ORBITAL CATEGORIES AND WEAK INDEXING SYSTEMS

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Abstract. We initiate the combinatorial study of the poset $\mathrm{wIndex}_{\mathcal{T}}$ of *weak $\mathcal{T}$-indexing systems*, consisting of composable collections of arities for $\mathcal{T}$-equivariant algebraic structures, where $\mathcal{T}$ is an orbital ∞-category, such as the orbit category of a finite group. In particular, we show that these are equivalent to *weak $\mathcal{T}$-indexing categories* and characterize various unitality conditions.

Within this sits a natural generalization $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}$ of Blumberg-Hill's *indexing systems*, consisting of arities for structures possessing binary operations and unit elements. We characterize the relationship between the posets of *unital weak indexing systems* and *indexing systems*, the latter remaining isomorphic to *transfer systems* on this level of generality. We use this to characterize the poset of unital C_{p^n}-weak indexing systems.

Contents

1. Introduction

Fix G a finite group. In [BH15], the notion of *$\mathcal{N}_\infty$-operads for G* was introduced, encapsulating a collection of *blueprints* for G-equivariantly commutative multiplicative structures on Mackey functors which possess underlying Green functors. They demonstrated that the ∞-category of $\mathcal{N}_\infty$-operads for G is an embedded sub-poset of the lattice of *indexing systems* Index_G.

Subsequently, the embedding $\mathcal{N}_\infty\text{−}\mathrm{Op}_G \subset \mathrm{Index}_G$ was shown to be an equivalence in several independent works [BP21; GW18; Rub21]; of particular note is the equivalent characterization of indexing systems as a poset of wide subcategories $\mathrm{IndexCat}_G \subset \mathrm{Sub}(\mathbb{F}_G)$ (referred to as *indexing categories* [BH18, § 3.2]) and the

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Changes from v1: fixed a typo.

observation that indexing categories only depend on their pullbacks to the subgroup lattice $\mathrm{Sub}_{\mathrm{Grp}}(G)$, the resulting embedded subposet

$$\begin{array}{ccccccc}
\mathrm{Index}_G & \xleftarrow{\sim} & \mathrm{IndexCat}_G & & \xrightarrow{\qquad\sim\qquad} & & \mathrm{Transf}_G \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathrm{FullSub}_G(\underline{\mathbb{F}}_G) & \xleftarrow[\mathbb{F}_{(-)}]{} & \mathrm{Sub}(\mathbb{F}_G) & \xrightarrow[(-)\cap\mathcal{O}_G]{} & \mathrm{Sub}(\mathcal{O}_G) & \xrightarrow[\text{p.b.}]{} & \mathrm{Sub}_{\mathrm{Poset}}\mathrm{Sub}_{\mathrm{Grp}}(G)
\end{array}$$

being referred to as *transfer systems* [BBR21; Rub19]. It is in this language that enumerative problems concerning $\mathcal{N}_\infty$-operads are often solved.

Noting that $\mathrm{Sub}_{\mathrm{Grp}}(\mathcal{O}_{C_{p^n}}) = [n+1]$, the transfer system approach was used in [BBR21] to prove that $\mathrm{Transf}_{C_{p^n}}$ is equivalent to the $(n+2)$nd associahedron K_{n+2}, where C_m is the cyclic group of order m. Furthermore, transfer systems have powered a large amount of further work on the topic; for instance, $\mathrm{Transf}_{C_{pqr}}$ is enumerated for p,q,r distinct primes in [BBPR20], with some indications on how to generalize this to arbitrary squarefree integers.

In this paper, we aim to demonstrate how one may extend this work in two ways:

(1) we will remove the assumption on indexing systems that they are closed under finite coproducts; on the side of algebra, we will see in [Ste25b] that this removes the assumption that algebras over the corresponding G-operad $\mathcal{N}_{I\infty}^{\otimes}$ in Mackey functors possess underlying Green functors;

(2) we will replace the orbit category $\mathcal{O}_G$ with an axiomatic version, called an *atomic orbital* ∞*-category*; this allows us to fluently describe equivariance under families and cofamilies, as well as extend to more general orbit categories, such as the finite-index orbit category of a compact Lie group.

For the former, when we assert a unitality assumption, we find that $\mathrm{wIndex}_G^{\mathrm{uni}}$ is finite when G is finite, and it can usually be explicitly described in terms of transfer systems and G-families (c.f. Theorem B and Corollary C). Moreover, unitality is compatible with joins (c.f. Proposition 2.57), and in [Ste25b] we will establish that joins compute tensor products of unital weak $\mathcal{N}_\infty$-operads.

We assure the skeptical reader that they may freely assume $\mathcal{T}$ is (the orbit category of) a G-family $\mathcal{F}$ and replace all instances of orbits $V \in \mathcal{T}$ with homogeneous G-spaces $[G/H]$ for $H \in \mathcal{F}$ (or with the subgroup $H \subset G$ itself, depending on which is contextually appropriate);[1] then, our results will only be novel in way (1). Regardless, we will now review the axiomatic setting of *(atomic) orbital* ∞*-categories.*

1.1. **Orbital ∞-categories.** We briefly review the setting introduced in [BDGNS16] generalizing the orbit category $\mathcal{O}_G$; we assume basic intuition for $\mathcal{O}_G$, consistent e.g. with the characterization in [Die09, § 1.2-1.3].

Construction 1.1 (c.f. [Gla17])**.** Given $\mathcal{T}$ an ∞-category[2], its *finite coproduct completion* is the full subcategory $\mathbb{F}_{\mathcal{T}} \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by finite coproducts of representable presheaves, where $\mathcal{S}$ denotes the ∞-category of spaces. ◁

Example 1.2. If G is a finite group, then $\mathbb{F}_{\mathcal{O}_G}$ is equivalent to the 1-category of finite G-sets; more generally, if $\mathcal{F} \subset \mathcal{O}_G$ is (the orbit category of) a G-family, then $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{O}_G}$ is the full subcategory spanned by finite G-sets S such that the stabilizer $\mathrm{stab}_G(x)$ lies in $\mathcal{F}$ for all $x \in S$. ◁

$\mathbb{F}_{\mathcal{T}}$ is *freely* generated by $\mathcal{T}$ under finite coproducts; in particular, given $S \in \mathbb{F}_{\mathcal{T}}$, there is a unique expression $S \simeq \bigoplus_{V \in \mathrm{Orb}(S)} V$ for some finite set of S*-orbits* $\mathrm{Orb}(S) \to \mathrm{Ob}\mathcal{T}$. Another important property of the finite coproduct completion is existence of equivalences [Gla17, Lem 2.14]

$$\mathbb{F}_{\mathcal{T},/S} \simeq \prod_{V \in \mathrm{Orb}(S)} \mathbb{F}_{\mathcal{T},/V}; \qquad\qquad \mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}.$$

[1] Throughohut this paper, a G*-family* will always refer to a subconjugacy closed collection of subgroups of G, That the reader understands weak indexing systems over G-families will become non-negotiable over the course of this paper, as we critically employ change of universe functors throughout the text, such as *Borelification*.

[2] Throughout this paper, we say ∞-categories to refer to $(\infty,1)$-categories as in [HTT], and we say n*-categories* to refer to $(n,1)$-categories, i.e. ∞-categories whose mapping spaces are $(n-1)$-truncated. 1-categories embed fully faithfully into ∞-categories, and the reader is free to safely assume all categorical terminology refer to 1-categories (and spaces as sets) *except* for the 2-category Cat_1 of 1-categories, which must be a 2-category in order for the definition of I-symmetric monoidal 1-categories to have coherences compatible with the ∞-categorical case.

We henceforth refer to $\mathbb{F}_{\mathcal{T},/V} \simeq \mathbb{F}_{\mathcal{T}_{/V}}$ as $\mathbb{F}_V$. Note that, in the case $\mathcal{T} = \mathcal{O}_G$, induction furnishes an equivalence $\mathcal{O}_{G,/[G/H]} \simeq \mathcal{O}_H$, so $\mathbb{F}_{[G/H]} \simeq \mathbb{F}_H$.

Fundamental to genuine-equivariant mathematics is the *effective Burnside category* $\mathsf{Span}(\mathbb{F}_G)$; for instance, the G-Mackey functors of [Dre71] may be presented as product-preserving functors $\mathsf{Span}(\mathbb{F}_G) \to \mathbf{Ab}$. In fact, the spectral Mackey functor theorem of [GM17] presents G-spectra as product-preserving functors of ∞-categories $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{Sp}$, a perspective which has been greatly exploited e.g. in [Bar14; BGS20].

In $\mathsf{Span}(\mathbb{F}_G)$, composition of morphisms is accomplished via the pullback

(1)
$$\begin{array}{ccccccc} & & & R_{fg} & & & \\ & & \swarrow & \lrcorner & \searrow & & \\ & R_g & & & & R_f & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ S & & & T & & & Q \end{array}$$

Indeed, given $\mathcal{T}$ an arbitrary ∞-category, the triple $(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ is *adequate* in the sense of [Bar14] if and only if $\mathbb{F}_{\mathcal{T}}$ has pullbacks, in which case the triple is *disjunctive*. Thus, Barwick's construction [Bar14, Def 5.5] defines an effective Burnside ∞-category $\mathsf{Span}(\mathbb{F}_{\mathcal{T}}) = A^{eff}(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$ precisely if $\mathcal{T}$ is *orbital* in the sense of the following definition.

Definition 1.3 ([Nar16, Def 4.1])**.** A (small) ∞-category $\mathcal{T}$ is *orbital* if $\mathbb{F}_{\mathcal{T}}$ has pullbacks; an orbital ∞-category $\mathcal{T}$ is *atomic* if all retracts in $\mathcal{T}$ are equivalences. ◁

If $\mathcal{T}$ is an orbital 1-category, then the effective Burnside ∞-category $\mathsf{Span}(\mathbb{F}_{\mathcal{T}})$ is a 2-category with objects the finite $\mathcal{T}$-sets, morphisms the spans of finite $\mathcal{T}$-sets, 2-cells the isomorphisms of spans, and composition defined by Eq. (1). We will not discuss the Burnside ∞-category for the main combinatorial results of this paper, but it factors greatly into the parallel study of *genuine equivariant algebra*, and hence in the forthcoming article [Ste25b]. Instead, we spend the rest of the subsection on examples.

Remark 1.4. We will show in Section 2.1 that, if $\mathcal{T}$ is an orbital ∞-category, then $\mathsf{ho}(\mathcal{T})$ is as well; furthermore, the main combinatorial objects of this paper are the same between $\mathcal{T}$ and $\mathsf{ho}(\mathcal{T})$. Hence the reader may uniformly assume that $\mathcal{T}$ is a 1-category, at the loss of essentially none of the combinatorics. ◁

Example 1.5. Given X a space considered as an ∞-category, X is atomic orbital; by [Gla18, Thm 2.13], the associated stable ∞-category is the Ando-Hopkins-Rezk ∞-category of parameterized spectra over X (c.f. [ABGHR14]). In particular, for $X = BG$, this recovers *spectra with G-action*. ◁

Example 1.6. Given P a meet semilattice, P is atomic orbital, as the meets in $\mathbb{F}_P$ are easily computed in terms of meets in P. ◁

Given G a topological group, let $\mathcal{S}_G$ denote the ∞-category of G-spaces, presented for instance by the simplicial localization of topological spaces with G-action at the maps inducing weak equivalences on point-set fixed points for each closed subgroup. Let $\mathcal{O}_G \subset \mathcal{S}_G$ denote the full subcategory spanned by the homogeneous G-spaces $[G/H]$ for $H \subset G$ a closed subgroup. We call this the *orbit ∞-category*.

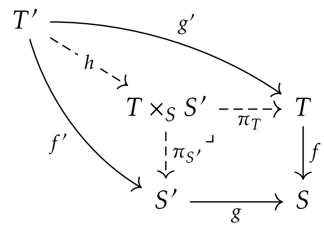
A famous issue with equivariant stable homotopy theory over an infinite group G is that the orbit ∞-category $\mathcal{O}_G$ is not *orbital*; the G-Burnside category does not exist, as $\mathbb{F}_G$ does not have pullbacks with which to define composition of spans, since the double coset formula constructs infinitely many elements in many such pullbacks. Nevertheless, this has been rectified in various homotopical contexts. One particularly lucid treatment due to Cnossen-Lenz-Linskens uses the slightly more general setting of *global homotopy theory*.

Definition 1.7 ([CLL23, Def 4.2.2, 4.3.2])**.** Given $\mathcal{P} \subset \mathcal{T}$ a wide subcategory of an ∞-category, we denote by $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}} := \mathbb{F}_{\mathcal{P}} \subset \mathbb{F}_{\mathcal{T}}$ the wide subcategory whose maps are induced by maps in $\mathcal{P}$. We say $\mathcal{P} \subset \mathcal{T}$ is an *orbital subcategory* if $\mathbb{F}_{\mathcal{T}}^{\mathcal{P}} \subset \mathbb{F}_{\mathcal{T}}$ is stable under pullbacks along arbitrary maps in $\mathbb{F}_{\mathcal{T}}$, and all such pullbacks exist. We say $\mathcal{P} \subset \mathcal{T}$ is additionally *atomic* if any morphism in $\mathcal{P}$ which admits a section in $\mathcal{T}$ is an equivalence. ◁

Note that an ∞-category is atomic orbital if and only if it's an atomic orbital subcategory of itself, so the orbital setting specializes the global setting. On the other hand, many global examples can be pulled back to the orbital setting.

Lemma 1.8. *Suppose $\mathcal{P} \subset \mathcal{T}$ is an atomic orbital subcategory. Then, $\mathcal{P}$ is atomic orbital as an ∞-category.*

Proof. First, assume we have a square in $\mathbb{F}_{\mathcal{P}} \simeq \mathbb{F}_{\mathcal{P}}^{\mathcal{T}}$; since $\mathcal{P} \subset \mathcal{T}$ is an orbital subcategory, we may extend our square to a pullback diagram $\mathbb{F}_{\mathcal{T}}$



To prove that $\mathcal{P}$ is orbital, it suffices to verify that the inner square is a pullback diagram lying in $\mathcal{P}$; to check that it lies in $\mathcal{P}$ we are tasked with verifying that $\pi_{S'}$ and π_T are in $\mathcal{P}$ and to check that it's a pullback we are tasked with verifying that h lies in $\mathcal{P}$. In fact, $\pi_{S'}$ and π_T are in $\mathcal{P}$ since $\mathcal{P} \subset \mathcal{T}$ is an orbital subcategory; h is then in $\mathcal{P}$ since atomic orbital subcategories are left cancellable by [CLL23, Lem 4.3.5].

We've proved that $\mathcal{P}$ is orbital. To see that $\mathcal{P}$ is atomic, note that this immediately follows from the second condition of Definition 1.7. □

To use this for equivariance over infinite groups, we make the following definition.

Definition 1.9. Given $\mathcal{T}$ an ∞-category, a *$\mathcal{T}$-family* is a full subcategory $\mathcal{F} \subset \mathcal{T}$ satisfying the condition that, given $V \to W$ a morphism with $W \in \mathcal{F}$, we have $V \in \mathcal{F}$. A *$\mathcal{T}$-cofamily* is a full subcategory $\mathcal{F}^{\perp} \subset \mathcal{T}$ such that $\mathcal{F}^{\perp,\mathrm{op}} \subset \mathcal{T}^{\mathrm{op}}$ is a $\mathcal{T}^{\mathrm{op}}$-family. ◁

Observation 1.10. Suppose $\mathcal{F} \subset \mathcal{T}$ is a full subcategory of an atomic orbital ∞-category satisfying the following conditions:

(a) whenever $U, W \in \mathcal{F}$ and there is a path $U \to V \to W$, we have $V \in \mathcal{F}$, and
(b) whenever $U, W \in \mathcal{F}$ and there is a cospan $U \to V \leftarrow W$, there is a span $U \leftarrow V' \to W$ with $V' \in \mathcal{F}$.

Then, the inclusion $\mathbb{F}_{\mathcal{F}} \subset \mathbb{F}_{\mathcal{T}}$ creates pullbacks; in particular, $\mathcal{F}$ is an atomic orbital ∞-category. Note that (a) is satisfied by all families and cofamilies, and (b) is satisfied by all families. ◁

Example 1.11. Let G be a Lie group and $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ the wide subcategory of the orbit ∞-category spanned by projections $[G/K] \to [G/H]$ corresponding with finite-index closed subgroup inclusions $K \subset H$. Then, by [CLL23, Ex 4.2.6], $\mathcal{O}_G^{f.i.} \subset \mathcal{O}_G$ is an orbital subcategory with pullbacks implemented by a double coset formula. In fact, it follows quickly from definition that it is atomic as well; hence $\mathcal{O}_G^{f.i.}$ is an atomic orbital ∞-category.

In fact, by Observation 1.10, the $\mathcal{O}_G^{f.i.}$-family $\mathcal{O}_G^{fin}$ spanned by *finite subgroups* is an atomic orbital ∞-category as well. In the case $G = \mathbb{T}$ this yields the *cyclonic orbit category*, so its stable homotopy theory is that of *cyclonic spectra*, i.e. *finitely genuine S^1-spectra* (c.f. [BG16, Thm 2.8]). ◁

Example 1.12. Given $H \subset G$ a closed subgroup, the $\mathcal{O}_G^{f.i.}$-cofamily $\mathcal{O}_{G,\leq[G/H]}^{f.i.}$ spanned by homogeneous G-spaces $[G/J]$ admitting a quotient map from $[G/H]$ satisfies the assumption of Observation 1.10, so it is atomic orbital; in the case $H = N \subset G$ is normal, it is equivalent to $\mathcal{O}_{G/N}^{f.i.}$. In any case, the associated stable homotopy theory is the value category of *H-geometric fixed points* with residual genuine G/H-structure (c.f. [Gla17]). ◁

1.2. Weak indexing systems and weak indexing categories.

Throughout the remainder of this introduction, we fix $\mathcal{T}$ an orbital ∞-category.

1.2.1. *Weak indexing systems.* In the case $\mathcal{T} = \mathcal{O}_G$ is the orbit category of a compact Lie group G, Elmendorf's theorem [DK84; Elm83] implies that the ∞-category of G-spaces is equivalent to the functor ∞-category

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}),$$

i.e. they are (homotopy coherent) *coefficient systems of spaces*. It is becoming traditional to allow G to act on the *category theory* surrounding genuine equivariant mathematics, culminating in the following definition.

Definition 1.13. The *2-category of small $\mathcal{T}$-1-categories* is the functor 2-category

$$\mathrm{Cat}_{\mathcal{T},1} := \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}_1) \simeq \mathrm{Fun}(h_2\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}_1),$$

where Cat_1 is the 2-category of small 1-categories and $h_2(-)$ denotes the homotopy 2-category. ◁

For the remainder of this paper, all $\mathcal{T}$-1-categories will be small, so we omit the word "small." We refer to the morphisms in $\mathrm{Cat}_{\mathcal{T},1}$ as *$\mathcal{T}$-functors*. Given a $\mathcal{T}$-1-category $\mathcal{C}$ and an object $V \in \mathcal{T}$, $\mathcal{C}$ has a *V-value* 1-category $\mathcal{C}_V := \mathcal{C}(V)$, and given a map $V \to W$ in $\mathcal{T}$, $\mathcal{C}$ has an associated *restriction functor* $\mathrm{Res}_V^W : \mathcal{C}_W \to \mathcal{C}_V$.

Example 1.14. By [NS22, Prop 2.5.1], the ∞-category $\mathcal{T}_{/V}$ is a 1-category, so $\mathbb{F}_V := \mathbb{F}_{\mathcal{T}_{/V}} \simeq \mathbb{F}_{\mathcal{T},/V}$ is a 1-category. Hence the functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending $V \mapsto \mathbb{F}_{\mathcal{T},/V}$ is a $\mathcal{T}$-1-category, which we call *the $\mathcal{T}$-1-category of finite $\mathcal{T}$-sets* and denote as $\underline{\mathbb{F}}_{\mathcal{T}}$. ◁

Notation 1.15. We refer to the terminal object $(V = V) \in \mathbb{F}_V$ as $*_V$ and call it the *contractible V-set*. We refer to the initial object $(\varnothing \to V) \in \mathbb{F}_V$ as $\varnothing_V$ and call it the *empty V-set*. ◁

Evaluation is functorial in the $\mathcal{T}$-1-category; indeed, a $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is just a collection of functors

$$F_V\colon \mathcal{C}_V \to \mathcal{D}_V$$

intertwining with restriction. We refer to a $\mathcal{T}$-functor whose V-values are fully faithful as a *fully faithful $\mathcal{T}$-functor*; if $\iota\colon \mathcal{C} \to \mathcal{D}$ is a fully faithful $\mathcal{T}$-functor, we say that $\mathcal{C}$ is a *full $\mathcal{T}$-subcategory of $\mathcal{D}$*. A full $\mathcal{T}$-subcategory of $\mathcal{D}$ is uniquely determined by an equivalence-closed and restriction-stable class of objects in $\mathcal{D}$; see [Sha23] for details.

Definition 1.16 (c.f. [HHR16, § 2.2.3])**.** Fix $\mathcal{C}$ a $\mathcal{T}$-1-category. The *induced V-set functor* $\mathrm{Ind}_U^V\colon \mathcal{C}_U \to \mathcal{C}_V$, if it exists, is the left adjoint to Res_U^V. Furthermore, given a V-set S and a tuple $(T_U)_{U \in \mathrm{Orb}(S)}$, the *$S$-indexed coproduct of* T_U is, if it exists, the element

$$\coprod_U^S T_U := \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^V T_U \in \mathcal{C}_V.$$

Dually, the *coinduced V-set* $\mathrm{CoInd}_U^V\colon \mathcal{C}_U \to \mathcal{C}_V$ is the right adjoint to Res_U^V (if it exists), and the S-indexed product is (if it exists), the element

$$\prod_U^S T_U := \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^V T_U \in \mathcal{C}_V. \qquad ◁$$

Example 1.17. Given a subgroup inclusion $K \subset H \subset G$, the associated functor $\mathbb{F}_H \to \mathbb{F}_K$ is restriction, and hence its left adjoint $\mathbb{F}_K \to \mathbb{F}_H$ is *G-set induction*, matching the *indexed coproducts* of [HHR16, § 2.2.3]. ◁

Given $S \in \mathbb{F}_V$, we write

$$\mathcal{C}_S := \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_V;$$

we say that $\mathcal{C}$ *strongly admits finite indexed coproducts* if $\coprod_U^S T_U$ always exists, in which case it is a functor

$$\coprod_U^S (-)\colon \mathcal{C}_S \to \mathcal{C}_V.$$

Remark 1.18. Given $S \in \mathbb{F}_V$, we may define the functor $\Delta^S\colon \mathcal{C}_V \to \mathcal{C}_S$ so that for each $U \in \mathrm{Orb}(S)$, the associated functor $\mathcal{C}_V \to \mathcal{C}_U$ is restriction along the composite map $U \to S \to V$. This is the rightwards horizontal composition in the following:

$$\mathcal{C}_V \underset{\prod_{U\in\mathrm{Orb}(S)}(-)}{\overset{\coprod_{U\in\mathrm{Orb}(S)}(-)}{\rightleftarrows}} \xrightarrow{\;\Delta\;} \prod_{U\in\mathrm{Orb}(S)} \mathcal{C}_V \underset{(\mathrm{CoInd}_U^V)}{\overset{(\mathrm{Ind}_U^V)}{\rightleftarrows}} \xrightarrow{\;(\mathrm{Res}_U^V)\;} \prod_{U\in\mathrm{Orb}(S)} \mathcal{C}_U$$

In particular, by composing adjoints, we acquire adjunctions $\coprod_U^S(-) \dashv \Delta^S \dashv \prod_U^S(-)$, i.e. we've constructed indexed (co)limits in the sense of [Sha22]. ◁

It follows from construction that $\underline{\mathbb{F}}_{\mathcal{T}}$ strongly admits finite indexed coproducts; indeed, $\mathbb{F}_{\mathcal{T},/V} = \mathbb{F}_{\mathcal{T}_{/V}}$ admits finite coproducts by definition, and $\mathcal{T}$-set induction along a map $f : V \to W$ is implemented by the postcomposition $f_! : \mathbb{F}_{\mathcal{T},/V} \to \mathbb{F}_{\mathcal{T},/W}$, as it participates in the categorical push-pull adjunction $f_! \dashv f^*$. Similarly, $\underline{\mathbb{F}}_{\mathcal{T}}$ strongly admits finite indexed products, so in particular, Res_U^V preserves coproducts.

Definition 1.19. Given a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ and a full $\mathcal{T}$-subcategory $\mathcal{E} \subset \mathcal{D}$, we say that $\mathcal{E}$ is *closed under $\mathcal{C}$-indexed coproducts* if, for all $S \in \mathcal{C}_V$ and $(T_U) \in \mathcal{E}_S$, the object $\coprod_U^S T_U$ exists and is in $\mathcal{E}_V$. ◁

Definition 1.20. We say that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is *closed under self-indexed coproducts* if it is closed under $\mathcal{C}$-indexed coproducts. ◁

Definition 1.21. Given $\mathcal{T}$ an orbital ∞-category, a *$\mathcal{T}$-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ with V-values $\mathbb{F}_{I,V} := (\underline{\mathbb{F}}_I)_V$ satisfying the following conditions:

(IS-a) whenever $\mathbb{F}_{I,V} \neq \varnothing$, we have $*_V \in \mathbb{F}_{I,V}$; and
(IS-b) $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

We denote by $\mathrm{wIndex}_{\mathcal{T}} \subset \mathrm{FullSub}_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ the embedded sub-poset spanned by $\mathcal{T}$-weak indexing systems. Moreover, we say that a $\mathcal{T}$-weak indexing system *has one color* if it satisfies the following condition:

(IS-i) for all $V \in \mathcal{T}$, we have $\mathbb{F}_{I,V} \neq \varnothing$;

these span an embedded subposet $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{oc}} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is almost essentially unital or (aE-unital) if it satisfies the following condition:

(IS-ii) for all noncontractible V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

An almost essentially unital $\mathcal{T}$-weak indexing system is *almost unital* if it has one color. These are denoted $\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. We say that a $\mathcal{T}$-weak indexing system is *essentially unital* (or E-unital) if it satisfies the following condition:

(IS-iii) for all V-sets $S \sqcup S' \in \mathbb{F}_{I,V}$, we have $S, S' \in \mathbb{F}_{I,V}$.

We say that an essentially unital $\mathcal{T}$-weak indexing system is *unital* if it has one color. We write $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \subset \mathrm{wIndex}_{\mathcal{T}}$. Lastly, a $\mathcal{T}$-weak indexing system is an *indexing system* if it satisfies the following condition:

(IS-iv) the subcategory $\mathbb{F}_{I,V} \subset \mathbb{F}_V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

We denote the resulting poset by $\mathrm{Index}_{\mathcal{T}} \subset \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$. ◁

Remark 1.22. The indexing systems of [BH15] are seen to be equivalent to ours when $\mathcal{T} = \mathcal{O}_G$ by unwinding definitions. The weak indexing systems of [BP21; Per18] are equivalent to our *unital* weak indexing systems when $\mathcal{T} = \mathcal{O}_G$ by [Per18, Rem 9.7] and [BP21, Rem 4.60]. ◁

In practice, we will find that non-aE-unital weak indexing systems are not well behaved, and questions involving aE-unital weak indexing systems are usually quickly reducible to the unital case; the reader is encouraged to focus primarily on unital weak indexing systems for this reason.

1.2.2. *Some examples.* We begin with some universal examples.

Example 1.23. The terminal $\mathcal{T}$-weak indexing system is $\underline{\mathbb{F}}_{\mathcal{T}}$; the initial $\mathcal{T}$-weak indexing system is the empty $\mathcal{T}$-subcategory; the initial one-color $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^{\mathrm{triv}} := \{*_V\}.$$ ◁

To understand the conditions of Definition 1.21, we introduce some invariants. Write

$$n \cdot S := \overbrace{S \sqcup \cdots \sqcup S}^{n\text{-fold}}.$$

Lemma 1.24. *Given $\underline{\mathbb{F}}_I$ a $\mathcal{T}$-weak indexing system, the following are $\mathcal{T}$-families:*

$$c(I) := \{V \in \mathcal{T} \mid *_V \in \mathbb{F}_{I,V}\}$$
$$v(I) := \{V \in \mathcal{T} \mid \varnothing_V \in \mathbb{F}_{I,V}\}$$
$$\nabla(I) := \{V \in \mathcal{T} \mid 2 \cdot *_V \in \mathbb{F}_{I,V}\}$$

Proof. This follows by noting that $\mathrm{Res}_U^V n \cdot *_V = n \cdot *_U$. □

We call $c(I)$ the *color family of* I, $\upsilon(I)$ the *unit family*, and $\nabla(I)$ the *fold map family*. Note that $c(I) \leq \upsilon(I) \cap \nabla(I)$; that is, Condition (IS-a) implies that whenever $\underline{\mathbb{F}}_I$ prescribes a unit or a fold map over V, it possesses a color over V. We will use the following lemma ubiquitously.

Lemma 1.25. *Let* $\underline{\mathbb{F}}_I$ *be a* $\mathcal{T}$*-weak indexing system.*

(1) $\underline{\mathbb{F}}_I$ *has one color if and only if* $c(I) = \mathcal{T}$.
(2) $\underline{\mathbb{F}}_I$ *is E-unital if and only if* $\upsilon(I) = c(I)$.
(3) $\underline{\mathbb{F}}_I$ *is unital if and only if* $\upsilon(I) = \mathcal{T}$.
(4) $\underline{\mathbb{F}}_I$ *is an indexing system if and only if* $\upsilon(I) \cap \nabla(I) = \mathcal{T}$.

Proof. (1) follows immediately by unwinding definitions. For (2), if $\underline{\mathbb{F}}_I$ is E-unital and $V \in c(I)$, then choosing $\varnothing_V \sqcup *_V \in \mathbb{F}_{I,V}$ yields $\varnothing_V \in \mathbb{F}_{I,V}$, i.e. $V \in \upsilon(I)$. Conversely, if $\upsilon(I) = c(I)$ and $S \sqcup S' \in \mathbb{F}_{I,V}$, then

$$S = \coprod_U^{S \sqcup S'} \chi_S(U), \qquad \text{where } \chi_S(U) := \begin{cases} *_U & U \in S \\ \varnothing_U & U \notin S \end{cases}$$

so $S \in \underline{\mathbb{F}}_I$, i.e. $\underline{\mathbb{F}}_I$ is E-unital. (3) follows by combining (1) and (2).

For (4), note that $\underline{\mathbb{F}}_I$ an indexing system implies that $\upsilon(I) \cap \nabla(I) = \mathcal{T}$ by taking nullary and binary coproducts of $*_V \in \mathbb{F}_{I,V}$. Conversely, if $\upsilon(I) \cap \nabla(I) = \mathcal{T}$, then by iterating binary coproducts $(n-1)$-times, we find that $n \cdot *_V = (*_V \sqcup (n-1) \cdot *_V) \in \mathbb{F}_{I,V}$ for all $V \in \mathcal{T}$ and $n \in \mathbb{N}$. Applying Condition (IS-b), we find that $\mathbb{F}_{I,V}$ is closed under n-ary coproducts for all $n \in \mathbb{N}$, i.e. $\underline{\mathbb{F}}_I$ is an indexing system. □

In fact, the proof of (2) shows more; we may use the same argument to show the following.

Lemma 1.26. $\underline{\mathbb{F}}_I$ *is aE-unital if and only if whenever* $S \in \mathbb{F}_{I,V}$ *is noncontractible,* $V \in \upsilon(I)$.

We may use c to reduce study of weak indexing systems to the one-color case via the following.

Construction 1.27. Given $\mathcal{F}$ a $\mathcal{T}$-family and $\underline{\mathbb{F}}_I$ an $\mathcal{F}$-weak indexing system, we may define the $\mathcal{T}$-weak indexing system $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I$ by

$$\left(E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_I\right)_V := \begin{cases} \mathbb{F}_{I,V} & V \in \mathcal{F}; \\ \varnothing & \text{otherwise.} \end{cases}$$

◁

This yields an embedding of posets $\mathrm{wIndex}_{\mathcal{F}} \to \mathrm{wIndex}_{\mathcal{T}}$. In Proposition 2.31, we prove the following.

Proposition 1.28. *The fiber of* $c\colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *is the image of* $E_{\mathcal{F}}^{\mathcal{T}}|_{\mathrm{oc}} : \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{oc}} \to \mathrm{wIndex}_{\mathcal{T}}$.

In particular, we find that $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}$ and $E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}}$ are terminal and initial among $c^{-1}(\mathcal{F})$.

Example 1.29. In [Ste25a] we define the *underlying* $\mathcal{T}$*-symmetric sequence* $\mathcal{O}(-)$ of a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$; the space $\mathcal{O}(S)$ parameterizes the S*-ary operations* endowed on an $\mathcal{O}$-algebra. We define the *arity support*

$$\mathbb{F}_{A\mathcal{O},V} := \{S \in \mathbb{F}_V \mid \mathcal{O}(S) \neq \varnothing\};$$

in [Ste25a], we show that this possesses a fully faithful right adjoint, making $\mathcal{T}$-weak indexing systems equivalent to *weak* $\mathcal{N}_\infty$*-*$\mathcal{T}$*-operads*, i.e. subterminal objects in the ∞-category of $\mathcal{T}$-operads. This inspires our naming; [Ste25a] establishes that $\underline{\mathbb{F}}_{A\mathrm{triv}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$ and $\underline{\mathbb{F}}_{A\mathrm{Comm}_{\mathcal{T}}} = \underline{\mathbb{F}}_{\mathcal{T}}$.

We may choose $\mathcal{T} = \mathcal{O}_G$, R an orthogonal G-representation, and $\mathbb{E}_R$ the little R-disks operad. This has arity support

$$\mathbb{F}_H^R := \mathbb{F}_{A\mathbb{E}_R,H} = \{S \in \mathbb{F}_H \mid \exists \text{ H-equivariant embedding } S \hookrightarrow R\}$$

(see [Hor19]). The unital weak indexing system $\underline{\mathbb{F}}_R$ is not always an indexing system; for instance, choosing $G = C_p$ and λ a 2-dimensional irreducible real orthogonal C_p-representation, we see by unwinding definitions that

$$\mathbb{F}_e^\lambda = \mathbb{F}_e, \qquad \mathbb{F}_{C_p}^\lambda = \left\{n \cdot [C_p/e] \mid n \in \mathbb{N}\right\} \sqcup \left\{*_{C_p} + n \cdot [C_p/e] \mid n \in \mathbb{N}\right\}.$$

In fact, a unital G-weak indexing system $\underline{\mathbb{F}}_I$ is an indexing system if and only if it contains $2 \cdot *_G$ (in which case, it must contain its restrictions $2 \cdot *_H$ for all $H \subset G$), and R admits a G-equivariant embedding of $2 \cdot *_G$ if and only if the inclusion $\{0\} \subset R^G$ is proper, i.e. R has positive-dimensional fixed points. Thus $\underline{\mathbb{F}}^R$ is not an indexing system when R has 0-dimensional fixed points. ◁

We will see in Section 2.3 that the construction $R \mapsto \underline{\mathbb{F}}^R$ is monotone and compatible with direct sums.

Example 1.30. The intial unital $\mathcal{T}$-weak indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^0$ is defined by

$$\mathbb{F}_{\mathcal{T},V}^0 \coloneqq \{\varnothing_V, *_V\};$$

the initial $\mathcal{T}$-indexing system $\underline{\mathbb{F}}_{\mathcal{T}}^\infty$ is defined by

$$\mathbb{F}_V^\infty \coloneqq \{n \cdot *_V \mid n \in \mathbb{N}\}.$$ ◁

Example 1.31. Let $\mathcal{T} = *$ be the terminal category. Then, a full subcategory $\underline{\mathbb{F}}_I \subset \mathbb{F}$ can be identified with a subset $n(I) \subset \mathbb{N}$, Condition (IS-a) with the condition that $n(I)$ is empty or contains 1, and Condition (IS-b) with the condition that $n(I)$ is closed under k-fold sums for all $k \in n(I)$. There are many such things; for instance, for each $n \in \mathbb{N}$, the set $\{1\} \cup \mathbb{N}_{\geq n} \subset \mathbb{N}$ gives a nonunital $*$-weak indexing system.

Nevertheless, if we assert that $0 \in n(I)$ (i.e. $\underline{\mathbb{F}}_I$ is unital), then $\underline{\mathbb{F}}_I$ is closed under summands, i.e. $n(I) \subset \mathbb{N}$ is lower-closed in $\mathbb{N}$. Thus we have the following computations for $\mathcal{T} = *$:

condition	poset
indexing system	$\mathbb{F}$
unital	$\mathbb{F}^0 \longrightarrow \mathbb{F}$
almost unital	$\mathbb{F}^{\mathrm{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
essentially unital	$\varnothing \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$
almost essentially unital	$\varnothing \longrightarrow \mathbb{F}^{\mathrm{triv}} \longrightarrow \mathbb{F}^0 \longrightarrow \mathbb{F}$

◁

Example 1.32. We will see in Corollary 2.4 that when X is a space, there is a canonical equivalence $\mathrm{wIndex}_X \simeq \mathrm{wIndex}_*$ respecting our various conditions. In particular, the computations for *Borel* equivariant weak indexing systems mirror those of Example 1.31. ◁

1.2.3. *Weak indexing categories.* With a wealth of examples under our belt, we now simplify the combinatorics.

Observation 1.33. Denote by $\mathrm{Ind}_V^{\mathcal{T}} S \to V$ the map corresponding with a finite V-set S under the equivalence $\mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T},/V}$. This equivalence implies that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is determined by the subgraph

$$I(\mathcal{C}) \coloneqq \left\{ \coprod_i \mathrm{Ind}_{V_i}^{\mathcal{T}} S_i \to V_i \;\middle|\; \forall i, \quad S \in \mathcal{C}_{V_i} \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

In other words, the construction I yields an embedding of posets

$$I(-) \colon \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Sub}_{\mathrm{graph}}(\mathbb{F}_{\mathcal{T}}).$$ ◁

We will prove the following in Section 2.2.

Theorem A. *Fix $\mathcal{T}$ an orbital ∞-category. Then, the image of the map $I(-)$ consists of the subcategories $I \subset \mathbb{F}_{\mathcal{T}}$ satisfying the following conditions*

(IC-a) (restriction-stability) I is stable under arbitrary pullbacks in $\mathbb{F}_{\mathcal{T}}$;
(IC-b) (Segal condition) the pair $T \to S$ and $T' \to S'$ are in I if and only if $T \sqcup T' \to S \sqcup S'$ is in I; and

Moreover, for all numbers n, condition (IS-n) of Definition 1.21 is equivalent to condition (IC-n) below:

(IC-i) (one color) I is wide; equivalently, I contains $\mathbb{F}_{\mathcal{T}}^{\simeq}$.
(IC-ii) (aE-unital) if $S \sqcup S' \to T$ is a non-isomorphism map in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iii) (E-unital) if $S \sqcup S' \to T$ is a map in I, then $S \to T$ and $S' \to T$ are in I.
(IC-iv) (indexing category) the fold maps $n \cdot V \to V$ are in I for all $n \in \mathbb{N}$ and $V \in \mathcal{T}$.

We refer to the image of $I(-)$ as the *weak indexing categories* $\mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_{\mathcal{T}})$. In general, we will refer to a generic weak indexing category as I and its corresponding weak indexing system as $\underline{\mathbb{F}}_I$. The following observations form the basis for the proof of Theorem A.

Observation 1.34. By a basic inductive argument, Condition (IC-b) is equivalent to the following condition:

(IC-b') $T \to S$ is in I if and only if $T_U = T \times_S U \to U$ is in I for all $U \in \mathrm{Orb}(S)$.

In particular, I is uniquely determined by the maps to orbits. ◁

Observation 1.35. By Observation 1.34, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the following condition:

(IC-a') for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{array}{ccc} T \times_V U & \longrightarrow & T \\ \big\downarrow{\scriptstyle \alpha'} & \lrcorner & \big\downarrow{\scriptstyle \alpha} \\ U & \longrightarrow & V \end{array} \tag{2}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◁

Remark 1.36. In view of Observations 1.34 and 1.35, Theorem A essentially boils down to the observation that composition in I corresponds with indexed coproducts in $\underline{\mathbb{F}}_I$ (see Observation 2.11), identity arrows on orbits correspond with contractible V-sets (by definition), and Condition (IC-a') for I corresponds with the condition that $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is restriction-stable, i.e. a full G-subcategory.

On the level of arity-supports for equivariant operads, composition of arrows in $A\mathcal{O}$ lifts to the formation of composite operations, identity arrows to the data of identity operations, Condition (IC-a') lifts to the restriction map from T-ary operations to $\mathrm{Res}^V_U T$-ary operations and Condition (IC-b') corresponds with the Segal condition for multimorphisms in a G-∞-operad. ◁

One of the major reasons for this formalism is the technology of *equivariant algebra*. If $\iota\colon I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable subcategory, then $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I)$ is an adequate triple in the sense of [Bar14], so we may form the span ∞-category

$$\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) := A^{eff}(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I),$$

whose forward maps are I and backwards maps are arbitrary. If $\mathcal{C}$ is an ∞-category, the ∞-category of *I-commutative monoids in* $\mathcal{C}$ is the product preserving functor ∞-category

$$\mathrm{CMon}_I(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}), \mathcal{C});$$

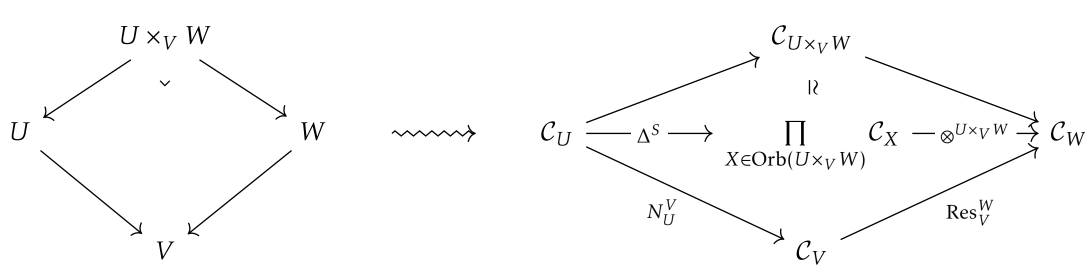
the *I-symmetric monoidal 1-categories* are

$$\mathrm{Cat}^{\otimes}_{I,1} := \mathrm{CMon}_I(\mathrm{Cat}_1),$$

where Cat_1 denotes the 2-category of 1-categories. These are a form of *I-symmetric monoidal Mackey functors* in the sense of [HH16].

$\mathcal{T}$-commutative monoids yield I-commutative monoids by neglect of structure.[3] By [Ste25b], a $\mathcal{T}$-1-category $\mathcal{D}$ with I-indexed coproducts possesses an essentially unique *cocartesian I-symmetric structure* $\mathcal{D}^{I-\sqcup}$ satisfying the property that its I-indexed tensor products implement I-indexed coproducts; a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \mathcal{D}$ is I-symmetric monoidal under this structure if and only if it's closed under I-indexed coproducts. Thus we may reinterpret Condition (IS-b) as stipulating that $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}^{I-\sqcup}$ is an I-symmetric monoidal full subcategory; we will see throughout this paper that indexed coproducts implement arities of composite operations.

Remark 1.37. If $\mathcal{C}$ is an I-symmetric monoidal category, $V \to W$ a map in I, and $U \to W$ a map in $\mathcal{T}$, then there is an associated commutative diagram



In particular, this encodes the double coset formula $\mathrm{Res}^V_W N^V_U R_U = \bigotimes_X^{U \times_V W} \mathrm{Res}^U_X R_U$.

[3] In particular, this is modeled by pullback along the product-preserving inclusion $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})$ induced by the inclusion of adequate triples $(\mathbb{F}_{c(I)}, \mathbb{F}_{c(I)}, I) \hookrightarrow (\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{T}})$.

In the case of the (co)cartesian structure this recovers a more traditional double coset formula: replacing U with some V-set S, we get the formula

$$\mathrm{Res}_V^W \coprod_U^S Z_U \simeq \coprod_X^{\mathrm{Res}_V^W S} \mathrm{Res}_X^{o(X)} Z_{o(X)},$$

where $o(X)$ is the orbit of S satisfying $X \subset \mathrm{Res}_V^W o(X) \subset \mathrm{Res}_V^W S$. ◃

1.3. **Unital weak indexing categories and transfer systems.** We now turn to transfer systems.

Definition 1.38. Given $\mathcal{T}$ an orbital ∞-category, an *orbital transfer system in $\mathcal{T}$* is a core-containing wide subcategory $\mathcal{T}^{\simeq} \subset R \subset \mathcal{T}$ satisfying the base change condition that for all $\mathcal{T}$ diagrams

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ U' & \longrightarrow & U \end{array}$$

whose associated $\mathbb{F}_{\mathcal{T}}$ map $V' \to V \times_U U'$ is a summand inclusion and with $\alpha \in R$, we have $\alpha' \in R$. The associated embedded sub-poset is denoted $\mathrm{Transf}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathrm{Cat}}(\mathcal{T})$. ◃

Observation 1.39. If I is a unital weak indexing category, the intersection $\mathfrak{R}(I) := I \cap \mathcal{T}$ is an orbital transfer system; hence it yields a monotone map

$$\mathfrak{R}(-)\colon \mathrm{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}.$$ ◃

We refer to the associated map $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \simeq \mathrm{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ by the same name. Transfer systems were first defined because of the following phenomenon.

Proposition 1.40 ([NS22, Rmk 2.4.9])**.** *$\mathfrak{R}(-)$ restricts to an equivalence*

$$\mathfrak{R}(-)\colon \mathrm{Index}_{\mathcal{T}} \xrightarrow{\sim} \mathrm{Transf}_{\mathcal{T}}.$$

Remark 1.41. In the case $\mathcal{T} = \mathcal{O}_G$, before Nardin-Shah's result, it was shown independently in [Rub19, Thm 3.7] and [BBR21, Cor 8] that pullback along the composite inclusion $\mathrm{Sub}_{\mathrm{Grp}}(G) \hookrightarrow \mathcal{O}_G \hookrightarrow \mathbb{F}_G$ yields an embedding $\mathrm{Index}_G \hookrightarrow \mathrm{Sub}_{\mathrm{Poset}}\big(\mathrm{Sub}_{\mathrm{Grp}}(G)\big)$ whose image is identified by those subposets which are closed under restriction and conjugation, which were called *G-transfer systems*; this and Proposition 1.40, together imply that pullback along the *homogeneous G-set* functor $\mathrm{Sub}_{\mathrm{Grp}}(G) \to \mathcal{O}_G$ induces an equivalence between the poset of G-transfer systems of [BBR21; Rub19] and the orbital $\mathcal{O}_G$-transfer systems of Definition 1.38. ◃

In view of Remark 1.41, we henceforth in this paper refer to orbital transfer systems simply as *transfer systems*, never referring to the other notion. Proposition 1.40 additionally allows for a reformulation of transfer systems which may be familiar to global equivariant homotopy theorists.

Observation 1.42. Let $\mathcal{T}$ be an orbital ∞-category. Then, a wide subcategory $R \subset \mathcal{T}$ is a transfer system if and only if it is an orbital subcategory in the sense of Definition 1.7; indeed, the axioms for an orbital subcategory encapsulate that of a transfer system, and give a transfer system, [NS22, Rmk 2.4.9] argues that $\mathbb{F}_{\mathcal{T}}^R$ is indexing category, so in particular it is pullback-stable.[4] Furthermore, if $\mathcal{T}$ is atomic orbital, then all of its orbital subcategories are atomic orbital, so in particular, weak indexing categories are equivalent to atomic orbital subcategories in this case. ◃

In Proposition 2.42, we will show that the composite

$$\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$$

is a fully faithful right adjoint to $\mathfrak{R}$, i.e. the poset of unital weak indexing systems possessing a given transfer system has a terminal object, given by the unique such indexing system. However, the fibers can be quite large; for instance, in Remark 2.47, we will see that $\mathfrak{R}$ also attains a fully faithful left adjoint, which is distinct from the right adjoint over all transfer systems when $\mathcal{T}$ has a terminal object (e.g. when $\mathcal{T} = \mathcal{O}_G$).

[4] In essence, the foundational difference between the orbital and global settings is that the orbital setting develops stable homotopy theory over a transfer system by specialization from the complete transfer system, whereas the global setting characterizes this directly; the latter strategy is more complicated, but allows for base categories which are not themselves orbital, such as the global indexing category.

The upshot is that unital weak indexing systems are not determined by their transitive V-sets. Nevertheless, we specify them by a small collection of data, for which we need the following definitions.

Definition 1.43. Denote by $\pi_0\mathcal{T}$ the set of isomorphism classes of objects in $\mathcal{T}$. Given $\mathcal{C}$ a $\mathcal{T}$-1-category, there is an underlying diagram $\mathrm{Ob}'\mathcal{C}\colon \pi_0\mathcal{T} \to \mathrm{Set}$; We refer to a $\pi_0\mathcal{T}$-graded subset of $\mathrm{Ob}'\mathcal{C}$ as a *$\mathcal{C}$-collection*. We will generally refer to $\underline{\mathbb{F}}_{\mathcal{T}}$-collections simply as *collections*. ◁

Construction 1.44. If $\mathcal{T}$ is an orbital ∞-category, then we define the collection of *sparse $\mathcal{T}$-sets* $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ to have V-value spanned by the V-sets

$$\varepsilon \cdot *_V \sqcup W_1 \sqcup \cdots \sqcup W_n,$$

for $\varepsilon \in \{0,1\}$ and $W_1,\dots,W_n \in \mathcal{T}_{/V}$ subject to the condition that there exist no maps $W_i \to W_j$ for $i \neq j$. ◁

Example 1.45. Let G be a finite group. Then, for (H) a conjugacy class of G, the *sparse H-sets* are precisely the H-sets of one of the following forms:

(1) $2 \cdot *_H$.
(2) $*_H \sqcup [H/K_1] \sqcup \cdots \sqcup [H/K_n]$ where none of $K_1,\dots,K_n$ are conjugate by elements of H.
(3) $[H/K_1] \sqcup \cdots \sqcup [H/K_n]$ where none of $K_1,\dots,K_n$ are conjugate by elements of H.

◁

Given $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}$, we may form the full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ generated by $\mathcal{C}^{\mathrm{sprs}}$ under iterated $\mathcal{C}^{\mathrm{sprs}}$-indexed coproducts We say that $\mathcal{C}^{\mathrm{sprs}}$ is *closed under applicable self-indexed coproducts* if $\mathcal{C}^{\mathrm{sprs}} = \mathcal{C} \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}$. We prove the following in Section 3.1.

Theorem B. *Suppose $\mathcal{T}$ is an atomic orbital ∞-category. Then, restriction along the inclusion $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \hookrightarrow \underline{\mathbb{F}}_{\mathcal{T}}$ yields an embedding of posets*

$$\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \subset \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}})$$

whose image is spanned by the aE-unital collections which are closed under applicable self-indexed coproducts.

Example 1.46. Let σ be the sign C_2-representation; following from Example 1.29, the sparse collection corresponding with $\mathbb{F}^{\sigma} = \mathbb{F}^{\infty\sigma}$ has nonequivariant part $\{2 \cdot *_e\}$ and C_2-equivariant part $\left\{[C_2/e], *_{C_p} + [C_2/e]\right\}$.

On the level of algebra, this corresponds with the fact that the data underlying an $\mathbb{E}_{\infty\sigma}$-algebra in a 1-category is generated from the underlying unital object $*_{C_2} \to A$ together with binary multiplication on A^e, a transfer $A^e \to A^{C_p}$, and a module structure map $A^{C_p} \otimes A^e \to A^{C_p}$, subject to conditions; we can see that nontransitive and nontrivial sparse C_2-sets must appear, as the module structure map is not determined by the remaining data.

This heavily contrasts the case of indexing systems; it is almost tautological that indexing systems are generated under binary coproducts by their orbits. ◁

In Remark 3.9, we will see that Theorem B is compatible with the conditions of Definition 1.21; namely, the conditions of almost unitality, essential unitality, unitality, and being an indexing system correspond with the same conditions on the sparse collection. We will prove in [Ste25b] that the aE-unital weak indexing systems are isomorphic to the poset of $\otimes$-idempotent weak $\mathcal{N}_\infty$-operads; this allows us to show that the poset of $\otimes$-idempotent weak $\mathcal{N}_\infty$ G-operads is finite whenever G is a finite group.

Remark 1.47. Let $\mathcal{T} = \mathcal{O}_G$ for G a finite group. By Theorem B, one may devise an inefficient algorithm to compute $\mathrm{wIndex}_G^{\mathrm{uni}}$. Namely, given a sparse collection $\mathcal{C}^{\mathrm{sprs}} \subset \underline{\mathbb{F}}_G^{\mathrm{sprs}}$, one may compute all of its self-indexed coproducts in finite time using the double coset formula in order to determine whether $\mathcal{C}^{\mathrm{sprs}}$ is closed under applicable self-indexed coproducts. One may simply iterate over the finite poset $\mathrm{Coll}(\underline{\mathbb{F}}_G^{\mathrm{sprs}})$, performing the above computation at each step, to determine the unital weak indexing systems. ◁

The above algorithm is quite inefficient; in practice, we instead prefer to divide and conquer, first computing Fam_G and Transf_G, then computing the fibers under $\aleph$ and ∇. We will state the result of this for $G = C_{p^n} = \mathbb{Z}/p^n\mathbb{Z}$, but first we need notation. Given $R \in \mathrm{Transf}_G$ for G Abelian, we define the families

$$\mathrm{Dom}(R) := \left\{ U \in \mathcal{O}_G \mid \ \exists U \to V \xrightarrow{f} W \ \text{ s.t. } \ f \in R - R^{\simeq} \right\};$$

$$\mathrm{Cod}(R) := \left\{ U \in \mathcal{O}_G \mid \ \exists V \xrightarrow{f} W \leftarrow U \ \text{ s.t. } \ f \in R - R^{\simeq} \right\}.$$

Given a full subcategory $\mathcal{F} \subset \mathcal{O}_G$ and a G-transfer system R, we denote by $\mathrm{Sieve}_R(\mathcal{F}) \subset \mathrm{Sub}_{\mathrm{graph}}(R)$ the poset of R-precomposition-closed and isomorphism-closed collections of maps in R whose codomains lie in $\mathcal{F}$ and satisfy the condition that, whenever $K \subset H$ is in R and $L \subset H$ lies in $\mathcal{F}$, the map $L \cap K \subset L$ is in R.

For $n \in \mathbb{N}$, we let K_n be the nth associahedron, i.e. the poset of parenthesizations of a string of length n. The main result of [BBR21] constructs an equivalence $\mathrm{Transf}_{C_{p^n}} \simeq K_{n+2}$, and it's not too hard to construct an equivalence $\mathrm{Fam}_{C_{p^n}} \simeq [n+2]$ for $[n+2]$ the total order on $n+2$ elements.

Corollary C. *Let p be a prime. Then, there is a map of posets*

$$(\aleph, \nabla) : \mathrm{wIndex}^{\mathrm{uni}}_{C_{p^n}} \to K_{n+2} \times [n+2]$$

with fibers satisfying

$$\aleph^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) = \begin{cases} \varnothing & \mathrm{Dom}(R) \not\leq \mathcal{F}; \\ * & \mathrm{Cod}(R) \leq \mathcal{F}; \\ \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) & \text{otherwise.} \end{cases}$$

Moreover, the associated surjection onto its image is a cocartesian fibration, with cocartesian transport computed along $R \leq R'$ given by the map

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \to \mathrm{Sieve}_{R'}(\mathrm{Cod}(R') - \mathcal{F})$$

sending $\mathfrak{S} \mapsto R^{\simeq} \cup \{J \subset K \subsetneq H \mid J \subset K \in R',\ K \subsetneq H \in \mathfrak{S}\}$ and cocartesian transport computed along $\mathcal{F} \leq \mathcal{F}'$ by the restriction

$$\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}').$$

This completely determines $\mathrm{wIndex}^{\mathrm{uni}}_{C_{p^n}}$. Nevertheless, we draw this explicitly for $n \leq 2$ in Section 3.

1.4. **Why (unital) weak indexing systems?** The author finds primarily weak indexing systems compelling for the following two reasons:

(1) once the algebraist is convinced that they want finite H-sets to index their G-equivariant algebraic structures, weak indexing systems are forced upon them, and our various conditions classify useful properties of algebraic theories;
(2) $\mathbb{E}_V$-spaces and $\mathbb{E}_V$-ring spectra naturally appear in algebraic topology, sometimes for V a representation which has *zero-dimensional fixed points*. As argued in Example 1.29, the associated G-operad $\mathbb{E}_V$ has arities supported only on a (unital) *weak* indexing system.

Hopefully this paper and [Ste25b] will demonstrate the first point handily; indeed, we will see in [Ste25b] that $\mathrm{wIndexCat}_{\mathcal{T}}$ occurs "in nature" as the poset of sub-terminal objects in the ∞-category $\mathrm{Op}_{\mathcal{T}}$ of $\mathcal{T}$-operads, and aE-unitality of I classifies the property that I satisfies a weak *Eckmann-Hilton argument*, i.e.

$$\mathrm{CAlg}_I \underline{\mathrm{CAlg}}^{\otimes}_I(\mathcal{C}) \xrightarrow{U} \mathrm{CAlg}_I(\mathcal{C})$$

is an equivalence.

The author's favorite example behind the second point is the sign C_2-representation σ; as explained above, its arity-support (which is shared with $\infty\sigma$) is *not* an indexing system. Furthermore, the evident conjectural extension of Dunn's additivity theorem [Dun88] in the equivariant setting would imply that $\mathbb{E}_\sigma^{\otimes\infty} \simeq \mathbb{E}_{\infty\sigma}$, and in [Ste25b] we argue that $\mathbb{E}_{\infty\sigma}$ is a *weak* $\mathcal{N}_\infty$-operad; one should expect $\mathbb{E}_{\infty\sigma}$-algebras to be relevant to constructions utilizing $\mathbb{E}_\sigma$ structures, such as Real topological Hochschild homology [AGH21, § 3].

1.5. **Notation and conventions.** There is an equivalence of categories between that of posets and that of categories whose hom sets have at most one point; we safely conflate these notions. In doing so, we use categorical terminology to describe posets.

A *sub-poset* of a poset P is an injective monotone map $P' \hookrightarrow P$, i.e. a relation on a subset of the elements of P refining the relation on P. A *embedded sub-poset* (or *full sub-poset*) is a sub-poset $P' \hookrightarrow P$ such that $x \leq_{P'} y$ if and only if $x \leq_P y$ for all $x, y \in P'$.

An *adjunction of posets* (or *monotone Galois connection*) is a pair of opposing monotone maps $L\colon P \rightleftarrows Q\colon R$ satisfying the condition that

$$Lx \leq_Q y \qquad \iff \qquad x \leq_P Ry \qquad \forall\, x \in P,\ y \in Q.$$

In this case, we refer to L as the *left adjoint* and R as the *right adjoint*, as L is uniquely determined by R and vice versa.

A *cocartesian fibration of posets* (or *Grothendieck opfibration*) is a monotone map $\pi\colon P \to Q$ satisfying the condition that, for all pairs $q \le q'$ and $p \in \pi^{-1}(q)$, there exists an element $t_q^{q'} p \in \pi^{-1}(q')$ characterized by the property

$$p \le p' \qquad \iff \qquad q' \le \pi(p') \;\text{ and }\; t_q^{q'} p \le p';$$

in this case, we note that $t_q^{q'}\colon \pi^{-1}(q) \to \pi^{-1}(q')$ is a monotone map, and we may express P as the set $\coprod_{q \in Q} \pi^{-1}(q)$ with relation determined entirely by the above formula.

Acknowledgements. I would like to thank Clark Barkwick for numerous helpful conversations on this topic; for instance, his skepticism at an early (erroneous) sketch of the classification of weak $\mathcal{N}_\infty$-operads motivated me to take a careful look at the combinatorics of weak indexing systems, which grew into this work. Additionally, I would like to thank Mike Hill for pointing out that the first version of this paper contained the redundant assumption that weak indexing categories be replete subcategories, which follows from pullback stability. I would be remiss to fail to mention that this project is closely linked with [Ste25a; Ste25b], about which many illuminating conversations were had with Clark Barwick, Dhilan Lahoti, Mike Hopkins, Piotr Pstrągowski, Maxime Ramzi, and Andy Senger.

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2. Weak indexing systems

This section concerns non-enumerative aspects of the study of weak indexing systems and weak indexing categories. We begin in Section 2.1 by recognizing weak indexing categories as indexed collections of weak indexing categories with respect to the slice categories of $\mathbb{F}_{\mathcal{T}}$ over orbits, allowing us to universally reduce structural statements about $\mathsf{wIndexCat}_{\mathcal{T}}$ to the case that $\mathcal{T}$ possesses a terminal object, so it is a 1-category. Using this, in Section 2.2, we prove Theorem A.

Following this, we dedicate some study to structural statements about $\mathsf{wIndex}_{\mathcal{T}}$, developing a litany of adjunctions and cocartesian fibrations involving it and its variants. We begin in Section 2.3 by developing the technology of *weak indexing system closures*, and using it to combinatorially characterize joins in the poset $\mathsf{wIndex}_{\mathcal{T}}$; as examples, we compute joins of the arity support $\underline{\mathbb{F}}^R$ of the little R-disks G-operad and characterize weak indexing system coinduction.

Next, in Section 2.4, we characterize the families c and v; the former is a fully faithful left and right adjoint (so we may reduce to the one-object case), and the latter has a fully faithful left adjoint, but interacts with joins in a complicated way. Following this, in Section 2.5, we characterize the map $\mathfrak{R}\colon \mathsf{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathsf{Transf}_{\mathcal{T}}$ of Observation 1.39, showing it possesses fully faithful left and right adjoints, which seldom agree; we then characterize ∇, showing that it has fully faithful left and right adjoints. We additionally develop another family ϵ, and use it to characterize adjoins and join-compatibility of the various conditions of Definition 1.21.

Lastly, in Section 2.6, we take a detour and generalize the theory of *compatible pairs of indexing systems* to the setting of weak indexing systems, showing that the multiplicative hull of a weak indexing system exists and is an indexing system.

2.1. Recovering weak indexing categories from their slice categories.

Recall that the poset of weak indexing categories $\mathsf{wIndexCat} \subset \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_{\mathcal{T}})$ is the embedded subposet spanned by those subcategories satisfying Conditions (IC-a) and (IC-b) of Theorem A; that is, they are pullback subcategories which are extended by coproducts from their maps to orbits.

We refer to $\mathcal{T}$-1-categories $\mathcal{C}$ whose V-values $\mathcal{C}_V$ are posets for all $V \in \mathcal{T}$ as *$\mathcal{T}$-posets*.

Construction 2.1. Given $\mathcal{C} \subset \mathbb{F}_{\mathcal{T}}$ a subcategory and $V \in \mathcal{T}$ an object, we write

$$\mathcal{C}_V := \left\{ f\colon \begin{array}{ccc} S & \xrightarrow{\widetilde{f}} & T \\ & \searrow \;\; \swarrow & \\ & V & \end{array} \;\middle|\; \widetilde{f} \in \mathcal{C} \right\} \subset \mathbb{F}_V;$$

that is, maps in $\mathcal{C}_V$ are maps over V whose underlying map in $\mathbb{F}_{\mathcal{T}}$ lies in $\mathcal{C}$. For every map $V \to W$, this yields a map $(-)_V : \mathrm{Sub}_{\mathrm{Cat}_W}(\mathbb{F}_W) \to \mathrm{Sub}_{\mathrm{Cat}_V}(\mathbb{F}_V)$, compatibly with composition. We let $\underline{\mathrm{Sub}}_{\underline{\mathrm{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$ be the resulting $\mathcal{T}$-poset ◁

Proposition 2.2. *If $I \subset \mathbb{F}_{\mathcal{T}}$ is a $\mathcal{T}$-weak indexing category, then $I_V \subset \mathbb{F}_V$ is a $\mathcal{T}_{/V}$-weak indexing category.*

Proof. Condition (IC-b) for I_V follows by unwinding definitions, noting that $\mathrm{Ind}_V^{\mathcal{T}} : \mathbb{F}_V \to \mathbb{F}_{\mathcal{T}}$ is coproduct-preserving. Lastly, Condition (IC-a) follows by unwinding definitions, noting that the pullback functor $\mathbb{F}_V \to \mathbb{F}_W$ is pullback-preserving for each $W \to V$. □

Proposition 2.2 lifts $\mathrm{wIndexCat}_{\mathcal{T}} \subset \mathrm{Sub}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$ to an embedded $\mathcal{T}$-subposet

$$\underline{\mathrm{wIndexCat}}_{\mathcal{T}} \subset \underline{\mathrm{Sub}}_{\underline{\mathrm{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}}).$$

Given a $\mathcal{T}$-poset $P : \mathcal{T}^{\mathrm{op}} \to \mathrm{Poset}$, we denote by $\Gamma^{\mathcal{T}} P$ the associated limit. There is a monotone map

$$\widetilde{\gamma} \colon \mathrm{Sub}_{\mathrm{Cat}}(\mathbb{F}_{\mathcal{T}}) \to \Gamma \underline{\mathrm{Sub}}_{\underline{\mathrm{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$$

defined by $\widetilde{\gamma}(\mathcal{C})_V := \mathcal{C}_V$. We may use $\widetilde{\gamma}$ to recover $\mathrm{wIndexCat}_{\mathcal{T}}$ from $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$.

Proposition 2.3. *$\widetilde{\gamma}$ restricts to an equivalence*

$$\gamma \colon \mathrm{wIndexCat}_{\mathcal{T}} \xrightarrow{\sim} \Gamma \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$$

Proof. Proposition 2.2 implies that $\widetilde{\gamma}$ restricts to a monotone map of posets $\gamma \colon \mathrm{wIndexCat}_{\mathcal{T}} \to \Gamma^{\mathcal{T}} \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$, so it suffices to prove that this is bijective. If $\gamma I = \gamma J$, then for a map $f : T \to V$, the canonical $\mathcal{T}_{/V}$-map $T \to *_V$ lies I_V if and only if it lies in J_V, so f lies in I if and only if it lies in J; thus Condition (IC-b') implies that $I = J$, so γ is injective.

It remains to prove that γ is surjective, so we fix $I_\bullet \in \Gamma^{\mathcal{T}} \underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Define the subcategory

$$I := \{T \to S \mid \forall U \in \mathrm{Orb}(S), \quad T \times_S U \to U \in I_U\} \subset \mathbb{F}_{\mathcal{T}}.$$

By definition, $\gamma I = I_\bullet$, so it suffices to verify that I is a weak indexing category. First note that I satisfies Condition (IC-b') by definition; additionally Condition (IC-a') is precisely the condition that $I_{(-)}$ is an element of $\underline{\mathrm{wIndexCat}}_{\mathcal{T}}$. Hence I is a $\mathcal{T}$-weak indexing system, proving that γ is an isomorphism. □

Noting that spaces (as ∞-categories) have *contractible* slice categories, this implies the following.

Corollary 2.4. *If X is a space, then the forgetful map $\mathrm{wIndex}_X \to \mathrm{wIndex}_*$ is an equivalence.*

We would like to use this to uniformly replace $\mathcal{T}$ with a 1-category, for which we need the following.

Example 2.5. The atomic orbital ∞-category $\mathcal{T}_{/V}$ has a terminal object; by [NS22, Prop 2.5.1], this implies that $\mathcal{T}_{/V}$ is a 1-category. In general for $F : J \to \mathcal{T}$ a diagram in an atomic orbital ∞-category indexed by a finite 1-category, $\mathcal{T}_{/J}$ is also a 1-category; in particular, the top arrow

$$\begin{array}{ccc} \mathcal{T}_{/J} & \longrightarrow & \mathrm{ho}(\mathcal{T})_{/J} \\ & \searrow & \wr\!\! \mid \\ & & \mathrm{ho}(\mathcal{T}_{/J}) \end{array}$$

is an equivalence. This implies that $\mathbb{F}_{\mathrm{ho}(\mathcal{T})}$ has pullbacks, i.e. $\mathrm{ho}(\mathcal{T})$ is orbital; because $\mathcal{T}$ is atomic, retracts in $\mathrm{ho}(\mathcal{T})$ are isomorphisms, i.e. $\mathrm{ho}(\mathcal{T})$ is atomic orbital. ◁

Using this and fact that the 1-category of posets is a 1-category, we an equivalence

$$\begin{array}{ccc} \mathrm{Sub}(\mathbb{F}_{\mathcal{T}}) & \xrightarrow{\quad \mathrm{ho} \quad} & \mathrm{Sub}(\mathbb{F}_{\mathrm{ho}(\mathcal{T})}) \\ \uparrow & & \uparrow \\ \mathrm{wIndexCat}_{\mathcal{T}} & \xrightarrow{\quad \sim \quad} & \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})} \\ \wr\!\! \mid & & \wr\!\! \mid \\ \lim_{V \in \mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathcal{T}_{/V}} & \xrightarrow{\sim} & \lim_{V \in \mathrm{ho}\mathcal{T}^{\mathrm{op}}} \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})_{/V}} \end{array}$$

In other words, we've observed the following.

Corollary 2.6. *The homotopy category construction restricts to an equivalence* $\mathrm{wIndexCat}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathrm{ho}(\mathcal{T})}$.

Using this, for the rest of the paper, we will assume that $\mathcal{T}$ is a 1-category.

2.2. Weak indexing categories vs weak indexing systems.

Construction 2.7. Given $I \subset \mathbb{F}_{\mathcal{T}}$ a subcategory, define the class of *I-admissible V-sets*

$$\mathbb{F}_{V,I} \coloneqq \left\{ S \;\middle|\; \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathbb{F}_V.$$

Taken altogether, we refer to the associated collection as $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. ◁

Recall the notation $I(-)$ used in Observation 1.33.

Observation 2.8. Given $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ a collection, we have $\mathbb{F}_{V,I(\mathcal{C})} \simeq \mathcal{C}$; conversely, if a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ satisfies Condition (IC-b), then $I(\underline{\mathbb{F}}_I) = I$. ◁

These are candidates for inverse maps $\mathrm{wIndex}_{\mathcal{T}} \rightleftarrows \mathrm{wIndexCat}_{\mathcal{T}}$, and they are well behaved:

Observation 2.9. Pullback-stable subcategories are *replete*, i.e. they contain all automorphisms of their objects. On the other hand, if $S \simeq S'$ as V-sets, then there exists an equivalence $\mathrm{Ind}_V^{\mathcal{T}} S \simeq \mathrm{Ind}_V^{\mathcal{T}} S'$ over V. Hence whenever $I \subset \mathbb{F}_{\mathcal{T}}$ is a pullback-stable subcategory and $S \in \underline{\mathbb{F}}_I$, the map $\mathrm{Ind}_V^{\mathcal{T}} S' \to V$ is in I, i.e. $\mathbb{F}_{V,I} \subset \mathbb{F}_V$ is closed under equivalence; these objects determine a unique full subcategory which we also call $\mathbb{F}_{V,I}$. On the other hand, if $\underline{\mathbb{F}}_I$ satisfies Condition (IS-a), this implies directly that $I(\underline{\mathbb{F}}_I)$ has identity arrows. ◁

Observation 2.10. By definition, the restriction functor $\mathrm{Res}_V^W : \mathbb{F}_W \to \mathbb{F}_V$ is implemented by the pullback

$$\begin{array}{ccc} \mathrm{Ind}_V^{\mathcal{T}} \mathrm{Res}_V^W S & \longrightarrow & \mathrm{Ind}_W^{\mathcal{T}} S \\ \big\downarrow & \lrcorner & \big\downarrow \\ V & \longrightarrow & W \end{array}$$

thus I satisfies Condition (IC-a') if and only if $\mathrm{Res}_V^W \mathbb{F}_{W,I} \subset \mathbb{F}_{V,I}$ for all maps $V \to W$; in particular, in this case, $\{\mathbb{F}_{V,I}\}_{V \in \mathcal{T}}$ corresponds with a unique full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$. ◁

The following is fundamental to passing between weak indexing categories and weak indexing systems.

Observation 2.11. Let $(T_U) \in \mathbb{F}_S$ be an S-tuple of elements of $\underline{\mathbb{F}}_{\mathcal{T}}$ for some $S \in \mathbb{F}_V$. Then, the indexed coproduct of (T_U) corresponds with the composite arrow

$$\mathrm{Ind}_V^{\mathcal{T}} \coprod_U^S T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_V^{\mathcal{T}} \mathrm{Ind}_U^V T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^{\mathcal{T}} T_U \to \mathrm{Ind}_V^{\mathcal{T}} S \to V; \tag{3}$$

in particular, if $I \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory satisfying Condition (IC-b) and (T_U) and S are I-admissible, both arrows in Eq. (3) lie in I, so the structure map of $\coprod_U^S T_U$ is in I, i.e. $\coprod_U^S T_U \in \underline{\mathbb{F}}_I$. In other words, Condition (IC-b) and the condition that $I \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory together imply that $\underline{\mathbb{F}}_I$ satisfies Condition (IS-b).

On the other hand, if $I \subset \mathbb{F}_{\mathcal{T}}$ is a subgraph satisfying Condition (IC-b) such that $\underline{\mathbb{F}}_I$ satisfies Condition (IS-b), then taking coproducts of Eq. (3) shows that I is closed under composition. If I additionally has identity arrows (e.g. if $\underline{\mathbb{F}}_I$ satisfies Condition (IS-a)), this implies that $I \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory. ◁

We are now ready to verify that $I(-)$ and $\underline{\mathbb{F}}_{(-)}$ restrict to maps between $\mathrm{wIndex}_{\mathcal{T}}$ and $\mathrm{wIndexCat}_{\mathcal{T}}$.

Proposition 2.12. *If $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a weak indexing system, then $I(\mathcal{C})$ is a weak indexing category.*

Proof. By Proposition 2.3, we may assume that $\mathcal{T}$ has a terminal object. By Observations 1.34 and 1.35, it suffices to verify that $I \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory satisfying Conditions (IC-a') and (IC-b'). Condition (IC-a') is verified by Observation 2.10; Condition (IC-b') follows immediately from construction; Observation 2.11 verifies that $I \subset \mathbb{F}_{\mathcal{T}}$ is a subcategory. □

Proposition 2.13. *If $I \subset \mathbb{F}_{\mathcal{T}}$ is a weak indexing category, then $\underline{\mathbb{F}}_I$ is a weak indexing system.*

Proof. Observations 2.9 and 2.10 verify that $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, and the fact that the identity arrow on V corresponds with the contractible V-set implies that whenever $\underline{\mathbb{F}}_{I,V} \neq \varnothing$ (i.e. $V \in I$), $*_V \in \underline{\mathbb{F}}_{I,V}$. Thus it suffices to verify that $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts; this is Observation 2.11. □

Having done this, we're poised to conclude that $I(-)$ and $\underline{\mathbb{F}}_{-}$ are inverse equivalences.

Proof of Theorem A. By Propositions 2.12 and 2.13, $I\colon \mathrm{wIndex}_{\mathcal{T}} \rightleftarrows \mathrm{wIndexCat}_{\mathcal{T}}\colon \underline{\mathbb{F}}_{(-)}$ are well defined monotone maps; by Observation 2.8, they are inverse to each other, so they are equivalences.

What remains is to verify that (IC-n) is equivalent to (IS-n) in Definition 1.21 and Theorem A. For $n = i$, this follows immediately by noting that $V \in I \iff \mathrm{id}_V \in I \iff *_V \in \mathbb{F}_{I,V} \iff \mathbb{F}_{I,V} \neq \varnothing$. For $n = ii$ and $n = iii$, this follows by unwinding definitions using Condition (IC-b'). For $n = iv$, this follows by noting that the fold map $n \cdot V \to V$ corresponds with the element $n \cdot *_V \in \mathbb{F}_V$. □

2.3. **Joins and coinduction.** We move on to intrinsic statements concerning $\mathrm{wIndex}_{\mathcal{T}}$.

2.3.1. *Prerequisites on adjunctions and cocartesian fibrations.* Recall that a monotone map $\pi\colon \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration (i.e. a Grothendieck opfibration) if and only if, for all related pairs $D \le D'$ in $\mathcal{D}$ and elements $C \in \pi^{-1}(D)$, there is an element $t_D^{D'} C \in \pi^{-1}(D')$ satisfying the property

$$\forall\, C' \text{ s.t. } D' \le \pi(C'), \qquad C \le C' \iff t_D^{D'} C \le C'$$

In this section, we relate these to adjunctions of posets (i.e. monotone Galois connections).

Lemma 2.14. *Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be a monotone map. The following are equivalent.*

(a) π possesses a fully faithful left adjoint L.
(b) For all $D \in \mathcal{D}$, the preimage $\pi^{-1}(\mathcal{D}_{\ge D})$ possesses an initial object $L(D)$ with $\pi L(D) = D$.
(c) For all $D \in \mathcal{D}$, the fiber $\pi^{-1}(D)$ has an initial object $L(D)$, and $D \le D'$ implies $L(D) \le L(D')$.

Furthermore, the element $L(D)$ agrees between these three constructions.

Proof. By definition, π has a left adjoint L if and only if there are initial objects in $\pi^{-1}(\mathcal{D}_{\le D})$, which are $L(D)$. By the usual category theoretic nonsense, L is fully faithful if and only if the unit relation $D \le \pi L(D)$ is an equality, i.e. $L(D) \in \pi^{-1}(D)$; hence (a) $\iff$ (b).

To see (b) $\iff$ (c), first note that

$$L(D) \le C' \iff D \le \pi(C') \iff L(D) \le L\pi(C');$$

if (b), then when $D = L(D) \le L\pi L(D') = D'$, we have $L(D) \le L(D')$, so (c). Conversely, if (c) and $L(D) \le C'$, then we have $D \le \pi(C')$, so D is initial in $\pi^{-1}(\mathcal{D}_{\le D})$, so (b). □

Proposition 2.15. *Suppose $\mathcal{C}$ has binary joins and $\pi\colon \mathcal{C} \to \mathcal{D}$ is a monotone map which is compatible with binary joins and possesses a fully faithful left adjoint L. Then, π is a cocartesian fibration with*

$$t_D^{D'} C = L(D') \vee C.$$

Proof. First note that

$$\pi(L(D') \vee C) = \pi L(D') \vee \pi(C) = D' \vee \pi(C) = D'.$$

Thus the property for cocartesian transport is given by

$$L(D') \vee C \le C' \iff L(D') \le C' \text{ and } C \le C';$$

indeed, when we restrict to the case $L(D') \le C'$ (i.e. $D' \le \pi(C')$), we then have $C \le C'$ if and only if $L(D') \vee C \le C'$, as desired. □

Remark 2.16. If π possesses a *right* adjoint R, then it is compatible with joins, as left adjoint functors are compatible with colimits.[5] The adjoint functor theorem for posets states the converse; indeed, R has arbitrary joins and π is compatible with joins, then its right adjoint is computed by

$$R(Z) = \bigvee_{\pi(Y) \le Z} Y.$$

Thus Proposition 2.15 may be weakened to state that whenever π has a left and right adjoint and the left is fully faithful, π is a cocartesian fibration with transport computed as stated. In fact, the left adjoint is fully faithful if and only if the right adjoint is fully faithful [DT87, Lem 1.3], so we may stipulate that either (or both) are fully faithful.

[5] We may see this directly in the binary case by noting that, for $X, Y \in \mathcal{C}$, the universal property for joins is satisfied by

$$\pi(X \vee Y) \le Z \iff X \vee Y \le R(Z) \iff X \le R(Z) \text{ and } Y \le R(Z) \iff \pi(X) \le Z \text{ and } \pi(Y) \le Z.$$

This is manifestly self-dual; in this setting, the dual of Proposition 2.15 implies that π is a cartesian fibration with cartesian transport given by $t_D^{D'} C = R(D) \wedge C$. We will not use this explicitly in this text, but the author suggests that homotopical combinatorialists keep this trick in mind. ◁

2.3.2. *Closures and joins of weak indexing systems.* The following construction will be used often.

Construction 2.17. Given collections $\mathcal{D}, \mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$, inductively define $\mathrm{Cl}_{\mathcal{D},0}(\mathcal{C}) := \mathcal{C}$ and

$$\mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})_V = \left\{ \coprod_U^S T_U \;\middle|\; (T_U) \in \mathrm{Cl}_{n-1}(\mathcal{C})_S,\;\; S \in \mathcal{D} \right\},$$

with $\mathrm{Cl}_{\mathcal{D},\infty}(\mathcal{C}) := \bigcup_n \mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})$ and $\mathrm{Cl}_n(\mathcal{C}) := \mathrm{Cl}_{\mathcal{C},n}(\mathcal{C})$. We call this the *$n$-step closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts*, or just the *closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts* when $n = \infty$. ◁

Proposition 2.18. *If $\mathcal{D}$ is a weak indexing system, then the canonical inclusion*

$$\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}) \subset \mathrm{Cl}_{\mathcal{D}}(\mathcal{C})$$

is an equality for all $\mathcal{C}$.

This follows immediately from the following lemma.

Lemma 2.19. *Fix an orbit $V \in \mathcal{T}$, a finite V-set $S \in \mathbb{F}_V$, and a finite S-set $(T_U) \in \mathbb{F}_S$. Write $T := \coprod_U^S T_U$. Then, there is a canonical natural equivalence*

$$\coprod_X^T (-) \simeq \coprod_U^S \coprod_X^{T_U} (-)$$

Proof. In view of Observation 2.11, this follows by composition of left adjoints to the composite functor

$$\Delta^T : \mathcal{C}_V \xrightarrow{\Delta^S} \mathcal{C}_S \xrightarrow{(\Delta^{T_U})} \mathcal{C}_T. \qquad \square$$

Observation 2.20. If $\mathcal{D}$ satisfies Condition (IS-a) and $c(\mathcal{D}) \supset c(\mathcal{C})$, then by taking $*_V$-indexed coproducts for all $V \in c(\mathcal{C})$, we find that $\mathcal{C} \subset \mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$. Similarly, if $\mathcal{C}$ satisfies Condition (IS-a) and $c(\mathcal{D}) \subset c(\mathcal{C})$, by taking indexed coproducts of $(*_U)$, we find that $\mathcal{C} \subset \mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})$. Combining these, if $\mathcal{C}$ and $\mathcal{D}$ satisfy Condition (IS-a) and $c(\mathcal{C}) = c(\mathcal{D})$ (e.g. they each have one color), then we have

$$\mathcal{C}, \mathcal{D} \subset \mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}).$$

Furthermore, note that $c(\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})) = c(\mathcal{C})$ in this situation, so $\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$ satisfies Condition (IS-a). ◁

Let $\mathsf{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}}) \subset \mathsf{FullSub}_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ denote the full subposet of elements satisfying Condition (IS-a).

Proposition 2.21. *The fully faithful map $\iota\colon \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathsf{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$ is right adjoint to Cl_∞.*

Proof. If $\mathrm{Cl}_\infty(\mathcal{C})$ is a weak indexing system, then it is clearly minimal among those containing $\mathcal{C}$, so it suffices to prove that it's a weak indexing system. By Observation 2.20, $\mathrm{Cl}_\infty(\mathcal{C})$ satisfies Condition (IS-a), so it suffices to verify Condition (IS-b).

In fact, by Lemma 2.19, we find that $\mathrm{Cl}_i(\mathcal{C})$-indexed coproducts of elements of $\mathrm{Cl}_j(\mathcal{C})$ are $\mathrm{Cl}_{i+1}(\mathcal{C})$-indexed coproducts of elements of $\mathrm{Cl}_{j-1}(\mathcal{C})$; applying this j-many times, we find that $\mathrm{Cl}_i(\mathcal{C})$-indexed coproducts of elements in $\mathrm{Cl}_j(\mathcal{C})$ are in $\mathrm{Cl}_\infty(\mathcal{C})$, so taking a union, we find that $\mathrm{Cl}_\infty(\mathcal{C})$ satisfies Condition (IS-b). $\square$

Define the rectified closure

$$\widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D}) = \mathrm{Cl}_{\mathcal{C} \cup \mathbb{F}^{\mathrm{triv}}_{c(\mathcal{D})},1}(\mathcal{D}) = \mathrm{Cl}_{\mathcal{C},1}(\mathcal{D}) \cup \mathcal{D};$$

the equalities follow from Observation 2.20, and in particular, when $c(\mathcal{C}) \supset c(\mathcal{D})$ we have $\mathrm{Cl}_{\mathcal{C},1}(\mathcal{D}) = \widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D})$. Similarly define $\widehat{\mathrm{Cl}}_{\mathcal{C}}(\mathcal{D}) := \mathcal{D} \cup \mathrm{Cl}_{\mathcal{C}}(\mathcal{D})$ and write $\widehat{\mathrm{Cl}}_I(-) := \widehat{\mathrm{Cl}}_{\underline{\mathbb{F}}_I}(-)$.

Proposition 2.22. $\mathrm{wIndex}_{\mathcal{T}}$ *is a lattice; the meets in* $\mathrm{wIndex}_{\mathcal{T}}$ *are intersections, and the joins are*

$$\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \bigcup_{n \in \mathbb{N}} \overbrace{\widehat{\mathrm{Cl}}_I \widehat{\mathrm{Cl}}_J \cdots \widehat{\mathrm{Cl}}_I \widehat{\mathrm{Cl}}_J}^{2n} (\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J).$$

Proof. By Proposition 2.21, $\mathrm{wIndex}_{\mathcal{T}}$ has meets computed in $\mathrm{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$, which are clearly given by intersections. Furthermore, Proposition 2.21 implies that $\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)$. Thus is suffices to note that, for arbitrary $\mathcal{C}, \mathcal{D}, \mathcal{E}$, we have

$$\widehat{\mathrm{Cl}}_{\mathcal{C}\cup\mathcal{D},\infty}(\mathcal{E}) = \bigcup_{n\in\mathbb{N}} \overbrace{\widehat{\mathrm{Cl}}_{\mathcal{C}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{D})}} \widehat{\mathrm{Cl}}_{\mathcal{D}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{C})}} \cdots \widehat{\mathrm{Cl}}_{\mathcal{C}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{D})}} \widehat{Cl}_{\mathcal{D}\cup\mathbb{F}^{\mathrm{triv}}_{c(\mathcal{C})}}}^{2n}(\mathcal{E}),$$

and set $\mathcal{C} = \underline{\mathbb{F}}_I$, $\mathcal{D} = \underline{\mathbb{F}}_J$, and $\mathcal{E} = \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$. $\square$

Remark 2.23. In fact, Proposition 2.21 constructs arbitrary meets in $\mathrm{wIndex}_{\mathcal{T}}$. Furthermore, chains in $\mathrm{wIndex}_{\mathcal{T}}$ have joins computed by unions; hence $\mathrm{wIndex}_{\mathcal{T}}$ is a complete lattice. ◃

Observation 2.24. Similarly, if $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a collection, then the full $\mathcal{T}$-subcategory $\widehat{\mathcal{C}}$ defined by

$$\widehat{\mathcal{C}}_V = \begin{cases} \{*_V\} \cup \bigcup_{V\to W} \mathrm{Res}^W_V \mathcal{C}_W & \mathcal{C}_V \neq \varnothing, \\ \varnothing & \mathcal{C}_V = \varnothing \end{cases}$$

is initial among full $\mathcal{T}$-subcategories containing $\mathcal{C}$ and satisfying Condition (IS-a) Combining adjunctions, we find that the fully faithful map $\iota : \mathrm{wIndex}_{\mathcal{T}} \hookrightarrow \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}})$ possesses a left adjoint $\mathrm{Cl}_\infty(\widehat{-})$, which we write simply as $\mathrm{Cl}_\infty(-)$ for brevity. ◃

Given $S \in \mathbb{F}_V$, let $\mathbb{F}_{I_S,V}$ be the closure of $\{*_V\}$ under S-indexed coproducts; more generally, let $\mathbb{F}_{I_S,W} := \bigcup_{f: W\to V} \mathrm{Res}^V_W \mathbb{F}_{I_S,V}$. Let $\underline{\mathbb{F}}_{I_S}$ be the collection defined by $\left(\underline{\mathbb{F}}_{I_S}\right)_W := \mathbb{F}_{I_S,W}$.

Proposition 2.25. *Given $S \in \mathbb{F}_V$, we have $\mathrm{Cl}_\infty(\{S\}) = \underline{\mathbb{F}}_{I_S}$.*

Proof. First, note that $\{S\} \subset \underline{\mathbb{F}}_{I_S} \subset \mathrm{Cl}_\infty(\{S\})$. By Proposition 2.21, it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is a weak indexing system. By construction, $\underline{\mathbb{F}}_{I_S} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory satisfying the property that

$$*_W \in \mathbb{F}_{I_S,W} \qquad \Longleftrightarrow \qquad \exists f : W \to V \qquad \Longleftrightarrow \qquad \mathbb{F}_{I_S,W} \neq \varnothing,$$

i.e. it satisfies Condition (IS-a). Hence it suffices to prove that $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts.

Lemma 2.19 implies that that if $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under T-indexed coproducts and X_U-indexed coproducts for $(X_U) \in \mathbb{F}_T$, then $\mathcal{C}$ is closed under $\coprod_U^T X_U$-indexed coproducts, as they are T-indexed coproducts of X_U-indexed coproducts; hence $\underline{\mathbb{F}}_{I_S}$ is closed under $\mathbb{F}_{I_S,V}$-indexed coproducts. Furthermore, Remark 1.37 implies that if $\mathcal{C}_W$ is generated under restrictions by $\mathcal{C}_U$ and $\mathcal{C}_U$ is closed under T-indexed coproducts, then $\mathcal{C}_W$ is closed under $\mathrm{Res}^U_W T$-indexed coproducts; hence $\underline{\mathbb{F}}_{I_S}$ is closed under self-indexed coproducts, as desired. $\square$

2.3.3. *Joins and $\mathbb{F}^R$.* Let G be a finite group and R a real orthogonal G-representation. Recall from Example 1.29 that there is a weak indexing system $\underline{\mathbb{F}}^R$ satisfying

$$\mathbb{F}^R_H = \{S \in \mathbb{F}_H \mid \exists\, H\text{-equivariant embedding } S \hookrightarrow R\}.$$

Observation 2.26. If $S \in \mathbb{F}^R_V$ and R is a subrepresentation of R', then the composite embedding $S \hookrightarrow R \hookrightarrow R'$ witnesses the membership $S \in \mathbb{F}^{R'}_V$; that is, $\underline{\mathbb{F}}^{(-)}$ is *monotone* under inclusions of subrepresentations. ◃

In particular, monotonicity yields relations $\underline{\mathbb{F}}^R, \underline{\mathbb{F}}^{R'} \subset \underline{\mathbb{F}}^{R\oplus R'}$, and hence a relation $\underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^{R'} \subset \underline{\mathbb{F}}^{R\oplus R'}$. We verify that this relation is an equality in the following argument; throughout the argument, when $x \in T$ is an element of an H-set, we will write $[x]_H$ for its orbit under the H-action.

Proposition 2.27. *For R, R' real orthogonal G-representations, we have $\underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^{R'} = \underline{\mathbb{F}}^{R\oplus R'}$.*

Proof. By the above argument, it suffices to verify the relation $\underline{\mathbb{F}}^{R\oplus R'} \subset \underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^R$. Let $S \in \mathbb{F}^{R\oplus R'}_H$ be a finite H-set embedding into $R \oplus R'$. The composite map $S \to R \oplus R' \to R$ possesses an image factorization

$$\begin{array}{ccc} S & \xhookrightarrow{\iota} & R \oplus R' \\ \downarrow\!\!\downarrow \psi & & \downarrow\!\!\downarrow \pi \\ S_R & \xhookrightarrow[\iota' = \mathrm{im}(\pi\iota)]{} & R \end{array}$$

Given $x \in S_R$, note that there is an isomorphism

$$\psi^{-1}[x]_H \simeq \mathrm{Ind}_{\mathsf{stab}_H(x)}^H \psi^{-1}(x),$$

where the $\mathsf{stab}_H(x)$ action on $\psi^{-1}(x)$ is restricted from the H-action on S. Furthermore, note that the fiber of $R \oplus R'$ over $(0, \iota'(x))$ is invariant under the $\mathsf{stab}_H(x)$ action and the resulting $\mathsf{stab}_H(x)$-space is taken isomorphically onto $R' \simeq \{0\} \oplus R'$ by $(-) - (0, \iota'(x))$; thus $\psi^{-1}(x)$ admits a $\mathsf{stab}_H(x)$-equivariant embedding into R'.

To summarize, we may make a choice of an element x_{K_i} in each orbit $[H/K_i] \subset S_R$ and apply the above argument to conclude that $S_R \in \mathbb{F}_H^R$, that $\psi^{-1}(x_{K_i}) \in \mathbb{F}_{K_i}^{R'}$, and that

$$S = \coprod_{[H/K_i] \in \mathrm{Orb}(S_R)} \psi^{-1}([H/K_i]) = \coprod_{[H/K_i] \in \mathrm{Orb}(S_R)} \mathrm{Ind}_{\mathsf{stab}_H(x)}^H \psi^{-1}\left(x_{K_i}\right) = \coprod_{K_i}^{S_R} \psi^{-1}(x_{K_i}).$$

In particular, this shows that

$$\underline{\mathbb{F}}^{R \oplus R'} \subset \mathrm{Cl}_{\underline{\mathbb{F}}_R}(\underline{\mathbb{F}}_{R'}) \subset \underline{\mathbb{F}}^R \vee \underline{\mathbb{F}}^{R'},$$

proving the proposition. □

2.3.4. *Coinduction.* If it exists, the right adjoint to $\mathrm{Res}_V^W : \mathsf{wIndex}_W \to \mathsf{wIndex}_V$ is denoted CoInd_V^W.

Proposition 2.28. *Let $\underline{\mathbb{F}}_I$ be a weak indexing system. Then, $\mathsf{CoInd}_V^W \underline{\mathbb{F}}_I$ exists and is computed by*

$$\left(\mathsf{CoInd}_V^W \underline{\mathbb{F}}_I\right)_U = \left\{ S \in \mathbb{F}_U \;\middle|\; \forall\, W \leftarrow U \leftarrow U' \to V,\ \mathrm{Res}_{U'}^U S \in \mathbb{F}_{I,U'} \right\}$$

Proof. Denote by $\mathcal{C}$ the right hand side of the above equation. Note that $\mathcal{C} \subset \underline{\mathbb{F}}_W$ is the maximum full $\mathcal{T}$-subcategory such that $\mathrm{Res}_V^W \mathcal{C} \leq \underline{\mathbb{F}}_I$. Indeed, if $S \in \mathbb{F}_U - \mathcal{C}_U$, then for some $U' \to V$, we have $\mathrm{Res}_{U'}^U S \notin \mathbb{F}_{I,U'}$; thus whenever $\underline{\mathbb{F}}_J \not\leq \mathrm{Res}_V^W \mathcal{C}$, we have $\underline{\mathbb{F}}_J \not\leq \underline{\mathbb{F}}_I$. Hence it suffices to prove that $\mathcal{C}$ is a weak indexing system.

First, suppose that $S \in \mathcal{C}_U$; then, $\mathrm{Res}_{U'}^U S \in \mathbb{F}_{I,U}$ for all $U' \to V$, so $*_{U'} = \mathrm{Res}_{U'}^U *_U \in \mathbb{F}_{I_U}$ for all $U' \to V$. Hence $*_U \in \mathcal{C}_U$, i.e. $\mathcal{C}$ satisfies Condition (IS-a). Now, fix $(T_X) \in \mathcal{C}_S$ an S-tuple. What remains is to verify that for all $U' \to V$,

$$\mathrm{Res}_{U'}^U \coprod_X^S T_X \simeq \coprod_{X'}^{\mathrm{Res}_{U'}^U S} \mathrm{Res}_{X'}^{o(X')} T_{o(X')} \in \mathbb{F}_{I,U'},$$

the equivalence coming from Remark 1.37. But by assumption, we have $\mathrm{Res}_{U'}^U S$, $\mathrm{Res}_{X'}^{o(X')} T_{o(X')} \in \underline{\mathbb{F}}_I$, so this is in $\underline{\mathbb{F}}_I$ by Condition (IS-b), as desired. □

We will use this in [Ste25b] to see that $\mathsf{CoInd}_V^W A\mathcal{O} = A\mathsf{CoInd}_V^W \mathcal{O}$ for all $\mathcal{T}$-operads $\mathcal{O}^{\otimes}$.

2.4. **The color and unit fibrations.** Recall the maps c, υ, and ∇ of Lemma 1.24 and $\aleph$ of Observation 1.39. In this subsection, we study c and υ, for which we start at the following observation.

Observation 2.29. By definition, we find that c, υ, ∇, and $\aleph$ are compatible with joins, in the sense that for each $F \in \{c, \upsilon, \nabla, \aleph\}$, and set of collections $(C_\alpha)_{\alpha \in A}$ we have an equality

$$\bigcup_{\alpha \in A} F(C_\alpha) = F\left(\bigcup_{\alpha \in A} C_\alpha\right). \qquad \triangleleft$$

Much of the following work concerns *joins* and these maps, beginning with c.

2.4.1. *The color-support fibration.* We will reduce the analysis of $\mathsf{wIndex}_{\mathcal{T}}$ to the one-color case.

Proposition 2.30. *The monotone map $c\colon \mathsf{wIndex}_{\mathcal{T}} \to \mathsf{Fam}_{\mathcal{T}}$ has a fully faithful left adjoint $\underline{\mathbb{F}}_{(-)}^{\mathrm{triv}}$ and a fully faithful right adjoint $\underline{\mathbb{F}}_{(-)}$.*

Proof. By Lemma 2.14 it suffices to note that $\underline{\mathbb{F}}_{c(\underline{\mathbb{F}}_I)}^{\mathrm{triv}} \leq \underline{\mathbb{F}}_I \leq \underline{\mathbb{F}}_{c(\underline{\mathbb{F}}_I)}$ for all $\mathcal{F}$, and that $\underline{\mathbb{F}}_{\mathcal{F}}^{\mathrm{triv}} \leq \underline{\mathbb{F}}_{\mathcal{F}'}^{\mathrm{triv}}$ and $\underline{\mathbb{F}}_{\mathcal{F}} \leq \underline{\mathbb{F}}_{\mathcal{F}'}$ whenever $\mathcal{F} \leq \mathcal{F}'$. □

The following proposition additionally follows by unwinding definitions.

Proposition 2.31. *The fiber* $c^{-1}(\mathrm{Fam}_{\mathcal{T},\le\mathcal{F}})$ *is equivalent to* $\mathrm{wIndex}_{\mathcal{F}}$*, and the associated fully faithful functor* $E^{\mathcal{T}}_{\mathcal{F}}\colon \mathrm{wIndex}_{\mathcal{F}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ *is left adjoint to* $\mathrm{Bor}^{\mathcal{T}}_{\mathcal{F}}(-) := (-)\cap \underline{\mathbb{F}}_{\mathcal{F}}$ *and has values given by*

$$E^{\mathcal{T}}_{\mathcal{F}}\mathcal{C}_V = \begin{cases}\mathcal{C}_V & V\in\mathcal{F};\\ \varnothing & \text{otherwise.}\end{cases}$$

In particular, the fiber $c^{-1}(\{\mathcal{F}\})$ *is the image of* $E^{\mathcal{T}}_{\mathcal{F}}\colon \mathrm{wIndex}^{\mathrm{oc}}_{\mathcal{F}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$*.*

Finally, in order to understand cocartesian transport, we make the following observation.

Observation 2.32. Since $\mathbb{F}^{\mathrm{triv}}_{\mathcal{F},V}$ is $*_V$ when $V\in\mathcal{F}$ and empty otherwise, a finite V-set X is a $\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}}$-indexed coproduct of elements in $\mathcal{C}$ if and only if $V\in\mathcal{F}$ and $X\in\mathcal{C}_V$. In other words, we have

$$\mathrm{Cl}_{I^{\mathrm{triv}}_{\mathcal{F}}}(\mathcal{C}) = \mathrm{Bor}^{\mathcal{T}}_{\mathcal{F}}(\mathcal{C}).$$

In fact, extending this logic, if $\mathrm{Bor}^{\mathcal{T}}_{c(I)}\mathcal{C}$ is closed under I-indexed coproducts, then we have $\mathrm{Cl}_I(\mathcal{C}) = \mathrm{Bor}^{\mathcal{T}}_{c(I)}\mathcal{C}$; hence $\widehat{\mathrm{Cl}}_I(\mathcal{C}) = \mathcal{C}$. In particular, applying Proposition 2.22, we find that

$$\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}} \vee \underline{\mathbb{F}}_I = \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}} \cup \underline{\mathbb{F}}_I. \qquad \triangleleft$$

Thus, applying Remark 2.16, Propositions 2.30 and 2.31, and Observation 2.32, we arrive at the following.

Corollary 2.33. *Let* $\mathcal{T}$ *be an orbital* ∞*-category.*

(1) The map $c\colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}^{oc}_{\mathcal{F}}$ *and with cocartesian transport along* $\mathcal{F}\le\mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}'} \vee E^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I$*.*

(2) The map $c\colon \mathrm{wIndex}^{E\mathrm{uni}}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{F}}$ *and cocartesian transport along* $\mathcal{F}\le\mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}'} \vee E^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I$*.*

(3) The map $c\colon \mathrm{wIndex}^{aE\mathrm{uni}}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $c^{-1}(\mathcal{F}) = \mathrm{wIndex}^{a\mathrm{uni}}_{\mathcal{F}}$ *and cocartesian transport along* $\mathcal{F}\le\mathcal{F}'$ *sending* $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{F}'} \vee E^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I$*.*

Remark 2.34. Entailed in this corollary is the statement that $\underline{\mathbb{F}}_I$ is E-unital if and only if $\underline{\mathbb{F}}_I = E^{\mathcal{T}}_{c(I)}\mathrm{Bor}^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}_I$ and $\mathrm{Bor}^{\mathcal{T}}_{c(I)}\underline{\mathbb{F}}_I$ is unital; in particular, we find that the E-unital weak indexing systems are those which come about by applying $E^{\mathcal{T}}_{(-)}$ to unital weak indexing systems. $\triangleleft$

2.4.2. *The unit fibration.* We study the map υ using the following.

Proposition 2.35. *The map* $\upsilon\colon \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *has fully faithful left adjoint given by* $E^{\mathcal{T}}_{-}\underline{\mathbb{F}}^0_{(-)}$*.*

Proof. In view of Lemma 2.14, we're tasked with proving that $E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^0_{\mathcal{F}} \in \upsilon^{-1}(\mathrm{Fam}_{\mathcal{T},\ge\mathcal{F}})$ is initial and $\upsilon\left(\underline{\mathbb{F}}^0_{\mathcal{F}}\right) = \mathcal{F}$, both of which follow by unwinding definitions. $\square$

Once again, we would like to simplify our expression for cocartesian transport.

Observation 2.36. Let $V\in\mathcal{F}$. Note that a V-set is an S-indexed coproduct of elements of $E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^0_{\mathcal{F}}$ if and only if it is a summand of S; in particular, if $\underline{\mathbb{F}}_I$ is closed under *nonempty* summands, then $\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}^0_{c(I)} = \mathrm{Cl}_I(\underline{\mathbb{F}}^0_{c(I)})$. In this case we have

$$\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^0_{cF} = \cdots \widehat{\mathrm{Cl}}_{\underline{\mathbb{F}}_I}\widehat{\mathrm{Cl}}_{E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^0_{\mathcal{F}}}(\underline{\mathbb{F}}_I \cup E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^0_{\mathcal{F}}) = \underline{\mathbb{F}}_I \cup E^{\mathcal{T}}_{\mathcal{F}}\underline{\mathbb{F}}^0_{\mathcal{F}}.$$

In particular, if $\underline{\mathbb{F}}_I$ is aE-unital, then it is closed under nonempty summands, so this applies. $\triangleleft$

We may use this to reduce enumerative problems from the almost unital setting (or the aE-unital setting in view of Corollary 2.33) to the unital setting.

Proposition 2.37. *The restricted map* $\upsilon_a\colon \mathrm{wIndex}^{a\mathrm{uni}}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ *is a cocartesian fibration with fiber* $\upsilon_a^{-1}(\mathcal{F}) = \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{F}}$ *embedded along* $\underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{T}} \cup E^{\mathcal{T}}_{\mathcal{F}}(-)$*. Moreover, the cocartesian transport map* $t^{\mathcal{F}'}_{\mathcal{F}}\colon \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{F}} \to \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{F}'}$ *is implemented by*

$$t^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I = \underline{\mathbb{F}}^0_{\mathcal{F}'} \cup E^{\mathcal{F}'}_{\mathcal{F}}\underline{\mathbb{F}}_I$$

Proof. The property $v_a^{-1}(\mathcal{F}) = \mathrm{wIndex}_{\mathcal{F}}^{\mathrm{uni}}$ follows by unwinding definitions using Lemma 1.25. For the remaining property, we're tasked with proving that $\underline{\mathbb{F}}^0_{\mathcal{F}'} \cup E_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I \in \mathrm{wIndex}_{\mathcal{F}'}^{\mathrm{uni}}$ is the initial unital $\mathcal{F}'$-weak indexing system which embeds $\underline{\mathbb{F}}_I$ after each are embedded into $\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}}$ along $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}} \cup E_{-}^{\mathcal{T}}(-)$. Unwinding definitions, this universal property is satisfied of $\underline{\mathbb{F}}^0_{\mathcal{F}'} \vee E_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I$; thus the proposition follows from Observation 2.36. □

The fibers of the unrestricted map v have terminal objects, which are sometimes useful counterexamples.

Proposition 2.38. *Given $\mathcal{F} \in \mathrm{Fam}_{\mathcal{T}}$, the fiber $v^{-1}(\mathcal{F})$ has a terminal object computed by*

$$\mathbb{F}_{\mathcal{F}^{\perp}-nu,V} = \begin{cases} \mathbb{F}_V & V \in \mathcal{F}; \\ \mathbb{F}_V - \{S \mid \forall\, U \in \mathrm{Orb}(S), U \in \mathcal{F}\} & V \notin \mathcal{F} \end{cases}$$

Proof. We begin by noting that $\underline{\mathbb{F}}_{\mathcal{F}^{\perp}-nu}$ contains all $\mathcal{T}$-weak indexing systems with unit family $\mathcal{F}$; indeed for contradiction, if $\underline{\mathbb{F}}_J$ satisfies $v(J) = \mathcal{F}$ and there is some $S \in \mathbb{F}_{J,V} - \mathbb{F}_{\mathcal{F}^{\perp}-nu,V}$, then we must have $U \in \mathcal{F} \subset v(J)$ for all $U \in \mathrm{Orb}(S)$ and $V \notin \mathcal{F}$, so

$$\coprod_U^S \varnothing_U = \varnothing_V \in \mathbb{F}_{J,V},$$

implying that $V \in v(J) - \mathcal{F}$ (which contradicts our assumption). Thus it suffices to verify that $\underline{\mathbb{F}}_{\mathcal{F}^{\perp}-nu}$ is a $\mathcal{T}$-weak indexing system. Since it contains all contractible V-sets, it suffices to prove that it's closed under self-indexed coproducts.

Fix some $S \in \mathbb{F}_{\mathcal{F}^{\perp}-nu,V}$ and $(T_U) \in \mathbb{F}_{\mathcal{F}^{\perp}-nu,S}$. If $V \in \mathcal{F}$, then there is nothing to prove, so suppose $V \notin \mathcal{F}$. Then, note that

$$\mathrm{Orb}\left(\coprod_U^S T_U\right) = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Orb}(T_U).$$

S must contain some orbit U outside of $\mathcal{F}$, and by assumption, T_U contains an orbit outside of $\mathcal{F}$; thus $\coprod_U^S T_U$ contains an orbit outside of $\mathcal{F}$, i.e. $\coprod_U^S T_U \in \underline{\mathbb{F}}_{\mathcal{F}^{\perp}-nu}$, as desired. □

Warning 2.39. v does not admit a right adjoint, as it is not even compatible with binary joins; for instance, if $\mathcal{T} = \mathcal{O}_G$, then note that the weak indexing system $\underline{\mathbb{F}}_{\varnothing^{\perp}-nu}$ consists of all nonempty H-sets, and $E_{BG}^G \underline{\mathbb{F}}_{BG}^0$ contains only the e-sets $\{\varnothing_e, *_e\}$. Nevertheless, the join $\underline{\mathbb{F}}_{\varnothing^{\perp}-nu,V} \vee E_{BG}^G \underline{\mathbb{F}}_{BG}^0$ contains the inductions $\mathrm{Ind}_e^H \varnothing_e = \varnothing_H$, so it is equal to the complete indexing system $\underline{\mathbb{F}}_G$. Thus when G is nontrivial, we have a proper family inclusion

$$v(\underline{\mathbb{F}}_{\varnothing^{\perp}-nu}) \cup v(E_{BG}^G \underline{\mathbb{F}}_{BG}^0) = BG \subsetneq \mathcal{O}_G = v(\underline{\mathbb{F}}_{\varnothing^{\perp}-nu} \vee E_{BG}^G \underline{\mathbb{F}}_{BG}^0). \qquad \triangleleft$$

Remark 2.40. Despite Warning 2.39, v is *lax*-compatible with joins, in the sense that there is a relation

$$v(I) \cup v(J) \leq v(I \vee J);$$

this follows by simply noting that $I \vee J$ contains I and J. In particular, by Lemma 1.24, we find that joins of unital weak indexing systems are unital. ◁

Observation 2.41. Despite Warning 2.39, v is compatible with joins *on aE-unital weak indexing systems*; indeed, if $\underline{\mathbb{F}}_I$ is aE-unital, then we have

$$\underline{\mathbb{F}}_I = E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{\mathrm{triv}} \cup E_{v(I)}^{\mathcal{T}} \mathrm{Bor}_{v(I)}^{\mathcal{T}} \underline{\mathbb{F}}_I,$$

so that

$$\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^{triv} \cup E_{c(J)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(J)}^{\mathrm{triv}} \cup E_{v(I)\cup v(J)}^{\mathcal{T}} \mathrm{Bor}_{v(I)\cup v(J)}^{\mathcal{T}} \left(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J\right).$$

Thus we have

$$v(I) \cup v(J) \leq v(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) = v\left(\mathrm{Bor}_{v(I)\cup v(J)}^{\mathcal{T}} \left(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J\right)\right) \leq v(I) \cup v(J). \qquad \triangleleft$$

2.5. **The transfer system and fold map fibrations.** We further reduce our classification using $\mathfrak{R}$ and ∇.

2.5.1. *The transfer system fibration.* Recall that the monotone map $\mathfrak{R}\colon \mathrm{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ is defined by $\mathfrak{R}(I) = I \cap \mathcal{T}$; we denote the composite $\mathrm{wIndex}_{\mathcal{T}} \simeq \mathrm{wIndexCat}_{\mathcal{T}} \to \mathrm{Transf}_{\mathcal{T}}$ as $\mathfrak{R}$ as well. Given R a transfer system, define the weak indexing system

$$\overline{\underline{\mathbb{F}}}_R \coloneqq \underline{\mathbb{F}}_{\mathcal{T}}^0 \vee \mathrm{Cl}_\infty\left(\left\{\mathrm{Res}_V^W U \mid U \to W \in R,\ \ V \to W \in \mathcal{T}\right\}\right)$$

Our main statements about $\mathfrak{R}$ will be the following proposition and its immediate corollary

Proposition 2.42. *The map of posets $\mathfrak{R}\colon \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Transf}_{\mathcal{T}}$ has fully faithful right adjoint given by the composite $\mathrm{Transf}_{\mathcal{T}} \simeq \mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ and fully faithful left adjoint given by $\overline{\underline{\mathbb{F}}}_{(-)}$.*

Corollary 2.43. *If I, J are unital weak indexing categories, then*

$$\mathfrak{R}(I) \vee \mathfrak{R}(J) = \mathfrak{R}(I \vee J) \qquad \text{and} \qquad \mathfrak{R}(I) \cap \mathfrak{R}(J) = \mathfrak{R}(I \cap J).$$

We begin with an easy technical lemma concerning closures and transfer systems.

Lemma 2.44. $\mathfrak{R}\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}) = \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\mathcal{D}),1}(\mathfrak{R}\mathcal{C})$.

Proof. Since $\mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\mathcal{D}),1}(\mathfrak{R}\mathcal{C}) \subset \mathfrak{R}\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$, it suffices to prove the opposite inclusion; indeed, whenever $\coprod_U^S T_U \in \mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$ is an orbit, there is exactly one T_U which is nonempty, in which case $\mathrm{Ind}_U^V T_U = \coprod_U^S T_U$, implying that T_U is an orbit, so that $\coprod_U^S T_U \in \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\mathcal{D}),1}(\mathfrak{R}\mathcal{C})$. □

We use this to give compatibility of $\mathfrak{R}$ with joins in a restricted setting.

Lemma 2.45. *If I, J unital satisfy $\mathfrak{R}(I) \le \mathfrak{R}(J)$, then $\mathfrak{R}(I \vee J) = \mathfrak{R}(J)$.*

Proof. Note that $\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$ is closed under I-indexed induction, so we have

$$\mathfrak{R}\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J,1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J),1}(\mathfrak{R}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)) = \mathfrak{R}\mathrm{Cl}_{\mathfrak{R}(J),1}(\mathfrak{R}(J)) = \mathfrak{R}(J).$$

Iterating this and taking a union, we find that

$$\mathfrak{R}(I \vee J) = \mathfrak{R}\mathrm{Cl}_{\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J,\infty}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \mathfrak{R}(J).$$ □

We additionally note the following.

Lemma 2.46. *$\overline{\underline{\mathbb{F}}}_R$ is initial in $\mathfrak{R}^{-1}(\mathrm{Transf}_{\mathcal{T},\ge R})$ and $\mathfrak{R}\overline{\underline{\mathbb{F}}}_R = R$.*

Proof. The only nontrivial part is showing that $\mathfrak{R}\overline{\underline{\mathbb{F}}}_R = R$; in fact, this follows by unwinding definitions and applying Lemma 2.44. □

Proof of Proposition 2.42. The left adjoint is Lemma 2.46, so we're left with proving that we've constructed the right adjoint. By Lemma 2.45, the indexing category $I_{\mathcal{T}}^\infty \vee I$ satisfies $\mathfrak{R}(I_{\mathcal{T}}^\infty \vee I) = \mathfrak{R}(I)$ and is an upper bound for I. In fact, by Proposition 1.40, $I_{\mathcal{F}}^\infty \vee I$ is the *unique* indexing system with $\mathfrak{R}(I \vee I_{\mathcal{F}}^\infty) = I$, and so it is an upper bound for all J with $\mathfrak{R}(I) = \mathfrak{R}(J)$. In fact, if $\mathfrak{R}(I) \ge \mathfrak{R}(J)$, then $J \le J \vee I \le I_{\mathcal{F}}^\infty \vee I$ by the same argument, so $I_{\mathcal{F}}^\infty \vee I$ satisfies the conditions of Lemma 2.14, as desired. □

Remark 2.47. If $\mathcal{T}$ is an atomic orbital ∞-category with a terminal object V, then $2 \cdot *_V$ is not in $\overline{\underline{\mathbb{F}}}_R$ for any R, since $2 \cdot *_V$ is not a summand in the restriction of any orbital W-sets for any $W \in \mathcal{T}$; indeed, since $\mathcal{T}$ is atomic, there are no non-isomorphisms $V \to W$, so this would require that $2 \cdot *_V$ is an orbit, but it is not. Hence $\overline{\underline{\mathbb{F}}}_R$ is not an indexing system; equivalently, $\mathfrak{R}^{-1}(R)$ has multiple elements. We may interpret this as saying that unital weak indexing systems are seldom determined by their transitive V-sets. ◁

2.5.2. *The fold map fibration.* Our first statement about ∇ is the following.

Proposition 2.48. *For all unital weak indexing systems $\underline{\mathbb{F}}_I$ and $\underline{\mathbb{F}}_J$, we have $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$.*

To prove this, we work through the formula in Proposition 2.22 one step at a time.

Lemma 2.49. *Suppose $\underline{\mathbb{F}}_I$ is unital. If $\nabla(\underline{\mathbb{F}}_I), \nabla(\mathcal{C}) \le \mathcal{F}'$, then $\nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C})) \le \mathcal{F}'$.*

Proof. Suppose $V \in \nabla(\mathrm{Cl}_{\underline{\mathbb{F}}_I,1}(\mathcal{C}))$, i.e. there exists some $S \in \mathbb{F}_{I,V}$ and some $(X_U) \in \mathcal{C}_S$ such that $\coprod_U^S X_U = 2 \cdot *_V$. We would like to prove that $V \in \mathcal{F}'$. Since $\underline{\mathbb{F}}_I$ is unital, writing $S = S_{ne} \sqcup S_\varnothing$ for $S_\varnothing$ the disjoint union of S-orbits over which X_U is empty, we have $S_{ne} \in \mathbb{F}_{I,V}$ and

$$\coprod_U^S X_U = \coprod_U^{S_{ne}} X_U;$$

hence we may replace S with S_{ne} and assume that X_U is nonempty for all U.

Note that, for all $U \in \mathrm{Orb}(S)$, we have $\mathrm{Ind}_U^V X_U = m \cdot *_V$ for some $m \geq 1$; in particular, this implies $U = V$. Hence $S = k \cdot *_V$ for some $k \geq 1$. Writing our decomposition as $S = \{1, \dots, k\}$ and $X_i = m_i *_V$, we find that $2 = \sum_{i=1}^k m_i$, so either $m_i = 2$ for some i or $k = 2$. In either case, we find $V \in \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\mathcal{C}) \subset \mathcal{F}'$, as desired. □

Proof of Proposition 2.48. By Observation 2.29, we have $\nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J) = \nabla(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J)$, so we are tasked with proving the opposite inclusion. By Lemma 2.49, we find inductively that $\nabla \mathrm{Cl}_{\underline{\mathbb{F}}_I,1} \mathrm{Cl}_{\underline{\mathbb{F}}_J,1} \cdots \mathrm{Cl}_{\underline{\mathbb{F}}_J,1}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$; applying Observation 2.29 to take a union, we find that $\nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) \leq \nabla(\underline{\mathbb{F}}_I) \cup \nabla(\underline{\mathbb{F}}_J)$, as desired. □

Now we're ready to use this to show that ∇ is a cocartesian fibration.

Proposition 2.50. *The restricted map $\nabla_u \colon \mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \to \mathrm{Fam}_{\mathcal{T}}$ has fully faithful left adjoint given by $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{-}^{\mathcal{T}} \underline{\mathbb{F}}_{(-)}^\infty$ and a fully faithful right adjoint; hence it is a cocartesian fibration, and the cocartesian transport map $t_{\mathcal{F}}^{\mathcal{F}'}$ is implemented by*

$$t_{\mathcal{F}}^{\mathcal{F}'} \underline{\mathbb{F}}_I \simeq \underline{\mathbb{F}}_I \vee E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^\infty$$

Proof. First note that Observation 2.29 and Proposition 2.48 together imply that $\nabla(-)$ is compatible with *arbitrary* joins; since $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}}$ has arbitrary joins, the adjoint functor theorem recalled in Remark 2.16 implies that $\nabla(-)$ has a right adjoint. In light of Remark 2.16, it thus suffices to prove that the monotone map $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{(-)}^{\mathcal{T}} \underline{\mathbb{F}}_{(-)}^\infty$ is a fully faithful left adjoint to ∇_u, or equivalently by Lemma 2.14, that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^\infty$ is an initial element of $\nabla_u^{-1}(\mathcal{F})$.

First note that it follows from Lemma 1.25 and Observation 2.36 that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^\infty$ is a weak indexing system; additionally, it follows from Proposition 2.48 that $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^\infty \in \nabla_u^{-1}(\mathcal{F})$, i.e. it's unital and has fold family $\mathcal{F}$. Lastly, it follows from Lemma 1.25 that every unital $\mathcal{T}$-weak indexing system with fold family $\mathcal{F}$ contains $\underline{\mathbb{F}}_{\mathcal{T}}^0 \cup E_{\mathcal{F}}^{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{F}}^\infty$, as desired. □

Remark 2.51. The author is not aware of an informative formula for the right adjoint to ∇_u, but there are interesting examples; for instance, if λ is a nontrivial irreducible real orthogonal C_p-representation, then we show in Section 3.2 that $\underline{\mathbb{F}}^\lambda$ is terminal among the C_p-weak indexing systems with fold maps over the trivial subgroup. In algebra, this may be interpreted as saying that $\mathbb{E}_{\lambda\infty}$ presents the terminal sub-C_p-commutative algebraic theory prescribing a multiplication on the underlying Borel type of a genuine C_p-object, but not on genuine C_p-fixed points. ◁

We would like to compute examples with many transfers and few fold maps.

Observation 2.52. Given $V \to W$ a map in $\mathcal{T}$, write $\underline{\mathbb{F}}_{I_{\mathrm{Ind}_V^W *_V}}$ for the weak indexing system of Proposition 2.25. In view of Observation 2.36, we may compute the associated fold family as

$$\nabla\left(\mathbb{F}_{\mathcal{T}}^0 \vee \underline{\mathbb{F}}_{I_U}\right) = \left\{U \in \mathcal{T} \;\middle|\; \exists U \to W \text{ s.t. } 2 \cdot *_U \subset \mathrm{Res}_U^W \mathrm{Ind}_V^W *_V\right\},$$

Furthermore, if R is a transfer system, then Propositions 2.25 and 2.42 yield an equality

$$\overline{\underline{\mathbb{F}}}_R = \underline{\mathbb{F}}_{\mathcal{T}}^0 \vee \bigvee_{V \to W \in R} \underline{\mathbb{F}}_{I_{\mathrm{Ind}_V^W *_V}} = \bigvee_{V \to W \in R} \underline{\mathbb{F}}_{\mathcal{T}}^0 \vee \underline{\mathbb{F}}_{I_{\mathrm{Ind}_V^W *_V}};$$

thus Proposition 2.48 yields

$$\begin{aligned}\nabla \underline{\mathbb{F}}_R &= \bigcup_{V \to W \in R} \nabla\left(\underline{\mathbb{F}}_{\mathcal{T}}^0 \vee \underline{\mathbb{F}}_{I_{\mathrm{Ind}_W^{\mathcal{T}} V}}\right)\\ &= \left\{U \in \mathcal{T} \mid \; \exists U \to W \xleftarrow{f} V \text{ s.t. } f \in R \text{ and } 2 \cdot *_U \subset \mathrm{Res}_U^W \mathrm{Ind}_V^W *_V\right\}.\end{aligned}$$

We write $\mathrm{Dom}(R) := \nabla \underline{\mathbb{F}}_R$ for the above expression. ◃

We may simplify this in a number of equivariant examples.

Remark 2.53. If $\mathcal{T} = \mathcal{F} \subset \mathcal{O}_G$ is a family of normal subgroups of a finite group (e.g. any family of subgroups of a finite Dedekind group), then for every pair of proper subgroup inclusions $H, K \subset J$, the double coset formula implies that $\mathrm{Res}_K^J \mathrm{Ind}_H^J *_H = |K\backslash J/H| \cdot [H/H \cap K]$. In particular, $2 *_H \subset \mathrm{Res}_K^J \mathrm{Ind}_H^J *_H$ if and only if $H \subset K$.

Unwinding definitions, we find in this case that $\mathrm{Dom}(R)$ is the family

$$\mathrm{Dom}(R) = \left\{K \in \mathcal{F} \;\middle|\; \exists K \to H \xrightarrow{f} G,\; f \in R,\; H \neq G\right\},$$

where we conflate $[G/K]$ with K; that is, it is the family generated by domains of nontrivial transfers in R. ◃

2.5.3. *The essence fibration.* Given $\underline{\mathbb{F}}_I$ a weak indexing system, define the *essence family*

$$\epsilon(I) := \{U \in \mathcal{T} \mid U \to V \text{ s.t. } \exists \mathbb{F}_{I,V} - \{*_V\} \neq \varnothing\}$$

so that $\underline{\mathbb{F}}_I$ is aE-unital if and only if $\epsilon(I) = \upsilon(I)$. This behaves similarly to c and ∇.

Lemma 2.54. *If $\epsilon(\mathcal{C}) \subset \epsilon(\mathcal{D})$, then*

$$\epsilon\left(\widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D})\right) = \epsilon(D).$$

Proof. Fix some non-contractible V-set $T \in \mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})$, and express it as an S-indexed colimit

$$T = \coprod_U^S T_U$$

for $S \in \mathcal{C}_V$ and $(T_U) \in \mathcal{D}_S$. Since T is non-contractible, either S is non-contractible or T_U is non-contractible; either way, this implies that $V \in \epsilon(D)$, so any U mapping to V is in $\epsilon(D)$. In other words, $\epsilon\left(\widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D})\right) \subset \epsilon(D)$. The opposite inclusion follows by the fact $D \subset \widehat{\mathrm{Cl}}_{\mathcal{C},1}(\mathcal{D})$. □

Observation 2.55. For all A-indexed diagrams in $\mathrm{wIndex}_{\mathcal{T}}$, we have $\epsilon\left(\bigcup_{\alpha \in A} \underline{\mathbb{F}}_{I_\alpha}\right) = \bigcup_{\alpha \in A} \epsilon\left(\underline{\mathbb{F}}_{I_\alpha}\right)$. ◃

Proposition 2.56. *ϵ is compatible with arbitrary joins.*

Proof. ϵ is clearly compatible with unions; hence it suffices to prove that it's compatible with binary joins. In fact, we may inductively prove using Lemma 2.54 that

$$\epsilon(\overbrace{\widehat{\mathrm{Cl}}_I \widehat{\mathrm{Cl}}_J \cdots \widehat{\mathrm{Cl}}_I \widehat{\mathrm{Cl}}_J}^{2n}(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)) = \epsilon(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J) = \epsilon(\underline{\mathbb{F}}_I) \cup \epsilon(\underline{\mathbb{F}}_J);$$

taking a union as $n \to \infty$ yields the desired statement. □

We're finally ready to round up localizations to our various conditions.

Proposition 2.57. *Let $\mathcal{T}$ be an orbital ∞-category.*

(1) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{Euni}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{\epsilon(I)}^0$.
(2) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{E\mathrm{uni}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee E_{c(I)}^{\mathcal{T}} \underline{\mathbb{F}}_{c(I)}^0$.
(3) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{oc}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{triv}}$.
(4) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{a\mathrm{uni}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\epsilon(\underline{\mathbb{F}}_I)}^0$.
(5) The inclusion $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{uni}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^0$.
(6) The inclusion $\mathrm{Index}_{\mathcal{T}} \hookrightarrow \mathrm{wIndex}_{\mathcal{T}}$ is right adjoint to $\underline{\mathbb{F}}_I \mapsto \underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}$.

Furthermore, each inclusion is additionally compatible with joins.

Proof. We begin with compatibility of each condition with joins. First, note by Propositions 2.30 and 2.56 that the maps $c, \epsilon : \mathrm{wIndex}_{\mathcal{T}} \to \mathrm{Fam}_{\mathcal{T}}$ are compatible with joins, by Remark 2.40 the map υ is lax-compatible with joins, and by Proposition 2.48, ∇ is compatible with joins of unital weak indexing systems. This implies that the conditions that $c(I) = \mathcal{T}$, that $\upsilon(I) = c(I)$, that $\upsilon(I) = \mathcal{T}$, and that $\nabla(I) \cap \upsilon(I) = \mathcal{T}$ are all compatible with joins, so we are left with proving that aE-unital weak indexing systems are closed under joins. But this follows by noting whenever I, J are aE-unital that

$$\epsilon(I \vee J) = \epsilon(I) \cup \epsilon(U) = \upsilon(I) \cup \upsilon(J) = \upsilon(I \vee J)$$

in view of Observation 2.41. Thus we are left with constructing left adjoints.

We begin by proving (1). By Lemma 2.14, we are tasked with verifying that $\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{\epsilon(I)}$ is initial among aE-unital weak indexing systems $\mathcal{C}$ satisfying the property that $\underline{\mathbb{F}}_I \leq \mathcal{C}$. In fact, if $\underline{\mathbb{F}}_I \leq \underline{\mathbb{F}}_J$ and $\underline{\mathbb{F}}_J$ is aE-unital, then $\epsilon(I) \leq \epsilon(J) = \upsilon(J)$ and $c(I) \leq c(J)$, so we have $E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^{\mathrm{triv}}_{c(I)}, E^{\mathcal{T}}_{\epsilon(I)} \underline{\mathbb{F}}^0_{\epsilon(I)} \leq \underline{\mathbb{F}}_J$. Taking a join, this implies that

$$\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{\epsilon(I)} = \underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^{\mathrm{triv}}_{c(I)} \vee E^{\mathcal{T}}_{\epsilon(I)} \underline{\mathbb{F}}^0_{\epsilon(I)} \leq \underline{\mathbb{F}}_J.$$

Thus we're left with verifying that $\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{\upsilon(I)}$ is aE-unital; in fact, we have

$$\upsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{\epsilon(I)}) \geq \upsilon\left(E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{\epsilon(I)}\right) = \epsilon(I),$$

and by Proposition 2.56 we have

$$\epsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}_{\epsilon(I)^0}) = \epsilon(I).$$

Together these imply that $\epsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{\epsilon(I)}) \geq \upsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{F}}_{c(I)} \underline{\mathbb{F}}^0_{\epsilon(I)})$, so it is aE-unital, proving (1).

The proof of (2) is analogous, instead concluding the relation $\upsilon(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{c(I)}) = c(\underline{\mathbb{F}}_I \vee E^{\mathcal{T}}_{c(I)} \underline{\mathbb{F}}^0_{c(I)})$ by the same argument, replacing Proposition 2.56 with Proposition 2.30. The proof of (3) is easier, as we only need to use Proposition 2.30 to verify that $c(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\mathrm{triv}}_{\mathcal{T}}) = \mathcal{T}$ Similarly, the proof of (6) uses Proposition 2.48 and Remark 2.40 to verify that $\mathcal{T} \geq \nabla(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{T}}) \cap \upsilon(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{T}}) \geq \mathcal{T}$. (4) follows by combining (1) and (3), and (5) follows by combining (1) and (2). $\square$

2.5.4. *The combined transfer-fold fibration.* We now combine ∇ and $\Re$.

Observation 2.58. By Lemma 2.46 and Observation 2.52, if $\mathrm{Dom}(R) \not\subset \mathcal{F}$, then $\Re^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ is empty. In fact, by Proposition 2.48 and Observation 2.52 we find that $\overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}} \in \mathcal{F}^{-1}(R) \cap \nabla^{-1}(\mathcal{F} \cup \mathrm{Dom}(R))$ is *initial*; in particular the condition $\mathrm{Dom}(R) \subset \mathcal{F}$ is necessary and sufficient for $\Re^{-1}(R) \cap \nabla^{-1}(\mathcal{F})$ to be nonempty. Furthermore, this is functorial in R and $\mathcal{F}$, since $\overline{\underline{\mathbb{F}}}_R \leq \overline{\underline{\mathbb{F}}}_{R'}$ and $\underline{\mathbb{F}}^{\infty}_{\mathcal{F}} \leq \underline{\mathbb{F}}^{\infty}_{\mathcal{F}'}$ whenever $R \leq R'$ and $\mathcal{F} \leq \mathcal{F}'$. $\triangleleft$

Define the embedded subposet $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} \subset \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ spanned by the pairs $(R, \mathcal{F})$ such that $\mathrm{Dom}(R) \subset \mathcal{F}$. Note that $(\Re, \nabla)$ is compatible with joins by Propositions 2.42 and 2.48, and joins of admissible pairs are admissible; in light of Lemma 2.14, we may rephrase this together with Observation 2.58 as follows.

Proposition 2.59. *The map* $(\Re, \nabla) : \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{T}} \to \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}}$ *has image* $(\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}}$ *and factors as the following diagram of join-preserving maps*

$$\begin{array}{ccc} \mathrm{wIndex}^{\mathrm{uni}}_{\mathcal{T}} & & \\ {\scriptstyle (\Re,\nabla)}\downarrow & \searrow^{(\Re,\nabla)} & \\ (\mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}})^{\mathrm{admsbl}} & \hookrightarrow & \mathrm{Transf}_{\mathcal{T}} \times \mathrm{Fam}_{\mathcal{T}} \end{array}$$

where the lefthand vertical map admits a fully faithful left adjoint computed by $(R, \mathcal{F}) \mapsto \overline{\underline{\mathbb{F}}}_R \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}}$. *Thus the left vertical map is a cocartesian fibration with cocartesian transport computed by*

$$t^{(R', \mathcal{F}')}_{(R, \mathcal{F})} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}'}.$$

2.6. **Compatible pairs of weak indexing systems.** We finish the section with a discussion of *compatible pairs of weak indexing systems*, generalizing the setting of [BH22].

Definition 2.60. A pair of one-object weak indexing categories (I_a, I_m) is *compatible* if $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is closed under I_m-indexed products, i.e. $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathbb{F}}_{\mathcal{T}}^{I_m-\times}$ is an I_m-symmetric monoidal full subcategory. ◃

We'd like to compare these with the notions from [CHLL24b], beginning with the following.

Observation 2.61. $\mathbb{F}_{\mathcal{T}}$ is *extensive* in the sense of [CHLL24b, Def 2.2.1]. Furthermore, a subcategory $I \subset \mathbb{F}_{\mathcal{T}}$ furnishes a *span pair* $(\mathbb{F}_{c(I)}, I)$ if and only if it satisfies Condition (IC-a); thus a span pair $(\mathbb{F}_{c(I)}, I)$ is *weakly extensive* in the sense of [CHLL24a, Def 2.2.1] if and only if I is a weak indexing category. Furthermore, by Lemma 1.25, a weakly extensive pair $(\mathbb{F}_{c(I)}, I)$ is *extensive* if and only if I is an indexing category. ◃

They have their own notion of compatibility, which generalizes ours.

Observation 2.62. A bispan triple $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ whose span pairs are weakly extensive is called a *semiring context* in [CHLL24a, Def 4.1.1] when the right adjoint $f_* \colon \mathbb{F}_{\mathcal{T},/X} \to \mathbb{F}_{\mathcal{T},/Y}$ to pullback along a map $f : X \to Y$ in I_m preserves morphisms whose image in $\mathbb{F}_{\mathcal{T}}$ lies in I_a; unwinding definitions, this is precisely the condition that (I_a, I_m) is a compatible pair of one-object weak indexing systems. ◃

Note that (I_a, I_m) is a compatible pair of *indexing categories* in the sense of [BH22, Def 3.4] if and only if it is a compatible pair of weak indexing categories such that I_a and I_m are both indexing categories. In this setting, we have argued that the triple $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ is a semiring context in the sense of [CHLL24a]. This is useful, as [CHLL24a, Thm 4.2.4] yields an operadic presentation for the associated theory of *bi-incomplete Tambara functors* valued in cocomplete cartesian closed ∞-categories.

Our main contribution to this is to concretely characterize the terminal (weak) indexing category $m(I)$ such that $(I, m(I))$ is a compatible pair, generalizing [BH22, Cor 6.19].

Proposition 2.63 (Multiplicative hull). *Given $\underline{\mathbb{F}}_I$ a one-object weak indexing system, the subcategories*

$$\mathbb{F}_{m(I),V} := \{S \in \mathbb{F}_V \mid \underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{T}} \text{ is closed under } S\text{-indexed products}\}$$

form an indexing system whose corresponding indexing category $m(I)$ is characterized by the property that, for all $I_m \in \mathrm{wIndex}_{\mathcal{T}}$, the pair (I, I_m) is compatible if and only if $I_m \le m(I)$.

Proof. It follows directly from construction that $I_m \le m(I)$ if and only if (I, I_m) is compatible. Furthermore, the $*_V$-indexed product functor is the identity, so $*_V \in \mathbb{F}_{m(I),V}$ for all V. Hence it suffices to prove that $\varnothing_V \in \mathbb{F}_{m(I),V}$ for all $V \in \mathcal{T}$ and that $\underline{\mathbb{F}}_{m(I)}$ is closed under binary coproducts and self-induction.

For the first statement, empty products are terminal objects (i.e. $*_V$), so $\varnothing_V \in \mathbb{F}_{m(I),V}$ for all $V \in \mathcal{T}$. For binary coprodcuts, note that Lemma 2.19 implies that $T \sqcup T'$-indexed products are equivalently presented as simply binary products of T- and T'-indexed products, so it suffices to prove that $\mathbb{F}_{I,V}$ is closed under binary products. Indeed, by distributivity of finite products and coproducts, we have

$$S \times S' = \coprod_{U \in \mathrm{Orb}(S)} U \times S' = \coprod_U^S \mathrm{Res}_U^V S',$$

which is in $\mathbb{F}_{I,V}$ by closure under restrictions and self-indexed coproducts. For self-induction, note that

$$\begin{aligned}
\prod_U^{\mathrm{Ind}_W^V S} T_U &= \prod_{U \in \mathrm{Orb}(\mathrm{Ind}_W^V S)} \mathrm{CoInd}_U^V T_U \\
&= \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_W^V \mathrm{CoInd}_U^W T_U \\
&= \mathrm{CoInd}_W^V \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^W T_U \\
&= \mathrm{CoInd}_W^V \prod_U^S T_U;
\end{aligned}$$

if S and $\mathrm{Ind}_W^V *_W$ are in $\underline{\mathbb{F}}_{m(I)}$, then this implies that $\prod_U^{\mathrm{Ind}_W^V S} T_U \in \mathbb{F}_{I,V}$ whenever $(T_U) \in \mathbb{F}_{I,\mathrm{Ind}_W^V S}$, so $\mathrm{Ind}_W^V S \in \mathbb{F}_{m(I),V}$. In other words, $\underline{\mathbb{F}}_{m(I)}$ is closed under self-induction, as desired. □

3. Enumerative results

Having developed the main beats of the theory of (unital) weak indexing systems in Section 2, we now turn to enumerating weak indexing systems under a number of unitality assumptions. In Section 3.1, we prove Theorem B; we use this in Section 3.2 to draw a Hasse diagram for $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$. Finally, in Section 3.3, we prove Corollary C and draw a Hasse diagram for $\mathrm{wIndex}_{C_{p^2}}^{\mathrm{uni}}$.

3.1. Sparsely indexed coproducts.

The following is the heart of our enumerative efforts.

Proposition 3.1. *If $\mathcal{T}$ is an atomic orbital ∞-category and $\underline{\mathbb{F}}_I$ is an aE-unital $\mathcal{T}$-weak indexing system, then*

$$\underline{\mathbb{F}}_I = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$$

In order to show this, given S an I-admissible V-set, we let

$$\mathrm{Istrp}(S) \coloneqq \{U \in \mathcal{T}_{/V} \mid \exists \text{ summand inclusion } U \hookrightarrow S\} \subset \mathcal{T}_{/V}$$

be the *isotropy category of* S. We will make a non-canonical choice of subcategory of $\mathrm{Istrp}(S)$ along which we break S into pieces with simpler isotropy.

Lemma 3.2. *There exists a full subcategory $\overline{\mathrm{Istrp}}(S) \subset \mathrm{Istrp}(S) \subset \mathcal{T}_{/V}$ along with the data of, for each $U \in \mathrm{Istrp}(S)$, a map*

$$f_U \colon U \to e(U)$$

subject to the following conditions:

(a) $e(U) \in \overline{\mathrm{Istrp}}(S)$ for all $U \in \mathrm{Istrp}(S)$;
(b) $e(U) \not\simeq V$ unless $U \simeq V$;
(c) f_V is an isomorphism; and
(d) there exist no maps $U \to W$ in $\overline{\mathrm{Istrp}}(S)$ whenever $V \not\simeq U \not\simeq W \not\simeq V$.

Proof of Lemma 3.2. First note that $\mathrm{Istrp}(S)$ together with the identity maps $f_U = \mathrm{id}_U$ satisfies conditions Properties (a) to (c). Given $\mathcal{C} \subset \mathrm{Istrp}(S)$ a full subcategory with the data f_U satisfying conditions Properties (a) to (c), let $b(\mathcal{C}) \in \mathbb{N}$ be the number of pairs of isomorphism classes $(U,W) \in \mathcal{C}^2$ with $V \not\simeq U \not\simeq W \not\simeq V$ such that there exists a map $U \to W$; the case $b(\mathcal{C}) = b(\mathrm{Istrp}(S))$ forms the base case in an inductive argument which constructs $(\mathcal{C},(f_U))$ satisfying Properties (a) to (c) with arbitrarily small $b(\mathcal{C})$.

Fix $(\mathcal{C},(f_U))$ satisfying conditions Properties (a) to (c) and $g \colon U' \to W$ a map in $\mathcal{C}$ with $V \not\simeq U' \not\simeq W \not\simeq V$. Note that $b(\mathcal{C} - \{U'\}) < b(\mathcal{C})$; furthermore, we may endow this with the structure $(\tilde{f}_U)$ by

$$\tilde{f}_U \coloneqq \begin{cases} \mathrm{id}_U & U \in \mathcal{C} - \{U'\}; \\ g \circ f_U & e(U) = U'; \\ f_U & \text{otherwise.} \end{cases}$$

By the assumption $W \in \mathcal{C}$, $(\widetilde{f_U})$ satisfies Property (a); by the assumption that $W \not\simeq V$, $(\widetilde{f_U})$ satisfies Property (b); by construction, $(\widetilde{f_V})$ satisfies Property (c). Thus we have performed the inductive step. Repeatedly applying this, we eventually arrive at $\mathcal{C}$ with $b(\mathcal{C}) = 0$, i.e. $(\mathcal{C},(f_U))$ satisfy Properties (a) to (d), as desired. □

Once and for all, we fix $\overline{\mathrm{Istrp}}(S)$ and (f_U) as in Lemma 3.2 for all $V \in \mathcal{T}$ and $S \in \mathbb{F}_V$. Using this, we define the V-set

$$\overline{S} \coloneqq \coprod_{W \in \overline{\mathrm{Istrp}}(S)} \mathrm{Ind}_W^V *_W .$$

and for all $W \in \overline{\mathrm{Istrp}}(S)$, we define the W-set

$$S_{(\overline{W})} \coloneqq \coprod_{\substack{U \in \mathrm{Orb}(S) \\ e(U) = W}} \mathrm{Ind}_U^W *_U$$

where the inductions are taken along f_U. These participate in a sequence of equivalences

$$
\begin{aligned}
S &\simeq \coprod_{W\in\overline{\mathrm{Istrp}}(S)} \coprod_{\substack{U\in\mathrm{Orb}(S)\\ e(U)=W}} \mathrm{Ind}_U^V *_U; && (4)\\
&\simeq \coprod_{W\in\overline{\mathrm{Istrp}}(S)} \mathrm{Ind}_W^V \coprod_{\substack{U\in\mathrm{Orb}(S)\\ e(U)=W}} \mathrm{Ind}_U^W *_U; && (5)\\
&\simeq \coprod_{W}^{\overline{\mathrm{Istrp}}(S)} S_{(\overline{W})};
\end{aligned}
$$

indeed the equivalence Eq. (4) follows from Property (a), and the equivalence Eq. (5) follows from the fact that f_U is a map over V. We've shown the following.

Lemma 3.3. *There is an equivalence* $S \simeq \coprod_W^{\overline{S}} S_{(\overline{W})}$.

Property (d) then implies that this is a sparsely indexed coproduct:

Lemma 3.4. $\overline{S}$ *is a sparsely indexed summand in* S*.*

To make use of this, we utilize the following lemmas; to do so, we write $S^V \subset S$ for the maximal V-subset of S of the form $n \cdot *_V$, and we refer to orbits of S^V as *fixed points of* S*.*

Lemma 3.5. *If* $\mathcal{T}$ *is an atomic orbital* ∞*-category, the* U*-set* $\mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$ *has a fixed point.*

Proof. We have a diagram

$$
\begin{array}{ccccc}
U & & \xrightarrow{\quad = \quad} & & U \\
& \searrow & & & \\
& & \mathrm{Ind}_U^{\mathcal{T}} \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U & \longrightarrow & U \\
& & \downarrow & \lrcorner & \downarrow \\
& \overset{=}{\longrightarrow} & U & \longrightarrow & V
\end{array}
$$

Taking slices over U, the lefthand triangle establishes $*_U$ as a retract of $\mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$, i.e. it is a retract of an orbital summand $*_U \rightleftarrows S \subset \mathrm{Res}_U^V \mathrm{Ind}_U^V *_U$. By the atomic assumption, this establishes $*_U = S$, as desired. ☐

Lemma 3.6. *When* $\underline{\mathbb{F}}_I$ *is an almost essentially unital weak indexing system and* $S \in \underline{\mathbb{F}}_I$*, we have* $S_{(\overline{W})}, \overline{S} \in \underline{\mathbb{F}}_I$*.*

Proof. Note that Lemma 3.5 provides a summand inclusion

$$
\begin{array}{ccc}
S_{(\overline{W})} & \hookrightarrow & \mathrm{Res}_W^V S \\
\wr\!\!\mid & & \wr\!\!\mid \\
\coprod\limits_{\substack{U\in\mathrm{Orb}(S)\\ e(U)=W}} \mathrm{Ind}_U^W *_U & \hookrightarrow & \coprod\limits_{\substack{U\in\mathrm{Orb}(S)\\ e(U)=W}} \mathrm{Res}_W^V \mathrm{Ind}_W^V \mathrm{Ind}_U^W *_U \sqcup \coprod\limits_{W'\in\overline{\mathrm{Istrp}}(S)-\{W\}} \mathrm{Ind}_{W'}^V S_{(\overline{W})}
\end{array}
\qquad (6)
$$

In particular, $\overline{S} \subset S$ and $S_{\overline{(W)}} \subset \mathrm{Res}_W^V S$ are nonempty summands of elements of $\underline{\mathbb{F}}_I$, so they are in $\underline{\mathbb{F}}_I$ by the assumption that it is almost essentially unital. ☐

We're now ready to prove that aE-unital weak indexing systems are generated by their sparse collections.

Proof of Proposition 3.1. First note that, since $n \cdot *_V \simeq *_V \sqcup (n-1) \cdot *_V$ and $2 \cdot *_V$ is sparse, the usual inductive argument shows that $\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty} \subset \mathrm{Cl}_{\infty}\left(\underline{\mathbb{F}}_I^{\mathrm{sprs}}\right)$. Hence it suffices to prove that $\underline{\mathbb{F}}_I$ is generated under sparsely I-indexed coproducts by $\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_{\mathcal{T}}^{\infty} \cap \underline{\mathbb{F}}_I\right)$.

Fix $S \in \mathbb{F}_{I,V}$. In the case $\mathrm{Ob}\,\overline{\mathrm{Istrp}}(S) = \{V\}$, Properties (b) and (c) imply that all orbits of S are equivalent to $*_V$, so $S \in \underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_{\mathcal{T}}^{\infty} \cap \underline{\mathbb{F}}_I\right)$; in the case $\mathrm{Ob}\,\overline{\mathrm{Istrp}}(S) = \{W\}$ for some $W \not\simeq V$, then by Lemmas 3.3, 3.4 and 3.6, we may replace S with $S_{(\overline{W})}$, which is a W-set with $W \in \overline{\mathrm{Istrp}}(S)$; in other words, it suffices to prove this in the case that $\left|\mathrm{Ob}\,\overline{\mathrm{Istrp}}(S)\right| > 1$.

We will prove the membership

$$S \in \mathrm{Cl}_{\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}}\left(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}\right)\right)$$

inductively on $|\mathrm{Orb}(S)|$. Note that $|\mathrm{Orb}(S)| \geq \left|\mathrm{Ob}\,\overline{\mathrm{Istrp}}(S)\right|$, so the above argument covers the base cases $|\mathrm{Orb}(S)| \in \{0,1\}$; we argue in the case $\left|\mathrm{Ob}\,\overline{\mathrm{Istrp}}(S)\right| \geq 2$ under the inductive assumption that the statement is true for all $T \in \underline{\mathbb{F}}_{\mathcal{T}}$ with $|\mathrm{Orb}(T)| < |\mathrm{Orb}(S)|$.

In this case, by the assumption $\left|\mathrm{Ob}\,\overline{\mathrm{Istrp}}(S)\right| \geq 2$, we have $\mathrm{Ind}_W^V S_{(\overline{W})} \subsetneq S$, so in particular, we have $\left|\mathrm{Orb}\left(S_{(\overline{W})}\right)\right| < |\mathrm{Orb}(S)|$. Since $S_{(\overline{W})} \in \mathbb{F}_{I,W}$ for each W, the inductive hypothesis and Lemma 3.6 guarantee

$$S_{(\overline{W})} \in \mathrm{Cl}_{\underline{\mathbb{F}}_I^{\mathrm{sprs}}}\left(\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_I \cap \underline{\mathbb{F}}_{\mathcal{T}}^{\infty}\right)\right)$$

for each W; Lemmas 3.3, 3.4 and 3.6 then witnesses the membership

$$S \in \mathrm{Cl}_{\underline{\mathbb{F}}_I^{\mathrm{sprs}}}\left(\underline{\mathbb{F}}_I^{\mathrm{sprs}} \cup \left(\underline{\mathbb{F}}_{\mathcal{T}}^{\infty} \cap \underline{\mathbb{F}}_I\right)\right) = \mathrm{Cl}_{\infty}\left(\underline{\mathbb{F}}_I^{\mathrm{sprs}}\right),$$

as desired. □

Proof of Theorem B. By Proposition 3.1, $(-)^{\mathrm{sprs}}$ is a section of $\mathrm{Cl}_\infty(-)$ and a right adjoint; this implies that $(-)^{\mathrm{sprs}}$ is an embedding by Lemma 2.14, with image spanned by those collections $\mathcal{C}$ satisfying $\mathcal{C} \simeq \mathrm{Cl}_\infty(\mathcal{C})^{\mathrm{sprs}}$. Unwinding definitions, this is what we set out to prove. □

Corollary 3.7. *If $\mathcal{T}$ is an atomic orbital ∞-category such that $\pi_0(\mathcal{T})$ is finite and $\mathcal{T}_{/V}$ is finite as a 1-category for all $V \in \pi_0(\mathcal{T})$, then there exist finitely many $\otimes$-idempotent weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads.*

Proof. In the forthcoming work [Ste25b] we prove that the $\otimes$-idempotent weak $\mathcal{N}_\infty$-$\mathcal{T}$-operads are the essential image of $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}}$ under $\mathcal{N}^{\otimes}_{(-)\infty}$, so we're tasked with proving that $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}}$ is finite. Theorem B yields an injective map

$$\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}} \hookrightarrow \prod_{V \in \pi_0 \mathcal{T}} \mathscr{P}(\mathrm{Ob}\,\underline{\mathbb{F}}_{\mathcal{T}_{/V}}^{\mathrm{sprs}}),$$

where $\mathscr{P}(-)$ denotes the power set. By assumption, $\underline{\mathbb{F}}_{\mathcal{T}_{/V}}^{\mathrm{sprs}}$ is finite, and hence $\mathscr{P}(\mathrm{Ob}\,\underline{\mathbb{F}}_{\mathcal{T}_{/V}}^{\mathrm{sprs}})$ is finite. Since $\pi_0\mathcal{T}$ is finite, this implies that the $\mathrm{wIndex}_{\mathcal{T}}^{aE\mathrm{uni}}$ injects into a finite poset, so it is finite. □

For instance, if G is finite, then there are finitely many subgroups of G, and hence finitely many transitive G-sets; this implies that $\pi_0\mathcal{O}_G$ is finite. Furthermore, since $\mathrm{Map}([G/H],[G/K])$ is a subquotient of G, it is finite as well, so $\mathcal{O}_G$ is finite as a 1-category; more generally, $\mathcal{O}_H \simeq \mathcal{O}_{G,/[G/H]}$ is finite for all $H \subset G$. Hence Corollary 3.7 specializes to the following.

Corollary 3.8. *If G is a finite group, then there exist finitely many $\otimes$-idempotent weak $\mathcal{N}_\infty$-G-operads.*

Remark 3.9. Note that the maps $\upsilon, c, \nabla, \aleph$ all factor as

$$\begin{array}{ccccc} \mathrm{wIndex}_{\mathcal{T}} & & \xrightarrow{\quad \upsilon, c, \nabla, \aleph \quad} & & \mathcal{C} \\ {\scriptstyle -\cap \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}}\downarrow & & & & \downarrow \\ \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{sprs}}) & \hookrightarrow & \mathrm{Coll}(\underline{\mathbb{F}}_{\mathcal{T}}) & \xrightarrow[\upsilon, c, \nabla, \aleph]{} & \mathcal{D} \end{array}$$

where $(\mathcal{C}, \mathcal{D}) = (\mathrm{Transf}_{\mathcal{T}}, \mathrm{Sub}_{\mathrm{Cat}}(\mathcal{T}))$ for $\aleph$ and $(\mathrm{Fam}_{\mathcal{T}}, \mathrm{FullSub}(\mathcal{T}))$ otherwise. Using Lemma 1.25, we find that:

(1) $\aleph(\underline{\mathbb{F}}_I) = \aleph(\underline{\mathbb{F}}_I^{\mathrm{sprs}})$.
(2) $\underline{\mathbb{F}}_I$ has one color if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ has one color.
(3) $\underline{\mathbb{F}}_I$ is essentially unital if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ is essentially unital.
(4) $\underline{\mathbb{F}}_I$ is unital if and only if $\underline{\mathbb{F}}_I^{\mathrm{sprs}}$ is unital.
(5) $\underline{\mathbb{F}}_I$ is an indexing system if and only if $\upsilon(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) \cap \nabla(\underline{\mathbb{F}}_I^{\mathrm{sprs}}) = \mathcal{T}$.

In particular, we may enumerate the associated posets using Theorem B. ◁

In fact, our description in terms of sparse V-sets is not as compact as it could be.

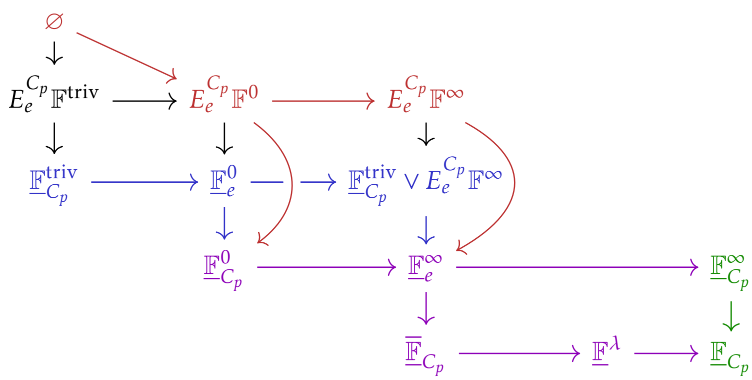
Observation 3.10. If $\underline{\mathbb{F}}_I$ is almost essentially unital and contains the sparse V-set $S = \varepsilon \cdot *_V \sqcup V_1 \sqcup \cdots \sqcup V_n$ and the transfer $U \to V_1$, then $\underline{\mathbb{F}}_I$ contains the sparse V-set $\varepsilon \cdot *_V \sqcup U \sqcup V_2 \sqcup \cdots \sqcup V_n$, as it's an S-indexed coproduct of elements of $\underline{\mathbb{F}}_I$. ◁

3.2. **Warmup: the (aE-)unital C_p-weak indexing systems.** The orbit category of the prime-order cyclic group C_p may be presented as follows:

$$\left\langle \; x \circlearrowright [C_p/e] \xrightarrow{\;r\;} *_{C_p} \;\middle|\; x^p = \mathrm{id}_{[C_p/e]}, \quad r = rx \right\rangle$$

It is easy to see that there are precisely two C_p-transfer systems: R_0 contains no transfers, and R_1 contains the transfer $e \to C_p$. Thus the poset Transf_{C_p} is $(R_0 \to R_1)$. Furthermore, there are exactly three C_p families, and the poset is $\left(\varnothing \to \{e\} \to \left\{e, C_p\right\}\right)$. We will use this to perform the following computation.

Theorem 3.11. *The poset* $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ *is presented by the following*



where $\left\{\underline{\mathbb{F}}_{C_p}^{\infty}, \underline{\mathbb{F}}_{C_p}\right\}$ *are the indexing systems,* $\left\{\underline{\mathbb{F}}_{C_p}^{0}, \underline{\mathbb{F}}_{e}^{\infty}, \overline{\underline{\mathbb{F}}}_{C_p}, \underline{\mathbb{F}}^{\lambda}\right\}$ *are the otherwise-unital weak indexing systems,* $\left\{\underline{\mathbb{F}}_{C_p}^{\mathrm{triv}}, \underline{\mathbb{F}}_{e}^{0}, \underline{\mathbb{F}}_{C_p}^{\mathrm{triv}} \vee E_e^{C_p}\mathbb{F}^{\infty}\right\}$ *are the otherwise-almost unital weak indexing systems, and* $\left\{\varnothing, E_e^{C_p}\mathbb{F}^{0}, E_e^{C_p}\mathbb{F}^{\infty}\right\}$ *are the otherwise-essentially unital weak indexing systems.*

Remark 3.12. Already, we see that none of $\mathrm{wIndex}_{C_p}^{\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{a\mathrm{uni}}$, $\mathrm{wIndex}_{C_p}^{E\mathrm{uni}}$, or $\mathrm{wIndex}_{C_p}^{aE\mathrm{uni}}$ are self-dual, since each embed the poset $\bullet \to \bullet \to \bullet \leftarrow \bullet$ as a cofamily, but none embed its dual as a family. This heavily contrasts the cases of $\mathrm{Index}_G = \mathrm{Transf}_G$ and of Fam_G, which are known to be self-dual for arbitrary abelian G by [FOOQW22].

Similarly, we may see that $\mathrm{wIndex}_{C_p}^{\mathrm{uni}} \subset \mathrm{wIndex}_{C_p}$ is a cofamily, as it consists of the elements which are at least $\underline{\mathbb{F}}_{C_p}^{0}$. However, its dual does not embed into wIndex_{C_p} as a family, since wIndex_{C_p} admits $\varnothing \to E_e^{C_p}\mathbb{F}^{\mathrm{triv}}$ as an initial sub-poset; hence wIndex_{C_p} is not self-dual either. ◁

Note that $\underline{\mathbb{F}}_{C_p}^{\infty} \subset \underline{\mathbb{F}}_{C_p}$ are C_p-indexing systems; Proposition 1.40 shows that this is the poset of indexing systems. This completely characterizes $\nabla^{-1}(\mathcal{T}) \cap \mathfrak{R}^{-1}(-)$, and we will extend this to arbitrary fibers. First, those with no transfers:

Observation 3.13. For any atomic orbital ∞-category $\mathcal{T}$, the map $\nabla \colon \mathfrak{R}^{-1}(\mathcal{T}^{\simeq}) \to \mathrm{Fam}_{\mathcal{T}}$ is an equivalence by Proposition 3.1; the fibers of this are

$$\nabla^{-1}(\mathcal{F}) \cap \mathfrak{R}^{-1}(\mathcal{T}^{\simeq}) = \left\{\underline{\mathbb{F}}_{\mathcal{F}}^{\infty}\right\}.$$ ◁

The only remaining case is $\nabla^{-1}(\{e\}) \cap \mathfrak{R}^{-1}(R_1)$. Unwinding definitions, we find that there are two options for unital sparse collections closed under applicable self-indexed coproducts with the specified transfers and fold maps; they each must have e-values given by $\{\varnothing_e, *_e, 2\cdot *_e\}$, and the two options for C_p-values are

$$\overline{\mathbb{F}}^{\mathrm{sprs}}_{C_p} = \left\{\varnothing_{C_p}, *_{C_p}, [C_p/e]\right\}, \qquad \mathbb{F}^{\lambda,\mathrm{sprs}}_{C_p} = \left\{\varnothing_{C_p}, *_{C_p}, [C_p/e], *_{C_p} \sqcup [C_p/e]\right\}.$$

Furthermore, in view of Corollary 2.4, we have $\mathrm{wIndex}^{\mathrm{uni}}_{BC_p} \simeq \mathrm{wIndex}^{\mathrm{uni}}_{*}$. Applying Example 1.31, we've arrived at the following computations:

$$\mathrm{wIndex}^{\mathrm{uni}}_{BC_p}: \qquad \underline{\mathbb{F}}^0 \longrightarrow \underline{\mathbb{F}}^\infty$$

$$\mathrm{wIndex}^{\mathrm{uni}}_{C_p}: \qquad \begin{array}{ccccc} \underline{\mathbb{F}}^0_{C_p} & \longrightarrow & \underline{\mathbb{F}}^\infty_e & \longrightarrow & \underline{\mathbb{F}}^\infty_{C_p} \\ & & \downarrow & & \downarrow \\ & & \overline{\underline{\mathbb{F}}}_{C_p} \longrightarrow \underline{\mathbb{F}}^\lambda & \longrightarrow & \underline{\mathbb{F}}_{C_p} \end{array}$$

Theorem 3.11 then follows by applying Corollary 2.33 and Proposition 2.37.

3.3. The fibers of the C_{p^n}-transfer-fold fibration.

Fix $\mathcal{T} = \mathcal{O}_{C_{p^n}}$ for some $n \in \mathbb{N}$.

Observation 3.14 ([Die09, Prop 1.3.1])**.** Fix $N \subset G$ a normal subgroup and $H \subset G$ another subgroup. Whenever $\mathrm{Map}([G/N],[G/H])$ is nonempty, evaluation at a point yields a bijection

$$\mathrm{Map}([G/N],[G/H]) \simeq G/H$$

whose right $\mathrm{Aut}_G([G/N]) \simeq G/N$-action is right multiplication by residues modulo H; furthermore, whenever $\mathrm{Map}([G/H],[G/N])$ is nonempty, it is similarly in bijective correspondence with G/N and with left G/N action given by left multiplication. In either case, the G/N action is transitive.

In particular, when $\mathcal{F} \subset \mathcal{O}_G$ is a collection of normal subgroups of G (e.g. any collection if G is a Dedekind group or an abelian group), an isomorphism-closed collection of arrows $\mathfrak{S}$ with codomains lying in $\mathcal{F}$ is determined by the corresponding inclusions $K \subset H$ such that the $\mathfrak{S}$ contains any (hence every) map $[G/K] \to [G/H]$. In this scenario, we will safely conflate these notions. ◁

Recall that when $\mathcal{F} \subset \mathcal{O}_{C_{p^n}}$ is a collection of orbits and R a C_{p^n}-transfer system, the term *R-sieves on $\mathcal{F}$* refers to subgraphs $\mathfrak{S} \subset R$ satisfying the following conditions:

(a) arrows in $\mathfrak{S}$ are closed under isomorphism;
(b) given an inclusion $K \subset H$ in $\mathfrak{S}$ and $L \subset H$ with $L \in \mathcal{F}$, the inclusion $L \cap K \subset L$ is in $\mathfrak{S}$;
(c) given an inclusion $K \subset H$ in $\mathfrak{S}$, we have $H \in \mathcal{F}$; and
(d) given inclusions $J \subset K$ in R and $K \subsetneq H$ in $\mathfrak{S}$, the composite $J \subset H$ is in $\mathfrak{S}$.

We will denote the full sub-poset of R-sieves on $\mathcal{F}$ by

$$\mathrm{Sieve}_R(\mathcal{F}) \subset \mathrm{Sub}_{\mathrm{Graph}}(R).$$

Given $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{C_{p^n}}$ an almost essentially unital weak indexing system, let $\mathscr{S}(\underline{\mathbb{F}}_I) \subset \mathfrak{R}(\underline{\mathbb{F}}_I)$ be the subgraph consisting of maps $U \to V$ with $V \in (\mathrm{Cod}(\mathfrak{R}(\underline{\mathbb{F}}_I)) - \nabla(\underline{\mathbb{F}}_I))$ such that $*_V \sqcup U \in \mathbb{F}_{I,V}$.

Proposition 3.15. *The restricted map* $\mathscr{S}\colon \mathfrak{R}^{-1}(R) \cap \nabla^{-1}(\mathcal{F}) \to \mathrm{Sub}_{\mathrm{Graph}}(\mathrm{Cod}(R))$ *is an embedding with image spanned by the R-sieves on $\mathcal{F}$.*

Proof. In view of Theorem B, a unital $\mathcal{T}$-weak indexing system lying over $(R,\mathcal{F})$ is determined by its nontrivial V-sets S such that:

- $S^V = *_V$;
- $S - S^V = U_1 \sqcup \cdots \sqcup U_n \neq \varnothing$ and there exist no maps $U_i \to U_j$ over V for $i \neq j$; and
- $V \in \mathrm{Cod}(R) - \mathcal{F}$.

In fact, since the subgroup lattice $\mathrm{Sub}_{\mathrm{Grp}}(\mathcal{O}_{C_{p^n}}) = [n+2]$ is a total order, such a sparse H-set is exactly an H-set of the form $*_H \sqcup [H/J]$ for some $J \subsetneq H$. Thus $\mathscr{S}$ is an embedding, so it suffices to characterize its image.

Condition (a) follows immediately for $\mathscr{S}(\underline{\mathbb{F}}_I)$ by the fact that $\underline{\mathbb{F}}_I$ is a full subcategory. Condition (b) follows by using the double coset formula to construct a summand inclusion $[L/L\cap K] \subset \mathrm{Res}^H_L [H/K]$, and thus a summand inclusion $*_L \sqcup [L/L\cap K] \subset \mathrm{Res}^H_L (*_H \sqcup [H/K])$. Condition (c) follows by construction. Condition (d) follows by noting that $*_H \sqcup [H/J]$ is a $*_H \sqcup [H/K]$-indexed coproduct of elements of $\underline{\mathbb{F}}_I$. Thus we've shown that $\mathrm{Im}\mathscr{S} \subset \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F})$, so it suffices to verify the opposite inclusion.

Fixing $\mathfrak{S}$ an R-sieve on $\mathrm{Cod}(R) - \mathcal{F}$, we define the collection $\underline{\mathbb{F}}_{\mathfrak{S}}$ by its values

$$\mathbb{F}_{\mathfrak{S},H} = \{S \mid \forall [H/K] \in \mathrm{Orb}(S),\ K \subset H \in R\}$$

when $H \in \mathcal{F}$, and

$$\begin{aligned}\mathbb{F}_{\mathfrak{S},H} = &\left\{ \coprod_i n_i \cdot [H/K_i] \;\middle|\; \forall i,\quad n_i \in \mathbb{N},\text{ and } K_i \subsetneq H \in R \right\} \\ &\cup \left\{ *_H \sqcup \coprod_i n_i \cdot [H/K_i] \;\middle|\; \forall i,\quad n_i \in \mathbb{N},\text{ and } K_i \subsetneq H \in \mathfrak{S} \right\}\end{aligned}$$

when $H \notin \mathcal{F}$. These are full subcategories by Condition (a), and they are restriction-stable (hence a full G-subcategory) by Condition (b). Furthermore, it follows immediately by definition that $\nabla(\underline{\mathbb{F}}_{\mathfrak{S}}) = \mathcal{F}$, that $\mathfrak{R}(\underline{\mathbb{F}}_{\mathfrak{S}}) = R$, that $v(\underline{\mathbb{F}}_{\mathfrak{S}}) = \mathcal{O}_{C_{p^n}} = c(\underline{\mathbb{F}}_{\mathfrak{S}})$, and by Condition (c) that $\mathscr{S}(\underline{\mathbb{F}}_{\mathfrak{S}}) = \mathfrak{S}$, so to conclude that $\underline{\mathbb{F}}_{\mathfrak{S}} \in \mathscr{S}^{-1}(\mathfrak{S})$ (and hence the proposition), it remains to show that $\underline{\mathbb{F}}_{\mathfrak{S}}$ is closed under self-indexed coproducts.

The cases $H \in \mathcal{F}$ or $T^H = \varnothing$. In either of these cases, we're tasked with proving that the orbital summands of T lie in $R_{/H}$. In any case, all orbital summands of T_{K_i} lie in $R_{/K_i}$ by assumption; since the orbital summands of S lie in $R_{/H}$ by assumption, all orbital summands of T are then R-indexed inductions of orbital summands of T_{K_i}. Unwinding definitions, we've argued that any orbital summand $[H/J]$ has structure map factoring as a composite $J \subset K \subset H$ of inclusions in R, so $J \subset H$ is in R, which is what we were trying to show.

The case $H \notin \mathcal{F}$ and $T^H \neq \varnothing$. Write $T = \coprod_{K_i}^S T_{K_i}$. Since T has a fixed point, S must as well; the decomposition $S = *_H \sqcup S'$ yields a decomposition $T = T_H \sqcup T'$ where $T_H \in \underline{\mathbb{F}}_{\mathfrak{S}}$ and T' is a coproduct of nontrivial $\mathfrak{S}$-indexed inductions of elements of $R_{/K_i}$. In particular, T' is fixed-point free, so $T^H = T_H^H \sqcup (T')^H = T_H^H = *_H$.

Fix $[H/K] \subset T$ a nontrivial orbital summand. We're tasked with proving that $K \subset H$ lies in $\mathfrak{S}$. The inclusion $[H/K] \subset T$ factors through an inclusion $[H/K] \subset T_H$ or $[H/K] \subset T'$. In the case $[H/K] \subset T_H$, the claim follows by unwinding definitions since $T_H \in \mathbb{F}_{S,H}$ has a fixed point. In the case $[H/K] \subset T'$, orbital summands of T' are nontrivial $\mathfrak{S}$-indexed inductions of $[K/J]$ for $J \subset K$ in R; hence they correspond with compositions $J \subset K \subsetneq H$, which lies in $\mathfrak{S}$ since $K \subsetneq H$ is in $\mathfrak{S}$ and $\mathfrak{S}$ is closed under precomposition with maps in R by Condition (d). To summarize, we've shown that $T^H = *_H$ and the nontrivial orbital summands of T lie in $\mathfrak{S}_{/H}$, so $T \in \mathbb{F}_{\mathfrak{S},H}$, and we are done. □

Proof of Corollary C. In view of [BBR21, Thm 25], the combined transfer-fold fibration has signature

$$(\mathfrak{R}, \nabla)\colon \mathrm{wIndex}^{\mathrm{uni}}_{C_{p^n}} \to K_{n+2} \times [n+2].$$

After Propositions 2.59 and 3.15, we've identified the fibers and proved that the restricted map is a cocartesian fibration. Thus it suffices to understand cocartesian transport, which is implemented by

$$t_{(R,\mathcal{F})}^{(R',\mathcal{F}')} \underline{\mathbb{F}}_I = \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \vee \underline{\mathbb{F}}^{\infty}_{\mathcal{F}'}$$

by Proposition 2.15, in terms of R-sieves. When $R = R'$, it is clear that this is given by the restriction $\mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \twoheadrightarrow \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}')$, so it suffices to characterize this in the case $\mathcal{F} = \mathcal{F}'$. Unwinding definitions, we're tasked with characterizing for which $K \hookrightarrow H$, we have

$$*_H + [H/K] \in \underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}.$$

Let $t_R^{R'}\colon \mathrm{Sieve}_R(\mathrm{Cod}(R) - \mathcal{F}) \hookrightarrow \mathrm{Sieve}_{R'}(\mathrm{Cod}(R') - \mathcal{F})$ be the map sending an R-sieve $\mathfrak{S}$ to the R'-sieve whose non-isomorphisms are the composites $J \subset K \subsetneq H$ with $K \subsetneq H \in \mathfrak{S} - \mathfrak{S}^{\simeq}$ and $J \subset K \in R'$. On one hand, note that, for all $J \subset K \subsetneq H$ in $t_R^{R'}\mathfrak{S}$, we have

$$*_H \sqcup [H/J] = *_H \sqcup \mathrm{Ind}_K^H [K/J],$$

i.e. $*_H \sqcup [H/J]$ is a $*_H \sqcup [H/K]$-indexed coproduct of elements of $\overline{\underline{\mathbb{F}}}_{R'}$; unwinding definitions, this implies that $\mathscr{S}\left(\underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}\right) \geq t_R^{R'} \mathscr{S}(\underline{\mathbb{F}}_I)$.

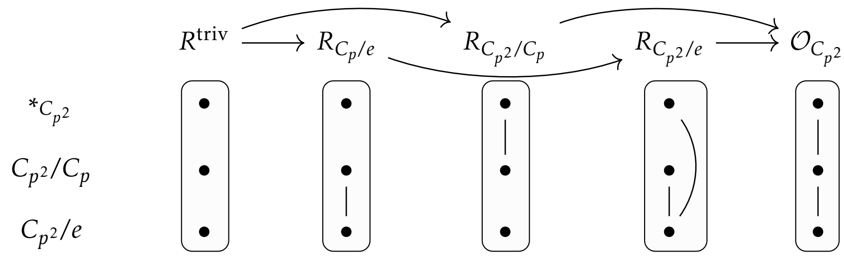
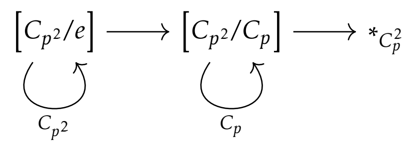


Figure 1. Pictured is the result of Rubin and Balchin-Barnes-Rotzheim's computation of $\mathrm{Transf}_{C_{p^2}}$.

On the other hand, note that $\underline{\mathbb{F}}_{t_R^{R'}\mathscr{S}(\underline{\mathbb{F}}_I)}$ is a unital weak indexing system containing both $\underline{\mathbb{F}}_I$ and $\overline{\underline{\mathbb{F}}}_{R'}$; this implies that $\underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'} \leq \underline{\mathbb{F}}_{t_R^{R'}\mathscr{S}(\underline{\mathbb{F}}_I)}$, so applying $\mathscr{S}$ together with the above inequality yields $\mathscr{S}\left(\underline{\mathbb{F}}_I \vee \overline{\underline{\mathbb{F}}}_{R'}\right) = t_R^{R'}\mathscr{S}(\underline{\mathbb{F}}_I)$, which is what we set out to prove. □

We finish by drawing this out for $n = 2$. We may illustrate $\mathcal{O}_{C_{p^2}}$ as follows



with $\mathrm{Map}([C_{p^2}/e],[C_{p^2}/C_p])$ a C_p-torsor and $\mathrm{Map}([C_{p^2}/C_p],*_{C_{p^2}}) = *$. The independent computations of [BBR21; Rub21] verify the that $\mathrm{Transf}_{C_{p^2}}$ agrees with Fig. 1.

Given $R \in \mathrm{Transf}_{C_{p^2}}$, we let $\underline{\mathbb{F}}_R$ be the corresponding indexing system. We will use Corollary C to compute $\mathrm{wIndex}^{\mathrm{uni}}_{C_{p^2}}$, which we will populate with examples from real representation theory. First, a simple lemma.

Lemma 3.16. *For all n and all real orthogonal C_{p^n}-representations V, the element*

$$\underline{\mathbb{F}}^V \in \aleph^{-1}\left(\aleph\underline{\mathbb{F}}^V\right) \cap \nabla^{-1}\left(\nabla\left(\underline{\mathbb{F}}^V\right)\right)$$

is terminal.

Proof. The observation $\mathrm{Conf}^H_{*_H+S}(V) = \mathrm{Conf}^H_S(V - \{0\})$ implies that $\mathscr{S}\left(\underline{\mathbb{F}}^V\right)$ is the complete $\aleph\left(\underline{\mathbb{F}}^V\right)$-sieve on $\mathrm{Cod}\left(\aleph\left(\underline{\mathbb{F}}^V\right)\right) - \nabla\left(\underline{\mathbb{F}}^V\right)$, so this follows from Corollary C. □

In view of this, to compute the position of $\underline{\mathbb{F}}^V$ in the classification of Corollary C, we need only compute its transfers and fold maps. Fix a distinguished generator $x \in C_{p^2}$.

Example 3.17. Let $\lambda_{C_{p^2}}$ be a 2-dimensional real orthogonal C_{p^2}-representation wherein x acts by a rotation of order p^2. Then, both $\lambda_{C_{p^2}}$ and $\mathrm{Res}^{C_{p^2}}_{C_p} \lambda_{C_{p^2}}$ have 0-dimensional fixed points, so they do not embed $2 \cdot *_{(-)}$; hence

$$\nabla\left(\underline{\mathbb{F}}^{\lambda_{C_{p^2}}}\right) = \{e\}.$$

The non-fixed points of $\lambda_{C_{p^2}}$ have orbit type $[C_{p^2}/e]$ and the non-fixed points of $\mathrm{Res}^{C_{p^2}}_{C_p} \lambda^{C_{p^2}}$ have orbit type $[C_p/e]$; together these imply that

$$\aleph\left(\underline{\mathbb{F}}^{\lambda_{C_{p^2}}}\right) = R_{C_{p^2}/e}$$

as in Fig. 1. ◁

Example 3.18. Similarly to Example 3.17, let λ_{C_p} be an irreducible C_{p^2}-representation wherein x acts by a rotation of order p; this is 1-dimensional (and the sign representation) if $p = 2$, and 2-dimensional if $p > 2$. Note that λ_{C_p} has 0-dimensional fixed points, but $\mathrm{Res}_{C_p}^{C_{p^2}} \lambda_{C_p}$ is trivial; hence

$$\nabla\left(\underline{\mathbb{F}}^{\lambda_{C_p}}\right) = \left\{e, C_p\right\}.$$

Furthermore, the orbit type of non-fixed points in λ_{C_p} is $[C_{p^2}/C_p]$; this implies that

$$\mathfrak{K}\left(\underline{\mathbb{F}}^{\lambda_{C_{p^2}}}\right) = R_{C_{p^2}/C_p}$$

as in Fig. 1. ◁

Note that $\overline{\underline{\mathbb{F}}}_R$ corresponds with the minimal R-sieve on $\mathrm{Cod}(R) - \mathrm{Dom}(R)$. Together with Examples 1.29, 3.17 and 3.18, this completely characterizes the image of the join generators of Fig. 2 under $(\mathfrak{K}, \nabla, \mathscr{S})$; since $\mathfrak{K}, \nabla, \mathscr{S}$ are compatible with joins, this completely characterizes the image of the entirety of Fig. 2 under $(\mathfrak{K}, \nabla, \mathscr{S})$. In fact, this is everything.

Corollary D. *The poset of unital C_{p^2}-weak indexing systems is presented by Fig. 2.*

What remains is to verify that Fig. 2 bijects onto the Sieve posets of Corollary C and that cocartesian transport as described by Corollary C is implemented by horizontal arrows. Cocartesian transport will follow simply by unwinding definitions.

When $R = \mathbb{F}_{C_{p^2}}^{\simeq}$ or $\mathcal{F} = \mathcal{O}_{C_{p^2}}$, the fibers are one point by Proposition 1.40 and Observation 3.13. The remaining one-point fiber $\mathfrak{K}^{-1}(R_{C_p/e}) \cap \nabla^{-1}(\left\{e, C_p\right\})$ is trivial since $\mathrm{Cod}(R_{C_p/e}) \subset \mathcal{F}$. The empty fibers in Fig. 2 follow from Corollary C,

The two-point fibers all follow from a similar consideration, which we may exemplify in the case $R = R_{C_{p^2}/C_p}$ and $\mathcal{F} = \left\{e, C_p\right\}$. In this case, the only orbit in $\mathrm{Cod}(R) - \mathcal{F}$ is $*_{C_{p^2}}$, and the only R-transfer with codomain $*_{C_{p^2}}$ is $C_p \subset C_{p^2}$. Thus there are exactly two R-sieves on $\mathrm{Cod}(R) - \mathcal{F}$, depending on whether or not they contain a transfer. The reader may easily verify that the other two-point fibers in Fig. 2 each also have only one applicable transfer.

The first example of a three-point fiber is $R = R_{C_{p^2}/e}$ and $\mathcal{F} = \{e\}$. In this instance, $\mathrm{Cod}(R) - \mathcal{F} = \left\{[C_{p^2}/C_p], *_{C_{p^2}}\right\}$, and all non-isomorphisms in R have codomain lying in $\mathrm{Cod}(R) - \mathcal{F}$; thus we are enumerating restriction and $R_{C_{p^2}/e}$-precomposition-closed subsets of $\left\{e \subset C_p, e \subset C_{p^2}\right\}$. In fact, there are no applicable precompositions, so the only condition comes from the fact that the restriction of $e \subset C_{p^2}$ to C_p is $e \subset C_p$, i.e. any R-sieve containing $e \subset C_{p^2}$ is complete. Thus there are three R-sieves on $\mathrm{Cod}(R) - \mathcal{F}$: the empty sieve, the complete sieve, and $\left\{e \subset C_p\right\}$.

The other example of a three-point fiber is $R = \mathcal{O}_{C_{p^2}}$ and $\mathcal{F} = \left\{e, C_p\right\}$. In this instance, $\mathrm{Cod}(R) - \mathcal{F} = \left\{*_{C_{p^2}}\right\}$, so we are considering restriction and precomposition-closed subsets of $\left\{e \subset C_{p^2}, C_p \subset C_{p^2}\right\}$. The only relevant condition is the precomposition condition; since $e \subset C_p$ is in R, if $C_p \subset C_{p^2}$ is in $\mathfrak{S}$, then $e \subset C_{p^2}$ is in $\mathfrak{S}$. Thus there are three R-sieves on $\mathrm{Cod}(R) - \mathcal{F}$: the empty sieve, the complete sieve, and $\left\{e \subset C_{p^2}\right\}$.

3.4. **Questions and future directions.** To stimulate further development in this area, we now pose a litany of questions concerning the structure and tabulation of weak indexing systems. The first arose to the author out of consternation concerning the apparent lack of structure arising in Fig. 2.

Question 3.19. Is there a closed form expression for $\mathrm{wIndex}_{\mathcal{O}_{C_{p^n}}}^{\mathrm{uni}}$ or $\left|\mathrm{wIndex}_{\mathcal{O}_{C_{p^n}}}^{\mathrm{uni}}\right|$? ◁

The author believes that, akin to the strategy employed in [BBR21], this may be solved by characterizing change-of-group functors such as restriction, Borelificaiton, and inflation. In particular, given $H \subset G$ a subgroup, the cofamily $\mathcal{O}_{G/H}$ consisting of transitive G-sets on which H acts trivially is an atomic orbital ∞-category, so it possesses a well-defined theory of weak indexing systems, which should participate in an adjunction

$$\mathrm{Infl}_H^G \colon \mathrm{wIndex}_{G/H} \rightleftarrows \mathrm{wIndex}_G \colon F_H^G,$$

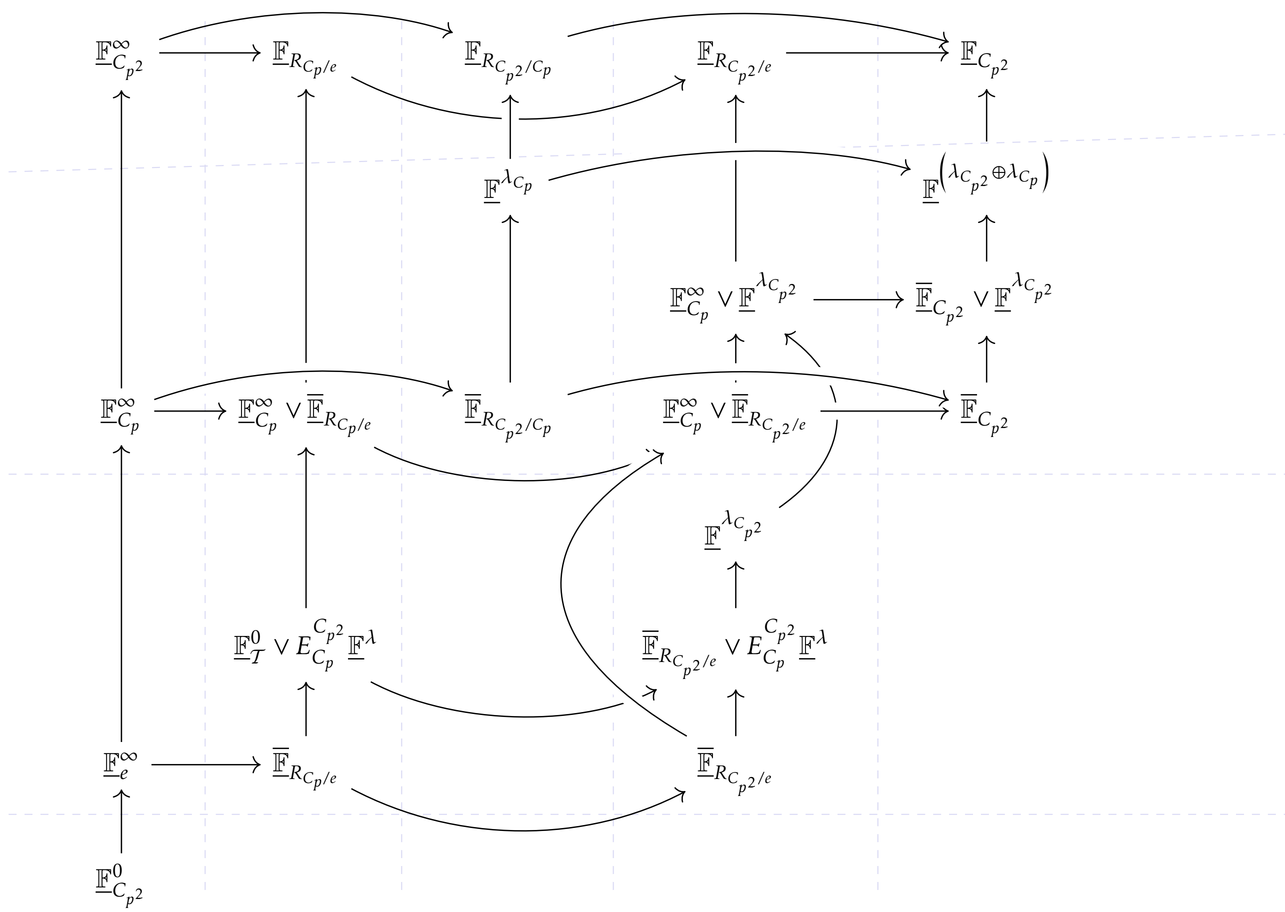


Figure 2. Pictured is a Hasse diagram for the poset of unital C_{p^2}-weak indexing systems. Dashed lines separate the fibers of the cocartesian fibration $(\aleph, \nabla)$.

where F_H^G metaphorically represents "fixed points with residual genuine $W_G(H)$-action," and literally sends $\underline{\mathbb{F}}_I$ to a $\mathcal{O}_{G/H}$-weak indexing system satisfying $F_H^G \mathbb{F}_{I,V} = \mathbb{F}_{I,V}$ for all $V \in \mathcal{O}_{G/H} \subset \mathcal{O}_G$. In the setting where $N \subset G$ is normal, $\mathcal{O}_{G/N}$ is canonically equivalent to the orbit category for the group G/N, so given a choice of a *normal* subgroup, this produces an inductive procedure: characterize $\mathcal{O}_G$ weak indexing systems by picking a normal subgroup and inductively characterizing weak indexing systems for $\mathcal{O}_{G,\geq N}$ (related to $\mathcal{O}_N$ by Proposition 2.3), weak indexing systems for $\mathcal{O}_{G/N}$, and the possible transfers from outside $\mathcal{O}_{G/N}$ to inside (as well as the possible additional data of H-sets S for which N acts trivially on G/H but not on $G/\mathsf{stab}_H(x)$ for all $x \in S$).

Outside of closed form expressions, the following question is evident as an extension of Corollary C.

Question 3.20. Is there a good combinatorial expression of $\nabla^{-1}(\mathcal{F}) \cap \aleph^{-1}(R)$ over an arbitrary abelian, dedekind, nilpotent, or general finite group? ◁

The author expects that our techniques may be extended to a similar sieve-based presentation for $\nabla^{-1}(\mathcal{F}) \cap f R^{-1}(R)$ over more general families of groups.

Another question arises by looking closely at Corollary D; we were able to tabulate all 21 unital C_{p^2}-weak indexing systems using only the examples $\underline{\mathbb{F}}_R$, $\overline{\underline{\mathbb{F}}}_R$, and $\underline{\mathbb{F}}^V$ together with joins and the functors $E_{(-)}^{C_{p^2}}$.[6] Thus we ask the following.

[6] To see this, note that $\underline{\mathbb{F}}^0_G$ is the arity support of the 0 G-representation and $\underline{\mathbb{F}}^{\infty}_G$ is the arity support of any positive-dimensional trivial G-representation.

Question 3.21. Which unital weak indexing systems are realizable via tensor products of $\left\{\mathbb{F}^V\right\}$ under various change of group functors? ◁

In particular, all recorded instances of the right adjoint to ∇ occur as the arity support $\underline{\mathbb{F}}^V$ of an $\mathbb{E}_V$-G-operad, so we ask the following.

Question 3.22. What is the right adjoint to ∇? Is it related to $\mathbb{E}_V$? ◁

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