ORBITAL CATEGORIES AND WEAK INDEXING SYSTEMS

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ABSTRACT. Fix $\mathcal{T}$ an atomic orbital ∞-category. In this exposé, we initiate the combinatorial study of the poset $\mathrm{wIndex}_{\mathcal{T}}$ of *weak $\mathcal{T}$-indexing systems*, which yields arities for equivariant algebraic structures which are closed under their own operations. Within this sits a natural orbital lift $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}$ of Blumberg-Hill's *indexing systems*, consisting of weak indexing systems which have all binary and nullary operations. For instance, we conclude from results of Balchin-Barnes-Roitzheim that the lattice of $C_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p$-indexing systems is equivalent to the infinite associahedron.

Along the way, we characterize the relationship between the posets of *unital weak indexing systems* and *indexing systems*, the latter remaining isomorphic to *transfer systems* on this level of generality. We use this to compute the poset of unital C_{p^N}-weak indexing systems for $N \in \mathbb{N} \cup \{\infty\}$.

CONTENTS

1. INTRODUCTION

Fix G a finite group. In [BH15], the notion of $\mathcal{N}_\infty$*-operads for* G was introduced, encapsulating a collection of *blueprints* for G-equivariantly commutative multiplicative structures on Mackey functors which possess underlying Green functors. They demonstrated that the ∞-category of $\mathcal{N}_\infty$-operads for G is an embedded sub-poset of the category of *indexing systems* Index_G.

Subsequently, the embedding $\mathcal{N}_\infty-\mathrm{Op}_G \subset \mathrm{Index}_G$ was shown to be an equivalence in several independent works [BP21; GW18; Rub21]; of particular interest is the equivalent redefinition of indexing systems as a poset of subcategories $\mathrm{Index}_G \subset \mathrm{Sub}(\mathbb{F}_G)$ (referred to as *indexing categories*) and the observation of Rubin

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that indexing categories only depend on their intersections with the orbit category $\mathcal{O}_G = \{G/H\} \subset \mathbb{F}_G$, the resulting embedded subposet

$$\begin{array}{ccccccc}
\mathrm{Index}_G & \xleftarrow{\ \sim\ } & \mathrm{IndexCat}_G & & \xrightarrow{\qquad\quad \sim \qquad\quad} & & \mathrm{Transf}_G \\
\big\downarrow & & \big\downarrow & & & & \big\downarrow \\
\mathrm{FullSub}_G(\underline{\mathbb{F}}_G) & \xleftarrow[\mathbb{F}_{(-)}]{} & \mathrm{Sub}(\mathbb{F}_G) & \xrightarrow[(-)\cap\mathcal{O}_G]{} & \mathrm{Sub}(\mathcal{O}_G) & \xrightarrow[\text{p.b.}]{} & \mathrm{Sub}_{\mathrm{Poset}}\mathrm{Sub}_{\mathrm{Grp}}(G)
\end{array}$$

being referred to as *transfer systems*. It is in this form that the burgeoning subfield of *homotopical combinatorics* (coined in [Bal+23], where it is related to finite model category theory) has attacked enumerative problems concerning $\mathcal{N}_\infty$-algebras.

Using the synonymous language of *norm maps* and noting that $\left[\mathcal{O}_{C_{p^n}}\right] = [n+1]$, this approach was used in [BBR21] to prove that $\mathrm{Transf}_{C_{p^n}}$ is equivalent to the $(n+1)$st associahedron K_{n+1}. Furthermore, this has powered a large amount of further work on the topic; for instance, $\mathrm{Transf}_{C_{pqr}}$ is enumerated for p, q, r distinct primes in [Bal+20], with some indications on how to generalize this to arbitrary squarefree integers.

In this paper, we aim to demonstrate how one may extend these enumerative efforts in two ways:

(1) we will remove the assumption of indexing systems that they are closed under coproducts; on the side of algebra, we see in [St24] this corresponds with removing the assumption that algebras over the corresponding G-operad $\mathcal{N}_{I\infty}$ in Mackey functors have underlying green functors.
(2) we will replace the orbit category $\mathcal{O}_G$ with an axiomatic version, called an *atomic orbital ∞-category*; this allows us to fluently describe equivariance under families and cofamilies, as well as extending to more general orbit categories, such as the finite-index orbit category of a compact Lie group.

For the former, we find that in Example 1.29 that the poset of *weak* indexing systems is always infinite; nevertheless, we find when we assert a unitality assumption that $\mathrm{wIndex}_G^{\mathrm{uni}}$ is finite when G is finite, and it can usually be explicitly described in terms of transfer systems and G-families (c.f. Theorem C and Corollary D). Moreover, this behaves well with joins (c.f. Proposition 2.42), and in [St24] we establish that this computes tensor products of unital weak $\mathcal{N}_\infty$-operads.

We assure the skeptical reader that they may freely assume $\mathcal{T}$ is (the orbit category of) a G-family. Nevertheless, we review our setup in the following.

1.1. Orbital categories.

We briefly review the setting introduced in [Bar+16].

Construction 1.1 (c.f. [Gla17])**.** Given $\mathcal{T}$ an ∞-category[1], its *finite coproduct completion* is the full subcategory $\mathbb{F}_\mathcal{T} \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by coproducts of representables. ◁

Example 1.2. If G is a finite group, then $\mathbb{F}_{\mathcal{O}_G}$ is equivalent to the category of finite G-sets; more generally, if $\mathcal{F} \subset \mathcal{O}_G$ is a subconjugacy-closed family of subgroups, then $\mathbb{F}_\mathcal{F} \subset \mathbb{F}_{\mathcal{O}_G}$ is equivalent to the subcategory of finite G-sets whose stabilizers lie in $\mathcal{F}$. ◁

Inspired by the above example, given $S \in \mathbb{F}_\mathcal{T}$, there is a canonical expression $S \simeq \bigoplus_I V$ for some elements $(V) \subset \mathcal{T}$. We refer to these (V) as *orbits*, and refer to the set of orbits of S as $\mathrm{Orb}(S)$. An important property of the finite coproduct completion is existence of equivalences

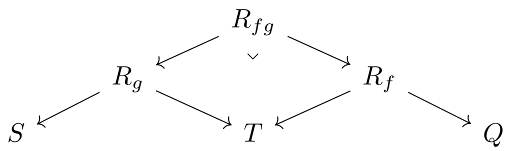
$$\mathbb{F}_{\mathcal{T},/S} \simeq \prod_{V \in \mathrm{Orb}(S)} \mathbb{F}_{\mathcal{T},/V}; \qquad\qquad \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}.$$

We henceforth refer to $\mathcal{T}_{/V}$ simply as $\underline{V}$, and $\mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\underline{V}}$ as $\mathbb{F}_V$. Note that, in the case $\mathcal{T} = \mathcal{O}_G$, induction furnishes an equivalence $\mathcal{O}_{G,/[G/H]} \simeq \mathcal{O}_H$, so $\mathbb{F}_{[G/H]} \simeq \mathbb{F}_H$.

Fundamental to representation theory is the *effective Burnside category*, $\mathrm{Span}(\mathbb{F}_G)$; for instance, G-Mackey functors may be presented as product-preserving functors $\mathrm{Span}(\mathbb{F}_G) \to \mathbf{Ab}$. In fact, the spectral Mackey functor theorem of [GM17] presents G-spectra as product-preserving functors of ∞-categories $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Sp}$, a perspective which has been greatly exploited e.g. in [Bar14; BGS20].

[1] 1-categories embed fully faithfully into ∞-categories, and the reader is free to safely assume all categorical terminology refer to 1-categories (and spaces as sets) if they so choose, at the expense of some examples regarding parameterization over spaces or non-discrete groups.

In $\mathrm{Span}(\mathbb{F}_G)$, composition of morphisms is accomplished via the pullback



Indeed, given $\mathcal{T}$ an arbitrary ∞-category, the triple $(\mathbb{F}_\mathcal{T}, \mathbb{F}_\mathcal{T}, \mathbb{F}_\mathcal{T})$ is *adequate* in the sense of [Bar14] if and only if $\mathbb{F}_\mathcal{T}$ has pullbacks, in which case the triple is *disjunctive*. Thus, Barwick's construction [Bar14, Def 5.5] defines a $\mathcal{T}$-effective Burnside ∞-category $\mathrm{Span}(\mathbb{F}_\mathcal{T}) = A^{eff}(\mathbb{F}_\mathcal{T}, \mathbb{F}_\mathcal{T}, \mathbb{F}_\mathcal{T})$ precisely if $\mathcal{T}$ is *orbital* in the sense of the following definition.

Definition 1.3 ([Nar16, Def 4.1])**.** An ∞-category is *orbital* if $\mathbb{F}_\mathcal{T}$ has pullbacks; an orbital ∞-category is *atomic* if all retracts in $\mathcal{T}$ are equivalences. ◁

We will not discuss the Burnside ∞-category in the remainder of this paper, as it is not crucial to our current combinatorics.

Remark 1.4. We show in Section 2.1 that, if $\mathcal{T}$ is an atomic orbital ∞-category, then $\mathrm{ho}(\mathcal{T})$ is as well, and the main combinatorial objects of this paper are the same between $\mathcal{T}$ and $\mathrm{ho}(\mathcal{T})$; hence the reader may uniformly assume that $\mathcal{T}$ is a 1-category, at the loss of essentially none of the combinatorics. ◁

Example 1.5. Given X a space considered as an ∞-category, X is atomic orbital; by [Gla18, Thm 2.13], the associated stable category is the Ando-Hopkins-Rezk category of parameterized spectra over X (c.f. [And+14]). ◁

Example 1.6. Given P a meet semilattice, P is atomic orbital; the associated stable category contains that of parameterized spectra over P. ◁

Given G a Lie group, let $\mathcal{S}_G$ denote the ∞-category presented by orthogonal G-spaces, and let $\mathcal{O}_G \subset \mathcal{S}_G$ denote the full subcategory spanned by the homogeneous G-spaces G/H for $H \subset G$ a closed subgroup. A famous issue with equivariant homotopy theory over positive-dimensional Lie groups is that $\mathcal{O}_G$ is not *orbital*; the G-Burnside category does not exist, as $\mathbb{F}_G$ does not have pullbacks with which to define composition of spans.

Nevertheless, this has been rectified in various contexts. One particularly lucid treatment due to [CLL23] uses the slightly more general setting of *global homotopy theory*.

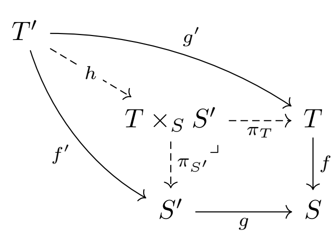
Definition 1.7 ([CLL23, Def 4.2.2, 4.3.2])**.** If $\mathcal{T}$ is an ∞-category, an *atomic orbital subcategory* of $\mathcal{T}$ is a wide subcategory $\mathcal{P} \subset \mathcal{T}$ satisfying the following conditions:

(1) Denote by $\mathbb{F}_\mathcal{T}^\mathcal{P} \subset \mathbb{F}_\mathcal{T}$ the wide subcategory consisting of morphisms which are disjoint unions of morphisms in $\mathcal{P}$. Then, $\mathbb{F}_\mathcal{T}^\mathcal{P}$ is stable under pullbacks along arbitrary maps in $\mathbb{F}_\mathcal{T}$, and all such pullbacks exist.
(2) Any morphism $A \to B$ in $\mathcal{P}$ admitting a section in $\mathcal{T}$ is an equivalence. ◁

An ∞-category is atomic orbital if and only if it's an atomic orbital subcategory of itself. We have a partial converse:

Lemma 1.8. *Suppose $\mathcal{P} \subset \mathcal{T}$ is an atomic orbital subcategory. Then, $\mathcal{P}$ is atomic orbital as an ∞-category.*

Proof. First, assume we have a square in $\mathbb{F}_\mathcal{P}$, which is canonically extended to be the outer square of the following $\mathcal{T}$-diagram



To prove that $\mathcal{P}$ is orbital, it suffices to verify that the inner square is a pullback, for which it suffices to check that all of the involved maps are in $\mathcal{P}$. First note that, $\pi_{S'}$ and π_T are in $\mathcal{P}$ since $\mathcal{P} \subset \mathcal{T}$ is orbital; h is then in $\mathcal{P}$ since atomic orbital subcategories are left cancellable by [CLL23, Lem 4.3.5], so we've proved that $\mathcal{P}$ is orbital. To see that $\mathcal{P}$ is atomic, note that this immediately follows from the second condition of Definition 1.7. □

Definition 1.9. Given $\mathcal{T}$ an ∞-category, a *$\mathcal{T}$-family* is a full subcategory $\mathcal{F} \subset \mathcal{T}$ satisfying the condition that, given $F : V \to W$ a morphism with $W \in \mathcal{F}$, we have $V \in \mathcal{T}$. A *$\mathcal{T}$-cofamily* is a full subcategory $\mathcal{F}^\perp \subset \mathcal{T}$ such that $\mathcal{F}^{\perp,\mathrm{op}} \subset \mathcal{T}$ is a $\mathcal{T}^{\mathrm{op}}$-family.

Given $\mathcal{T}$ an ∞-category, an *interval family* of $\mathcal{T}$ is an intersection of a family and a cofamily; equivalently, it is a full subcategory $\mathcal{F}$ with the property that whenever $U, W \in \mathcal{F}$ and there is a path $U \to V \to W$, we have $V \in \mathcal{F}$. ◁

Observation 1.10. If $\mathcal{F} \subset \mathcal{T}$ is an interval family in an atomic orbital ∞-category satisfying the condition that, for all cospans $U \to W \leftarrow V \in \mathcal{T}$ with $U, W \in \mathcal{F}$, there is a span $U \leftarrow W' \to V$ with $W \in \mathcal{F}$, then the inclusion $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{T}}$ creates pullbacks. In particular, $\mathcal{F}$ is an atomic orbital ∞-category. ◁

Example 1.11. Let G be a Lie group and $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ the wide subcategory of the orbit ∞-category spanned by projections $G/K \to G/H$ corresponding with finite-index closed subgroup inclusions $K \subset H$. Then, by [CLL23, Ex 4.2.6], $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ is an orbital subcategory. In fact, it follows quickly from definition that it is atomic as well; hence $\mathcal{O}_G^{f.i.}$ is an atomic orbital ∞-category. The pullbacks in $\mathbb{F}_G^{f.i.}$ are computed by a double coset formula.

In fact, by Observation 1.10, the $\mathcal{O}_G$ interval families consisting of *finite subgroups* and of *finite-index closed subgroups* are atomic orbital ∞-categories as well. The former in the case $G = \mathbb{T}$ yields the *cyclonic orbit category* of [BG16]. ◁

Example 1.12. Given $H \subset G$ a closed subgroup, the cofamily $\mathcal{O}_{G,\geq[G/H]}^{f.i.}$ spanned by homogeneous G-spaces G/J admitting a quotient map from G/H satisfies the assumptions of Observation 1.10, so it is atomic orbital; in the case $H = N \subset G$ is normal, it is equivalent to $\mathcal{O}_{G/N}^{f.i.}$. In any case, the associated stable homotopy theory is the value category of *H-geometric fixed points* with residual genuine G/H-structure (c.f. [Gla17]). ◁

1.2. **Weak indexing systems and weak indexing categories.** Throughout the remainder of this introduction, we fix $\mathcal{T}$ an atomic orbital ∞-category. In the case $\mathcal{T} = \mathcal{O}_G$ is the orbit category of a compact Lie group G, Elmendorf's theorem [DK84; Elm83] implies that the ∞-category of G-spaces is equivalent to the functor ∞-category

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}),$$

i.e. they are (homotopy coherent) *indexing systems of spaces*. It has become traditional to allow G to act on the *category theory* surrounding equivariant homotopy theory, culminating in the following definition.

Definition 1.13. The *2-category of $\mathcal{T}$-1-categories* is the functor 2-category[2]

$$\mathbf{Cat}_{\mathcal{T},1} \coloneqq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_1) \simeq \mathrm{Fun}(h_2\mathcal{T}^{\mathrm{op}}, \mathbf{Cat}_1),$$

where $\mathbf{Cat}_1$ is the 2-category of 1-categories. ◁

We refer to the morphisms in $\mathbf{Cat}_{\mathcal{T},1}$ as *$\mathcal{T}$-functors*. Given a $\mathcal{T}$-1-category $\mathcal{C}$ and an object $V \in \mathcal{T}$, there is a *V-value* 1-category $\mathcal{C}_V \coloneqq \mathcal{C}(V)$, and given a map $V \to W$ in $\mathcal{T}$, there is an associated *restriction functor* $\mathcal{C}_W \to \mathcal{C}_V$.

Example 1.14. By [NS22, Prop 2.5.1], the ∞-category $\underline{V}$ is a 1-category, so $\mathbb{F}_V \simeq \mathbb{F}_{\underline{V}} \simeq \mathbb{F}_{\mathcal{T},/V}$ is a 1-category. Hence the functor $\mathcal{T}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ sending $V \mapsto \mathbb{F}_{\mathcal{T},/V}$ is a $\mathcal{T}$-1-category, which we call $\underline{\mathbb{F}}_{\mathcal{T}}$. ◁

Evaluation is functorial in the $\mathcal{T}$-category; given a $\mathcal{T}$-functor $\mathcal{C} \to \mathcal{D}$, there is a canonical functor

$$\mathrm{Res}_V^W : \mathcal{C}_V \to \mathcal{D}_V.$$

We refer to a $\mathcal{T}$-functor whose V-values are fully faithful as a *fully faithful $\mathcal{T}$-functor*; if $\iota : \mathcal{C} \to \mathcal{D}$ is a fully faithful $\mathcal{T}$-functor, we say that $\mathcal{C}$ is a *full $\mathcal{T}$-subcategory of $\mathcal{D}$*. A full $\mathcal{T}$-subcategory of $\mathcal{D}$ is uniquely determined by an equivalence-closed and restriction-stable class of objects in $\mathcal{D}$; see [Sha23] for details.

[2] Throughout this paper, *n-category* will mean $(n,1)$-category, i.e. ∞-category whose mapping spaces are $(n-1)$-truncated.

Definition 1.15 (c.f. [HHR16, § 2.2.3])**.** Fix $\mathcal{C}$ a $\mathcal{T}$-1-category. The functor $\mathrm{Ind}_U^V : \mathcal{C}_U \to \mathcal{C}_V$, if it exists, is the left adjoint to Res_U^V. Furthermore, given a V-set S and a tuple $(T_U)_{U \in \mathrm{Orb}(S)}$, the *$S$-indexed coproduct of T_U* is, if it exists, the element

$$\coprod_U^S T_U \coloneqq \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U \in \mathcal{C}_W.$$

Dually, $\mathrm{CoInd}_U^V : \mathcal{C}_U \to \mathcal{C}_V$ denote the right adjoint to Res_U^V (if it exists), and the S-indexed product is (if it exists), the element

$$\prod_U^S T_U \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^V T_U \in \mathcal{C}_U. \qquad \triangleleft$$

Example 1.16. Given a subgroup inclusion $K \subset H \subset G$, the associated functor $\mathbb{F}_H \to \mathbb{F}_K$ is restriction, and hence its left adjoint $\mathbb{F}_K \to \mathbb{F}_H$ is *G-set induction*, matching the *indexed coproducdcts* of [HHR16, § 2.2.3]. $\triangleleft$

Given $S \in \mathbb{F}_V$, we write

$$\mathcal{C}_S \coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_V;$$

we say that $\mathcal{C}$ *strongly admits finite coproducts* if $\coprod_U^S T_U$ always exists, in which case it amounts to a functor

$$\coprod_{-}^S (-) : \mathcal{C}_S \to \mathcal{C}_V.$$

It follows from construction that $\underline{\mathbb{F}}_{\mathcal{T}}$ strongly admits finite coproducts.

Definition 1.17. Given a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ and a full $\mathcal{T}$-subcategory $\mathcal{E} \subset \mathcal{D}$, we say that $\mathcal{E}$ is *closed under $\mathcal{C}$-indexed coproducts* if, for all $S \in \mathcal{C}_V$ and $(T_U) \in \mathcal{E}_S$, we have $\coprod_U^S T_U \in \mathcal{E}_V$. $\triangleleft$

Definition 1.18. We say that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is *closed under self-indexed coproducts* if it is closed under $\mathcal{C}$-indexed coproducts. $\triangleleft$

Definition 1.19. Given $\mathcal{T}$ an orbital category, a *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ satisfying the following conditions:

(IS-a) Whenever $\underline{\mathbb{F}}_{I,V} \neq \varnothing$, we have $*_V \in \underline{\mathbb{F}}_{I,V}$.
(IS-b) $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

We denote by $\mathrm{wIndex}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}_{\mathcal{T}}}(\mathbb{F}_{\mathcal{T}})$ the embedded sub-poset spanned by $\mathcal{T}$-weak indexing systems. Moreover, we say that a $\mathcal{T}$-weak indexing system *has one color* if it satisfies the following condition

(IS-i) For all $V \in \mathcal{T}$, we have $\underline{\mathbb{F}}_{I,V} \neq \varnothing$;

these span an embedded subposet $\mathrm{wIndex}_{\mathcal{T}}^{oc} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is almost E-unital if it satisfies the condition

(IS-ii) For all noncontractible V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

An almost E-unital $\mathcal{T}$-weak indexing system is *almost unital* if it has one color. These are denoted $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is *E-unital* if it satisfies the condition

(IS-iii) For all $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

and an E-unital $\mathcal{T}$-weak indexing system is *unital* if it has one color. We write $\mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. Lastly, a $\mathcal{T}$-weak indexing system is an *indexing system* if it satisfies the following condition.

(IS-iv) The subcategory $\mathbb{F}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

We denote the resulting poset by $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$. $\triangleleft$

Remark 1.20. The indexing systems of [BH15] are seen to be equivalent to ours when $\mathcal{T} = \mathcal{O}_G$ by unwinding definitions. The weak indexing systems of [BP21; Per18] are equivalent to our *unital* weak indexing systems in this case by [Per18, Rem 9.7] and [BP21, Rem 4.60]. $\triangleleft$

In practice, we will find that non-almost E-unital weak indexing systems are not well behaved, and questions involving almost E-unital weak indexing systems are usually quickly reducible to the unital case; the non-combinatorial user is encouraged to focus primarily on unital weak indexing systems for this reason.

Example 1.21. The terminal $\mathcal{T}$-weak indexing system is $\underline{\mathbb{F}}_{\mathcal{T}}$; the initial $\mathcal{T}$-weak indexing system is the empty subcategory; the initial one-color $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^{\mathrm{triv}} \coloneqq \mathbb{F}_V^{\simeq}. \qquad \triangleleft$$

Remark 1.22. In [St24] we define the *underlying $\mathcal{T}$-symmetric sequence* $\mathcal{O}(-)$ of a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$; $\mathcal{O}^{\otimes}$ parameterizes a type of equivariant multiplicative structures, and the space $\mathcal{O}(S)$ parameterizes the *S-ary operations* endowed on an $\mathcal{O}$-algebra. There we define the *arity support*

$$\mathbb{F}_{A\mathcal{O},V} \coloneqq \{S \in \mathbb{F}_V \mid \mathcal{O}(S) \neq \varnothing\};$$

in [St24], we show that this possesses a fully faithful right adjoint, making $\mathcal{T}$-weak indexing systems equivalent to *weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads*, i.e. subterminal objects in the ∞-category of $\mathcal{T}$-operads.

This inspires our naming; [St24] establishes that $\underline{\mathbb{F}}_{A\mathrm{triv}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ and $\underline{\mathbb{F}}_{A\mathrm{Comm}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}$. $\triangleleft$

Proposition 1.23. *Given $\underline{\mathbb{F}}_I$ a $\mathcal{T}$-weak indexing system, the following are $\mathcal{T}$-families:*

$$\begin{aligned} c(I) &\coloneqq \{V \in \mathcal{T} \mid *_V \in \mathbb{F}_{I,V}\} \\ \upsilon(I) &\coloneqq \{V \in \mathcal{T} \mid \varnothing_V \in \mathbb{F}_{I,V}\} \\ \nabla(I) &\coloneqq \{V \in \mathcal{T} \mid 2*_V \in \mathbb{F}_{I,V}\} \end{aligned}$$

Proof. This follows by noting that $\mathrm{Res}_U^V n \cdot *_V = n \cdot *_U$ and weak indexing systems are restriciton-stable. $\square$

Note that $c(I) \leq \upsilon(I) \cap \nabla(I)$. The following lemma will be used ubiquitously.

Lemma 1.24. *Let $\underline{\mathbb{F}}_I$ be a $\mathcal{T}$-weak indexing system.*

(1) $\underline{\mathbb{F}}_I$ has one-color if and only if $c(I) = \mathcal{T}$.
(2) $\underline{\mathbb{F}}_I$ is E-unital if and only if $\upsilon(I) = c(I)$.
(3) $\underline{\mathbb{F}}_I$ is unital if and only if $\upsilon(I) = \mathcal{T}$.
(4) $\underline{\mathbb{F}}_I$ is an indexing system if and only if $\upsilon(I) \cap \nabla(I) = \mathcal{T}$.

Proof. (1) follows immediately by unwinding definitions. For (2), if $\underline{\mathbb{F}}_I$ is E-unital and $V \in c(I)$, then choosing $\varnothing_V \sqcup *_V \in \mathbb{F}_{I,V}$ yields $\varnothing_V \in \mathbb{F}_{I,V}$, i.e. $V \in \upsilon(I)$. Conversely, if $\upsilon(I) = c(I)$ and $S \coprod S' \in \mathbb{F}_{i,V}$, then

$$S = \coprod_U^S \varnothing_U \sqcup \coprod_U^S *_U \in \mathbb{F}_{I,V},$$

so $\underline{\mathbb{F}}_I$ is E-unital. (3) follows by combining (1) and (2).

For (4), note that $\underline{\mathbb{F}}_I$ an indexing system implies that $\upsilon(I) \cap \nabla(I) = \mathcal{T}$ by choosing $n = 0$ and $n = 2$ in Condition (IS-iv). Conversely, if $\upsilon(I) \cap \nabla(I) = \mathcal{T}$, then we already know Condition (IS-iv) when $n = 2$; in fact, by iterating binary coproducts $(n-1)$-times, we find that $n*_V = (*_V \coprod (n-1)*_V) \in \mathbb{F}_{I,V}$ for all V, so $\underline{\mathbb{F}}_I$ is an indexing system. $\square$

Construction 1.25. Given $\mathcal{F}$ a $\mathcal{T}$-family and $\underline{\mathbb{F}}_I$ an $\mathcal{F}$-weak indexing system, we may define the $\mathcal{T}$-weak indexing system $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I$ by

$$\left(E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I\right)_V \coloneqq \begin{cases} \mathbb{F}_{I,V} & V \in \mathcal{F}; \\ \varnothing & \text{otherwise.} \end{cases}$$

this is an injective monotone map $\mathrm{wIndex}_{\mathcal{F}} \to \mathrm{wIndex}_{\mathcal{T}}$. $\triangleleft$

Proposition 1.26. *The fiber of $c : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ is the image of $E_{\mathcal{F}}^{\mathcal{T}}|_{oc} : \mathrm{wIndex}_{\mathcal{F}}^{oc} \to \mathrm{wIndex}_{\mathcal{T}}$.*

In particular, we find that $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}$ and $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\simeq}$ are terminal and initial among $c^{-1}(\mathcal{F})$.

Example 1.27. The intial unital $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^0$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^0 \coloneqq \{\varnothing_V, *_V\};$$

we see in [St24] that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_0}$. $\triangleleft$

Example 1.28. The initial $\mathcal{T}$-indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^{\infty}$ is defined by

$$\mathbb{F}_V^{\infty} \coloneqq \{n \cdot *_V \mid n \in \mathbb{N}\};$$

we see in [St24] that this is equal to $\underline{\mathbb{F}}_{A\mathbb{E}_\infty}$. ◁

Example 1.29. Let $\mathcal{T} = *$ be the terminal category. Then, a full subcategory $\underline{\mathbb{F}}_I \subset \mathbb{F}$ can be identified with a subset $n(I) \subset \mathbb{N}$, Condition (IS-a) with the condition that $n(I)$ is nonempty or contains 1, and condition Condition (IS-b) with the condition that $n(I)$ is closed under k-fold sums for all $k \in n(I)$. There are many such things; for instance, for each $n \in \mathbb{N}$, the set $\{1\} \cup n\mathbb{N} \subset \mathbb{N}$ gives a nonunital $*$-weak indexing system.

Nevertheless, if we assert that $\varnothing \in n(I)$ (i.e. $\underline{\mathbb{F}}_I$ is unital), then $n(I)$ is closed under summands, i.e. it is lower-closed in $\mathbb{N}$. Thus we have the following computations for $\mathcal{T} = *$:

condition	poset
indexing system	$\mathbb{F}$
unital	$\mathbb{F}^0 \longrightarrow \mathbb{F}$
almost-unital	$\mathbb{F}^{\mathrm{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
E-unital	$\varnothing \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
almost-E-unital	$\varnothing \longrightarrow \mathbb{F}^{\mathrm{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$

◁

Example 1.30. We will see in Corollary 2.6 that when X is a space, there is a canonical equivalence $\mathrm{wIndex}_X \simeq \mathrm{wIndex}_*$ respecting our various conditions. In particular, the computations for *Borel* equivariant weak indexing systems mirror those of Example 1.29. ◁

Example 1.31. Choosing $\mathcal{T} = \mathcal{O}_{C_p}$ with standard representation λ, we show that in [St24] that the *little $\infty\lambda$-disks C_p-operad* has arity support

$$\mathbb{F}_{A\mathbb{E}_{\infty\lambda},e} = \mathbb{F}_e, \qquad \mathbb{F}_{A\mathbb{E}_{\infty\lambda},C_p} = \{n \cdot [C_p/e] \mid n \in \mathbb{N}\} \sqcup \left\{*_{C_p} + n \cdot [C_p/e] \mid n \in \mathbb{N}\right\};$$

in particular, this unital weak indexing system corresponds with an interesting algebraic theory and it is *not* an indexing system. ◁

With a wealth of examples under our belt, we begin on the road towards other perspectives on weak indexing systems.

Observation 1.32. Denote by $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ the map corresponding a V-set S under the equivalence $\mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T},/V}$. This equivalence implies a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is determined by its subgraph

$$I(\mathcal{C}) \coloneqq \left\{ \coprod_i \mathrm{Ind}_{V_i}^{\mathcal{T}} S_i \to V_i \;\middle|\; \forall i, \quad S \in \mathcal{C}_{V_i} \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

In other words, the construction I yields an embedding of posets

$$I(-) : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Sub}_{\mathrm{graph}}(\mathbb{F}_{\mathcal{T}}).$$ ◁

Theorem A. *The image of $I(-)$ consists of the subcategories $I \subset \mathcal{C}$ satisfying the following conditions*

(IC-a) (restrictions) I is stable under arbitrary pullbacks in $\mathbb{F}_{\mathcal{T}}$;
(IC-b) (segal condition) the pair $T \to S$ and $T' \to S'$ are in I if and only if $T \coprod T' \to S \coprod S'$ is in I; and
(IC-c) ($\Sigma_{\mathcal{T}}$-action) if $S \in I$, then all automorphisms of S are in I.

moreover, for all numbers n, condition (IS-n) of Definition 1.19 is equivalent to condition (IC-n) below:

(IC-i) (one color) I is wide; equivalently, I contains $\mathbb{F}_{\mathcal{T}}^{\simeq}$.
(IC-ii) (aE-unital) if $S \coprod S' \to T$ is a non-isomorphism identity in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iii) (E-unital) if $S \coprod S' \to T$ is in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iv) (indexing category) the fold maps $n \cdot V \to V$ are in I for all $n \in \mathbb{N}$ and $V \in \mathcal{T}$.

We refer to the image of $I(-)$ as the *weak indexing categories* $\mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}})$. In general, we will refer to a generic weak indexing category as I and its corresponding weak indexing system as $\underline{\mathbb{F}}_I$.

The following observations form the basis for the proof of Theorem A.

Observation 1.33. By a basic inductive argument, Condition (IC-b) is equivalent to the following condition:
(IC-b') $S \to T$ is in I if and only if $S_U = S \times_T U \to U$ is in I for all $U \in \mathrm{Orb}(T)$.

in particular, I is uniquely determined by the maps to orbits. ◁

Observation 1.34. By Observation 1.33, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the following condition:

(IC-a') for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{array}{ccc} T \times_V U & \longrightarrow & T \\ \downarrow{\alpha'} & \lrcorner & \downarrow{\alpha} \\ U & \longrightarrow & V \end{array} \tag{1}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◁

One of the major reasons for this formalism is the technology of *equivariant algebra*. If $\iota : I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable subcategory write $\mathbb{F}_{c(I)}$ for the coproduct closure of the essential image of ι. Then $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I)$ is an adequate triple, so we may form the span category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \coloneqq A^{eff}(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I),$$

whose forward maps are I and backwards maps are arbitrary. If $\mathcal{C}$ is an ∞-category, the category of *I-commutative monoids* is the product preserving functor category

$$\mathrm{CMon}_I(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C});$$

the *I-symmetric monoidal* 1*-categories* are

$$\mathbf{Cat}_{I,1}^{\otimes} \coloneqq \mathrm{CMon}_I(\mathbf{Cat}_1),$$

where $\mathbf{Cat}_1$ denotes the 2-category of 1-categories. These are a form of *I-symmetric monoidal Mackey functors*.

$\mathcal{T}$-commutative monoids yields I-commutative monoids by neglect of structure. By [St24], a full $\mathcal{T}$-subcategory of a cocartesian I-symmetric monoidal category $\mathcal{C} \subset \mathcal{D}^{I-\sqcup}$ is I-symmetric monoidal if and only if it's closed under I-indexed coproducts. Hence we have the following.

Corollary B. *Fix a collection of objects $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ containing the contractible $c(I)$-sets and $I \subset \mathbb{F}_{\mathcal{T}}$ the corresponding collection of maps satisfying Condition (IC-b). Then, the following conditions are equivalent:*

(1) I is a weak indexing category;
(2) $\underline{\mathbb{F}}_I$ is a weak indexing system;
(3) $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}^{I-\sqcup}$ is an I-symmetric monoidal subcategory.

We explore this further in [St24].

1.3. Weak indexing categories and transfer systems.

Definition 1.35. Given $\mathcal{T}$ an orbital category, an *orbital transfer system in* $\mathcal{T}$ is a core-containing subcategory $\mathcal{T}^{\simeq} \subset R \subset \mathcal{T}$ which is stable under *base change* in the sense that for all $\mathcal{T}$ digarams

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow{\alpha'} & & \downarrow{\alpha} \\ U' & \longrightarrow & U \end{array}$$

whose associated $\mathbb{F}_{\mathcal{T}}$ map $V' \to V \times_U U$ is a summand inclusion, if $\alpha \in R$, we have $\alpha' \in R$. The associated embedded sub-poset is

$$\mathrm{Transf}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}}).$$ ◁

Observation 1.36. If I is a unital weak indexing category, the intersection $\mathfrak{R}(I) \coloneqq I \cap \mathcal{T}$ is an orbital transfer system; hence it yields a monotone map

$$\mathfrak{R}(-) : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}.$$ ◁

Proposition 1.37 ([NS22, Rmk 2.4.9])**.** $\mathfrak{R}(-)$ *restricts to an equivalence*

$$\mathfrak{R}(-) : \mathrm{Index}_{\mathcal{T}} \xrightarrow{\sim} \mathrm{Transf}_{\mathcal{T}}.$$

Remark 1.38. In the case $\mathcal{T} = \mathcal{O}_G$, it was shown independently in [Rub19, Thm 3.7] and [BBR21, Cor 8] that the composite inclusion $\mathrm{Sub}_{\mathbf{Grp}}(G) \hookrightarrow \mathcal{O}_G \hookrightarrow \mathbb{F}_G$ induces an embedding $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathbf{Poset}}(\mathrm{Sub}_{\mathbf{Grp}}(G))$ whose image is identified by those subposets which are closed under restriction and conjugation, which were called *G-transfer systems*; this and Proposition 1.37, together imply that pullback along the *homogeneous G-set* functor $\mathrm{Sub}_{\mathbf{Grp}}(G) \to \mathcal{O}_G$ induces an equivalence between the poset of G-transfer systems of [BBR21; Rub19] and the orbital $\mathcal{O}_G$-transfer systems of Definition 1.35. ◁

In view of Remark 1.38, we henceforth in this paper refer to orbital transfer systems simply as *transfer systems*, never referring to the other notion.

In Theorem 2.26, we fact show that the composite

$$\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$$

is a fully faithful right adjoint to $\mathfrak{R}$, i.e. the poset of unital weak indexing systems possessing a given transfer system has a terminal object, given by the unique such indexing system. However, we show that the fibers can be quite large; for instance, in 2.31, we will see that $\mathfrak{R}$ also attains a fully faithful left adjoint, which is distinct from the right adjoint over all transfer systems when $\mathcal{T}$ has a terminal object (e.g. when $\mathcal{T} = \mathcal{O}_G$).

The upshot is that unital weak indexing systems are not determined by their transitive V-sets. Nevertheless, we may say a bit more, after introducing some terminology.

Definition 1.39. We say that $\mathcal{T}$ *has no self-normalizing transfers* if for all non-isomorphisms $f : V \to W$, there is a summand inclusion $2*_V \subset \mathrm{Res}_V^W \mathrm{Ind}_V^W *_V$. ◁

Example 1.40. If G is a finite group, then the following conditions are equivalent:

(1) G is nilpotent.
(2) $\mathcal{O}_G$ has no self-normalizing transfers.

To see this, note that the double coset formula implies the fixed point formula

$$\left(\mathrm{Res}_J^H \mathrm{Ind}_J^H *_J\right)^J = N_J(H);$$

thus $\mathcal{O}_G$ has no self-normalizing transfers if and only if, for all $H \subsetneq J$, H is not self-normalizing in J. But the condition that proper subgroups of H are non-self-normalizing is equivalent to the condition that H is nilpotent; thus $\mathcal{O}_G$ has no self-normalizing transfers if and only if all subgroups of G are nilpotent, which is equivalent to G itself being nilpotent. ◁

Construction 1.41. If $\mathcal{T}$ is an orbital ∞-category, then we define the collection of objects $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ to have V-value spanned by the V-sets

$$\varepsilon *_V + W_1 + \cdots + W_n,$$

for $\epsilon \in \{0,1\}$ and $W_1, \ldots, W_n \in \underline{V}$ subject to the condition that there exist no maps $W_i \to W_j$ for $i \neq j$. ◁

Example 1.42. Let G be a finite group. Then, for $(H) \subset G$ a conjugacy class of G, the *sparse H-sets* are precisely the H-sets

$$\epsilon *_H + K_1 + \cdots + K_n,$$

where none of the conjugacy classes $(K_1), \ldots, (K_n)$ include into each other. ◁

Given $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}$, we may form the full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ generated by $\mathcal{C}^{\mathrm{sprs}}$ under $\mathcal{C}^{\mathrm{sprs}}$-indexed colimits. We say that $\mathcal{C}^{\mathrm{sprs}}$ is *closed under applicable self-indexed coproducts* if $\mathcal{C}^{\mathrm{sprs}} = \mathcal{C} \cap \underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{sprs}}$. Similarly, we define $\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2}$ to consist of the objects admitting at most two orbits of each type.

Theorem C. *Restriciton along the inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T}}$ yields an embedding of posets*

$$\underline{\mathbb{F}}_{\mathcal{T}} \hookrightarrow \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\leq 2}),$$

with image the collections closed under applicable self-indexed coproducts. Furthermore, if $\mathcal{T}$ has no self-normalizing transfers, then restriction along the inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T}}$ yields an embedding of posets

$$\mathrm{wIndex}_{\mathcal{T}} \subset \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}})$$

whose image is spanned by collections which are closed under applicable self-indexed coproducts.

Corollary 1.43. *If $\mathcal{T}$ is an orbital ∞-category such that $\pi_0(\mathcal{T})$ is finite and $\mathcal{T}_{/V}$ is finite as a 1-category for all $V \in \pi_0(\mathcal{T})$, then there exist finitely many $\otimes$-idempotent weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads.*

Remark 1.44. Let $\mathcal{T} = \mathcal{O}_G$ for G a nilpotent group. By Theorem C, one may devise an inefficient algorithm to compute $\mathrm{wIndex}_G^{\mathrm{uni}}$. Namely, given a collections sparse collection $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_G^{\mathrm{sprs}}$, one may compute all of its self-indexed coproducts in finite time using the double coset formula in order to determine whether $\mathcal{C}^{\mathrm{sprs}}$ is closed under applicable self-indexed coproducts. One may simply iterate over the finite poset $\mathrm{Coll}(\underline{\mathbb{F}}_G^{\mathrm{sprs}})$, performing the above computation at each step to determine which collections correspond with unital weak indexing systems. ◁

The above algorithm is quite inefficient; in practice, we instead prefer to divide and conquer, first computing Fam_G and Transf_G, then computing the fibers under $\mathfrak{R}$ and ∇. We will do this for $G = C_{p^N}$, but first we need notation. Given $R \in \mathrm{Transf}_G$, we define the families

$$\mathrm{Dom}(R) := \left\{ U \in \mathcal{O}_G \mid \quad \exists U \rightarrow V \xrightarrow{f} W \;\text{ s.t. }\; f \in R \right\};$$
$$\mathrm{Cod}(R) := \left\{ U \in \mathcal{O}_G \mid \quad \exists V \xrightarrow{f} W \leftarrow U \;\text{ s.t. }\; f \in R \right\}.$$

Given a full subcategory $\mathcal{F} \subset \mathcal{O}_G$ and a G-transfer system T, we denote by $\mathrm{Sieve}_T(\mathcal{F})$ the poset of precomposition-closed wide subcategories of $T \cap \mathcal{F}$.

Corollary D. *Fix $N \in \mathbb{N} \cup \{\infty\}$. Then, there is a cocartesian fibration*

$$(\mathfrak{R}, \nabla) : \mathrm{wIndex}_{C_{p^N}}^{\mathrm{uni}} \rightarrow K_N \times [N]$$

with fibers satisfying

$$\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) = \begin{cases} \varnothing & \mathrm{Dom}(R) \not\leq \mathcal{F}; \\ * & \mathrm{Cod}(R) \leq \mathcal{F}; \\ \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) & \text{otherwise.} \end{cases}$$

Moreover, cocartesian transport is computed along $R \leq R'$ by the inclusion

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \hookrightarrow \mathrm{Sieve}_{R'}(\mathrm{Cod}(R') - \mathcal{F})$$

and computed along $\mathcal{F} \leq \mathcal{F}'$ by the restriction

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}')$$

This completely determines $\mathrm{wIndex}_{C_{p^N}}^{\mathrm{uni}}$. Nevertheless, we draw this explicitly for $N \leq 2$ in Section 3.

1.4. **Why weak indexing systems?**

1.5. **Notation and conventions.** There is an equivalence of categories between that of posets and that of categories whose hom sets have at most one point; we safely conflate these notions. In doing so, we use categorical terminology to describe posets.

A *sub-poset* of a poset P is an injective map $P' \hookrightarrow P$, i.e. a relation on a subset of the elements of P refining the relation on P. A *embedded sub-poset* (or *full sub-poset*) is a sub-poset $P' \hookrightarrow P$ such that $x \leq_{P'} y$ if and only if $x \leq_P y$ for all $x, y \in P'$.

An *adjunction of posets* (or *monotone Galois connection*) is a pair of opposing monotone maps $L : P \rightleftarrows Q : R$ satisfying the condition that

$$Lx \leq_Q y \qquad \iff \qquad x \leq_P Ry \quad \forall\, x \in P,\; y \in Q.$$

In this case, we refer to L as the *left adjoint* and R as the *right adjoint*, as L is uniquely determined by R and vice versa.

A *cocartesian fibration of posets* is a monotone map $\pi : P \rightarrow Q$ satisfying the condition that, for all pairs $q \leq q'$ and $p \in \pi^{-1}(q)$, there exists an element $t_q^{q'} p \in \pi^{-1}(q')$ characterized by the property

$$p \leq p' \qquad \iff \qquad t_q^{q'} p \leq p' \quad \forall\, p' \in \pi^{-1}(q');$$

in this case, we note that $t_q^{q'} : \pi^{-1}(q) \rightarrow \pi^{-1}(q')$ is a monotone map and the relation on P is entirely determined by Q and the maps $t_q^{q'}$.

Acknowledgements.

2. Weak indexing systems

2.1. **Recovering weak indexing categories from their slice categories.** Recall that the poset of weak indexing systems $\mathrm{wIndexCat} \subset \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}})$ is the embedded subposet spanned by those subcategories satisfying Conditions (IC-a) to (IC-c) of Theorem A.

Proposition 2.1. *If I is a $\mathcal{T}$-weak indexing category then, $I_V \coloneqq I_{/V}$ is a $\underline{V}$-weak indexing category.*

Proof. Condition (IC-c) for I_V follows quickly by noting that automorphisms I_V have underlying automorphisms, and Condition (IC-b) for I_V follows by unwinding definitions, noting that $\mathbb{F}_V \to \mathbb{F}_{\mathcal{T}}$ is coproduct-preserving. Lastly, Condition (IC-a) follows by unwinding definitions, noting that the pullback functor $\mathbb{F}_V \to \mathbb{F}_W$ is pullback-preserving for each $W \to V$ since $\mathbb{F}_{\mathcal{T}}$ is an ∞-topos. □

Construction 2.2. We denote the embedded $\mathcal{T}$-subposet with V-values the $\underline{V}$-weak indexing systems by $\underline{\mathrm{wIndexCat}}_{\mathcal{T}} \subset \underline{\mathrm{Sub}}_{\underline{\mathbf{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$. ◁

There is a monotone map of posets

$$\gamma : \mathrm{Sub}_{\mathbf{Cat}}(\mathbb{F}_{\mathcal{T}}) \to \Gamma \underline{\mathrm{Sub}}_{\underline{\mathbf{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$$

satisfying $\gamma(\mathcal{C})_V \simeq \mathcal{C}_{/V}$ with functoriality supplied by pullback. The primary proposition of this subsection recovers $\mathrm{wIndexCat}_{\mathcal{T}}$ from $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$.

Theorem 2.3. *γ restricts to an equivalence*

$$\gamma : \mathrm{wIndexCat}_{\mathcal{T}} \xrightarrow{\sim} \Gamma \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$$

Proof. Proposition 2.1 implies that γ restricts to a monotone map of posets $\gamma_W : \mathrm{wIndexCat}_{\mathcal{T}} \to \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$, so it suffices to prove that this is bijective. In fact, it quickly follows from Condition (IC-b) that γ_W is injective, so it suffices to prove that it is surjective.

To do so, fix $I_{(-)} \in \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Define the subcategory

$$I \coloneqq \{T \to S \mid \forall U \in \mathrm{Orb}(S), \quad T \times_S V \to V \in I_V\} \subset \mathbb{F}_{\mathcal{T}};$$

note that I satisfies Condition (IC-b) by definition. Furthermore, since any automorphism of V is isomorphic to $*_V \in \mathbb{F}_V$, the subcategory I satisfies Condition (IC-c). Lastly, Condition (IC-a) is precisely the condition that $I_{(-)}$ is an element of $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Hence I is a $\mathcal{T}$-weak indexing system, proving that γ_W is an isomorphism of posets. □

Remark 2.4. The atomic orbital ∞-category $\underline{V}$ has a terminal object; by [NS22, Prop 2.5.1], this implies that $\underline{V}$ is a 1-category. In general for $F : J \to \mathcal{T}$ a diagram in an atomic orbital ∞-category indexed by a finite 1-category, $\mathcal{T}_{/J}$ is also a 1-category; in particular, the top arrow

$$\begin{array}{ccc} \mathcal{T}_{/J} & \longrightarrow & \mathrm{ho}(\mathcal{T})_{/J} \\ & \searrow & \wr\| \\ & & \mathrm{ho}(\mathcal{T}_{/J}) \end{array}$$

is an equivalence. This implies that $\mathbb{F}_{\mathrm{ho}\mathcal{T}}$ has pullbacks, i.e. $\mathrm{ho}(\mathcal{T})$ is orbital; because $\mathcal{T}$ is atomic, retracts in $\mathrm{ho}(\mathcal{T})$ are isomorphisms, i.e. $\mathrm{ho}(\mathcal{T})$ is atomic orbital. ◁

Using this and fact that the 1-category of posets is a 1-category, we an equivalence

$$\begin{array}{ccc} \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}) & \xrightarrow{\mathrm{ho}} & \mathrm{Sub}(\mathbb{F}_{\mathrm{ho}(\mathcal{T})}) \\ \uparrow & & \uparrow \\ \mathrm{wIndexCat}_{\mathcal{T}} & \xrightarrow{\sim} & \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})} \\ \wr\| & & \wr\| \\ \lim_{V \in \mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathcal{T}_{/V}} & \xrightarrow{\sim} & \lim_{V \in \mathrm{ho}\mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})_{/V}} \end{array}$$

i.e. the following.

Corollary 2.5. *The homotopy category construction yields an equivalence* $\mathrm{wIndexCat}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})}$.

Using this, for the rest of the paper, we will assume that $\mathcal{T}$ is a 1-category.

Corollary 2.6. *If X is a space, then the forgetful map* $\mathrm{wIndex}_X \to \mathrm{wIndex}_*$ *is an equivalence.*

2.2. Weak indexing categories vs weak indexing systems.

Construction 2.7. Given $I \subset \mathbb{F}_{\mathcal{T}}$ a subgraph, define the class of *I-admissible V-sets*

$$\mathbb{F}_{V,I} := \left\{ S \mid \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathbb{F}_V.$$

Taken altogether, we refer to this as $\underline{\mathbb{F}}_I$. ◁

Observation 2.8. Given $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ a collection of objects, we have $\mathbb{F}_{V,I(\mathcal{C})} \simeq \mathcal{C}$; conversely, if $I \subset \mathbb{F}_{\mathcal{T}}$ satisfies Condition (IC-b), then $I(\underline{\mathbb{F}}_I) = I$. ◁

Observation 2.9. If $S \simeq S'$, then there exists an equivalence $\psi : \mathrm{Ind}_V^{\mathcal{T}} S \simeq \mathrm{Ind}_V^{\mathcal{T}} S'$ over V. Hence whenever I satisfies Condition (IC-c), ψ is in I, and I is a category, so this implies that $\mathrm{Ind}_V^{\mathcal{T}} S' \to V$ is in I, i.e. $\mathbb{F}_{V,I} \subset \mathbb{F}_V$ is closed under equivalence; these objects determine a unique full subcategory, which we henceforth refer to by the same name.

Conversely, if $\underline{\mathbb{F}}_I$ is a $\mathcal{T}$-weak indexing system and $\mathcal{T}$ has a terminal object $*_{\mathcal{T}}$, then the fact that $\underline{\mathbb{F}}_I$ contains all automorphisms immediately implies that $I(\underline{\mathbb{F}}_I)$ contains all automorphisms. ◁

Observation 2.10. By definition, the restriction map $\mathbb{F}_V \to \mathbb{F}_W$ is implemented by the pullback

$$\begin{array}{ccc} \mathrm{Ind}_W^{\mathcal{T}} \mathrm{Res}_W^V S & \longrightarrow & \mathrm{Ind}_V^{\mathcal{T}} S \\ \downarrow & \lrcorner & \downarrow \\ W & \longrightarrow & V \end{array}$$

Condition (IC-a) then yields that $\mathrm{Res}_W^V \mathbb{F}_{V,I} \subset \mathbb{F}_{W,I}$; hence in the presence of Condition (IC-b), $\{\mathbb{F}_{V,I}\}_{V \in \mathcal{T}}$ correspond with a unique full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$.

Conversely, the above argument shows that a collection of full subcategories $\mathbb{F}_{I,V} \subset \mathbb{F}_V$ are a full $\mathcal{T}$-subcategory if and only if $I(\mathbb{F}_{I,V})$ satisfies Condition (IC-a'). ◁

Proposition 2.11. *If $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a weak indexing system, then $I(\mathcal{C})$ is a weak indexing category.*

Proof. By Observations 1.33 and 1.34, it suffices to verify Conditions (IC-a'), (IC-b') and (IC-c). Note that $I(\mathcal{C})$ is compatible with restrictions, so by Theorem 2.3, it suffices to prove this individually for each $\underline{V}$, and hence we may assume $\mathcal{T}$ has a terminal object. Condition (IC-a') is verified in this case by Observation 2.10; Condition (IC-b') follows immediately from construction; Condition (IC-c) is verified in Observation 2.9. □

Proposition 2.12. *If I is a weak indexing category, then $\underline{\mathbb{F}}_I$ is a weak indexing system.*

Proof. Observations 2.9 and 2.10 verify that $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, and the fact that the identity arrow on V corresponds with the contractible V-set implies that whenever $\underline{\mathbb{F}}_{I,V} \neq \varnothing$ (i.e. $V \in I$), $*_V \in \underline{\mathbb{F}}_{I,V}$. Thus it suffices to verify that $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

Let $(T_U) \in \underline{\mathbb{F}}_{I,S}$ be an S-tuple in $\underline{\mathbb{F}}_I$ for some $S \in \mathbb{F}_{I,V}$. Then, the indexed coproduct of (T_U) corresponds with the composite arrow

$$\mathrm{Ind}_V^{\mathcal{T}} \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^{\mathcal{T}} T_U \to S \to V;$$

the left arrow is in I by Condition (IC-b) applied to the structure maps for each T_U and the right arrow is in I by assumption. Thus the composite is in I, i.e. $\coprod_U^S T_U \in \underline{\mathbb{F}}_I$, as desired. □

Proof of Theorem A. By Propositions 2.11 and 2.12, $I : \mathrm{wIndex}_{\mathcal{T}} \rightleftarrows \mathrm{wIndexCat}_{\mathcal{T}} : \underline{\mathbb{F}}_{(-)}$ are well defined monotone maps; by Observation 2.8, they are inverse to each other, so I is an isomorphism onto its image $\mathrm{wIndexCat}_{\mathcal{T}}$.

What remains is to verify that (IC-n) is equivalent to (IS-n) in Definition 1.19 and Theorem A. For $n = i$, this follows immediately by noting that $V \in I \iff \mathrm{id}_V \in I \iff *_V \in \mathbb{F}_{I,V} \iff \mathbb{F}_{I,V} \neq \varnothing$. For $n = ii$ and $n = iii$, this follows by unwinding definitions using by Condition (IC-b'). For $n = iv$, this follows by noting that the fold map $n \cdot V \to V$ corresponds with the element $n \cdot *_V \in \mathbb{F}_V$. □

2.3. Joins, closures, color-support, and color-borelification.

2.3.1. *Prerequisites on cocartesian fibrations.* Recall that a monotone map $\pi : \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration if and only if, for all related pairs $D \leq D'$ and elements $C \in \pi^{-1}(D)$, there is an element $t_D^{D'} C \in \pi^{-1}(D')$ satisfying the property

$$C \leq C' \qquad \iff \qquad t_C^{C'} C \leq C' \quad \forall\, D' \leq \pi(C') \in \pi^{-1}(D').$$

Proposition 2.13. *Suppose $\pi : \mathcal{C} \to \mathcal{D}$ is a monotone map possessing a left adjoint L and $\mathcal{C}$ has binary joins. Then, π is a cocartesian fibration with*

$$t_C^{C'} C = L(D) \vee C.$$

Proof. This follows immediately from the propery

$$L(D') \vee C \leq C' \iff L(D') \leq C' \text{ and } C \leq C',$$

noting that $L(D') \leq C'$ by assumption. ☐

Lemma 2.14. *Let $\pi : \mathcal{C} \to \mathcal{D}$ be a monotone map. The following are equivalent.*

- *(a) π possesses a fully faithful left adjoint L.*
- *(b) For all $D \in \mathcal{D}$, the preimage $\pi^{-1}(\mathcal{D}_{\geq D})$ possesses an initial object $L(D)$ with $\pi L(D) = D$.*
- *(c) For all $D \in \mathcal{D}$, the fiber $\pi^{-1}(D)$ has an initial object $L(D)$, and $D \leq D'$ implies $L(D) \leq L(D')$.*

Proof. By definition, π has a left adjoint L if and only if there are initial objects to $\pi^{-1}(\mathcal{D}_{\leq D})$, which are $L(D)$. By the usual category theoretic nonsense, L is fully faithful if and only if the unit relation $D \leq \pi L(D)$ is an equiality, i.e. $L(D) \in \pi^{-1}(D)$; hence (a) $\iff$ (b). To see (b) $\iff$ (c), it follows to note that when (c), $L(D) \leq C'$ if and only if $D \leq C'$ if and only if $L(D) \leq L\pi(C')$. ☐

2.3.2. *Closures and joints of weak indexing systems.*

Construction 2.15. Given $\mathcal{D}, \mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ full $\mathcal{T}$-subcategories, inductively define $\mathrm{Cl}_{\mathcal{D},0}(\mathcal{C}) \coloneqq \mathcal{C}$ and

$$\mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})_V = \left\{ \coprod_U^S T_U \mid (T_U) \in \mathrm{Cl}_{n-1}(\mathcal{C})_S, \;\; S \in \mathcal{D} \right\},$$

with $\mathrm{Cl}_{\mathcal{D},\infty}(\mathcal{C}) \coloneqq \bigcup_n \mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})$. and $\mathrm{Cl}_\infty(\mathcal{C}) \coloneqq \mathrm{Cl}_{\mathcal{C},\infty}(\mathcal{C})$. We call this the *$n$-step closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts.* ◁

Observation 2.16. If $\mathcal{D}$ is a weak indexing system, then the canonical inclusion

$$\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}) \subset \mathrm{Cl}_{\mathcal{D}}(\mathcal{C})$$

is an equality for all $\mathcal{C}$. ◁

Let $\mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}}) \subset \mathrm{FullSub}_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ denote the full subposet of elements satisfying Condition (IS-a).

Lemma 2.17. *The fully faithful map $\iota : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ is right adjoint to Cl_∞.*

Proof. If $\mathrm{Cl}(\mathcal{C})$ is a weak indexing system, then it is clearly minimal among those containing $\mathcal{C}_S$, so it suffices to prove that it's a weak indexing system. Note that $\mathrm{Cl}(\mathcal{C})_V \neq \varnothing$ iff $\mathcal{C}_V \neq \varnothing$ iff $*_V \in \mathcal{C}_V$ iff $*_V = \coprod_{*_V}^{*_V} *_V \in \mathrm{Cl}(\mathcal{C})_V$, so it suffices to prove that $\mathrm{Cl}(\mathcal{C})$ is closed under self-indexed coproducts.

In fact, if by a basic inductive argument, we find that $\mathrm{Cl}(\mathcal{C})_i$-indexed coproducts of elements of $\mathrm{Cl}(\mathcal{C})_j$ lie in $\mathrm{Cl}(\mathcal{C})_{i+j} \subset \mathrm{Cl}(\mathcal{C})$, so the result follows by taking a union. ☐

Given $S \in \mathbb{F}_V$, let $\mathbb{F}_{I_S,V}$ be the closure of $\{*_V\}$ under S-indexed coproducts; more generally, let $\mathbb{F}_{I_S,W} \coloneqq \bigcup_{f:W\to V} \mathrm{Res}_W^V \mathbb{F}_{I_S,V}$, and let $\left(\underline{\mathbb{F}}_{I_S}\right)_W \coloneqq \mathbb{F}_{I_S,W}$.

Proposition 2.18. *Given $S \in \mathbb{F}_V$, we have $\mathrm{Cl}_\infty(\{S\}) = \underline{\mathbb{F}}_{I_S}$.*

Proof. First, note that $\underline{\mathbb{F}}_{I_S} \subset \mathrm{Cl}_\infty(\{S\})$. By Lemma 2.17, it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is weak indexing system containing S.

By construction, $\underline{\mathbb{F}}_{I_S}$ is a full $\mathcal{T}$-full subcategory satisfying the property that

$$*_W \in \mathbb{F}_{I_S,W} \qquad \iff \qquad \exists f : W \to V \qquad \iff \qquad \varnothing \neq \mathbb{F}_{I_S,W}.$$

Hence it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts.

First, note that if $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under T-indexed coproducts X_U-indexed coproducts for $X \in \mathbb{F}_U$ for all $U \in \mathrm{Orb}(T)$, then $\mathcal{C}$ is closed under $\coprod_U^T X_U$-indexed coproducts; hence $\mathbb{F}_{I_S,V}$ is closed under $\mathbb{F}_{I_S,V}$-indexed coproducts.

Second, note that if $\mathcal{C}_W$ is generated under restrictions by $\mathcal{C}_U$ and $\mathcal{C}_U$ is closed under T-indexed coproducts, then $\mathcal{C}_W$ is closed under $\mathrm{Res}_W^U T$-indexed coproducts; hence $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts, as desired. □

Proposition 2.19. $\mathrm{wIndex}_{\mathcal{T}}$ *is a lattice; the meets in* $\mathrm{wIndex}_{\mathcal{T}}$ *are intersections, and the joins are*

$$\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \bigcup_{n \in \mathbb{N}} \overbrace{\mathrm{Cl}_I \mathrm{Cl}_J \cdots \mathrm{Cl}_I \mathrm{Cl}_J}^{2n} (\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J).$$

Proof. By Lemma 2.17, $\mathrm{wIndex}_{\mathcal{T}}$ has meets computed in $\mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$, which are clearly given by intersections. Furthermore, Lemma 2.17 implies that $\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)$. Thus is suffices to note that, for arbitrary $\mathcal{C}, \mathcal{D}, \mathcal{E}$, we have

$$\mathrm{Cl}_{\mathcal{C} \cup \mathcal{D}, \infty}(\mathcal{E}) = \bigcup_{n \in \mathbb{N}} \overbrace{\mathrm{Cl}_{\mathcal{C}} \mathrm{Cl}_{\mathcal{D}} \cdots \mathrm{Cl}_{\mathcal{C}} \mathrm{Cl}_{\mathcal{D}}}^{2n} (\mathcal{E}),$$

and set $\mathcal{C} = \underline{\mathbb{F}}_I$, $\mathcal{D} = \underline{\mathbb{F}}_J$, and $\mathcal{E} = \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$. □

Given $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$, we define the family

$$c(\mathcal{C}) \coloneqq \{ V \in \mathcal{T} \mid \mathcal{C}_V \neq \varnothing \}.$$

Observation 2.20. For any $\mathcal{C}$, we have $\mathrm{Cl}_{\underline{\mathbb{F}}^{\mathrm{triv}}_{c(\mathcal{C})}}(\mathcal{C}) = \mathcal{C}$. ◁

2.3.3. *The color-support fibration.*

Proposition 2.21. *The monotone map* $c : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *has a fully faithful left adjoint* $\underline{\mathbb{F}}^{\mathrm{triv}}_{(-)}$ *and a fully faithful right adjoint* $\underline{\mathbb{F}}_{(-)}$.

Proof. By Lemma 2.14 it suffices to note that $\underline{\mathbb{F}}^{\mathrm{triv}}_{c(\underline{\mathbb{F}}_I)} \leq \underline{\mathbb{F}}_I \leq \underline{\mathbb{F}}_{c(\underline{\mathbb{F}}_I)}$. □

The following proposition follows by unwinding definitions.

Proposition 2.22. *The fiber* $c^{-1}(\mathrm{Fam}_{\mathcal{T}, \leq \mathcal{F}})$ *is equivalent to* $\mathrm{wIndex}_{\mathcal{F}}$*, and the associated fully faithful functor* $E_{\mathcal{F}}^{\mathcal{T}} : \mathrm{wIndex}_{\mathcal{F}} \hookrightarrow \mathrm{wIndex}_{\mathcal{F}}$ *is left adjoint to* $\mathrm{Bor}_{\mathcal{F}}^{\mathcal{T}} \coloneqq (-) \cap \underline{\mathbb{F}}_{\mathcal{F}}$.

Proposition 2.23. *Let* $\mathcal{T}$ *be an orbital category.*

(1) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \to \mathrm{wIndex}_{\mathcal{T}}$ *is left adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(\underline{\mathbb{F}}_I)} \underline{\mathbb{F}}^0_{\upsilon(\underline{\mathbb{F}}_I)}$.
(2) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \to \mathrm{wIndex}_{\mathcal{T}}$ *is left adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(\underline{\mathbb{F}}_I)} \underline{\mathbb{F}}^0_{c(\underline{\mathbb{F}}_I)}$.
(3) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \to \mathrm{wIndex}_{\mathcal{T}}$ *is left adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^0_{\upsilon(\underline{\mathbb{F}}_I)}$.
(4) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{wIndex}_{\mathcal{T}}$ *is left adjoint to* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^0_{\mathcal{T}}$.

Corollary 2.24. *Let* $\mathcal{T}$ *be an orbital category.*

(1) The map $c : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{oc}$ *and with cocartesian transport along* $\mathcal{F} \leq \mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}'}$.
(2) The map $c : \mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ *and cocartesian transport along* $\mathcal{F} \leq \mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}'}$.
(3) The map $c : \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{a\mathrm{uni}}$ *and cocartesian transport along* $\mathcal{F} \leq \mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}'}$.

Remark 2.25. Entailed in this corollary is the statement that $\underline{\mathbb{F}}_I$ is E-unital if and only if $\underline{\mathbb{F}}_I = E^{\mathcal{T}}_{c(I)} \mathrm{Bor}^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}_I$; in particular, we find that the E-unital weak indexing systems are those which are E of unital weak indexing systems. ◁

2.4. **The transfer system fibration.** Recall that the monotone map $\mathfrak{R} : \mathrm{wIndexCat}_{\mathcal{T}} \to \mathrm{Transf}_{\mathcal{T}}$ is defined by $\mathfrak{R}(I) = I \cap \mathcal{T}$; we denote the composite $\mathrm{wIndex}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathcal{T}} \to \mathrm{Transf}_{\mathcal{T}}$ as $\mathfrak{R}$ as well.

Given R a transfer system, define the subcategory

$$\overline{\mathbb{F}}_{R,V} \coloneqq \mathrm{Cl}_\infty(\{\mathrm{Res}_V^W U \mid U \to W \in R\})_V.$$

This subsection is primarily dedicated to proving the following.

Theorem 2.26. *The map of posets* $\mathfrak{R} : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ *has fully faithful right adjoint given by the composite* $\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ *and fully faithful left adjoint given by* $\overline{\underline{\mathbb{F}}}_{(-)}$.

Corollary 2.27. *If* I, J *are unital weak indexing categories, then* $\mathfrak{R}(I) \vee \mathfrak{R}(J) = \mathfrak{R}(I \vee J)$.

In particular, this immediately implies that $\mathfrak{R}$ is compatible with meets and joins. Our first step in proving this is verifying a restricted compatibility with joins.

Proposition 2.28. *If* I, J *unital satisfy* $\mathfrak{R}(I) \le \mathfrak{R}(J)$, *then* $\mathfrak{R}(I \vee J) = \mathfrak{R}(J)$.

This breaks down to the following easy technical lemma.

Lemma 2.29. $\mathfrak{R}(\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})) = \mathfrak{R}(\mathrm{Cl}_{R(\mathcal{D}),1}\mathfrak{R}\mathcal{C})$.

Proof. It suffices to note that whenever $\coprod_U^S T_U$ is an orbit, there is exactly one T_U which is nonempty, in which case $\mathrm{Ind}_U^V T_U = \coprod_U^S T_U$, implying that T_U is an orbit. □

Proof of Proposition 2.28. Note that $\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$ is closed under J-indexed induction, so we have

$$\mathfrak{R}(\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J,1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)) = \mathfrak{R}(\mathrm{Cl}_{\mathfrak{R}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J),1}(\mathfrak{r}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J))) = \mathfrak{R}(\mathrm{Cl}_{\mathfrak{R}(J),1}(\mathfrak{R}(J)))) = \mathfrak{R}(J).$$

Iterating this and taking a union, we find that

$$\mathfrak{R}(I \vee J) = \mathfrak{R}\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J,\infty}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \mathfrak{r}(J).$$ □

Proposition 2.30. $\overline{\underline{\mathbb{F}}}_R$ *is the terminal element of* $\mathfrak{R}^{-1}(R)$.

Proof. The only nontrivial part is showing that $\mathfrak{R}(\overline{\underline{\mathbb{F}}}_R) = R$; in fact, this follows by unwinding definitions and applying Lemma 2.29. □

Proof of Theorem 2.26. By Proposition 2.28, the indexing system $I_{\mathcal{T}}^\infty \vee I$ satisfies $\mathfrak{R}(I_{\mathcal{T}}^\infty \vee I) = \mathfrak{R}(I)$, and is an upper bound for I. In fact, by Proposition 1.37, this is the *unique* indexing system over $\mathfrak{R}(I)$, so it is automatically terminal. This and Proposition 2.30 together imply the theorem by Lemma 2.14. □

Remark 2.31. If $\mathcal{T}$ has a terminal object V, then $2*_V$ is not in $\overline{\underline{\mathbb{F}}}_R$ for any R, since $2*_V$ is not a summand in the restriction of any transitive W-sets for any $W \in \mathcal{T}$. Hence $\overline{\underline{\mathbb{F}}}_R$ is not an indexing system, or equivalently, $\mathfrak{R}^{-1}(R)$ has multiple elements. We may interpret this as saying that unital weak indexing systems are seldom determined by their transitive V-sets. ◁

2.5. **The unit and fold map fibrations.**

2.5.1. *The unit fibration.*

Proposition 2.32. *The map* $\upsilon : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *has fully faithful left adjoint given by* $\underline{\mathbb{F}}^0_{(-)}$.

Furthermore, $\underline{\mathbb{F}}^0_{\mathcal{F}}$ is almost-unital.

Corollary 2.33. *The restricted map* $\upsilon_a : \mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $\upsilon_a^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ *embedded along* $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}} \cup E_{\mathcal{F}}^{\mathcal{T}}(-)$. *Moreover, the cocartesian transport map* $t_{\mathcal{F}}^{\mathcal{F}'} : \upsilon_a^{-1}(\mathcal{F}) \to \upsilon_a^{-1}(\mathcal{F}')$ *is implemented by*

$$t_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}^0_{\mathcal{F}'} \cup E_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I$$

2.5.2. *The fold map fibration.*

Proposition 2.34. *The map $\nabla_u : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ has fully faithful left adjoint given by $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup \underline{\mathbb{F}}_{(-)}^{\infty}$; hence it is a cocartesian fibration, and the cocartesian transport map $t_{\mathcal{F}}^{\mathcal{F}'}$ is implemented by*

$$t_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I \simeq \underline{\mathbb{F}}_{\mathcal{F}}^{\infty} \vee \underline{\mathbb{F}}_I.$$

Lemma 2.35. *Suppose $\underline{\mathbb{F}}_I$ is unital. If $\nabla(\underline{\mathbb{F}}_I), \nabla(\mathcal{C}) \leq \mathcal{F}'$, then $\nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C})) \leq \mathcal{F}'$.*

Proof. Suppose $V \in \nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C}))$, i.e. there exists some $S \in \underline{\mathbb{F}}_I$, $(X_U) \in \mathcal{C}_S$, and $n \geq 2$ such that $\coprod_U^S X_U = n *_V$. We would like to prove that $V \in \mathcal{F}'$. Since $\underline{\mathbb{F}}_I$ is unital, we may "remove" any empty X_U and replace S with its summand consisting of orbits over which X_U is nonempty, hence assume WLOG that X_U is nonempty for all U.

Note that $\mathrm{Ind}_U^V X_U = m *_V$ for some m; in particular, this implies $U = V$. Hence $S = k *_V$ for some k. Writing our decomposition as $S = \{1, \ldots, n\}$ and $X_i = m_i *_V$, we find that $n = \sum_{i=1}^k m_i \geq 2$, so either $m_i \geq 2$ for some i or $k \geq 2$. In either case, we find $V \in \mathcal{F}'$, as desired. □

Observation 2.36. For any nonempty set of collections $(\mathcal{C}_i)_{i \in I}$, it follows by unwinding definitions that we have $\nabla\left(\bigcup_{i \in I} \mathcal{C}_i\right) = \bigcup_{i \in I} \nabla\left(\mathcal{C}_i\right)$. ◁

Proposition 2.37. $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$.

Proof. By **??**, we have $\nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J)$, so we prove the opposite inclusion. By Lemma 2.35, we find inductively that $\nabla \mathrm{Cl}_{\underline{\mathbb{F}}_I,1} \mathrm{Cl}_{\underline{\mathbb{F}}_J,1} \cdots \mathrm{Cl}_{\underline{\mathbb{F}}_J,1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$; applying Observation 2.36 to take a union, we find that $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$, as desired. □

Remark 2.38. The fibers of ∇ are all nonempty by Proposition 2.34; by Observation 2.36 and Proposition 2.37, $\nabla^{-1}(\mathcal{F})$ is closed under *arbitrary* joins, so it has a terminal object, i.e. ∇ possesses a fully faithful right adjoint.

The author is not aware of a general formula for this, but there are interesting examples; for instance, if λ is a nontrivial irreducible real orthogonal C_p-representation, then we show in [St24] that the arity support $A\lambda$ of the C_p-weak $\mathcal{N}_\infty$-operad $\mathbb{E}_{\lambda\infty}$ is terminal among the C_p-weak indexing systems with fold maps over the trivial subgroup. In algebra, this may be intrerpreted as saying that $\mathbb{E}_{\lambda\infty}$ presents the terminal sub-C_p-commutative algebraic theory prescribing fold maps on the underlying Borel type of a genuine C_p-object, but not on genuine C_p-fixed points. ◁

We would like to compute examples with many transfers and few folds.

Observation 2.39. If R is a transfer system, then unwinding definitions, we find

$$\nabla \overline{\underline{\mathbb{F}}}_R = \mathrm{Dom}(R) := \left\{ U \in \mathcal{T} \mid \;\; \exists U \to W \xleftarrow{f} V \text{ s.t. } f \in R \text{ and } 2*_U \subset \mathrm{Res}_U^W \mathrm{Ind}_V^W *_V \right\}.$$

◁

Remark 2.40. If $\mathcal{T} = \mathcal{F} \subset \mathcal{O}_G$ is a family of normal subgroups of a finite group (e.g. $\mathcal{T} = \mathcal{O}_G$ and G is a Dedekind group), then for every pair of proper subgroup inclusion $H, K \subset J$, the double coset formula implies that $\mathrm{Res}_K^J \mathrm{Ind}_H^J *_H = [K \backslash J / H] \cdot H / H \cap K$. In particular, $2*_H \subset \mathrm{Res}_K^J \mathrm{Ind}_H^J *_H$ if and only if $H \subset K$.

Unwinding definitions, we find in this case that $\mathrm{Dom}(R)$ is the family

$$\mathrm{Dom}(R) = \left\{ U \in \mathcal{F} \mid \exists U \to V \xrightarrow{f} W \mid f \in R - R^{\simeq} \right\},$$

i.e. it is the family of subgroups generated by domains of nontrivial transfers. ◁

2.5.3. *The combined transfer-fold fibration.*

Observation 2.41. By Proposition 2.30 and Observation 2.39, If $\mathrm{Dom}(R) \not\subset \mathcal{F}$, then $\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ is empty. In fact, by Proposition 2.37 and Observation 2.39 we find that $\overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}_{\mathcal{F}}^{\infty} \in \mathcal{F}^{-1}(R) \cap \nabla^{-1}(\mathcal{F} \cup \mathrm{Dom}(R))$ is *initial*; in particular the condition $\mathrm{Dom}(R) \subset \mathcal{F}$ is necessary and sufficient for $\mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ to be empty. ◁

Define the embedded subposet $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} \subset \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ spanned by the pairs $(R, \mathcal{F})$ such that $\mathrm{Dom}(R) \leq \mathcal{F}$. In light of Lemma 2.14, we may rephrase Observation 2.41 as follows.

Proposition 2.42. *The map* $(\mathfrak{R},\nabla) : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ *has image* $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}}$*, and factors as*

$$\begin{array}{ccc} \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} & & \\ \downarrow{\scriptstyle(\mathfrak{R},\nabla)} & \searrow^{(\mathfrak{R},\nabla)} & \\ (\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} & \hookrightarrow & \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}} \end{array}$$

where the lefthand map admits a fully faithful left adjoint computed by $(R,\mathcal{F}) \mapsto \overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}_{\mathcal{F}}^{\infty}$.

Corollary 2.43. *The unrestricted map* $(\mathfrak{R},\nabla) : \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration, with cocartesian transport along nonempty fibers given by the union*

$$t_{(R,\mathcal{F})}^{(R',\mathcal{F}')} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}_{\mathcal{F}'}^{\infty}.$$

2.6. **Compatible pairs of weak indexing systems.**

Proposition 2.44. *If* I_m, I_a *are weak indexing categories, the following conditions are equivalent:*

- *(a)* $\underline{\mathbb{F}}_{I_a}$ *admits* I_m*-indexed products.*
- *(b)* $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ *is a bispan triple in the sense of [EH23, Def 2.4.3].*

Proof. Note that $\mathbb{F}_{\mathcal{T}}$ is an ∞-topos, as it is a localization of a presheaf topos. Hence [EH23, Rmk 2.4.7] and [EH23, Lem 2.4.6] imply that $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ is a bispan triple if and only if, for all maps $T \to S$ in I_m, the pullback

$$\begin{array}{ccc} I_{a,/T} & \xrightarrow{f^*} & I_{a,S} \\ \wr\| & & \wr\| \\ \prod_{U \in S} \mathbb{F}_{I_a,U}^{\times T_U} & \xrightarrow{f^*} & \prod_{U \in S} \mathbb{F}_{I_a,U} \end{array}$$

has a right adjoint; unwinding definitions, this is true if and only if I_a admits I_m-indexed products. □

Definition 2.45. A pair of one-object weak indexing categories (I_a, I_m) is *compatible* if $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under I_m-indexed products, i.e. $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\times}$ is an I_m-symmetric monoidal subcategory inclusion. ◁

Given a compatible pair (I_a, I_m), Proposition 2.44 and [EH23, Notn 2.5.11] yield an ∞-category

$$P_{I_a,I_m}^{\mathcal{T}} \coloneqq \mathrm{Bispan}_{I_m,I_a}(\mathbb{F}_{\mathcal{T}}),$$

whose homotopy category recovers the category P_{I_a,I_m}^G of [BH22] when I_a, I_m are $\mathcal{O}_G$-indexing systems. Furthermore, this is compatible with restrictions, and hence it yields a $\mathcal{T}$-category $\underline{P}_{I_a,I_m}^{\mathcal{T}}$ equipped with a core-preserving and I_m-product-preserving $\mathcal{T}$-functor

$$\iota : \mathrm{Span}_{I_a}(\underline{\mathbb{F}}_{\mathcal{T}}) \to \underline{P}_{I_a,I_m}^{\mathcal{T}}.$$

Together, this defines a pair of ∞-categories

$$\mathrm{Mack}_{I_a}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathrm{Span}_{I_a}(\mathbb{F}_{\mathcal{T}}), \mathcal{C}),$$
$$\mathrm{Tamb}_{I_a,I_m}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(P_{I_a,I_m}^{\mathcal{T}}, \mathcal{C}),$$

together with a forgetful functor

$$U : \mathrm{Tamb}_{I_a,I_m}(\mathcal{C}) \to \mathrm{Mack}_{I_a}(\mathcal{C}),$$

the codomain being modelled by $\mathrm{CAlg}_{I_a}\left(\underline{\mathrm{CoFr}}\,\mathcal{C}^{I_a-\times}\right)$ in [St24]. Furthermore, in [St24], we will define on $\mathrm{Span}_{I_a}(\underline{\mathbb{F}}_{\mathcal{T}})$ a *smash product* I_m*-symmetric monoidal structure* (restricted from the case that I_a is complete), which we will show to induce a *Day convolution* I_m*-symmetric monoidal structure* on $\mathrm{Mack}_{I_a}(\mathcal{C})$. We expect that, generalizing work of [Cha24], there is an equivalence of $\mathcal{T}$-categories $\mathrm{CAlg}_{I_m}(\underline{\mathrm{Mack}}_{I_a}(\mathcal{C})) \simeq \mathrm{Tamb}_{I_a,I_m}(\mathcal{C})$ over $\mathrm{Mack}_{I_a}(\mathcal{C})$.

Remark 2.46. Let $\mathrm{Comp}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{oc,\times 2}$ be the poset of compatible pairs, so that $(I_m, I_a) \leq (I'_m, I'_a)$ if and only if $I_m \leq I'_m$ and $I_a \leq I'_a$. Then, note that $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{I'_a}$ is an I_m-symmetric monoidal subcategory inclusion, so we have a product preserving subcategory inclusion $P^{\mathcal{T}}_{I_a,I_m} \to P^{\mathcal{T}}_{I'_a,I'_m}$. Hence these yield functoriality

$$\mathrm{CAlg}_{I'_m} \mathrm{Mack}_{I'_a}(\mathcal{C}) \to \mathrm{CAlg}_{I_m} \mathrm{Mack}_{I'_a}(\mathcal{C}) \to \mathrm{Mack}_{I_a}(\mathcal{C}),$$
$$\mathrm{Tamb}_{I'_m,I'_a}(\mathcal{C}) \to \mathrm{Tamb}_{I_m,I_a}(\mathcal{C}),$$

i.e. both $\mathrm{CAlg}_{(-)}\mathrm{Mack}_{(-)}(\mathcal{C})$ and $\mathrm{Tamb}_{-,-}(\mathcal{C})$ are functors out of $\mathrm{Comp}_{\mathcal{T}}$. The equivalence of [Cha24] is natural under this; we expect that the homotopical version of this equivalence will be natural as well. ◁

This will greatly be simplified by the following.

Proposition 2.47 (Multiplicative hull)**.** *Given $\underline{\mathbb{F}}_I$ a one-object weak indexing system, the subcategories*

$$\mathbb{F}_{m(I),V} \coloneqq \{S \in \mathbb{F}_V \mid \underline{\mathbb{F}}_I \text{ closed under } S\text{-indexed products}\}$$

form an indexing system characterized by the property that, for all $I_m \in \mathrm{wIndex}_{\mathcal{T}}$, the pair (I, I_m) is compatible if and only if $I_m \leq m(I)$.

Proof of Proposition 2.47. It follows directly from construction that $I_m \leq m(I)$ if and only if (I, I_m) is compatible. Furthermore, the $*_V$-indexed product functor is the identity, so $*_V \in \mathbb{F}_{m(I),V}$ for all V. Hence it suffices to prove that $n*_V \in \mathbb{F}_{m(I),V}$ for all $n \neq 1$ and $\mathbb{F}_{m(I)}$ and $\underline{\mathbb{F}}_{m(I)}$ is closed under self-induction.

For the first statement, empty products are terminal objects (i.e. $*_V$), so $\varnothing_V \in \mathbb{F}_{m(I),V}$ for all V. Hence it suffices to prove that $2*_V \in \mathbb{F}_{m(I),V}$, i.e. $\mathbb{F}_{I,V}$ is closed under binary products. By distributivity of products and coproducts, we have

$$S \times S' = \coprod_{U \in \mathrm{Orb}(S)} U \times S' = \coprod_U^S \mathrm{Res}_U^V S',$$

which is in $\mathbb{F}_{I,V}$ by closure under self-indexed coproducts.

For the second statement, it suffices to note that

$$\coprod_U^{\mathrm{Ind}_U^V S} T_U = \mathrm{Ind}_U^V \coprod_U^S T_U, = \coprod_U^{\mathrm{Ind}_U^V *_U} \coprod_U^S T_U$$

which is in $\mathbb{F}_{I,V}$ by closure under self-indexed coproducts. □

The situation with fixed I_m and varying I_a is more complicated, and has been studied for indexing systems in [BH22]; we do not study it here.

3. Computational results

3.1. **Sparsely indexed coproducts.** Let $\mathrm{Istrp}(S) \coloneqq \{U \in \underline{V} \mid \exists \text{ summand inclusion } U \hookrightarrow S\}$ be the *isotropy poset*, and given $V \in \mathrm{Istrp}(S)$, write $S_{(U)}$ for the maximal summand of S which is a multiple of U. Furthermore, write

$$\overline{S} \coloneqq \coprod_{U \in \mathrm{Istrp}(S)} U.$$

Proposition 3.1. *If $\mathcal{T}$ is unital, then*

$$\underline{\mathbb{F}}_I = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2}).$$

Proof. Note that

$$S = \coprod_U^{\overline{S}} \mathrm{Res}_U^V S_{(U)},$$

and $\overline{S} \in \underline{\mathbb{F}}_I^{\leq 2}$; hence it suffices to prove that $S_{(U)} \in \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2})$ for each S, U. In fact, since $S_{(U)}$ is a $U \in \underline{\mathbb{F}}_I^{\leq 2}$-indexed coproduct of $\mathrm{Res}_U^V S_{(U)} = n*_U$, it suffices to prove that $n*_U \in \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2})$, subject to the condition that $n \geq 2$ implies that $2*_U \in \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\leq 2})$. But this follows from the argument in Lemma 1.24. □

Proposition 3.2. *If $\mathcal{T}$ has no self-normalizing transfers and $\underline{\mathbb{F}}_I$ is a unital $\mathcal{T}$-weak indexing system, then*

$$\underline{\mathbb{F}}_I = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$$

Proof. Since $\mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \subset \underline{\mathbb{F}}_I$, it suffices to prove the opposite inclusion. We first note that $\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^\infty \subset \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$ by Proposition 3.1. Hence it suffices to prove that $\underline{\mathbb{F}}_I$ is generated under sparse and trivially self-indexed colimits by $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup \underline{\mathbb{F}}_{\nabla(I)}^\infty$. In fact, if $n{*_V} \in \mathbb{F}_{I,V}$ for some (hence all) $n \geq 2$, we immediately find by unitality that $\mathbb{F}_{I,V}$ is generated under trivially self-indexed coproducts by its orbits, which are sparse. Hence it suffices to prove this in the case $2{*_V} \notin \mathbb{F}_{I,V}$.

Fix some $S \in \mathbb{F}_{I,V}$, and recall that $\overline{S} \in \mathbb{F}_{I,V}^{\mathrm{sprs}}$. Furthermore, we have $\mathrm{Res}_U^V S_{(U)} \in \mathbb{F}_{I,U}^{\mathrm{sprs}} \cup \mathbb{F}_{\nabla(I),U}^\infty$. Hence S is a sparse colimit of elements of $\mathbb{F}_{I,U}^{\mathrm{sprs}} \cup \mathbb{F}_{\nabla(I),U}^\infty$, as desired. □

Proof of Theorem C. By Proposition 3.1, $(-)^{\leq 2}$ is a section of $\mathrm{Cl}_\infty(-)$ and a left adjoint; this implies that $(-)^{\leq 2}$ is an embedding by Lemma 2.14, with image spanned by those collections $\mathcal{C}$ satisfying $\mathcal{C} \simeq \mathrm{Cl}_\infty(\mathcal{C})^{\leq}$. Unwinding definitions, this is what we set out to prove. The second statement follows by an identical argument using Proposition 3.2. □

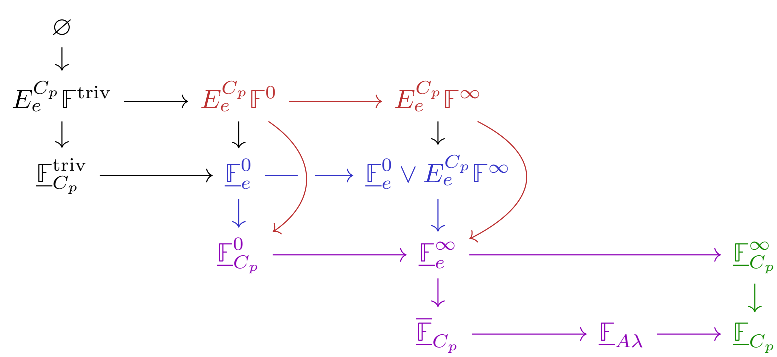
Observation 3.3. If $\underline{\mathbb{F}}_I$ contains the sparse V-set $S = \varepsilon *_V + V_1 + \cdots + V_n$ and the transfer $U \to V_1$, then $\underline{\mathbb{F}}_I$ contains the sparse V-set $\varepsilon *_V + U + \cdots + V_n$; hence it is likely that the description in terms of sparse V-sets is not as compact as it could be. We exploit this for C_{p^N} in the following sections. ◁

3.2. **Warmup: the (almost-E-)-unital C_p-weak indexing systems.** The orbit category of the prime cyclic group $C_p = \langle x \mid x^p \rangle$ may be presented as follows:

$$\left\langle \; \tau \circlearrowright [C_p] \xrightarrow{r_{e,C_p}} *_{C_p} \;\middle|\; \tau^p = \mathrm{id}_{[C_p]}, \quad r_{e,C_p} = r_{e,C_p}\tau \; \right\rangle$$

It is easy to see that there are precisely two C_p-transfer systems: R_0 contains no transfers, and R_1 contains the transfer $e \to C_p$. Thus the poset Transf_{C_p} is $\mathcal{O}_{C_p}^{\simeq} \to \mathcal{O}_{C_p}$. Furthermore, there are exactly three C_p families, and the poset is $\varnothing \to \{e\} \to \{e, C_p\}$.

Theorem 3.4. *The poset $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ is presented by the following*



where $\left\{\underline{\mathbb{F}}_{C_p}^\infty, \underline{\mathbb{F}}_{C_p}\right\}$ *are the indexing systems,* $\left\{\underline{\mathbb{F}}_{C_p}^0, \underline{\mathbb{F}}_e^\infty, \overline{\underline{\mathbb{F}}}_{C_p}, A_\lambda\right\}$ *are the otherwise-unital weak indexing systems,* $\left\{\underline{\mathbb{F}}_e^0, \underline{\mathbb{F}}_e^0 \vee E_e^{C_p}\underline{\mathbb{F}}^\infty\right\}$ *are the otherwise almost-unital weak indexing systems, and* $\left\{E_e^{C_p}\mathbb{F}^0, E_e^{C_p}\mathbb{F}^\infty\right\}$ *are the otherwise E-unital weak indexing systems.*

Remark 3.5. Already, we see that none of $\mathrm{wIndex}_{C_p}^{\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{a\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{E\mathrm{uni}}$, or $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ are self-dual, since each embed the poset $\bullet \to \bullet \to \bullet \leftarrow$ as a cofamily, but none embed its dual as a family. This heavily contrasts the cases of $\mathrm{Index}_G = \mathrm{Transf}_G$ and Fam_G, which are known to be self-dual for arbitrary abelian G by [Fra+22]. ◁

The indexing systems correspond with transfer systems, and it's easy to see that $\mathcal{O}_{C_p}^{\simeq} \to \mathcal{O}_{C_p}$ is the poset of C_p-transfer systems; hence $\underline{\mathbb{F}}_{C_p}^\infty \to \underline{\mathbb{F}}_{C_p}$ is the poset of C_p weak indexing systems, i.e. we've completely characterized $\nabla^{-1}(\mathcal{T}) \cap \mathfrak{R}^{-1}(-)$.

We may extend this to unital weak indexing systems. First, those with no transfers:

Observation 3.6. The map $\nabla : \mathfrak{R}^{-1}(\mathcal{T}^{\simeq}) \to \mathrm{Fam}_{\mathcal{T}}$ is an equivalence. ◁

The only remaining case is $\nabla^{-1}(\{e\}) \cap \mathfrak{R}^{-1}(\mathcal{T})$. Unwinding definitions, we find that there are two options for sparse collections satisfying **??** with the specified transfers: $\overline{\underline{\mathbb{F}}}_{C_p}^{\mathrm{sprs}}$ and $\underline{\mathbb{F}}_{A\lambda}^{\mathrm{sprs}}$. We've already verified that these are both weak indexing systems, so we are done computing the unital weak indexing systems.

Furthermore, in view of Corollary 2.6, we have $\mathrm{wIndex}_{BC_p}^{\mathrm{uni}} \simeq \mathrm{wIndex}_{*}^{\mathrm{uni}}$. Applying **??**, we've arrived at the following computations:

$$\mathrm{wIndex}_{BC_p}^{\mathrm{uni}} : \qquad \underline{\mathbb{F}}^{0} \longrightarrow \underline{\mathbb{F}}^{\infty}$$

$$\mathrm{wIndex}_{C_p}^{\mathrm{uni}} : \qquad \begin{array}{ccccc} \underline{\mathbb{F}}_{C_p}^{0} & \longrightarrow & \underline{\mathbb{F}}_{e}^{\infty} & \longrightarrow & \underline{\mathbb{F}}_{C_p}^{\infty} \\ & & \downarrow & & \downarrow \\ & & \overline{\underline{\mathbb{F}}}_{C_p} & \longrightarrow \ \underline{\mathbb{F}}_{A\lambda} \ \longrightarrow & \underline{\mathbb{F}}_{C_p} \end{array}$$

Theorem 3.4 then follows by applying Corollaries 2.24 and 2.33.

3.3. **The fibers of the C_{p^N}-transfer-fold fibration.** Recall that when $\mathcal{F} \subset \mathcal{O}_{C_{p^N}}$ is a collection of objects and R a C_{p^N}-transfer system, we say that a *R-sieve on $\mathcal{F}$* is a precomposition-closed wide subcategory of $R \cap \mathcal{F}$.

Let $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{C_{p^N}}$ be a collection of objects which is sparsely closed under self-indexed coproducts. Let $S(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \subset \mathrm{Cod}(\mathfrak{R}(\underline{\mathbb{F}}_I^{\mathrm{sprs}})) - \nabla(\underline{\mathbb{F}}_I)$ be the wide subcategory consisting of maps $U \to V$ such that $*_V + U \in \mathbb{F}_{I,V}^{\mathrm{sprs}}$.

Proposition 3.7. *The induced map $S : \mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) \to \mathrm{Sub}_{\mathbf{Cat}}(\mathrm{Cod}(\mathfrak{R}(\underline{\mathbb{F}}_I)) - \mathcal{F})$ is embedding with image the R-sieves.*

Proof. First, note that a unital $\mathcal{T}$-weak indexing system lying over $(R, \mathcal{F})$ is determined by its sparse V-sets containing a trivial locus of size 1 and a nonempty nontrivial locus, and for $V \notin \mathcal{F}$. Thus we may restrict fully faithfully to just these sparse V-sets for $V \in \mathrm{Cod}(\mathfrak{R}(\underline{\mathbb{F}}_I)) - \mathcal{F}$.

In fact, since $\left[\mathcal{O}_{C_{p^N}}\right]$ is a total order, such a sparse V-set is exactly a V-set of the form $*_V + U$ for some $U \neq V$. Thus S is an embedding, so it suffices to characterize its image. This follows by noting that closure under sparse self-induction is precisely the characteristic that $S(\underline{\mathbb{F}}_I)$ is closed under precomposition along maps in R, i.e. it is an R-sieve. □

In order to prove Corollary D, we need to identify $\mathrm{Transf}_{C_{p^N}}$; this was already done in [BBR21] when N is finite, and the infinite case follows immediately from e.g. Theorem 2.3.

Proposition 3.8 ([BBR21, Thm 25])**.** *For $N \in \mathbb{N} \cup \{\infty\}$, there is an equivalence of posets*

$$K_{N+1} \simeq \mathrm{Transf}_{C_{p^N}},$$

the left side denoting the Nth associahedron.

Proof of Corollary D . In view of Proposition 3.8, the combined transfer-fold fibration maps $(\mathfrak{R}, \nabla) : \mathrm{wIndex}_{C_{p^N}}^{\mathrm{uni}} \to K_{N+1} \times [N+1]$ After Propositions 2.42, 3.7 and 3.8, we've identified the fibers. Thus it suffices to understand cocartesian transport, which is implemented by

$$t_{(R,\mathcal{F})}^{(R',\mathcal{F}')} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}_{\mathcal{F}'}^{\infty}$$

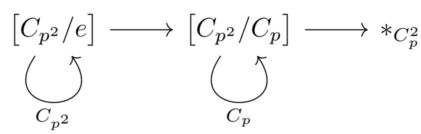
by Proposition 2.13, in terms of R-sieves. When $R = R'$, it is clear that this is given by the restriction $\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}')$, so it suffices to characterize this in the case $\mathcal{F} = \mathcal{F}'$. Unwinding definitions, we're tasked with characterizing for which $U \hookrightarrow V$, we have

$$*_V + U \in \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}.$$

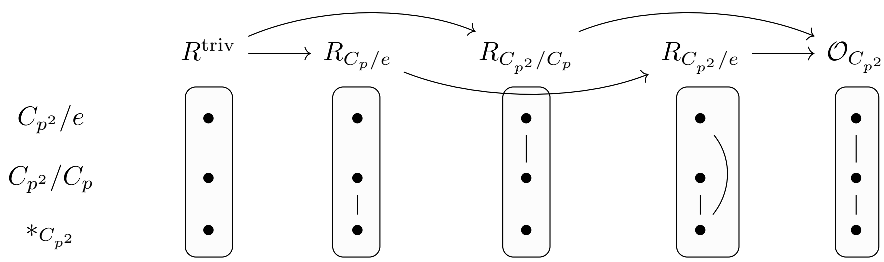
By Theorem C, it suffices to characterise which of these are presented as sparse indexed coproducts of elements of $\underline{\mathbb{F}}_I$ and $\underline{\mathbb{F}}_{R'}$; Certainly the closure of the sieve for $\underline{\mathbb{F}}_I$ under precomposition along elements of R'

is presented by sparse indexed coproducts of such elements; in turn, any sprase indexed coproduct ends up in such a form, proving the theorem. □

We finish by drawing this out for $N = 2$. We may illustrate $\mathcal{O}_{C_{p^2}}$ as follows.

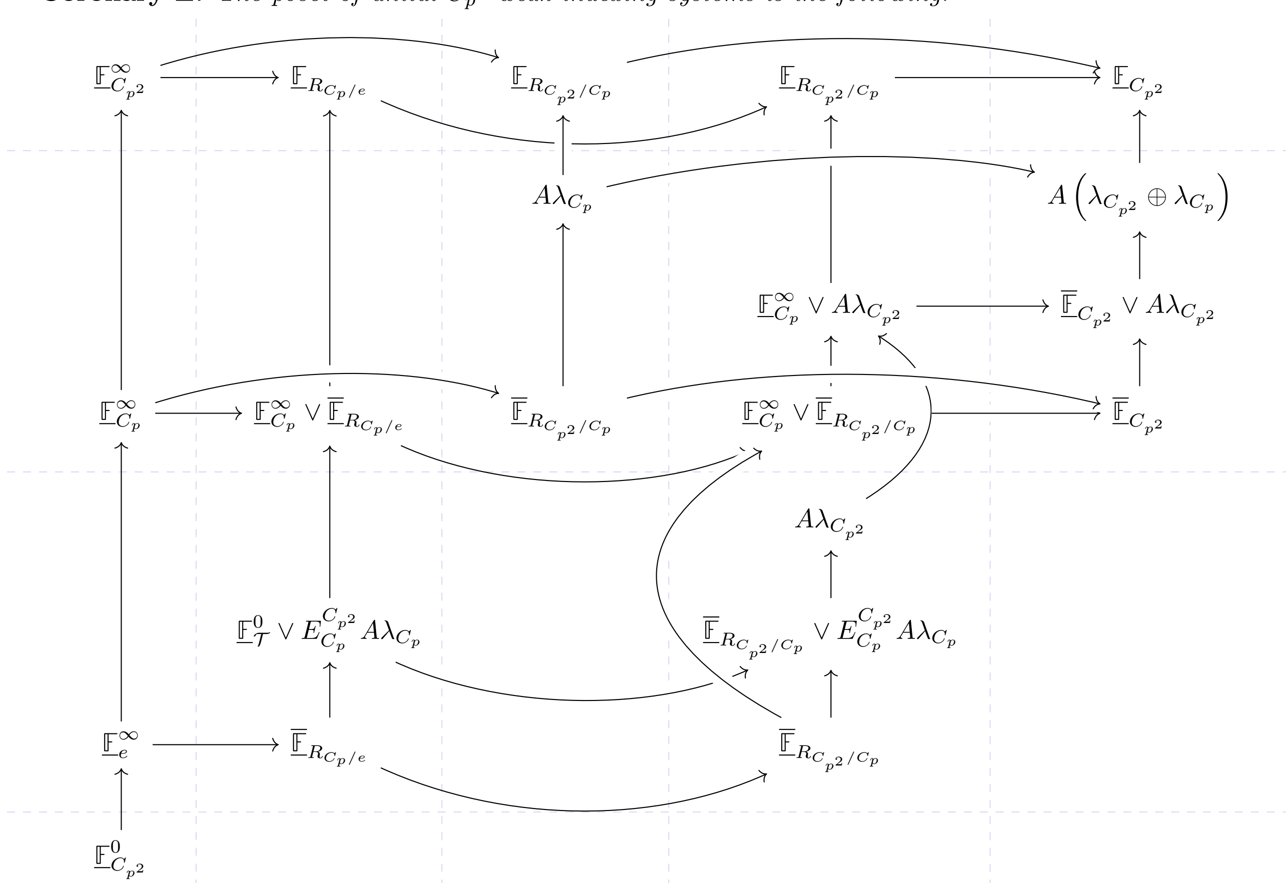


Then, the independent computations of [BBR21; Rub21] verify the that the following 5 transfer systems are the elements of $\mathrm{Transf}_{C_{p^2}}$



Given $R \in \mathrm{Transf}_{C_{p^2}}$, we let $\underline{\mathbb{F}}_R$ be the corresponding indexing system.

Corollary E. *The poset of unital C_{p^2}-weak indexing systems is the following:*



3.4. Questions and future directions.

Question 3.9. Is there a closed form expression for $\mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{O}_{C_{p^N}}}$ or $\left|\mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{O}_{C_{p^N}}}\right|$? ◁

Question 3.10. Is there a good combinatorial expression of $\nabla^{-1}(\mathcal{F}) \cap \mathfrak{R}^{-1}(R)$ over an arbitrary dedekind, nilpotent, or general finite group? ◁

Question 3.11. Which unital weak indexing systems are realizable via tensor products of the image of $\mathbb{E}_V$ operads under various change of group functors? ◁

Question 3.12. What is the right adjoint to ∇? Is it related to $\mathbb{E}_V$? ◁

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