Rushil Mallarapu — ∞-Operads

April 7^{th}, 2022

$(\infty, 1)$-Seminar — SC 232

Intro & Outline

- Thank organizers & attendees
- So the goal is for today to cover the def'n & intuition for ∞-operads
- This means its going to mostly be definitions, but that's just how HA 2.1 is.
- Before I get into it, let me say something about the problem we're trying to solve:

<u>Ex.</u> Tensor Products

- Do <u>not</u> arise via some "product" - construction is noncanonical.
- We actually specify $U \otimes V \longrightarrow W$; this is enough!
  - Easily find $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$!
- So, instead of specifying operation, lets specify (set) of maps these operations represent

2.0.0.1 <u>Def</u> Let $(C, \otimes)$ by SMC. We define category $C^{\otimes}$ w/

1. obj: n-tuples of obj. of C, e.g. $[C_1, \ldots, C_n]$
2. morphism: $[C_1, \ldots, C_n] \longrightarrow [C_1', \ldots, C_m']$ is
   - (a) partial map $\alpha : \{\ast, 1, 2, \ldots, n\} \to \{\ast, 1, 2, \ldots, m\}$ $(\alpha(\ast) = \ast)$
   - (b) maps $\{ f_j : \bigotimes_{i \in \alpha^{-1}(j)} C_i \longrightarrow C_j' \}_{1 \leq j \leq m}$
3. Composition

$$\begin{tikzcd} {[C_n]} \arrow[rr, "gf"] \arrow[dr, "f"', "{(\alpha, f_i)}"'] & & {[C''_\ell]} \\ & {[C'_m]} \arrow[ur, "g"', "{(\beta, g_i)}"'] & \end{tikzcd}$$

via $(\beta\alpha)$, & maps

$$\bigotimes_{(\beta\alpha)^{-1}k} C_i \simeq \bigotimes_{j \in \beta^{-1}(k)} \bigotimes_{i \in \alpha^{-1}(j)} C_i \xrightarrow{f_j} \bigotimes_{j \in \beta^{-1}(k)} C_j' \xrightarrow{g_j} C_k''$$

for $1 \le k \le \ell$.

- Really, we should think of such a category as coming w/ a natural functor to Fin_* — pointed sets!

2.0.0.2 **Def** The category $\underline{Fin_*}$ has objects pointed sets $\langle n \rangle = \{*, 1, 2, \dots, n\}$ & maps which preserve the basept. Think of $*$ as a Maybe type.

- Also, $p^i : \langle n \rangle \to \langle 1 \rangle$, $1 \le i \le n$ def'n $p^i(j) = \begin{cases} * & j \ne i \\ 1 & j = i \end{cases}$

- Punchline: $\mathcal{C}^\otimes \longrightarrow Fin_*$ is an op-fibration, which effectively means we can take an object $[C_n] \in \mathcal{C}^\otimes$, map $f : \langle n \rangle \to \langle m \rangle$, & lift universally to a map $[C_n] \to [C'_m]$.
  - → via F giving isos $C'_j \simeq \bigotimes_{f^{-1}(j)} C_i$ $\forall$ $1 \le j \le m$.
  - → maps in Fin "determine" higher composition & coherence
- In fact, we can make this work backward, & get an SMC out of an op-fibration $\mathcal{D} \to Fin_*$

  (Work out) $\underset{\text{unit}}{\mathcal{D}_0}$, $\mathcal{D}_2 \to \mathcal{D}_1$, symm $\sigma : [1,2] \to [2,1]$
- Takeaway is that this is the better perspective on organizing a symmetric monoidal struct on ∞-cats, & the point of ∞-operads is to encapsulate all this data!

1: Operads & ∞-Operads

- First, some classical review:

2.1.1.1 **Def** A typed operad (colored/multicat) $\mathcal{O}$, is the following:

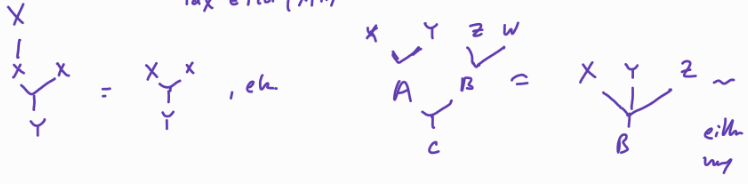
① Collection of types $X, Y, \dots$ "objects"

② $\forall$ finite sets I, set $Mul(X_I, Y)$ $\quad X_I = \{X_i\}_{i \in I}$

③ Composition map $\prod_J Mul(X_{I_j}, Y_j) \times Mul(Y_J, Z) \longrightarrow Mul(X_I, Z)$

④ Units, & associativity!

$id_X \in Mul(X, X)$



2.1.1.- Variants:

- Simplicial operad - Mul is simplicial set
- Operadification $\dashv$ $(\mathcal{C}\mathrm{Op} \mapsto \mathcal{C}\mathrm{at})$ via $\mathrm{Mul}(X_{\cdots}, Y) = \begin{cases} \mathrm{Hom}(X_1, Y) & n=1 \\ \emptyset & \text{else} \end{cases}$
- By above discussion, if $\mathcal{C}$ is SMC,

  i.e. $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, then $\mathcal{C}$ is operad via

  $\mathrm{Mul}(X_I, Y) = \mathrm{Hom}(\otimes_I X_i, Y)$ $\to$ recover $\otimes$ via Yoneda!
- Operad is typed operad w/ 1 color! $\mathcal{O}(n) \to \mathcal{O}(m)$!
- We can repeat prev. construction to get $\mathcal{O}^{\otimes}$, w/

  some punchline. $\to$ flip definition!

<u>Def</u> Say $f: \langle n \rangle \to \langle m \rangle$ <u>inert</u> if, $i \neq *$, $f^{-1}(i)$ is singleton.

2.1.1.10 <u>Def</u> An <u>∞-operad</u> is a functor $p: \mathcal{O}^{\otimes} \longrightarrow N(\mathrm{Fin}_*)$ s.t.

1. $\forall f: \langle m \rangle \to \langle n \rangle$ inert & $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, then is a

   p-coCart lift $\bar{f}: C \to C'$; functor $\mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$.

2. $C \in \mathcal{O}_{\langle m \rangle}$, $C' \in \mathcal{O}_{\langle n \rangle}$, $f: \langle m \rangle \to \langle n \rangle$, $\mathrm{Map}^f(C, C')$ ctd comp.

   of $\mathrm{Map}(C, C')$ over f; choose p-coCart maps

   $C' \to C_i'$ over $p^i: \langle n \rangle \to \langle 1 \rangle$, then induced

   $$\mathrm{Map}^f(C, C') \longrightarrow \prod_{1 \le i \le n} \mathrm{Map}^{p^i f}(C, C_i')$$ is

   htpy equiv.

3. $\forall$ finite $C_1, C_2, \dots, C_n \in \mathcal{O}_{\langle 1 \rangle}$, $\exists$ obj. $C \in \mathcal{O}_{\langle n \rangle}$

   & $p$cc's $C \to C_i$ over p^i!

<u>Remarks</u>

- Usually just write $\mathcal{O} := \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ as underlying ∞-cat,

  & $\mathcal{O}^{\otimes}$ for ∞-operad.

2.1.1.14 - Get canonical equiv. $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^n$

  $\to$ obj. in $\mathcal{O}^{\otimes}$ are seq. of obj. in $\mathcal{O}$.

2.1.1.13 - $p: \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ is fibration (HTT magic)

  $\hookrightarrow$ HTT 2.3.1.5, 2.4.6.5, 2.4.1.5

<u>Ex.</u>

2.1.1.18-20

- $\mathrm{Comm}^{\otimes} = N(\mathrm{Fin}_*) \xrightarrow{id} N(\mathrm{Fin}_*)$ — "commutative ∞-operad"
- $\mathrm{Fin}_*^{inj} \subset \mathrm{Fin}_*$; $\mathbb{E}_0^{\otimes} := N(\mathrm{Fin}_*^{inj})$ — "unital ∞-operad"
- $\mathrm{Triv}: (\mathrm{Fin}_*, \mathrm{inert})$; $\mathcal{T}\mathrm{riv}^{\otimes} := N(\mathrm{Triv})$ - "trivial ∞-operad"

... nerve of colored operads.

- Get all usual generalizations to simp. operads: we can repeat Segal constr., & take simplicial nerve;

2.1.1.23 This is <u>operadic nerve</u> $N^{\otimes}(\mathcal{O})$

- Say typed op. is fibrant if all mult-sets fibrant.

2.1.1.27 <u>Prop</u> | If $\mathcal{O}$ is fibrant simplicial typed op, then $N^{\otimes}(\mathcal{O})$ is ∞-operad!

<u>Pf.</u> (1) $\mathcal{O}^{\otimes}$ fibrant simp. cat, so $N^{\otimes}(\mathcal{O})$ is ∞-cat.

If $C = (m, [C_1, \dots, C_m]) \in N^{\otimes}(\mathcal{O})$, $\alpha : \langle m \rangle \to \langle n \rangle$ inert, then have canonical map

$$C \longrightarrow C' = (\langle n \rangle, [C_{\alpha^{-1}(i)}]), \text{ which}$$

is edge $\tilde{\alpha}$ over α.

By HTT 2.4.1.10, $\tilde{\alpha}$ is p-coCart! $(p : N^{\otimes}(\mathcal{O}) \to N(Fin_*))$

(2) Specifically, we get p-coCart maps $\tilde{\alpha}^{i,m} : C \to (\langle 1 \rangle, C_i)$ covering ρ^i! We want them to define p-limit diagram $\langle m \rangle^{\circ \triangleleft} \longrightarrow N^{\otimes}(\mathcal{O})$; i.e. $\forall\ D = (n, [D_n])$ in $N\mathcal{O}$, $\beta : \langle n \rangle \to \langle m \rangle$, canonical map

$$\mathrm{Map}^{\beta}(D, C) \longrightarrow \prod_{1 \le i \le m} \mathrm{Map}^{\rho^i \beta}(D, C_i) \text{ is htpy}$$

equiv; this is iso of sSets!!

(3) Finally, need functors $\rho^i_!$ to induce essentially surj. map

$$N^{\otimes}(\mathcal{O}) \times_{N(Fin_*)} \{\langle m \rangle\} \longrightarrow \prod_{1 \le i \le m} N(\mathcal{O})$$

which works bc $N^{\otimes}(\mathcal{O}) \times_{N(Fin_*)} \{\langle m \rangle\}$ is canonically iso to $N(\mathcal{O})^m$! ∎

2: ∞-Operad Maps

2.1.2.1 <u>Def</u> | A morphism $f : \langle n \rangle \to \langle m \rangle$ is <u>active</u> if $f^{-1}\{*\} = \{*\}$

- In Fin_*:

$$\begin{tikzcd} \langle n \rangle \arrow[rr, "f"] \arrow[dr, "f''"'] & & \langle m \rangle \\ & \cdot \arrow[ur, "f'"'] & \end{tikzcd}$$

inert quotient   active inj.

2.1.2.3 <u>Def</u> | Let $p : \mathcal{O}^{\otimes} \to N(Fin_*)$ be ∞-operad. $f \in \mathcal{O}^{\otimes}$ is

① <u>inert</u> if $p(f)$ inert, f p-coCart

② <u>active</u> if $p(f)$ active.

- This determines factorization system on $\mathcal{O}^{\otimes}$ via abstract ∞-nonsense allowing us to lift certain FSs against inner fibrations. (HA 2.1.2.5)

2.1.2.7 <u>Def</u> A map of ∞-operads is map $f: \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ s.t.

(1) TFDC

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{O}'^{\otimes} \arrow[dl, "q"] \\ & N(\mathrm{Fin}_*) & \end{tikzcd}$$

(2) f preserves inert morphisms.

Let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ be subcat of $\mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ consisting of ∞-op. maps.

2.1.2.9 <u>Rmk</u> In fact, only need f to preserve inert lifts of ρ^i's!

<u>Def</u> $q: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ of ∞-ops is <u>fibration of ∞-ops</u> if a categorical fibration.

2.1.2.12 <u>Prop</u> Let $\mathcal{O}^{\otimes}$ be ∞-op. $q: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ coCart fib. Then TFAE:

① Composite $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ makes $\mathcal{C}^{\otimes}$ ∞-op,

② $\forall\, T = T_1 \oplus \dots \oplus T_n \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, inert maps $T \to T_i$ induce equiv.

$$\mathcal{C}^{\otimes}_T \simeq \prod_{1 \le i \le n} \mathcal{C}^{\otimes}_{T_i}$$

<u>Pf.</u> $\Rightarrow$ ① p preserves inert morphisms; by HTT 2.4.1.3, we can turn p-coCart lifts of $q(f)$ into q-coCart lifts, as needed.

① Cat htpy diagram:

$$\begin{tikzcd} \mathcal{C}^{\otimes}_{\langle n \rangle} \arrow[r] \arrow[d, "s"] & \mathcal{O}^{\otimes}_{\langle n \rangle} \arrow[d, "s"] \\ \mathcal{C}^n \arrow[r] & \mathcal{O}^n \end{tikzcd}$$

b/c both ∞-ops.

Passing to htpy fibers over $T \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, we get desired equiv!

$\Leftarrow$ ① $C \in \mathcal{C}^{\otimes}$, inert $\alpha: q(C) \to \langle n \rangle$. Can lift to $\tilde{\alpha}: p(C) \to X$, & b/c p is coCart, we can lift to p-coCart $\bar{\alpha}: C \to \bar{X}$, which by HTT 2.4.1.3 is q-coCart!

② Let $C \in \mathcal{C}^{\otimes}_{\langle m \rangle}$, $C' \in \mathcal{C}^{\otimes}_{\langle n \rangle}$, $f: \langle m \rangle \to \langle n \rangle$,

$T = p(C)$, $T' = p(C')$, choose $g_i: T' \to T_i'$ over ρ^i,

& p-coCart $\bar{g}_i: C' \to C_i'$ over g_i. Let $f_i = \rho^i f$

Get diagram

$$\begin{tikzcd} \mathrm{Map}^{t}(C, C') \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft_H"] & \prod \mathrm{Map}^{f_i}(C, C'_i) \arrow[d] \\ \mathrm{Map}^{f}(T, T') \arrow[r, "\sim"] & \prod \mathrm{Map}^{f_i}(T, T'_i) \end{tikzcd}$$

Bottom map $\simeq$ b/c $\mathcal{O}^{\otimes}$ ∞-operad.

To get desired equiv, want $\simeq$ of fibers over

any lift $h: T \to T'$ of f!

Let $D = h_! C$, $D_i = g_{i!} D$. Then, this map is

$$\mathrm{Map}_{\mathcal{C}^{\otimes}_{T}}(D, C) \longrightarrow \prod_{1 \le i \le n} \mathrm{Map}_{\mathcal{C}^{\otimes}_{T_i}}(D_i, C_i)$$

which is $\simeq$ by assumption (b)!

(3) Fix objects $\{C_i \in \mathcal{C}^{\otimes}_{\langle 1 \rangle}\}$ & set $T_i = p(C_i)$. b/c $\mathcal{O}^{\otimes}$

∞-op, choose cover $f: T \to T_i$, & by (b), we

lift to $\bar{f}_i: C \to C_i$; by HTT 2.4.1.3. $\bar{f}_i$ is q-coCart! ☒

2.1.2.13 .14 **Def** Such a map is a <u>coCart fib of ∞-operads</u>. It "exhibits" $\mathcal{C}^{\otimes}$ as <u>$\mathcal{O}$-monoidal ∞-cat!</u>

Also, denote $\mathcal{C} := \mathcal{C}^{\otimes} \otimes_{\mathcal{O}^{\otimes}} \mathcal{O}$, call $\mathcal{C}$ an $\mathcal{O}$-monoidal ∞-cat.

2.1.2.18 **Def** A sym. monoidal ∞-cat is ∞-cat w/ coCart fib

of ∞-operads $\mathcal{C}^{\otimes} \longrightarrow N(\mathrm{Fin}_*)$

Ops: $\langle 0 \rangle \to \langle 1 \rangle$ $\quad$ $\langle 2 \rangle \to \langle 1 \rangle$

$\Delta^0 \to \mathcal{C}$ $\quad$ $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$

+ <u>all</u> higher coherence!

Or, just coCart fib w/ equiv btwn $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{n}$ $(\mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1 \rangle})$

2.1.2.22 **Prop** Let $q: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be inner fib on ∞-operads. TFAE

(1) q is inner fib of ∞-operads.

(2) If $C \in \mathcal{C}^{\otimes}$, $f: q(C) \to X$ inert in $\mathcal{O}^{\otimes}$, $\exists$ inert lift

$\bar{f}: C \to \bar{X}$.

In this case, inert morphism in $\mathcal{C}^{\otimes}$ are those which are

q-coCart & have inert image.

Pf: Simplicial argument: $2 \Rightarrow 1$ by HTT 2.4.6.5, & $1 \Rightarrow 2$ as

$$\begin{tikzcd} & X' \arrow[d, "g"] \\ q(C) \arrow[ur, "q(f')"] \arrow[r] & X \end{tikzcd}$$

equiv, as f, $q(f')$ lifts of D_0!

$f_0 = \operatorname{im} f \in N(\mathrm{Fin}_*)$, f' lift to $\mathcal{C}^\otimes$

$C \to \bar{X}'$

& lift to $\mathcal{C}^\otimes$ b/c q inner! ⊠

3. Algebra Objects

2.1.3.1 **Def** | Let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be fib of ∞-operads, & suppose we have ∞-operads $\alpha : \mathcal{O}'^\otimes \to \mathcal{O}^\otimes$.

$$\begin{tikzcd} \mathcal{O}'^\otimes \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & \mathcal{C}^\otimes \arrow[dl, "p"] \\ & \mathcal{O}^\otimes & \end{tikzcd}$$

$f \in \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)$

(1) $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ full subcat of ∞-op. maps. ← fiber over α of $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \xrightarrow{p_\circ} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O})$

(2) If $\underset{\alpha = id}{\mathcal{O} = \mathcal{O}'}$. $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$

(3) $\mathcal{O} = \mathcal{O}' = N(\mathrm{Fin}_*)$. $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C})$,

cat of <u>comm alg. objs in $\mathcal{C}$!</u>

<u>Rmk.</u>

· Call $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ cat of $\mathcal{O}'$-alg obj. of $\mathcal{C}$.

<u>Ex.</u> $\mathcal{C}$ SMC, $N(\mathcal{C})$ sym. mon ∞-cat;

$\mathrm{Alg}_{N(\mathrm{Fin}_*)} \mathcal{C} \simeq \mathrm{CAlg}(N(\mathcal{C}))$ — comm alg. obj. of $\mathcal{C}$;

ie obj A w/ unit & mult maps

$$1 \to A, \qquad A \otimes A \to A.$$

2.1.3.4 <u>Rmk.</u> If $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is coCart fib of ∞-ops. $K \in \mathrm{sSet}$,

Skip

then $\mathrm{Fun}(K, \mathcal{C}^\otimes) \otimes_{\mathrm{Fun}(K, \mathcal{O}^\otimes)} \mathcal{O}^\otimes \to \mathcal{O}^\otimes$ is coCart fib.

$\forall X \in \mathcal{O}$, iso $\mathcal{D}_X = \mathrm{Fun}(K, \mathcal{C}_X)$ & compute $\otimes_F$ ptwise.

$$\Rightarrow \quad \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \simeq \mathrm{Fun}(K, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$$

• If $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ are sym. mon. ∞-cats. An ∞-op. map $F \in \mathrm{Alg}_{\mathcal{C}}(\mathcal{D})$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which is compatible w/ SM str in sense that we have maps

$$FC \otimes FC' \to F(C \otimes C'), \quad 1 \to F(1) \quad \text{(lax sense)}$$

2.1.3.7 **Def** | Let $\mathcal{O}^\otimes$ be ∞-operad, let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ & $q : \mathcal{D}^\otimes \to \mathcal{O}^\otimes$ coCart fib. of ∞-operads. We say $F \in \mathrm{Alg}_{\mathcal{C}} \mathcal{D}$ is an <u>$\mathcal{O}$-monoidal functor</u> if it carries p-coCart morphism to q-coCart morphism.

$\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathcal{C},\mathcal{D})$ is full subcat of $\mathcal{O}$-monoidal functors.

When $\mathcal{O}^{\otimes} = N(\mathrm{Fin}_*)$, $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ is sym. monoidal functors.

2.1.3.8 Rmk. Let $F: \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be $\mathcal{O}$-monoidal btwn $\mathcal{O}$-monoidal ∞-cats; TFAE

① F is equivalence

② Underlying map $\mathcal{C} \to \mathcal{D}$ is equivalence.

③ $\forall\, X \in \mathcal{O}$, map $\mathcal{C}_X \to \mathcal{D}_X$ is equiv.

2.1.3.9 Prop. Let $p: \mathcal{O}^{\otimes} \to \mathbb{E}_0^{\otimes}$ be fib, consider $\mathcal{O}' \to \mathbb{E}_0^{\otimes}$ induced by $\alpha: \langle 0 \rangle \to \langle 1 \rangle$ of $\mathrm{Fin}_*^{\mathrm{inj}}$. Then, restriction is trivial Kan fib:

$$\theta: \mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{O}) \longrightarrow \mathrm{Fun}_{\mathbb{E}_0^{\otimes}}(\mathcal{O}', \mathcal{O}^{\otimes})$$

Pf. Long & messy.

2.1.3.10 $\Rightarrow$ If $\mathcal{C}^{\otimes}$ is SM∞, then $\mathbb{E}_0$-alg. obj. is obj. $A \in \mathcal{C}$ w/ map $1 \to A$, via $\mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{C}) \to \mathrm{Fun}_{\mathbb{E}_0}(\mathbb{1}, \mathcal{C})$!

$\mathrm{Alg}_{/\mathbb{E}_0}(\mathcal{C})$ – unital objects!

4. Preoperads & Top. operads

- We can organize all ∞-op. into simp. cat $\mathcal{O}p_{\infty}^{\Delta}$, w/ 0-simplices (small) ∞-operads & $\mathrm{Map}_{\mathcal{O}p_{\infty}^{\Delta}}(\mathcal{O}, \mathcal{O}')$ the Kan cplx $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')^{\simeq}$ of ∞-op. maps.

Def ‖ $\mathcal{O}p_{\infty} = N(\mathcal{O}p_{\infty}^{\Delta})$ – ∞-cat of ∞-operads.

HA 2.1.4 goes on to show $\mathcal{O}p_{\infty}$ is presentable by combinatorial simplicial model cat of ∞-preoperads.

One last thing: there is a paper by Heuts, Hinich, & Moerdijk that gives a zig-zag of Quillen equivalences btwn the theory of ∞-operads as we have it, & a theory based on "dendroidal sets." I'll stop here, but what you should take away from this is that if you get topological operads, e.g. little cubes, then the step to ∞-operads isn't too bad.

Thanks & questions!