.

Similar to the endomorphism Lie algebra, we have an endomorphism dg Lie algebra:

> **Example 3.3:**
> Let (V, d_V) be a chain complex. The graded vector space $\mathrm{End}(V) = \bigoplus_n \mathrm{Hom}^n(V,V)$, where $\mathrm{Hom}^n(V,V)$ consists of endomorphisms which shift degree by n, can be endowed with a differential
> $$d_{\mathrm{End}\,V} = [d_V, -] : f \mapsto d_V \circ f - (-1)^{|f|} f \circ d_V.$$
> This has a natural structure as a dg associative k-algebra, and the commutator bracket gives it a structure as a dg k-Lie algebra.

We can form a chain and cochain form of the Chevalley-Eilenberg complex for dg Lie algebras, but to avoid repetition, we'll now introduce L_∞-algebras, and define the Chevalley-Eilenberg complexes in that setting.

3.2 L_∞-algebras.

Definition 3.4. Let $\mathrm{Unshuff}(i, n-i) \subset S_n$, called the *unshuffle permutations* denote the subset of S_n given by permutations σ who have an index i such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. Given such a permutation and a homogeneous element $x_1 \otimes \cdots \otimes x_n \in \mathfrak{g}^{\otimes n}$, define the *alternating Koszul sign* $\chi(\sigma, x_1, \ldots, x_n)$ by the following on simple transpositions:

$$\chi((i \quad i+1), x_1, \ldots, x_n) = (-1)^{|x_i||x_{i+1}|+1}$$

and extend it by multiplying signs when composing permutations.

This is the appropriate sign rule to define our homotopical generalization of dg Lie algebras:

Definition 3.5. An L_∞-*algebra* is a $\mathbb{Z}$-graded, projective R-module $\mathfrak{g}$ equipped with a sequence of multilinear maps of cohomological degree $2 - n$

$$\ell_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$$

satisfying the following properties:

(i) *(graded antisymmetry)* for each $1 \le i, i+1 \le n$, we have

$$\ell_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (-1)^{|x_i||x_{i+1}|+1}\ell_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n).$$

(ii) *(strong homotopy Jacobi identity)*:

$$0 = \sum_{i=1}^{n}(-1)^i \sum_{\sigma \in \mathrm{Unshuff}(i,n-i)} \chi(\sigma, x_1, \ldots, x_n) \cdot \ell_{n-i+1}(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}).$$

We'll now unravel the strong homotopy Jacobi identity in low dimensions; call this identity in the n-ary case the *n-Jacobi rule*. Then, the 1-Jacobi rule simply says

$$\ell_1(\ell_1(x)) = 0,$$

i.e. the map $d := \ell_1$ is a differential.

Write $[x_1, \ldots, x_n] := \ell_n(x_1, \ldots, x_n)$. The 2-Jacobi rule states that

$$0 = [dx, y] - [dy, x] + d[x, y]$$

which recovers the graded Leibniz rule.

The 3-Jacobi rule says that

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = d[x_1, x_2, x_3] + [dx_1, x_2, x_3] + [dx_2, x_1, x_3] + [dx_3, x_1, x_2].$$

i.e. the Jacobi rule holds *up to other brackets*; if $d = 0$ or $\ell_3 = 0$ then it holds on the nose.

The higher Jacobi rules are complicated, but can be shown to be implied by the 3-Jacobi rule in the case that $\ell_n = 0$ for $n \ge 3$; hence dg Lie algebras are the same thing as L_∞ algebras with vanishing higher brackets. In particular, ordinary Lie algebras are the same thing as L_∞ algebras concentrated at degree zero.

3.3 The Chevalley-Eilenberg complexes for L_∞-algebras

As in the Lie algebra case, we will define two Chevalley-Eilenberg complexes.

Definition 3.6. For $\mathfrak{g}$ an L_∞ algebra, the *Chevalley-Eilenberg complex for homology* $C_*^{\mathrm{Lie}}(\mathfrak{g})$ is the cdgca

$$\mathrm{Sym}_R(\mathfrak{g}[1]) = \bigoplus_{n=0}^{\infty} \left((\mathfrak{g}[1])^{\otimes n}\right)_{S_n}$$

equipped with the coderivation d whose restriction to cogenerators $d_n : \mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ are precisely the higher brackets ℓ_n.

Before defining the other complex, we take a detour to define maps of L_∞ algebras $\mathfrak{g} \rightsquigarrow \mathfrak{h}$. Though the perspective of L_∞ algebras as *homotopy Lie algebras*, we won't want to just define these on the nose, but instead as homotopy coherent stacks of maps $\mathrm{Sym}^n(\mathfrak{g}) \to \mathfrak{h}$. Luckily, the dgca structure on $C_*^{\mathrm{Lie}}(\mathfrak{g})$ summarizes all of the higher brackets in only one structure (the differential), so we have a convenient language to succinctly define homotopy coherent maps of L_∞ algebras:

Definition 3.7. A *map of L_∞ algebras* $F : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is given by a map of cdgcas $F : C_*^{\mathrm{Lie}}(\mathfrak{g}) \to C_*^{\mathrm{Lie}}(\mathfrak{h})$.

In order to define the cdga $C_{\mathrm{Lie}}^*(\mathfrak{g})$, we need to define the right notion of the dual, which we will take graded R-linearly:

$$\mathfrak{g}^\vee := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_R(\mathfrak{g}_n, R)[n].$$

With this notation, we can define the other complex, with one special quirk:

Definition 3.8. The *Chevalley-Eilenberg complex for cohomology* $C_{\mathrm{Lie}}^*(\mathfrak{g})$ is the completed cdga

$$\widehat{\mathrm{Sym}}_R(\mathfrak{g}^\vee[1]) = \prod_{n=0}^\infty \left(\left(\mathfrak{g}[1]^\vee \right)^{\otimes n} \right)_{S_n}.$$

Here the completion is taken with respect to the ideal generated by $\mathfrak{g}[1]^\vee$.

As in [3], we adopt the philosophy that all commutative algebras are the functions on some "space," so we interpret $C_{\mathrm{Lie}}^*(\mathfrak{g})$ as the functions on some "space" $B\mathfrak{g}$, and $C_*^{\mathrm{Lie}}(\mathfrak{g})$ as distributions on $B\mathfrak{g}$. The we view the natural pairing between $C_{\mathrm{Lie}}^*(\mathfrak{g})$ and $C_*^{\mathrm{Lie}}(\mathfrak{g})$ as the pairing between functions and distributions. We denote by $\mathrm{Der}(\mathfrak{g})$ the module $C_{\mathrm{Lie}}^*(\mathfrak{g}, \mathfrak{g}[1])$, i.e. the completion of $\mathrm{Sym}_R(\mathfrak{g}^\vee[1]) \otimes \mathfrak{g}[1]$, called the *derivations of* $\mathfrak{g}$. This complex is naturally a dg Lie algebra, and acts canonically on $C_{\mathrm{Lie}}^*\mathfrak{g}$ by the Lie derivative.

3.4 Revisiting Lie algebra cohomology

By examining the (classical) Chevalley-Eilenberg complex, we arrive at the following characterization of cohomology of (ordinary) Lie algebras:

- Assume that $\mathfrak{g}$ is finite dimensional. Then, $C_{\mathrm{Lie}}^*(\mathfrak{g}) \simeq \mathrm{Hom}(C_*^{\mathrm{Lie}}(\mathfrak{g}), M)$, and hence

$$H^1(\mathfrak{g}; M) = \mathrm{Der}(\mathfrak{g}, M) / \mathrm{Ider}(\mathfrak{g}, M).$$

  i.e. the first cohomology is the group of derivations $\mathfrak{g} \to M$ modulo the group of inner derivations, which are given by bracketing by a single element.

- From a similar perspective, one may note that $H^2(\mathfrak{g}; M)$ is the group of equivalence classes of Lie algebra extensions

$$0 \to M \to \mathfrak{h} \to \mathfrak{g} \to 0$$

  under the Baer sum.

- As we increase degree, a Lie algebra cochain supplies maps $\ell_n : \mathfrak{g}^{\otimes n} \to M$, which are assumed to satisfy a version of the strong homotopy Jacobi identity on the nose. In fact, for $n \geq 3$, this descends to an isomorphism between $H^n(\mathfrak{g}; M)$ and the set of isomorphism classes of L_∞ algebra structures on the graded vector space $\mathfrak{g} \oplus M[2-n]$. For more detail, see [1] (noting that they use an opposing sign convention).

References

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