Constructions on ∞-operads: subcategories, overcategories, envelopes, and tensor products

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Abstract

We describe several methods of constructing ∞-operads and $\mathcal{O}$-monoidal ∞-categories from other ones. We first show that a $\otimes$-closed full subcategory of an $\mathcal{O}$-monoidal ∞-category is canonically $\mathcal{O}$-monoidal. We then state that a slice category of an $\mathcal{O}$-monoidal ∞-category over a $\mathcal{O}$-algebra is canonically $\mathcal{O}$-monoidal.

We go on to construct coproducts in Op_∞. We then construct the Boardman-Vogt tensor product for operads in **Set** and for preoperads, and hence we present the symmetric monoidal ∞-category $\mathrm{Op}_\infty^\otimes$. We sketch the Eckmann Hilton argument in **Set**, and its homotopy-coherent generalization, called Dunn additivity. Time permitting, we describe the constructions of the monoidal envelope and Day convolution.

Contents

These notes are currently rough. We'll cover most of HA §2.2 to some extent.

In what follows, we generally fix $\mathcal{O}^\otimes$ an ∞-operad and $\mathcal{C}^\otimes$ an $\mathcal{O}^\infty$-monoidal ∞-category. Recall that, for $\mathcal{M}$ a symmetric monoidal category, a full subcategory of $\mathcal{M}$ inherits a compatible symmetric monoidal structure iff $\mathrm{ob}\,\mathcal{M}$ is closed under the monoidal product; in Section ref , we state an analog of this for $\mathcal{O}^\infty$-monoidal ∞-categories.

Similarly, recall that, for $A \in \mathcal{M}$ a commutative monoid and M finitely cocomplete, the overcategory $\mathcal{M}_{/A}$ has a symmetric monoidal structure given by a combination of the external tensor product with multiplication:

$$X \otimes Y \to A \otimes A \xrightarrow{\mu} A.$$

In Section ref , we prove results containing this for ∞-operads; when the underlying category $\mathcal{O}$ is contractible, $\mathcal{C}^\otimes_{p/}$ and $\mathcal{C}^\otimes_{/p}$ are given the structure of operads with fibrations to $\mathcal{O}^\otimes$. Assuming one believes that $\mathrm{Fun}(X, Y)$ can be made symmetric monoidal when Y is, this and the previous section will allow us to construct a symmetric monoidal structure on $\mathbf{Alg}_\mathcal{O}(\mathcal{C})$.

In Section ref , we briefly discuss coproducts of ∞-operads.

There is a forgetful ∞-functor $\mathbf{Cat}_\infty^\otimes \to \mathrm{Op}_\infty$, and many constructions of symmetric monoidal ∞-categories are lifted from constructions of operads. In Section ref , we construct a left adjoint to this forgetful functor, called the $\mathcal{O}$-monoidal envelope.

The main thrust of this talk is Section ref , as it is the least familiar; using the object-wise tensor product (of functor categories), $\mathbf{Alg}_\mathcal{O}(\mathcal{C})$ is itself a symmetric monoidal ∞-category, and hence an operad. We define a symmetric monoidal tensor product, called the *Boardman-Vogt tensor product*, on Op_∞ which endows it with a structure akin to a closed monoidal category; in particular, this is the essentially unique monoidal product on Op_∞ satisfying the adjunction

$$\begin{array}{ccc} \mathbf{Alg}_{\mathcal{O}'\otimes\mathcal{O}}(\mathcal{C}) & \xrightarrow{\sim} & \mathbf{Alg}_{\mathcal{O}'}(\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})) \\ \| & & \| \\ \underline{\mathrm{Op}}_\infty(\mathcal{O}'\otimes\mathcal{O},\mathcal{C}) & \xrightarrow{\sim} & \underline{\mathrm{Op}}_\infty\Big(\mathcal{O}', \underline{\mathrm{Op}}_\infty(\mathcal{O},\mathcal{C})\Big)\, n \end{array}$$

This is the derived monoidal product of a symmetric monoidal model category of preoperads and, conjecturally, a more classical symmetric monoidal model category of either dendroidal sets or topological/simplicial operads. In this language we state *Dunn additivity*, i.e. the computation that

$$\mathbb{E}_n \otimes \mathbb{E}_1 = \mathbb{E}_{n+1}, \qquad \text{i.e.} \qquad \mathbf{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C}) = \mathbf{Alg}_{\mathbb{E}_1}\mathbf{Alg}_{\mathbb{E}_n}(\mathcal{C}),$$

an unreasonably useful result.

We finish up by discussing the Day convolution monoidal structure on $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ for $\mathcal{C},\mathcal{D}$ symmetric monoidal ∞-categories.

1 Familiar closure properties for monoidal categories

1.1 A $\otimes$-closed full subcategory of an $\mathcal{O}$-monoidal ∞-category is $\mathcal{O}$-monoidal

Let $p : \mathcal{C}^\otimes \to \mathcal{O}$ be a coCartesian fibration of ∞-operads, let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory stable under equivalence, and let $\mathcal{D}^\otimes \subset \mathcal{C}^\otimes$ be the full subcategory of $\mathcal{C}^\otimes$ spanned by objects of the form $D_1 \oplus \cdots D_n$ where each object belongs to $\mathcal{D}$. The following proposition is more or less obvious when thinking hard about the definition of ∞-operads:

Proposition 1.1. *$D^\otimes$ is an ∞-operad, and the inclusion $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ is a map of ∞-operads.*

The legumes[1] of this section is in the following proposition.HA 2.2.1.1

Proposition 1.2 (Tensor closed subcategories of $\mathcal{O}$-monoidal ∞-categories.)**.** *Let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a coCarteisan fibration of ∞-operads and let $\mathcal{D}, \mathcal{D}^\otimes$ be as above. Suppose that, for every operation $\theta \in \{\mathrm{X_i}\}, \mathrm{Y}$, the induced functor*

$$\theta_* : \prod_i \mathcal{C}_{X_i} \to \mathcal{C}_Y$$

descends to a functor $\prod_i \mathcal{D}_{X_i} \to \mathcal{D}_Y$. Then,

(i) The restricted map $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ is a coCartesian fibration of ∞-operads.

(ii) The inclusion $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ is an $\mathcal{O}$-monoidal functor.

(iii) Suppose that for each object $X \in \mathcal{O}$, the inclusion $\mathcal{D}_X \subset \mathcal{C}_X$ admits a right adjoint L_X> Then, there exists a functor $L^\otimes : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ which is a right adjoint of the inclusion $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ with counit projecting to degenerate edges of $\mathcal{O}^\otimes$; further, $L^\otimes$ is a map of ∞-operads.

(iv) $L^\otimes$ induces the right adjoint to the inclusion $\mathbf{Alg}_{/\mathcal{O}}(\mathcal{D}) \hookrightarrow \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$.

This has a plain english Corollary:

Corollary 1.3. *Suppose $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category and $\mathcal{D} \subset \mathcal{C}$ is a full subcategory closed under equivalence and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Then, there is a canonical sub-symmetric monoidal ∞-category $\mathcal{D}^\otimes \hookrightarrow \mathcal{C}^\otimes$ whose underlying category is $\mathcal{D}$; in particular, $\mathcal{D}$ is canonically symmetric monoidal.*

Suppose $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a coCartesian fibration of ∞-operads, suppose that $\mathcal{C}_X$ is stable for each X. Suppose further that every operation $\theta \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}, Y)$ induces an exact functor

$$\theta_* : \prod_{i\in I} \mathcal{C}_{X_i} \to \mathcal{C}_Y.$$

Suppose even further that we're given a family of t-structures $(\mathcal{C}_{X,\geq 0}, \mathcal{C}_{X,\leq 0})$ preserved by each θ_*. Let $\mathcal{C}^\otimes_{\leq 0} \subset \mathcal{C}^\otimes$ be the full subcategory spanned by objects $C \in \mathcal{C}^\otimes$ such that, for every $C' \in \mathcal{C}_X$ and every inert morphism $C \to C'$, the object C' belongs to $\mathcal{C}_{X,\geq 0}$.

[1]This is an ethical replacement for the "meat" of the section.

Corollary 1.4 (Monoidal structure on connective objects)**.** *The induced map $\mathcal{C}^{\otimes}_{\geq 0} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.*

We refer to HA §2.2.1 for the proof of the third part; the first and second part follow by the tensor closure, as this essentially says that the coCartesian lifts with chosen domains $\mathcal{D}^{\otimes}$ are contained in $\mathcal{D}^{\otimes}$ (and are hence coCartesian and preserved by the inclusion).

Localization????

1.2 Slices of an $\mathcal{O}$-monoidal ∞-category over a commutative algebra are $\mathcal{O}$-monoidal; slices of an operad over $\mathcal{O}$ are operads over $\mathcal{O}$

First, let's form a notion of relative slicing:

Definition 1.5. Let $q : X \to S$ be a map of simplicial sets, and suppose we're given a diagram

$$\begin{array}{ccc} & \overset{p}{\nearrow} & X \\ & & \downarrow{\scriptstyle q} \\ S \times K & \longrightarrow & S \end{array}$$

Define the simplicial set $q' : X_{pS/} \to S$ over S universally such that there is a natural bijection between $\mathrm{Fun}_S(Y, X_{pS/})$ with the commutative diagrams

$$\begin{array}{ccccc} Y \times K & \hookrightarrow & Y \times K^{\triangleright} & \twoheadrightarrow & Y \\ {\scriptstyle q_Y \times \mathrm{id}_K}\downarrow & & \downarrow & & \downarrow{\scriptstyle q_Y} \\ S \times K & \xrightarrow{p} & X & \xrightarrow{q_X} & S \end{array}$$

where the top left arrow is inclusion and the top right arrow is projection to the "tip" of K.

We define the relative overcategory $X_{/pS}$ by replacing $\triangleright$ with $\triangleleft$. This notion recovers traditional overcategories and undercategories when $S = *$.

Remark. The only piece of data supplied in the functor represented by $X_{pS/}$ is the arrow highlighted in red. We may read this definition as saying that a map $\varphi : Y \to X_{pS/}$ is the same as a map $\tilde{\varphi} : Y \times K^{\triangleright} \to X$ from "points of Y together with a K-shape extended by a cone point to the right," satisfying the conditions that

1. The restriction of the map φ to $Y \times K$ factors through the diagram $S \times K \xrightarrow{p} X$.
2. Pushing forward to the cone point yields is compatible with the structure maps to the base simplicial set; this implies that the restricted map $Y \times \{*\} \to X$ is a functor over S.

For $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a fibration of ∞-operads and $p : K \to \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$ a diagram, there is an adjoint map

$$K \to \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{O}^{\otimes}) \qquad \rightsquigarrow \qquad K \times \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes} \quad \text{over } \mathcal{O}^{\otimes}$$

and hence we may form the slices $C^{\otimes}_{/p\mathcal{O}}$ and $\mathcal{C}^{\otimes}_{p\mathcal{O}/}$. In the case $p = A \in \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$, we denote this by $\mathcal{C}^{\otimes}_{A\mathcal{O}/}$.

We can now state the mushrooms and potatoes of this section, from HA §2.2.2.4:

Theorem 1.6. *Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞ operads and let $p : K \to \mathbf{Alg}_{/\mathcal{O}}(\mathcal{C})$ be a diagram. Then,*

(1) the maps $\mathcal{C}^{\otimes}_{p\mathcal{O}/} \to \mathcal{O}^{\otimes} \leftarrow \mathcal{C}^{\otimes}_{/p\mathcal{O}}$ are fibrations of ∞-operadds.

(2) A morphism in $\mathcal{C}^{\otimes}_{p\mathcal{O}/}$ is inert if and only if its image in $\mathcal{C}^{\otimes}$ is inert; the same is true of $\mathcal{C}^{\otimes}_{/p\mathcal{O}}$.

(3) If q is a coCartesian fibration, then $\mathcal{C}^{\otimes}_{/p\mathcal{O}} \to \mathcal{O}^{\otimes}$ is a coCaretian fibration. If in addition, $p(k) : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal functor for each vertex $k \in K$, then $\mathcal{C}^{\otimes}_{p\mathcal{O}/} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads.

We get a plain english corollary.

Proposition 1.7. *Suppose $\mathcal{C}$ is an ∞-operad and $A \in C\mathbf{Alg}(\mathcal{C})$ a commutative algebra object. Then, each of $\mathcal{C}^{\otimes}_{A/}$ and $\mathcal{C}^{\otimes}_{/A}$ are ∞-operads. If $\mathcal{C}$ is a symmetric monoidal ∞-category, then $\mathcal{C}_{/A}$ is a symmetric monoidal ∞-category.*

No proof? *megamind meme*

1.3 An application: the pointwise monoidal structure On $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$

Let $\mathcal{C}$ be a category and $\mathcal{D}$ a symmetric monoidal category. There is a symmetric monoidal structure on the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ given by

$$(F \otimes G)(-) := F(-) \otimes G(-).$$

We refer to this as the *pointwise (symmetric) monoidal structure*. Our goal is to repeat this construction in the ∞-operadic setting, and use our formalism for restriction and slicing to descend this to a pointwise monoidal structure on $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$.

This is pedagogical in nature; we frequently need to define an ∞-operad structure on $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$ for $\mathcal{C}^{\otimes}$ an ∞-operad (not necessarily a symmetric monoidal ∞-category). Our construction is generalized in HA §3.2.4, which we will cover later in this seminar. As such, proceed with caution; I haven't seen this worked out elsewhere, so mistakes are relatively more likely here than in material covered in HA.

Construction 1.8. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $\mathcal{O}$ an ∞-category. Let $\theta : N(\mathrm{Fin}_*) \to \mathbf{Cat}_\infty$ be the functor corresponding to $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ via the Grothendieck construction.

Define the functor $\Delta_{\mathcal{O}} : N(\mathrm{Fin}_*) \to \mathbf{Cat}_\infty^{\mathrm{op}}$ to be the functor sending

$$\Delta_{\mathcal{O}}(\langle n \rangle) := \mathcal{O}^n$$

and acting on $f : \langle n \rangle \to \langle m \rangle$ via the "traditional" diagonal on fibers.

There is a functor

$$\varphi : N(\mathrm{Fin}_*) \xrightarrow{\Delta_{\mathcal{O}} \times \theta} \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathbf{Cat}_\infty \xrightarrow{\mathrm{Fun}(-,-)} \mathbf{Cat}_\infty$$

such that the induced map

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{O}^n, \mathcal{C}^n) & \xrightarrow{\prod \rho_i} & \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \\ \| & & \| \\ \varphi(n) & \longrightarrow & \varphi(1) \end{array}$$

is an equivalence. The *pointwise symmetric monoidal category of functors* is the associated symmetric monoidal ∞-category, written $\mathrm{Fun}_{\mathrm{ptws}}(\mathcal{O}, \mathcal{C})^{\otimes}$.

The *pointwise symmetric monoidal category of algebras* is the full symmetric monoidal ∞-subcategory

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \subset \mathrm{Fun}_{\mathrm{ptws}}(\mathcal{C}, \mathcal{O})^{\otimes}.$$

formed on the functors over $N(\mathrm{Fin}_*)$ preserving inert morphisms.[2]

Note that the underlying category of $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$, avoiding notational and conceptual confusion.

2 New constructions on operads

We first construct the coproduct of ∞-operads; classically, the BV-tensor product of operads is constructed as a quotient of this construction.

2.1 Coproducts of ∞-operads

Suppose we have a coproduct functor $\coprod : \mathrm{Op}_\infty^{\times 2} \to \mathrm{Op}_\infty$; this is uniquely determined by the identity

$$\begin{aligned} \mathbf{Alg}_{\mathcal{O} \coprod \mathcal{O}'}(\mathcal{C})^{\simeq} &= \mathrm{Op}_\infty(\mathcal{O} \coprod \mathcal{O}', \mathcal{C}) \\ &\simeq \mathrm{Op}_\infty(\mathcal{O}, \mathcal{C}) \times \mathrm{Op}_\infty(\mathcal{O}', \mathcal{C}) \\ &= \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^{\simeq} \times \mathbf{Alg}_{\mathcal{O}'}(\mathcal{C})^{\simeq}. \end{aligned}$$

That is, we're looking for an operad $\mathcal{O} \coprod \mathcal{O}'$ whose algebras are precisely the pairs of $\mathcal{O}$-algebras and $\mathcal{O}'$-algebras; we will construct this as the restriction of a product

$$\boxtimes : \mathbf{Cat}_{\infty,/N(\mathrm{Fin}_*)}^{\times 2} \to \mathbf{Cat}_{\infty,/N(\mathrm{Fin}_*)}.$$

But first, we need an auxiliary construction.

[2] Check that this makes sense!

Construction 2.1. The category Sub has:

- objects given by triples $(\langle n\rangle, S, T)$ where $n \in \mathrm{Fin}_*$, and S, T are pointed subsets forming a partition of $\langle n\rangle$, and
- morphisms $(\langle n\rangle, S, T) \to (\langle n'\rangle, S, T)$ a morphism of triples in Fin_*.

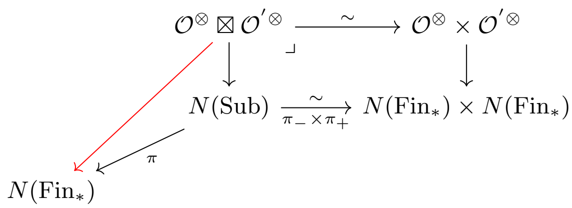
There is a triple of functors $\pi, \pi_-, \pi_+ : \mathrm{Sub} \to \mathrm{Fin}_*$ given by

$$\pi_-(\langle n\rangle, S, T) = [S] \qquad \pi(\langle n\rangle, S, T) = \langle n\rangle \qquad \pi_+(\langle n\rangle, S, T) = [T]$$

Note that $\pi_- \times \pi_+ : \mathrm{Sub} \to \mathrm{Fin}_*^{\times 2}$ is an equivalence.

With this done, we can construct the product $\boxtimes$;

Construction 2.2. Given maps $\mathcal{O}^\otimes \to N(\mathrm{Fin}_*)$ and $\mathcal{O}'^\otimes \to N(\mathrm{Fin}_*)$, define the map $\mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes \to \mathrm{Fin}$ to be the highlighted composition in the following diagram:



Choosing a pullback functor, this can be assembled into an honest bifunctor, completing the construction.

Now that we've made the construction, let's simply list the reasons why we care; there is a morphism $\iota_- : \mathcal{O}^\otimes \to \mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes$ given by including $\mathcal{O}$ as the fiber over $*$ of π_+, and a similar embedding ι_+ of $\mathcal{O}'$, piecing into a diagram

$$\mathcal{O}^\otimes \hookrightarrow \mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes \hookleftarrow \mathcal{O}'^\otimes \tag{1}$$

Theorem 2.3. *Let $\mathcal{O}^\otimes, \mathcal{O}'^\otimes$ be ∞-operads. Then, $\mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes$ is an ∞-operad, and Diagram (1) induces an equivalence of ∞-categories*

$$\mathbf{Alg}_{\mathcal{O}^\otimes \boxtimes \mathcal{O}'^\otimes}(\mathcal{C}) \xrightarrow{\sim} \mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \times \mathbf{Alg}_{\mathcal{O}'}(\mathcal{C}),$$

natural in $\mathcal{C}$. In particular, Diagram (1) is a colimit diagram in Op_∞.

proof here, maybe

2.2 The Boardman-Vogt tensor product in the ∞-operadic setting, and Dunn additivity

For the sake of staying grounded, let's first work out a closed monoidal structure on the operads in **Set**.

The BV tensor product of operads in Set. Let $\mathcal{O}$ and $\mathcal{O}'$ be operads in **Set**. Given an *ideal in an operad* $I \subset \mathcal{O}$, we may describe the quotient operad $\mathcal{O}/I$ in exactly the way one might expect. We use this:

Construction 2.4. The *Boardman-Vogt tensor product of $\mathcal{O}$ and $\mathcal{O}'$* is the quotient of the coproduct $\mathcal{O} \coprod \mathcal{O}'$ by the *interchange law:*

$$\mathcal{O} \otimes \mathcal{O}' := \frac{\mathcal{O} \coprod \mathcal{O}'}{(\gamma(\theta; \theta', \dots, \theta')) - \gamma(\theta'; \theta, \dots, \theta) \cdot \sigma_{k,k'})}$$

where γ is composition in the coproduct, $\theta \in \mathcal{O}(k)$, $\theta' \in \mathcal{O}'(k')$, and $\sigma_{k,k'} \in S_{kk'}$ is the permutation which "exchanges rows and columns."

This was first constructed in the BV reference , at the same time as the instantiation of much of the rest of modern homotopy theory. The following is a classical result, and the internal hom is not too hard to see:

Theorem 2.5. *The Boardman-Vogt tensor product endows on* Op *the structure of a closed monoidal category, with internal hom given by $\mathbf{Alg}_{(-)}(-)$.*

With this done, we can give a universal algebraic version of the *Eckmann Hilton argument*:

Theorem 2.6 (Eckmann-Hilton argument)**.**

$$\mathrm{Assoc} \otimes \mathrm{Assoc} \simeq \mathrm{Comm};$$

in particular, if an object in a symmetric monoidal category bears two monoid structures which distribute over each other, then the two structures agree and are commutative, i.e. the natural functor is an equivalence:

$$\mathrm{Comm}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Assoc}\,\mathrm{Assoc}(\mathcal{C})$$

.

In the more-classical setting, this extends to a notion of an enriched BV tensor product on operads in a suitably nice monoidal model category (say, $\mathcal{M}$ such a category satisfying the monoid axiom and everything else we want). Unfortunately, this does not satisfy the pushout product axiom.[3] After introducing a monoidal model structure on preoperads, we'll shout out that the model for dendroidal sets is simply better, and (homotopy-symmetric) strong-monoidally equivalent to our category of preoperads.

The BV tensor product of preoperads Let $\mathcal{C}^{\otimes}, \mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}$ be ∞-operads. We'd like to introduce a process akin to passing from the direct sum of vector spaces to the tensor product. To do so, we'll consider the analog of bilinear maps; first, we need an auxiliary construction, which is likely familiar.

Construction 2.7. Suppose $\mathcal{C}$ is a finitely bicomplete symmetric monoidal category with terminal object. Then, the *category of pointed objects* is the slice category

$$\mathcal{C}_* := \mathcal{C}_{I/}.$$

This is endowed with a symmetric monoidal structure given by

$$X \wedge Y := \mathrm{Cofib}_{\mathcal{C}}\left(X \otimes \{*_Y\} \coprod \{*_X\} \otimes Y \to X \otimes Y\right)$$

In the special case that $\mathcal{C} = \mathrm{Fin}$ is the category of finite sets, we call $\wedge$ the *smash product* of finite pointed sets.

This endows on $N(\mathrm{Fin}_*)$ the structure of a simplicial monoid. We use this:

Definition 2.8. A *bifunctor of operads* is a map $f : \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes}$ sending each pair of inert morphisms to an inert morphism, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}^{\otimes} \times \mathcal{O}'^{\otimes} & \longrightarrow & \mathcal{C}^{\otimes} \\ \downarrow & & \downarrow \\ N(\mathrm{Fin}_*) \times N(\mathrm{Fin}_*) & \xrightarrow{\wedge} & N(\mathrm{Fin}_*) \end{array}$$

These form a full subcategory of a functor category, called $\mathrm{BiFun}(\mathcal{O}, \mathcal{O}'; \mathcal{C})$.

Bilinear maps, when corepresented by a bifunctor, form half of a closed monoidal structure on Vect_k, through noting by currying that

$$\mathrm{BiHom}(V, V'; W) \simeq \mathrm{Hom}(V, \mathrm{Hom}(V', W)).$$

We make a bold claim, which we can only currently understand in the case that $\mathcal{C}$ is a symmetric monoidal ∞-category. The following argument is not in HA as far as I can find, but it is heavily implied that Lurie believes it to be true.

Claim. *There is a natural equivalence of functors*

$$\mathrm{BiFun}(\mathcal{O}, \mathcal{O}'; \mathcal{C}) \simeq \mathbf{Alg}_{\mathcal{O}}\mathbf{Alg}_{\mathcal{O}'}(\mathcal{C})$$

Proof idea. We push through the adjunction: an object in $\mathbf{Alg}_{\mathcal{O}}\mathbf{Alg}_{\mathcal{O}'}(\mathcal{C})$ corresponds with:

$$\varphi : \mathcal{O} \to \mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathcal{O}', \mathcal{C}) \qquad \rightsquigarrow \qquad \tilde{\varphi} : \mathcal{O} \times \mathcal{O}' \to \mathcal{C} \quad \text{over } N(\mathrm{Fin}_*).$$

The functor φ must preserve inert morphims and be valued in functors preserving inert morphisms; this is equivalent to $\tilde{\varphi}$ preserving pairs of inert morphisms.[4] ∎

[3] See the following math overflow post.

[4] I don't know if this is actually true!!!

This gives some hope that we could form a closed symmetric monoidal structure on Op_∞. The easiest way to go about doing this is to present it as the simplicial localization of a symmetric monoidal model category. Let $\mathcal{P}\,\mathrm{Op}_\infty \subset s\mathbf{Set}^+_{/N(\mathrm{Fin}_*)}$ be the full subcategory of marked simplicial sets whose markings lie above inert morphisms. We call these *∞-preoperads*. Every ∞-operad $\mathcal{O}^\otimes$ has an underlying ∞-preoperad $\mathcal{O}^{\otimes,\natural}$ whose marked morphisms are the inert morphisms. The following is the core of HA §2.1.4.

Theorem 2.9. *There exists a left proper combinatorial simplicial model structure on $\mathcal{P}\,\mathrm{Op}_\infty$ characterized by the following structure:*

- *A morphism $f : \overline{X} \to \overline{Y}$ in $\mathcal{P}\,\mathrm{Op}_\infty$ is a weak equivalence iff for every ∞-operad $\mathcal{O}^\otimes$, the induced map*
$$\mathrm{Map}_{\mathcal{P}\,\mathrm{Op}_\infty}(\overline{Y}, \mathcal{O}^{\otimes,\natural}) \to \mathrm{Map}_{\mathcal{P}\,\mathrm{Op}_\infty}(\overline{X}, \mathcal{O}^{\otimes}, \natural)$$
is a weak homotopy equivalence.
- *The fibrant objects of $\mathcal{P}\,\mathrm{Op}_\infty$ are precisely those of the form $\mathcal{O}^{\otimes,\natural}$ for $\mathcal{O}^\otimes$ some ∞-operad.*

This model presents Op_∞.

Construction 2.10. Let $\overline{X} = (X, M)$ and $\overline{Y} = (Y, N)$ by ∞-preoperads; then, the preoperad $\overline{X} \otimes \overline{Y}$ is defined by
$$(X \times Y, M \times M').$$
with structure map
$$X \times Y \to N(\mathrm{Fin}_*) \times N(\mathrm{Fin}_*) \xrightarrow{\wedge} N(\mathrm{Fin}_*).$$

The following theorem is not so new, and the proof is not enlightening.

Theorem 2.11 ($\otimes$ presents the BV tensor product)**.** *The functor $\otimes$ endows $\mathcal{P}\,\mathrm{Op}_\infty$ with the structure of a monoidal model category; the left-derived monoidal structure on* Op *restricts to a functor*
$$\otimes : \mathrm{Op}_\infty \times \mathrm{Op}_\infty \to \mathrm{Op}_\infty$$
such that
$$\mathbf{Alg}_{\mathcal{O}\otimes\mathcal{O}'}(\mathcal{C}) = \mathrm{BiFun}(\mathcal{O}, \mathcal{O}'; \mathcal{C}).$$

Remark. We have two remarks. First, the monoidal structure on $\mathcal{P}\,\mathrm{Op}_\infty$ is not symmetric[5]. However, its derived functor satisfies a symmetric universal property, so it must be symmetric.

Second, monoidal model categories are in fact monoidal closed, with right-Quillen internal hom; hence there is a derived functor
$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})^\otimes := \mathbb{R}\,\mathrm{Hom}_{\mathcal{P}\,\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{C})$$
satisfying an analog of Claim ref ; hence when $\mathcal{C}$ is a symmetric monoidal ∞-category, we recover the pointwise tensor product of algebras.

Corollary 2.12. *There exists a symmetric monoidal ∞-category $\mathrm{Op}_\infty^\otimes$ with underying category Op_∞ such that $\otimes : \mathrm{Op}_\infty \times \mathrm{Op}_\infty \to \mathrm{Op}_\infty$ corepresents* BiFun*.*

This enables us to talk about a generalization of Dunn additivity! Let $\mathbb{E}_n$ refer to the little n-cubes operad, realized as the operadic nerve of the usual construction. It is a general fact[6] that the ∞-category of algebras over a Σ-cofibrant topological operad agrees with the ∞-category of algebras over its operadic nerve, so we are safe to equivocate between these two. Note that $\mathbb{E}_1^\otimes \simeq \mathrm{Assoc}^\otimes$ and $\mathbb{E}_\infty^\otimes \simeq \mathrm{Comm}^\otimes$.

Theorem 2.13 (Dunn additivity)**.** $\mathbb{E}_k^\otimes \otimes \mathbb{E}_\ell^\otimes \simeq \mathbb{E}_{k+\ell}^\otimes$.

Proof idea. It suffices to construct a weak equivalence of topological operads $\alpha : \mathbb{E}_1^{\otimes k} \xrightarrow{\sim} \mathbb{E}_k$. We can construct this map rather easily; there are n inclusions $\mathbb{E}_1 \hookrightarrow \mathbb{E}_k$ "along the axes," and α may be defined to be the map induced by these.[7] We just have to prove that α is a weak equivalence, which proceeds in two steps:

1. α induces an isomorphism of $\mathbb{E}_1^{\otimes k}$ onto a suboperad $\mathbb{E}_k^{\mathrm{decom}} \subset \mathbb{E}_k$.
2. the operad $\mathbb{E}_k^{\mathrm{decom}}$ consists of the *decomposable elements of* $\mathbb{E}_k$, i.e. the little cube diagrams such that each axis has a perpendicular hyperplane with cubes on each side and intersecting the interior of no cubes; "shrinking cubes" yields a local deformation retract $\mathbb{E}_k \xrightarrow{\sim} \mathbb{E}_k^{\mathrm{decom}]}$ witnessing that the inclusion is a weak equivalence.

[5] flesh this out
[6] cite???
[7] This is essentially that the tensor product corepresents BiFun.

The second part is a continuous analog of Eckmann-Hilton; the first part is hard, and the reader can read Dunn for a proof. ☐

Remark. This statement is true both as cofibrant topological operads and as ∞-operads; unfortunately, in the literature, $\mathbb{E}_k$ operads are often allowed to only be Σ-cofibrant, in which case this theorem may fail.

2.3 The $\mathcal{O}$-monoidal envelope of a ∞-operad

Construction 2.14. Let $\mathcal{O}$ be an ∞-operad and $\mathcal{C}^{\otimes} \to \mathcal{O}$ a fibration of ∞-operads. For $\mathrm{Act}(\mathcal{O}^{\otimes}) \subset \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})$ the full subcategory of active morphisms, the *$\mathcal{O}$-monoidal envelope of* $\mathcal{C}^{\otimes}$ is the fiber product

$$\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^{\otimes} := \mathcal{C}^{\otimes} \times_{\mathrm{Fun}(\{0\},\mathcal{O}^{\otimes})} \mathrm{Act}\left(\mathcal{O}^{\otimes}\right)$$

When $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}$, we simply write $\mathrm{Env}(\mathcal{C})^{\otimes}$.

This may be viewed as the ∞-category of pairs (C, α) where $C \in \mathcal{C}^{\otimes}$ and $\alpha : p(C) \to X$ is an active morphism in $\mathcal{O}^{\otimes}$.

The following is the tempeh of this section:

Proposition 2.15. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads.*

(1) Evaluation at $\{1\} \subset \Delta^1$ together with p induces a coCartesian fibration of operads

$$p' : \mathrm{Env}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{O}^{\otimes}.$$

That is, $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})$ is an $\mathcal{O}$-monoidal ∞-category.

(2) The inclusion $\iota : \mathcal{C}^{\otimes} \hookrightarrow \mathrm{Env}_{\mathcal{O}}(\mathcal{C})$ is fully faithful.

(3) Let $\mathcal{D}$ be an $\mathcal{O}$-monoidal ∞-category. The inclusion $\iota : \mathcal{C}^{\otimes} \hookrightarrow \mathrm{Env}_{\mathcal{O}}(\mathcal{C})$ induces an equivalence

$$\mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}), \mathcal{D}) \to \mathbf{Alg}_{\mathcal{C}}(\mathcal{D}).$$

Note that the underlying ∞-category $\mathrm{Env}(\mathcal{C})^{\mathrm{op}}$ is identified with the active morphisms in $\mathcal{C}^{\otimes}$; we can informally view $\mathrm{Env}(\mathcal{C})^{\otimes}$ as consisting of a monoidal structure on $\mathcal{C}^{\otimes}_{\mathrm{act}}$.

We get the following plain-english corollary.

Corollary 2.16. *Let $\mathcal{C}$ be an ∞-operad. Then, there exists a fully faithful embedding of $\mathcal{C}$ into a symmetric monoidal ∞-category, which is left adjoint to the inclusion of symmetric monoidal ∞-categories into ∞-operads.*

2.4 Day convolution

We will again delve first into classical results in the 1-categorical setting, then upgrade to the ∞-categoriecal setting.

Day convolution in the 1-categorical setting Let $\mathcal{C}, \mathcal{D}$ be two symmetric monoidal categories with $\mathcal{D}$ cocomplete, preserving small colimits separately in each variable. We may define a monoidal product $\circledast : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ pointwise via a left Kan extension:

$$\begin{array}{ccccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{F \times G} & \mathcal{D} \times \mathcal{D} & \xrightarrow{\otimes_{\mathcal{D}}} & \mathcal{D} \\ \big\downarrow{\scriptstyle \otimes_{\mathcal{C}}} & & & \nearrow{\scriptstyle F \circledast G := \mathrm{Lan}} & \\ \mathcal{C} & & & & \end{array}$$

Cocompleteness of $\mathcal{D}$ ensures that this exists, and it is computed by a colimit:

$$(F \circledast G)(C) := \mathrm{colim}_{C_0 \otimes_{\mathcal{C}} C_1 \to C}\, F(C_0) \otimes_{\mathcal{D}} G(C_1)$$

In the case that $\mathcal{D} = \mathcal{V}$ is bicomplete and monoidal closed, and $\mathcal{C}$ is a tensored $\mathcal{V}$-category, we can repeat this construction in the enriched setting, yielding a $\mathcal{V}$-enriched monoidal functor category. This is the setting in which Day convolution was originally conceived, and in this setting we have a coend formula for the defining Left Kan extension:

$$F \circledast G(-) = \int^{c,c'} \underline{\mathcal{C}}(c \otimes c', -) \otimes F(c) \otimes G(c').$$

For δ a kernel, it is the analogy between this and the convolution formula

$$f * g(t) := \int_{-\infty}^{\infty} f(t-x)g(x) = \int_{\mathbb{R}^2} \delta_{x=y+t} f(y)g(x)$$

that gives Day convolution its name.

Note that this computation still applies when $\mathcal{D}$ is not given by $\mathcal{V}$, but is instead given by a tensored $\mathcal{V}$-category.

Let's mention a few examples:

Example 2.17:
Let $\mathcal{V}$ be a closed bicomplete symmetric monoidal category, and let M be a small monoid, realized as a discrete monoidal $\mathcal{V}$-category. Let $\mathrm{gr}^M \mathcal{V}$ be the category of $\mathcal{V}$-enriched functors $M \to \mathcal{V}$, i.e. the M-indexed collection of objects in $\mathcal{V}$. This admits a Day convolution monoidal product, who ends up being computed by

$$\begin{aligned}(X_\bullet \circledast Y_\bullet)_n &= \int^{m,m'} M(m \otimes m', n) \otimes X_m \otimes Y_{m'} \\ &= \bigoplus_{m+m'=n} X_m \otimes Y_n.\end{aligned}$$

Example 2.18:
Let Σ be the symmetric monoidal $\mathbf{Top}_*$-category having objects given by $\mathbb{N}$ and hom objects given by

$$\mathcal{O}(n,m) = \begin{cases} \Sigma_{n,+} & n = m \\ * & \text{otherwise} \end{cases}$$

with the obvious composition, which acts via addition in $\mathbb{N}$ on objects, and which (when nontrivial) acts on hom objects by block inclusion

$$\Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}.$$

A *symmetric sequence in* $\mathbf{Top}_*$ is a $\mathbf{Top}_*$-functor $\Sigma \to \mathbf{Top}_*$, i.e. a sequence of spaces with Σ_+-actions which are associative and unital. This is endowed with a day convolution symmetric monoidal product.

The sphere spectrum $\mathbb{S} : n \mapsto S^n = \left(S^1\right)^{\wedge n}$ has the structure of a symmetric sequence via the Σ_n action permuting indices. In fact, the sphere spectrum has an evident structure of a commutative monoid in symmetric sequences in $\mathbf{Top}_*$; hence we have a symmetric monoidal category $\mathcal{S}^\Sigma := \mathbf{Mod}_{\mathbb{S}}(\mathrm{Fun}(\Sigma, \mathbf{Top}_*))$, called *symmetric spectra.*

It is known that this may be refined to the structure of a symmetric monoidal model category, who presents the symmetric monoidal ∞-category of spectra that we're familiar with.

One useful fact about Day convolution is as follows:

Proposition 2.19. *The category $C\mathbf{Alg}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}), \circledast)$ of commutative algebra objects for Day convolution is equivalent to the category of lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.*

As an application of this, we could show that $\mathbb{S}$ is a commutative monoid in symmetric sequences directly by using the isomorphism $S^n \wedge S^m = S^{nm}$ and the twist map to give $\mathbb{S}$ a lax symmetric monoidal structure.

Day convolution in the ∞-categorical setting This section will be brief, and consist mostly of a statement:

Theorem 2.20 (Glasman, Lurie)**.** *Let $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ be $\mathcal{O}$-monoidal ∞-categories, and let κ be an uncountable regular cardinal such that the following are satisfied:*

(a) For each object $X \in \mathcal{O}$, the ∞-category $\mathcal{C}_X$ is essentially κ-small.
(b) For each object $Y \in \mathcal{O}$, the ∞-category $\mathcal{D}_Y$ admits κ-small colimits.
(c) For each operation $\theta \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}, Y)$, the associated tensor product functor $\prod_i \mathcal{D}_{X_i} \to \mathcal{D}_Y$ preserves κ-small colimits separately in each variable.

Then, there exists an $\mathcal{O}$-monoidal ∞-category $\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})^{\otimes}$ *satisfying the following conditions:*

1. *For each object $X \in \mathcal{O}$, there is a canonical equivalence of categories*

$$\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})_X \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}_X,\mathcal{D}_X).$$

2. *There is a canonical equivalence of categories*

$$\mathbf{Alg}_{/\mathcal{O}}\left(\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})\right) \xrightarrow{\sim} \mathbf{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$$

from $\mathcal{O}$-algebras in $\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})^{\otimes}$ *to lax $\mathcal{O}$-monoidal functors $\mathcal{C} \to \mathcal{D}$.*

Moreover, when $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}$, the underlying tensor product functor

$$\circledast : \mathrm{Fun}(\mathcal{C},\mathcal{D}) \times \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

is given by Day convolution.