ON TENSOR PRODUCTS OF EQUIVARIANT COMMUTATIVE OPERADS

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Abstract. We lift the Boardman-Vogt tensor product to a symmetric monoidal closed G-∞-category $\mathrm{Op}_G^{\otimes}$ of $\mathcal{O}_G$-∞-operads. Using this, we construct a G-colocalizing subcategory

$$\mathcal{N}_{(-)\infty} : \underline{\mathrm{wIndex}}_G \hookrightarrow \underline{\mathrm{Op}}_G$$

called the *G-poset of weak $\mathcal{N}_\infty$-G-operads* whose colocalization functor constructs *arity support* weak indexing system.

We precisely characterize the weak $\mathcal{N}_\infty$-G-operads whose tensor powers are weak $\mathcal{N}_\infty$, called the *aE-unital weak $\mathcal{N}_\infty$-G-operads*. We show that the G-subcategory

$$\underline{\mathrm{wIndex}}_G^{aE-\mathrm{uni},\otimes} \hookrightarrow \underline{\mathrm{Op}}_G^{\otimes}$$

is pointwise-symmetric monoidal and combinatorially characterize its tensor products; in particular, the full G-subcategory of *unital weak $\mathcal{N}_\infty$-G-operads* is cocartesian symmetric monoidal, i.e. its tensor products are joins of weak indexing systems.

As a special case, we recognize Blumberg-Hill's $\mathcal{N}_\infty$-operads as a symmetric monoidal sub-poset $\mathrm{Index}_G^{\vee} \subset \mathrm{wIndex}_G^{\mathrm{uni},\vee}$ confirming a conjecture of Blumberg-Hill. In particular, for I, J unital weak indexing systems and $\mathcal{C}$ an $I \vee J$-symmetric monoidal ∞-category, we construct a canonical $I \vee J$-symmetric monoidal equivalence

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes} \mathcal{C} \simeq \underline{\mathrm{CAlg}}_{I \vee J}^{\otimes} \mathcal{C}.$$

Contents

Introduction

Summary of main results. Homotopy-coherent algebraic structures in genuine-equivariant mathematics are naturally founded in the notion of *G-commutative monoids.* In the context of this paper, the *∞-category of G-commutative monoids* in an ∞-category $\mathcal{D}$ is the ∞-category of product-preserving functors

$$\mathrm{CMon}_G(\mathcal{D}) := \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_G), \mathcal{D}),$$

where $\mathbb{F}_G$ denotes the category of finite G-sets.[1] The ∞-category of *small G-symmetric monoidal ∞-categories* is $\mathbf{Cat}_G^\otimes \deg \mathrm{CMon}_G(\mathbf{Cat})$, where **Cat** denotes the ∞-category of small ∞-categories.

Given $\mathcal{C}^\otimes \in \mathbf{Cat}_G^\otimes$ a G-symmetric monoidal ∞-category, the product-preserving functor

$$\iota_H : \mathrm{Span}(\mathbb{F}) \xrightarrow{* \mapsto G/H} \mathrm{Span}(\mathbb{F}_G)$$

constructs a symmetric monoidal ∞-category $\mathcal{C}_H^\otimes := \iota_H^* \mathcal{C}^\otimes$ whose underlying ∞-category $\mathcal{C}_H$ is the *value* of $\mathcal{C}^\otimes$ on the orbit G/H.[2] For all subgroups $K \subset H \subset G$, the covariant and contravariant functoriality of $\mathcal{C}^\otimes$ then yield a lax-symmetric monoidal *restriction* and symmetric-monoidal *norm* functor

$$\mathrm{Res}_K^H : \mathcal{C}_H^\otimes \to \mathcal{C}_K^\otimes,$$
$$N_K^H : \mathcal{C}_K^\otimes \to \mathcal{C}_H^\otimes.$$

We use this structure to encode *algebras* in $\mathcal{C}^\otimes$, for which we need a notion of *G-operads.*

Various notions of G-operad have been introduced for this. In Section 2.3 we introduce an ∞-category Op_G of *$\mathcal{O}_G$-∞-operads* (henceforth G-operads) equivalent to that of [NS22]. Given $\mathcal{O}^\otimes \in \mathrm{Op}_G$ a G-operad and $S \in \mathbb{F}_H$ an H-set for some $H \subset G$, we construct a *space of S-ary operations* $\mathcal{O}(S)$, together with *operadic composition maps*

$$\mathcal{O}(S) \otimes \bigotimes_{H/K_i \in \mathrm{Orb}(S)} \mathcal{O}(T_i) \to \mathcal{O}\left(\coprod_{H/K_i \in \mathrm{Orb}(S)} \mathrm{Ind}_{K_i}^H T_i \right), \tag{1}$$

operadic restriction maps

$$\mathcal{O}(S) \to \mathcal{O}(\mathrm{Res}_K^H S), \tag{2}$$

[1] In this paper we will call ∞-categories *∞-categories* and 0-truncated ∞-categories *1-categories.* We hope this prevents avoidable confusion in older readers.

[2] In this paper, "orbits" refer to transitive G-sets, i.e. objects of the orbit category $\mathcal{O}_G$.

and *equivariant symmetric group action*

$$\mathrm{Aut}_H(S) \times \mathcal{O}(S) \to \mathcal{O}(S). \tag{3}$$

Eqs. (2) and (3) together ascend to a structure of a G-*symmetric sequence*; we go on to show in Corollary 2.68 that this structure is *monadic* under a reducedness assumption.

Definition. We say that $\mathcal{O}^\otimes$ *has at least one color* if $\mathcal{O}(*_H)$ is nonempty for all subgroups $H \subset G$, and we say $\mathcal{O}^\otimes$ *has at most one color* if $\mathcal{O}(*_H) \in \{*, \emptyset\}$ for all $H \subset G$. We say that $\mathcal{O}^\otimes$ *has one color* if it has at least one color and at most one color. ◁

When $\mathcal{O}^\otimes$ has one color, an $\mathcal{O}$-algebra in the G-symmetric monoidal ∞-category $\mathcal{C}^\otimes$ can intuitively be viewed[3] as a tuple $(X_H \in \mathcal{C}_H^{BW_G(H)})_{G/H \in \mathcal{O}_G}$ satisfying $X_K \simeq \mathrm{Res}^H_K X_H$, together with $\mathcal{O}(S)$-actions

$$\mu_S : \mathcal{O}(S) \otimes X_H^{\otimes S} \to X_H \tag{4}$$

for all $S \in \mathbb{F}_H$ and $H \subset G$, homotopy-coherently compatible with the maps Eqs. (1) to (3), where we write

$$X_H^{\otimes S} := \bigotimes_{H/K \in \mathrm{Orb}(S)} N_K^H \mathrm{Res}^H_K X_H.$$

for the *indexed tensor products* in $\mathcal{C}^\otimes$. In this paper, we are concerned with defining *indexed tensor products* of $\mathcal{O}$-algebras, as well as $\mathcal{P}$-algebras in the resulting G-symmetric monoidal ∞-category. Mirroring the nonequivariant case, we will accomplish this by realizing the operad of $\mathcal{O}$-alebras in $\mathcal{P}$ as the *internal hom* with respect to a symmetric monoidal structure on the ∞-category of G-operads.

In order to characterize this tensor product, we will relate it to a tensor product on the category of G-symmetric monoidal ∞-categories. In Section 1.1 we define the ∞-category of G-∞-*categories* to be

$$\mathbf{Cat}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Cat}).$$

As a weakening of the notion of G-symmetric monoidal ∞-categories, we define a *symmetric monoidal* G-∞-*category* to be a commutative monoid object in $\mathbf{Cat}_G$. The restriction structure between the ∞-categories $\mathrm{CMon}_G(\mathcal{C})$ is summarized defining a G-∞-category $\underline{\mathrm{CMon}}_G(\mathcal{C})$ with the values

$$\left(\underline{\mathrm{CMon}}_G(\mathcal{C})\right)_H := \mathrm{CMon}_H(\mathcal{C}).$$

This G-∞-category underlies a symmetric monoidal G-∞-category satisfying an analogous universal property to [GGN15, Thm 5.1], whose free functor depends on the G category of coefficient systems

$$\left(\underline{\mathrm{Coeff}}_G\mathcal{C}\right)_H := \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \mathcal{C}).$$

Theorem A. *If $\mathcal{C}$ is a presentable ∞-category, then there exists a unique symmetric monoidal structure $\underline{\mathrm{CMon}}_G^{\otimes-\mathrm{mode}}(\mathcal{C})$ on $\underline{\mathrm{CMon}}_G(\mathcal{C})$ such that the free G-commutative monoid G-functor*

$$\underline{\mathrm{Coeff}}_G\mathcal{C} \to \underline{\mathrm{CMon}}_G(\mathcal{C})$$

possesses a (necessarily unique) symmetric monoidal structure.

In Section 1.3, we generalize the above theorem generalizes directly to the setting of G-*presentable* ∞-*categories* as developed in [Hil24]. We use this to define the coherences on a *Boardman-Vogt symmetric monoidal structure on G-categories.*

Theorem B. *There exists a unique symmetric monoidal structure $\underline{\mathrm{Op}}_G^\otimes$ on $\underline{\mathrm{Op}}_G$ attaining a (necessarily unique) symmetric monoidal structure on the fully faithful G-functor*

$$\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup}} : \underline{\mathrm{Op}}_G^\otimes \to \underline{\mathrm{CMon}}_G^{\otimes-\mathrm{mode}}(\mathbf{Cat})_{/\underline{\mathbb{F}}_{\mathcal{T}}},$$

Furthermore, $\underline{\mathrm{Op}}_G^\otimes$ satisfies the following.

[3] Throughout this paper, we let the *orbit category* $\mathcal{O}_G \subset \mathbb{F}_G$ be the full subcategory spanned by transitive G-sets G/H.

(1) *In the case $G = e$, there is a canonical symmetric monoidal equivalence $\mathrm{Op}_e^{\otimes} \simeq \mathrm{Op}_\infty^{\otimes}$, where the codomain has the symmetric monoidal structure of [BS24a]; in particular, the underlying tensor product is equivalent to that of [BV73; HM23; HA].*[4]
(2) *the underlying tensor functor $- \otimes^{\mathrm{BV}} \mathcal{O} : \mathrm{Op}_G \to \mathrm{Op}_G$ possesses a right adjoint $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(-)$, whose underlying G-∞-category is the G-∞-category of algebras $\underline{\mathbf{Alg}}_{\mathcal{O}}(-)$; the associated ∞-category is the ∞-category of algebras $\mathbf{Alg}_{\mathcal{O}}(-)$.*
(3) *The unit of $\mathrm{Op}_G^{\otimes}$ is the G-operad $\mathrm{triv}_{\mathcal{T}}^{\otimes}$ defined in [NS22]; hence $\underline{\mathbf{Alg}}_{\mathrm{triv}_{\mathcal{T}}}^{\otimes}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}$.*
(4) *When $\mathcal{C}^{\otimes}$ is G-symmetric monoidal, $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is G-symmteric monoidal; furthermore, when $\mathcal{O}^{\otimes}$ has one object, the forgetful functor*
$$\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}$$
is G-symmetric monoidal.
(5) *When $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a G-symmetric monoidal functor, the induced lax G-symmetric monoidal functor*
$$\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C}) \to \underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{D})$$
is G-symmetric monoidal.

Remark. In analogy to [BV73], in Observation 2.27 we interpret algebras over the tensor product $\mathcal{O}^{\otimes} \otimes^{\mathrm{BV}} \mathcal{P}^{\otimes}$ in a G-symmetric monoidal category $\mathcal{C}^{\otimes}$ as *Bifunctors of G-operads $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \to \mathcal{C}^{\otimes}$*; unwinding definitions in the case $\mathcal{C}^{\otimes}$ is G-symmetric monoidal (e.g. as in Section 5.2), we interpret these as *interchanging pairs of $\mathcal{O}$- and $\mathcal{P}$-algebras structures on an object of $\mathcal{C}$*; we show that this fully determines $\otimes^{\mathrm{BV}}$ in Corollary 4.7.

Furthermore, by Yoneda's lemma, the $\mathcal{T}$-operad $\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ itself is determined by the property that its $\mathcal{O}$-algebras are interchanging pairs of $\mathcal{O}$- and $\mathcal{P}$-algebra structures on an object in $\mathcal{C}$; we show in Philosophical remark 4.1 that G-symmetric monoidal ∞-categories are determined by their underlying G-operads, so this fully determines $\underline{\mathbf{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{C})$ as a G-symmetric monoidal ∞-category. ◁

Remark. After this introduction, we replace $\mathcal{O}_G$ with an atomic orbital ∞-category $\mathcal{T}$ for the remainder of the paper; we prove Theorem B as well as other theorems in this introduction in this setting, greatly generalizing the stated results, at the cost of intuition. ◁

Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_G^{\mathrm{oc}}$ a G-operad with one color and $\psi : T \to S$ a map of finite H-sets, we also define the *space of multimorphisms*[5]
$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) := \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_S U).$$
We then define the subcategory[6] $A\mathcal{O} \subset \mathbb{F}_G$ of *$\mathcal{O}$-admissible maps* by
$$A\mathcal{O} := \left\{ \psi : T \to S \;\middle|\; \mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) \neq \varnothing \right\} \subset \mathbb{F}_G.$$
In essence, taking tensor products of Eq. (4) yields an action
$$\mathrm{Mul}_{\mathcal{O}}^{\psi}(T;S) \otimes X_H^{\otimes T} \to X_H^{\otimes S},$$
and $A\mathcal{O}$ consists of the *pairs of equivariant arities* over which this produces structure on X.

The fact that $\varnothing$ accepts no maps from nonempty sets potentially obstructs construction of maps as in Eqs. (1) and (2), so $A\mathcal{O}$ can't be an *arbitrary* subcategory. In Corollary 3.10, we combinatorially characterize the image of A in $\mathrm{Sub}(\mathbb{F}_G)$ as the poset wIndex_G of *weak indexing systems*, a weakened variant of the notion introduced in [BP21] which is studied combinatorially in upcoming work of the author [St].

[4] The author shares the perspective of [BHS22, Rem p.4] that the construction of [HA] has coherences which are delicate, inaccessible, and as far as she knows, yet to be applied anywhere. On the other hand, the coherences of the construction of this paper and [BHS22] are canonically determined by symmetric monoidality of the sliced G-symmetric monoidal envelope, whose underlying functor is in frequent use.

[5] We only make the assumption that $\mathcal{O}^{\otimes}$ has one color for ease of exposition; throughout the remainder of text following the introduction, we will not make this assumption.

[6] Throughout this paper, we say *subobject* to mean monomorphism in the sense of [HTT, § 5.5.6]; in the case the ambient ∞-category is a 1-category, this agrees with the traditional notion.

In the case our objects are in the ∞-category **Cat** of small ∞-categories, we call this a *subcategory*; in the case that the containing ∞-category is a 1-category, this is canonically expressed as a *wide subcategory of a full subcategory*, and it is uniquely determined by its morphisms, so we will implicitly identify subcategories of $\mathcal{C}$ a 1-category with their corresponding subsets of $\mathrm{Mor}(\mathcal{C})$.

We say that a G-operad $\mathcal{O}^{\otimes}$ is *E-unital* if

$$\mathcal{O}(\varnothing_V) \in \begin{cases} * & \mathcal{O}(*_V) \neq \varnothing; \\ \varnothing & \mathcal{O}(*_V) = \varnothing. \end{cases}$$

We say that $\mathcal{O}^{\otimes}$ is *unital* if it is E-unital and has at least one color. We denote the full subcategory spanned by unital G-operads by $\mathrm{Op}_G^{\mathrm{uni}} \subset \mathrm{Op}_G$.

Theorem C. *The following posets are each equivalent:*

1. *The poset* $\mathrm{Sub}_{\mathbf{Cat}_G^{\otimes}}(\mathbb{F}_G^{\amalg})$ *of G-symmetric monoidal subcategories of* $\mathbb{F}_G^{\amalg}$.
2. *The poset* $\mathrm{Sub}_{\mathrm{Op}_G}(\mathrm{Comm}_G)$ *of* sub-commutative G-operads.
3. *The poset* $\left\{\mathrm{triv}_{\varnothing}^{\otimes}\right\} \star \mathrm{Sub}_{\mathrm{CAlgOp}_G}(\mathrm{Comm}_G)$, *where the right subposet consists of nonempty sub-commutative G-operads possessing split codiagonal natural transformations*

$$\begin{array}{ccc} & \mathbf{Alg}_{\mathcal{O}}\underline{\mathbf{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) & \\ \nearrow & & \searrow^{U} \\ \mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) & = & \mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \end{array}$$

4. *The poset* $\mathrm{Op}_{G,\leq -1}$ *of* G-(-1)-operads.
5. *The image* $A(\mathrm{Op}_G) \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$
6. *The sub-poset* $\mathrm{wIndex}_G \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G)$ *spanned by subcategories $I \subset \mathbb{F}_{\mathcal{T}}$ which are closed under base change and automorphisms and satisfy the Segal condition that*
$$T \to S \in I \qquad \Longleftrightarrow \qquad \forall U \in \mathrm{Orb}(S), \quad T \times_S U \to U \in I$$
7. *The sub-poset* $\mathrm{Sub}^{\mathrm{full}}_{\mathbf{Cat}_G}(\underline{\mathbb{F}}_G)$ *spanned by full G-subcategories $\mathcal{C} \subset \underline{\mathbb{F}}_G$ which are closed under self-indexed coproducts and have $*_H \in \mathcal{C}_H$ whenever $\mathcal{C}_H \neq \varnothing$.*

Furthermore, there are a equalities of sub-posets

$$\mathrm{Index}_G = \mathrm{wIndex}_{G,\geq A\mathbb{E}_\infty} = \mathrm{wIndex}^{\mathrm{uni}}_{G,\geq A\mathbb{E}_\infty} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_G),$$

where Index_G *denotes the* indexing systems *of [BP21; GW18; Rub21a].*

References. The *sliced G-symmetric monoidal envelope* is shown to implement an equivalence between (1) and (2) in [BHS22], which we recall in Corollary 2.62. We then characterize the image of A, constructing an equivalence between (5) and (6) in Proposition 2.35 and Corollary 3.10. The equivalence between (6) and (7) is handled in [St].

(4) and (5) in Corollary 3.10 by constructing a fully faithful right adjoint to

$$A : \mathrm{Op}_{\mathcal{T}} \rightleftarrows \mathrm{wIndex}_G : \mathcal{N}_{(-)\infty}. \tag{5}$$

Along the way, in Remark 3.9 we show that (2) and (4) are equivalent as subcategories. Finally, the equivalence between (2) and (3) is observed to follow from the computation of the arity-support of Boardman-Vogt tensor products in Observation 5.3. □

Under the assumption that $\mathcal{O}^{\otimes}$ is *reduced* (i.e. unital and one-colored), by [St], the information of $A\mathcal{O}$ may be understood as simply specifying the colors over which $\mathcal{O}^{\otimes}$ prescribes a binary multiplication

$$X_H^{\otimes 2} \to X_H$$

and the subgroup inclusions $K \hookrightarrow H$ over which $\mathcal{O}^{\otimes}$ prescribes a transfer

$$X_K \to X_H.$$

We call the operads $\mathcal{N}_{I\infty}^{\otimes}$ constructed by Eq. (5) *weak $\mathcal{N}_\infty$-operads*. In general, by Theorem C, we find that a slice category $\mathrm{Op}_{G,/\mathcal{O}^{\otimes}} \to \mathrm{Op}_G$ is a full subcategory if and only if $\mathcal{O}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad, in which case we write

$$\mathrm{Op}_I := \mathrm{Op}_{G,/\mathcal{N}_{I\infty}^{\otimes}} \simeq A^{-1}(\mathrm{wIndex}_{G,\leq I});$$

explicitly, maps $\mathcal{P}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$ are a *property* of $\mathcal{O}^{\otimes}$, and this property is the support condition $A\mathcal{P} \leq I$.

We may understand $\mathcal{N}_{I\infty}^{\otimes}$ in a hands-on manner in a number of ways; for instance, it is constructed explicitly in Proposition 2.35. On the other hand, the equivalence between conditions (4) and (6) of Theorem C shows that $\mathcal{N}_{I\infty}^{\otimes}$ is uniquely identified by the property

$$\mathcal{N}_{I\infty}(S) = \begin{cases} * & \mathrm{Ind}_H^G S \to G/H \quad \text{is in } I; \\ \varnothing & \text{otherwise.} \end{cases} \tag{6}$$

Alternatively, we may see this indirectly using the existence of free G-operads on symmetric sequences (see Corollary 2.68).

In fact, there are many weak $\mathcal{N}_\infty$-G-operads of interest outside of the world of $\mathcal{N}_\infty$-G-operads:

Example. Given $\mathcal{F} \subset \mathcal{O}_G^{\mathrm{op}}$ a G-family, the operad $\mathrm{triv}_{\mathcal{F}}^{\otimes} := \mathcal{N}_{\mathbb{F}_{\mathcal{F}}^{\simeq}}^{\otimes}$ is characterized by a natural equivalence

$$\underline{\mathbf{Alg}}_{\mathrm{triv}_{\mathcal{F}}}^{\otimes}(\mathcal{C}) = \mathrm{Bor}_{\mathcal{F}}^G(\mathcal{C}^{\otimes}),$$

in Proposition 2.51, where $\mathrm{Bor}_{\mathcal{F}}^G$ is the $\mathcal{F}$*-Borelification* discussed in Section 3.3. ◁

Given I a unital weak indexing system, in Corollary 4.12, we characterize the ∞*-category of I-commutative monoids* in $\mathcal{C}$ a complete ∞-category as

$$\mathrm{CMon}_I(\mathcal{C}) := \mathbf{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}^{\times}) \simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_G), \mathcal{C}),$$

where $\mathrm{Span}_I(\mathbb{F}_G) \subset \mathrm{Span}(\mathbb{F}_G)$ is the subcategory whose forward maps are in I; we define the ∞*-category of I-symmetric monoidal ∞-categories* as

$$\mathbf{Cat}_I^{\otimes} := \mathrm{CMon}_I(\mathbf{Cat}).$$

We also show in Proposition 2.41 that I-symmetric monoidal ∞-categories have underlying I-operads; for $\mathcal{C} \in \mathbf{Cat}_I^{\otimes}$, we define the ∞-category of *I-commutative algebras* in $\mathcal{C}$ as

$$\mathrm{CAlg}_I(\mathcal{C}) := \mathbf{Alg}_{\mathcal{N}_{I\infty}}(\mathcal{C}).$$

We show in Corollary 3.16, that analogs of Theorem B (5) and (6) hold for I-commutative algebra objects in I-symmetric monoidal categories.

We go on to compute the I-indexed tensor products in $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ under a distributivity assumption; they are *I-cocartesian*, in the sense that their I-indexed tensor products are indexed coproducts (c.f. Section 4.3), under a weak unitality assumption.

To that end, we say that a G-operad is *almost-E-unital* (henceforth aE-unital) if, whenever $\mathcal{O}(S) \neq \varnothing$ for some noncontractible $S \in \mathbb{F}_H$, we have $\mathcal{O}(\varnothing_H) = *$. In the a$E$-unital setting, we show that weak $\mathcal{N}_\infty$-G-operads are precisely the G-operads whose algebras have cocartesian tensor products indexed over their support.

Theorem D. *Let $\mathcal{O}^{\otimes}$ be an aE-unital G-operad. Then, the following properties are equivalenent.*

(a) The $A\mathcal{O}$-symmetric monoidal ∞-category $\underline{\mathbf{Alg}}_{\mathcal{O}}^{\otimes}\underline{\mathcal{S}}_G$ is $A\mathcal{O}$-cocartesian.

(b) The unique map $\mathcal{O}^{\otimes} \to \mathcal{N}_{A\mathcal{O}\infty}^{\otimes}$ is an equivalence.

Furthermore, $\underline{\mathrm{CAlg}}_I^{\otimes}\mathcal{C}$ is I-cocartesian for any distributive I-symmetric monoidal ∞-category $\mathcal{C}$ and almost-E-unital weak indexing system I.

We say that an I-operad $\mathcal{O}^{\otimes}$ is *reduced* if the (unique) map $\mathcal{O}^{\otimes} \to \mathcal{N}_{I\infty}$ induces equivalences

$$\mathcal{O}(S) \simeq \mathcal{N}_{I\infty}(S) \qquad \forall \;\; S \in \mathbb{F}_H \quad \text{empty or contractible}$$

(c.f. Eq. (6)). We characterize algebras in cocartesian I-symmetric monoidal categories in Theorem 2.54, and from this Theorem D entirely characterizes the tensor products of reduced I-operads with $\mathcal{N}_{I\infty}^{\otimes}$ in the almost-E-unital setting.

Corollary E. *$\mathcal{N}_{I\infty}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes}$ is a weak $\mathcal{N}_\infty$-operad if and only if I is aE-unital. In this case, if $\mathcal{O}^{\otimes}$ is a reduced I-operad, then the unique map*

$$\mathcal{O}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes} \to \mathcal{N}_{I\infty}^{\otimes}$$

is an equivalence.

The only part of Corollary E which doesn't follow directly from Theorem D is the statement that I non-aE-unital implies that $\mathcal{N}_{I\infty}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes}$ is not weak $\mathcal{N}_\infty$; we prove this in Section 5.1.

This immediately characterizes many tensor products of weak $\mathcal{N}_\infty$-operads, since $\mathcal{N}_{I\infty}$ is a J-operad whenever $I \leq J$. We go on to completely characterize indexed tensor products of almost-E-unital weak $\mathcal{N}_\infty$-operads, confirming Conjecture 6.27 of [BH15].

Theorem F. *The functor $\mathcal{N}_{(-)\infty}^{\otimes} : \mathrm{wIndex}_G \to \mathrm{Op}_G$ lifts to a G-colocalizing subcategory inclusion*

$$\underline{\mathrm{wIndex}}_G \underset{A}{\overset{\mathcal{N}_{(-)\infty}^{\otimes}}{\rightleftarrows}} \underline{\mathrm{Op}}_G \qquad (\perp)$$

whose restriction $\underline{\mathrm{wIndex}}_G^{\mathrm{wuni}} \subset \underline{\mathrm{Op}}_G$ is symmetric monoidal. Furthermore, the resulting tensor product on $\mathrm{wIndex}_G^{\mathrm{wuni},\otimes}$ is computed by the join

$$\begin{aligned} I \otimes J &= \mathrm{Bor}_{\mathrm{cSupp}(I\cap J)}^{G}(I \vee J) \\ &= \left\{ \coprod_i (\psi_i : T_i \to S_i) \;\middle|\; \forall i, \quad S_i \in I \cap J, \quad \text{and} \quad \psi_i \in I \cup J \right\}. \end{aligned}$$

In particular, when I and J are almost-E-unital weak indexing systems, we have

$$\begin{aligned} \mathcal{N}_{I\infty}^{\otimes} \otimes \mathcal{N}_{J\infty}^{\otimes} &\simeq \mathcal{N}_{(I\vee J)\infty}^{\otimes} \otimes \mathrm{triv}_{\mathrm{cSupp}(I\cap J)}^{\otimes} \\ \mathcal{N}_{I\infty}^{\otimes} \times \mathcal{N}_{J\infty}^{\otimes} &\simeq \mathcal{N}_{(I\cap J)\infty}^{\otimes} \\ \mathrm{Res}_H^G \mathcal{N}_{I\infty}^{\otimes} &\simeq \mathcal{N}_{\mathrm{Res}_H^G I\infty}^{\otimes} \\ \mathrm{CoInd}_H^G \mathcal{N}_{I\infty}^{\otimes} &\simeq \mathcal{N}_{\mathrm{CoInd}_H^G I\infty}^{\otimes}. \end{aligned}$$

Hence norms of I-commutative algebras are $\mathrm{CoInd}_H^G I$-commutative algebras, and when I, J are unital, we have

$$\underline{\mathrm{CAlg}}_I^{\otimes} \underline{\mathrm{CAlg}}_J^{\otimes}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{I\vee J}(\mathcal{C}). \tag{7}$$

We offer various additional corollaries in Sections 3.5 and 4.5 concerning lifts of various functors in equivariant homotopy theory to functors between categories of I-commutative algebras; included among these are equivariant factorization homology and equivariant algebraic K-theory. We go on to state a family of conjectures concerning further properties of equivariant higher algebra in Section 5.5.

Notation and conventions.

Acknowledgements.

1. Equivariant symmetric monoidal categories

1.1. **Recollections on $\mathcal{T}$-∞-categories.** In this section, we briefly summarize some relevant elements of parameterized and equivariant higher category theory. There have been many developments in this theory outside of what is summarized here; further details can be found in the work of Barwick-Dotto-Glasman-Nardin-Shah [Bar+16a; Bar+16b; Nar16; Sha22; Sha23], Cnossen-Lenz-Linskens [CLL23a; CLL23b; CLL24; Lin24; LNP22], and Hilman [Hil24].

We view this setting of *orbital ∞-categories* as a natural home for higher algebra centered around categories of spans (see [Nar16, § 4]), generalizing the orbit categories families of subgroups of profinite groups. We summarize the dictionary between these settings in ref . The reader who is exclusively interested in equivariant homotopy theory is encouraged to assume every orbital ∞-category is the orbit category of a (pro-)finite group, or of a family of subgroups; these are always epiorbital.